EECS 16B Notes

Spring 2020

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Lecture 1

16A review and prerequisites

1.1 The language of circuits

Electrical circuits are models, specifically, abstractions of underlying physics-based descriptions of realities that govern behavior of an electrical system under analysis. Mathematically, circuits are collections of *nodes* joined by *branch elements*. Between every pair of adjacent nodes there is a *voltage difference*, measured in volts, as well as a *current*, measured in amps. You should be able to explain, both in approximate physical terms, and, if possible, by a mechanical analog, what voltage and current are. Given a circuit drawing, you should be able to write a comprehensive set of voltage-current constraints that fully predicts what is happening in the circuit. For a well-posed circuit model with N nodes, one preferred method is Nodal Analysis, which involves writing $N-1$ linearly independent KCL node equations, and incorporating KVL and element branch consraints while writing the node equations.

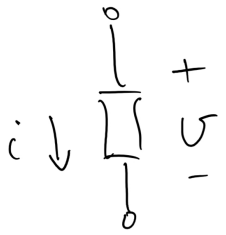

Figure 1.1: Current and voltage annotated on a passive element.

Understand how current and voltage are annotated on a circuit. Our terms are "voltage *across* branch element X" and "current *through* branch element X." The phrases "voltage *through*. . . " or "current *across*. . . " do not make sense. Understand, as shown in Fig. 1.1, that the reference directions for voltage

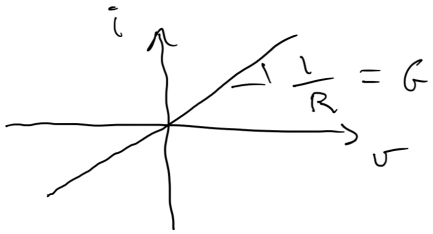

Figure 1.2: I-V characteristic of a resistor.

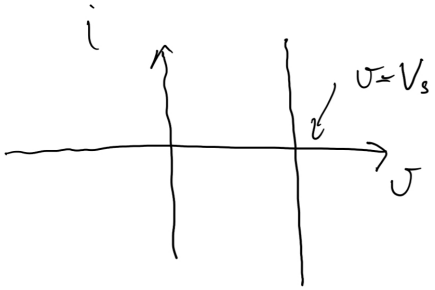

Figure 1.3: I-V characteristic of a voltage source.

and current are such that power absorbed by a circuit element is given by the formula vi.

1.2 Current-voltage characteristic

Resistor

As shown in Fig. 1.2, resistors enforce a proportionality relationship between current and voltage:

$$V = RI \tag{1.1}$$

$$I = GV \tag{1.2}$$

The ratio V/I is called *resistance*. The ratio I/V is called *conductance*.

Voltage source

As shown in Fig. 1.3, a voltage source will provide any current (or none at all) to maintain its target voltage.

Current source

As shown in Fig. 1.4, a current source will provide any voltage (or none at all) to maintain its target current.

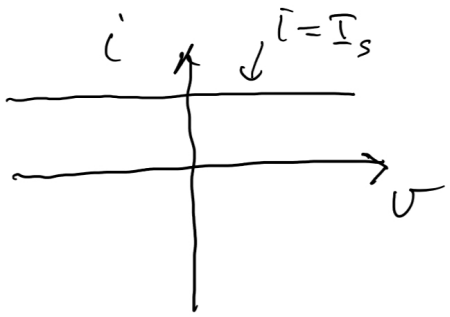

Figure 1.4: I-V characteristic of a current source.

Circuit-solving techniques

Be familiar with the following methods for solving circuits:

- Series elements, e.g. two resistors in series
- Parallel elements, e.g. two resistors in parallel.
- Voltage and current dividers
- Kirchoff's voltage and current laws
- Norton and Thévenin equivalent circuits
- Nodal analysis
- Power calculations

1.3 Linear algebra

Know what a vector is. Know what eigenvalues and eigenvectors are, and know how to solve for the eigenvalues and eigenvectors of a matrix, by solving for the null space of $A - \lambda I$, where λ is an indeterminate. Know why this technique works.

Lecture 2

Transistor Circuits

2.1 MOSFET behavior at a low level

Transistors are nonlinear circuit elements that are integral to building digital electronics. We'll focus on a class of transistor called MOSFET (*metal-oxide semiconductor field-effect transistor*), of which there are two types, NMOS and PMOS. For the most part, we will view MOSFETs from a digital perspective as voltage-controlled switches (more on that later), but we'll first have a look at the analog world under the hood.

The physical makeup of a MOSFET is shown in Figure 2.1. It is a device built on a silicon substrate with three terminals: *source* (S), *drain* (D), and *gate* (G). What makes a transistor a transistor is 2) mediated by gate voltage. (No current enters the gate of a MOSFET: $I_G = 0$.) 1) a current-voltage characteristic between drain and source, These quantities are labeled on Figure 2.2. Notice that voltages are understood with reference to their difference from V_S, so:

- D-S current-voltage characteristic is between I_D and $v_{DS} = v_D - v_S$,
- parameterized by $v_{GS} = v_G - v_S$.

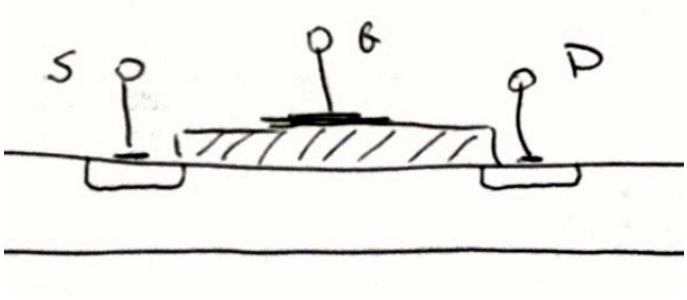

Figure 2.1: Physical construction of a simple MOSFET.

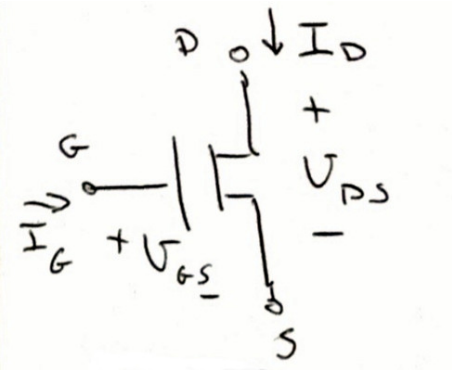

Figure 2.2: Currents and voltages labeled on an NMOS transistor.

