## 1.2 Affine toric varieties from monomial maps

Let $T \simeq (\mathbb{C}^*)^n$ be a torus with character lattice $M$ and fix $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$.

We consider the map

$$\phi_{\mathcal{A}} : T \longrightarrow \mathbb{C}^s \text{ given by } \phi_{\mathcal{A}}(t) = (\chi^{m_i}(t))_{i=1,\dots,s}$$

The closure of the image is

$$Y_{\mathcal{A}} = \overline{\operatorname{im} \phi_{\mathcal{A}}} \subset \mathbb{C}^s.$$

<u>Prop</u>: $Y_{\mathcal{A}}$ is an affine toric variety whose dense torus has character lattice $\mathbb{Z}\mathcal{A}$.

Fact: for a morphism $\phi : T_1 \to T_2$ of tori that is a group homomorphism, $\phi(T_1)$ is a closed subtorus of $T_2$.

<u>proof</u> $T' = \operatorname{im} \phi_{\mathcal{A}}$ is a closed subtorus of $(\mathbb{C}^*)^s$, since $\phi_{\mathcal{A}}$ can be viewed as a map $T \to (\mathbb{C}^*)^s$.

it's the complement of a subvariety!

Hence $Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s = T'$, so that $Y_{\mathcal{A}} = \overline{T'}$ is irreducible with a Zariski dense torus.

$t \in T' \subset (\mathbb{C}^*)^s$ acts on $\mathbb{C}^s$ and takes varieties to varieties. Hence

$$T' = t \cdot T' \subset t \cdot Y_{\mathcal{A}},$$

and since $Y_{\mathcal{A}} = \overline{T'}$, we find $Y_{\mathcal{A}} \subset t \cdot Y_{\mathcal{A}}$.
replacing $t$ with $t^{-1}$, we see that $Y_{\mathcal{A}} \subset t^{-1} \cdot Y_{\mathcal{A}}$ and hence $t \cdot Y_{\mathcal{A}} \subset Y_{\mathcal{A}} \subset t \cdot Y_{\mathcal{A}}$.

To see what the character lattice of $Y_{\mathcal{A}}$ is, we consider the diagrams

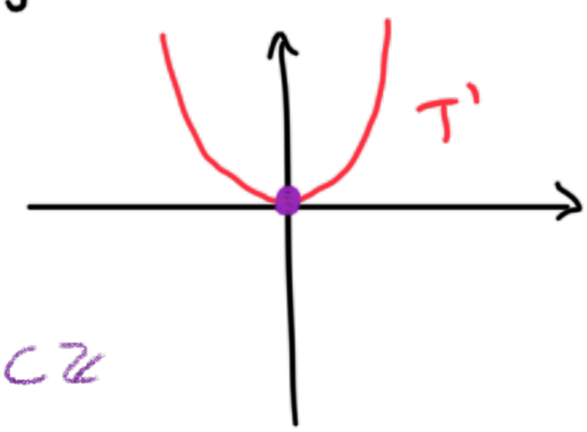
$$\begin{tikzcd} T \arrow[r] \arrow[dr, two heads] & (\mathbb{C}^*)^s \\ & T' \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} M & \mathbb{Z}^s \arrow[l] \arrow[d] \\ & M' \arrow[ul, hook] \end{tikzcd}$$

which shows that the char. lattice $M' \simeq \mathbb{Z}\mathcal{A}$. $\square$

Concretely: collect the lattice points in $\mathcal{A}$ in the columns of a matrix $A \in \mathbb{Z}^{n\times s}$. Then $\dim Y_{\mathcal{A}} = \operatorname{rank} A$.

