Discrete Probability Distributions
Lecture: 3

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Outline

Random variables

Random Variable

A random variable is a variable that takes on numerical values realized by the outcomes in the sample space generated by a random experiment

Discrete Random variable

A random variable is a discrete random variable if it can take on no more than a countable number of values

Continous Random variable

A random variable is a continuous random variable if it can take any value in an interval, e.g., we are interested in the day's high temperature. The random variable, temperature, is measured on a continuum and so is said to be continuous

Other examples include:

- The yearly income for a family
- The change in the price of a share of IBM common stock in a month
- The amount of oil imported into the United States in a particular month

Outline

Probability distribution function for discrete random variables

Probability distribution function:

The probability that random variable X takes specific value x is denoted $P(X = x)$, for all values of x.

- X is used to denote the discrete random variable and the corresponding lowercase letter, x, to denote a possible value
- The term probability distribution is used to represent probability distribution in general
- This representation might be algebraic, graphical, or tabular

Example

We look at the sales of macbooks in computer shop in one day. The probability distribution of sales of macboobs is given by the following table:

x	0	1	2	3
P(x)	0.1	0.2	0.4	0.3

Probability distribution function for discrete variables

Example (Cont)

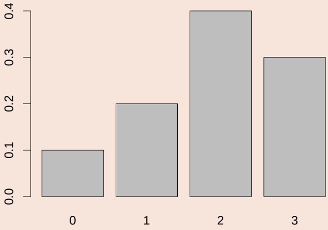
Discrete random variable $(X) = (x = 0, x = 1, x = 2, x = 3)$

Whereas the probability of X is; $P(X = 0.1, 0.2, 0.4, 0.3)$

The probability distribution function says that the probability of selling 0 macbooks is 10%, and so on.

We can also graphically show the probability distribution:

```
px = c(0.1, 0.2, 0.4, 0.3);        barplot(px, names.arg = c(0,1,2,3))
```



Required Properties of Probability Distribution for Discrete Random Variables

Let X be a discrete random variable with probability distribution $P(x)$. Then:

- 1. $0 \leq P(x) \leq 1$ for any value x, and
- 2. the individual probabilities sum to 1, that is,

$$\sum_x P(x) = 1$$

- Property 1 merely states that probabilities cannot be negative or exceed 1
- Property 2 means the sum of the probabilities for x must sum to 1

Cumulative probability disctribution

The cumulative probability distribution, $F(x_0)$, of a random variable X, represents the probability that X does not exceed the value x_0, as a function of x_0. That is,

$$F(x_0) = P(X \leq x_0)$$

Example:

- Bayern AutoGroup is a car dealer in Aalborg. Based on an analysis of its sales history, the managers know that on any single day the number of BMW cars sold can vary from 0 to 5. The probability distribution function of car sales is given below:

x	0	1	2	3	4	5
$P(x)$	0.15	0.3	0.2	0.2	0.1	0.05
$F(x)$	0.15	0.45	0.65	0.85	0.95	1

- The random variable, X, takes on the values of x indicated in the first column, and the probability distribution, $P(x)$, is defined in the second column.
- The third column contains the cumulative distribution, $F(x)$.

Outline

Properties of discrete random variables

Expected Value of a Discrete Random Variable:

The expected value, $E[X]$, of a discrete random variable X is defined as:

$$E[X] = \mu = \sum_x P(x)x$$

The expected value of a random variable is also called its mean and is denoted μ.

Variance and SD of a Discrete Random Variable:

Let X be a discrete random variable. The expectation of the squared deviations about the mean, $(X - \mu)$, is called the variance, denoted as σ^2 and given by:

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 P(x)$$

The variance of a discrete random variable X can also be expressed as:

$$\sigma^2 = E[X^2] - \mu^2 = \sum_x x^2 P(x) - \mu^2$$

The **standard deviation**, σ, is the positive square root of the variance

Properties of discrete random variables

Example:

Reconsider the example of Bayern AutoGroup car dealer in Aalborg:

x	0	1	2	3	4	5
$P(x)$	0.15	0.3	0.2	0.2	0.1	0.05
$F(x)$	0.15	0.45	0.65	0.85	0.95	1

What is the expected value and variance for this probability distribution?

Calculating the expected Mean:

$$\mu_x = E[X] = \sum_x P(x)x = 0(0.15) + 1(0.3) + 2(0.2) + 3(0.2) + 4(0.1) + 5(0.05) = 1.95$$

Calculating the variance:

$$\begin{aligned}\sigma_x^2 &= (0-1.95)^2(0.15) + (1-1.95)^2(0.3) + (2-1.95)^2(0.2) + (3-1.95)^2(0.2) + \\ &= (4-1.95)^2(0.1) + (5-1.95)^2(0.05) = 1.94\end{aligned}$$

Properties of discrete random variables

Expected Value of a Discrete Random Variable in R:

```
x = c(0, 1, 2, 3, 4, 5);           px = c(0.15, 0.3, 0.2, 0.2, 0.1, 0.05)
```

Now we can simply calculate the mean in R:

```
mean = sum(x * px)
mean

## [1] 1.95
```

