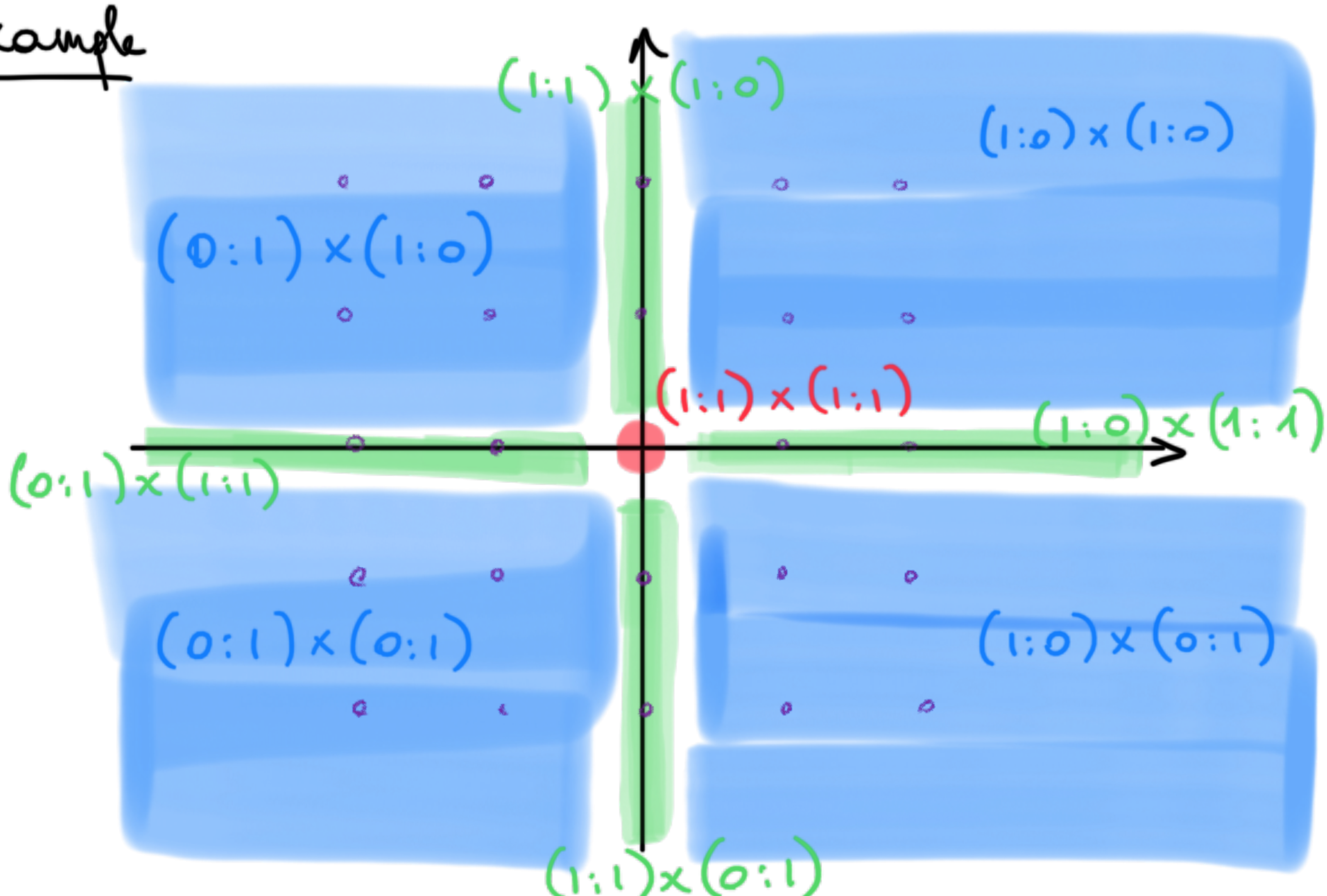
3.3 Limits of one-parameter subgroups

Example



$$X_{\Sigma'} = \mathbb{P}^1 \times \mathbb{P}^1 \qquad T_{\mathbb{P}^1\times\mathbb{P}^1} = \{ (1:t_1)\times(1:t_2) \mid (t_1,t_2) \in (\mathbb{C}^*)^2 \}.$$

$$u = (a,b), \quad \lambda^u(\underset{\in\,\mathbb{C}^*}{t}) = (1:t^a)\times(1:t^b) \in \mathbb{P}^1\times\mathbb{P}^1$$

