GSM Analysis of TE_{11p} Mode Cylindrical Cavity

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1 Introduction

The purpose of these notes is to document a method to rapidly and accurately compute transmission amplitude for a resonant cavity like that shown in Fig. 1. At one end of the cavity, following a transition from rectangular to circular guide, is a thin annular, metal spacer (shown in gray in the figure). It has the effect of slightly reducing the inner radius of the circular waveguide comprising the body of the cavity. This region of reduced radius is labeled "CWG I" in part (b) of the figure. Following the spacer, in the slightly larger radius guide, is a thin disk of dielectric material (shown in yellow in part (a) of the figure), followed by a region of empty guide ("CWG II") terminated with a final transition to rectangular guide.

The structure will be analyzed by cascading the generalized scattering matrices (GSMs) that characterize each discontinuity and each region of uniform waveguide. For most of the structure, it will be sufficient to track only a single waveguide mode, the TE_{11} mode. But for the radius step followed by the thin dielectric disk, a full multi-modal GSM analysis will be used. This is illustrated in Fig. 2.

2 Theory

2.1 Mode Matching at a Junction of Two Circular Waveguides

In this section we restrict consideration to the junction of a pair of semi-infinite circular waveguides, which may be filled with different materials. The mode match-

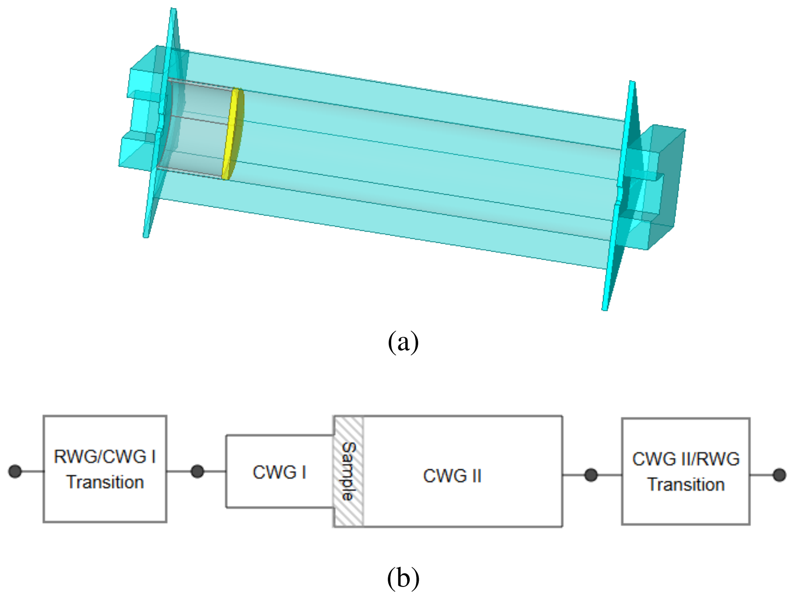

Figure 1: The structure to be analyzed: (a) Cutaway view of 3D model. (b) Schematic.

ing formulation developed here most closely follows the notation of James [1]. We assume that the structure is excited at either end by an incident field consisting of a single TE_{11} mode, so that only $m = 1$ modes as defined in Appendix A can exist at any location in the structure. For these modes, the azimuthal variation is limited to be proportional to either $\cos\phi$ or $\sin\phi$.

In the interior of either uniform CWG segment, the entire (finite) set of TE and TM $m = 1$ modes under consideration is sorted into order of increasing cutoff frequency and the modes in this set are indexed using a single natural number, say, m. The modes are orthogonal and normalized in accordance with (31).

The two uniform CWGs are concentric with the z-axis, of radii R_{I} and $R_{\mathrm{II}} \geq R_{\mathrm{I}}$. The junction occurs at the $z = 0$ plane and the waveguide with smaller (larger) radius occupies the half-space $z < 0$ ($z > 0$).

Near the junction the transverse fields in Region I ($z < 0$) are written as a sum of the modes of waveguide I:

$$\boldsymbol{E}^{\mathrm{I}}(\rho, \phi, z) = \sum_{m=1}^{M_{\mathrm{I}}} (a_m^{\mathrm{I}} e^{-\gamma_m^{\mathrm{I}} z} + b_m^{\mathrm{I}} e^{\gamma_m^{\mathrm{I}} z}) \boldsymbol{e}_m^{\mathrm{I}}(\rho, \phi) \tag{1a}$$

$$\boldsymbol{H}^{\mathrm{I}}(\rho, \phi, z) = \sum_{m=1}^{M_{\mathrm{I}}} (a_m^{\mathrm{I}} e^{-\gamma_m^{\mathrm{I}} z} - b_m^{\mathrm{I}} e^{\gamma_m^{\mathrm{I}} z}) \boldsymbol{h}_m^{\mathrm{I}}(\rho, \phi). \tag{1b}$$

