§2 : Projective toric varieties

A <u>projective toric variety</u> is an irreducible projective variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset, such that $T \times T \to T$ extends to an algebraic action $T \times X \to X$.

<u>Example</u> : $\mathbb{P}^n$ is a projective toric variety with torus

$$T_{\mathbb{P}^n} = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0 \ , \forall i \}$$
$$= \{ (1 : t_1 : \cdots : t_n) \in \mathbb{P}^n \mid t \in (\mathbb{C}^*)^n \}$$
$$\simeq (\mathbb{C}^*)^n .$$

Recall : $T_{\mathbb{P}^n} \simeq (\mathbb{C}^*)^{n+1} / \mathbb{C}^*$, with quotient map

$$\pi : (\mathbb{C}^*)^{n+1} \longrightarrow T_{\mathbb{P}^n} \ , \quad (t_0, \dots, t_n) \mapsto (t_0 : \cdots : t_n) .$$

$\Rightarrow$ the character lattice M_n of $T_{\mathbb{P}^n}$ can be viewed as a sublattice of $\mathbb{Z}^{n+1}$.

$$1 \longrightarrow \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{n+1} \xrightarrow{\pi} T_{\mathbb{P}^n} \longrightarrow 1$$

gives $\quad 0 \longrightarrow M_n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0$

$$M_n = \{ (a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^{n} a_i = 0 \}$$

2.1 Projective toric varieties from monomial maps and toric ideals

Recall: $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$ gives

$$\phi_{\mathcal{A}} : T \longrightarrow \mathbb{C}^s, \quad t \mapsto (\chi^{m_i}(t))_{i=1,\dots,s}.$$

We compose this with $\pi : (\mathbb{C}^*)^s \to \mathbb{P}^{s-1}$ to get a map to projective space:

$$X_{\mathcal{A}} = \overline{(\pi \circ \phi_{\mathcal{A}})(T)} \subset \mathbb{P}^{s-1}.$$

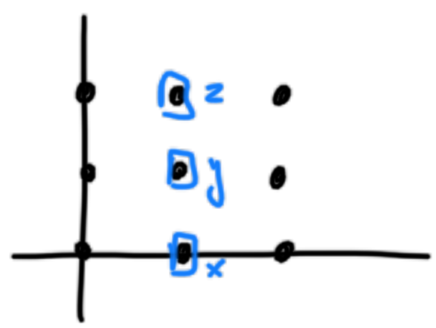
Example: $T = (\mathbb{C}^*)^n$, $X_{\mathcal{A}}$ is the closure of the image of

$$t \mapsto (t^{m_1} : \dots : t^{m_s}).$$

Prop: $X_{\mathcal{A}}$ is a projective toric variety

proof: similar to "$Y_{\mathcal{A}}$ is an affine toric variety." $\square$

We've seen examples where $Y_{\mathcal{A}} \subset \mathbb{C}^s$ is given by homogeneous equations, so that it is naturally the cone over $X_{\mathcal{A}}$:



$y^2 - xz$

$\dim Y_{\mathcal{A}} = 2$

$\dim X_{\mathcal{A}} = 1.$

let $L = \ker \hat{\phi}_{\mathcal{A}}$, where $\hat{\phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M$

$$(a_1, \dots, a_s) \longmapsto \sum a_i m_i$$

and let $I_L = I(Y_{\mathcal{A}}) = \langle x^a - x^b \mid a - b \in L \rangle$.

<u>Prop</u>: I_L is homogeneous if and only if there is $u \in N$ such that $\langle u, m_i \rangle = c \neq 0$ for $i = 1, \dots, s$.

$\iff$ the m_i lie on an affine hyperplane

$\iff (1, 1, \dots, 1) \in \mathrm{Row}_{\mathbb{Q}}(A)$.

<u>proof</u>: (case $M = \mathbb{Z}^n$) I_L homogeneous means

$$\sum_{i=1}^{s} (\ell_+)_i - \sum_{i=1}^{s} (\ell_-)_i = \langle (1, \dots, 1), \ell \rangle = 0 \quad \forall \ell \in L.$$

Therefore $(1,\dots,1) \in (L\otimes\mathbb{Q})^{\perp} = \mathrm{Row}_{\mathbb{Q}}(A)$.

Conversely, $(1,\dots,1) \in \mathrm{Row}_{\mathbb{Q}}(A)$ means $\sum a_i = \sum b_i$

for every $a-b \in L$, so that I_L is homogeneous. $\square$.

Note: in this case $\dim Y_{\mathcal{A}} = \dim X_{\mathcal{A}} + 1$, and

$Y_{\mathcal{A}}$ is the affine cone over $X_{\mathcal{A}}$.