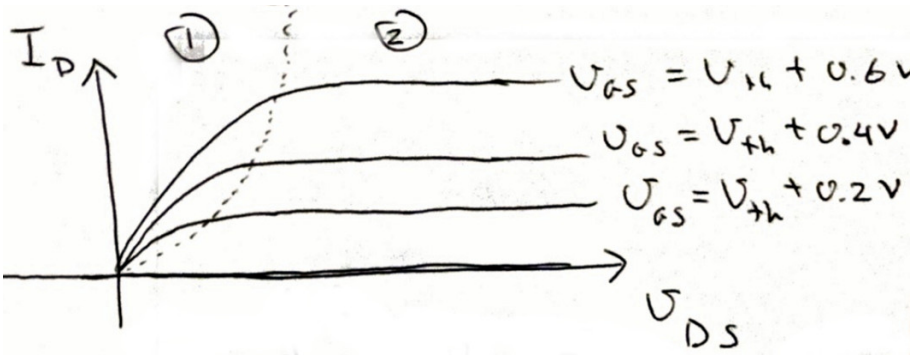

Figure 2.3: I-V characteristic of an NMOS transistor at different values of v_{GS}.

The role of v_{GS} in NMOS

Figure 2.3 depicts several current-voltage characteristics of an nmos, parameterized by v_{GS}. There's a lot happening on this graph in both the vertical and horizontal directions. Here's a self-guided tour:

- Notice the horizontal line lying along the positive v_{DS}-axis. This is the plot of the I-V characteristic when $v_{GS} < v_{t,n}$, where $v_{t,n} > 0$ is the threshold voltage for an NMOS transistor. The current-voltage characteristic is $I = 0$, the transistor is behaving as a current source corresponding to zero current—in other words, it's an open circuit. The transistor is "off." [1]
- Notice that three I-V curves, parameterized by how much v_{GS} exceeds $v_{t,n}$, lie above the line $I = 0$. Each of them is intersected by what looks like the eastern half of a dotted upward-facing parabola rising from the origin. This parabola divides the quadrant into two regions, one left and

[1] English semantics for "on" and "off" in circuits can be counterintuitive. An open circuit/switch is off, and vice versa.

region #	on/off?	v_{GS} predicate	v_{DS} predicate	name
0	off	$v_{GS} < v_{t,n}$	any	
1	on	$v_{GS} > v_{t,n}$	low	"linear region"
2	on	$v_{GS} > v_{t,n}$	high	"saturation"

Figure 2.4: Regions of an NMOS I-V characteristic.

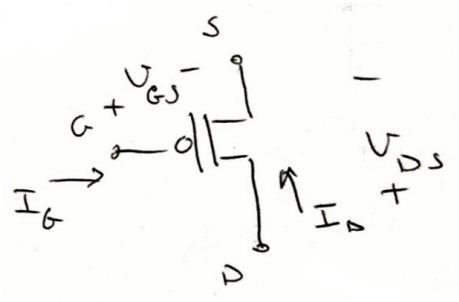

Figure 2.5: Currents and voltages labeled on a PMOS transistor.

one right. The left region is called region 1; the right region is called region 2.

- Focus on region 1, which is called the Linear Region. Notice that in region 1 near the origin, I_D and v_{DS} are proportional for every value of v_{GS}. The slope $G = I_D/v_{DS}$ increases for higher values of v_{GS}. This means that the D-S resistance $R = G^{-1}$ transitions from ∞ to a finite (perhaps small) value as v_{GS} increases past $v_{t,n}$. A resistor that can alternate between finite and infinite resistance is called a switch: in the Linear Region the transistor is a voltage-controlled switch.
- Focus on region 2, which is called Saturation, Here the I_D increases only very weakly as v_{DS} increases.For a given V_{DS}, I_D increases with increasing v_{GS}: the transistor behaves approximately as a voltage-controlled current source!

These characteristics are summarized in Figure 2.4. Regions 0 and 1 can be used to implement a switch. Region 2 is used for analog electronics—dependent sources, amplifiers, etc.

PMOS transistors: opposite of NMOS

Another kind of MOSFET is the PMOS. They have a similar construction as NMOS transistors, but their behavior is opposite, and for the "on" condition of $V_{GS} < V_{t,p}$, $V_{t,p} < 0$. Figure 2.5 and Figure 2.6 are the counterparts of Figure 2.2 and Figure 2.3, respectively.

For most of this class, we'll use more idealized models of these transistors in digital logic settings. In the voltage-controlled switch perspective, NMOS

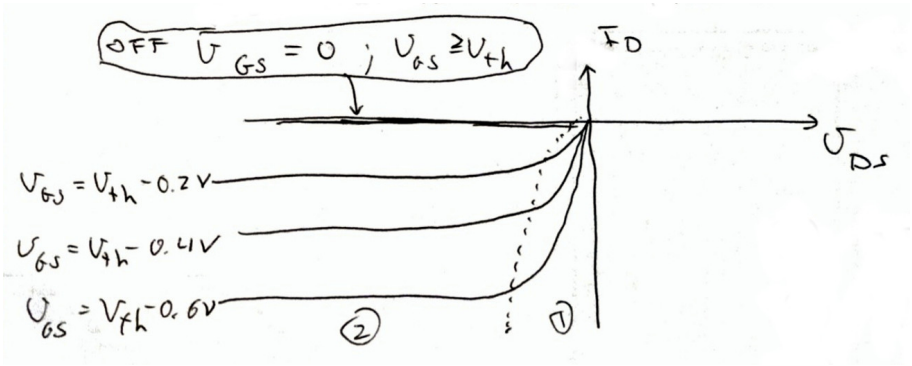

Figure 2.6: I-V characteristic of a PMOS transistor at different values of v_{GS}.

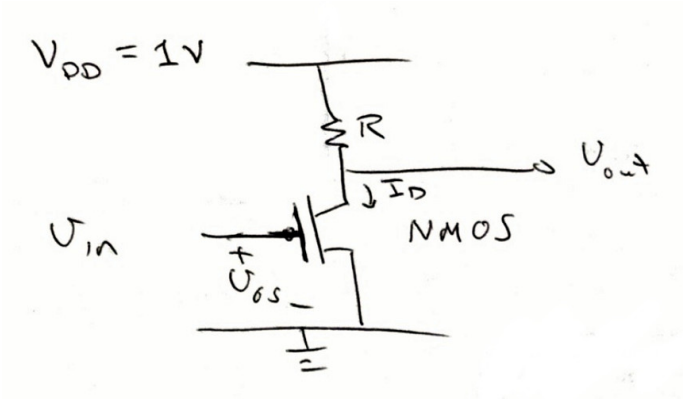

Figure 2.7: A inverter built using an NMOS transistor.

transistors open at lower voltages and close at higher voltages, and PMOS transistors close at lower voltages and open at high ones.

2.2 An NMOS inverter

One building block we need to understand digital logic is the *inverter*, which is a circuit that outputs a high voltage when its input is a low voltage, and vice versa. The high voltage represents a digital value of 1 (true), and the low voltage represents a digital value of 0 (false).

