§3: Abstract toric varieties

3.1 Gluing affine varieties

Consider a finite collection $\{V_\alpha\}_\alpha$ of affine varieties and suppose that for all α, β, we have Zariski open subsets $V_{\beta\alpha} \subset V_\alpha$ and isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \simeq V_{\alpha\beta}$ satisfying

1. $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all α, β.
2. $\left.\begin{array}{l} g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta} \\ g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha} \text{ on } V_{\alpha\beta} \cap V_{\gamma\beta} \end{array}\right\}$ for all α, β, γ

Let $Y = \bigsqcup_\alpha V_\alpha$ (disj. union) and define the equivalence relation [$a \sim b$ iff $a \in V_\alpha$, $b \in V_\beta$ and $b = g_{\beta\alpha}(a)$] on Y.

The quotient space $X = Y/\sim$ with the quotient topology is the abstract variety defined by the gluing data $\{V_\alpha\}, \{g_{\alpha\beta}\}$.

Zariski topology

or simply "variety"

For each α, let $\mathcal{U}_\alpha = \{[a] \mid a \in V_\alpha\} \subset X$.

and $h_\alpha : V_\alpha \xrightarrow{\sim} \mathcal{U}_\alpha$, $h_\alpha(a) = [a]$

$\mathcal{U}_\alpha \subset X$ is open, and X locally looks like an affine variety

Example: $V_0 = \mathbb{C} = \operatorname{Specm} \mathbb{C}[u]$

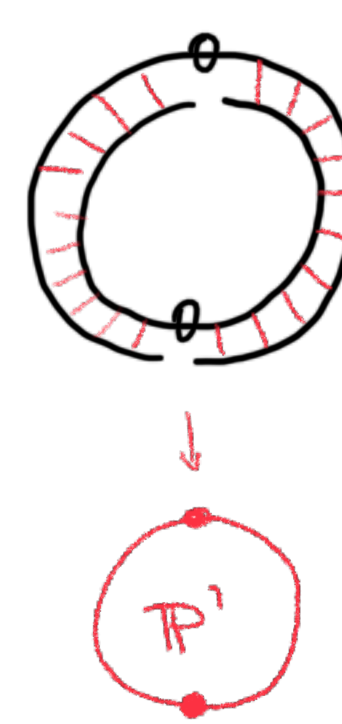
$V_1 = \mathbb{C} = \operatorname{Specm} \mathbb{C}[v]$

$V_{10} = V_0 \setminus V(u) = \operatorname{Specm} \mathbb{C}[u]_u$

$V_{01} = V_1 \setminus V(v) = \operatorname{Specm} \mathbb{C}[v]_v$

$g_{10} : V_{10} \longrightarrow V_{01}, \quad x \longmapsto \frac{1}{x}$

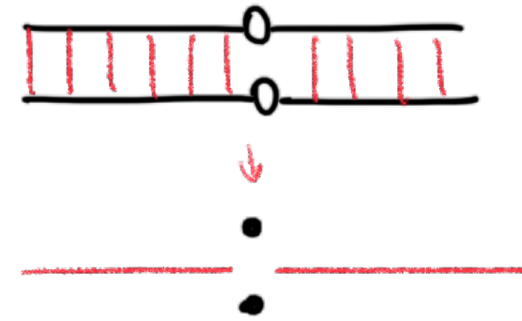
$g_{01} : V_{01} \longrightarrow V_{10}, \quad y \longmapsto \frac{1}{y}$



Example : V_0, V_1, V_{10}, V_{10} as above, but

$$g_{10} : V_{10} \longrightarrow V_{01} \quad , \quad x \longmapsto x$$

$$g_{01} : V_{01} \longrightarrow V_{10} \quad , \quad y \longmapsto y \, .$$



Here X is again a union of $\mathbb{C}^*$ and 2 points, but the classical topology on X is not Hausdorff $\Longleftrightarrow$ not <u>separated</u>.