We can make the points in $\mathcal{A} \subset \mathbb{Z}^n$ lie on an

affine hyperplane by adding a row of ones to A:

$$t \mapsto (t^{m_1} : \dots : t^{m_s}) \rightsquigarrow (t_0, t) \mapsto (t_0 t^{m_1} : \dots : t_0 t^{m_s})$$

$$\mathcal{A} = \{m_1, \dots, m_s\} \in \mathbb{Z}^n \rightsquigarrow \hat{\mathcal{A}} = \{(1,m_1), \dots, (1,m_s)\}$$

For $L = \ker \hat{\Phi}_{\hat{\mathcal{A}}}$, we have

$$I(Y_{\hat{\mathcal{A}}}) = I(X_{\hat{\mathcal{A}}}) = I(X_{\mathcal{A}}) = I_L$$

Prop: The character lattice of the torus $T_{X_{\mathcal{A}}}$ of $X_{\mathcal{A}}$ is

$$\mathbb{Z}'\mathcal{A} = \left\{ \sum_{i=1}^{s} a_i m_i \;\middle|\; a_i \in \mathbb{Z},\ \sum a_i = 0 \right\}.$$

Note: $\mathbb{Z}'\mathcal{A}$ is called the lattice <u>affinely generated</u> by $\mathcal{A}$. It is equal to $\mathbb{Z}\cdot(\mathcal{A}-m_i)$, for any i, since

$$a_1 m_1 + \dots + \widehat{a_i m_i} + \dots + a_s m_s - \Big(\sum_{j\neq i}' a_j\Big) m_i = \sum_{j\neq i}' a_j (m_j - m_i).$$

<u>proof</u>:

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & T_{\mathbb{P}^{s-1}} \\ & T_{X_{\mathcal{A}}} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} M & M_{s-1} \arrow[l] \arrow[d] \\ & M' \arrow[ul] \end{tikzcd} \qquad \square.$$

<u>Cor</u>: $\dim X_{\mathcal{A}} = \begin{cases} \operatorname{rank} \mathbb{Z}\mathcal{A} - 1 & \text{if } I(Y_{\mathcal{A}}) \text{ is homogeneous} \\ \operatorname{rank} \mathbb{Z}\mathcal{A} & \text{otherwise.} \end{cases}$

<u>Exercise</u>: Let $\mathcal{A}$ be given by . Show that

$T_{Y_{\mathcal{A}}} \to T_{X_{\mathcal{A}}}$ is 2-to-1.

<u>2.2 Kushnirenko's theorem</u>

Let $\mathcal{A} = \{m_i\}_{i=1}^{s} \subset \mathbb{Z}^n$ be such that $\mathbb{Z}'\cdot\mathcal{A} = \mathbb{Z}^n$.

The <u>convex hull</u> of $\mathcal{A}$ is

$$\operatorname{Conv}(\mathcal{A}) = \Big\{ \sum_{i=1}^{s} \lambda_i m_i \;\Big|\; \lambda_i \in \mathbb{R}_{\geq 0}, \sum \lambda_i = 1 \Big\} \subset \mathbb{R}^n.$$

<u>Theorem</u> (Kushnirenko) The degree of $X_{\mathcal{A}}$ is the normalized volume $n!\,\mathrm{Vol}(\mathrm{Conv}(\mathcal{A}))$.

Our proof is based on the notes [Sottile '17]. It uses a series of auxiliary results, starting with a classic:

<u>Theorem</u> (Hilbert) Let $X \subset \mathbb{P}^{s-1}$ be a projective variety with coordinate ring $\mathbb{C}[X] = \dfrac{\mathbb{C}[x_1, \dots, x_s]}{I(X)}$.

For $d \gg 0$, the <u>Hilbert function</u>

$$HF_X : \mathbb{Z} \to \mathbb{N}, \quad d \mapsto \dim_{\mathbb{C}} \mathbb{C}[X]_d$$

is a polynomial (called the <u>Hilbert polynomial</u>) with leading term $\dfrac{\deg(X)}{\dim(X)!} \cdot d^{\dim(X)}$.

