Orbit-Cone correspondence:

(a) $\{\text{cones } \sigma \text{ in } \Sigma\} \longleftrightarrow \{T\text{-orbits } O(\sigma) \subset X_\Sigma\}$.

(b) $\dim \sigma = n - \dim O(\sigma)$

(c) $\mathcal{U}_\sigma = \bigsqcup_{\tau \leq \sigma} O(\tau)$

(d) $\overline{O(\tau)} = \bigsqcup_{\tau \leq \sigma} O(\sigma)$

3.6 Toric morphisms (between abstract toric varieties)

Def: Let $X_{\Sigma_1}, X_{\Sigma_2}$ be normal toric varieties, with dense tori T_1, T_2. A morphism $\phi: X_{\Sigma_1} \to X_{\Sigma_2}$ is toric if $\phi(T_1) \subset T_2$ and $\phi_{|T_1}$ is a group homomorphism.

Lemma A toric morphism $\phi: X_{\Sigma_1} \to X_{\Sigma_2}$ is equivariant. that is $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$.

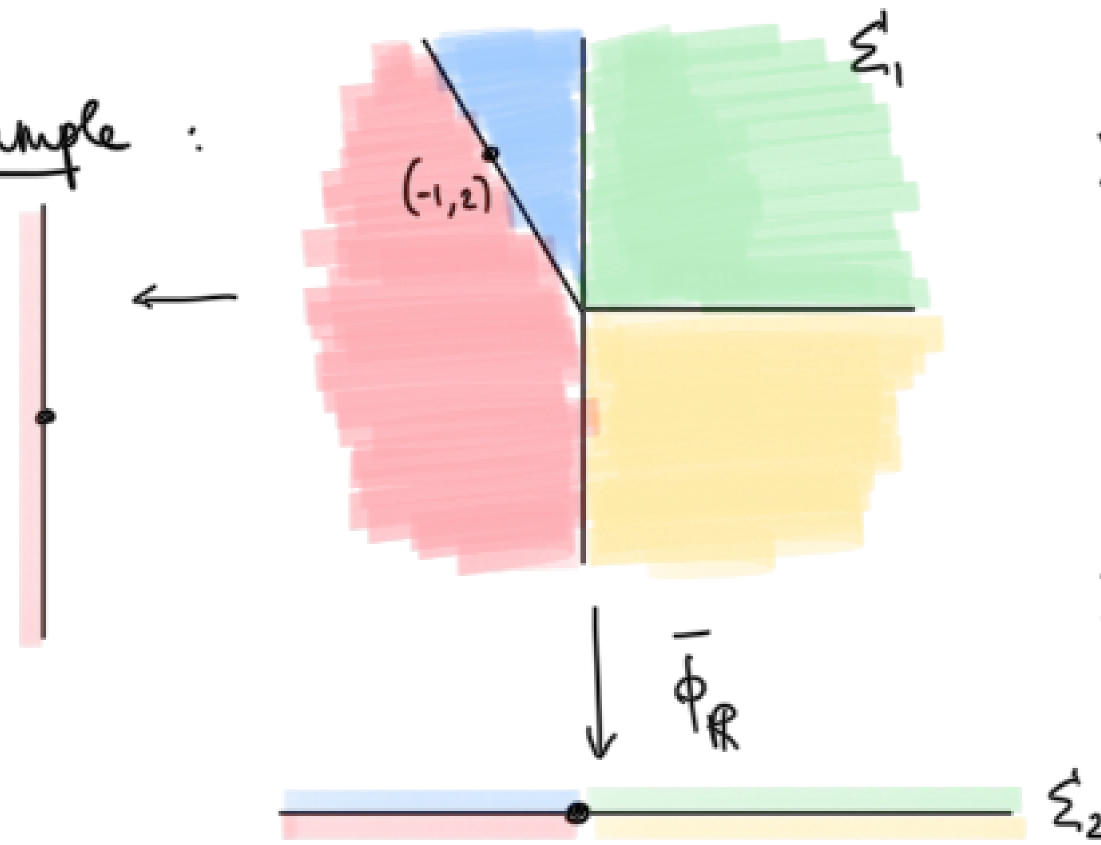
proof

$$\begin{array}{ccc} T_1 \times T_1 & \longrightarrow & T_1 \\ \downarrow & & \downarrow \phi_{|T_1} \\ T_2 \times T_2 & \longrightarrow & T_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} T_1 \times X_{\Sigma_1} & \longrightarrow & X_{\Sigma_1} \\ \downarrow \phi_{|T_1} \times \phi & & \downarrow \phi \\ T_2 \times X_{\Sigma_2} & \longrightarrow & X_{\Sigma_2} \end{array} \quad \square$$

Def

Let Σ_i be a fan in $(N_i)_{\mathbb{R}}$, $i=1,2$. A $\mathbb{Z}$-linear map $\overline{\phi}: N_1 \to N_2$ is <u>compatible</u> with Σ_1, Σ_2 if for each $\sigma_1 \in \Sigma_1$, there is $\sigma_2 \in \Sigma_2$ s.t. $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subset \sigma_2$.

<u>Example</u> :



X_{Σ_1} = Hirzebruch surface

$\downarrow \phi$

$X_{\Sigma_2} = \mathbb{P}^1$

<u>Lemma/def</u> Let $X = \bigcup_i \mathcal{U}_i$ and Y be varieties, and let $\phi_i : \mathcal{U}_i \to Y$ be morphisms. A morphism $\phi: X \to Y$ is <u>glued</u> from the ϕ_i if $\phi_{|\mathcal{U}_i} = \phi_i$. Such a ϕ exists iff $\phi_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \phi_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$.