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Let $X_1 = \bigcup_\alpha \mathcal{U}_\alpha$, $X_2 = \bigcup_\beta \mathcal{U}'_\beta$ be abstract varieties. A <u>morphism</u> $\Phi : X_1 \longrightarrow X_2$ is a Zariski continuous map such that

$$\Phi_{|\mathcal{U}_\alpha \cap \Phi^{-1}(\mathcal{U}'_\beta)} : \mathcal{U}_\alpha \cap \Phi^{-1}(\mathcal{U}'_\beta) \longrightarrow \mathcal{U}'_\beta$$

is a morphism of open subsets of affine varieties. (Def 3.0.3)

The product $X_1 \times X_2$ is an abstract variety as well, obtained by gluing $\mathcal{U}_\alpha \times \mathcal{U}'_\beta$ in the appropriate way.

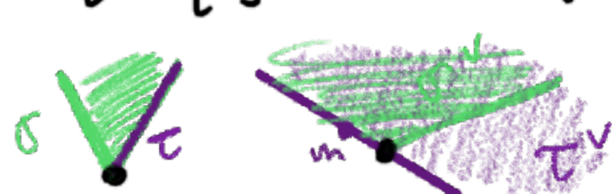
§ 3.2 Toric varieties from fans

A toric variety is an irreducible abstract variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action $\underbrace{T \times X \to X}_{\text{morphism of abstract varieties.}}$.

Recall that a fan Σ in $N_\mathbb{R}$ is a collection of cones that fit together in a nice way. Such a fan Σ encodes the gluing data for constructing X.

fact 1: faces of σ give affine open subsets of U_σ:

Let $\tau \preceq \sigma$, $\tau, \sigma \in \Sigma$ and $\tau = \sigma \cap H_m$, with $m \in \sigma^\vee$. Then $S_\tau = S_\sigma + \mathbb{Z}(-m)$ and $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$. $\Rightarrow U_\tau = (U_\sigma)_{\chi^m}$



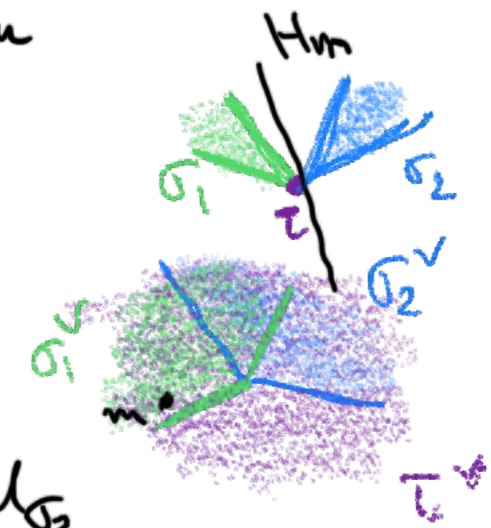
Fact 2: Common faces give common affine open subsets:

Let $\tau = \sigma_1 \cap \sigma_2$, $\sigma_1, \sigma_2, \tau \in \Sigma_1'$. Then

$$\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m$$

for some $m \in \sigma_1^\vee \cap (-\sigma_2^\vee) \cap M$.

$$\mathcal{U}_{\sigma_1} \supseteq (\mathcal{U}_{\sigma_1})_{\chi^m} \simeq \mathcal{U}_\tau \simeq (\mathcal{U}_{\sigma_2})_{\chi^{-m}} \subseteq \mathcal{U}_{\sigma_2}$$



GLUING DATA : • $\{\mathcal{U}_\sigma\}_{\sigma \in \Sigma_1'}$

• for $\sigma_1, \sigma_2 \in \Sigma_1'$, $\sigma_1 \cap \sigma_2 = \sigma_1 \cap H_m = \sigma_2 \cap H_m$

$$g_{\sigma_2,\sigma_1} : (\mathcal{U}_{\sigma_1})_{\chi^m} \simeq (\mathcal{U}_{\sigma_2})_{\chi^{-m}}$$

