Hypothesis tests
Lecture: 9

Hamid Raza
Assistant Professor in Economics
raza@business.aau.dk

Aalborg University

Statistics - Statistik

Outline

Concept of hypothesis

- We begin the hypothesis-testing procedure by considering a value for a population probability distribution parameter such as the mean, μ, the variance, s^2, or the proportion, P.
- We start with a hypothesis about the parameter - called the **null hypothesis** - that hold unless there is strong evidence against this null hypothesis.
- If we reject the null hypothesis, then the second hypothesis, named the **alternative hypothesis**, will be accepted.
- **Note:** if we fail to reject the null hypothesis, we cannot necessarily conclude that the null hypothesis is correct. It can also mean that the alternative hypothesis is correct, but our test procedure is not strong enough to reject the null hypothesis

Concept of hypothesis

Concept of hypothesis: Example

- We assume that students enrolled in Cand. Oecon study 175 hours (per subject) on average during a semester in Denmark, so our null hypothesis is defined as follows:
$$H_0 : \mu = 175$$
- The null hypothesis is that the population parameter μ is equal to a specific value of 175.
- In this example, a possible alternative hypothesis is that the mean of population fall in a range of values greater than 175 hours:
$$H_1 : \mu > 175$$
This is known as **one-sided right (or upper) tailed test**
- In this example, another possible alternative hypothesis is that the mean of population fall in a range of values less than 175 hours:
$$H_1 : \mu < 175$$
This is known as **one-sided left (or lower) tailed test**
- Another type of alternate hypothesis would be, **two-sided composite alternative hypothesis:**
$$H_1 : \mu \neq 175$$

Outline

One-sided test for mean: population variance known

One-sided test for mean: General procdeure when population variance is given

- We assume that data is a sample from population which is $N(\mu, \sigma^2)$
- The mean for a selected sample is: $\bar{x}$
- Assume the null hypothesis is: $H_0 : \mu = \mu_0$
- Alternative hypothesis can be: $H_1 : \mu < \mu_0$ (lower tailed)
- OR alternative hypothesis can be: $H_1 : \mu > \mu_0$ (upper tailed)
- z score is calculated as: $z = \frac{\bar{x} - \mu_0}{se}$, where $se = \frac{\sigma}{\sqrt{n}}$
- **Decision rule**
  - Reject H_0: if $z < -z_\alpha$ (for left (or lower) tailed test)
    or
  - Reject H_0: if $z > z_\alpha$ (for right (or upper) tailed test)
- **Decision rule based on p-values**
  - For lower tailed test, we get p-value= “lower tail probability in the normal distribution”.
  - For upper tailed test, we get p-value= “upper tail probability in the normal distribution”

One-sided test for mean: population variance known

One-sided test for mean: population variance given (EXAMPLE)

- We assume study hours of students in cand. oecon are normally distributed: $N(\mu, \sigma)$, with a population mean $\mu = 175$ and $\sigma = 5$. Assume I come up with a new strategy of providing an incentive for students so they can work harder (say giving them a free drink after the lecture).
- Assume by the end of the semester, I collect a sample of 20 students from my statistics class and find that the study hours for these students has a mean $\bar{x} = 173$. Using 0.05 significance level, I test whether giving free dinks to students has lowered their study hours as compared to the population?
- Assume the null hypothesis is: $H_0 : \mu = 175$
- Alternative hypothesis can be: $H_1 : \mu < 175$
- Since, population variance is known, we calculate z score:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{173 - 175}{5/\sqrt{20}} = -1.788$$

- The corresponding z-score is -1.788, so what is our conclusion?
- Think of the normal distribution, where would this value fall in a standard normal distribution?