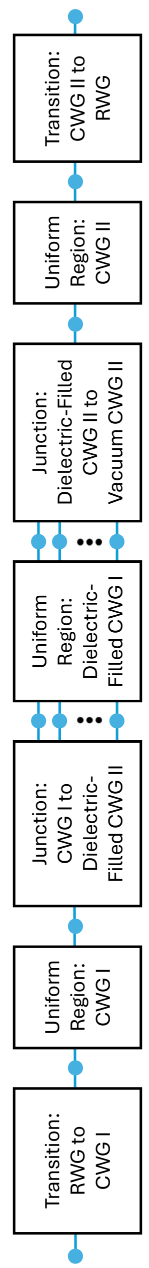

Figure 2: Diagram illustrating each GSM network to be cascaded. Single-mode 2×2 scattering matrices are shown as 2-port networks.

For Region II ($z > 0$), we again write the fields in terms of incident and reflected traveling waves:

$$\boldsymbol{E}^{\mathrm{II}}(\rho, \phi, z) = \sum_{m=1}^{M_{\mathrm{II}}} (a_m^{\mathrm{II}} e^{\gamma_m^{\mathrm{II}} z} + b_m^{\mathrm{II}} e^{-\gamma_m^{\mathrm{II}} z}) \boldsymbol{e}_m^{\mathrm{II}}(\rho, \phi), \tag{2a}$$

$$\boldsymbol{H}^{\mathrm{II}}(\rho, \phi, z) = \sum_{m=1}^{M_{\mathrm{II}}} (-a_m^{\mathrm{II}} e^{\gamma_m^{\mathrm{II}} z} + b_m^{\mathrm{II}} e^{\gamma_m^{\mathrm{II}} z}) \boldsymbol{h}_m^{\mathrm{II}}(\rho, \phi). \tag{2b}$$

To find the scattering matrix of the junction, we enforce the equality of the tangential electric and magnetic fields in a weighted average sense, using the modes of the larger guide as testing or weighting functions. First the continuity of tangential electric field is tested. In the following, the disk of radius R is denoted as C_R.

$$\begin{aligned} \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{E}^{\mathrm{I}}(x, y, 0) \times \boldsymbol{h}_m^{\mathrm{II}}(\rho, \phi) \cdot \hat{\boldsymbol{z}}\, dS &= \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{E}^{\mathrm{II}}(x, y, 0) \times \boldsymbol{h}_m^{\mathrm{II}}(\rho, \phi) \cdot \hat{\boldsymbol{z}}\, dS \\ &= \iint_{C_{R_{\mathrm{II}}}} \boldsymbol{E}^{\mathrm{II}}(x, y, 0) \times \boldsymbol{h}_m^{\mathrm{II}}(\rho, \phi) \cdot \hat{\boldsymbol{z}}\, dS. \end{aligned} \tag{3}$$

The final equality in (3) above follows from the fact that the tangential electric field $\boldsymbol{E}^{\mathrm{II}}(x, y, 0)$ is zero over the region $C_{R_{\mathrm{II}}} - C_{R_{\mathrm{I}}}$ so that the integral can be extended over the larger disk. Inserting (1) and (2) into (3) and using the orthogonality of the modes yields

$$\sum_{n=1}^{M_{\mathrm{I}}} \left(a_n^{\mathrm{I}} + b_n^{\mathrm{I}}\right) P_{mn} = a_m^{\mathrm{II}} + b_m^{\mathrm{II}}, \quad m = 1, 2, \ldots, M_{\mathrm{II}}, \tag{4}$$

where

$$P_{mn} = \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{e}_n^{\mathrm{I}} \times \boldsymbol{h}_m^{\mathrm{II}} \cdot \hat{\boldsymbol{z}}\, dS. \tag{5}$$

Next, we enforce continuity of tangential magnetic field over the common aperture:

$$\iint_{C_{R_{\mathrm{I}}}} \boldsymbol{e}_m^{\mathrm{I}} \times \boldsymbol{H}^{\mathrm{I}}(x, y, 0) \cdot \hat{\boldsymbol{z}}\, dS = \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{e}_m^{\mathrm{I}} \times \boldsymbol{H}^{\mathrm{II}}(x, y, 0) \cdot \hat{\boldsymbol{z}}\, dS \tag{6}$$

which, upon expanding the fields into modal series and invoking orthogonality, yields

$$a_m^{\mathrm{I}} - b_m^{\mathrm{I}} = \sum_{n=1}^{M_{\mathrm{II}}} \left(b_n^{\mathrm{II}} - a_n^{\mathrm{II}}\right) P_{nm}, \quad m = 1, 2, \ldots, M_{\mathrm{I}}. \tag{7}$$