<u>Lemma</u>: The coordinate ring of $X_{\mathcal{A}}$ is $\mathbb{C}[\mathbb{N}\hat{\mathcal{A}}]$, with grading

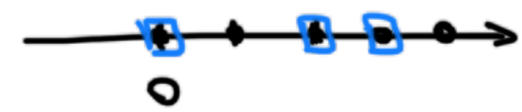
$$\mathbb{C}[\mathbb{N}\hat{\mathcal{A}}] = \bigoplus_{d \in \mathbb{Z}} \mathbb{C}[\mathbb{N}\hat{\mathcal{A}}]_d, \quad \mathbb{C}[\mathbb{N}\hat{\mathcal{A}}]_d = \bigoplus_{m \in d\mathcal{A}} \mathbb{C} \cdot x^{(d,m)}$$

$\left\{ \sum_{j=1}^{d} m_j \mid m_j \in \mathcal{A} \right\}$

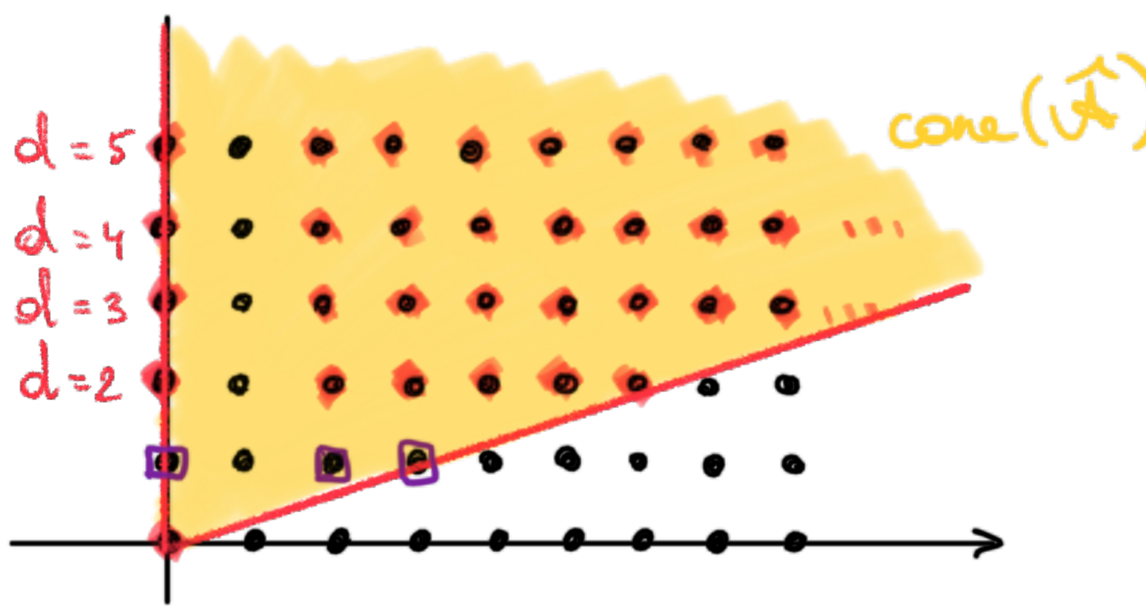
proof : $\mathbb{C}[X_{\mathcal{A}}] = \mathbb{C}[Y_{\hat{\mathcal{A}}}] = \dfrac{\mathbb{C}[x_1, \dots, x_s]}{I(Y_{\hat{\mathcal{A}}})} \simeq \mathbb{C}[\mathbb{N}\hat{\mathcal{A}}]$,

where the last isomorphism is $x_i + I(Y_{\hat{\mathcal{A}}}) \mapsto \chi^{(1, m_i)}$.

Example $\mathcal{A} = \{0, 2, 3\} \subset \mathbb{Z}$



$\hat{\mathcal{A}} = \{(1,0), (1,2), (1,3)\} \subset \mathbb{Z}^2$



d	0	1	2	3	4	5	...
$HF_{X_{\mathcal{A}}}(d)$	1	3	6	9	12	15	...

$\rightarrow HP_{X_{\mathcal{A}}} = 3d$.

Exercise : $I(X_{\mathcal{A}}) = \langle xz^2 - y^3 \rangle$.

Corollary : $HF_{X_{\mathcal{A}}}(d) = |d \cdot \mathcal{A}|$.

The <u>Ehrhart polynomial</u> E_P of a lattice polytope $P \subset \mathbb{R}^n$ is given by $E_P : \mathbb{N} \to \mathbb{N}$, $d \longmapsto |dP \cap \mathbb{Z}^n|$.

<u>Theorem</u> The degree of E_P is $\dim P$. If $\dim P = n$, the leading term is $\mathrm{Vol}(P)\, d^n$.

<u>Proposition</u> : $HF_{X_{\mathcal{A}}}(d) \leq E_{\mathrm{Conv}(\mathcal{A})}(d)$ for all $d \in \mathbb{N}$.

