HW3_LASSO

November 20, 2023

Convex Optimization

Homework 3 : LASSO

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Question 1

The lagrangian of the LASSO problem, with change of variable $u = Xw - y$ and dual variable v, is

$$L(w, u, v) = \frac{1}{2}\|u\|_2^2 + \lambda\|w\|_1 + v^T(Xw - y - u)$$

According to previous homework, maximizing the lagrangian over w, u gives us the dual function $g(v) = -\frac{v^T v}{2} - y^T v - I\left(\frac{X^T v}{\lambda}\right)$ where

$$I(x) = 0 \text{ if } \|x\|_\infty \leq 1 \text{ and } +\infty \text{ otherwise.}$$

Hence, the dual of LASSO is :

$$\max_v -\frac{v^T v}{2} - y^T v$$

such that

$$\|X^T v\|_\infty \leqslant \lambda$$

$\iff$

$$\begin{pmatrix} X^T \\ -X^T \end{pmatrix} v \leqslant \lambda$$

It's exactly a quadratic problem with :

$$Q = \frac{1}{2} I_n$$
$$p = y$$
$$A = \begin{pmatrix} X^T \\ -X^T \end{pmatrix}$$
$$b = (\lambda, \cdots, \lambda)^T$$

Derivations of gradient and hessian of for the log-barrier objective function We consider the approximation of the quadratic program via logarithmic barrier. The objective function is :

$$f(v) = t(v^T Q v + p^T v) - \sum_{i=1}^{m} \log(b_i - a_i^T v)$$

The gradient of f is equal to

$$\nabla f(v) = 2tQv + tp + \sum_{i=1}^{m} \frac{a_i}{(b_i - a_i^T v)}$$

The hessian of f is

$$\nabla^2 f(v) = 2tQ + \sum_{i=1}^{m} \frac{a_i a_i^T}{(b_i - a_i^T v)^2}$$

Question 2

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp
```

[ ]:
```
def f0(Q, p, v) :
    """Compute the quadratic form vTQv + pTv"""

    vQv = v@Q@v
    pv = p@v
    return vQv + pv

def f(Q, p, A, b, v, t) :
    """Compute the quadratic form with a log-barrier penalty"""
    Av = A@v
    vQv = v@Q@v
    pv = p@v
    z = (b-Av)
    if np.any(z<0) :
        return np.inf
    return t*vQv + t*pv - np.sum(np.log(z))
```

```python
def H(Q, p, A, b, v, t) :
    """Compute the Hessian of f"""
    Av = A@v
    m, _ = np.shape(A)
    z = 1 / (Av-b)**2
    s = np.einsum("ij, ik, i -> jk", A, A, z)
    return 2*t*Q + s

def grad(Q, p, A, b, v, t) :
    """Compute the gradient of f"""
    Qv = Q@v
    Av = A@v
    z = 1 / (b-Av)
    return 2*t*Qv + t*p + np.einsum("ki, k -> i", A, z)

def centering_step(Q, p, A, b, t, v0, eps) :
    """Centering step of the log-barrier method for a quadratic program"""
    V = [v0]
    v = np.copy(v0)
    alpha = 0.1
    beta = 0.7
    u = 1

    # Newton's method
    Hessian = H(Q, p, A, b, v, t)
    g = grad(Q, p, A, b, v, t)
    Hinv = np.linalg.inv(Hessian)
    delta_v = -Hinv@g
    decrement = -g@delta_v/2

    k=0
    while (decrement >= eps) and (k<20):
        k+=1
        fv = f(Q, p, A, b, v, t)
        u = 1

        # Backtracking line search
        while (f(Q, p, A, b, v + u*delta_v, t) > (fv - alpha*u*g@delta_v)) :
            u*= beta

        # Update
        v+=u*delta_v
        Hessian = H(Q, p, A, b, v, t)
        g = grad(Q, p, A, b, v, t)
        Hinv = np.linalg.inv(Hessian)
        delta_v = -Hinv@g
        decrement = -g@delta_v/2
```