<u>Theorem</u> Let Σ_i be a fan in $(N_i)_{\mathbb{R}}$, $i=1,2$.

(a) If $\bar{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1, Σ_2, then it induces a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ satisfying $\phi_{|T_1} = \bar{\phi} \otimes \mathrm{id} : u \otimes t \to \bar{\phi}(u) \otimes t$.

(b) If $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$, compatible with Σ_1, Σ_2.

<u>proof</u> (a) $\sigma_i \in \Sigma_i$, $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subset \sigma_2$. We have seen that $\bar{\phi}$ induces a morphism $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2}$ (if $\bar{\phi}$ is given by a matrix F, this comes from $\hat{\phi}(m) = F^T m$.)

These agree on overlaps, so they glue to $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$.

ϕ is toric since $\phi_{\{0\}} : T_1 \to T_2$ is the group homomorphism $\bar{\phi} \otimes \mathrm{id} : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \to N_2 \otimes \mathbb{C}^*$.

(in coordinates : $t \mapsto (t^{F_{ij}})_i$.

(b) $\phi_{|T_1}$ is a group homomorphism. It induces $\bar{\phi}: N_1 \to N_2$, by sending $n \in N_1$ to the cocharacter

$$\phi_{|T_1} \circ \lambda^n : \mathbb{C}^* \longrightarrow T_2 .$$

Since ϕ is equivariant, it sends the orbit $O(\sigma_1)$ into an orbit $O(\sigma_2)$, $\sigma_i \in \Sigma_i$. To show that $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subset \sigma_2$ it is enough to show that $\phi(\mathcal{U}_{\sigma_1}) \subset \mathcal{U}_{\sigma_2}$.

By OCC, $\mathcal{U}_{\sigma_1} = \bigsqcup_{\tau_1 \leq \sigma_1} O(\tau_1)$, $\mathcal{U}_{\sigma_2} = \bigsqcup_{\tau_2 \leq \sigma_2} O(\tau_2)$.

We need to show that $\phi(O(\tau_1)) \subset O(\tau_2)$ for some face $\tau_2 \leq \sigma_2$.

