1.7 Normality and smoothness

An integral domain R is normal if it is integrally closed in its field of fractions K:

if $x \in K$ satisfies $x^d + r_{d-1} x^{d-1} + \dots + r_0 = 0$ with $r_i \in R$, then $x \in R$.

An irreducible affine variety V is normal if $\mathbb{C}[V]$ is normal.

An affine semigroup $S \subset M$ is saturated if $km \in S$ for some $k \in \mathbb{Z}_{>0} \Rightarrow m \in S$.

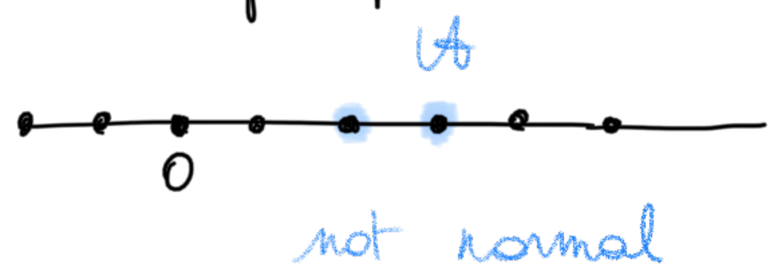
Theorem: Let V be an affine toric variety with torus T (with lattices M, N). TFAE:

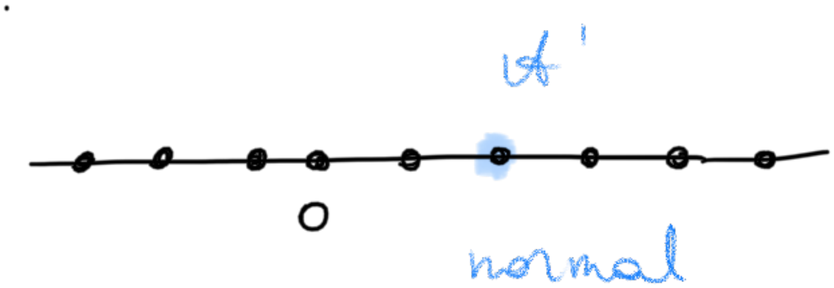
(a) V is normal

(b) $V = \operatorname{Spec} \mathbb{C}[S]$ where $S \subset M$ is a saturated affine semigroup

(c) $V = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ for a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$.

<u>Example</u>: non-normal affine toric varieties correspond to semigroups with "holes".





We'd like to understand which affine toric varieties are smooth.

As smooth $\Rightarrow$ normal, the only candidates are $U_\sigma = \operatorname{Specm}(\mathbb{C}[S_\sigma])$, for σ a RCPC.

<u>Theorem</u> An affine toric variety V is smooth iff $V = \operatorname{Specm} \mathbb{C}[S_\sigma] = U_\sigma$ for some smooth RCPC $\sigma \subset N_{\mathbb{R}}$.

proof σ smooth $\Rightarrow U_\sigma$ smooth follows from a previous example.

Suppose U_σ is smooth and $\dim \sigma = n$. Then the fixed point corresponding to $\underset{m}{\underset{\shortparallel}{\langle \chi^m \mid m \in S_\sigma \setminus \{0\}\rangle}}$ is smooth in U_σ, hence its tangent space has dimension n: $\dim_{\mathbb{C}} m/m^2 = n$. Observe that

$$m = \bigoplus_{m \in S_\sigma \setminus \{0\}} \mathbb{C} \cdot \chi^m = \bigoplus_{m \text{ irred}} \mathbb{C} \cdot \chi^m \oplus \bigoplus_{m \text{ reducible}} \mathbb{C} \cdot \chi^m$$

$$= \bigoplus_{m \in \mathcal{H}} \mathbb{C} \cdot \chi^m \oplus \underset{m^2!}{\underset{\shortparallel}{\bigoplus_{m \text{ reducible}} \mathbb{C} \cdot \chi^m}}$$

$\Rightarrow \dim_{\mathbb{C}} m/m^2 = |\mathcal{H}|$.

$\Rightarrow \sigma^\vee$ has a Hilbert basis consisting of n elts

$\Rightarrow \sigma^\vee$ is smooth $\Rightarrow \sigma$ is smooth.

$\dim \sigma < n \Rightarrow$ exercise. $\square$.

1.8 Toric morphisms

Consider $V_i = \operatorname{Specm} \mathbb{C}[S_i]$, $i = 1,2$ affine toric varieties. A morphism $\phi : V_1 \longrightarrow V_2$ is <u>toric</u> if $\phi^* : \mathbb{C}[S_2] \longrightarrow \mathbb{C}[S_1]$ is induced by a semi-group homomorphism $\hat{\phi} : S_2 \to S_1$.