One-sided test for mean: population variance given (EXAMPLE)

- **Decision Rule:**
  - The corresponding z-score is -1.788, and z_α = -1.645 is the corresponding lower-tail value for the standard normal at 0.05 level

```
alpha=0.05
qnorm(alpha) # note: for upper tailed test we write qnorm(1-alpha)

## [1] -1.645
```

  - Thus we reject the null hypothesis in the favour of alternative hypothesis because our z score is to the left of -1.645 (i.e., z $<$ -z$_\alpha$)
  - Do we reject the null hypothesis when $\alpha = 0.01$:

```
alpha=0.01
qnorm(alpha, mean = 0, sd = 1)

## [1] -2.326
```

  - Our z score (-1.788) is to the right of -2.326. Therefore, we cannot reject the null hypothesis if we set $\alpha = 0.01$ for decision making rule
  - To make our decision rule easier, we directly calculate the p-values:

```
pnorm(-1.788) # note for upper tailed test, we will write: 1 - pnorm(1.788)

## [1] 0.03689
```

    P value (3%) clearly tells us that we can reject the null hypothesis at 5% significance level but not at 1% significance level

- **Thought experiment: Imagine if the mean was 174.5 hours. In that case, would we be getting closer to (accepting) the null hypothesis?**

Two-sided test for mean: population variance known

Two-sided test for mean: General procdeure when population variance is given

- We assume that data is a sample from population which is $N(\mu, \sigma^2)$
- The mean for a selected sample is: $\bar{x}$
- Assume the null hypothesis is: $H_0 : \mu = \mu_0$
- Alternative hypothesis can be: $H_1 : \mu \neq \mu_0$
- z score is calculated as: $z = \frac{\bar{x} - \mu_0}{se}$, where $se = \frac{\sigma}{\sqrt{n}}$
- **Decision rule**
    - Reject H_0: if $z < -z_{\alpha/2}$
      or
    - Reject H_0: if $z > z_{\alpha/2}$
- **Decision rule based on p-values**
    - We calculate: p-value = 2 times "upper tail probability of |z| of normal dist."

Two-sided test for mean: population variance known

Two-sided test for mean: population variance known (Example)

- Again, we assume study hours for cand.oecon are normally distributed: $N(\mu, \sigma)$, with population mean $\mu = 175$ and $\sigma = 5$. Assume a sample of 20 students is selected. The mean of the sample $\bar{x} = 173$.
- Assume the null hypothesis is: $H_0 : \mu = 175$
- Alternative hypothesis can be: $H_1 : \mu \neq 175$
- Since, population variance is known, we calculate z score:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{173 - 175}{5/\sqrt{20}} = -1.788$$

- The corresponding z-score is -1.788, so what is our conclusion?

Two-sided test for mean: population variance known

Two-sided test for mean: when population variance is given (EXAMPLE)

- **Decision Rule:**
  - Our z-score is -1.788, and $z_{\alpha/2} =$ -1.96 is the corresponding value for the standard normal at $\alpha = 0.05$ level

```
alpha=0.05
qnorm(alpha/2)

## [1] -1.96
```

  - Thus we fail to reject the null hypothesis because our z score falls between ± 1.96
  - For a two sided test we get p values from a normal dist:

```
2*pnorm(-1.788)

## [1] 0.07378
```

You can see p-value (7%) is higher than 5% significance level, so we fail to reject the null hypothesis.

Outline

Hypothesis test for mean (population variance unknown)

- In the previous section, we worked with examples where population variance was given
- When population variance was given, we used standard normal distribution in order to make decisions whether acceptance or rejection range (we also choose significance level α for making a final decision)
- In most practical examples, population variance is not known but we can still use hypothesis testing
- **What do we do when population variance is not known? And what family of distributions do we use in that case?**

One-sided t-test for mean: population variance unknown

- We assume that data is a sample from population which is $N(\mu, \sigma^2)$
- The mean and standard deviation for the sample are: $\bar{x}$ and s based on n observations
- Assume the null hypothesis is, $H_0 : \mu = \mu_0$
- Alternative hypothesis can be, $H_1 : \mu > \mu_0$ or $(\mu < \mu_0)$
- Test statistic is calculated as: $t = \frac{\bar{x} - \mu_0}{se}$ where $se = \frac{s}{\sqrt{n}}$
  Note: now we use t statistics instead of z score AND replace σ with standard dev of sample s because we do not know σ)
- **Decision rule**
  - Reject H_0: if $t < -t_{n-1,\alpha}$ (for left (or lower) tailed test)
    or
  - , Reject H_0: if $t > t_{n-1,\alpha}$ (for right (or upper tailed test))
- **Decision rule based on p-values**
  - For lower tailed test, we get p-value= "lower tail probability in t dist. with df degrees of freedom".
  - For upper tailed test, we get p-value= "lower tail probability in t dist. with df degrees of freedom'