Equations (4) and (7) can be written in matrix form as follows:

$$\boldsymbol{P}(\boldsymbol{a}^{\mathrm{I}} + \boldsymbol{b}^{\mathrm{I}}) = \boldsymbol{a}^{\mathrm{II}} + \boldsymbol{b}^{\mathrm{II}} \tag{8}$$

$$\boldsymbol{a}^{\mathrm{I}} - \boldsymbol{b}^{\mathrm{I}} = \boldsymbol{P}^{\top}(\boldsymbol{b}^{\mathrm{II}} - \boldsymbol{a}^{\mathrm{II}}), \tag{9}$$

where $\boldsymbol{P}^{\top}$ is the transpose of $\boldsymbol{P}$. Multiplying (9) by $\boldsymbol{P}$ and adding the result to (8) eliminates $\boldsymbol{b}^{\mathrm{I}}$ and allows one to solve for $\boldsymbol{b}^{\mathrm{II}}$ in terms of $\boldsymbol{a}^{\mathrm{I}}$ and $\boldsymbol{a}^{\mathrm{II}}$. Similarly, multiplying (8) by $\boldsymbol{P}^{\top}$ and subtracting the result from (9) eliminates $\boldsymbol{b}^{\mathrm{II}}$ and allows one to solve for $\boldsymbol{b}^{\mathrm{I}}$ in terms of $\boldsymbol{a}^{\mathrm{I}}$ and $\boldsymbol{a}^{\mathrm{II}}$. After performing these operations, one can write the generalized[1] scattering relation of the junction as

$$\begin{bmatrix} \boldsymbol{b}^{\mathrm{I}} \\ \boldsymbol{b}^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}^{\mathrm{I,I}} & \boldsymbol{S}^{\mathrm{I,II}} \\ \boldsymbol{S}^{\mathrm{II,I}} & \boldsymbol{S}^{\mathrm{II,II}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}^{\mathrm{I}} \\ \boldsymbol{a}^{\mathrm{II}} \end{bmatrix} \tag{10}$$

where

$$\boldsymbol{S}^{\mathrm{I,I}} = (\boldsymbol{I} + \boldsymbol{P}^{\top}\boldsymbol{P})^{-1}(\boldsymbol{I} - \boldsymbol{P}^{\top}\boldsymbol{P}), \tag{11a}$$

$$\boldsymbol{S}^{\mathrm{I,II}} = 2(\boldsymbol{I} + \boldsymbol{P}^{\top}\boldsymbol{P})^{-1}\boldsymbol{P}^{\top}, \tag{11b}$$

$$\boldsymbol{S}^{\mathrm{II,I}} = 2(\boldsymbol{I} + \boldsymbol{P}\boldsymbol{P}^{\top})^{-1}\boldsymbol{P}, \tag{11c}$$

$$\boldsymbol{S}^{\mathrm{II,II}} = -(\boldsymbol{I} + \boldsymbol{P}\boldsymbol{P}^{\top})^{-1}(\boldsymbol{I} - \boldsymbol{P}\boldsymbol{P}^{\top}), \tag{11d}$$

and $\boldsymbol{I}$ is the identity matrix. Since the scattering matrix of a waveguide junction is symmetric, we must have that $\boldsymbol{S}^{\mathrm{I,I}}$ and $\boldsymbol{S}^{\mathrm{II,II}}$ are symmetric matrices, and that $\boldsymbol{S}^{\mathrm{II,I}}$ is the transpose of $\boldsymbol{S}^{\mathrm{I,II}}$.

2.1.1 Efficient Evaluation of the Mode Coupling Coefficients

The scattering matrices in (11) will be required at many different frequencies. It is possible to compute a universal set of coupling coefficients that can be used for any frequency. The procedure for doing so is outlined in this section.