It's possible to build an inverter using an NMOS transistor, as shown in Figure 2.7. The high voltage is called V_{DD}, which stands for the voltage supplied by the high power rail, and in this example has a value of 1 volt.[2] In this example, our reference voltage will be ground—0 volts.

[2]For obscure historical reasons.

v_{in}	v_{out}
0	V_{DD}
V_{DD}	0

Figure 2.8: Truth table of the NMOS inverter.

Analysis

- (Case $v_{\text{in}} = 0$) The transistor, as a switch, is off. As a result, the terminal v_{out} is connected directly to V_{DD} by a resistor. Because no current flows into the voltage terminal, by Ohm's law there can be no voltage drop across the resistor. Therefore $v_{\text{out}} = V_{\text{DD}}$.
- (Case $v_{\text{in}} = V_{\text{DD}}$) The transistor, as a switch, is on. The terminal v_{out} has a short to ground, so $v_{\text{out}} = 0$.

Figure 2.8 shows the truth table of this circuit and verifies that this circuit is indeed an inverter.

Power consumption

When $v_{\text{in}} = 0$, the circuit consumes no power, as we have established that there is no current through the resistor between V_{DD} and v_{out}. When $v_{\text{in}} = V_{\text{DD}}$, there is a path from V_{DD} through the resistor, then the transistor, to ground. The circuit consumes power $VI = V_{\text{DD}}^2/R$. While this might not necessarily be a lot, in computing applications with countless transistors, it adds up, and moving heat away from a dense circuit poses engineering challenges. Dense digital circuits were made possible by the discovery of the CMOS inverter architecture, which avoids a path from V_{DD} to ground.

2.3 A CMOS inverter

Figure 2.9 shows an inverter circuit that exemplifies the CMOS design strategy of using PMOS and NMOS transistors together.

Analysis

- (Case $v_{\text{in}} = V_{\text{DD}}$)
  - The PMOS having as its source V_{DD} and v_{out} as its drain $V_{\text{GS,1}} = 0$, which is higher than $V_{\text{t,p}}$. Therefore there is no path from V_{DD} to v_{out}.
  - The NMOS having v_{out} as its drain and ground as its source has $V_{\text{GS,2}} = V_{\text{DD}}$, which is higher than $V_{\text{t,n}}$. Therefore, due to the terminal's short to ground, $v_{\text{out}} = 0$.

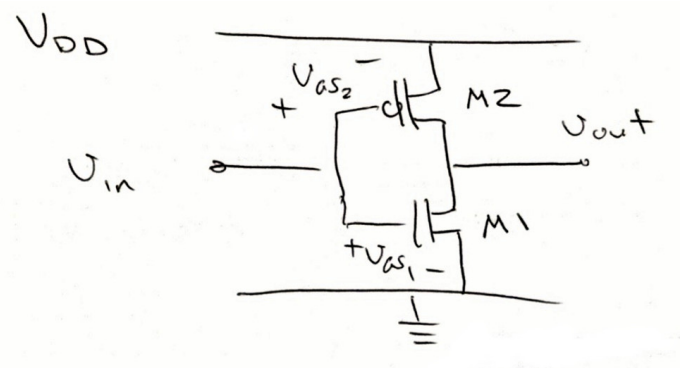

Figure 2.9: A inverter built using the CMOS design..

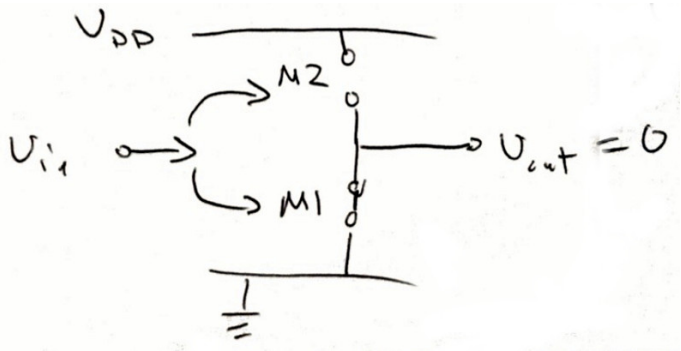

Figure 2.10: Equivalent circuit of Figure 2.9 when $v_{\text{in}} = V_{\text{DD}}$.

The equivalent circuit once the switch model has been applied is shown in Figure 2.10.

- (Case $v_{\text{in}} = 0$)
  - The PMOS, having $V_{\text{GS},1} = -V_{\text{DD}} < V_{\text{t,p}}$, turns on.
  - The NMOS, having $V_{\text{GS},2} = 0 < V_{\text{t,n}}$, turns off.

  Therefore $V_{\text{out}} = V_{\text{DD}}$.

Power consumption

All currents are zero in this model, so no power is consumed.

2.4 A CMOS inverter chain with capacitance

Contrary to our last conclusion, inverters in real electronics certainly do consume some power. We'll pretend digital circuits are chains of inverters (Figure 2.11)—

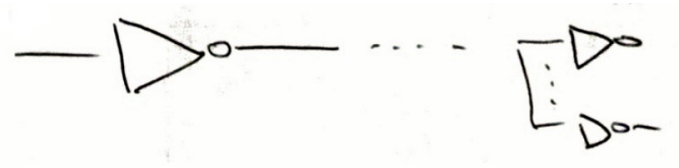

Figure 2.11: A chain of inverters, which is kind of similar to a computer.

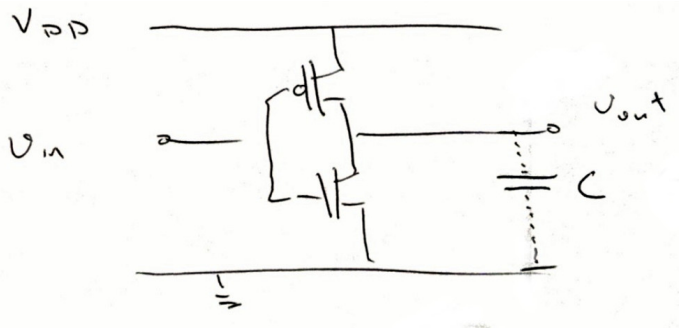

Figure 2.12: An inverter taken from a chain with a capacitor modeling the next stage.

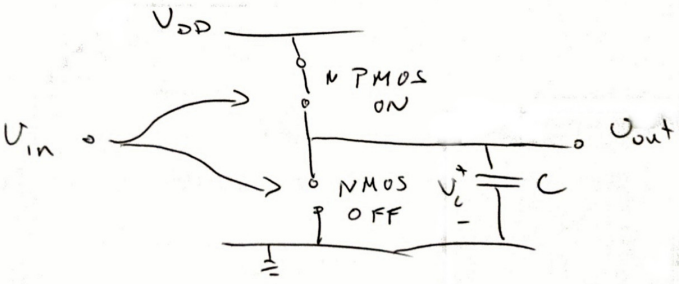

Figure 2.13: An inverter outputting V_{DD} with load capacitor.

although this model won't teach you how to build a computer, it is close enough to real CMOS networks to illustrate when and where power is expended.