One-sided t-test for mean

One-sided t-test for mean: Example (read example no. 9.4 in Newbold)

Assume the sample size (n=134), the sample mean $\bar{x}$ is 3593 and the sample standard deviation s is 4919. Let's say, using $\alpha = 0.05$, we are supposed to test the following:

- null hypothesis: $H_0 : \mu = 2400$
- alternative hypothesis: $H_1 : \mu > 2400$ (this is a case of one sided upper tailed t test)
- Since we do not know the population standard deviation σ, we will rely on our sample standard deviation s and use a t statistics:

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{3593 - 2400}{4919/\sqrt{134}} = 2.81$$

- Our t statistics is 2.81, so what is our conclusion?

One-sided t-test for mean:

One-sided t-test for mean: (EXAMPLE)

- **Decision Rule:**
  - Our t statistics is 2.81, and $t_{n-1,\alpha} = 1.65$ is the corresponding upper-tail value for the t distribution with df=133 and $\alpha = 0.05$

```
alpha=0.05
qt(1-alpha, df=133)

## [1] 1.656
```

  - Thus we reject the null hypothesis in the favour of alternative hypothesis because our t statistics is greater than 1.65 (i.e., t $>$ t$_{n-1,\alpha}$)
  - Do we reject the null hypothesis when we set $\alpha = 0.01$:

```
alpha=0.01
qt(1-alpha, df=133)

## [1] 2.355
```

  - We still reject the null hypothesis
  - To make our decision rule easier, we directly calculate the p-values:

```
1-pt(2.81, df=133)

## [1] 0.002851
```

Two-sided t-test for mean: population variance unknown

Two-sided t-test for mean

- We assume that data is a sample from population which is $N(\mu, \sigma^2)$
- The mean and standard deviation for the sample are: $\bar{x}$ and s based on n observations
- Assume the null hypothesis is $H_0 : \mu = \mu_0$
- Alternative hypothesis can be $H_1 : \mu \neq \mu_0$
- t statistics is calculated as: $t = \frac{\bar{x} - \mu_0}{se}$, where $se = \frac{s}{\sqrt{n}}$
- **Decision rule**
  - Reject H_0: if $t < -t_{n-1,\alpha/2}$
    or
  - Reject H_0: if $t > t_{n-1,\alpha/2}$
- **Decision rule based on p-values**
  - We calculate p-value = 2 times "upper tail probability of |t|". The probability is calculated in the t-distribution with df degrees of freedom

two-sided t-test for mean: population variance unknown

two-sided t-test for mean: Example

Assume the sample size (n=134), the sample mean $\bar{x}$ is 3593 and the sample standard deviation s is 4919. Let's say, using $\alpha = 0.05$, we are supposed to test the following:

- null hypothesis: $H_0 : \mu = 2400$
- alternative hypothesis: $H_1 : \mu \neq 2400$ (this is a case of two sided t test)
- Since we do not know the population standard deviation σ, we will rely on our sample standard deviation s and use a t statistics:

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{3593 - 2400}{4919/\sqrt{134}} = 2.81$$

- Our t statistics is 2.81, so what is our conclusion?

Two-sided test for mean

two-sided t-test for mean: Example (cont)

- **Decision Rule:**
  - Our t statistics is 2.81, and $t_{n-1,\alpha/2} = 1.97$ is the corresponding value for the t distribution at $\alpha = 0.05$ level

```
alpha=0.05
qt(1-alpha/2, df=133)

## [1] 1.978
```

  - Thus we reject the null hypothesis because our t statistics is extreme than ± 1.97
  - For a two sided test we get p values from a t dist:

```
2*(1-pt(2.81, df=133))

## [1] 0.005703
```

You can see p-value (0.5%) is much lower than 5% significance level, so we reject the null hypothesis.

