§5: Toric varieties as GIT quotients

Let $X = X_\Sigma$ be a normal toric variety corresponding to the fan Σ in $N_\mathbb{R}$. Assume: no torus factors.

Our goal is to describe X as a quotient

$$X = (\mathbb{C}^k \setminus Z)/G$$

where
- $k = |\Sigma(1)|$,
- Z is a union of coordinate subspaces.
- G is a group acting algebraically on $\mathbb{C}^k \setminus Z$ (it is a quasi-torus).

<u>Example</u> Let $X = \mathbb{P}^2$. X can be realized as

$$X = (\mathbb{C}^{3 = k} \setminus \underbrace{\{0\}}_{= Z}) / \underbrace{\mathbb{C}^*}_{= G}$$

where $\mathbb{C}^* \times (\mathbb{C}^3 \setminus \{0\}) \to \mathbb{C}^3 \setminus \{0\}$ is $\lambda \cdot (x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$.

This action extends to $\mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{C}^3$.

Subvarieties of $\mathbb{P}^2$ are given by <u>homogeneous</u> ideals in $S = \mathbb{C}[x_1, x_2, x_3]$, with respect to the grading

$$S = \bigoplus_{\alpha \in \mathbb{Z}} S_\alpha .$$

The grading is such that, for $f \in S$ homogeneous, $V_{\mathbb{C}^3}(f)$ is stable under the $\mathbb{C}^*$ action:

$$f(x) = 0 \implies f(\lambda \cdot x) = 0 .$$

So homogeneous ideals $I \subset S$ have well-defined zero sets $V_X(I)$ in X.

There is a distinguished homogeneous ideal $B \subset S$ whose variety $V_X(B) = \emptyset$: $B = \langle x_1, x_2, x_3 \rangle$.

ALGEBRA		GEOMETRY.
S	$\xrightarrow{\mathrm{Specm}(\cdot)}$	$\mathbb{C}^3$
B	$\xrightarrow{V_{\mathbb{C}^3}(\cdot)}$	$\{0\}$
$\mathbb{Z}$	$\xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)}$	$\mathbb{C}^*$

The map sending a G-orbit in $\mathbb{C}^3 \setminus \{0\}$ to its corresponding point in X is

$$\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2, \quad (x_1, x_2, x_3) \mapsto (x_1 : x_2 : x_3).$$

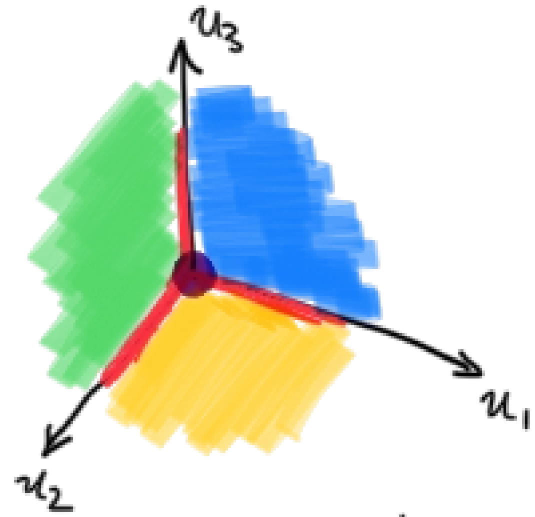
This satisfies $\pi|_{T_{\mathbb{C}^3\setminus\{0\}}}(T_{\mathbb{C}^3\setminus\{0\}}) \subset T_{\mathbb{P}^2}$ and $\pi|_{T_{\mathbb{C}^3\setminus\{0\}}}$ is a group homomorphism.

$\Rightarrow$ π is a toric morphism!

Lg $\Rightarrow$ it comes from a linear map $\mathbb{R}^3 \to \mathbb{R}^2$, compatible with the fans $\Sigma'_{\mathbb{C}^3\setminus\{0\}}$, $\Sigma_{\mathbb{P}^2}$.

$$(x_1, x_2, x_3) \longmapsto \left(\frac{x_1}{x_3} : \frac{x_2}{x_3} : 1\right)$$

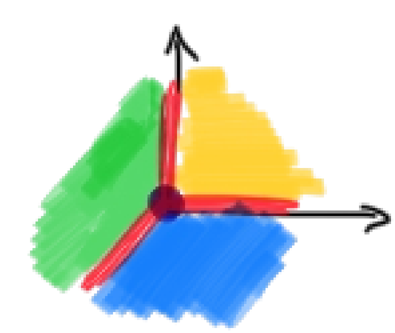
$\mathbb{R}^3$



$$F = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$\longrightarrow$

$\mathbb{R}^2$



(Note that F consists of the rays of Σ.) Δ

Let $\Sigma'(1) = \{\rho_1, \dots, \rho_k\}$ and let $u_i \in N$ be the primitive ray generator of ρ_i. This gives

$$F = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix} \in \mathbb{Z}^{n \times k}.$$

We view this as a map $\mathbb{R}^k \to \mathbb{R}^n$.

