An intermezzo on cones

To a lattice $N \simeq \mathbb{Z}^n$ we associate a real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$. The dual vector space $M_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) \simeq M \otimes_{\mathbb{Z}} \mathbb{R}$ comes from the dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

A <u>convex polyhedral cone</u> (CPC) in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{cone}(S) = \left\{ \sum_{u \in S}' \lambda_u u \;\middle|\; \lambda_u \in \mathbb{R}_{\geq 0} \right\} \subset N_{\mathbb{R}},$$

where $S \subset N_{\mathbb{R}}$ is a finite subset.
↳ set of 'generators'

A CPC σ is <u>rational</u> if $\sigma = \mathrm{cone}(S)$ for $S \subset N$.
(finite)

Given a CPC $\sigma \subset N_{\mathbb{R}}$, its <u>dual cone</u> is

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \}$$

FACT: $\sigma^\vee$ is a CPC and $(\sigma^\vee)^\vee = \sigma$.

Let $H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle = 0\} \subset N_{\mathbb{R}}$,

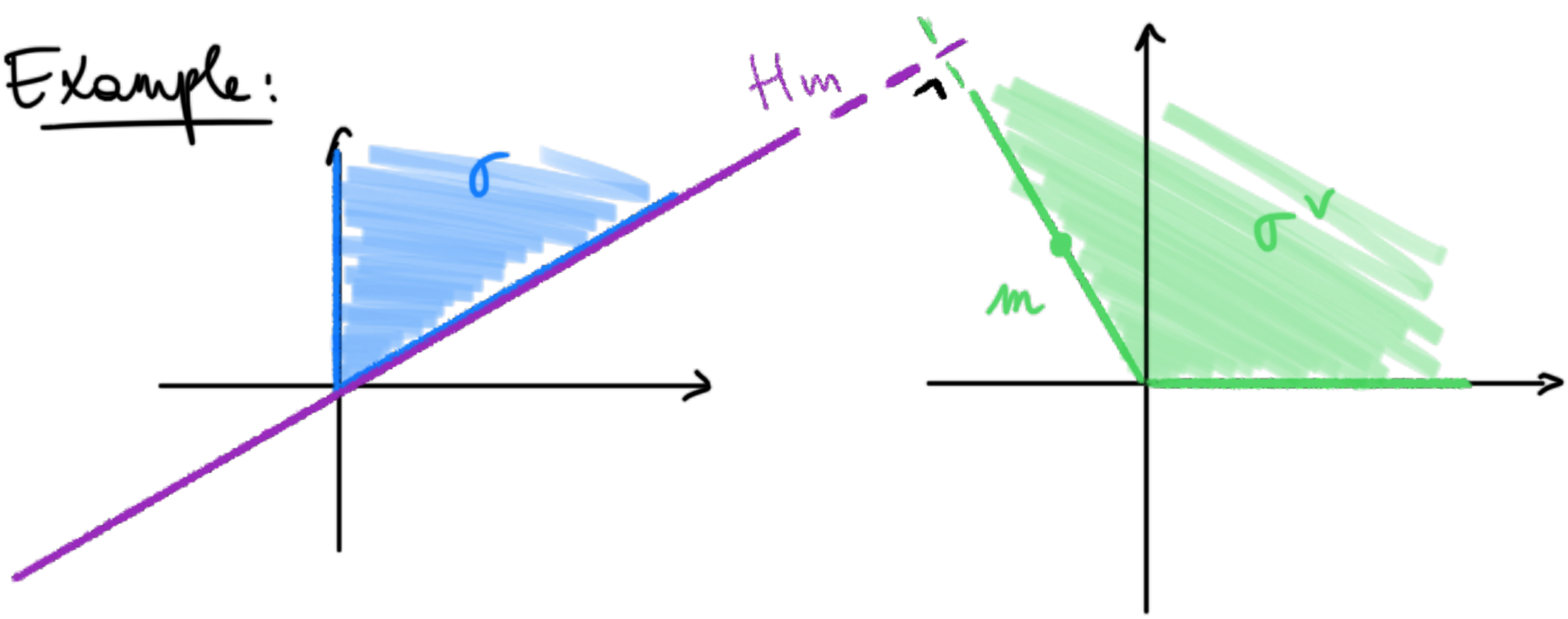
$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle \geq 0\} \subset N_{\mathbb{R}}.$$

A face τ of σ is given by

$$\tau = \sigma \cap H_m \text{ for some } m \in \sigma^\vee.$$

We write $\tau \preceq \sigma$ and $\tau \prec \sigma$ if $\tau \neq \sigma$.