$\lim_{t\to 0} \lambda^u(t)$ depends on u.

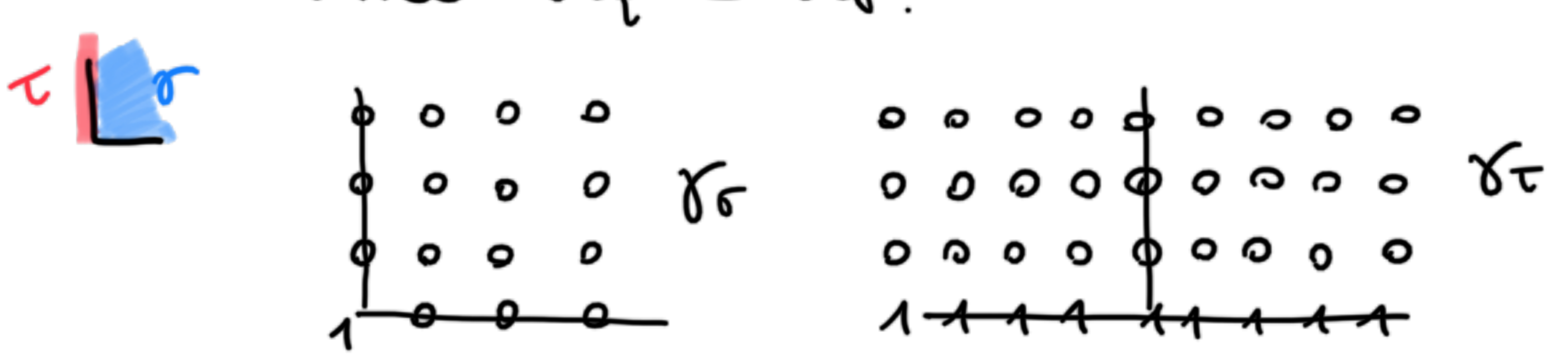
σ corresponds to a T-orbit O if $\lim_{t\to 0} \lambda^u(t) \in O$ for all $u \in \mathrm{RelInt}(\sigma)$.

Recall: points of $\mathcal{U}_\sigma$ are semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$. ($\sigma$ strongly convex RCPC in $N_\mathbb{R} \simeq \mathbb{R}^n$)

Def: The distinguished point of the affine toric variety $\mathcal{U}_\sigma$ is

$$\gamma_\sigma : m \in S_\sigma \longmapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

- The point γ_σ is fixed under the torus action on $\mathcal{U}_\sigma$ iff $\dim \sigma = n$.
- If $\tau \preceq \sigma$, $\gamma_\tau \in \mathcal{U}_\sigma$ (restricted to S_σ) since $\mathcal{U}_\tau \subset \mathcal{U}_\sigma$.



Prop: Let σ be a strongly convex RCPC in $N_\mathbb{R}$ and $u \in N$. Then $u \in \sigma \iff \lim_{t \to 0} \lambda^u(t)$ exists in $\mathcal{U}_\sigma$.