Outline

P value

P value

Getting p-value is the most popular procedure for considering the test of the null hypothesis in statisticis

The p-value is the probability of obtaining a value of the test statistic as extreme as or more extreme than the actual value obtained when the null hypothesis is true

p-value is the smallest significance level at which a null hypothesis can be rejected, given the observed sample statistic

- The smaller the p-value is, the less we trust H_0
- What is a small p-value? If it is below 5% we say it is significant at the 5% level

Significance level

Significance level

In practice it can be necessary to decide that at what p-value we are going to reject H_0

- The decision can be made if we have decided on a so-called α-level, known as the **significance level** of the test
- We reject H_0, if p-value is less than or equal to α
- We typically use 5% or 1% significance levels.

Outline

Hypothesis test for proportion

Hypothesis test for proportion

- Assume a random sample of n observations from a population that has a proportion P
- Let $\hat{p}$ be the sample proportion
- Null hypothesis: $H_0 : P = P_0$ where (P_0 is unknown)
- **One sided alternative hypothesis:** $H_1 : P < P_0$ OR $(P > P_0)$
- **Two sided alternative hypothesis:** $H_1 : P \neq P_0$
- z score is calculated as:

  $z = (\hat{p} - P_0)/se$, where $se = \sqrt{P_0(1 - P_0)/n}$
- **Decision rule**
  - Reject H_0: if $z < -z_{\alpha/2}$
  - OR Reject H_0: if $z > z_{\alpha/2}$

Hypothesis test for proportion

Hypothesis test for proportion

when $n\hat{p}(1-\hat{p})$ is larger than 5, we know that $\hat{p}$ follows a normal distribution (approximately). Therefore, our decision rules are as follows:

- **Decision rule (for one sided test)**
  - Reject H_0: if $z < -z_\alpha$ (for left (or lower) tailed test)
  - Reject H_0: if $z > z_\alpha$ (for right (or upper) tailed test)
- **Decision rule (for two sided test)**
  - Reject H_0: if $z < -z_{\alpha/2}$
    or
  - Reject H_0: if $z > z_{\alpha/2}$
- Note: the process of obtaining the p values is the same as discussed earlier (for z_α and $z_\alpha/2$)

Two-sided test for proportion

Two-sided test for proportion: Example

- In relation to the problem of financing public service, a random sample of 1200 individuals were asked whether they preferred welfare cuts or tax increases.

- 52% preferred tax increases. Do they represent a majority?
  - Sample with $n = 1200$ observations and estimated proportion $\hat{p} = 0.52$
  - Let's say we test the following null hypothesis: $H_0 : P = P_0 = 0.5$
  - Alternative hypothesis can be: $H_1 : P \neq 0.5$

- z score is calculated as: $z = \frac{(\hat{p} - P_0)}{se} = \frac{(0.52 - 0.5)}{se}$

  where $se = \sqrt{P_0(1 - P_0)/n} = \sqrt{(0.5)(0.5)/1200} = 0.014$

- Our z-score is 1.39, so what is our conclusion using significance level $\alpha = 0.05$?

Two-sided test for proportion

Two-sided test for proportion: Example (cont)

- **Decision Rule:**
  - Our z-score is 1.31, and $z_{\alpha/2} = 1.96$ is the corresponding value for the standard normal at $\alpha = 0.05$ level

```
alpha=0.05
qnorm(1- alpha/2)

## [1] 1.96
```

  - Thus we fail to reject the null hypothesis because our z score falls between ± 1.96
  - We can directly obtain p values for a two sided test:

```
2*(1-pnorm(1.31))

## [1] 0.1902
```

You can see p-value (19%) is higher than 5% significance level, so we fail to reject the null hypothesis.