<u>Example:</u>

$$\phi_{\mathcal{A}} : T \longrightarrow (\mathbb{C}^*)^s$$
$$\hat{\phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M$$

<u>Prop</u> Let T_i be the dense torus of V_i, with (co-) character lattice M_i (N_i). Then

$\phi : V_1 \to V_2$ is toric iff $\phi(T_1) \subseteq T_2$ and $\phi_{|T_1} : T_1 \longrightarrow T_2$ is a group homom.

proof : Let $V_i = \operatorname{Specm}(\mathbb{C}[S_i])$ so that

$M_i = \mathbb{Z}\cdot S_i$. If ϕ is toric, $\hat{\phi} : S_2 \to S_1$

extends to a group homomorphism $\hat{\phi} : M_2 \to M_1$:

$$\begin{tikzcd} S_2 \arrow[r, "\hat{\phi}"] \arrow[d, hook] & S_1 \arrow[d, hook] \\ M_2 \arrow[r, "\hat{\phi}"] & M_1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbb{C}[S_2] \arrow[r, "\phi^*"] \arrow[d, hook] & \mathbb{C}[S_1] \arrow[d, hook] \\ \mathbb{C}[M_2] \arrow[r] & \mathbb{C}[M_1] \end{tikzcd} \xrightarrow[\rightsquigarrow]{\operatorname{Specm}} \begin{tikzcd} V_2 & V_1 \arrow[l, "\phi"'] \\ T_2 \arrow[u, hook] & T_1 \arrow[l] \arrow[u, hook] \end{tikzcd}$$

We see that $\phi(T_1) \subset T_2$, and $\phi_{|T_1} : T_1 \to T_2$

is a group homomorphism obtained by taking

$\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ of $M_2 \to M_1$.

$$\operatorname{Hom}_{\mathbb{Z}}(M_i, \mathbb{C}^*) \simeq T_i$$

via $\quad (m_i \mapsto \chi^{m_i}(t)) \sim t$

$$\left(M_2 \xrightarrow{\hat{\phi}} M_1\right) \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)} \left(T_2 \longleftarrow T_1\right)$$

$$\begin{aligned} (m_2 \mapsto \chi^{\hat{\phi}(m_2)}(t)) &\longleftarrow\!\shortmid (m_1 \mapsto \chi^{m_1}(t)) \\ \parallel \\ (m_2 \mapsto \chi^{m_2}(\phi(t))). \end{aligned}$$

Conversely, if $\phi(T_1) \subset T_2$ and $\phi_{|T_1}$ is a group homomorphism, the purple diagram induces the yellow diagram, $\mathbb{C}[M_2] \to \mathbb{C}[M_1]$ comes from a group homomorphism and $\phi^*(\mathbb{C}[S_2]) \subset \mathbb{C}[S_1]$ completes the proof. $\square$.

Prop : A toric morphism $\phi : V_1 \to V_2$ of affine toric varieties V_1, V_2 is equivariant, i.e.

$$\phi(t \cdot p) = \phi(t) \cdot \phi(p) \quad , \quad t \in T_1,\ p \in V_1.$$

proof:

$$\begin{array}{ccc} T_1 \times T_1 & \longrightarrow & T_1 \\ \downarrow \phi\times\phi & & \downarrow \phi \\ T_2 \times T_2 & \longrightarrow & T_2 \end{array} \quad \text{extends to} \quad \begin{array}{ccc} T_1 \times V_1 & \longrightarrow & V_1 \\ \downarrow \phi\times\phi & & \downarrow \phi \\ T_2 \times V_2 & \longrightarrow & V_2 \end{array} \quad \square.$$

Let $\bar{\phi} : \underset{\simeq \mathbb{Z}^{n_1}}{N_1} \to \underset{\simeq \mathbb{Z}^{n_2}}{N_2}$ be a homomorphism of lattices.

This induces a group homomorphism $\phi : T_1 \to T_2$

of tori $T_i = N_i \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^{n_i}$, which is also a morphism.