If $u \in \mathrm{Relint}(\sigma)$, $\lim_{t \to 0} \lambda^u(t) = \gamma_\sigma$.

proof $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ

$$\iff \lim_{t\to 0} \chi^m(\lambda^u(t)) \text{ exists in } \mathbb{C} \text{ for all } m \in \sigma^\vee \cap M.$$

$$\iff \lim_{t\to 0} t^{\langle u,m\rangle} \text{ exists in } \mathbb{C},\ \forall m \in \sigma^\vee \cap M$$

$$\iff \langle m, u\rangle \geq 0 \text{ for all } m \in \sigma^\vee \cap M$$

$$\iff u \in (\sigma^\vee)^\vee = \sigma$$

If $u \in \sigma$, $\lim_{t\to 0} \lambda^u(t)$ is the point $m \mapsto \lim_{t\to 0} t^{\langle u,m\rangle}$.

$$u \in \operatorname{Relint}\sigma \Rightarrow \begin{cases} \langle u, m\rangle > 0, & m \in (\sigma^\vee \setminus \sigma^\perp) \cap M \\ \langle u, m\rangle = 0, & m \in \sigma^\perp \cap M. \end{cases} \quad \square.$$

$\rightsquigarrow$ we can recover Σ from limits of 1-parameter subgroups! See Example 3.2.3.

3.4 The Orbit-Cone correspondence

We define $O(\sigma) = T \cdot \gamma_\sigma \subset X_\Sigma$.

These orbits are tori themselves.

Let $N_\sigma = \mathbb{Z}\cdot(\sigma\cap N)$ and $N(\sigma) = N/N_\sigma$.

The perfect pairing

$$\sigma^\perp \cap M \times N(\sigma) \to \mathbb{Z}$$

identifies $\sigma^\perp\cap M = N(\sigma)^\vee$, hence

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp\cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}$$

where $T_{N(\sigma)} = N(\sigma)\otimes_{\mathbb{Z}} \mathbb{C}^*$ is the torus of $N(\sigma)$.

$$[u]\otimes t \sim \left(m \longmapsto \chi^m(\lambda^u(t)) = t^{\langle u,m\rangle}\right)$$

(*) <u>Lemma</u> Let $\sigma\subset N_{\mathbb{R}}$ be a strongly convex RCPC.

$$O(\sigma) = \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m)\neq 0 \iff m\in\sigma^\perp\cap M\}.$$

(*)

$$\simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp\cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}.$$

proof Let $O' = (*)$. Clearly $\gamma_\sigma \in O'$ and O' is T-invariant, since $t\cdot(m\mapsto\gamma(m)) = (m\mapsto\chi^m(t)\gamma(m))$

If $\gamma\in O'$, then $\gamma_{|\sigma^\perp\cap M} : \sigma^\perp\cap M \to \mathbb{C}^*$ is a group homomorphism. Conversely, if $\hat\gamma : \sigma^\perp\cap M\to\mathbb{C}^*$

is a group homomorphism, then $\gamma : m \longmapsto \begin{cases} \hat{\gamma}(m), & m \in \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases}$

is in $O(\sigma)$. Hence $O' \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$.

We now show $O' = O(\sigma)$. Tensoring $N \to N(\sigma) \to 0$

with $\mathbb{C}^*$ gives $T \xrightarrow{\text{“}T_N\text{”}} T_{N(\sigma)} \to 1$. So T acts

transitively on $T_{N(\sigma)}$, and this action is compatible

with the bijections $T_{N(\sigma)} \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq O'$:

$t \in T$, $\sum_i [e_i] \otimes t_i' \in T_{N(\sigma)} \sim \gamma : m \mapsto \chi^m(t_i')$

$$t \cdot \left(\sum [e_i] \otimes t_i' \right) = \sum_i [e_i] \otimes t_i t_i'$$

$$\sim \quad m \mapsto \prod (t_i t_i')^{\langle e_i, m \rangle}$$

$$= \chi^m(t) \cdot \gamma(m)$$

$\square$

Theorem: The Orbit-Cone correspondence. Let $X_{\Sigma'}$ be the toric variety of the fan Σ' in $N_{\mathbb{R}} \simeq \mathbb{R}^n$.

