Recall that $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$ gives a proj. toric variety $X_{\mathcal{A}} \subset \mathbb{P}^{s-1}$ given by $X_{\mathcal{A}} = \overline{(\pi \circ \phi_{\mathcal{A}})(T)}$.

2.3 Affine pieces of a projective toric variety

Let $U_i = \mathbb{P}^{s-1} \setminus V(x_i)$ for $i = 1, \dots, s$

Recall $U_i \simeq \mathbb{C}^{s-1}$ via

$$(x_1 : \dots : x_s) \longmapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_s}{x_i} \right)$$

and $T_{\mathbb{P}^{s-1}} \subset \bigcap_{i=1}^{s} U_i$. $\longleftarrow$

$$\text{Since } T_{X_{\mathcal{A}}} = X_{\mathcal{A}} \cap T_{\mathbb{P}^{s-1}} \subset \underline{X_{\mathcal{A}} \cap U_i}$$

$\underline{X_{\mathcal{A}} \cap U_i}$ is the Zariski closure of $T_{X_{\mathcal{A}}}$ in $U_i \simeq \mathbb{C}^{s-1}$

$$T \xrightarrow{\pi \circ \phi_{\mathcal{A}}} U_i \xrightarrow{\sim} \mathbb{C}^{s-1}$$

$$t \longmapsto \left(\chi^{m_1 - m_i}(t), \dots, \chi^{m_{i-1} - m_i}(t), \chi^{m_{i+1} - m_i}(t), \dots, \chi^{m_s - m_i}(t) \right)$$

Proposition $X_{\mathcal{A}} \cap U_i$ is the affine toric variety $Y_{\mathcal{A}_i}$ where $\mathcal{A}_i = \mathcal{A} - m_i = \{m_1 - m_i, \dots, m_s - m_i\}$.

That is $X_{\mathcal{A}} \cap U_i \simeq \operatorname{Specm}(\mathbb{C}[S_i])$ where

$$S_i = \mathbb{N}\mathcal{A}_i$$

Rmk The char. lattice of the torus of $X_{\mathcal{A}} \cap U_i \simeq Y_{\mathcal{A}_i}$ is $\mathbb{Z}\mathcal{A}_i$

and $\mathbb{Z}\cdot\mathcal{A}_i = \mathbb{Z}\cdot(\mathcal{A} - m_i) = \mathbb{Z}'\mathcal{A}$. $\forall i$.

We describe how the affine pieces $X_{\mathcal{A}} \cap U_i$ patch together, i.e. describe

$$X_{\mathcal{A}} \cap U_i \supset \underline{X_{\mathcal{A}} \cap U_i \cap U_j} \subset X_{\mathcal{A}} \cap U_j$$

$$X_{\mathcal{A}} \cap U_i \cap U_j = \left\{ \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_s}{x_i} \right) \in \underline{X_{\mathcal{A}} \cap U_i} \text{ s.t. } \underline{\frac{x_j}{x_i} \neq 0} \right\}$$

$$= \operatorname{Specm} \mathbb{C}[S_i]_{\chi^{m_j - m_i}}$$

$X_{\mathcal{A}} \cap U_i \cap U_j \quad \subset \quad X_{\mathcal{A}} \cap U_i$ is given by

$$\mathbb{C}[S_i]_{\chi^{m_j - m_i}} \supset \mathbb{C}[S_i]$$

$$X_{\mathcal{A}} \cap U_i \supset X_{\mathcal{A}} \cap U_i \cap U_j \subset X_{\mathcal{A}} \cap U_j$$

$$\mathbb{C}[S_i] \subset \mathbb{C}[S_i]_{\chi^{m_j - m_i}} = \mathbb{C}[S_j]_{\chi^{m_i - m_j}} \supset \mathbb{C}[S_j].$$

More connections with $P = \mathrm{Conv}(\mathcal{A})$:

- $\dim P$ is the dimension of the smallest affine subspace containing P, which is also $\dim X_{\mathcal{A}}$.
- The vertices of P give a very efficient affine open cover of $X_{\mathcal{A}}$.