We will concentrate our analysis on just one stage of the CMOS inverter chain. A single inverter is shown in Figure 2.12, with a capacitor between v_{out} and ground to model the next stage's load. Figure 2.13 shows the equivalent circuit when the output of this inverter settles at V_{DD}.

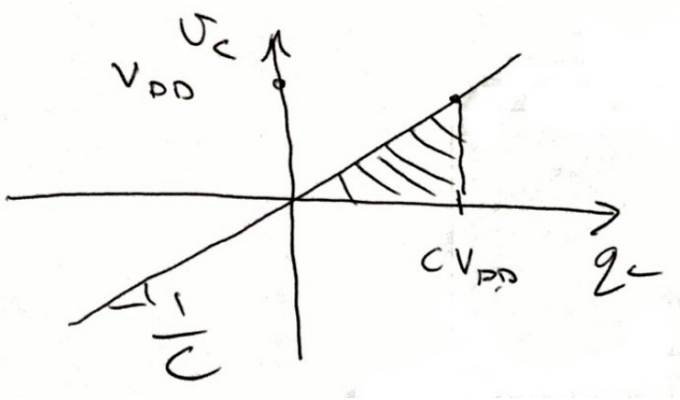

Figure 2.14: Energy stored in a capacitor can be computed by an integral under the $V = Q/C$ curve.

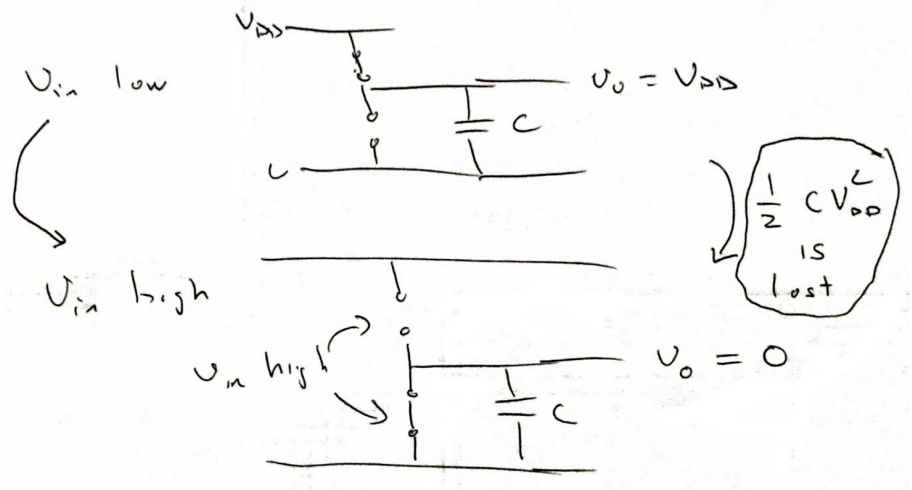

Figure 2.15: CMOS load capacitor forced from voltage V_{DD} to 0.

Potential energy in a capacitor

The energy stored in the capacitor when it has voltage V_{DD} is given by the formula

$$E_{cap} = \frac{1}{2} C V_{DD}^2, \tag{2.1}$$

which can be derived by using the facts that 1) that voltage is energy per unit charge and 2) a capacitor obeys $Q = CV$, and integrating through the total charge stored in the capacitor: $\int_0^{CV_{DD}} v_C \, dq$ (Figure 2.14).

When the inverter's input changes from low to high, the output must change

from V_{DD} to 0 (Figure 2.15). That means that the load capacitor must discharge fully, burning $\frac{1}{2}CV_{DD}^2$ of potential energy as heat.

Total energy supplied

Even though the capacitor only stores and discharges $\frac{1}{2}CV_{DD}^2$, an up-down cycle costs CV_{DD}^2. This is because the voltage source must offer $Q = CV_{DD}$ of charge at V_{DD} energy per unit charge. Where does this go? Let's follow the energy as the output changes from 0 to 1 and back to 0.

1. $(q_C = 0, v_C = 0)$
2. Voltage source loads CV_{DD} of charge at V_{DD} energy per unit charge, at a total expense of CV_{DD}^2. Half of its energy output is burned by "parasitic" resistance en route to the capacitor, and the other half is stored in the capacitor.
3. $(q_C = CV_{DD}, v_C = V_{DD})$
4. Transistors toggle, and the capacitor drains, generating $\frac{1}{2}CV_{DD}^2$ of heat on the pull-down circuit.
5. $(q_C = 0, v_C = 0)$

Where does the energy in a device go?

With reference to our chain-of-inverters model, power consumption in digital devices is mainly explained by three phenomena:

- If the inverter flips every cycle at a clock speed of f_s, the circuit will burn $f_s CV_{DD}^2$ charging its capacitors.
- Leakage: a transistor that's "off" isn't 100% off, and a small amount of current flows and burns some energy.
- Short-circuit current (smaller): when the input is flipping between 0 and 1, there's a very short instant during which both transistors may be *on*, and some current flows through the momentary V_{DD}-ground short.

Lecture 3

Transient Analysis

(For this lecture, a MOSFET transistor is considered to transition between "on" and "off" at $v_{GS} = \frac{1}{2}V_{DD}$.)

We'll enrich our analog model of MOSFETs as voltage-controlled switches by acknowledging capacitance between the MOSFET's gate and source. Figure 3.1 and Figure 3.2 depict NMOS and PMOS transistors in this model.

Figure 3.3 summarizes the three levels of abstraction with which we are able to reason about CMOS inverters. On the very left is a digital symbol for an inverter that hides how the inverter works. In the center is the construction of an inverter using complementary MOSFETs. On the right is a fairly faithful analog representation of an inverter that will allow us to interrogate the assumptions that, thus far, have enabled us to treat the analog circuit as a digital one.

3.1 RC transient in an inverter chain

Let's return to the case study of a chain of inverters, this time focusing on just two consecutive inverters. In Figure 3.4 three wires are labeled as follows:

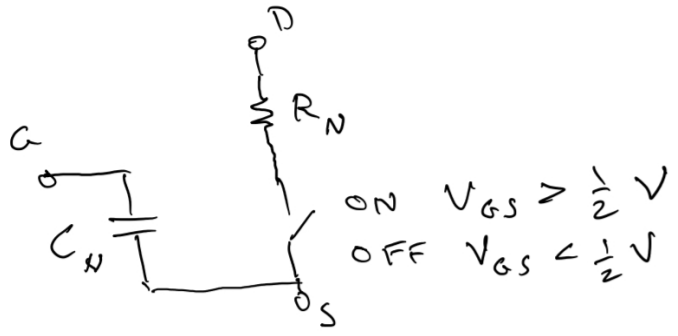

Figure 3.1: Model of NMOS transistor with G-S capacitance.