a) There is a bijective correspondence

$$\{\text{cones } \sigma \in \Sigma_1'\} \xleftrightarrow{1:1} \{T\text{-orbits in } X_{\Sigma'}\}$$
$$\sigma \longleftrightarrow O(\sigma)$$

b) For each $\sigma \in \Sigma$, $\dim \sigma + \dim O(\sigma) = n$

c) The affine open subset $\mathcal{U}_\sigma$ is the (disj.) union of orbits $\mathcal{U}_\sigma = \bigsqcup_{\tau \leq \sigma} O(\tau)$.

d) $\tau \leq \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$ and

$$\overline{O(\tau)} = \bigsqcup_{\tau \leq \sigma} O(\sigma).$$

Here the closure is in both Zariski and classical topology.

proof a) Let $O \subset X_\Sigma$ be a T-orbit. Since X_Σ is covered by the T-invariant $\mathcal{U}_\sigma$, there is a minimal cone $\sigma \in \Sigma$ s.t. $O \subset \mathcal{U}_\sigma$. Claim: $O = O(\sigma)$.

Let $\gamma \in O$. For some $\tau \leq \sigma$

$$\{ m \in S_\sigma \mid \gamma(m) \neq 0 \} = \sigma^\vee \cap \tau^\perp \cap M$$

(since all such m's must lie on a face of $\sigma^\vee$).

This means that γ extends to a semigroup homomorphism $S_\tau \to \mathbb{C}$, and hence $O \subset \mathcal{U}_\tau \Rightarrow \tau = \sigma$ by minimality.

The claim follows, by our lemma (*).

b) Direct consequence of (*).

c) $\mathcal{U}_\sigma$ is T-invariant, so it's a union of orbits.

$O(\tau) \subset \mathcal{U}_\tau \subset \mathcal{U}_\sigma$. The proof of part (a) implies that any orbit $O \subset \mathcal{U}_\sigma$ is $O(\tau)$ for some face $\tau \preceq \sigma$.

d) We show that if $\tau \preceq \sigma$, $O(\sigma) \subset \overline{O(\tau)}$ in the classical topology. First of all, note that $\overline{O(\tau)}$ is T-invariant, so it suffices to show that $\overline{O(\tau)} \cap O(\sigma) \neq \phi$.

↳ by continuity of $T \times O(\tau) \to O(\tau)$

Here is where we use limits of one-parameter subgroups.

Let $u \in \mathrm{Relint}(\sigma)$ and $\gamma(t) = \lambda^u(t) \cdot \gamma_\tau$.

Note that $\gamma(t): m \longmapsto t^{\langle u,m\rangle} \gamma_\tau(m)$

Hence $\lim\limits_{t\to 0} \gamma(t): m \longmapsto \lim\limits_{t\to 0} t^{\langle u,m\rangle} \gamma_\tau(m)$

$$= \begin{cases} 0 & m \in (\sigma^\vee \setminus \sigma^\perp) \cap M \\ \gamma_\tau(m), & m \in \sigma^\perp \cap M \end{cases}$$

Since $\sigma^\perp \subset \tau^\perp$, $\gamma(0) = \lim_{t\to 0} \gamma(t) = \gamma_\sigma \in O(\sigma)$.

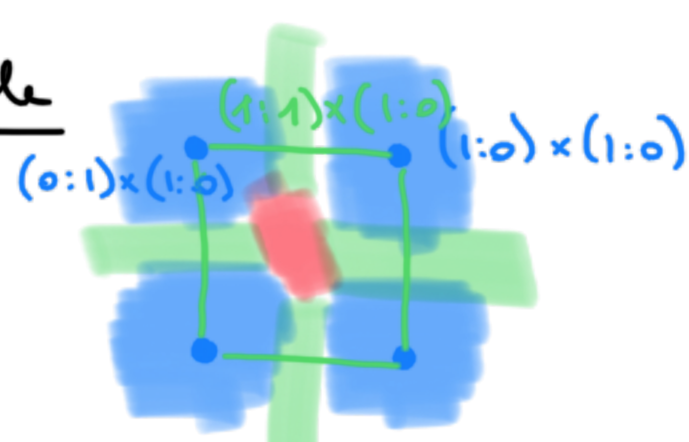
By construction, $\gamma(0)$ is also in $\overline{O(\tau)}$.

