Recall that $\mathrm{Div}_0(X) \subset \mathrm{CDiv}(X) \subset \mathrm{Div}(X)$.

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Div}_0(X)$$

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/\mathrm{Div}_0(X)$$

For the normal toric variety X_Σ, we have

(*)
$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

(when enough rays)

primitive ray generator

$$m \longmapsto \mathrm{div}(\chi^m) = \sum_\rho \langle u_\rho, m \rangle \cdot D_\rho .$$

Let $\mathrm{Div}_T(X_\Sigma) = \bigoplus_\rho \mathbb{Z} \cdot D_\rho \subset \mathrm{Div}(X_\Sigma)$

4.3 Cartier divisors on X_Σ.

① Let $D \in \mathrm{CDiv}(X_\Sigma)$ be a Cartier divisor.

By (*), D is linearly equivalent to $\sum_\rho a_\rho D_\rho$,

with $\sum_\rho a_\rho D_\rho \in \mathrm{CDiv}_T(X_\Sigma) = \mathrm{Div}_T(X_\Sigma) \cap \mathrm{CDiv}(X_\Sigma)$.

② For $m \in M$, $\mathrm{div}(\chi^m) \in \mathrm{CDiv}_T(X_\Sigma)$.

(*), ① and ② imply :

Theorem: We have an exact sequence

$$M \xrightarrow{\mathrm{div}} \mathrm{CDiv}_T(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0.$$

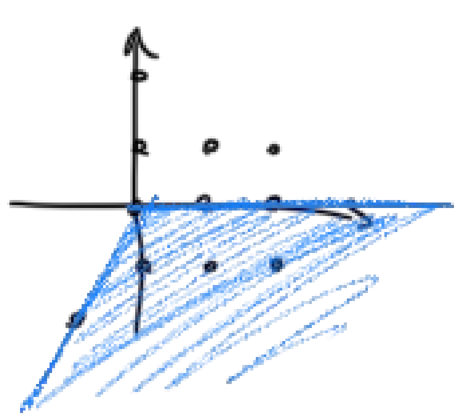
Moreover, if $\mathrm{span}_{\mathbb{R}}\{u_\rho, \rho \in \Sigma(1)\} = N_{\mathbb{R}}$, we have

$$0 \longrightarrow M \longrightarrow \mathrm{CDiv}_T(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0. \quad \square$$

Our goal is to describe $\mathrm{CDiv}(X_\Sigma)$ and $\mathrm{Pic}(X_\Sigma)$ explicitly. We start with $X_\Sigma = U_\sigma$.

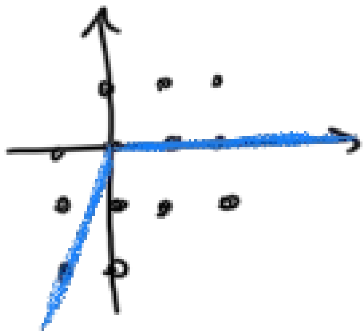
Proposition (4.2.2 in [CLS]). Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex RCPC. Then $\mathrm{Pic}(U_\sigma) = 0$. That is, $M \to \mathrm{CDiv}_T(X_\Sigma)$ is surjective.

Example



$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}} \mathbb{Z}^2$$

$$\mathrm{Cl}(U_\sigma) = \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Pic}(U_\sigma) = 0$$



$$\mathrm{Cl}(X_\Sigma) = \mathrm{Pic}(X_\Sigma) = \mathbb{Z}/2\mathbb{Z}.$$

<u>Proposition</u> If Σ contains a cone of dimension n, then $\mathrm{Pic}(X_\Sigma)$ is free.

<u>proof</u> Since $0 \to M \to \mathrm{CDiv}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$, it suffices to show that $kD = \mathrm{div}(\chi^m)$, implies $D = \mathrm{div}(\chi^{m'})$, $k \in \mathbb{Z}_{>0}$.

Let $D = \sum_\rho a_\rho D_\rho$ and $\sigma \in \Sigma(n)$. By our previous proposition,

$$D_{|U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho = \mathrm{div}(\chi^{m'})_{|U_\sigma}$$

for some $m' \in M$.

Therefore $a_\rho = \langle u_\rho, m' \rangle$ for $\rho \in \sigma(1)$.

On the other hand, $kD = \mathrm{div}(\chi^m)$ implies

$$k a_\rho = \langle u_\rho, m \rangle \text{ for } \rho \in \Sigma(1).$$

Hence $\langle u_\rho, km' \rangle = \langle u_\rho, m \rangle$ for $\rho \in \sigma(1)$,

so that $km' = m$ and

$$\begin{aligned} \operatorname{div}(\chi^{m'}) &= \sum_\rho{}' \langle u_\rho, m' \rangle D_\rho \\ &= \frac{1}{k} \sum_\rho{}' \langle u_\rho, m \rangle D_\rho \\ &= D. \end{aligned}$$

$\square$.

When Σ_1 is smooth, we know that $Cl(X_\Sigma) = Pic(X_\Sigma)$.

