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Mathematics

Mathematics in Science and Society

To what extent does Wittgenstein prove that Mathematics is Invented?

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1 Introduction

Ludwig Wittgenstein's 'Tractatus Logico Phiolosophicus' [Wittgenstein, L. 1922], is one of the earlier works in Wittgenstein's academic career. This essay will focus on how far Wittgenstein's early formalist views [Simons, P 2009, p. 291] go to argue that mathematics is purely invented. This question will be addressed in two parts; mathematics is intrinsically linked to the concepts both of number and logic, so it is necessary to disentangle the two and consider their invention separately. Bertrand Russell's set theoretic definition of number (effectively from Frege's) will be contrasted with Wittgenstein's logical, inductive definition, in order to ascertain which of the two approaches is more 'natural' and if naturalism is the right criterion.

In the second section, mathematics is discussed as a language. Even if a Platonist argument can be established [Bouveresse, J. 2005], Wittgenstein raises questions that 'human mathematics' (for the lack of a better term) can never be separated from the languages we use to think, write or describe it [Wittgenstein, L. 1939]. Subsequently the essay will consider to what extent mathematical intuition can be trusted, with reference to Kurt Gödel. Finally the conclusion is reached that while we can never answer the question directly, we can see if mathematics and language are inextricable, and numbers are invented too, then we have a strong argument for a formalistic approach to mathematics.

2 Are Numbers Invented?

Wittgenstein's definition is actually quite simple for any mathematician to understand, as it just the concept of induction - *"The general form of the cardinal number is:* $[0, \xi, \xi + 1]$*"* [Wittgenstein, L. 1922, 6.03]. In this we clearly see a basis case, zero, and the inductive 'step' between two numbers. Wittgenstein chooses to formalize this as *"the exponent of an operation"* [Wittgenstein, L. 1922, 6.021], so instead of just defining it as addition (which would already require a concept of number) he bases it on his earlier definition of what he calls the *"general truth function"* [Wittgenstein, L. 1922, 6.0], which is the form that all propositions take. Explicitly in 6.02 he defines the 'step' as:

$$\Omega'\Omega^{\nu\prime}x = \Omega^{\nu+1}x$$

,

so that every subsequent application of the operation (Ω) increments the exponent ($\nu \rightarrow \nu + 1$), thereby creating cardinal numbers.

His formalist views become increasingly apparent; in Wittgenstein's eyes the world is logic, and so a definition of any concept should be able to be defined logically [Wittgenstein, L. 1922, 1.13, 2.18, 3]. He then succinctly addresses this issue with his assertion "Mathematics is a method of logic" [Wittgenstein, L. 1922, 6.234]. Wittgenstein, especially as a trained engineer, was a very functional man. Not only did he need to see the use in an object or concept, but also in his writing on language and games he states that it is only in the utility of language that words have any meaning *"what is thinkable is also possible"* [Wittgenstein, L. 1922, 3.02]. In formalizing cardinal numbers as above, we can see these ideas are in their infancy with regards to mathematics, but it is nevertheless clear that Wittgenstein has a clear stance, opposite Russell:

> *"The Cardinal Number of a is defined as the class of all classes similar to a. This definition is due to Frege."* [Whitehead, A., Russell, B. 1912, Vol. 2, p. 4].

Clearly Russell bases his construction of a cardinal number in set theory, but the meaning of his exact quote might be unclear. The argument he essentially follows is that if there's a one-to-one mapping, or bijection, between elements of two sets, then they must have the same 'number' of elements (notice we do not define the numbers 0 or 1 first). From this we have the number five, say, as the quality that all sets containing five elements share. Strict ordering of numbers in this sense arises if we have two sets and make an attempt at a bijection; the set that has all of its elements 'used up' first is clearly the smaller set. Evolutionarily speaking, this might be the construction we would be inclined to use - if one is asked to think of the number 3, the typical response is to imagine a set of 3 things. Whilst the well ordered nature of these sets comes somewhat after the definition, we can separate the idea of number from the idea of counting. The work of natural scientists such as Koehler [Davis, H., Memmott, J. 1982, p. 551] can identify ways in which animals recognize number. Indeed, it is logical that it would be evolutionarily desirable to be able to distinguish between more or less food, and identifying where there are fewest predators. Birds can typically identify the 'size' of a set, but mammals much closer to us genetically have a stronger concept of number, as Charles Ferster notes on chimpanzees:

> *"We succeeded in bringing out the abstract property of number by varying the size, shape, arrangement and other physical properties of the objects presented, so that the chimpanzee could give the correct answer consistently only if it singled out the number of objects as the important property. (Ferster, 1964, p. 105)."* [Davis, H., Memmott, J. 1982, p. 553]

The importance of a trans-species definition of number is that it is hard to argue that it is not a priori. Wittgenstein's definition of cardinal numbers relies on induction, and *"The so-called law of induction cannot in any case be a logical law, for it is obviously a significant proposition. And therefore it cannot be a law a priori either."* [Wittgenstein, L. 1922, 6.31]. Russell says *"a cardinal number appertains to a class, not to the members of the class."* [Whitehead, A., Russell, B. 1912, Vol. 2, p. 4]. So if we choose to define number set-theoretically, we then have an a priori definition of number, as property of a general set, rather than something we have to know through experience or induce logically as the exponent of an operation. Wittgenstein also says *"The concept number is nothing else than that which is common to all numbers, the general form of number."* [Wittgenstein, L. 1922, 6.022], which is in clear disagreement to Russell's (Frege's) definition in which number is a property of a set.

3 Mathematical Language and Intuition

Wittgenstein also runs into problems with his construction of number when considering infinity. A set theoretic definition of numbers like Cantor's (effectively the same as Frege/Russell's) admits the possibility of infinities in a consistent way [Dauben, J.W., 1990]. Whilst Wittgenstein could, according to Russell, extend his system to include transfinite numbers, he in fact chooses not to as his formalism turns to finitism [Marion, M. 1995]. It is his view that the word infinity is not a 'number word' as an infinite set is defined by a recursive rule, whereas a finite set can be described wholly by the elements it contains. This is a good example of Wittgenstein's ethos; that there exist propositions that cannot be fully described by our language, and it is there that inconsistencies and naïve assumptions form. Moreover, it is his view that language will always fail to do so, and that we must resign ourselves to this fate.

> *"Everything, therefore, which is involved in the very idea of the expressiveness of language must remain incapable of being expressed in language, and is, therefore, inexpressible in a perfectly precise sense. This inexpressible contains, according to Mr Wittgenstein, the whole of logic and philosophy."* [Wittgenstein, L. 1922, (Foreword by Russell, B.) p.18]

We have a native language in which we discuss mathematics e.g. English or German, and we have a mathematical language in which we perform computation and proof. Wittgenstein is simply being wary. As the two have separate (although possibly overlapping) imprecisions and inexpressible statements, is it possible that our intuition could falter, as we have no unbiased way to discuss, think or calculate? Wittgenstein emphatically says yes, and from that the argument for this essay forms thusly: if our mathematical language and our mathematical prose cannot be separated, then in some respect the mathematics we perform is as invented as our language is. Furthermore, if numbers are also invented then any mathematics performed with numbers must also be invented.

Non-Euclidean geometry is a prime example of a situation where a standard was established intuitively because of the world around us [Coxeter, H.S.M., 1998, Chapter 1]. Indeed Euclid's parallel postulate seems to hold in any experiment one could perform, but mathematically speaking, other avenues of possibility have been missed, due to the axiomatic nature of statements such as the parallel postulate. Complex numbers have only been in use for around three hundred years [Nahin, P.J.,2010, p.6], illustrating that what we can intuit from the real world is not enough for good mathematical intuition. The term imaginary when applied here causes contention between some mathematicians, and there are many others where the meaning we assume from our native language is not the exact mathematical meaning, causing us conceptual problems when discussing these topics.