Example:



If $\{m_1, \dots, m_s\} \subset M$ generate $\sigma^\vee$, then

$$\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+,$$

and vice versa.

- faces are CPCs
- $\tau \preceq \sigma,\ \tau' \preceq \sigma \Rightarrow \tau \cap \tau' \preceq \sigma$
- $\tau \preceq \sigma,\ \tau' \preceq \tau \Rightarrow \tau' \preceq \sigma$
- If $\tau \preceq \sigma$, $v, w \in \sigma$, then

  $$v + w \in \tau \Rightarrow v \in \tau \text{ and } w \in \tau.$$
- there is a bijective, inclusion-reversing correspondence between faces of σ and faces of $\sigma^\vee$:

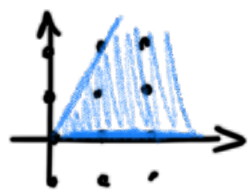
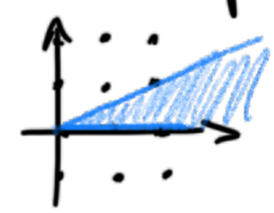
A CPC is strongly convex if $\{0\}$ is a face of σ. Equivalently $\sigma \cap (-\sigma) = \{0\}$, or $\dim \sigma^\vee = n$.

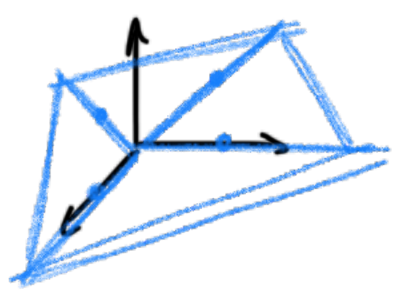
A rational CPC (RCPC) in $N_{\mathbb{R}}$ is smooth if its minimal generators form part of a $\mathbb{Z}$-basis of N.

A RCPC in $N_{\mathbb{R}}$ is <u>simplicial</u> if its minimal generators are $\mathbb{R}$-linearly independent.

(obviously : smooth $\Rightarrow$ simplicial)

Examples: • all 2-dim cones are simplicial.

•  is not smooth,  is.

•  $\mathrm{Cone}(\{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\})$ is not simplicial.

1.5 Cones and affine toric varieties

Let σ be a RCPC. Consider the semigroup

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

<u>Gordan's lemma</u>: S_σ is affine.

Theorem: Let $\sigma \subset N_{\mathbb{R}}$ be a RCPC with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$\mathcal{U}_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety. Furthermore,

$$N \otimes_{\mathbb{Z}} \mathbb{C}^* = T_N$$

$$\dim \mathcal{U}_\sigma = n \iff \text{the dense torus of } \mathcal{U}_\sigma \text{ is } T_N$$

$$\iff \sigma \text{ is strongly convex.}$$

proof: The first statement is obvious. We also know that $\dim \mathcal{U}_\sigma = \operatorname{rank} \mathbb{Z}S_\sigma$. The key is to show $\operatorname{rank} \mathbb{Z}S_\sigma = n \iff \mathbb{Z}S_\sigma = M$

$$\iff \sigma \text{ strongly convex} \quad \square$$

Example: $\sigma = \mathrm{Cone}(e_1, \dots, e_r) \subset \mathbb{R}^n$, $r \leq n$.

Then $\sigma^\vee = \mathrm{Cone}(e_1', \dots, e_r', \pm e_{r+1}', \dots, \pm e_n')$

and $\mathcal{U}_\sigma = \mathrm{Specm}\, \mathbb{C}[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_n^{\pm 1}]$

$$= \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}.$$

$\Rightarrow$ $\mathcal{U}_\sigma$ looks like this for all smooth cones σ.

Example : $\mathcal{A} = \{(1,0), (0,1), (1,1)\} \subset \mathbb{Z}^2$

$$I = \langle xy - z \rangle \subset \mathbb{C}[x,y,z]$$

$$S = \mathbb{N}\mathcal{A}$$

$\sigma =$  (looks the same as its dual)

give the same affine toric variety, isom. to $\mathbb{C}^2$.

How to find generators of S_σ ?

$m \in S_\sigma$ is called <u>irreducible</u> if $m = m' + m''$ with $m', m'' \in S_\sigma$ implies $m' = 0$ or $m'' = 0$.

<u>Prop</u> : Let $\mathcal{H} = \{ m \in S_\sigma \mid m \text{ is irreducible} \}$.