This is an "$\Longleftrightarrow$" in our setting:

Prop X_Σ is smooth $\Longleftrightarrow$ $CDiv(X_\Sigma) = Div(X_\Sigma)$.

proof We show $\Longleftarrow$. The restriction $Cl(X_\Sigma) \to Cl(U_\sigma)$ is surjective, hence $CDiv(U_\sigma) = Div(U_\sigma)$.

We also have $Pic(U_\sigma) = 0$, so that

$$M \longrightarrow Div_T(U_\sigma) = \bigoplus_{\rho \in \sigma(1)} \mathbb{Z} \cdot D_\rho$$

is surjective. Choosing coordinates, this is the matrix

$F^T = \begin{bmatrix} - u_1 - \\ \vdots \\ - u_s - \end{bmatrix}$, where $\sigma(1) = \{\rho_1, \dots, \rho_s\}$.

F^T is surjective iff $\{u_1, \ldots, u_s\}$ can be extended to a basis of N (Smith normal form). $\square$

SKIP during lectures, present as exercise

<u>Proposition</u> TFAE:

(a) Every Weil divisor on X_Σ has a positive integer multiple that is Cartier.

(b) $\mathrm{Pic}(X_\Sigma)$ has finite index in $\mathrm{Cl}(X_\Sigma)$.

(c) X_Σ is simplicial.

<u>proof</u> Exercise 4.2.2. $\square$.

recap

By definition, a torus invariant Cartier divisor $D = \sum_\rho a_\rho D_\rho$ is locally principal. We consider the covering $X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$. The principal divisor $D|_{U_\sigma}$ on U_σ is that of a character.

we now answer which characters are s.t. $\mathrm{div}(\chi^m) = D|_{U_\sigma}$.

Theorem Let $D = \sum_\rho' a_\rho D_\rho \in \mathrm{Div}_T(X_\Sigma)$. TFAE:

(a) D is Cartier.

(b) D is principal on U_σ for all $\sigma \in \Sigma$.

(c) For each $\sigma \in \Sigma$, there is $m_\sigma \in M$ s.t. $\langle u_\rho, m_\sigma \rangle = -a_\rho$ for all $\rho \in \sigma(1)$ called "Cartier data"

(d) Idem for each $\sigma \in \Sigma_{max}$.

Moreover, if $D \in \mathrm{CDiv}_T(X_\Sigma)$ and $\{m_\sigma\}_{\sigma\in\Sigma}$ as in (c),

(1) m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$.

(2) If $\tau \preceq \sigma$, then $m_\sigma = m_\tau \bmod M(\tau)$

proof We already know (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (c) follows from the fact that m_σ will do for all faces of σ.

For (1), suppose $\langle u_\rho, m_\sigma \rangle = \langle u_\rho, m_\sigma' \rangle = -a_\rho$ for all $\rho \in \sigma(1)$, then

$$\langle u_\rho, m'_\sigma - m_\sigma \rangle = 0 \quad \text{for all } \rho \in \sigma(1)$$

$$\iff \langle u, m'_\sigma - m_\sigma \rangle = 0 \quad \text{for all } u \in \sigma$$

$$\iff m'_\sigma - m_\sigma \in \sigma^\perp \cap M = M(\sigma).$$

This implies $m_\sigma = m_\tau \bmod M(\tau)$ by our observation that m_σ works for m_τ. $\square$

<u>Rmks</u>: in the notation of this theorem, $D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})$. The minus signs come from our notation for polytopes: $\langle u_\rho, m \rangle + a_\rho \geq 0$. If $\sigma \in \Sigma(n)$, m_σ is uniquely determined.

We can regard m_σ as a uniquely determined element of $M/M(\sigma)$. If $\tau \preceq \sigma$, the canonical map $M/M(\sigma) \to M/M(\tau)$ sends m_σ to m_τ.

This gives an explicit description of $\mathrm{CDiv}_T(X_\Sigma)$.

Let $\Sigma_{max} = \{\sigma_1, \dots, \sigma_r\}$ and

$$\phi : \bigoplus_i M/M(\sigma_i) \longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)$$
$$(m_i)_i \longmapsto (m_i - m_j)_{i<j}$$

<u>Proposition</u> : $\mathrm{CDiv}_T(X_\Sigma) \simeq \ker \phi$.

<u>Example : a Cartier divisor on X_P.</u>

Let $P = \{m \in M_{\mathbb{R}} \mid \langle u_F, m\rangle + a_F \geq 0, F \in \mathcal{F}\}$ be a full-dimensional polytope with facets $\mathcal{F}$. The u_F are the inward pointing facet normals.

$X_P = X_{\Sigma_P}$, with Σ_P the normal fan of P.