Kurt Gödel, the famous mathematician and logician, believed that mathematical intuition is the *"equivalent of what perception is for physical objects"*, and while we abstract or idealize objects that exist in the world, there really does exist a mathematical world independent of ours [Bouveresse, J. 2005, p. 58]. Wittgenstein believed abstract existence like this is ridiculous, but he would have agreed with the fact that we can never attain the exact 'form' of a particular idea. *"Mechanics is an attempt to construct according to a single plan all true propositions which we need for the description of the world."* [Wittgenstein, L. 1922, 6.343]. The key words here are attempt, and description. Wittgenstein sees a distinction between the mathematical world (if it exists) and the world which we inhabit. We can at best only describe the world with mathematics, Wittgenstein says, rather than using whatever tools we can to discover the mathematics that is already there. *"In life it is never a mathematical proposition which we need, but we use mathematical propositions only in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics."* [Wittgenstein, L. 1922, 6.211]. Given a proposition about the real world, we can translate it into mathematical language and perform calculation by means of substitution. We then formulate an answer and then attempt to translate that answer back into the real world. However we cannot make any direct statement about the world from the mathematics, as *"Mathematics is a logical method"* [Wittgenstein, L. 1922, 6.2] and *"the propositions of logic are tautologies."* [Wittgenstein, L. 1922, 6.1].

So, we are stuck with limitations of using our language to describe mathematics insofar in that there is no way to think about mathematics other than in a language. If I were to state a suitably complex proposition in Chinese prose, it may have significantly different meaning in English. As above, the method for inferring conclusions involves a translation between natural language and mathematical language. It is largely in this translation that we develop problems - we must intuit what our answer means, and how it actually relates to the state of affairs at hand. But what of the calculations, how might they obfuscate the true meaning of a proposition? Mathematics today takes place in the decimal world. This a non-trivial choice, based only on us having ten fingers on which to count [Pickover, C.A., 2003, p. 62], and so has nothing to do with the actual mathematics we are trying to investigate. The choice raises problems; the question of whether there are infinitely many occurrences of a particular digit in the decimal expansion of an irrational number like $\sqrt{2}$ is still unknown, and it is even unknown how valuable an answer would be. Whether we take Wittgenstein's view that mathematics is at a detriment because of this intuition or Gödel's view that it is integral to mathematical progress, using an arbitrary base system for our numbers may make, or solve, problems that would nor exist in another system. This is, as much as the choice of language we use, inescapable in how we view mathematics. It shapes the questions we ask and the answers we obtain.

4 Conclusion

In this essay cardinal numbers have been defined in two separate ways, the strongest evidence lying in favour of Frege and Russell's set theoretic definition, which seems to imply numbers are not just invented. Wittgenstein's inductive definition makes no distinction between counting and number (strict ordering of sets is sufficient but not necessary in order to define the 'number' of elements in a set) so it seems logical that we should favour an a priori, set-theoretic definition of number, if only by Occam's razor [Wittgenstein, L. 1922, 3.328]. Sets exist in nature, hence number is not a concept invented by humanity. However it is intuitive leaps such as this about which Wittgenstein is wary, but if we are not so quick to doubt our senses then we can agree with Gödel that we have useful intuition and we are striving towards an ideal version of mathematics that actually exists somewhere.

And so it seems that there is a distinction to be made between the mathematical ideal and mathematics that we have created and know today. In this case it will most likely never be certain whether the ideal mathematics is invented or not. The mathematics that we have created has been invented, because it is at best an approximation to this mathematical ideal. Mathematics is made up of tautological propositions of logic about arbitrary symbols or numbers. The symbolic nature is effectively just shorthand for mathematical prose, which performs the same job in common speech. If one is of the opinion that language is invented, as indeed Wittgenstein is, then the mathematics we have created must be by definition invented, as it is inextricably linked to and defined in terms of said language, regardless of whether there exists a platonic ideal towards which we strive.

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