(a) $\mathcal{H}$ is finite and generates S_σ.

(b) $\mathcal{H}$ contains the ray generators of $\sigma^\vee$.

(c) $\mathcal{H}$ is the minimal generating set w.r.t. inclusion

$\mathcal{H}$ is called the <u>Hilbert basis</u> of S_σ.

1.6 Points on affine toric varieties

Prop Let $V = \mathrm{Specm}(\mathbb{C}[S])$ for an affine semigroup $S \subset M$. Then there are 1:1 correspondences between

(a) points $p \in V$

(b) maximal ideals $m_p \in \mathbb{C}[S]$

(c) semigroup homomorphisms $S \to \mathbb{C}$.

image of 0 is 1 !!

proof (a) $\xleftrightarrow{1:1}$ (b) is standard and holds for any affine variety.

(a) $\longrightarrow$ (c) given $p \in V$, define $S \to \mathbb{C}$ by $m \mapsto \chi^m(p)$.

(c) $\longrightarrow$ (b) given a semigroup homomorphism $\gamma : S \to \mathbb{C}$, we obtain an induced map of $\mathbb{C}$-algebras $\mathbb{C}[S] \longrightarrow \mathbb{C}$ which is surjective $(\gamma(0) = 1)$.

Call its kernel $\mathfrak{m}_p$. This is maximal because $0 \to \mathfrak{m}_p \to \mathbb{C}[S] \to \mathbb{C} \to 0$ is exact.

(c) $\to$ (a) (more concretely) Let $S = \mathbb{N}\mathcal{A}$ with $\mathcal{A} = \{m_1, \dots, m_s\}$. Let

$p = (\gamma(m_1), \dots, \gamma(m_s)) \in \mathbb{C}^s$. If

$\alpha, \beta \in \mathbb{N}^s$ are such that $\alpha - \beta \in \ker \hat{\phi}_{\mathcal{A}}$,

we have
$$p^\alpha - p^\beta = \prod_{i=1}^{s} \gamma(m_i)^{\alpha_i} - \prod_{i=1}^{s} \gamma(m_i)^{\beta_i}$$
$$= \gamma\left(\textstyle\sum \alpha_i m_i\right) - \gamma\left(\textstyle\sum \beta_i m_i\right)$$
$$= 0$$

hence $p \in Y_{\mathcal{A}}$. Moreover, in step ((a) $\to$ (c)) we associated the semigroup homomorphism $m_i \mapsto \chi^{m_i}(p) = p_i = \gamma(m_i)$ to p. $\square$.

<u>Example</u> : intrinsic description of the group action :

$$p \sim (m \mapsto \gamma(m)) \implies t \cdot p \sim (m \mapsto \chi^m(t)\,\gamma(m))$$

Exercise 1.3.1

<u>Prop</u> Let V be an affine toric variety :

$$V = \mathrm{Specm}(\mathbb{C}[S]) = Y_{\mathcal{A}} \subset \mathbb{C}^s \quad (\mathcal{A} = S \setminus \{0\}).$$

(a) The torus action has a fixed point if and only if S is pointed. In this case, the fixed point is $\gamma : m \longmapsto \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$ unique!

(b) The torus action has a fixed point if and only if $0 \in Y_{\mathcal{A}}$, in which case it is 0.

<u>proof</u> (a) $\chi^m(t)\,\gamma(m) = \gamma(m) \quad \forall t$

$$\Rightarrow \quad m = 0 \ \&\ \gamma(m) = 1 \ \text{ or } \ m \neq 0,\ \gamma(m) = 0$$

(b) follows straightforwardly from (a), and the fact that $0 \in \mathbb{C}^s$ is fixed by $(\mathbb{C}^*)^s \curvearrowright \mathbb{C}^s$.

<u>Cor</u>: Let U_σ be the affine toric variety corresponding to a strongly convex RCPC. Then the torus action has a fixed point iff $\dim \sigma = \dim N_{\mathbb{R}}$. In this case, the fixed point is given by the max. ideal $\langle \chi^m \mid m \in S_\sigma \setminus \{0\} \rangle \subseteq \mathbb{C}[S_\sigma]$.

<u>Exercises</u>: 1.3.1 , 1.3.2 , 1.3.3

Let $\sigma = \mathrm{Cone}((1,2,3,4,5),(2,1,5,4,3),(5,4,3,2,1),$

$(3,2,1,5,4),(1,5,4,3,2))$

Embed U_σ in the smallest possible ambient affine space. How small is this? Is U_σ smooth? What's its dimension? How many binomials generate the toric ideal?