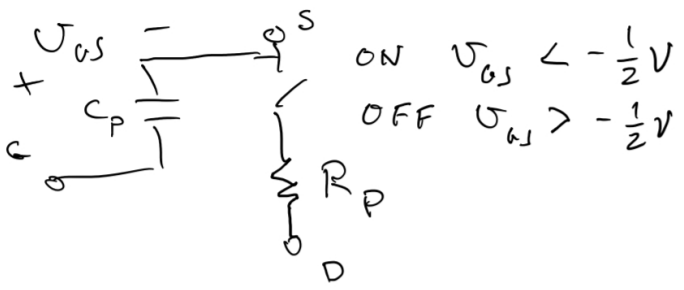

Figure 3.2: Model of pmos transistor with G-S capacitance.

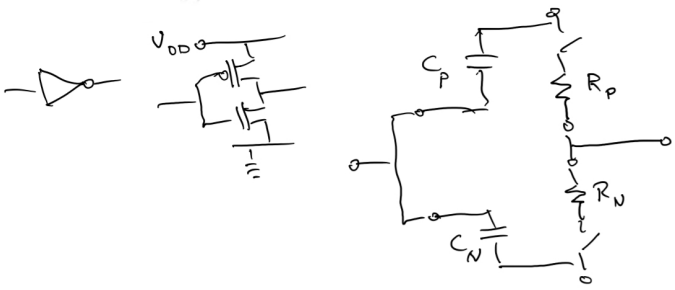

Figure 3.3: A cmos inverter at three levels of abstraction.

Figure 3.4: Two consecutive cmos inverters, part of a longer chain.

- v_{in} is the input to the first inverter,
- v_{o_1} is the output of the first inverter (and the input to the second), and
- v_{o_2} is the output of the second.

The digital logic interpretation is that v_{o_2} is the double negation of v_{in}, that is, $v_{\text{o}_2} = v_{\text{in}}$.

We will study what happens when v_{in} is driven by the input depicted in Figure 3.5. It will begin having remained at 0 for a long time, change to v_{DD} at time t_1, then return to 0 at time $t_2 > t_1$. Figure 3.6 shows the actions of the switches of the first inverter's transistors at times t_1 and t_2. For the rest of this section, we'll just concentrate on what happens to v_{o_1}.

Figure 3.5: Input signal to the first inverter of Figure 3.4

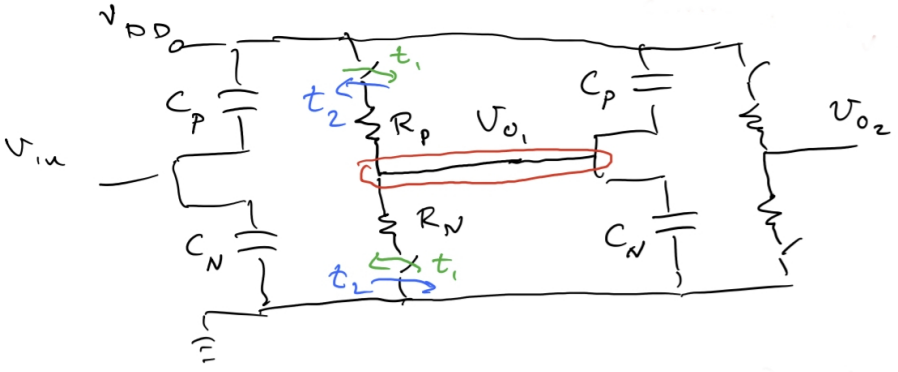

Figure 3.6: Analog redrawing of Figure 3.4, showing switch actions of the first inverter, as well as a distinguished node.

Before t_1

As $v_{\text{in}} = 0$ well before t_1, we can assume that the circuit has settled, and the output of the first inverter is v_{DD}.

After t_1, before t_2

At t_1, the pull-up switch opens, and the pull-down switch closes. KCL applied to the distinguished (red) middle node of Figure 3.6 requires the outgoing currents to sum to zero. Using Ohm's Law once and the capacitor current-voltage relationship twice, we have the following equation:

$$\frac{v_{\text{o}_1}}{R_N} + C_N \frac{\text{d}}{\text{d}t} v_{\text{o}_1} + C_P \frac{\text{d}}{\text{d}t} \left(v_{\text{o}_1} - v_{\text{DD}}\right) = 0 \tag{3.1}$$

$$\frac{\text{d}}{\text{d}t} v_{\text{o}_1} + \frac{1}{R_N \left(C_N + C_P\right)} v_{\text{o}_1} = 0 \tag{3.2}$$

This is a differential equation that we will analyze with initial condition $v_{\text{o}_1}(t_1) = V_{\text{DD}}$. For equations of this sort we will identify a characteristic quantity τ as follows:

$$\tau = R_N \left(C_N + C_P\right) \tag{3.3}$$

The International System of Units means that τ is measured in Ohm-Farads, or seconds. For this reason, τ is called the *time constant* of the system. A time constant on the order of tens of picoseconds is considered state-of-the-art for modern devices, arising from resistances on the order of kiloOhms and capacitances on the order of femtofarads. Rewriting using τ,

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{o}_1} = -\frac{1}{\tau} v_{\mathrm{o}_1} \tag{3.4}$$

We will refer to the constant of proportionality between $\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{o}_1}$ and v_{o_1} as λ.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{o}_1} = \lambda v_{\mathrm{o}_1} \tag{3.5}$$

There are many heuristic techniques to propose a solution to this differential equation. One of them is called Separation of Variables, which involves equations such as $\int \frac{\mathrm{d}v_{\mathrm{o}_1}}{v_{\mathrm{o}_1}} = \int \lambda \, \mathrm{d}t$. The resulting solution form, where A is a constant that remains to be determined, is all that you will need to know about this variety of differential equation:

$$v_{\mathrm{o}_1}(t) = Ae^{\lambda t} \tag{3.6}$$

(As an aside, you can verify that $v_{\mathrm{o}_1}(t) = Ae^{\lambda t}$ is a solution— differentiating both sides with respect to t results in $\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{o}_1}(t) = A\lambda e^{\lambda t} = \lambda(Ae^{\lambda t})$.) Our next goal is to determine A. We can do so by choosing A to meet the initial condition $v_{\mathrm{o}_1}(t_1) = V_{\mathrm{DD}}$. Substituting $v_{\mathrm{o}_1}(t) = Ae^{\lambda t}$,

$$Ae^{\lambda t_1} = V_{\mathrm{DD}} \tag{3.7}$$

$$A = V_{\mathrm{DD}} e^{-\lambda t_1} \tag{3.8}$$

$$v_{\mathrm{o}_1} = \left(V_{\mathrm{DD}} e^{-\lambda t_1} \right) e^{\lambda t} \tag{3.9}$$

$$= V_{\mathrm{DD}} e^{-\left(\frac{t - t_1}{\tau}\right)} \tag{3.10}$$

Figure 3.7 is a sketch of v_{in} and v_{o_1} after t_1 and before t_2. Notice that v_{o_1} doesn't immediately jump to 0 like the digital model assumes. Rather, v_{o_1} decays exponentially toward 0 at a rate predicted by τ. Discharging a capacitor takes time, and digital devices' clock speed is limited by how quickly binary values settle in between logic gates.