We begin with (5), writing the integrand in terms of the modal electric fields:

$$P_{mn} = \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{e}_n^{\mathrm{I}} \times \boldsymbol{h}_m^{\mathrm{II}} \cdot \hat{\boldsymbol{z}}\, dS = Y_m^{\mathrm{II}} \iint_{C_{R_{\mathrm{I}}}} \boldsymbol{e}_n^{\mathrm{I}}(\rho, \phi) \cdot \boldsymbol{e}_m^{\mathrm{II}}(\rho, \phi)\, dS. \tag{12}$$

We now define the unit radius, frequency-independent modal fields $\boldsymbol{\varepsilon}_{mn}^{\mathrm{TE}}$ and $\boldsymbol{\varepsilon}_{mn}^{\mathrm{TM}}$ to be the mnth TE or TM modal electric field that would exist in a unit radius waveguide, divided by the square root of the modal impedance. From the formulas

[1] "Generalized" because it also includes scattering coefficients for cut-off modes.

of Appendix A we see that the explicit forms of the frequency-independent, unit-radius modes under consideration (those with azimuthal index m fixed at 1) are:

$$\boldsymbol{\varepsilon}_{1n}^{\mathrm{TE}}(\rho,\phi) = \frac{\sqrt{2}}{J_1(\chi'_{1n})\sqrt{\pi\left(\chi'^2_{1n}-1\right)}}\left[\hat{\boldsymbol{\phi}} J'_1(\chi'_{1n}\rho)\chi'_{1n}\sin\phi - \hat{\boldsymbol{\rho}}\frac{J_1(\chi'_{1n}\rho)}{\rho}\cos\phi\right], \tag{13a}$$

$$\boldsymbol{\varepsilon}_{1n}^{\mathrm{TM}}(\rho,\phi) = \sqrt{\frac{2}{\pi}}\frac{1}{J_2(\chi_{1n})}\left[\hat{\boldsymbol{\phi}}\frac{J_1(\chi_{1n}\rho)}{\chi_{1n}\rho}\sin\phi - \hat{\boldsymbol{\rho}} J'_1(\chi_{1n}\rho)\cos\phi\right]. \tag{13b}$$

Continuing with our convention that a single modal subscript without accompanying superscript denotes a particular choice of azimuthal and radial mode indices and mode type (TE or TM), Eq. (12) can be written in terms of $\boldsymbol{\varepsilon}_m$ and $\boldsymbol{\varepsilon}_n$ as

$$\begin{aligned} P_{mn} &= Y_m^{\mathrm{II}} \iint_{C_{R_{\mathrm{I}}}} \frac{\sqrt{Z_n^{\mathrm{I}}}}{R_{\mathrm{I}}}\boldsymbol{\varepsilon}_n(\rho/R_{\mathrm{I}},\phi)\cdot\frac{\sqrt{Z_m^{\mathrm{II}}}}{R_{\mathrm{II}}}\boldsymbol{\varepsilon}_m(\rho/R_{\mathrm{II}},\phi)\,dS \\ &= \frac{\sqrt{Z_n^{\mathrm{I}}}}{R_{\mathrm{I}}R_{\mathrm{II}}\sqrt{Z_m^{\mathrm{II}}}}\int_0^{2\pi}\int_0^{R_{\mathrm{I}}}\boldsymbol{\varepsilon}_n(\rho/R_{\mathrm{I}},\phi)\cdot\boldsymbol{\varepsilon}_m(\rho/R_{\mathrm{II}},\phi)\,\rho\,d\rho\,d\phi \end{aligned} \tag{14}$$

Note that for complex numbers Z_1 and Z_2, $\sqrt{Z_1 Z_2} \neq \sqrt{Z_1}\sqrt{Z_2}$ and $\sqrt{Z_1/Z_2} \neq \sqrt{Z_1}/\sqrt{Z_2}$ in general. Thus the radicals in the above equation must not be combined. We now make the change of variables $\rho' = \rho/R_{\mathrm{I}}$, so that $dS = \rho\,d\rho\,d\phi = R_{\mathrm{I}}^2\rho'\,d\rho'\,d\phi$, and the region of integration transforms to the unit disk C_1. If we define $t \equiv R_{\mathrm{II}}/R_{\mathrm{I}}$ we can write

$$P_{mn} = \frac{\sqrt{Z_n^{\mathrm{I}}}}{\sqrt{Z_m^{\mathrm{II}}}}\frac{1}{t}\iint_{C_1}\boldsymbol{\varepsilon}_n(\rho,\phi)\cdot\boldsymbol{\varepsilon}_m\left(\rho/t,\phi\right)\,dS = \frac{\sqrt{Z_n^{\mathrm{I}}}}{\sqrt{Z_m^{\mathrm{II}}}}\kappa_{mn}(t), \tag{15}$$

where the coupling coefficient defined as

$$\kappa_{mn}(t) \equiv \frac{1}{t}\iint_{C_1}\boldsymbol{\varepsilon}_n(\rho,\phi)\cdot\boldsymbol{\varepsilon}_m\left(\rho/t,\phi\right)\,dS, \qquad t \geq 1 \tag{16}$$

is seen to be independent of frequency, and a function only of the radius ratio $t = R_{\mathrm{II}}/R_{\mathrm{I}}$. Therefore, for a particular step ratio these coefficients can be precomputed once only, for all necessary m and n indices. Of course, for $t = 1$ we have the closed-form result