$(\Sigma_P)_{max}$ is the set of cones σ_v corresponding to the vertices v of P. We have

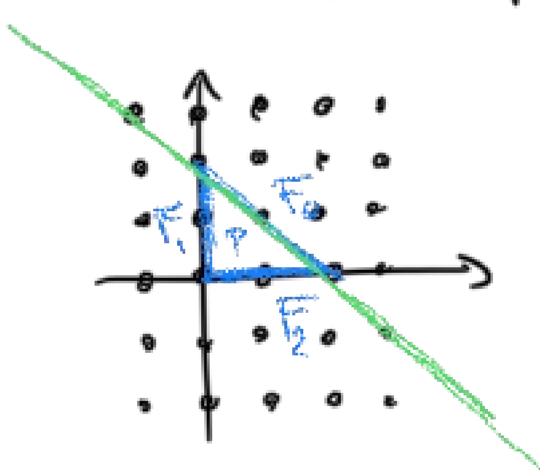
$$\langle u_F, v\rangle + a_F = 0, \quad \forall F \ni v$$
i.e. $\forall u_F \subset \sigma_v$

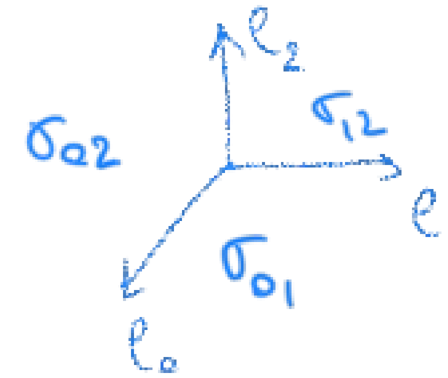
so that the vertices of P are the Cartier data for the divisor $D_P = \sum'_{F \in \mathcal{F}} a_F D_F \in \mathrm{Div}_T(X_P)$.

P and $P + m$ give the same toric variety, but different divisors on it: $D_{P+m} = D_P + \mathrm{div}(\chi^m)$
$$\sim D_P.$$

The divisor class of D_P corresponds to translates of P.

Example





gives divisor on $\mathbb{P}^2$

$$D_P = 2 D_0 \qquad \left(P = \left\{ m \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \geq 0 \right\} \right)$$

$$(2,0) = m_{\sigma_{02}} \quad : \quad \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{(2,0) + \mathbb{Z} \cdot (1,-1)} = m_{e_0} \quad : \quad (-1 \ -1) \begin{pmatrix} 2 + a \\ 0 - a \end{pmatrix} + 2 = 0.$$

4.4 The sheaf of a divisor

A Weil divisor D on a normal variety X determines a sheaf of $\mathcal{O}_X$-modules.

The structure sheaf $\mathcal{O}_X$ is defined by

$$\mathcal{U} \mapsto \mathcal{O}_X(\mathcal{U}) = \{ f \in \mathbb{C}(X)^* \mid \operatorname{div}(f)_{|\mathcal{U}} \geq 0 \} \cup \{0\}.$$

Similarly, we define the sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{U} \longmapsto \mathcal{O}_X(D)(\mathcal{U}) = \{ f \in \mathbb{C}(X)^* \mid (\operatorname{div}(f) + D)_{|\mathcal{U}} \geq 0 \} \cup \{0\}.$$

notation: $\Gamma(\mathcal{U}, \mathcal{O}_X(D)) = \mathcal{O}_X(D)(\mathcal{U})$.

<u>Lemma</u> Consider the action of T on $\mathbb{C}[M]$ given by $t \cdot f = (p \longmapsto f(t^{-1} \cdot p))$. If $A \subset \mathbb{C}[M]$ is a subvector space stable under this action, then

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

<u>proof</u> See lemma 1.1.16 in [CLS]. $\square$

<u>Proposition</u> Let $D \in \mathrm{Div}_T(X_\Sigma)$. Then

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi^m)+D \geq 0} \mathbb{C} \cdot \chi^m .$$

<u>proof</u> $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \Longleftrightarrow \mathrm{div}(f) + D \geq 0$

$$\Longrightarrow \mathrm{div}(f)_{|T} \geq 0$$

$$\Longrightarrow f \in \mathbb{C}[M].$$

Hence $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subset \mathbb{C}[M]$ is a subspace, stable under the action of T on $\mathbb{C}[M]$. $\square$

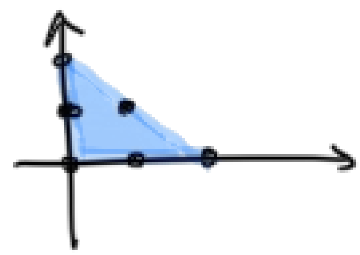
If $D = \sum_\rho a_\rho D_\rho$, we have

$$\{m \mid \mathrm{div}(\chi^m) + D \geq 0\} = \{m \mid \langle u_\rho, m\rangle + a_\rho \geq 0\}$$

This is a <u>polyhedron</u> called P_D:

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

Example : $X_\Sigma = \mathbb{P}^2$, $D = 2D_0$

$P_D =$ 

$\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2D_0))$

$\simeq$ polynomials of degree ≤ 2.

$$a + bt_1 + ct_2 + d\,t_1t_2 + e\,t_1^2 + f\,t_2^2$$

$$\sim \frac{a x_0^2 + b x_0x_1 + c\, x_0x_2 + d\, x_1x_2 + e\, x_1^2 + f\, x_2^2}{x_0^2}$$

This clearly also works for $D = 2D_i$. The reason is $D_i \sim D_j \Rightarrow \Gamma(X, \mathcal{O}_X(D_i)) \simeq \Gamma(X, \mathcal{O}_X(D_j))$.