<u>Example</u> $(n=1)$ $\quad \mathcal{A} = \{2, 4\} \subset \mathbb{Z}$

$$\phi_{\mathcal{A}}(t) = (t^2, t^4)$$



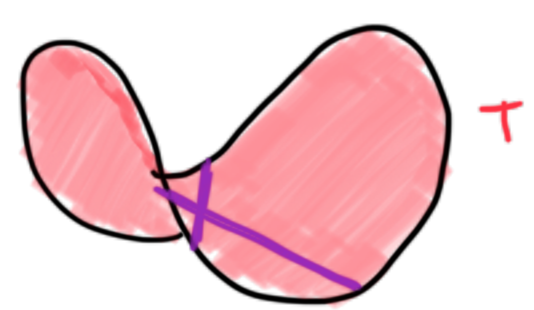

$$\phi_{\mathcal{A}} : \mathbb{C}^* \to (\mathbb{C}^*)^2 \rightsquigarrow A : \mathbb{Z}^2 \to \mathbb{Z}$$

$\operatorname{im} A = 2\mathbb{Z} \subset \mathbb{Z}$

<u>Example</u> $(n=2)$ $\quad \mathcal{A} = \{(1,0), (0,1), (1,1)\}$

$$\phi_{\mathcal{A}}(t_1, t_2) = (t_1, t_2, t_1 t_2)$$

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has rk 2.




In general an affine toric variety is a torus with something 'extra'.

Example: $(n=2)$ $\mathcal{A} = \{(1,1), (2,2), (3,3)\}$

$Y_{\mathcal{A}} \subset \mathbb{C}^3$ is the twisted cubic curve.

$\operatorname{rk} A = 1$.

!!! We might have $Y_{\mathcal{A}} = Y_{\mathcal{A}'}$ for $\mathcal{A} \neq \mathcal{A}'$, see Ex. 1.1.6. in CLS.

## 1.3 Toric Ideals

goal: describe the ideal $I(Y_{\mathcal{A}}) \subset \mathbb{C}[x_1, \dots, x_s]$.

example: find eqns vanishing on the image of

$$(t_1, t_2) \mapsto (\underset{x}{t_1}, \underset{y}{t_2}, \underset{z}{t_1 t_2})$$

$I(Y_{\mathcal{A}}) = \langle z - xy \rangle$. $\rightsquigarrow$ find relations between the exponents $m_i \in \mathcal{A}$.

The map $\Phi_{\mathcal{A}}$ induces a map of character lattices $\hat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M$ which sends $e_i$ to $m_i$.

The relations we are looking for are given by its kernel.

<u>Examples</u>: $T = (\mathbb{C}^*)^n$, $M = \mathbb{Z}^n$, $\hat{\phi}_{\mathcal{A}} = A$

$n=1$, $\mathcal{A} = \{2,3\}$, $\hat{\phi}_{\mathcal{A}} = [2,3] : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^1$

$$(3,-2) \in \ker \hat{\phi}_{\mathcal{A}} \rightsquigarrow x^3 - y^2$$

$n=1$, $\mathcal{A} = \{1,2,3\}$, $\hat{\phi}_{\mathcal{A}} = [1,2,3] : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^1$

$$(1,-2,1),\ (1,1,-1),\ (2,-1,0)$$

$n=2$, $\mathcal{A} = \{(1,0),(0,1),(1,1)\}$

$$\hat{\phi}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$$

$$(1,1,-1) \in \ker \hat{\phi}_{\mathcal{A}} \rightsquigarrow x^1 y^1 - z^1$$

Let $l \in \mathbb{Z}^s$ be an element of $\ker \hat{\phi}_{\mathcal{A}}$:

$$l = (l_1, \dots, l_s), \quad \sum l_i m_i = 0.$$

set $\ell_+ = \sum_{l_i>0} l_i e_i$ , $\ell_- = \sum_{l_i<0} -l_i e_i$.

It is easy to see that $x^{\ell_+} - x^{\ell_-}$ vanishes on im $\phi_{\mathcal{A}}$.