$$\kappa_{mn}(1) = \delta_{mn}. \tag{17}$$

2.1.2 Explicit Formulas for the κ Coefficients

In the remaining formulas of this section, we employ the convention that the subscripts q and q' on the left-hand sides of the equations are mode enumeration indices with corresponding radial mode indices $n = n(q)$ and $n' = n'(q')$. All modes under consideration have the azimuthal index fixed at unity. We therefore omit this index from the numbers $\chi_n \equiv \chi_{1n}$ and $\chi'_n \equiv \chi'_{1n}$. The mode type (TE or TM) will be stated explicitly. For example, the notation $\kappa^{\mathrm{TETM}}_{qq'}$ means that mode q is TE and mode q' is TM. The following identity from [2] is employed in this section:

$$
\begin{aligned}
I(a,b) &\equiv \int_0^1 \left[J_1'(ax)J_1'(bx) + \frac{1}{abx^2} J_1(ax)J_1(bx) \right] x\,dx \\
&= \begin{cases} \dfrac{bJ_1'(a)J_1(b) - aJ_1(a)J_1'(b)}{b^2-a^2}, & (a \neq b) \\ \dfrac{aJ_1(a)J_1'(a) + a^2 J'^2(a) - a^2 J_1(a)J_1''(a)}{2a^2} & (a = b) \end{cases} \\
&= \begin{cases} \dfrac{aJ_1(a)[J_1(b)/b - J_0(b)] - bJ_1(b)[J_1(a)/a - J_0(a)]}{b^2-a^2}, & (a \neq b) \\ \dfrac{(a^2-2)J_1^2(a) + a^2 J_0^2(a)}{2a^2}. & (a = b). \end{cases}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\kappa^{\mathrm{TETE}}_{qq'}(t) &= \frac{1}{t}\int_0^{2\pi}\int_0^1 \boldsymbol{\varepsilon}^{\mathrm{TE}}_{n'}(\rho,\phi)\cdot\boldsymbol{\varepsilon}^{\mathrm{TE}}_{n}(\rho/t,\phi)\,\rho\,d\rho\,d\phi \\
&= \frac{2\chi'_n\chi'_{n'}}{tJ_1(\chi'_n)J_1(\chi'_{n'})\sqrt{\chi'^2_n - 1}\sqrt{\chi'^2_{n'}-1}} \times \\
&\qquad \int_0^1 \left\{ J_1'(\chi'_{n'}\rho)J_1'(\chi'_n\rho/t) + \frac{J_1(\chi'_{n'}\rho)J_1(\chi'_n\rho/t)}{\rho^2\chi'_{n'}\chi'_n/t} \right\} \rho\,d\rho \\
&= \frac{2\chi'_n\chi'_{n'}I(\chi'_{n'},\chi'_n/t)}{tJ_1(\chi'_n)J_1(\chi'_{n'})\sqrt{\chi'^2_n - 1}\sqrt{\chi'^2_{n'}-1}},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\kappa^{\mathrm{TETM}}_{qq'}(t) &= \frac{1}{t}\int_0^{2\pi}\int_0^1 \boldsymbol{\varepsilon}^{\mathrm{TM}}_{n'}(\rho,\phi)\cdot\boldsymbol{\varepsilon}^{\mathrm{TE}}_{n}(\rho/t,\phi)\,\rho\,d\rho\,d\phi \\
&= \frac{2}{J_1(\chi'_n)J_2(\chi_{n'})\chi_{n'}\sqrt{\chi'^2_n - 1}} \times
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left[\chi_{n'} J_1'(\chi_{n'}\rho) J_1\big(\frac{\chi_n'\rho}{t}\big) + \frac{\chi_n'}{t} J_1(\chi_{n'}\rho) J_1'\big(\frac{\chi_n'\rho}{t}\big)\right] d\rho \\
&= \frac{2}{J_1(\chi_n') J_2(\chi_{n'}) \chi_{n'} \sqrt{\chi_n'^2 - 1}} \int_0^1 \frac{\partial}{\partial\rho}\left[J_1(\chi_{n'}\rho) J_1\big(\frac{\chi_n'\rho}{t}\big)\right] d\rho \\
&= \frac{2}{J_1(\chi_n') J_2(\chi_{n'}) \chi_{n'} \sqrt{\chi_n'^2 - 1}} \left[J_1(\chi_{n'}\rho) J_1\big(\frac{\chi_n'\rho}{t}\big)\right]\Bigg|_0^1 = 0
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\kappa_{qq'}^{\mathrm{TMTE}}(t) &= \frac{1}{t}\int_0^{2\pi}\int_0^1 \boldsymbol{\varepsilon}_{n'}^{\mathrm{TE}}(\rho,\phi)\cdot\boldsymbol{\varepsilon}_n^{\mathrm{TM}}(\rho/t,\phi)\,\rho\,d\rho\,d\phi \\
&= \frac{2}{\chi_n\sqrt{\chi_{n'}'^2 - 1} J_2(\chi_n) J_1(\chi_{n'}')} \times \\
&\quad \int_0^1 \left[\frac{\chi_n}{t} J_1'\big(\frac{\chi_n\rho}{t}\big) J_1(\chi_{n'}'\rho) + \chi_{n'}' J_1'(\chi'\rho)_{n'}) J_1\big(\frac{\chi_n\rho}{t}\big)\right] d\rho \\
&= \frac{2}{\chi_n\sqrt{\chi_{n'}'^2 - 1} J_2(\chi_n) J_1(\chi_{n'}')} \left[J_1\big(\frac{\chi_n\rho}{t}\big) J_1(\chi_{n'}'\rho)\right]\Bigg|_{\rho=0}^{\rho=1} \\
&= \frac{2J_1(\chi_n/t) J_1(\chi_{n'}')}{\chi_n\sqrt{\chi_{n'}'^2 - 1} J_2(\chi_n) J_1(\chi_{n'}')}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\kappa_{qq'}^{\mathrm{TMTM}}(t) &= \frac{1}{t}\int_0^{2\pi}\int_0^1 \boldsymbol{\varepsilon}_{n'}^{\mathrm{TM}}(\rho,\phi)\cdot\boldsymbol{\varepsilon}_n^{\mathrm{TM}}(\rho/t,\phi)\,\rho\,d\rho\,d\phi \\
&= \frac{2}{tJ_2(\chi_n)J_2(\chi_{n'})}\int_0^1 \left[J_1'\big(\frac{\chi_n\rho}{t}\big) J_1'(\chi_{n'}\rho) + \frac{tJ_1\big(\frac{x\chi_n}{t}\big) J_1(\chi_{n'}\rho)}{\rho^2\chi_n\chi_{n'}}\right]\rho\,d\rho \\
&= \frac{2I(\chi_{n'},\chi_n/t)}{tJ_2(\chi_n)J_2(\chi_{n'})}.
\end{aligned}
\tag{22}
$$