Proposition given $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$, let $P = \mathrm{Conv}(\mathcal{A}) \subset M_{\mathbb{R}}$. Then

$$X_{\mathcal{A}} = \bigcup_{i=1}^{s} (X_{\mathcal{A}} \cap U_i) = \bigcup_{\substack{m_i \text{ is a} \\ \text{vtx of } P}} (X_{\mathcal{A}} \cap U_i).$$

proof: Let $\mathcal{V} = \{i \mid m_i \text{ is a vertex of } P\}$.

Strategy: $X_{\mathcal{A}} \cap U_j \subset X_{\mathcal{A}} \cap U_i$ for some $i \in \mathcal{V}$.

$$P \cap M_{\mathbb{Q}} = \{ \sum_{i \in \mathcal{V}}' r_i m_i \mid r_i \in \mathbb{Q}_{\geq 0},\ \sum' r_i = 1 \}.$$

$$m_j = \sum_{i \in \mathcal{V}}' r_i m_i \implies \underline{k}\, m_j = \sum_{i \in \mathcal{V}}' k_i m_i, \quad \sum_{\substack{k_i \geq 0 \\ k > 0}} k_i = k$$

$$\implies \sum_{i \in \mathcal{V}}' k_i (m_j - m_i) = 0$$

choose i such that $k_i > 0$, then $m_j - m_i \in S_j = \mathbb{N}\mathcal{A}_j$

because $$k_i(m_j - m_i) = \sum_{i' \in \mathcal{V} \setminus \{i\}}' k_{i'} (m_{i'} - m_j).$$

$$m_j - m_i = \sum_{i' \in \mathcal{V} \setminus \{i\}}' k_{i'} (m_{i'} - m_j) + k_i (m_i - m_j)$$
$$\in \mathbb{N}\, S_j$$

$$m_j - m_i \in S_j \quad , \quad m_i - m_j \in S_j$$

$$\Rightarrow \chi^{m_i - m_j} \text{ is invertible in } \mathbb{C}[S_j]$$

$$\Rightarrow \mathbb{C}[S_j]_{\chi^{m_i - m_j}} = \mathbb{C}[S_j]$$

$$\begin{aligned} \mathrm{Specm}\left(\mathbb{C}[S_j]_{\chi^{m_i-m_j}}\right) &= X_{\mathcal{A}} \cap U_i \cap U_j \\ &= \mathrm{Specm}\left(\mathbb{C}[S_j]\right) \\ &= X_{\mathcal{A}} \cap U_j . \end{aligned}$$

$$X_{\mathcal{A}} \cap U_j \subset U_i . \qquad \square$$

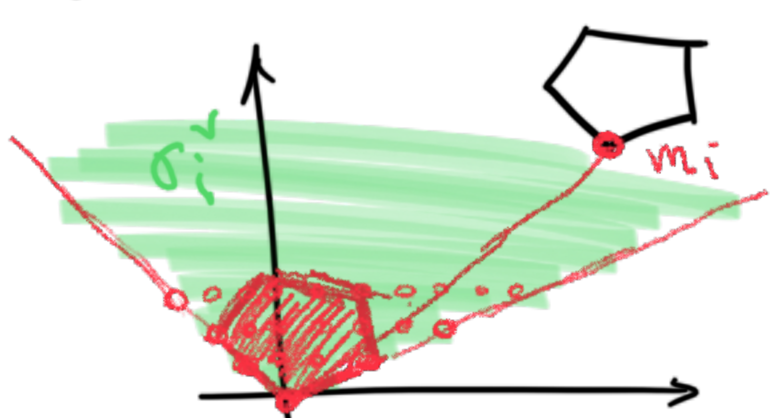
2.4 Projective toric varieties from very ample polytopes

A lattice polytope $P \subset M_{\mathbb{R}}$ is <u>very ample</u> if for every vertex $m \in P$, the semigroup $\mathbb{N}(P \cap M - m)$ is saturated in the lattice M.