<u>Proposition</u> : The ideal of $Y_{\mathcal{A}} \subset \mathbb{C}^s$ is given by

$$I(Y_{\mathcal{A}}) = \langle x^{\ell_+} - x^{\ell_-} \mid \ell \in \ker \hat{\phi}_{\mathcal{A}} \rangle \subset \mathbb{C}[x_1,\dots,x_s].$$

<u>proof</u> : based on Groebner basis and really nice.

See [CLS, p15]. Try exercise 1.1.2.

TAKE AWAY : affine toric varieties are defined by prime ideals, generated by binomials. Such ideals are called <u>toric ideals</u>.

(RMK : each $Y_{\mathcal{A}}$ contains $(1,\dots,1) \in \mathbb{C}^s$).

## 1.4 Affine semigroups

An <u>affine semigroup</u> is a set $S$ with a binary operation '$+$' such that

- $+$ is associative

usual requirement for a semigroup.

- $S$ has an identity element $0 \in S$
- $+$ is commutative
- $S$ is finitely generated: there is a finite subset $\mathcal{A} \subset S$ such that
$$\mathbb{N}\mathcal{A} = \{ \sum_{m \in \mathcal{A}} a_m m \mid a_m \in \mathbb{N} \} = S.$$
- $S$ can be embedded in a lattice $M$.

Up to isomorphism, all affine semigroups are of the form $\mathbb{N}\mathcal{A}$ for some finite subset $\mathcal{A} \subset M$ of a lattice.

The <u>semigroup algebra</u> $\mathbb{C}[S]$ associated to an affine semigroup $S \subset M$ is the $\mathbb{C}$-algebra

$$\mathbb{C}[S] = \left\{ \sum_{m \in S}' c_m \chi^m \;\middle|\; c_m \in \mathbb{C},\ \underset{\text{nonzero}}{\text{finitely many}} \right\}$$

with multiplication induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

If $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$, then

$$\mathbb{C}[\mathbb{N}\mathcal{A}] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}].$$

<u>Example</u>: $\mathbb{C}[M] \simeq \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the coordinate ring of $T_N$.

<u>Prop</u>: Let $S \subset M$ be an affine semigroup. Then

(a) $\mathbb{C}[S]$ is an integral domain and a finitely generated $\mathbb{C}$-algebra.

(b) $\operatorname{Specm}(\mathbb{C}[S])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}S$.

<u>proof</u>: (a) $\mathbb{C}[S] \subset \mathbb{C}[M] \Rightarrow \mathbb{C}[S]$ I.D.

$$S = \mathbb{N}\mathcal{A} \Rightarrow \mathbb{C}[S] \text{ f.g.}$$

(b) Let $\phi^*_{\mathcal{A}} : \mathbb{C}[x_1, \dots, x_s] \to \mathbb{C}[M]$ be the pullback of $\phi_{\mathcal{A}} : T \to \mathbb{C}^s$.

There is a short exact sequence

$$0 \to I(Y_{\mathcal{A}}) \hookrightarrow \mathbb{C}[x_1, \dots, x_s] \xrightarrow{\phi^*_{\mathcal{A}}} \mathbb{C}[S] \to 0,$$

from which $\mathbb{C}[S] \simeq \mathbb{C}[x_1, \dots, x_s]/I(Y_{\mathcal{A}})$.

Since $\mathbb{Z}\mathcal{A} = \mathbb{Z}S$, the torus of $\mathrm{Spec}(\mathbb{C}[S])$ is $\mathbb{Z}S$. □

## EQUIVALENCE OF CONSTRUCTIONS.

<u>Theorem</u>. Let $V$ be an affine variety. TFAE:

(a) $V$ is an affine toric variety

(b) $V = Y_{\mathcal{A}}$ for a finite subset $\mathcal{A}$ of a lattice.

(c) $V$ is defined by a toric ideal.

(d) $V = \mathrm{Specm}(\mathbb{C}[S])$ for an affine semigroup $S$.

suggested exercises : 1.1.1 ((c)-(e) requires Gröbner bases)

1.1.2

1.1.4

1.1.6

1.1.10

1.1.11

1.1.12