INTRODUCTION TO TORIC GEOMETRY

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What are toric varieties?

- geometric objects defined by combinatorial information.
- subclass of algebraic varieties which is very well understood.

Why toric varieties?

- they provide a rich source of examples
- test cases for theorems and conjectures
- they provide a nice introduction to more general AG, polyhedral geometry, tropical geometry, geometric invariant theory, ...
- they pop up naturally when solving systems of sparse polynomial equations.
- they are still very popular.

When and where toric varieties?

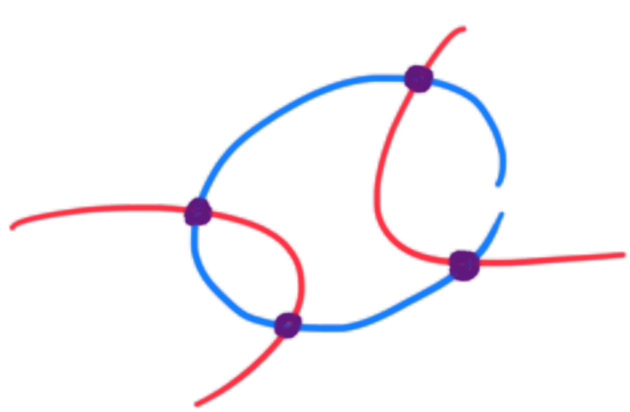
Wednesdays at 10AM in G3 10.

I encourage you to do a weekly round of exercises! I'm happy to make suggestions and answer questions.

A motivational example

> Slogan: toric varieties are those varieties that are parametrized by monomials.

$$f_1 = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2$$
$$f_2 = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2$$



Bézout: $f_1 = f_2 = 0$ is expected to have 4 solutions in $(\mathbb{C}^*)^2$.

first toric variety. (affine)

define $\phi_\Delta : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^5$ → second toric variety (first projective)

$(x, y) \longmapsto (1 : x : y : xy : x^2 : y^2)$

2nd projective toric variety. ← $X_\Delta = \overline{\operatorname{im} \phi_\Delta} \subset \mathbb{P}^5$

$$L_\Delta = \left\{ \begin{array}{l} a_0 u_0 + a_1 u_1 + \dots + a_5 u_5 = 0 \\ b_0 u_0 + b_1 u_1 + \dots + b_5 u_5 = 0 \end{array} \right\} \subset \mathbb{P}^5$$

A solution to $f_1 = f_2 = 0$ in $(\mathbb{C}^*)^2$ is a point in

$$\operatorname{im} \phi_\Delta \cap L_\Delta \subset X_\Delta \cap L_\Delta$$

← solutions of $F_1 = F_2 = 0$ on $\mathbb{P}^2 \simeq X$.

usually an equality.

$$|X_\Delta \cap L_\Delta| = \deg X_\Delta = \deg \nu_2(\mathbb{P}^2) = 4$$

$$f_1 = a_0 + a_1 x + a_2 y + a_3 xy$$
$$f_2 = b_0 + b_1 x + b_2 y + b_4 xy$$

$f_1 = f_2 = 0$ usually has only 2 solutions in $(\mathbb{C}^*)^2$

What about Bézout?

$$F_1 = a_0 Z^2 + a_1 XZ + a_2 YZ + a_3 XY$$
$$F_2 = b_0 Z^2 + b_1 XZ + b_2 YZ + b_3 XY$$

$F_1 = F_2 = 0$ has solutions $(X:Y:Z) = \begin{cases} (0:1:0) \\ (1:0:0) \end{cases}$

The sparse monomial support forces two solutions to lie on the boundary of $(\mathbb{C}^*)^2$ in $\mathbb{P}^2$.