Assume $\dim P = n$ and P is very ample. We associate to P the projective toric variety $X_{P\cap M}$.

Theorem For every vertex $m_i \in P\cap M$ ($=\mathcal{A}$), $X_{P\cap M} \cap U_i$ is the normal affine toric variety $\mathcal{U}_{\sigma_i} = \operatorname{Specm}(\mathbb{C}[\sigma_i^\vee \cap M])$, where $\sigma_i^\vee = \operatorname{Cone}(P\cap M - m_i)$. Here $\dim \sigma_i = n$ and σ_i is strongly convex. The torus of $X_{P\cap M}$ has character lattice M (so it's torus is T).

proof $\operatorname{Cone}(P\cap M - m_i)$ is strongly convex of dimension $\dim P$. Hence both σ_i and $\sigma_i^\vee$ are strongly convex RCPCs of dim n.



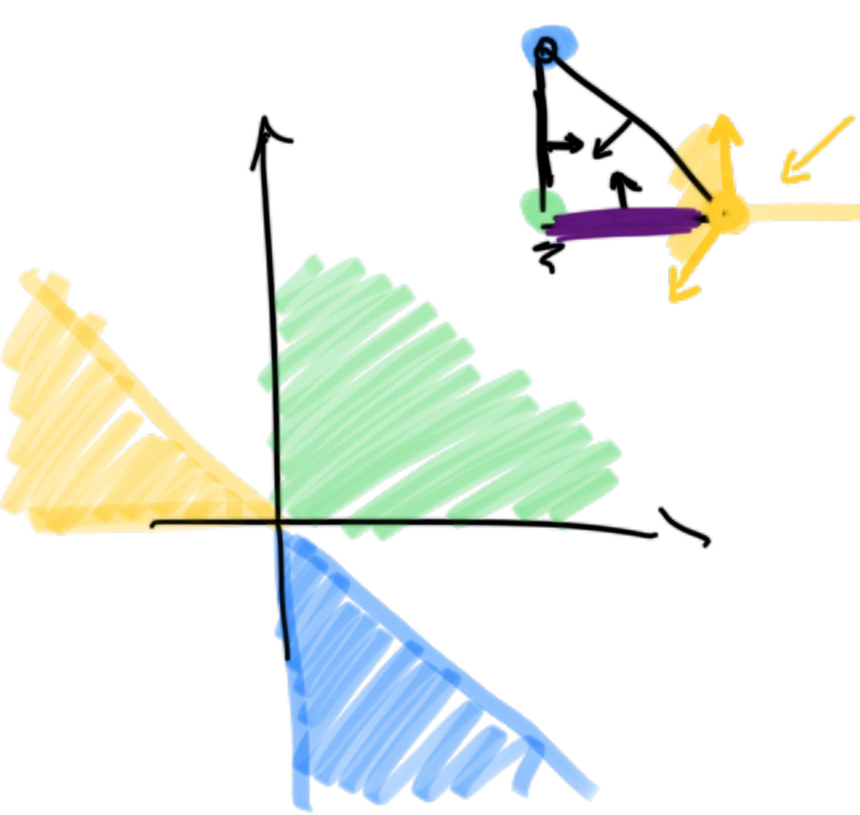
by very ample $\mathbb{N}\cdot(P\cap M - m_i) = \sigma_i^\vee \cap M$.

