pspipe notes : generalized covariance

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1 Combinatoric : on the full sky

X,Y,W,Z denote $\{T, E, B\}$.
α, β, ν, μ denote the different detectors arrays.
i,j,k,l denotes the split number.
s_a, s_b, s_c, s_d denote the different surveys. Let's write the estimator assuming full sky and no beam, we will include complexity later.

$$C_\ell^{X^{s_a}_{\alpha,i} Y^{s_b}_{\beta,j}} = \frac{1}{2\ell+1} \sum_m a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} (1 - \delta_{s_a s_b} \delta_{ij}). \tag{1}$$

We average all cross split power spectra

$$C_{\text{cross},\ell}^{X^{s_a}_{\alpha} Y^{s_b}_{\beta}} = \frac{1}{n_c^{s_a s_b}} \sum_{i=1}^{n^{s_a}_{\text{split}}} \sum_{j=1}^{n^{s_b}_{\text{split}}} \frac{1}{2\ell+1} \sum_m a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} (1 - \delta_{s_a s_b} \delta_{ij}). \tag{2}$$

$n^{s_a}_{\text{split}}$ is the number of splits of the data of the survey s_a and $n_c^{s_a s_b}$ is the number of individual cross split power spectra between the survey n^{s_a} and n^{s_b}.

$$n_c^{s_a s_b} = \sum_{i=1}^{n^{s_a}_{\text{split}}} \sum_{j=1}^{n^{s_b}_{\text{split}}} (1 - \delta_{s_a s_b} \delta_{ij}) = n^{s_a}_{\text{split}} (n^{s_b}_{\text{split}} - \delta_{s_a s_b}). \tag{3}$$

The role of the delta function is to remove any auto-power spectrum. We can compute the covariance of any mean cross power spectrum as follow

$$\Xi^{X^{s_a}_{\alpha} Y^{s_b}_{\beta} W^{s_c}_{\gamma} Z^{s_d}_{\eta}} = \langle (C_{\text{cross},\ell}^{X^{s_a}_{\alpha} Y^{s_b}_{\beta}} - C_\ell)(C_{\text{cross},\ell}^{W^{s_c}_{\gamma} Z^{s_d}_{\eta}} - C_\ell) \rangle = \langle C_{\text{cross},\ell}^{X^{s_a}_{\alpha} Y^{s_b}_{\beta}} C_{\text{cross},\ell}^{W^{s_c}_{\gamma} Z^{s_d}_{\eta}} \rangle - C_\ell^{X_\alpha Y_\beta} C_\ell^{W_\gamma Z_\eta}. \tag{4}$$

Replacing the estimate of the cross spectra $\hat{C}$ by their explicit expression we get

$$\langle C_{\text{cross},\ell}^{X^{s_a}_{\alpha} Y^{s_b}_{\beta}} C_{\text{cross},\ell}^{W^{s_c}_{\gamma} Z^{s_d}_{\eta}} \rangle = \frac{1}{(2\ell+1)^2} \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} \sum_{mm'} \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{W^{s_c}_{\gamma,k}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle (1 - \delta_{s_a s_b} \delta_{ij})(1 - \delta_{s_c s_d} \delta_{kl}). \tag{5}$$

Since the $a_{\ell m}$ follow a gaussian distribution, we can then expand the four point function using the Wick theorem

$$\langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{W^{s_c}_{\gamma,k}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle = \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} \rangle \langle a_{\ell m'}^{W^{s_c}_{\gamma,k}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle + \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m'}^{W^{s_c}_{\gamma,k}} \rangle \langle a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle + \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle \langle a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{W^{s_c}_{\gamma,k}} \rangle$$

and the covariance matrix become a sum of four terms :