2.1.3 GSM of a Uniform Section of Waveguide

The generalized scattering matrix of a uniform section of waveguide of radius R and length L is the matrix

$$
\begin{bmatrix} \mathbf{0} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma} & \mathbf{0} \end{bmatrix}
\tag{23}
$$

where all blocks are square matrices of dimension M (the total number of modes considered in the guide), and

$$
\boldsymbol{\gamma} = \mathrm{diag}(e^{-\gamma_1 L}, e^{-\gamma_2 L}, \ldots, e^{-\gamma_M L}).
\tag{24}
$$

2.1.4 GSM of a Cascade

Here we present the formulas for the GSM $\boldsymbol{S}$ of an interconnection of two waveguide structures whose individual GSMs are known. The number of modes used in Region II of the first (leftmost) structure must equal the number of modes used in Region I of the second (rightmost) structure, since these are in fact the same region. Let the GSM of the first structure be $\boldsymbol{A}$ and that of the second be $\boldsymbol{B}$. The formula for the interconnection is well known [3–5] and is reproduced here for convenience:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}^{\mathrm{I,I}} & \boldsymbol{S}^{\mathrm{I,II}} \\ \boldsymbol{S}^{\mathrm{II,I}} & \boldsymbol{S}^{\mathrm{II,II}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{\mathrm{I,I}} + \boldsymbol{A}^{\mathrm{I,II}}\boldsymbol{B}^{\mathrm{I,I}}\boldsymbol{G}^{AB}\boldsymbol{A}^{\mathrm{II,I}} & \boldsymbol{A}^{\mathrm{I,II}}\boldsymbol{G}^{BA}\boldsymbol{B}^{\mathrm{I,II}} \\ \boldsymbol{B}^{\mathrm{II,I}}\boldsymbol{G}^{AB}\boldsymbol{A}^{\mathrm{II,I}} & \boldsymbol{B}^{\mathrm{II,II}} + \boldsymbol{B}^{\mathrm{II,I}}\boldsymbol{A}^{\mathrm{II,II}}\boldsymbol{G}^{BA}\boldsymbol{B}^{\mathrm{I,II}} \end{bmatrix}, \tag{25}$$

where

$$\boldsymbol{G}^{AB} \equiv (\boldsymbol{I} - \boldsymbol{A}^{\mathrm{II,II}}\boldsymbol{B}^{\mathrm{I,I}})^{-1}, \qquad \boldsymbol{G}^{BA} \equiv (\boldsymbol{I} - \boldsymbol{B}^{\mathrm{I,I}}\boldsymbol{A}^{\mathrm{II,II}})^{-1}. \tag{26}$$