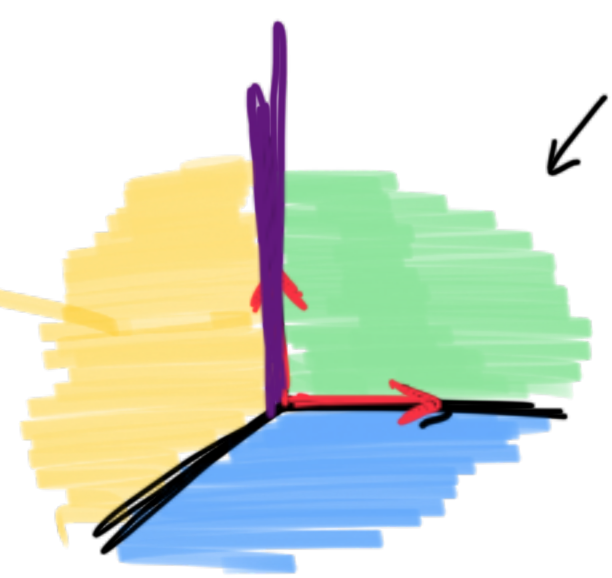
T is the torus of U_{σ_i} $\Rightarrow$ T is the torus of $X_{P\cap M}$. $\square$.

2.5 The normal fan

A *fan* is a finite collection Σ of strongly cvx RCPCs such that

- $\sigma \in \Sigma$, $\tau \leq \sigma \Rightarrow \tau \in \Sigma$
- $\tau = \sigma \cap \sigma'$, $\sigma, \sigma' \in \Sigma \rightarrow \tau \leq \sigma$, $\tau \leq \sigma'$.





If $P \subset M_{\mathbb{R}}$ is given by

$$P = \{ m \in M_{\mathbb{R}} \mid \langle u_F, m \rangle + a_F \geq 0 \text{ for } F \text{ a facet of } P \}$$

The <u>normal fan</u> of P is

$$\Sigma_P = \{ \sigma_Q \mid \underline{Q \preceq P} \}, \quad \sigma_Q = \operatorname{Cone}(u_F,\ Q \subset F)$$

<u>Properties</u> : $P \subset M_{\mathbb{R}}$ full-dimensional, $Q \preceq P$

- $\dim Q + \dim \sigma_Q = n$
- $N_{\mathbb{R}} = \bigcup_{v \text{ vertex of } P} \sigma_v = \bigcup_{Q \preceq P} \sigma_Q$ (<u>complete</u> fan)
- for any $\underline{m \in M}$, $k \in \mathbb{N} \setminus \{0\}$

$$\Sigma_{kP+m} = \Sigma_P .$$

Proposition Let $P \subset M_{\mathbb{R}}$ be a full-dim. very ample polytope. Let $v \neq w$ be vertices of P and let Q be the smallest face of P containing v, w.