```
        V.append(v)

    return V

def barr_method(Q, p, A, b, v0, eps, mu=10) :
    """Implement the log-barrier method for a quadratic program"""
    V = [v0]
    m, _ = np.shape(A)
    t = 1
    vt = np.copy(v0)
    while m/t > eps :
        vt = centering_step(Q, p, A, b, t, vt, eps)
        V+=vt
        t *= mu
        vt = vt[-1]


    return V
```

[ ]:
```
# Parameters and data
#np.random.seed(42)
Lambda = 10
d = 500
n = 10
X = np.random.normal(size=(n, d)) # / 4
y = np.random.normal(size=n)+15 #/ 4

eps=1e-9
Q = np.eye(n)/2
p = y
A = np.zeros((2*d, n))
A[:d,:]= X.T
A[d:,:] = -X.T
b = Lambda*np.ones(2*d)
v = np.random.normal(size=n)
```

[ ]:
```
np.linalg.norm(X.T@y)
```

[ ]: 957.9665166432452

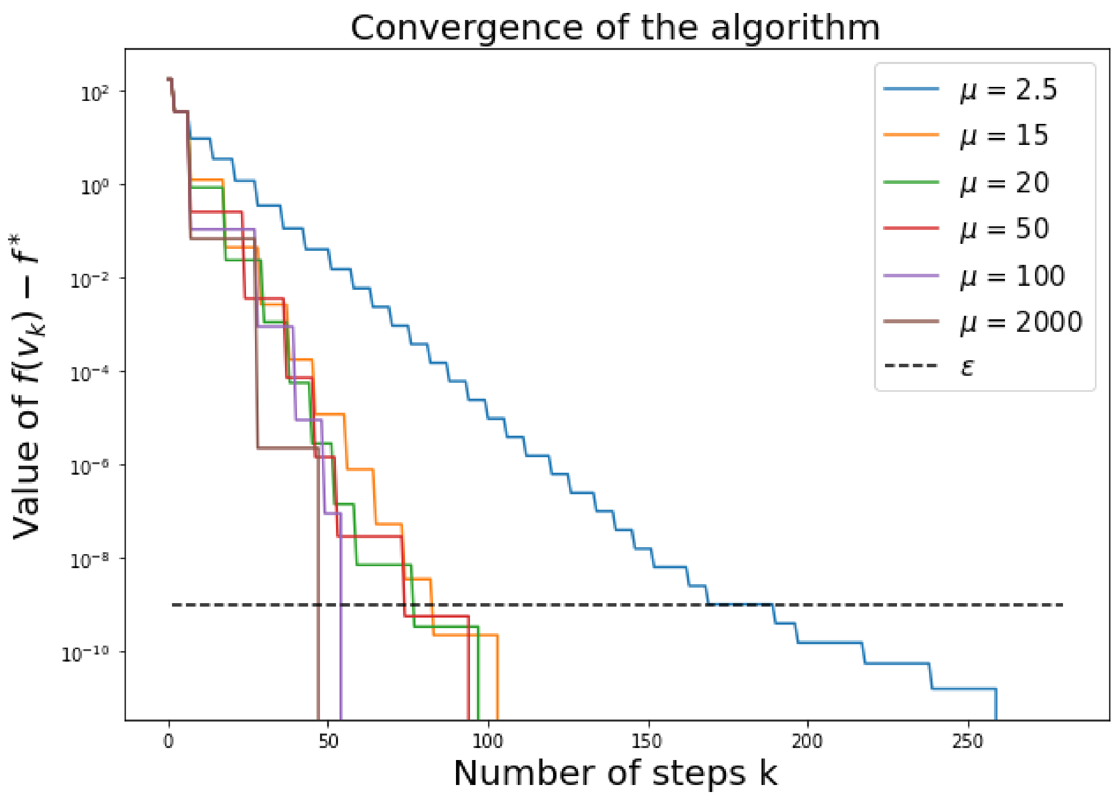
[ ]:
```
fig = plt.figure(figsize=(10, 7))

L=-np.inf
for mu in [2.5, 15, 20, 50, 100, 2000] :
    V = barr_method(Q, p, A, b, v0=np.zeros(n), eps=eps, mu=mu)
    if len(V)>L :
        L = len(V)
```

```
    x = []
    for v in V :
        x.append(f0(Q, p, v))
    print(len(V))
    print("Value of the optimum : {:.4f}".format(f0(Q, p, V[-1])))
    plt.plot(np.array(x)-np.min(x))
    plt.yscale("log")
    plt.xlabel("Number of steps k", fontsize=20)
    plt.ylabel("Value of $f(v_k) - f^*$", fontsize=20)
    plt.title("Convergence of the algorithm", fontsize=20)

plt.hlines(eps, 1, L, linestyles='--', color="k", label="eps")
plt.legend(["$\mu$ = {}".format(mu) for mu in [2.5, 15, 20, 50, 100,␣
 ↪2000]]+["$\epsilon$"], fontsize=15)
plt.show()
```

```
280
Value of the optimum : -182.2625
124
Value of the optimum : -182.2625
118
Value of the optimum : -182.2625
115
Value of the optimum : -182.2625
70
Value of the optimum : -182.2625
68
Value of the optimum : -182.2625
```



Let's check if our implementation of Newton's method is correct, using `cvxpy` library.