Consider the fan of $\mathbb{C}^k$, which is $\mathbb{R}^k_{\geq 0}$ and all its faces. We define Σ' as the subfan of all cones whose image under F is contained in a cone of Σ. By construction, F is compatible with Σ', Σ.

We obtain a toric morphism

$$\pi : X_{\Sigma'} \longrightarrow X_{\Sigma}.$$

Here $X_{\Sigma'} = \mathbb{C}^k \setminus Z$.

<u>Def</u> A subset $C \subseteq \Sigma'(1)$ is a <u>primitive collection</u> if

(a) $C \subsetneq \sigma(1)$ for all $\sigma \in \Sigma$

(b) For every proper subset $C' \subsetneq C$, there is $\sigma \in \Sigma$ with $C' \subseteq \sigma(1)$.

Let $S = \mathbb{C}[x_1, \ldots, x_k]$. For each $\sigma \in \Sigma'$,

we define the square-free monomial

$$x^{\hat{\sigma}} = \prod_{\rho_i \notin \sigma} x_i .$$

<u>Proposition</u> The <u>base locus</u> Z is given by

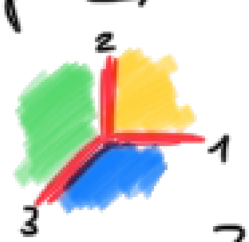
$$Z = \bigcup_{\substack{\text{primitive} \\ \text{collections} \\ C}} V(x_i \mid \rho_i \in C)$$

This is the affine variety $V_{\mathbb{C}^k}(B)$ of the <u>irrelevant ideal</u>

$$B = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma' \rangle = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma'_{max} \rangle . \quad \square$$

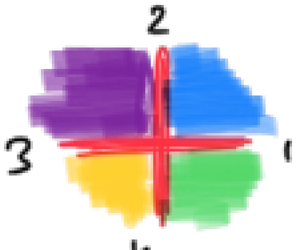
<u>Rmk</u> B is sometimes called the Stanley-Reisner ideal of Σ'.

<u>Example</u> :



$$B = \langle x_1, x_2, x_3 \rangle$$

$$Z = V_{\mathbb{C}^3}(B) = V(x_1, x_2, x_3)$$



$$\langle x_1 x_4 , x_3 x_4 , x_1 x_2 , x_2 x_3 \rangle$$

$$Z = V_{\mathbb{C}^4}(x_1, x_3) \cup V_{\mathbb{C}^4}(x_2, x_4) .$$

That covers $\mathbb{C}^k \setminus Z$. We now identify the group G.

Since $\pi : \mathbb{C}^k \setminus Z \longrightarrow X_\Sigma$ is the toric morphism coming from $F : \mathbb{R}^k \to \mathbb{R}^n$, its restriction to the torus $(\mathbb{C}^*)^k$ is given by

$$\pi_{|(\mathbb{C}^*)^k} : (\mathbb{C}^*)^k \to (\mathbb{C}^*)^n, \quad t \mapsto (t^{F_{1,:}}, \dots, t^{F_{n,:}}).$$

The kernel of this group homomorphism is

$$G = \{ \lambda \in (\mathbb{C}^*)^k \mid \lambda^{F_{i,:}} = 1 \ , \ i = 1, \dots, n \}.$$

$G \subset (\mathbb{C}^*)^k$ is a subgroup, so it acts on $\mathbb{C}^k \setminus Z$.

Moreover, by equivariance

$$\pi(\lambda \cdot x) = \pi(\lambda) \cdot \pi(x)$$
$$= \pi(x), \quad \forall \lambda \in G.$$

$\Rightarrow$ π is constant on G-orbits.