$$\begin{aligned}
\Xi^{X^{s_a}_{\alpha} Y^{s_b}_{\beta} W^{s_c}_{\gamma} Z^{s_d}_{\eta}} &= \frac{1}{(2\ell+1)^2} \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} \sum_{mm'} \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m}^{Y^{s_b}_{\beta,j}*} \rangle \langle a_{\ell m'}^{W^{s_c}_{\gamma,k}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle (1 - \delta_{s_a s_b} \delta_{ij})(1 - \delta_{s_c s_d} \delta_{kl}) \\
&+ \frac{1}{(2\ell+1)^2} \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} \sum_{mm'} \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m'}^{W^{s_c}_{\gamma,k}} \rangle \langle a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle (1 - \delta_{s_a s_b} \delta_{ij})(1 - \delta_{s_c s_d} \delta_{kl}) \\
&+ \frac{1}{(2\ell+1)^2} \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} \sum_{mm'} \langle a_{\ell m}^{X^{s_a}_{\alpha,i}} a_{\ell m'}^{Z^{s_d}_{\eta,l}*} \rangle \langle a_{\ell m}^{Y^{s_b}_{\beta,j}*} a_{\ell m'}^{W^{s_c}_{\gamma,k}} \rangle (1 - \delta_{s_a s_b} \delta_{ij})(1 - \delta_{s_c s_d} \delta_{kl}) \\
&- C_\ell^{X_\alpha Y_\beta} C_\ell^{W_\gamma Z_\eta}.
\end{aligned} \tag{6}$$

Each contribution can be easily computed, we first have to expand

$$\begin{aligned}\sum_{mm'} \langle a_{\ell m}^{X_{\alpha,i}^{s_a}} a_{\ell m}^{Y_{\beta,j}^{s_b}*} \rangle \langle a_{\ell m'}^{W_{\gamma,k}^{s_c}} a_{\ell m'}^{Z_{\eta,l}^{s_d}*} \rangle &= (2\ell+1)^2 C_\ell^{X_{\alpha,i}^{s_a} Y_{\beta,j}^{s_b}} C_\ell^{W_{\gamma,l}^{s_c} Z_{\eta,l}^{s_d}} \\ &= (2\ell+1)^2 (C_\ell^{X_\alpha Y_\beta} + N_{\ell,s_a}^{X_\alpha Y_\beta} \delta_{ij}\delta_{s_a s_b})(C_\ell^{W_\gamma Z_\eta} + N_{\ell,s_c}^{W_\gamma Z_\eta}\delta_{kl}\delta_{s_c s_d}) \quad (7)\end{aligned}$$

where each C_ℓ is written as the sum of the underlying power spectrum and a noise bias term N_ℓ. The first term of the covariance matrix becomes

$$\frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} (C_\ell^{X_\alpha Y_\beta} + N_{\ell,s_a}^{X_\alpha Y_\beta} \delta_{ij}\delta_{s_a s_b})(C_\ell^{W_\gamma Z_\eta} + N_{\ell,s_c}^{W_\gamma Z_\eta}\delta_{kl}\delta_{s_c s_d})(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) \quad (8)$$

which is simply equal to $C_\ell^{X_\alpha Y_\beta} C_\ell^{W_\gamma Z_\eta}$. This is easy to see because any contribution of the form $\sum_{ij} \delta_{s_a s_b}\delta_{ij}(1-\delta_{s_a s_b}\delta_{ij})$ is going to be zero. The covariance matrix thus simplify to the sum of two terms,

$$\Xi^{X_\alpha^{s_a} Y_\beta^{s_b} W_\gamma^{s_c} Z_\eta^{s_d}} = \frac{1}{2\ell+1}\left(T_{Y_\beta^{s_b} Z_\eta^{s_d}}^{X_\alpha^{s_a} W_\gamma^{s_c}} + T_{Y_\beta^{s_b} W_\gamma^{s_c}}^{X_\alpha^{s_a} Z_\eta^{s_d}}\right) \quad (9)$$