```
v = cp.Variable(n)
prob = cp.Problem(cp.Minimize(cp.quad_form(v, Q) + p.T @ v),
                  [A @ v <= b])
print("Value of the optimum : {:.4f}".format(prob.solve()))
print("Optimal point :", v.value)
print("Optimal point obtained with a log-barrier method :", V[-1])
```

```
Value of the optimum : -182.2625
Optimal point : [-1.48163596 -2.24122506  0.08928414 -1.60312274 -1.42664813
-2.04950161
 -1.26367456 -0.67457764 -1.13191262 -1.64197554]
Optimal point obtained with a log-barrier method : [-1.48163596 -2.24122506
0.08928414 -1.60312273 -1.42664813 -2.04950161
 -1.26367456 -0.67457764 -1.13191262 -1.64197554]
```

We can see above that our implementation is correct.

What is a good choice for μ ?

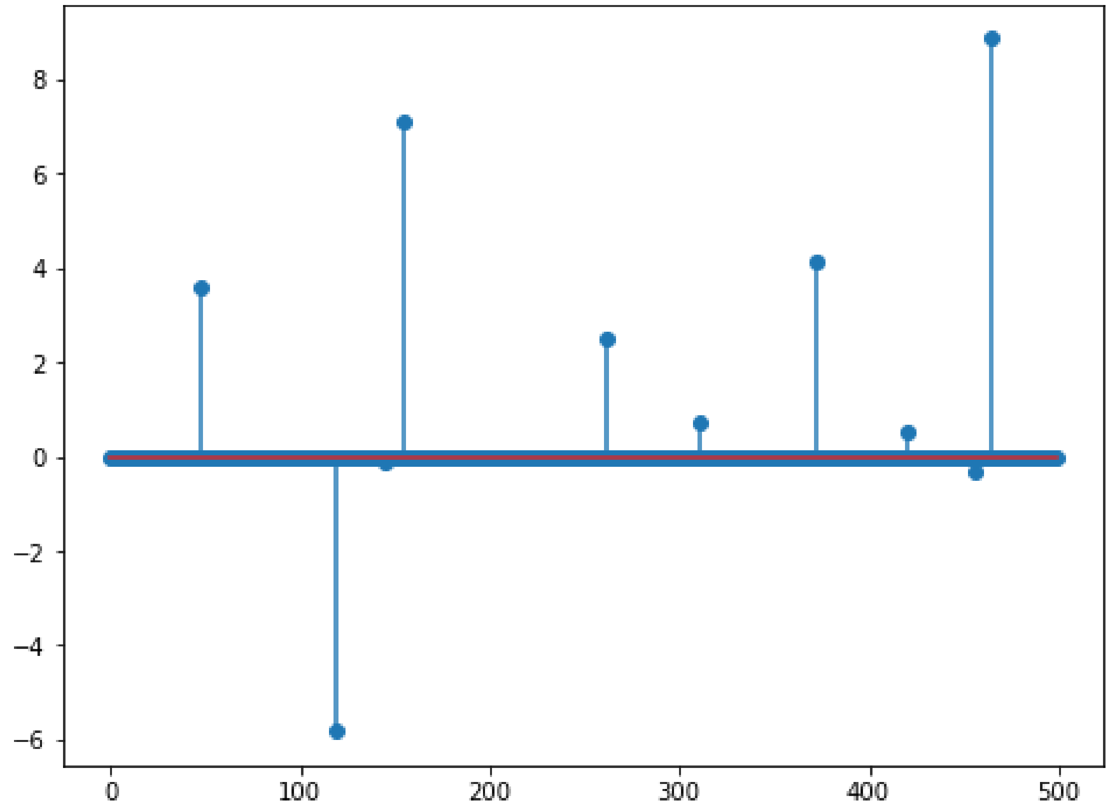
Heuristically, an appropriate choice for μ would be a $\mu \geqslant 20$. Indeed, for $\mu \geqslant 20$, the total number of steps of the method doesn't change and gives us a linear convergence with a high slope.

Recovering w^* from v^*

We want to recover w^* from v^*. First, let's take a look at cvxpy solution. We have a spare solution as expected.

[ ]:
```
w = cp.Variable(d)
prob = cp.Problem(cp.Minimize(cp.norm2(y - X@w)**2 + Lambda*cp.norm1(w)),
                 )
print("Value of the optimum : {:.4f}".format(prob.solve(solver=cp.ECOS)))
fig = plt.figure(figsize=(8, 6))
plt.stem(w.value)
plt.show()
```

