§4: Divisors on toric varieties

4.1 Some background on divisors.

Let X be an irreducible (abstract) variety. A *prime divisor* $D \subset X$ is an irreducible subvariety of codim 1.

To a rational function $\phi \in \mathbb{C}(X)^*$, we want to associate a $\mathbb{Z}$-linear combination of divisors $\sum_i a_i D_i$ encoding the "order of vanishing of ϕ along D_i".

This works best when X is *normal*.

For a prime divisor $D \subseteq X$, we define

$$\mathcal{O}_{X,D} = \{ \phi \in \mathbb{C}(X) \mid \phi \text{ is defined on } \mathcal{U}, \mathcal{U} \cap D \neq \emptyset \}.$$

Since X is irreducible, $\mathbb{C}(X) = \mathbb{C}(\mathcal{U})$, and if $\mathcal{U} \cap D \neq \emptyset$, $\mathcal{O}_{X,D} = \mathcal{O}_{\mathcal{U}, \mathcal{U} \cap D}$. Hence, we may assume $X = \operatorname{Specm}(R)$.

In this case,

$$\{\text{prime divisors of } X\} \overset{1:1}{\longleftrightarrow} \{\text{codim } 1 \text{ prime ideals of } R\}$$

Let $\mathfrak{p} = I(D)$. We have

$$\mathcal{O}_{X,D} = \{ \tfrac{f}{g} \in K \mid f, g \in R,\ g \notin \mathfrak{p} \}.$$
$$= R_{\mathfrak{p}}$$

field of fractions

This is a local ring with maximal ideal $\mathfrak{p} R_{\mathfrak{p}}$.

Proposition: Let R be a normal domain and $\mathfrak{p} \subset R$ a codimension one prime ideal. Then there exists a group homomorphism $\nu_D : K^* \longrightarrow \mathbb{Z}$ $(D = V(\mathfrak{p}))$ such that

- $\nu_D(f+g) \geq \min(\nu_D(f), \nu_D(g))$ when $f, g, f+g \in K^*$. (note also $\nu_D(fg) = \nu_D(f) + \nu_D(g)$)
- $R_{\mathfrak{p}} = \{ f \in K^* \mid \nu_D(f) \geq 0 \} \cup \{0\}$.

ν_D is a discrete valuation and $R_{\mathfrak{p}} \subset K$ is the corresponding discrete valuation ring (DVR).

In the situation of this proposition, $R_{\mathfrak{p}}$ is a PID, and all ideals are of the form $\langle \pi^k \rangle$, where

$$\mathfrak{p} R_{\mathfrak{p}} = \langle \pi \rangle = \{ f \in R_{\mathfrak{p}} \mid v_D(f) > 0 \}.$$

For $f \in R_{\mathfrak{p}}^*$, we have $v_D(f) = k$ where k is the largest integer for which $f \in \langle \pi^k \rangle$. If $f \in K^* \setminus R_{\mathfrak{p}}$, $v_D(f) = -k$ where k is the largest integer for which $f^{-1} \in \langle \pi^k \rangle$.

<u>Corollary</u> Let X be a normal variety and $D \subset X$ a prime divisor, then there is a discrete valuation

$$v_D : \mathbb{C}(X)^* \longrightarrow \mathbb{Z}$$

with DVR $\mathcal{O}_{X,D}$.

We say that f $\begin{cases} \text{vanishes with order } v_D(f) \text{ along } D. \\ \text{has a pole of order } |v_D(f)| \end{cases}$

Let $\mathrm{Div}(X)$ be the free abelian group generated by the prime divisors on X. A Weil divisor is an element $D = \sum' a_i D_i$ (finite sum) of $\mathrm{Div}(X)$. D is effective if $a_i \geq 0$ for all i. The support of D is $\bigcup_{a_i \neq 0} D_i$.

Let X be a normal variety, the divisor of $f \in \mathbb{C}(X)^*$ is

$$\mathrm{div}(f) = \sum_{D \subset X}' v_D(f) \cdot D.$$

This is a Weil divisor (lemma 4.0.9). All Weil divisors of the form $\mathrm{div}(f)$ are called principal divisors. They form a subgroup $\mathrm{Div}_0(X) \subset \mathrm{Div}(X)$.

Example Let $X = \mathbb{P}^n$, $D_i = V(x_i)$, $i = 0, \dots, n$.

$$D_i - D_j \in \mathrm{Div}_0(X), \quad D_i \in \mathrm{Div}(X) \setminus \mathrm{Div}_0(X).$$

Rmk: $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$, $\mathrm{div}(f^{-1}) = -\mathrm{div}(f)$.

so $\mathrm{Div}_0(X)$ is indeed a subgroup.

Two divisors D, E are <u>linearly equivalent</u> if

$D - E \in \mathrm{Div}_0(X)$.