we focus on one of them

$$T_{Y_\beta^{s_b} Z_\eta^{s_d}}^{X_\alpha^{s_a} W_\gamma^{s_c}} = \frac{1}{(2\ell+1)} \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl}\sum_{mm'} \langle a_{\ell m}^{X_{\alpha,i}^{s_a}} a_{\ell m'}^{W_{\gamma,k}^{s_c}} \rangle \langle a_{\ell m}^{Y_{\beta,j}^{s_b}*} a_{\ell m'}^{Z_{\eta,l}^{s_d}*} \rangle (1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}), \quad (10)$$

expanding

$$\begin{aligned}\sum_{mm'} \langle a_{\ell m}^{X_{\alpha,i}^{s_a}} a_{\ell m'}^{W_{\gamma,k}^{s_c}} \rangle \langle a_{\ell m}^{Y_{\beta,j}^{s_b}*} a_{\ell m'}^{Z_{\eta,l}^{s_d}*} \rangle &= (2\ell+1) C_\ell^{X_{\alpha,i}^{s_a} W_{\gamma,k}^{s_c}} C_\ell^{Y_{\beta,j}^{s_b} Z_{\eta,l}^{s_d}} \\ &= (2\ell+1)(C_\ell^{X_\alpha W_\gamma} + N_{\ell,s_a}^{X_\alpha W_\gamma}\delta_{ik}\delta_{s_a s_c})(C_\ell^{Y_\beta Z_\eta} + N_{\ell,s_b}^{Y_\beta Z_\eta}\delta_{jl}\delta_{s_b s_d}) \quad (11)\end{aligned}$$

we get

$$\begin{aligned}T_{Y_\beta^{s_b} Z_\eta^{s_d}}^{X_\alpha^{s_a} W_\gamma^{s_c}} &= \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} (C_\ell^{X_\alpha W_\gamma} + N_{\ell,s_a}^{X_\alpha W_\gamma}\delta_{ik}\delta_{s_a s_c})(C_\ell^{Y_\beta Z_\eta} + N_{\ell,s_b}^{Y_\beta Z_\eta}\delta_{jl}\delta_{s_b s_d})(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) \\ &= \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} C_\ell^{X_\alpha W_\gamma} C_\ell^{Y_\beta Z_\eta}(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) \\ &+ \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} (C_\ell^{X_\alpha W_\gamma} N_{\ell,s_b}^{Y_\beta Z_\eta}\delta_{jl}\delta_{s_b s_d} + C_\ell^{Y_\beta Z_\eta} N_{\ell,s_a}^{X_\alpha W_\gamma}\delta_{ik}\delta_{s_a s_c})(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) \\ &+ \frac{1}{n_c^{s_a s_b} n_c^{s_c s_d}} \sum_{ijkl} N_{\ell,s_a}^{X_\alpha W_\gamma}\delta_{ik}\delta_{s_a s_c} N_{\ell,s_b}^{Y_\beta Z_\eta}\delta_{jl}\delta_{s_b s_d}(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}). \quad (12)\end{aligned}$$

The remaining work is to compute sum of δ function

$$\begin{aligned}\textstyle\sum_{ijkl} \quad & \delta_{jl}\delta_{s_b s_d}(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) = \\ \textstyle\sum_{ijkl} \quad & \delta_{jl}\delta_{s_b s_d} - \delta_{jl}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{ij} - \delta_{jl}\delta_{s_b s_d}\delta_{s_c s_d}\delta_{kl} + \delta_{jl}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{ij}\delta_{s_c s_d}\delta_{kl} = \\ & n_{\rm split}^{s_a} n_{\rm split}^{s_c} n_{\rm split}^{s_b}\delta_{s_b s_d} - n_{\rm split}^{s_b} n_{\rm split}^{s_c}\delta_{s_b s_d}\delta_{s_a s_b} - n_{\rm split}^{s_a} n_{\rm split}^{s_b}\delta_{s_b s_d}\delta_{s_c s_d} + n_{\rm split}^{s_b}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{s_c s_d} = \\ & n_{\rm split}^{s_b}(n_{\rm split}^{s_a} n_{\rm split}^{s_c}\delta_{s_b s_d} - n_{\rm split}^{s_c}\delta_{s_a s_b s_d} - n_{\rm split}^{s_a}\delta_{s_b s_d s_c} + \delta_{s_a s_b s_d s_d}) = \\ & f_{s_b s_d}^{s_a s_c} n_c^{s_a s_b} n_c^{s_c s_d}, \quad (13)\end{aligned}$$