```
Value of the optimum : 340.8627
```



How can we recover w^* from v^* ?

We have the following relation between w^* and v^* from the primal feasibility and computation of dual :

$$v^* = Xw^* - y \tag{1}$$

From the extended KKT conditions, we have :

$$0 \in \partial_{w_i} L(w^*, u^*, v^*) = \lambda \cdot \partial_{w_i^*} \|w^*\|_1 + (X^T v^*)_i$$

Morover, we have

$$\lambda \cdot \partial_{w_i^*} \|w^*\|_1 = -(X^T v^*)_i$$

where

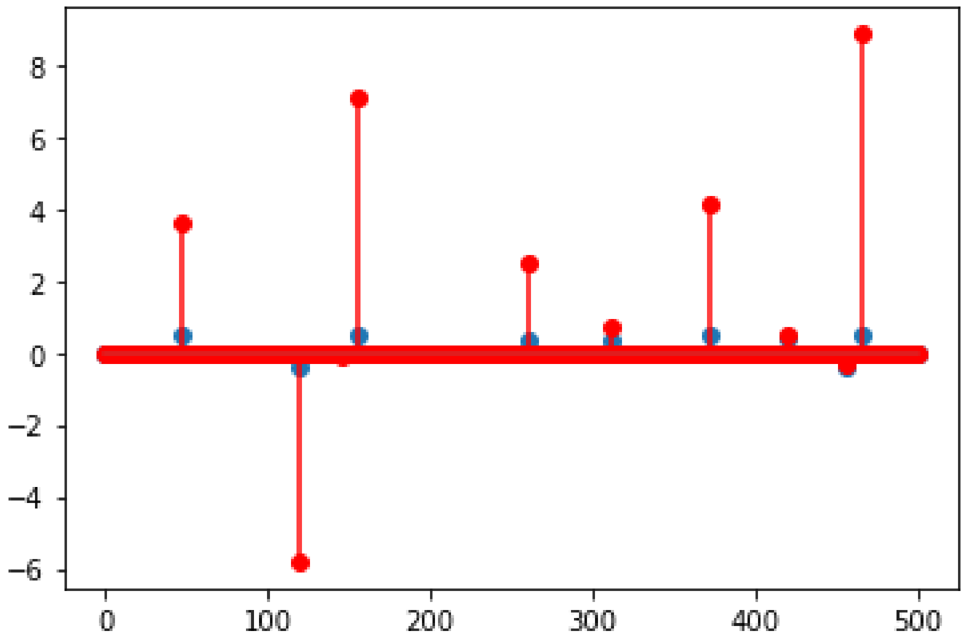
$$\partial_{x_i^*} \|x\|_1 = \begin{cases} 1 & \text{if } x_i > 0 \\ -1 & \text{if } x_i < 0 \\ [-1, 1] & \text{if } x_i = 0 \end{cases}$$

Hence,

$$w_i^* < 0 \text{ if } (X^T v^*)_i = \lambda$$
$$w_i^* > 0 \text{ if } (X^T v^*)_i = -\lambda$$
$$w_i = 0 \text{ if } (X^T v^*)_i \in (-\lambda, \lambda)$$

Hence, we're searching for a w^* such that $w^* = X^+(v^* + y)$ and the previous conditions on the signs hold.

[ ]:
```
V = barr_method(Q, p, A, b, v0=np.zeros(n), eps=1e-9, mu=150)
signs = np.zeros(d)
signs[np.isclose(X.T@V[-1], Lambda)] = -1
signs[np.isclose(X.T@V[-1], -Lambda)] = +1
w_star = np.linalg.pinv(X)@((V[-1]+y))
plt.stem(w_star*np.abs(signs))
stem_color = 'r'  # blue
marker_color = 'r'  # red
plt.stem(w.value, linefmt=f'{stem_color}', markerfmt=f'{marker_color}o')
plt.show()
```



We recover the signs of the lasso optimizer but not the true values.