<u>Example</u> $f = c(x-a_1)^{m_1} \cdots (x-a_r)^{m_r} \in \mathbb{C}[x]$

(a_i distinct complex numbers).

$$\mathrm{div}(f) = \sum_{i=1}^{r} m_i \cdot \{a_i\}$$ (f viewed as a rational function on $\mathbb{C}$)

$$\mathrm{div}(f) = \sum_{i=1}^{r} m_i \cdot \{a_i\} - \left(\sum_{i=1}^{r} m_i\right) \cdot \{\infty\}$$

(f viewed as a rational function on $\mathbb{P}^1$).

A Weil divisor D is <u>Cartier</u> if it is <u>locally principal</u>, meaning that there is an open cover $\{U_i\}$ of X s.t. $D|_{U_i} \in \mathrm{Div}_0(U_i)$.

Here $D_{|\mathcal{U}} = \left(\sum a_i D_i\right)_{|\mathcal{U}} = \sum_{D_i \cap \mathcal{U} \neq \phi} a_i (D_i \cap \mathcal{U})$.

Cartier divisors form a subgroup $CDiv(X) \subset Div(X)$, and all principal divisors are Cartier:

$$Div_0(X) \subset CDiv(X) \underset{=}{\subset} Div(X).$$

= if X is smooth.

Example $\mathbb{P}^1 = \underset{x_0 \neq 0}{\mathcal{U}_0} \cup \underset{x_1 \neq 0}{\mathcal{U}_1}$ (so $CDiv(\mathbb{P}^1) = Div(\mathbb{P}^1)$)

$$= (\mathbb{P}^1 \setminus D_0) \cup (\mathbb{P}^1 \setminus D_1)$$

D_1 is not principal, but it is locally principal:

$$D_1|_{\mathcal{U}_0} = div\left(\frac{x_1}{x_0}\right), \quad D_1|_{\mathcal{U}_1} = div(1).$$

The <u>divisor class group</u> of a normal variety X is $Cl(X) = Div(X)/_{Div_0(X)}$. Its <u>Picard group</u> is $Pic(X) = CDiv(X)/_{Div_0(X)}$.

<u>Theorem</u> : Let R be a UFD and $X = \mathrm{Spec}(R)$. Then

(a) R is normal and every codim 1 prime ideal is principal.

(b) $Cl(X) = 0$.

<u>proof</u> (a) UFD $\Rightarrow$ normal (exercise). Let $\mathfrak{p} \subset R$ be codim 1 and $f \in \mathfrak{p} \setminus \{0\}$. Then $f = c \prod_{i=1}^{s} f_i^{a_i}$, f_i prime, c unit. Since $\mathfrak{p}$ is prime, $f_i \in \mathfrak{p}$ for some i, and since $\mathfrak{p}$ is codim 1, $\mathfrak{p} = \langle f_i \rangle$.

(b) Let D_i be a prime divisor, then $\mathfrak{p}_i = I(D_i)$ is principal by (a): $\mathfrak{p}_i = \langle f_i \rangle$. Then $D = \sum_{i=1}^{s} a_i D_i$, then $D = \mathrm{div}\left(\prod_{i=1}^{s} f_i^{a_i}\right)$. ($\nu_{D_i}(f_i) = 1$ since f_i generates the maximal ideal $\mathfrak{p}_i R_{\mathfrak{p}_i} \subset R_{\mathfrak{p}_i}$). $\square$.

<u>Example</u> : $Cl(\mathbb{C}^n) = 0$.

<u>Exercise</u> : $Cl(X) \to Cl(U)$: $D \mapsto D|_U$ is well-defined.

Theorem: Let X be a normal variety and $\mathcal{U} \subset X$ a nonempty open subset. Let $D_1, \dots, D_s$ be the irreducible components of $X \setminus \mathcal{U}$ that are prime divisors. Then

$$\bigoplus_{i=1}^{s} \mathbb{Z} \cdot D_i \longrightarrow Cl(X) \longrightarrow Cl(\mathcal{U}) \longrightarrow 0$$

is exact. Here $\sum' a_i D_i \mapsto [\sum' a_i D_i] \mapsto [\sum' a_i D_{i|\mathcal{U}}]$.

proof: exactness at $Cl(\mathcal{U})$: $D' = \sum' a_i D_i' \in Div(\mathcal{U})$ is the restriction of $D = \sum' a_i D_i$ where $D_i = \bar{D_i'}$.

at $Cl(X)$: clearly, the composition is zero. Suppose $[D]$ restricts to zero in $Cl(\mathcal{U})$. Then $D_{|\mathcal{U}}$ is the divisor of $f \in \mathbb{C}(\mathcal{U})^*$. Since $\mathbb{C}(\mathcal{U}) = \mathbb{C}(X)$, the divisor of f in $Div(X)$ restricts to the divisor of f in $Div(\mathcal{U})$:

$$D_{|\mathcal{U}} = div(f)_{|\mathcal{U}}.$$

Hence $(D - div(f))_{|\mathcal{U}} = 0$, hence $D - div(f) \in \bigoplus_{i=1}^{s} \mathbb{Z} \cdot D_i$. $\square$

Example: $\mathbb{Z}\{\infty\} \hookrightarrow Cl(\mathbb{P}^1) \longrightarrow \underset{\overset{\parallel}{0}}{Cl(\mathbb{C})} \rightarrow 0$.