After t_2

We will try to write a differential equation describing the evolution of v_{o_1} at time t_2 and beyond. Figure 3.6 shows that at time t_2, the pull-up switch closes, and the pull-down switch opens. KCL applied to the same central node yields

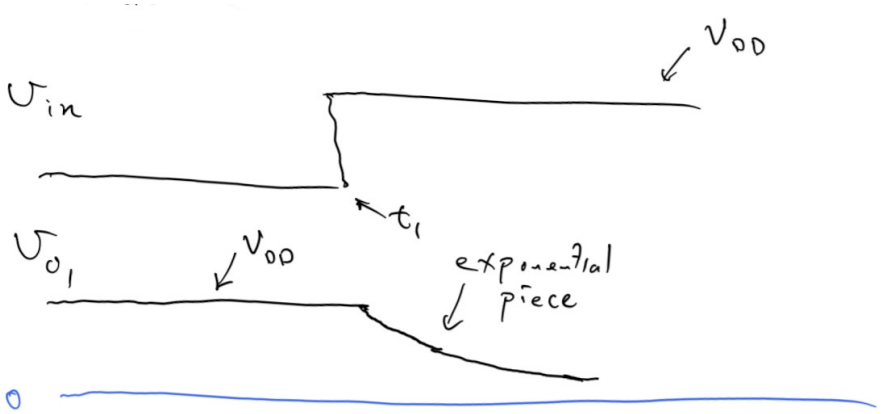

Figure 3.7: Sketch of transient from t_1 to t_2 in Figure 3.6.

the following differential equation:

$$\frac{v_{o_1} - V_{DD}}{R_P} + (C_P + C_N)\frac{d}{dt}v_{o_1} = 0 \tag{3.11}$$

$$\frac{d}{dt}v_{o_1} + \frac{1}{R_P(C_P + C_N)}v_{o_1} = \frac{v_{DD}}{R_P(C_P + C_N)} \tag{3.12}$$

The previous solution for v_{o_1}, which is valid up until time t_2, may be evaluated at t_2 for a boundary condition valid past t_2:

$$v_{o_1}(t_2) = V_{DD}e^{-\left(\frac{t_2 - t_1}{\tau}\right)} \tag{3.13}$$

A solution for v_{o_1} from t_2 onwards is:

$$v_{o_1} = V_{DD} + \left(v_{o_1}(t_2) - V_{DD}\right)e^{-\left(\frac{t - t_2}{\tau_P}\right)}, \tag{3.14}$$

where $\tau_P = R_P(C_P + C_N)$.

3.2 Uniqueness

We solved a differential equation. Differential equations are universal and ubiquitous in science and engineering.

A theorem states that a large class of differential equations with boundary conditions have unique solutions. These differential equations are of the form

$$\frac{d}{dt}x = f(x,t), \quad x(0) = x_0, \tag{3.15}$$

where

1. for all values of t, $f(x,t)$ is differentiable with respect to x and $\left|\frac{\partial f}{\partial x}(x,t)\right| < M$ for some nonnegative real number M; and
2. for all values of x, $f(x,t)$ has a finite number of discontinuities in t in any unit interval $[t_0, t_0+1]$.

If these conditions hold, then our differential equation has a unique solution.

Note that these conditions are in fact quite loose, and are more than enough to certify that unique solutions exist to differential equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x = f(x) = \lambda x$. It is important that we have proofs of existence and uniqueness because methods such as Separation of Variables are not inherently rigorous. Only once we have verifed that a proposed solution satisfies the differential equation and boundary condition may we claim that it is a solution. Because these problems have unique solutions, we may be certain that the model we are using is physically deterministic—it tells precisely what must happen, not just what *may* happen.

Lecture 4

Differential equations with inputs

4.1 RC with exponential input

In this section we will derive, in a more hands-on way, the behavior of an RC circuit forced by an exponential input. If you have ever used an ampwith knobs for treble and bass (Figure 4.1), then you have interacted with two circuits similar to the one shown in Figure 4.2. The resistor with a arrow is a variable resistor, or potentiometer,[1] that might be controlled by one of the amp's knobs.

In Figure 4.2,

- v_{in} represents the amp's analog input,
- v_o is used to drive the speakers after subsequent amplification, and
- R represents the setting on one of the potentiometers.

By studying the distinguished (green) node, we can write the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_o(t) = -\frac{1}{RC} v_o(t) + \frac{1}{RC} v_{\text{in}}(t). \tag{4.1}$$

[1] Electric guitars use this circuit component, which guitarists call "pots," to blend the pickups' signals.

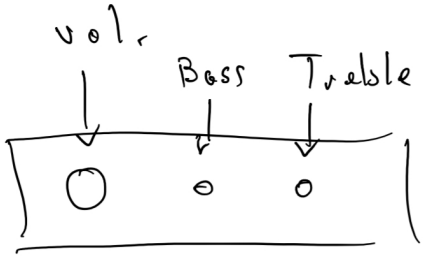

Figure 4.1: An amp with three knobs to adjust playback.

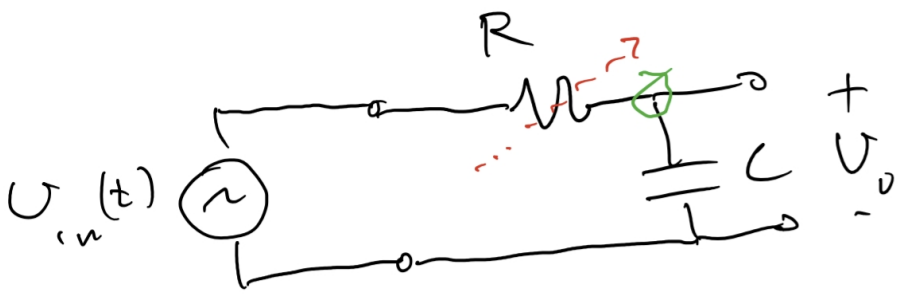

Figure 4.2: RC circuit as a filter.