Variance of Discrete Random Variable in R:

We can simply calculate the variance in R:

```
mean <- sum(x * px)
sum((x - mean)^2 * px)

## [1] 1.948
```

Properties of discrete random variables

Expected Value of Functions of Random Variables

Let X be a discrete random variable with probability distribution $P(x)$, and let $g(X)$ be some function of X. Then the expected value, $E[g(X)]$, of that function is defined as follows:

$$E[g(X)] = \sum_{x} g(x)P(x)$$

Summary of Properties for Linear Functions of a Random Variable

Consider a new random variable Y, defined by the following:

$$Y = a + bX$$

- Let X be a random variable with mean μ_x and variance σ_x^2
- let a and b be any constant fixed numbers.
- **Then, the mean of Y is:**

$$\mu_y = E[a + bX] = a + b\mu_x$$

- **And the variance of Y is:**

$$\sigma_y^2 = Var[a + bX] = b^2\sigma_x^2$$

Properties of discrete random variables

Example:

You are responsible for a construction project. Lets say the material for the project costs 25000 DKK and the labour costs 900 DKK per day, i.e, total cost:

$$C = 25000 + 900X$$

(note that cost is a linear function of X)

- where X is the no. of days to complete the project
- Assume you are given the probability of days (px) required to completing this kind of a project as follows:

x (no. of days to complete a project)	10	11	12	13	14
px	0.1	0.3	0.3	0.2	0.1

- Find the mean of X (no. of days to complete a project) and C (total cost)?
- Find the variance of X (no. of days to complete a project) and C (total cost)?

Properties of discrete random variables

Expected Value of Functions of Random Variables

Solution in R:

```
x = c(10, 11, 12, 13, 14);          px = c(0.1, 0.3, 0.3, 0.2, 0.1);          c = 25000 + 900*x
```

We can easily calculate the mean of X:

```
mean_x = sum(x * px)
mean_x

## [1] 11.9
```

We can also easily calculate the mean of total cost C; ($\mu_y = E[a + bX] = a + b\mu_x$)

```
mean_c = 25000 + 900 * mean_x
mean_c

## [1] 35710
```

We can easily calculate the variance of total cost C; ($\sigma_y^2 = Var[a + bX] = b^2\sigma_x^2$)

```
var_x = sum((x - mean_x)^2*px) # first calculate the variance of X
var_c = (900^2)*(var_x)
var_c

## [1] 1044900
```

Outline

Binomial Distribution

What is a binomial distribution?

The binomial distribution is a discrete probability distribution. It describes the outcome of n independent trials in an experiment. Each trial is assumed to have **only two outcomes, either success or failure**

- Let P denote the probability of success, and, the probability of failure $(1 - P)$.
- Assume a random variable X. If X takes the value 1 then the outcome of the experiment is success and 0 otherwise.

  The probability distribution of this random variable is then:

  $$P(1) = P \text{ and } P(0) = (1 - P)$$

  This distribution is known as the **Bernoulli distribution**

Binomial Distribution

For example, what is the probability of getting 1 head in 2 trials?

- First, we calculate the number of possible outcomes from this experiment:

$$[HH, HT, TH, TT]$$

- Now we look at the number of possible outcomes in which one head appears. Here, we have only 2 possible outcomes in which we get one head in 2 trials $[HT, TH]$.
- So the probability of getting 1 head in 2 trials is $2/4 = 0.5$

Example in R:

To compute the binomial probabilities in R, we use dbinom. The probability of getting 1 head in two coin tosses is:

```
x <- 1;                    dbinom(x, size = 2, prob = 0.5)

## [1] 0.5
```

The probabilities of getting 0, 1, 2 heads in two trials is:

```
x <- 0:2;                  dbinom(x, size = 2, prob = 0.5)

## [1] 0.25 0.50 0.25
```

Binomial Distribution

In the coin toss example involving the probability of one head in 2 trials, it was very easy to calculate the no. of heads. But what if we have many independent bernoulli trials, e.g., what is the probability of getting 9 heads in 50 trials?

We generalize the result for any combination of trials (n) and no. of success (x)

Number of Sequences with x Successes in n Trials:

The number of sequences with x successes in n independent trials is

$$C_x^n = \frac{n!}{x!(n-x)!}$$

This is read as "n choose x"

- For example, we want to calculate 2 heads in 5 bernoulli trials:

  $$C_x^n = \frac{n!}{x!(n-x)!} = C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5.4.3.2.1}{2.1(3.2.1)} = 10$$

  The event "x" successes (which is head in our example) resulting from ($n = 5$) trials can occur in 10 mutually exclusive ways
- In R, we can simply type:

```
choose(5, 2)

## [1] 10
```

Binomial Distribution

Binomial distribution

Suppose that a random experiment can result in two possible mutually exclusive and collectively exhaustive outcomes:

1. "success"
2. "failure"

P is the probability of a success in a single trial.

If n independent trials are carried out, the distribution of the number of resulting successes, x, is called the **binomial distribution**.