Outline

Hypothesis test for variance

Hypothesis test for variance

we will develop procedures for testing the population variance, σ^2, based on the sample variance, s^2, computed using a random sample of n observations from a normally distributed population

- The null hypothesis is that the population variance is equal to some specified value, that is:
- Null hypothesis, $H_0 : \sigma^2 = \sigma_0^2$
- **One sided alternative hypothesis**, $H_1 : \sigma^2 < \sigma_0^2$ OR $\sigma^2 > \sigma_0^2$
- **Two sided alternative hypothesis**, $H_1 : \sigma^2 \neq \sigma_0^2$
- Test statistic (measures how big the measured variance is relative to the hypothesized value):

  $\chi^2_{n-1} = \frac{s^2(n-1)}{\sigma_0^2}$ has a chi-square distribution with $(n-1)$ degrees of freedom, where

  sample variance: $s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_i - \bar{x})^2$

Hypothesis test for variance

Hypothesis test for variance

- **Decision rule (for one sided test)**
  - Reject H_0: if $\chi^2_{n-1} > \chi^2_{n-1,\alpha}$ (upper tailed test)
    or
  - Reject H_0: if if $\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}$ (lower tailed test)
- **Decision rule (for two sided test)**
  - Reject H_0: if $\chi^2_{n-1} > \chi^2_{n-1,\alpha/2}$
    or
  - Reject H_0: if $\chi^2_{n-1} < \chi^2_{n-1,1-\alpha/2}$

Hypothesis test for variance

Hypothesis test for variance

- We have a sample with n $= 25$ observations and sample variance $s^2 = 0.8659$.
- The hypotheses:
  - $H_0 : \sigma^2 = 1$
  - $H_1 : \sigma^2 < 1$
- Test statistics:

$$\chi^2_{n-1} = \frac{(25-1)0.86}{1} = 20.78$$

- Using $\alpha = 0.05$, we calculate $\chi^2_{n-1,\alpha}$:

```
alpha=0.05
qchisq(alpha, df = 24)

## [1] 13.85
```

- You can also check this value in Table no. 7 in the appendix (check df $= 24$, α=0.05).
  - Unfortunately, these calculations work opposite in R and the way they are reported in statistical books, e.g., for lower tail, we directly write ($\alpha = 0.05$) in R, whereas in the book it can be obtained by looking at $\alpha = 0.95$
  - For upper tail we write ($1 - \alpha = 0.95$) in R, whereas in the book it can be obtained by looking at significance level $\alpha = 0.05$

Hypothesis test for variance

Hypothesis test for variance

- In our example, we cannot reject the null hypothesis because our calculated test statistics is NOT EXTREME than the critical value for a lower tailed test
- We can directly get p-value:

```
pchisq(20.78, df = 24)

## [1] 0.3483
```

- P value is 34%, which is much higher than our 5% significance level therefore H_0 cannot be reject

Outline

Exercise

1. We assume students study hours are normally distributed: $N(\mu, \sigma)$, with mean $\mu = 175$ and $\sigma = 5$. Assume a sample of 20 students is selected. The mean of the sample $\bar{x} = 176$.
   - Using 0.10, 0.05, and 0.01 significance level, test whether the mean of study hours is statistically higher from the population mean? (use one sided test).
   - Calculate the corresponding p values in this example.
   - Calculate the 95% confidence interval of mean
2. Suppose that we want to test the hypothesis with a significance level of .05 that the climate has changed since industrializatoin. Suppose that the mean temperature throughout history is 50 degrees. During the last 49 years, the mean temperature has been 51 degrees with a standard deviation of 2 degrees. What can we conclude?
3. Determine the lower and upper tail critical values of χ^2_{n-1} for each of the following:
   - $\alpha = 0.01, n = 26$
   - $\alpha = 0.05, n = 17$
   - $\alpha = 0.10, n = 14$
4. We have a sample with n = 25 observations and sample variance $s^2 = 5.5$. Using $\alpha = 0.05$, test the following:
   - $H_0 : \sigma^2 = 6$
   - $H_1 : \sigma^2 \neq 6$