Exercise: these satisfy ① and ②

The resulting abstract variety is denoted by $X_{\Sigma_1'}$

Theorem: Let Σ_1' be a fan in $N_{\mathbb{R}}$. The variety $X_{\Sigma_1'}$ is a normal toric variety.

proof : each σ is strongly convex (by def. of fan).
Hence they all share the face $\{0\} \subset N_{\mathbb{R}}$; and
$T \simeq (\mathbb{C}^*)^n = \operatorname{Specm}(\mathbb{C}[M]) \subset U_\sigma$, $\forall \sigma \in \Sigma$.
all of these tori are identified by the gluing, so
$T \subset X_\Sigma$ is a Zariski open subset. The actions
$T \times U_\sigma \to U_\sigma$ are compatible on overlaps,
and therefore glue to a morphism $T \times X_\Sigma \to X_\Sigma$.
X_Σ is irreducible since each U_σ is irreducible and
contains T. X_Σ is normal since all U_σ are. $\square$

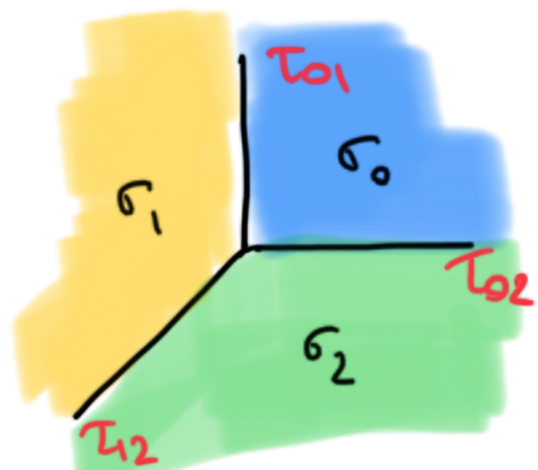
RMK : if $\tau \leq \sigma$, $U_\tau \subseteq U_\sigma$, so only need max cones to cover X_Σ.

Thm X_Σ is separated. (Ex. 3.1.2)

Example : Let P be a full-dimensional polytope with normal fan Σ_P. Then $X_P \simeq X_{\Sigma_P}$.

Theorem: Let X be a normal, separated toric variety with torus $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Then there exists a fan Σ' in $N_{\mathbb{R}}$ such that $X \simeq X_{\Sigma'}$.

Example:



$$\mathcal{U}_{\sigma_0} = \operatorname{Specm} \mathbb{C}[x, y] \simeq \mathbb{C}^2$$

$$\mathcal{U}_{\sigma_1} = \operatorname{Specm} \mathbb{C}[x^{-1}, x^{-1}y] \simeq \mathbb{C}^2$$

$$\mathcal{U}_{\sigma_2} = \operatorname{Specm} \mathbb{C}[xy^{-1}, y^{-1}] \simeq \mathbb{C}^2$$

$$\mathcal{U}_{\tau_{01}} = \underbrace{\operatorname{Specm} \mathbb{C}[x, y]_x}_{= \, \mathcal{U}_{\sigma_1, \sigma_0}} \simeq \underbrace{\operatorname{Specm} \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}}}_{= \, \mathcal{U}_{\sigma_0, \sigma_1}}$$

isomorphism: $x \longmapsto (x^{-1})^{-1}$

$$y \longmapsto x^{-1}y \cdot (x^{-1})^{-1}$$

$$x^{-1} \longmapsto x^{-1}$$

$$\mathbb{C}\left[\frac{x_1}{x_0},\frac{x_2}{x_0}\right]_{\frac{x_1}{x_0}} \simeq \mathbb{C}\left[\frac{x_0}{x_1},\frac{x_2}{x_1}\right]_{\frac{x_0}{x_1}}$$

usual glueing of $\mathbb{P}^2$:

$$\frac{x_1}{x_0} \longmapsto \left(\frac{x_0}{x_1}\right)^{-1}$$

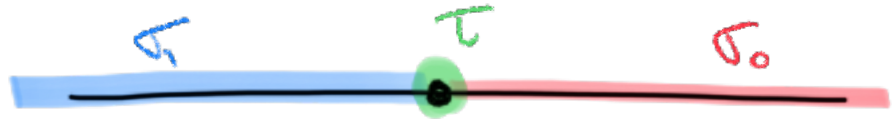
$$\frac{x_2}{x_0} \longmapsto \frac{x_2}{x_1} \Big/ \frac{x_0}{x_1}$$

$$\left(\frac{x_1}{x_0}\right)^{-1} \longmapsto \frac{x_0}{x_1}$$

$$\Longrightarrow X_{\Sigma_1'} = X_{\Delta_2} = \mathbb{P}^2.$$

Exercise: Similarly, $\mathbb{P}^1 \times \mathbb{P}^1$ is X_{Σ_1} for Σ': 

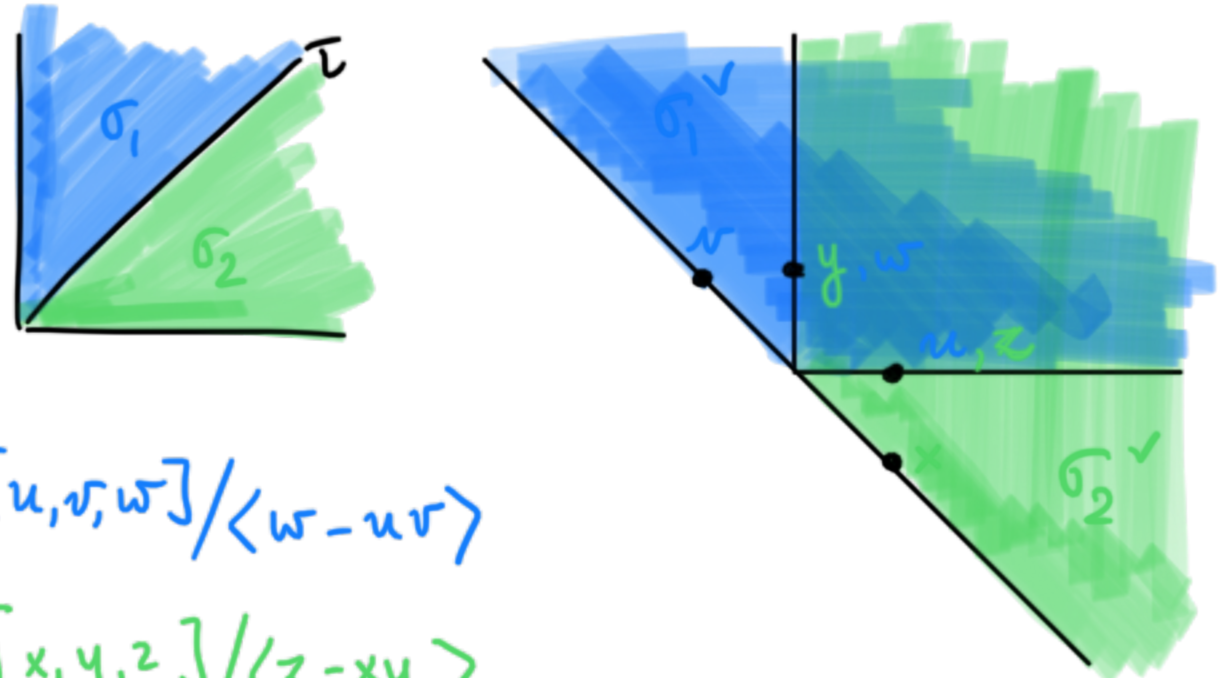
Example: if $N = \mathbb{Z}$, there are only 3 cones in $N_{\mathbb{R}}$:



$$\Sigma_1' = \begin{cases} \{\tau\} & \longrightarrow X_{\Sigma'} = \mathbb{C}^* \\ \{\sigma_0,\tau\},\{\tau,\sigma_1\} & \longrightarrow X_{\Sigma'} = \mathbb{C} \\ \{\sigma_0,\sigma_1,\tau\} & \longrightarrow X_{\Sigma'} = \mathbb{P}^1 \end{cases}$$