Let τ_2 be such that $\phi(O(\tau_1)) \subset O(\tau_2)$.

By OCC, $O(\sigma_1) \subset \overline{O(\tau_1)}$. By continuity, $\phi(\overline{O(\tau_1)}) \subset \overline{O(\tau_2)}$. Hence $O(\sigma_2) \subset \overline{O(\tau_2)}$, and the statement follows from OCC. $\square$.

Example : Let $N_i = \mathbb{Z}^2$, $i=1,2$ and $\bar{\phi} : u \mapsto l \cdot u$

This is compatible with the fan of $\mathbb{P}^2$. The matrix is $F = \begin{bmatrix} l & \\ & l \end{bmatrix}$, $(t_1, t_2) \mapsto (t_1^l : t_2^l)$ is the restriction of ϕ to the torus. Globally, ϕ is given by $\phi((x_0 : x_1 : x_2)) = (x_0^l : x_1^l : x_2^l)$.

Exercise : Compute ϕ in our previous example in coordinates.

SUBLATTICES OF FINITE INDEX.

Proposition : Let $N' \subset N$ be a sublattice of finite index, Σ a fan in $(N')_{\mathbb{R}} = N_{\mathbb{R}}$ and $G = N/N'$. Then $\bar{\phi} : N' \hookrightarrow N$ induces the morphism

$$\phi : X_{\Sigma, N'} \longrightarrow X_{\Sigma, N}$$

that presents $X_{\Sigma, N}$ as $X_{\Sigma, N'}/G$.

(see Ex. 3.3.8).

TORUS FACTORS

<u>Theorem</u> Let Σ be a fan in $N_{\mathbb{R}}$. TFAE:

(a) $X_\Sigma \simeq X_{\Sigma'} \times (\mathbb{C}^*)^r$ (X_Σ <u>has a torus factor</u>).

(b) There is a nonconstant morphism $X_\Sigma \to \mathbb{C}^*$

(c) The rays $\rho \in \Sigma(1)$ do not span $N_{\mathbb{R}}$

<u>proof</u>: (a) $\Rightarrow$ (b) $\quad X_\Sigma \to (\mathbb{C}^*)^r \xrightarrow{\chi^m} \mathbb{C}^*$.

(b) $\Rightarrow$ (c) $\phi: X_\Sigma \to \mathbb{C}^*$ non-constant implies $\phi_{|T}: T \to \mathbb{C}^*$ non-constant, hence $\phi_{|T} = c \cdot \chi^{m \neq 0}$. Multiplying by c^{-1}, we may assume $\phi_{|T} = \chi^m$. The corresponding map $\bar{\phi}$ is given by $\bar{\phi}(u) = \langle u, m \rangle$. Since $\bar{\phi}$ is compatible with Σ and $\{0\}$, we have $u_\rho \in \ker \bar{\phi}$ $\forall \rho \in \Sigma(1)$.

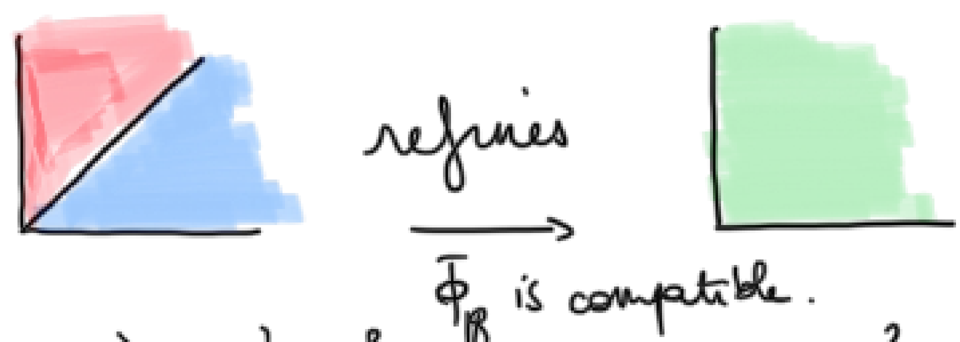
Therefore, the u_ρ are $\mathbb{R}$-linearly dependent.