Then $\underline{X_{P\cap M} \cap U_v \cap U_w = \mathcal{U}_{\sigma_Q}}$

$$= \mathrm{Specm}\left(\mathbb{C}[\sigma_Q^\vee \cap M]\right)$$

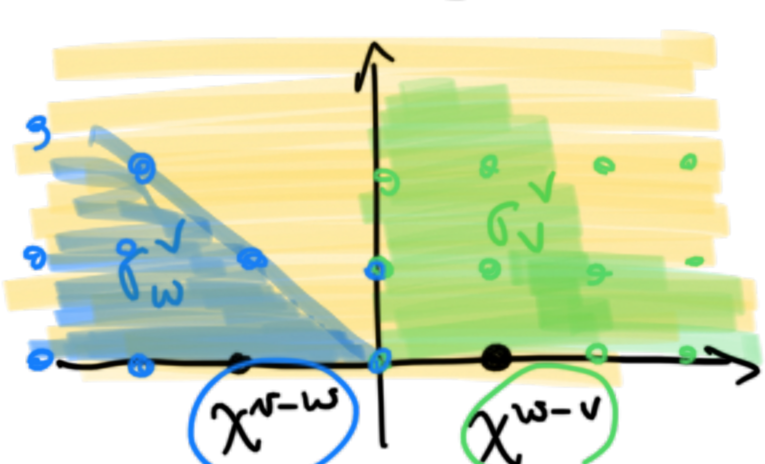
and

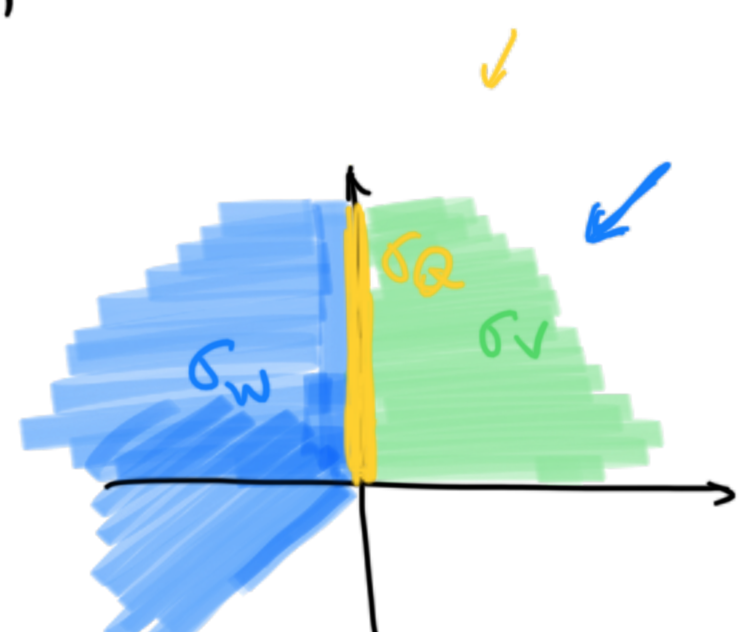
$$X_{P\cap M} \cap U_v \supset X_{P\cap M} \cap U_v \cap U_w \subset X_{P\cap M} \cap U_w$$

can be written

$$\longrightarrow \mathcal{U}_{\sigma_v} \supseteq (\mathcal{U}_{\sigma_v})_{\chi^{w-v}} = (\mathcal{U}_{\sigma_w})_{\chi^{v-w}} \subseteq \mathcal{U}_{\sigma_w}.$$

Example





$$\mathbb{C}[\sigma_v^\vee \cap M] = \mathbb{C}[x, y]$$

$$\mathbb{C}[\sigma_w^\vee \cap M] = \mathbb{C}[x^{-1}, x^{-1}y]$$

$$\mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} = \mathbb{C}[x, y]_x$$

$$\pi \circ \phi_{P\cap M} : (t_1, t_2) \mapsto (1 : t_1 : t_2)$$

$$\overline{(\pi \circ \phi_{P\cap M})((\mathbb{C}^*)^2)} = \mathbb{P}^2$$

<u>Conclusion</u>: the normal fan Σ_P completely encodes how $X_{P\cap M}$ is covered by affine toric varieties and how these intersect.

2.6 The toric variety of a polytope ...

... is the projective toric variety encoded by its normal fan.

Given $P \subset M_{\mathbb{R}}$, let $X_P = X_{(kP) \cap M}$ such that kP is very ample.

Rmks : • a very ample polytope $P \subset M_{\mathbb{R}}$ gives a normal projective toric variety $X_{P \cap M}$, meaning that $X_{P\cap M}$ is covered by normal affine varieties.

• If $P \subset M_{\mathbb{R}}$ is normal, $X_{P\cap M}$ is projectively normal, meaning that the affine cone $Y_{\widehat{P \cap M}}$ is normal.

• very ample $\underset{\not\Longrightarrow}{\Longleftarrow}$ normal (Ex 2.2.20)

- $P \subset M_{\mathbb{R}}$, kP is normal (and, in particular, very ample) for all

$$k \geq n-1.$$

- For any polytope P, X_P is normal.
- if kP and $k'P$ are normal, then

$$X_{(kP)\cap M} \simeq X_{(k'P)\cap M} \simeq X_P.$$

- X_P is smooth $\iff$ Σ_P is smooth (all cones are smooth)

<u>Exercise</u> glue $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ using normal fans of  and 

Describe "Veronese embeddings" of $\mathbb{P}^2$ and Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ in this language.