Computations

Problem formulation

Let the domain of interest be the cube of dimensions $Lx \times Ly \times Lz = 1 \times 1 \times 1$. Let h and k be the spatial and temporal resolutions, such that $1 = Nx \times h = Ny \times h = Nz \times h$. We assume that the membrane is rigidly clamped at its boundaries, meaning Neumann boundary conditions will be used. Similarly, the drum cavity is assumed to be rigid too.

Finite difference methods

Membrane scheme

Let w be the displacement on the membrane, $w = w(x, y, t) = w^n_{l,m}$ where $x = hl$, $y = hm$ and $t = kn$. The scheme we are using is

$$\delta_{tt} w = \gamma^2 \delta_{\Delta_2} w - 2\sigma_0 \delta_{t.} w$$

where

- $\gamma = c/L$ with c the celerity of the wave and L a characteristic length. In our case, $L = 1$.
- $\sigma_0 = 6 \log 10 / T_{60}$ with T_{60} the 60 dB decay time (seconds)
  After expanding the scheme, we arrive at the recursion

$$w^{n+1}_{l,m} = \frac{2}{2k\sigma_0 + 1} u^n_{l,m} + \frac{k^2\gamma^2}{2k\sigma_0 + 1} \mathcal{D}_2 w^n_{l,m} + \frac{2\sigma_0 k - 1}{2\sigma_0 k + 1} w^{n-1}_{l,m}$$

  It is important to note that, in order for the stability of this scheme to be valid, one must have $\lambda \leq 1/\sqrt{2}$ with $\lambda = \gamma k/h$ the Courant number.
  Here, $\mathcal{D}_2$ refers to the matrix operator of the 2D Laplacian approximation under Neumann conditions. It can be decomposed as blocks, and is of shape $Nx + 1 \times Nx + 1$ blocks, each of size $Ny + 1 \times Ny + 1$. Still thinking of it as blocks, the super and sub diagonals are identity blocks. The diagonal is composed of blocks of the following shape:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -3 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -3 & 1 \\ & & & & 1 & -2 \end{pmatrix} \text{ or } \begin{pmatrix} -3 & 1 & & & & \\ 1 & -4 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -4 & 1 \\ & & & & 1 & -3 \end{pmatrix}$$

  depending if they correspond to edge or interior points.

Cavity scheme

In the context of the 3D study, we will denote Ψ the variation in the acoustic field inside of the drum cavity. We will also write

$$\Psi = \Psi(x, y, z, t) = \Psi^n_{l,m,p}$$

where $x = lh, y = mh, z = ph$ and $t = kn$.
The drum cavity being a relatively small volume, we assume that the propagation of sound in it follows the 3D wave equation, which we approximate using the following scheme:

$$\delta_{tt}\Psi = \gamma^2 \delta_{\Delta_3}\Psi$$

Stability condition

To determine the 3D stability condition, we consider the ansatz $\Psi^n_{l,m,p} = z^n \exp\left[jh\left(l\beta_x + m\beta_y + p\beta_z\right)\right]$. The left-hand term of the equation becomes

$$\begin{aligned}\delta_{tt}\Psi &= \frac{1}{k^2}\left[z^{n+1} - 2z^n + z^{n-1}\right]\exp\left[jh\left(l\beta_x + m\beta_y + p\beta_z\right)\right] \\ &= \frac{z^n}{k^2}\left[z - 2 + z^{-1}\right]\exp\left(j\Phi\right)\end{aligned}$$

The right-hand term is

$$\begin{aligned}\delta_{\Delta_3}\Psi &= (\delta_{xx} + \delta_{yy} + \delta_{zz})\Psi \\ &= z^n\left(A_x + A_y + A_z\right)\end{aligned}$$

Developing the x-wise term gives

$$\begin{aligned}A_x &= \frac{1}{h^2}\left[e^{jh(l+1)\beta_x} - 2e^{jhl\beta_x} + e^{jh(l-1)\beta_x}\right] \\ &= \frac{e^{jhl\beta_x}}{h^2}(2\cos(h\beta_x) - 2)\end{aligned}$$