Theorem Consider the action of G on $\mathbb{C}^k \setminus Z$. There is a one-to-one correspondence

$$\{\text{closed } G\text{-orbits in } \mathbb{C}^k \setminus Z\} \overset{1:1}{\longleftrightarrow} \{\text{points in } X\}$$

Moreover, on an open subset $\mathcal{U} \subset X$ s.t. $\operatorname{codim}_X(X \setminus \mathcal{U}) \geq 3$, we have that

$$\{G\text{-orbits in } \mathbb{C}^k \setminus Z\} \overset{1:1}{\longleftrightarrow} \{\text{points in } \mathcal{U}\}.$$

These correspondences are realized by the toric morphism $\pi : \mathbb{C}^k \setminus Z \longrightarrow X$ coming from $F = [u_1 \cdots u_k]$.

proof See [CLS, thm 5.1.11] or Cox's original (very nice!) paper "The homogeneous coordinate ring of a toric variety". $\square$

RMK: The subvariety $X \setminus \mathcal{U}$ is the union $\bigcup_{\substack{\sigma \in \Sigma \\ \sigma \text{ non-simplicial}}} O(\sigma)$. Hence, if X is simplicial, $\mathcal{U} = X$ and the quotient is geometric.

<u>Example</u> 

$Z = V_{\mathbb{C}^4}(x_1, x_3) \cup V_{\mathbb{C}^4}(x_2, x_4)$.

$$F = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad G = \{ t \in (\mathbb{C}^*)^4 \mid t_1 t_3^{-1} = t_2 t_4^{-1} = 1 \}$$
$$= \{ (\lambda, \mu, \lambda, \mu) \mid (\lambda, \mu) \in (\mathbb{C}^*)^2 \}$$

$G \simeq (\mathbb{C}^*)^2$ acts on $\mathbb{C}^4 \setminus Z$ by

$$(\lambda, \mu) \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \mu x_2, \lambda x_3, \mu x_4)$$

$$X = (\mathbb{C}^4 \setminus Z)/(\mathbb{C}^*)^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

$$(x_0 : x_1) \times (y_0 : y_1) = \pi(x_1, y_1, x_0, y_0). \quad \square$$

<u>RMK</u> Recall that there is a SES

$$0 \longrightarrow M \xrightarrow{F^T} \mathbb{Z}^k \longrightarrow Cl(X) \longrightarrow 0.$$

Taking $Hom_{\mathbb{Z}}(-, \mathbb{C}^*)$ gives

$$1 \longrightarrow Hom_{\mathbb{Z}}(Cl(X), \mathbb{C}^*) \longrightarrow (\mathbb{C}^*)^k \xrightarrow{\pi|_{(\mathbb{C}^*)^k}} (\mathbb{C}^*)^n \longrightarrow 1$$

which shows $G = Hom_{\mathbb{Z}}(Cl(X), \mathbb{C}^*)$.

We now discuss the algebra of this construction.

We would like to grade $S = \mathbb{C}[x_1, \dots, x_k]$ such that homogeneous elements have well-defined zero sets on X.

The grading is by the class group $Cl(X)$:

for $\alpha = \left[\sum_{i=1}^{k} a_i D_i\right] \in Cl(X)$, set

$$(*) \quad S_\alpha = \bigoplus_{F^T m + a \geq 0} \mathbb{C} \cdot x^{F^T m + a} \subseteq S.$$

The sum is over all lattice points in the polyhedron P_D associated to the representative divisor $D = \sum a_i D_i$.

Exercise: this $(*)$ does not depend on the choice of representative.

Let $f \in S_\alpha$, $\lambda \in G$. We have

$$f(\lambda \cdot x) = \sum_{m \in P_D \cap M} c_m (\lambda \cdot x)^{F^T m + a} = \sum_{m \in P_D \cap M} c_m \lambda^{F^T m + a} x^{F^T m + a}$$

$$= \lambda^a f(x) \quad \text{since } \lambda^{F^T m} = 1!$$

Therefore $f(x)=0 \Rightarrow f(\lambda \cdot x)=0$, and $V_{\mathbb{C}^k}(f)$ is stable under the action of G. We define

$$V_X(f) = \{ p \in X \mid f(z)=0 \text{ for some } z \in \pi^{-1}(p) \}.$$

More generally, for a homogeneous ideal I,

$$V_X(I) = \{ p \in X \mid p \in V_X(f) \text{ for all } f \in I \text{ homog.} \}.$$

The ring S, with its grading by $\mathcal{Cl}(X)$ and its irrelevant ideal B is called the <u>Cox ring</u> or homogeneous coordinate ring of X.

We generalize our table:

	ALGEBRA		GEOMETRY	
Cox ring	S	$\xrightarrow{\mathrm{Specm}(\cdot)}$	$\mathbb{C}^k$	total coordinate space
irrelevant ideal	B	$\xrightarrow{V_{\mathbb{C}^k}(\cdot)}$	Z	base locus
class group	$\mathcal{Cl}(X)$	$\xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)}$	G	reductive group