Conversely, suppose $O(\sigma) \subset \overline{O(\tau)}$. This implies $O(\tau) \subset U_\sigma$, as otherwise $U_\sigma \cap O(\tau) = \phi$ and hence $U_\sigma \cap \overline{O(\tau)} = \phi$. Then $\tau \leq \sigma$ by part a. Zariski topology $\Rightarrow$ exercise $\square$

A cone $\sigma \in \Sigma$ gives
- an affine open subset U_σ
- a torus orbit $O(\sigma) \subset U_\sigma$.

The relation between these two is point c).

<u>Example</u>



$\mathbb{P}^1 \times \mathbb{P}^1$ has

4 torus fixed points

4 1-dim'l torus orbits

1 dense torus orbit.

Example: as a particular case of d) for $\tau = \{0\}$, we have $X_\Sigma = \bigsqcup_{\sigma \in \Sigma} O(\sigma)$.

3.5 Orbit closures as toric varieties

Notation: $V(\tau) = \overline{O(\tau)}$.

We have seen that $V(\tau)$ has an open subset $O(\tau) \simeq T_{N(\tau)}$. In fact, $V(\tau)$ is a normal toric variety corresponding to a fan in $N(\tau)_{\mathbb{R}}$.

The T-orbits in

$$V(\tau) = \bigsqcup_{\tau \leq \sigma} O(\sigma)$$

can be viewed as $T_{N(\tau)}$-orbits:

$$\begin{tikzcd} T \times O(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & O(\sigma) \\ T_{N(\tau)} \times O(\sigma) \arrow[ur] & \end{tikzcd}$$

$$((u \otimes t), \gamma) \mapsto (([u] \otimes t), \gamma)$$

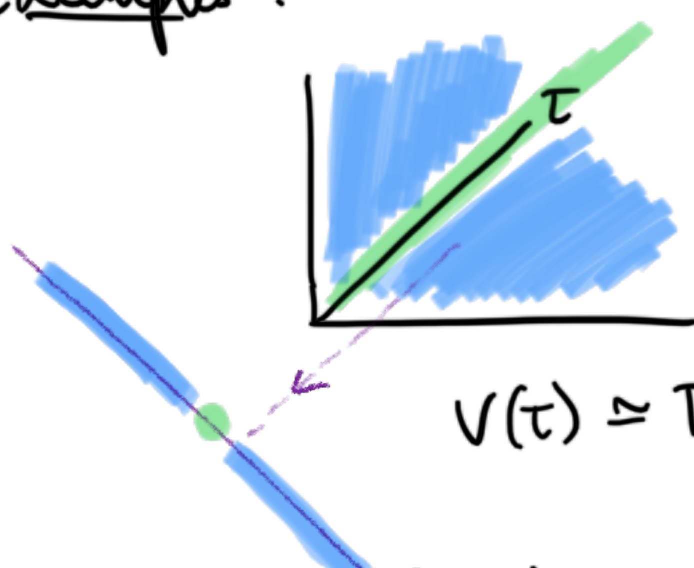
Prop: Let $\mathrm{Star}(\tau) = \{ \overline{\sigma} \subset N(\tau)_{\mathbb{R}} \mid \tau \leq \sigma \in \Sigma' \}$

where $\overline{\sigma}$ is the image of σ under $N_{\mathbb{R}} \longrightarrow N(\tau)_{\mathbb{R}}$.