Probability distribution function for Binomial variable

The probability distribution function for the binomial random variable $X = x$ is as follows:

$$P(x) = \frac{n!}{x!(n-x)!} P^x (1-p)^{n-x}$$

Binomial Distribution

Binomial distribution: Example

Suppose that a property dealer, John, has 5 contacts, and he believes that for each contact the probability of making a sale is 0.40.

- a. Find the probability that he makes 1 sale
- b. What is the probability that he makes at most 1 sale
- c. Find the probability that he makes between 2 and 4 sales (inclusive).
- d. Graph the probability distribution function.

Solution (a): In this example, $n = 5$, $x = 1$ is the success (i.e., making one sale)
The probabiliy of making 1 sale is:

$$P(x) = C_x^n P^x (1 - P)^{n-x} = \frac{5!}{1!(5-1)!} 0.4^1 (1 - 0.4)^{5-1} = 0.25$$

a. We can very easily compute this in R:

```
dbinom(1, size = 5, prob = 0.4)
```

```
## [1] 0.2592
```

Binomial Distribution

Binomial distribution: Example

b. We can calculate the probability that he makes at most 1 sale by adding the probabilities of 0 and 1 sale.

```
dbinom(0, size = 5, prob = 0.4) + dbinom(1, size = 5, prob = 0.4)

## [1] 0.337
```

c. We can calculate the probabilities that he makes between 2 and 4 sales (inclusive).

```
p2 = dbinom(2, size = 5, prob = 0.4)
p3 = dbinom(3, size = 5, prob = 0.4)
p4 = dbinom(4, size = 5, prob = 0.4)
p2 + p3 + p4

## [1] 0.6528
```

Note: we can calculate all the probabilities

```
dbinom(0:5, size = 5, prob = 0.4)

## [1] 0.07776 0.25920 0.34560 0.23040 0.07680 0.01024
```

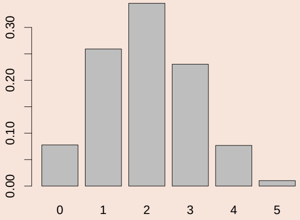
Binomial Distribution

Binomial distribution: Plots in R

We can also plot the probability distribution in this example:

```
x = dbinom(0:5, size = 5, prob = 0.4)
```

```
barplot(x, names.arg = c(0,1,2,3,4,5))
```



Binomial Distribution

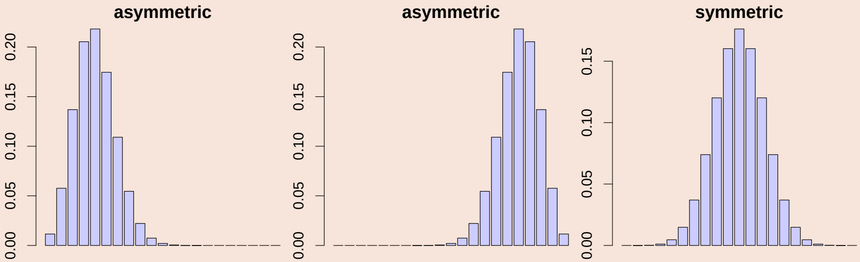
Binomial distribution: Plots in R

When $p \to 0.5$, the distribution becomes symmetric.

Lets try a few plots in R (note the probabilities):

```
x <- dbinom(0:20, size = 20, prob = 0.2)
y <- dbinom(0:20, size = 20, prob = 0.8)
z <- dbinom(0:20, size = 20, prob = 0.5);
```

```
barplot(x, col = rgb(0.8, 0.8, 1), main = "asymmetric")
barplot(y, col = rgb(0.8, 0.8, 1), main = "asymmetric")
barplot(z, col = rgb(0.8, 0.8, 1), main = "symmetric")
```



Binomial Distribution

Cummulative binomial distributions in R

We can also very easily calculate cummulative binomial distributions in R:

In the earlier, example when $n = 5$, and probability of making a sale is 0.4, we compute binomial distribution as well as cummulative binomial distribution

```
x <- dbinom(0:5, size = 5, prob = 0.4)
x

## [1] 0.07776 0.25920 0.34560 0.23040 0.07680 0.01024
```

Now we calculate cummulative binomial distribution:

```
x <- pbinom(0:5, size = 5, prob = 0.4)
x

## [1] 0.07776 0.33696 0.68256 0.91296 0.98976 1.00000
```

Note: that the probabilities sum to 1

Binomial Distribution

Mean and Variance of binomial distribution:

As binomial probability is a discrete probability distribution, the same properties of expected value and variance are applied.

Mean and Variance:

$$E[X] = \mu_x = nP$$

and Variance is:

$$\sigma_x^2 = E[(X - \mu_x)^2] = nP(1 - P)$$

For example, we want to calculate the expected value of heads (mean) in 2 coin tosses

```
x <- 0:2
px = dbinom(x, size = 2, prob = 0.5)
sum(x * px)

## [1] 1
```

Outline

Exercise

Exercise:

- Calculate the probability of 13 heads in 30 independent bernoulli trials
- Calculate the binomial distribution of the above experiment
- Calculate the cummulative distribution of the above experiment
- Calculate the mean and variance of heads in 30 bernoulli trials