We will also consider $\overline{\phi}_{\mathbb{R}} = \overline{\phi} \otimes_{\mathbb{Z}} \mathbb{R} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$

Example: $N_i = \mathbb{Z}^{n_i}$, $T_i = (\mathbb{C}^*)^{n_i}$

$\overline{\phi} : \mathbb{Z}^{n_1} \to \mathbb{Z}^{n_2}$ given by a $n_2 \times n_1$ matrix F.

then $\phi : (\mathbb{C}^*)^{n_1} \to (\mathbb{C}^*)^{n_2}$ is given by

$$\phi(t) = \left(t^{F_{1,:}}, \dots, t^{F_{n_2,:}} \right).$$

$\overline{\phi}_{\mathbb{R}}$ is the $\mathbb{R}$-linear map represented by F.

When does ϕ extend to a map of toric varieties $U_{\sigma_1} \longrightarrow U_{\sigma_2}$?

The map $\hat{\phi} : \mathbb{Z}^{n_2 = M_2} \to \mathbb{Z}^{n_1 = M_1}$ is given by F^T.

We need that a character on U_{σ_2} pulls back to a character on U_{σ_1}.

That is, we need that

$$\hat{\phi}(m_2) \in \sigma_1^\vee \cap M_1 \quad \text{for all } m_2 \in \sigma_2^\vee \cap M_2.$$

$$\langle \overset{\shortparallel}{F^T m_2}, u_1 \rangle \geq 0 \quad \forall\, u_1 \in \sigma_1,\ m_2 \in \sigma_2^\vee$$

$$\iff \langle m_2, F u_1 \rangle \geq 0 \quad \forall\, u_1 \in \sigma_1,\ m_2 \in \sigma_2^\vee$$

$$\iff F u_1 \in (\sigma_2^\vee)^\vee = \sigma_2, \quad \forall\, u_1 \in \sigma_1$$

$\rightsquigarrow$ we need $\bar{\phi}_{\mathbb{R}}$ to be "compatible" with the cones σ_1 and σ_2.

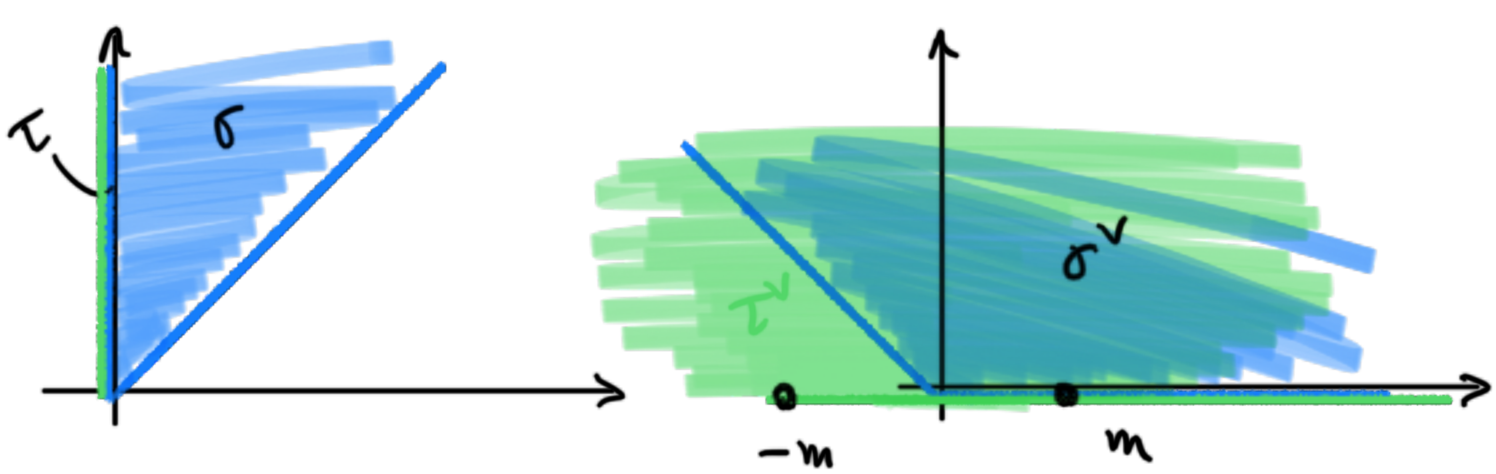
<u>Prop</u>: Let $\sigma_i \subset (N_i)_{\mathbb{R}}$ be strongly convex RCPCs, and let $\bar{\phi}: N_1 \to N_2$ be a lattice homomorphism. Then $\phi: T_1 \to T_2$ extends to $\phi: U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.

Example: affine open subsets from faces.