From which we can deduce that

$$A_y = \frac{e^{jhm\beta_y}}{h^2}(2\cos(h\beta_y) - 2) \qquad A_z = \frac{e^{jhp\beta_z}}{h^2}(2\cos(h\beta_z) - 2)$$

Meaning that

$$\delta_{\Delta_3}\Psi = \frac{2}{h^2}[\cos(h\beta_x) + \cos(h\beta_y) + \cos(h\beta_z) - 3]$$

And so the PDE becomes

$$\frac{z^n}{k^2}\left[z - 2 + z^{-1}\right] = \frac{2\gamma^2}{h^2}[\cos(h\beta_x) + \cos(h\beta_y) + \cos(h\beta_z) - 3]z^n$$

Now denoting $\lambda = \gamma k/h$, we can write this equation as

$$
\begin{aligned}
z+z^{-1}-2 &= 2\lambda^2\left[\underbrace{\cos(h\beta_x)+\cos(h\beta_y)+\cos(h\beta_z)}_{\zeta}-3\right] \\
\implies z+z^{-1} &= 2+2\lambda^2\left(\zeta-3\right)
\end{aligned}
$$

To ensure the stability of the scheme, we need $|z|<1$, i.e. $z+z^{-1}\in[-2,2]$, i.e. $2+2\lambda^2\left(\zeta-3\right)\in[-2,2]$. The worst-case scenario happens for $\zeta=-3$ and we then have

$$
\begin{aligned}
2-12\lambda^2 &\geq -2 \\
\implies \lambda^2 &\leq \frac{1}{3} \\
\implies &\boxed{\lambda\leq\frac{1}{\sqrt{3}}}
\end{aligned}
$$

Finite difference scheme

Just as in the 2D case, expanding the scheme leads to the following recursion

$$
\Psi_{l,m,p}^{n+1}=2\Psi_{l,m,p}^{n}+\lambda^2\underline{\mathcal{D}}_3\Psi_{l,m,p}^{n}-\Psi_{l,m,p}^{n-1}
$$

where, similarly as before, $\underline{\mathcal{D}}_3$ refers to the matrix operator of the 3D Laplacian approximation under Neumann conditions. This matrix can be expressed using $\underline{\mathcal{D}}_2$, as it is composed of $Nz+1\times Nz+1$ blocks, each of size $(Ny+1)(Nx+1)\times(Ny+1)(Nx+1)$. The super and sub-diagonals of $\underline{\mathcal{D}}_3$ are identities, and the diagonal blocks are variations of $\underline{\mathcal{D}}_2$: interior blocks are 2D Laplacian matrix from which we subtracted the identity, while the first and last blocks are $\underline{\mathcal{D}}_2-2I$.

Modal analysis

In this section, we are computing solutions to the PDEs using modal analysis.

Membrane

Assuming a solution to the lossless 2D wave equation to be of the form

$$
u(x,y,t)=e^{j\omega t}U(x,y)
$$

One may write, plugging this solution back into the PDE, that

$$
-\omega^2U(x,y)=\gamma^2\;\nabla^2U(x,y)
$$

where ∇^2 refers to the 2D Laplacian operator. We will assume that U is separable, meaning we can write it as $U(x,y)=X(x)\times Y(y)$, as well as the fact that $U(x,y)$ is defined over a rectangular domain with Neumann boundary conditions. We can re-write the previous equation as

$$
\nabla^2U+k^2U=0\qquad k^2=\frac{\gamma^2}{\omega^2}
$$

or, replacing U by $X\times Y$,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2 \implies \lambda^2 + \mu^2 = k^2$$

This leads to a system of two equations to solve:

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 & X'(0) = X'(1) = 0 \quad (1) \\ Y''(y) + \mu^2 Y(y) = 0 & Y'(0) = Y'(1) = 0 \quad (2) \end{cases}$$

We will detail the computations for (1). The general form of the solution is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

The boundary conditions give us that $X'(0) = 0 \implies B = 0$ and so that $X(x) = A\cos(\lambda x)$, and that $X'(1) = 0 \implies \lambda_n = n\pi$. The choice of A is arbitrary, thus we will consider $A = 1$. The solutions are then

$$X(x) = \cos(n\pi x) \qquad Y(y) = \cos(m\pi y)$$

This means that for every pair of integers $(n, m) \in \mathbb{N}^{*2}$, we have

$$\lambda_n = n\pi \qquad \mu_m = m\pi \qquad k_{n,m}^2 = \lambda_n^2 + \mu_m^2 = \pi^2(n^2 + m^2)$$