We'll constrain v_{in} to have the following form:

$$v_{\text{in}}(t) = V_{\text{in}} e^{st}. \tag{4.2}$$

While it seems that this form is arbitrary, it will prove insightful, because e^{st} is an *eigenfunction* for input-output behavior of this circuit, i.e. we expect

$$v_o(t) = V_o e^{st}. \tag{4.3}$$

We can determine V_o by substituting our parameterization of v_o into Equation 4.1, whose LHS...

$$\frac{\mathrm{d}}{\mathrm{d}t} v_o(t) = \frac{\mathrm{d}}{\mathrm{d}t} V_o e^{st} \tag{4.4}$$

$$= s V_o e^{st} \tag{4.5}$$

... is equated with the RHS:

$$s V_o e^{st} = -\frac{1}{RC} V_o e^{st} + \frac{1}{RC} V_{\text{in}} e^{st}. \tag{4.6}$$

Now we can isolate V_o.

$$s V_0 + \frac{1}{RC} V_o = \frac{1}{RC} V_{\text{in}} \tag{4.7}$$

$$V_o = \left(\frac{1}{RC}\right)\left(\frac{1}{s + \frac{1}{RC}}\right) V_{\text{in}} \tag{4.8}$$

Substituting $\lambda = -\frac{1}{RC}$,

$$V_o = \frac{1}{1 - \frac{s}{\lambda}} V_{\text{in}} \tag{4.9}$$

All together, our solution for $v_o(t)$ is the following:

$$v_o(t) = V_o e^{st} = \frac{1}{1 - \frac{s}{\lambda}} V_{\text{in}} e^{st}. \tag{4.10}$$

Suppose that we have an initial condition for v_o at time 0.

$$v_o|_{t=0} = v_1 \tag{4.11}$$

Then our solution, taking this fact into account, will be

$$v_o(t) = Ae^{-\frac{t}{RC}} + \frac{1}{RC}\left(\frac{V_{\text{in}}e^{st}}{s + \frac{1}{RC}}\right), \tag{4.12}$$

where A remains to be determined, viz. by evaluating both sides at $t = 0$:

$$v_1 = A + \frac{1}{RC}\left(\frac{V_{\text{in}}}{s + \frac{1}{RC}}\right) \tag{4.13}$$

$$A = v_1 - \frac{1}{RC}\left(\frac{V_{\text{in}}}{s + \frac{1}{RC}}\right). \tag{4.14}$$

This concludes our example. A solution to a linear differential equation will, generally, have the following structure:

$$v(t) = v_{\text{homogeneous}}(t) + v_{\text{particular}}(t), \tag{4.15}$$

where $v_{\text{homogeneous}}(t)$ corresponds to the initial condition, and $v_{\text{particular}}(t)$ to the input term.

(4.16)

4.2 General scalar differential equation

We will verify that the following general differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = \lambda x(t) + u(t); \quad x(t_0) = x_0 \tag{4.17}$$

has the following solution, which is a sum of a homogeneous and a particular term:

$$x(t) = e^{\lambda(t-t_0)}x_0 + \int_{t_0}^{t} e^{\lambda(t-\tau)}u(\tau)\,\mathrm{d}\tau. \tag{4.18}$$

We can check the initial condition $x(t_0) = x_0$: the former term evaluates to x_0 and the latter to 0. Next, we can verify that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$ holds by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = \left\{\lambda e^{\lambda(t-t_0)}x_0\right\} + u(t) + \left\{\int_{t_0}^{t} \lambda e^{\lambda(t-\tau)}u(\tau)\,\mathrm{d}\tau\right\} \tag{4.19}$$

The two terms in curly braces sum to $\lambda x(t)$, so Equation 4.17 is satisfied.

Lecture 5

Vector differential equations and second-order circuits

5.1 Guess-and-check for RC filter with cosine input

Last lecture we derived the folowing equation modeling the input-output properties of an amp: (where R is set by a potentiometer)

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{\text{out}}(t) = -\frac{1}{RC} v_{\text{out}}(t) + \frac{1}{RC} v_{\text{in}}(t); \quad v_{\text{out}}|_{t_0} = V \tag{5.1}$$

In this section we will try to determine the result in v_{out} when v_{in} has the following sinusoidal form:

$$v_{\text{in}}(t) = V_{\text{in}} \cos(\omega t) \tag{5.2}$$

This defines a sinusoid with amplitude V_{in} and a frequency of ω, which is angular frequency, in rad/s. Angular frequency is related to cycles/second by $\omega = 2\pi f$, where f is in units of Hz.

We can solve for v_{out} by guessing that the particular solution—-the summand that corresponds to v_{in}—has the form $A\cos\left(\omega t + \phi\right)$. The second summand of v_{out} is the homogeneous solution, which corresponds to the initial condition. It has the form $Be^{-\frac{1}{RC}(t-t_0)}$.

$$v_{\text{out}}(t) = A\cos\left(\omega t + \phi\right) + Be^{-\frac{1}{RC}(t-t_0)} \tag{5.3}$$

Substitution into the differential equation and initial conditions result in the following constants:

$$A = \frac{V_{\text{in}}}{\sqrt{\omega^2 (RC)^2 + 1}} \tag{5.4}$$

$$\phi = -\tan^{-1}(\omega RC) \tag{5.5}$$

$$B = v_{\text{out}}|_{t_0} - A\cos\left(\omega t_0 + \phi\right) \tag{5.6}$$

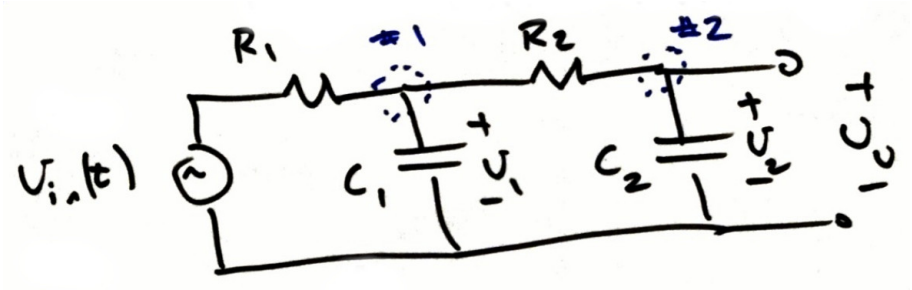

Figure 5.1: Filter with two resistors and two capacitors.