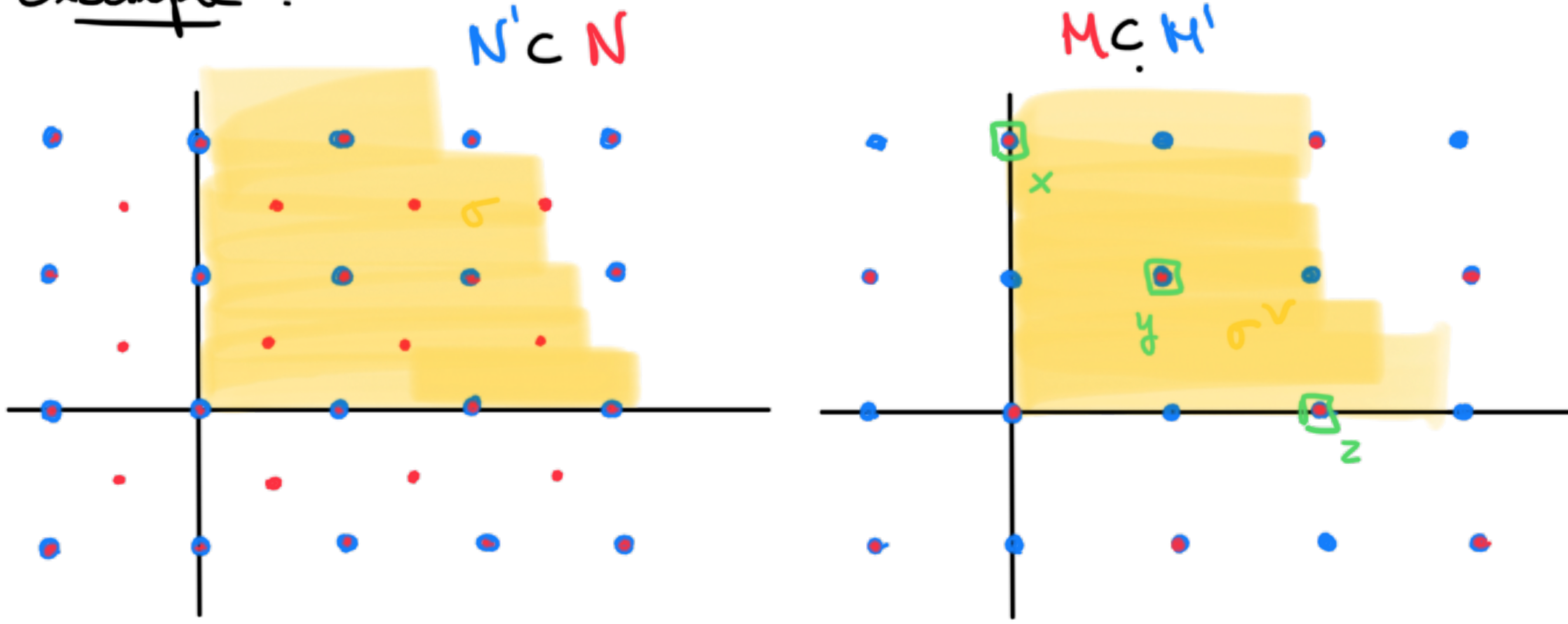
Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex RCPC, and let $\tau \preceq \sigma$ be a face. There is $m \in \sigma^\vee \cap M$ s.t. $\tau = \sigma \cap H_m$. The identity map $\bar{\phi} : N \to N$ is compatible with $\tau \subset \sigma$.

It induces $\phi : U_\tau \hookrightarrow U_\sigma$. The pullback is the inclusion $\phi^* : \mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\tau]$

prop $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$.



Example :



$\mathcal{U}_{\sigma,N'} = \mathbb{C}^2$, $\mathcal{U}_{\sigma,N}$ is not smooth !

the Hilbert basis for $\sigma^\vee \subset M$ has 3 elements

$N' \subset N$ induces a toric morphism $\phi : \mathbb{C}^2 \to \mathcal{U}_{\sigma,N}$

given by $\mathbb{C}[S_{\sigma,N}] \subset \mathbb{C}[S_{\sigma,N'}]$

$$\begin{array}{ccc} \mathbb{C}[S_{\sigma,N}] & \subset & \mathbb{C}[S_{\sigma,N'}] \\ \cong & & \cong \\ \mathbb{C}[x,y,z]/\langle xz-y^2\rangle & \hookrightarrow & \mathbb{C}[s,t] \end{array}$$

$$x \longmapsto t^2$$
$$y \longmapsto st$$
$$z \longmapsto s^2$$

Exercises : 1.3.4

1.3.11

1.3.12

1.3.14