$$\operatorname{im} \phi_\Delta \cap L_\square \subsetneq X_\Delta \cap L_\square$$

never equality

$X_\Delta \cong \mathbb{P}^2$

define $\phi_\square : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^3$

$$(x,y) \longmapsto (1 : x : y : xy)$$

$X_\square = \overline{\operatorname{im} \phi_\square} \subset \mathbb{P}^3$ is given by $xw - yz = 0$

$$L_\square = \left\{ \begin{array}{l} a_0 u_0 + a_1 u_1 + a_2 u_2 + a_3 u_3 = 0 \\ b_0 u_0 + b_1 u_1 + b_2 u_2 + b_3 u_3 = 0 \end{array} \right\} \subset \mathbb{P}^3$$

$$\operatorname{im} \phi_\square \cap L_\square \subseteq X_\square \cap L_\square$$

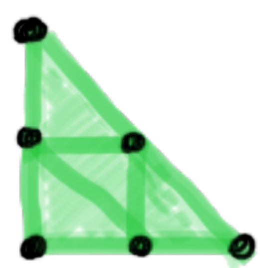
almost always equality.

$$|X_\square \cap L_\square| = \deg X_\square = \deg \sigma(\mathbb{P}^1 \times \mathbb{P}^1) = 2.$$

$$\left(X_\square = \{ u_0 u_3 - u_1 u_2 = 0 \} \right).$$

The new equations are naturally interpreted as equations on $\mathbb{P}^1 \times \mathbb{P}^1$, rather than $\mathbb{P}^2$!

Geometric information from combinatorial data:



dimension = 2 dimension = 2

degree = 4 degree = 2

Kushnirenko's theorem.

Check this in other cases using your favourite software.

§1 : AFFINE TORIC VARIETIES.

1.0 Tori and their lattices

A **torus** T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from this isomorphism.

Characters of T are morphisms $T \to \mathbb{C}^*$ that are group homomorphisms.

Prop : The characters of $(\mathbb{C}^*)^n$ are given by

$$(t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$$

for $(a_1, \dots, a_n) \in \mathbb{Z}^n$.

Cor : The characters of a torus $T \simeq (\mathbb{C}^*)^n$ form a **lattice** $M \simeq \mathbb{Z}^n$

↳ a finitely generated, free abelian group.

Conventionally, $m \in M$ gives $\chi^m : T \to \mathbb{C}^*$.

The dual lattice N of the character lattice M is called the cocharacter lattice or the lattice of one-parameter subgroups of T.

$$N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

It is given by morphisms $\mathbb{C}^* \to T$ that are group homomorphisms.

Prop: The one-parameter subgroups of $(\mathbb{C}^*)^n$ are
$$t \mapsto (t^{u_1}, \dots, t^{u_n})$$
for $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$.

Picking an isomorphism $T \overset{\phi}{\simeq} (\mathbb{C}^*)^n$ fixes a basis $e_1, \dots, e_n$ of M: $\chi^{e_i}(t) = \phi(t)_i$ and a dual basis $e_1', \dots, e_n'$ of N: $\lambda^{e_i'}(t) = (1, \dots, \underset{\substack{\uparrow \\ i\text{-th spot.}}}{t}, \dots, 1)$.

So it fixes isomorphisms $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$.

The natural pairing $M \times N \to \mathbb{Z}$

$$(m, n) \mapsto \chi^m \circ \lambda^n \quad (t \mapsto t^l,\ l \in \mathbb{Z})$$

is given in coordinates by $\underbrace{\langle m, n \rangle = m \cdot n}_{\text{the usual dot-product in } \mathbb{Z}^n} = l$

1.1 Definition of an affine toric variety

An affine toric variety is an irreducible affine variety V containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski dense open subset such that the action of T on itself extends to an algebraic action of T on V.

an action $T \times V \to V$ given by a morphism.

Examples:

- $(\mathbb{C}^*)^n$, $\mathbb{C}^n$
- $V(x^3 - y^2)$ $\quad (t, (x, y)) \mapsto (t^2 x, t^3 y)$
- $V(xy - zw)$ $\quad ((t_1, t_2, t_3), (x, y, z, w)) \mapsto (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w)$