<u>proof</u> Immediate from $d\mathcal{A} \subset dP \cap \mathbb{Z}^n$ $\square$

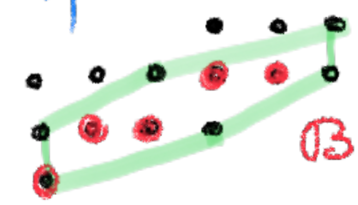
Let $\sigma^\vee = \mathrm{Cone}(\hat{\mathcal{A}}) \subset \mathbb{R}^{n+1}$ and $S_\sigma = \sigma^\vee \cap \mathbb{Z}^{n+1}$.

If $\underbrace{S_\sigma = \mathbb{N}\hat{\mathcal{A}}}_{(*)}$, then $|d \cdot \mathcal{A}| = |d \cdot P \cap \mathbb{Z}^n|$, $d \in \mathbb{N}$, so $HF_{X_{\mathcal{A}}}(d) = E_P(d)$, $d \in \mathbb{N}$ and this proves Kushnirenko's theorem. If $(*)$ holds, $X_{\mathcal{A}}$ is called <u>projectively normal</u>. In general, $S_\sigma \supset \mathbb{N}\hat{\mathcal{A}}$.

<u>Lemma</u> : There exists $m \in \mathbb{N}\hat{\mathcal{A}}$ such that

$$m + S_\sigma \subset \mathbb{N}\hat{\mathcal{A}}.$$

proof: Let $\mathcal{B} = \left\{ \sum_{i=1}^{s_1} \lambda_i (1, m_i) \in \mathbb{Z}^{n+1} \;\middle|\; \lambda_i \in [0,1) \cap \mathbb{Q} \right\}$

(This is the origin + the interior points of a zonotope, in our previous example: )

for each $b \in \mathcal{B}$, fix $a_i(b) \in \mathbb{Z}$ such that

$$b = \sum_{i=1}^{s_1} a_i(b)(1, m_i)$$

This is possible because $\mathbb{Z}'\mathcal{A} = \mathbb{Z}^n$

Fix $\nu \in \mathbb{N}$ s.t. $-\nu \leq a_i(b)$, $\forall b \in \mathcal{B}$, $i = 1, \dots, s$.

We define $m = \nu \cdot \sum_{i=1}^{s_1} (1, m_i)$.

If $m' \in m + S_\sigma$, we have $m' - m = \sum_{i=1}^{s_1} \alpha_i (1, m_i)$

for some $\alpha_i \in \mathbb{Q}_{\geq 0}$. Set $\alpha_i = \lambda_i + \gamma_i$, $\lambda_i \in [0,1) \cap \mathbb{Q}$, $\gamma_i \in \mathbb{N}$.

Then $m' - m = \sum \lambda_i (1, m_i) + \sum \gamma_i (1, m_i) = \underbrace{b}_{\in \mathcal{B}} + \underbrace{c}_{\in \mathbb{N}\hat{\mathcal{A}}}$

so that $m' = m + \sum a_i(b)(1, m_i) + c$

$$= \sum{}'(a_i(b) + \nu)(1, m_i) + c \in \mathbb{N}\hat{\mathcal{A}}. \square$$

<u>proof</u> (of Kushnirenko's theorem) By the previous lemma: for $d \gg 0$

$$\left| (d - \nu \cdot |\mathcal{A}|) \cdot \mathrm{Conv}(\mathcal{A}) \cap \mathbb{Z}^n \right| \leq |d \cdot \mathcal{A}| \leq \left| d \cdot \mathrm{Conv}(\mathcal{A}) \cap \mathbb{Z}^n \right|$$

$$E_{\mathrm{Conv}(\mathcal{A})}(d - \nu |\mathcal{A}|) \leq HF_{X_{\mathcal{A}}}(d) \leq E_{\mathrm{Conv}(\mathcal{A})}(d)$$

So these are all polynomials with the same leading term.

Hence $\mathrm{Vol}(\mathrm{Conv}(\mathcal{A})) = \dfrac{\deg(X_{\mathcal{A}})}{n!}$. $\square$.

<u>Corollary</u>: for generic coefficients $c_{i,j}$, the system of equations $f_1 = \dots = f_n = 0$ with

$$f_j = \sum_{i=1}^{s} c_{i,j} \cdot t^{m_i} \in \mathbb{C}[\mathbb{Z}^n]$$

has $n! \, \mathrm{Vol}(\mathrm{Conv}(\mathcal{A}))$ solutions.

Exercises: 2.1.4

2.1.6

For $\mathcal{A} = \{(0,0), (1,0), (0,1), (2,1), (1,2)\}$, compute $\deg(\mathcal{A})$ via Kushnirenko's theorem, and verify using the last corollary in this lecture.

Now delete the point $(0,0)$. Explain what happens.

Please read Sec 2.2. in [CLS].