The general solution is thus

$$\begin{aligned} U(x, y) &= X(x)Y(y) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} \cos(n\pi x)\cos(m\pi y) \end{aligned}$$

And this solution satisfies the 2D wave equation is each of the modes (n, m) satisfies

$$\pi^2(n^2 + m^2) = \frac{\omega^2}{\gamma^2} \qquad \text{or} \qquad k_{n,m}^2 = k^2$$

This means that the pulsations of the system are

$$\omega_{n,m} = \gamma\pi\sqrt{n^2 + m^2}$$

The coefficients $A_{n,m}$ depend on the initial state of the system. Assuming the general case where $U(x, y, 0) = f(x, y)$, then these coefficients are given by Fourier analysis:

$$A_{n,m} = 4\int_0^1 \int_0^1 f(x, y)\cos(n\pi x)\cos(m\pi y)\, dx\, dy$$

And the final solution is

$$|u(x, y, t)| = U(x, y)$$

Now, we are interested in the case where there is a damping term to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \gamma^2 \nabla^2 u - 2\sigma_0 \frac{\partial u}{\partial t}$$

From what we have done above, we can assume that

$$u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m}(t)\Phi_{n,m}(x,y)$$

Where $\Phi_{n,m}(x,y) = \cos(n\pi x)\cos(m\pi y)$ which satisfies the Neumann boundary conditions over $[0,1] \times [0,1]$. Noticing that

$$\nabla^2 \Phi_{n,m} = -k_{n,m}^2 \Phi_{n,m} \qquad k_{n,m}^2 = \pi^2(n^2 + m^2)$$

we can plug back the solution into the PDE, which means that for each mode (n,m),

$$\begin{aligned} &A''_{n,m}(t)\Phi_{n,m} + 2\sigma_0 A'_{n,m}(t)\Phi_{n,m} = -\gamma^2 k_{n,m}^2 \Phi_{n,m} A_{n,m}(t) \\ \implies &A''_{n,m}(t) + 2\sigma_0 A'_{n,m}(t) + \gamma^2 k_{n,m}^2 A_{n,m}(t) = 0 \end{aligned}$$

Which is the form of a classic damped harmonic oscillator. Its characteristic equation is

$$s^2 + 2\sigma_0 s + \gamma^2 k_{n,m}^2 = 0 \implies s = -\sigma_0 \pm \sqrt{\sigma_0^2 - \gamma^2 k_{n,m}^2}$$

And from there, three cases arise:

- Under-damped: $\gamma^2 k_{n,m}^2 > \sigma_0^2$ and the behaviour is oscillatory with decay
- Critically damped: $\gamma^2 k_{n,m}^2 = \sigma_0^2$
- Over-damped: $\gamma^2 k_{n,m}^2 < \sigma_0^2$

We will assume the first case and thus can write

$$A_{n,m} = C_{n,m} e^{-\sigma_0 t}\cos(\omega_{n,m} t) \qquad \omega_{n,m} = \sqrt{\gamma^2 k_{n,m}^2 - \sigma_0^2}$$

Assuming zero initial velocity (as the initial condition will most likely be a raised cosine distribution), we have that $A'_{n,m}(0) = 0$ and thus

$$u(x,y,0) = f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m}(0)\Phi_{n,m}$$

From which we can deduce that $C_{n,m} = A_{n,m}(0)$, and thus that

$$\begin{aligned} A_{n,m}(t) &= e^{-\sigma_0 t}\cos(\omega_{n,m} t) \times 4 \int_0^1 \int_0^1 f(x,y)\cos(n\pi x)\cos(m\pi y)\, dx\, dy \\ &= e^{-\sigma_0 t}\cos(\omega_{n,m} t) A_{n,m} \end{aligned}$$

Which gives us the final solution

$$\boxed{u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} e^{-\sigma_0 t}\cos(\omega_{n,m} t)\cos(n\pi x)\cos(m\pi y)}$$