(Σ' is the normal fan of a line segment )

Example :



$U_{\sigma_1} = \operatorname{Specm} \mathbb{C}[u,v,w]/\langle w - uv\rangle$

$U_{\sigma_1} = \operatorname{Specm} \mathbb{C}[x,y,z]/\langle z - xy\rangle$

$$U_\tau = \operatorname{Specm}\left(\mathbb{C}[u,v,w]/\langle w-uv\rangle\right)_{[v]} = \operatorname{Specm}\left(\mathbb{C}[x,y,z]/\langle z-xy\rangle\right)_{[x]}$$

$$g_{\sigma_2,\sigma_1}^{\ast} : x \mapsto v^{-1},\ y \mapsto w,\ z \mapsto u$$

$$X_{\Sigma_1'} = \{x_0 z - x_1 y = 0\} \subset \mathbb{P}^1 \times \mathbb{C}^2,\quad X_{\Sigma'} = Bl_0(\mathbb{C}^2).$$

Prop : Let $\Sigma_1' \subset (N_1)_\mathbb{R}$, $\Sigma_2' \subset (N_2)_\mathbb{R}$ be fans. Then

$$\Sigma_1' \times \Sigma_2' = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i'\}$$

is a fan in $(N_1)_\mathbb{R} \times (N_2)_\mathbb{R} = (N_1 \times N_2)_\mathbb{R}$ and

$$X_{\Sigma_1' \times \Sigma_2'} \simeq X_{\Sigma_1'} \times X_{\Sigma_2'}.$$

RMK: if $P_1 \subset (M_1)_\mathbb{R}$, $P_2 \subset (M_2)_\mathbb{R}$ are polytopes, their normal fans satisfy $\Sigma_{P_1}' \times \Sigma_{P_2}' = \Sigma_{P_1 \times P_2}'$, hence $X_{P_1 \times P_2} \simeq X_{P_1} \times X_{P_2}$.

Example : $\mathbb{P}^1 = X$, $\mathbb{P}^1 \times \mathbb{P}^1 = X$  $= X$  $\times X$ 

Properties of Σ' vs properties of $X_{\Sigma'}$:

Σ_1'	$X_{\Sigma'}$
smooth	smooth
simplicial	simplicial (def.), i.e. $X_{\Sigma'}$ is an orbifold
complete: $\bigcup_{\sigma \in \Sigma_1'} \sigma = N_{\mathbb{R}}$	compact in classical topology.

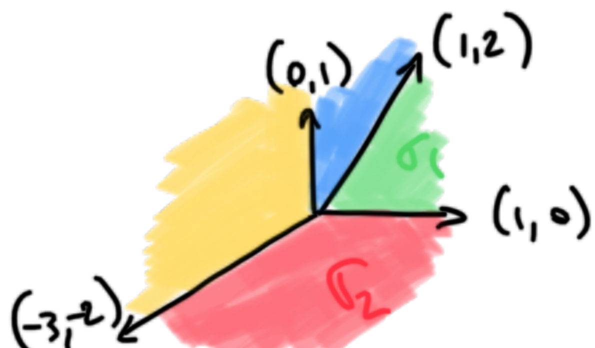
Exercises : 3.1.1 Check compatibility of g_{σ_2,σ_1}

3.1.3 Projective space

3.1.5 If you like blow-ups

3.1.7 $\mathbb{P}^1 \times \mathbb{C}^k$

Consider the complete fan Σ_1':



embed $X_{\Sigma'}$ in projective space

identify smooth affine pieces

embed U_{σ_1} , U_{σ_2} and compute g_{σ_2,σ_1} in coordinates