$Cl(\mathbb{P}^1) \simeq \mathbb{Z}$.

4.2 Weil divisors on toric varieties

$X = X_\Sigma$, normal toric variety coming from a fan.

Let $\Sigma(1)$ be the set of rays of Σ. By the OCC, these give torus invariant prime divisors $D_\rho = \overline{O(\rho)}$, $\rho \in \Sigma(1)$. We want to compute the valuation map

$$\nu_\rho = \nu_{D_\rho} : \mathbb{C}(X_\Sigma)^* \longrightarrow \mathbb{Z} \quad \text{on characters.}$$

(i.e., we compute the order of vanishing of Laurent monomials along the boundary of $T \subset X_\Sigma$).

<u>Proposition</u> Let $u_\rho \in N$ be the primitive ray generator of $\rho \in \Sigma(1)$. We have $\nu_\rho(\chi^m) = \langle u_\rho, m \rangle$.

<u>proof</u> Let $e_1 = u_\rho, e_2, \dots, e_n$ be a basis of N.

The affine toric variety $U_\rho = U_{\mathrm{Cone}(e_1)}$ is

$$\mathbb{C} \times (\mathbb{C}^*)^{n-1} = \operatorname{Specm} \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

and $U_\rho \cap D_\rho$ is defined by $x_1 = 0$.

Hence $\mathcal{O}_{X_{\Sigma},D_\rho} = \mathcal{O}_{U_\rho, U_\rho \cap D_\rho} = \mathbb{C}[x_1,\dots,x_n]_{\langle x_1\rangle}$

and $\nu_\rho(f) = k$ means $f = x_1^k \frac{g}{h}$, $g,h \in \mathbb{C}[x_1,\dots,x_n] \setminus \langle x_1\rangle$.

Let $m_1,\dots,m_n$ be the dual basis of $e_1,\dots,e_n$.

If $m = \sum a_i m_i$, then $\langle e_i, m\rangle = a_i$. So

$$\chi^m = x_1^{\langle e_1, m\rangle} \cdots x_n^{\langle e_n, m\rangle}, \text{ and therefore}$$

$\nu_\rho(\chi^m) = \langle e_1, m\rangle = \langle u_\rho, m\rangle$. $\square$

<u>Corollary</u>: $\mathrm{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle u_\rho, m\rangle D_\rho$.

<u>Theorem</u> There is an exact sequence

$$M \xrightarrow{m \mapsto \mathrm{div}(\chi^m)} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \xrightarrow{D \mapsto [D]} \mathrm{Cl}(X_\Sigma) \longrightarrow 0.$$

Furthermore, this extends to a short exact sequence

$$0 \to M \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

if and only if $\{u_\rho, \rho\in\Sigma(1)\}$ spans $N_{\mathbb{R}}$.

i.e., no torus factors.

Prop 4.0.16

Lemma: Let X be a normal variety and $f \in \mathbb{C}(X)^*$.

(a) $\mathrm{div}(f) \geq 0$ iff $f : X \to \mathbb{C}$ is a morphism.

(b) $\mathrm{div}(f) = 0$ iff $f : X \to \mathbb{C}^*$ is a morphism.

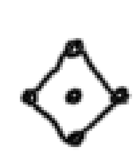
of the theorem

proof The sequence $\bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho \to Cl(X_\Sigma) \to Cl(T) \to 0$ is exact by our previous theorem. We have also seen that $Cl(T) = 0$, so the $[D_\rho]$ generate $Cl(X_\Sigma)$.

exactness at $\bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho$: composition is clearly zero.

suppose $[D] = 0$ for some $D \in \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho$. Then $D = \mathrm{div}(f)$, and $\mathrm{div}(f)_{|T} = 0$. This implies that $f : T \to \mathbb{C}^*$ is a morphism $(*)$, and hence $f = c\chi^m$, so that $D = \mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m)$.

Exactness at M if $\{u_\rho\}$ $\mathbb{R}$-linearly indep : exercise. $\square$

Exercise, $Cl(\mathbb{P}^2)$? $Cl(X_P)$, $P =$  ? $Cl(Bl_0(\mathbb{C}^2))$?
$Cl(\mathbb{P}^1 \times \mathbb{P}^1)$?