and

$$\begin{aligned}\textstyle\sum_{ijkl} \quad & \delta_{ik}\delta_{jl}\delta_{s_a s_c}\delta_{s_b s_d}(1-\delta_{s_a s_b}\delta_{ij})(1-\delta_{s_c s_d}\delta_{kl}) = \\ & \sum_{ijkl} \delta_{ik}\delta_{jl}\delta_{s_a s_c}\delta_{s_b s_d} - \delta_{ik}\delta_{jl}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{ij} - \delta_{ik}\delta_{jl}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_c s_d}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{ij}\delta_{s_c s_d}\delta_{kl} = \\ & n_{\rm split}^{s_a} n_{\rm split}^{s_b}\delta_{s_a s_c}\delta_{s_b s_d} - n_{\rm split}^{s_a}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_a s_b} - n_{\rm split}^{s_a}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_a s_b} + n_{\rm split}^{s_a}\delta_{s_a s_c}\delta_{s_b s_d}\delta_{s_a s_b}\delta_{s_c s_d} = \\ & n_{\rm split}^{s_a}(n_{\rm split}^{s_b}\delta_{s_a s_c}\delta_{s_b s_d} - \delta_{s_a s_b s_c s_d}) = \\ & g_{s_a s_c, s_b s_d} n_c^{s_a s_b} n_c^{s_c s_d} \quad (14)\end{aligned}$$

With these expressions we can write

$$T^{X_\alpha^{s_a} W_\gamma^{s_c}}_{Y_\beta^{s_b} Z_\eta^{s_d}} = \left(C_\ell^{X_\alpha W_\gamma} C_\ell^{Y_\beta Z_\eta} + f^{s_a s_c}_{s_b s_d} C_\ell^{X_\alpha W_\gamma} N_{\ell, s_b}^{Y_\beta Z_\eta} + f^{s_b s_d}_{s_a s_c} C_\ell^{Y_\beta Z_\eta} N_{\ell, s_a}^{X_\alpha W_\gamma} + g_{s_a s_c, s_b s_d} N_{\ell, s_a}^{X_\alpha W_\gamma} N_{\ell, s_b}^{Y_\beta Z_\eta} \right) \quad (15)$$

$$\begin{aligned} f^{s_a s_c}_{s_b s_d} &= \frac{n^{s_b}_{\rm split}(n^{s_a}_{\rm split} n^{s_c}_{\rm split} \delta_{s_b s_d} - n^{s_c}_{\rm split}\delta_{s_a s_b s_d} - n^{s_a}_{\rm split}\delta_{s_b s_d s_c} + \delta_{s_a s_b s_d s_d})}{n^{s_a}_{\rm split} n^{s_c}_{\rm split}(n^{s_b}_{\rm split} - \delta_{s_a s_b})(n^{s_d}_{\rm split} - \delta_{s_c s_d})} \\ g_{s_a s_c, s_b s_d} &= \frac{n^{s_a}_{\rm split}(n^{s_b}_{\rm split}\delta_{s_a s_c}\delta_{s_b s_d} - \delta_{s_a s_b s_c s_d})}{n^{s_a}_{\rm split} n^{s_c}_{\rm split}(n^{s_b}_{\rm split} - \delta_{s_a s_b})(n^{s_d}_{\rm split} - \delta_{s_c s_d})} \end{aligned} \quad (16)$$

