"Kernel methods in machine learning"
Homework 1
Due February 7th, 2024, 1:30pm

to be submitted on gradescope (check your mailbox)

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Exercice 1. Function and kernel boundedness
Consider a p.d. kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $K(x, z) \leq b^2$ for all x, z in $\mathcal{X}$. Show that $\|f\|_\infty = \sup_{x \in \mathcal{X}} |f(x)| \leq b$ for any function f in the unit ball of the corresponding RKHS.

Exercice 2. Kernels encoding equivalence classes.
Consider a similarity measure $K : \mathcal{X} \times \mathcal{X} \to \{0, 1\}$ with $K(x, x) = 1$ for all x in $\mathcal{X}$. Prove that K is p.d. if and only if, for all x, x', x'' in $\mathcal{X}$,

- $K(x, x') = 1 \Leftrightarrow K(x', x) = 1$, and
- $K(x, x') = K(x', x'') = 1 \Rightarrow K(x, x'') = 1$.

Exercice 3. RKHS
In this exercise, we will describe the RKHS of kernels obtained by various constructions. By "describing", it means writing in simple mathematical terms the set of functions forming the RKHS, and the associated inner product.

1. Let K_1 and K_2 be two positive definite kernels on a set $\mathcal{X}$, and α, β two positive scalars. Show that $\alpha K_1 + \beta K_2$ is positive definite, and describe its RKHS.

2. Let $\mathcal{X}$ be a set and $\mathcal{F}$ be a Hilbert space. Let $\Psi : \mathcal{X} \to \mathcal{F}$, and $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be:

$$\forall x, x' \in \mathcal{X}, \quad K(x, x') = \langle \Psi(x), \Psi(x') \rangle_{\mathcal{F}} \, .$$

Show that K is a positive definite kernel on $\mathcal{X}$, and describe its RKHS.

3. Prove that for any p.d. kernel K on a space $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ belongs to the RKHS $\mathcal{H}$ with kernel K if and only if there exists $\lambda > 0$ such that $K(\mathbf{x}, \mathbf{x}') - \lambda f(\mathbf{x}) f(\mathbf{x}')$ is p.d.