Device A is a Uniform Waveguide In the case where device A is just a section of waveguide of length L, we have

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma} & \boldsymbol{0} \end{bmatrix} \tag{27}$$

where $\boldsymbol{\gamma}$ is defined in Equation (24). $\boldsymbol{G}^{AB}$ and $\boldsymbol{G}^{BA}$ both reduce to the identity matrix, and the formula given in (25) for the composite scattering matrix simplifies to

$$\begin{bmatrix} \boldsymbol{S}^{\mathrm{I,I}} & \boldsymbol{S}^{\mathrm{I,II}} \\ \boldsymbol{S}^{\mathrm{II,I}} & \boldsymbol{S}^{\mathrm{II,II}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\gamma}\boldsymbol{B}^{\mathrm{I,I}}\boldsymbol{\gamma} & \boldsymbol{\gamma}\boldsymbol{B}^{\mathrm{I,II}} \\ \boldsymbol{B}^{\mathrm{II,I}}\boldsymbol{\gamma} & \boldsymbol{B}^{\mathrm{II,II}}. \end{bmatrix} \tag{28}$$

Device B is a Uniform Waveguide In the case where device B is just a uniform guide of length L, we have

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma} & \boldsymbol{0} \end{bmatrix}, \tag{29}$$

$\boldsymbol{G}^{AB}$ and $\boldsymbol{G}^{BA}$ both reduce to the identity matrix, and the formula given in (25) for the composite scattering matrix simplifies to

$$\begin{bmatrix} \boldsymbol{S}^{\mathrm{I,I}} & \boldsymbol{S}^{\mathrm{I,II}} \\ \boldsymbol{S}^{\mathrm{II,I}} & \boldsymbol{S}^{\mathrm{II,II}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{\mathrm{I,I}} & \boldsymbol{A}^{\mathrm{I,II}}\boldsymbol{\gamma} \\ \boldsymbol{\gamma}\boldsymbol{A}^{\mathrm{II,I}} & \boldsymbol{\gamma}\boldsymbol{A}^{\mathrm{II,II}}\boldsymbol{\gamma}. \end{bmatrix} \tag{30}$$

A Circular Waveguide Modes

We consider a circular waveguide of radius a. The modes are defined as in the horizontally polarized case of [6, pp. 66–72], except that the normalization is adjusted so that above or below cutoff

$$\int_0^a \int_0^{2\pi} \boldsymbol{e}^{p}_{mn} \times \boldsymbol{h}^{p'}_{m'n'} \cdot \hat{\boldsymbol{z}}\, \rho\, d\phi\, d\rho = P_0 \delta_{mm'} \delta_{nn'} \delta_{pp'} \tag{31}$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ are the z-independent, transverse (to z) portions of the electric and magnetic fields, respectively, associated with a single mode of the waveguide, the superscript p or p' is 1 for TE modes and 2 for TM modes, m and m' are nonnegative integers, n and n' are positive integers, and $\delta_{mm'}$ is the Kronecker delta function.

A.1 TE Modal Fields

For m and n positive integers, the TE_{mn} modal electric field is

$$\boldsymbol{e}^{\mathrm{TE}}_{mn}(\rho, \phi) = \hat{\boldsymbol{\rho}} a^{\mathrm{TE}}_{mn} \frac{J_m(\chi'_{mn}\rho/a)}{\rho} \cos m\phi + \hat{\boldsymbol{\phi}} b^{\mathrm{TE}}_{mn} J'_m(\chi'_{mn}\rho/a) \sin m\phi. \tag{32}$$

In (32), J_m is the Bessel function of the first kind of order m, J'_m is the derivative of the Bessel function with respect to its argument, χ'_{mn} is the nth nonvanishing zero of J'_m,

$$a^{\mathrm{TE}}_{mn} = -\left[(Z_0)^{\mathrm{TE}}_{mn}\right]^{1/2} \sqrt{\frac{\varepsilon_m}{\pi}} \frac{m}{\sqrt{\chi'^2_{mn} - m^2}} \frac{1}{J_m(\chi'_{mn})}, \tag{33}$$

$$b^{\mathrm{TE}}_{mn} = \left[(Z_0)^{\mathrm{TE}}_{mn}\right]^{1/2} \sqrt{\frac{\varepsilon_m}{\pi}} \frac{\chi'_{mn}}{\sqrt{\chi'^2_{mn} - m^2}} \frac{1}{a J_m(\chi'_{mn})}, \tag{34}$$