Remains to show (c) $\Rightarrow$ (a), see Prop 3.3.9 in [CLS] $\square$.

REFINEMENTS.

A fan Σ' <u>refines</u> Σ_1 if every cone of Σ' is contained in a cone of Σ_1 and $|\Sigma'| = |\Sigma_1|$.

<u>Example</u>:



$V(x_0 y - x_1 x) \subset \mathbb{P}^1 \times \mathbb{C}^2$ $\qquad\qquad$ $\mathbb{C}^2$

Corresponds to the "blow-down" morphism $Bl_0(\mathbb{C}^2) \to \mathbb{C}^2$.

Let σ be a smooth n-dim. cone of Σ_1. The <u>star subdivision</u> $\Sigma^*(\sigma)$ of Σ_1 along σ is given by $\Sigma_1 \setminus \{\sigma\} \cup \Sigma'(\sigma)$ where $\Sigma'(\sigma)$ is constructed as follows. Let $\sigma = \mathrm{Cone}(u_1, \dots, u_n)$. Set $u_0 = u_1 + \dots + u_n$. $\Sigma_1'(\sigma)$ is the fan of all $\mathrm{Cone}(u_0, \dots, \hat{u}_i, \dots, u_n)$, $i \neq 0$, and all their faces.

<u>Example</u> $\quad \Sigma_1$

$\Sigma^*(\sigma)$

Proposition $\Sigma_1'^{+}(\sigma)$ refines Σ_1', and the induced toric morphism $\phi : X_{\Sigma_1'^{+}(\sigma)} \to X_{\Sigma_1'}$ makes $X_{\Sigma_1'^{+}(\sigma)}$ the blowup of $X_{\Sigma_1'}$ at γ_σ.

Proof By restricting ϕ, we may assume $\Sigma_1' = \{\sigma + \text{faces}\}$.

$\phi : X_{\Sigma_1'^{+}(\sigma)} \to U_\sigma \overset{\simeq \mathbb{C}^n}{}$ comes from the identity map $\overline{\phi} = id$ on N. The gluing construction shows that the affine pieces of $X_{\Sigma_1'^{+}(\sigma)}$ are those of $Bl_0(\mathbb{C}^n)$. $\square$.

Lemma : Let $\phi : X_\Sigma \to X_{\Sigma'}$ be the toric morphism coming from $\overline{\phi} : N \to N'$. Given $\sigma \in \Sigma$, let $\sigma' \in \Sigma'$ be the minimal cone s.t. $\overline{\phi}_{\mathbb{R}}(\sigma) \subset \sigma'$, then

(a) $\phi(\gamma_\sigma) = \gamma_{\sigma'}$

(b) $\phi(O(\sigma)) \subseteq O(\sigma')$ and $\phi(V(\sigma)) \subseteq V(\sigma')$

(c) $\phi_{|V(\sigma)} : V(\sigma) \to V(\sigma')$ is a toric morphism.

proof (a) $\bar{\phi}(u) \in \mathrm{Relint}(\sigma')$ if $u \in \mathrm{Relint}(\sigma)$ since σ' is minimal. Therefore

$$\phi(\gamma_\sigma) = \phi\left(\lim_{t\to 0} \lambda^u(t)\right) = \lim_{t\to 0} \phi(\lambda^u(t))$$
$$= \lim_{t\to 0} \lambda^{\bar{\phi}(u)}(t)$$
$$= \gamma_{\sigma'}.$$

(b) follows from (a).

(c) By equivariance, $\phi_{|O(\sigma)} : O(\sigma) \to O(\sigma')$ is a group homomorphism. $\square$.