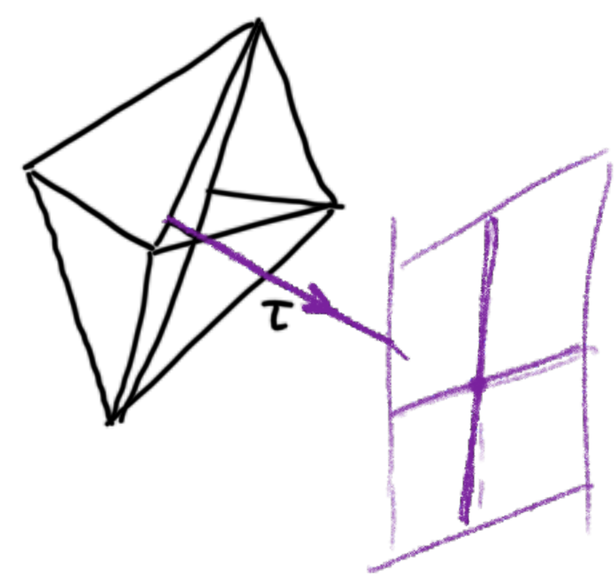
Then $\mathrm{Star}(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$ and

$$V(\tau) = \overline{O(\tau)} \simeq X_{\mathrm{Star}(\tau)}.$$

Examples:



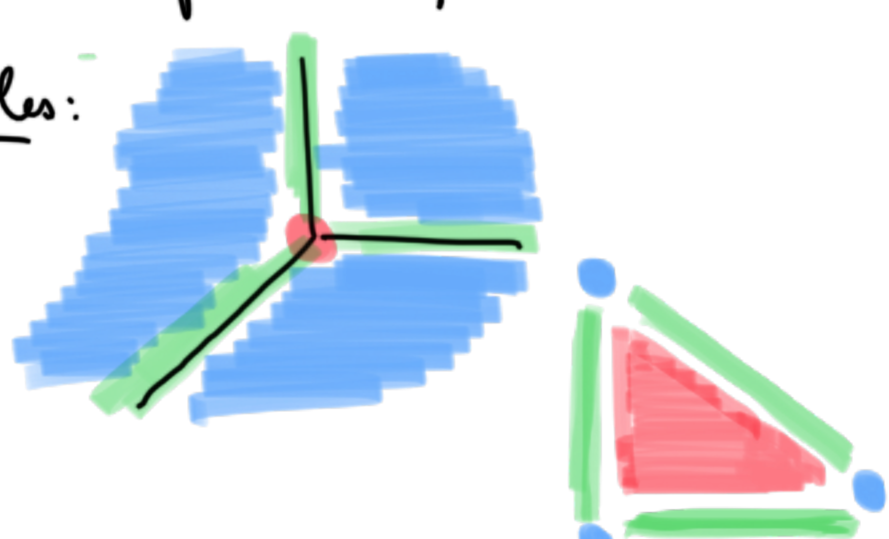
$V(\tau) \simeq \mathbb{P}^1$

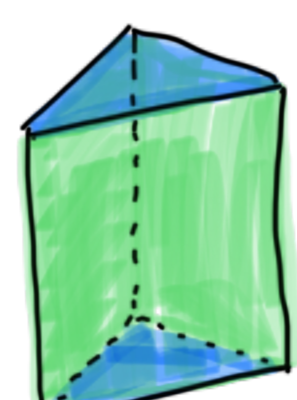


$V(\tau) \simeq \mathbb{P}^1 \times \mathbb{P}^1$

Prop: if $\Sigma' = \Sigma'_P$ is the normal fan of a full dimensional polytope $P \subset M_{\mathbb{R}}$, then torus orbits correspond to faces $Q \leq P$ and $V(\sigma_Q) \simeq X_Q$.

Examples:





$X_P \simeq \mathbb{P}^1 \times \mathbb{P}^2$

two torus orbits look like $\mathbb{P}^2$

three look like $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercises : • 3.2.3

- describe the orbit-cone correspondence for $\mathbb{P}^2$ in detail. What are the distinguished points in homogeneous coordinates?
- same for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. what is the polytope?