$$\varepsilon_m = 2 - \delta_{m0} = \begin{cases} 1 & (m = 0) \\ 2 & (m \neq 0) \end{cases}, \tag{35}$$

$$(Z_0)^{\mathrm{TE}}_{mn} = 1 \big/ (Y_0)^{\mathrm{TE}}_{mn} = \frac{\omega\mu}{\beta^{\mathrm{TE}}_{mn}} = \frac{j\omega\mu}{\gamma^{\mathrm{TE}}_{mn}}, \tag{36}$$

$$\beta^{\mathrm{TE}}_{mn} = \sqrt{k^2 - \left(\chi'_{mn}/a\right)^2} = -j\gamma^{\mathrm{TE}}_{mn}. \tag{37}$$

The square root in (37) is selected so that β_{mn} is in the fourth quadrant of the complex plane.

The modal magnetic field is related to the electric field through the modal wave admittance.

$$\boldsymbol{h}^{\mathrm{TE}}_{mn}(\rho, \phi) = (Y_0)^{\mathrm{TE}}_{mn}\, \hat{\boldsymbol{z}} \times \boldsymbol{e}^{\mathrm{TE}}_{mn}(\rho, \phi). \tag{38}$$

A.2 TM Modal Fields

With m a nonnegative integer and n a positive integer the TM_{mn} modal electric field is

$$\boldsymbol{e}_{mn}^{\mathrm{TM}}(\rho,\phi) = \hat{\boldsymbol{\rho}} a_{mn}^{\mathrm{TM}} J_m'(\chi_{mn}\rho/a)\cos m\phi + \hat{\boldsymbol{\phi}} b_{mn}^{\mathrm{TM}} \frac{J_m(\chi_{mn}\rho/a)}{\rho}\sin m\phi, \quad (39)$$

where χ_{mn} is the nth nonvanishing zero of J_m,

$$a_{mn}^{\mathrm{TM}} = -\left[(Z_0)_{mn}^{\mathrm{TM}}\right]^{1/2}\sqrt{\frac{\varepsilon_m}{\pi}}\frac{1}{a J_{m+1}(\chi_{mn})}, \quad (40)$$

$$b_{mn}^{\mathrm{TM}} = \left[(Z_0)_{mn}^{\mathrm{TM}}\right]^{1/2}\sqrt{\frac{\varepsilon_m}{\pi}}\frac{m}{\chi_{mn}}\frac{1}{J_{m+1}(\chi_{mn})}, \quad (41)$$

$$(Z_0)_{mn}^{\mathrm{TM}} = 1\Big/(Y_0)_{mn}^{\mathrm{TM}} = \frac{\beta_{mn}^{\mathrm{TM}}}{\omega\epsilon} = \frac{\gamma_{mn}^{\mathrm{TM}}}{j\omega\epsilon}, \quad (42)$$

$$\beta_{mn}^{\mathrm{TM}} = \sqrt{k^2 - (\chi_{mn}/a)^2} = -j\gamma_{mn}^{\mathrm{TM}}. \quad (43)$$

The modal magnetic field is again related to the electric field through the modal wave admittance.

$$\boldsymbol{h}_{mn}^{\mathrm{TM}}(\rho,\phi) = (Y_0)_{mn}^{\mathrm{TM}}\,\hat{\boldsymbol{z}} \times \boldsymbol{e}_{mn}^{\mathrm{TM}}(\rho,\phi). \quad (44)$$

References

[1] G. James, "Analysis and design of TE_{11}-to-HE_{11} corrugated cylindrical waveguide mode converters," *IEEE Trans. Microwave Theory Tech.*, vol. 29, pp. 1059–1066, Oct 1981.

[2] V. P. Pyati, "Comments on "Useful Bessel function identities and integrals," *IEEE Trans. Microwave Theory Tech.*, vol. 42, p. 1283, July 1994.

[3] P. S. Simon, "PSSFSS theory documentation," tech. report, May 2021.

[4] R. C. Rumpf, "Improved formulation of scattering matrices for semi-analytical methods that is consistent with convention," *Progress In Electromagnetics Research B*, vol. 35, pp. 241–261, 2011.

[5] Wikipedia contributors, "Redheffer star product — Wikipedia, the free encyclopedia." `https://en.wikipedia.org/w/index.php?title=Redheffer_star_product&oldid=1290145331`, 2025. [Online; accessed 4-June-2025].

[6] N. Marcuvitz, ed., *Waveguide Handbook.* Boston Tech. Pub., 1964.