2 Beam covariance

let's compute the form of the beam covariance, we assume T = P beam, and that the beam do not depend on split, we therefore only keep one index (e.g α) to denote the array band

$$\Xi^{\alpha\beta\gamma\eta} = \langle (C_\ell^{\alpha\beta} - \langle C_\ell^{\alpha\beta}\rangle)(C_\ell^{\gamma\eta} - \langle C_\ell^{\gamma\eta}\rangle)\rangle = \langle C_\ell^{\alpha\beta} C_\ell^{\gamma\eta}\rangle - \langle C_\ell^{\alpha\beta}\rangle\langle C_\ell^{\gamma\eta}\rangle \quad (17)$$

Let's assume the beam we measure is denoted by B^α_ℓ and that the true beam is $B^\alpha_\ell + \delta B^\alpha_\ell$. Now let's denote C_ℓ the actual spectra on the sky, we can notice that our estimate for the beam deconvolved spectrum is biased

$$\langle C_\ell^{\alpha\beta}\rangle = \left\langle \frac{B^\alpha_\ell + \delta B^\alpha_\ell}{B^\alpha_\ell} \frac{B^\beta_\ell + \delta B^\beta_\ell}{B^\beta_\ell} \right\rangle C_\ell = \left\langle \left(1 + \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\right)\left(1 + \frac{\delta B^\beta_\ell}{B^\beta_\ell}\right)\right\rangle C_\ell = \left(1 + \left\langle \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\frac{\delta B^\beta_\ell}{B^\beta_\ell}\right\rangle\right) C_\ell \quad (18)$$

Now let's compute the covariance

$$\begin{aligned} \Xi^{\alpha\beta\gamma\eta} &= \left\langle \left(1 + \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\right)\left(1 + \frac{\delta B^\beta_\ell}{B^\beta_\ell}\right)\left(1 + \frac{\delta B^\gamma_\ell}{B^\gamma_\ell}\right)\left(1 + \frac{\delta B^\eta_\ell}{B^\eta_\ell}\right)\right\rangle C_\ell^2 \\ &- \left(1 + \left\langle \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\frac{\delta B^\beta_\ell}{B^\beta_\ell}\right\rangle\right)\left(1 + \left\langle \frac{\delta B^\gamma_\ell}{B^\gamma_\ell}\frac{\delta B^\eta_\ell}{B^\eta_\ell}\right\rangle\right) C_\ell^2 \\ &\sim \left(\left\langle \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\frac{\delta B^\gamma_\ell}{B^\gamma_\ell}\right\rangle + \left\langle \frac{\delta B^\alpha_\ell}{B^\alpha_\ell}\frac{\delta B^\eta_\ell}{B^\eta_\ell}\right\rangle + \left\langle \frac{\delta B^\beta_\ell}{B^\beta_\ell}\frac{\delta B^\gamma_\ell}{B^\gamma_\ell}\right\rangle + \left\langle \frac{\delta B^\beta_\ell}{B^\beta_\ell}\frac{\delta B^\eta_\ell}{B^\eta_\ell}\right\rangle\right) C_\ell^2 \end{aligned} \quad (19)$$

where we dropped all term of order >2 in beam errors. Assuming no correlation in the beam measurement from different arrays :

$$\Xi^{\alpha\beta\gamma\eta} = \left[(\delta_{\alpha\gamma} + \delta_{\alpha\eta}) \left\langle \frac{\delta B^\alpha_\ell \delta B^\alpha_\ell}{B^\alpha_\ell B^\alpha_\ell}\right\rangle + (\delta_{\beta\gamma} + \delta_{\beta\eta}) \left\langle \frac{\delta B^\beta_\ell \delta B^\beta_\ell}{B^\beta_\ell B^\beta_\ell}\right\rangle \right] C_\ell^2 \quad (20)$$