5.2 Second-order filter with two capacitors

Perhaps a "better" filter could be constructed by using two capacitors and two resistors instead of just one. Figure 5.1 depicts the proposed circuit, which is a "second-order circuit" or "second-order" filter, with values $C_1 = C_2 = 1\ \mu\text{F}$, $R_1 = \frac{1}{3}\ \text{M}\Omega$, and $R_2 = \frac{1}{2}\ \text{M}\Omega$. KCL at the two dotted-circled upper nodes yields:

$$C_1 \frac{\mathrm{d}}{\mathrm{d}t} v_1 + \frac{v_1 - v_{\text{in}}(t)}{R_1} + \frac{v_1 - v_2}{R_2} = 0 \tag{5.7}$$

$$C_2 \frac{\mathrm{d}}{\mathrm{d}t} v_2 + \frac{v_2 - v_1}{R_2} = 0 \tag{5.8}$$

In order to view this system of differential equations in state-space form, we will isolate derivatives on the LHS and emphasize that the RHS consists of linear combinations of v_1, v_2, and $v_{\text{in}}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_1 = -v_1 \left(\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{1}{C_1} \right) + v_2 \left(\frac{1}{R_2 C_1} \right) + v_{\text{in}}(t) \left(\frac{1}{R_1 C_1} \right) \tag{5.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_2 = v_1 \left(\frac{1}{R_2 C_2} \right) - v_2 \left(\frac{1}{R_2 C_2} \right) \tag{5.10}$$

Written in matrix-vector form with physical parameters substituted,

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} v_{\text{in}}(t) \tag{5.11}$$

5.3 General state-space linear ODEs

Generally, a system of linear differential equations similar to the one derived above has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x} = A\vec{x} + \vec{b}u(t), \tag{5.12}$$

where $\vec{x}$ is a vector and A is a 2×2 matrix.

Suppose that A has an eigenvector $\vec{v}$ for an eigenvalue λ. We propose the following solution to the homogeneous problem $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x}$:

$$\vec{x}(t) = \vec{v}e^{\lambda t} \tag{5.13}$$

and verify that "$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}$" and "$A\vec{x}$" for this candidate solution are equal:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{v}e^{\lambda t}\right) = \lambda\vec{v}e^{\lambda t} \tag{5.14}$$

$$A\left(\vec{v}e^{\lambda t}\right) = \lambda\vec{v}e^{\lambda t} \tag{5.15}$$

Detour: diagonalization of A

Let's additionally assume that A has two linearly independent eigenvectors:

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \tag{5.16}$$

$$A\vec{v}_2 = \lambda_2\vec{v}_2 \tag{5.17}$$

These two relationships can be expressed simultaneously using matrices that consolidate the eigenvectors (side by side) and eigenvalues (on a diagonal):

$$A\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \tag{5.18}$$

Calling the former two matrices V and the latter Λ,

$$AV = V\Lambda \tag{5.19}$$

Because we chose two linearly independent eigenvectors to constitute V, V is invertible. Stating A in terms of its eigenvectors and eigenvalues is called the *eigenvector-eigenvalue decomposition* of A:

$$A = V\Lambda V^{-1} \tag{5.20}$$

Second-order homogeneous solution from modes

Generally, $\vec{x}(0)$ will be a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{x}(0) = \tilde{x}_1(0)\vec{v}_1 + \tilde{x}_2(0)\vec{v}_2 \tag{5.21}$$

These coefficients can be solved by inverting V:

$$\begin{bmatrix}\tilde{x}_1(0)\\ \tilde{x}_2(0)\end{bmatrix} = V^{-1}\vec{x}(0) \tag{5.22}$$

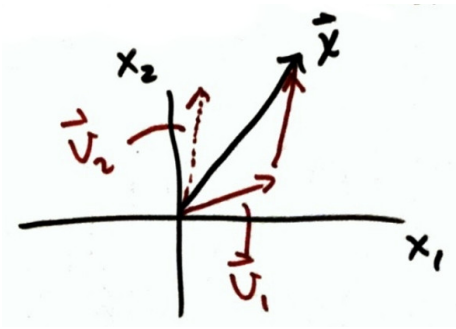

Figure 5.2: Decomposition of $\vec{x}$ along eigenbasis directions $\vec{v}_1$ and $\vec{v}_2$.

We can build a homogeneous solution for $\vec{x}(t)$ by superposing one-dimensional solutions in each eigenvector's respective direction:

$$\vec{x}(t) = \vec{v}_1 e^{\lambda_1 t}\tilde{x}_1(0) + \vec{v}_2 e^{\lambda_2 t}\tilde{x}_2(0) \tag{5.23}$$

$$= V \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} \tag{5.24}$$

To verify the initial condition, we can observe that the diagonal matrix of exponentials becomes an identity matrix at time 0:

$$\vec{x}(0) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix}, \tag{5.25}$$

which is true by construction (Equation 5.21).

Modal decomposition

In the previous section, we wrote $\vec{x}(0)$ in eigenbasis-aligned coordinates $\tilde{x}_1(0)$ and $\tilde{x}_2(0)$. In this section, we will follow $\tilde{x}_1$ and $\tilde{x}_2$ as functions of t. Recall that the eigenbasis-aligned coordinates are defined as follows:

$$\vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = V\tilde{\vec{x}}. \tag{5.26}$$

In reverse,

$$\tilde{\vec{x}} = V^{-1}\vec{x}. \tag{5.27}$$

We can use the Chain Rule to obtain a differential equation for $\vec{\tilde{x}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}} = V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}x \tag{5.28}$$

$$= V^{-1}\left(A\vec{x} + \vec{b}u\right) \tag{5.29}$$

$$= V^{-1}AV\vec{\tilde{x}} + V^{-1}\vec{b}u \tag{5.30}$$

$$= \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\vec{\tilde{x}} + \vec{\tilde{b}}u, \quad \vec{\tilde{b}} = V^{-1}b \tag{5.31}$$

This vector differential equation is effectively scalar in each variable, in which scalar techniques can be applied separately. The separation of x into its eigenbasis-aligned components is called *modal decomposition*; $\tilde{x}_1\vec{v}_1$ and $\tilde{x}_2\vec{v}_2$ are the two *modes* of this system.

Lecture 6

Diagonalization to solve vector differential equations

In the last lecture, a second-order low-pass filter circuit using two resistors and two capacitors led us to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x} + \vec{b}u, \tag{6.1}$$

where $\vec{x} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ and $\vec{x}(0)$ or $\vec{x}(t_0)$ is known. We represented $\vec{x}$ as a linear combination of A's eigenvectors $\vec{v}_1$ (for eigenvalue λ_1) and $\vec{v}_2$ (for eigenvalue λ_2):

$$\vec{x} = \vec{v}_1\tilde{x}_1 + \vec{v}_2\tilde{x}_2 \tag{6.2}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = V\vec{\tilde{x}} \tag{6.3}$$

We will assume that λ_1 and λ_2 are distinct, which implies that A has an invertible matrix of linearly independent eigenvectors V.We established that

$$AV = V\Lambda, \quad V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda 2 \end{bmatrix}. \tag{6.4}$$

These findings are summarized in Figure 6.1, which shows how $\vec{x}$, $A\vec{x}$, $\vec{\tilde{x}}$, and $\Lambda\vec{\tilde{x}}$ are related by matrix multiplication (along arrows).

6.1 Solution technique

A system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x} + \vec{b}u; \quad \vec{x}(t_0) \tag{6.5}$$

is solved as follows:

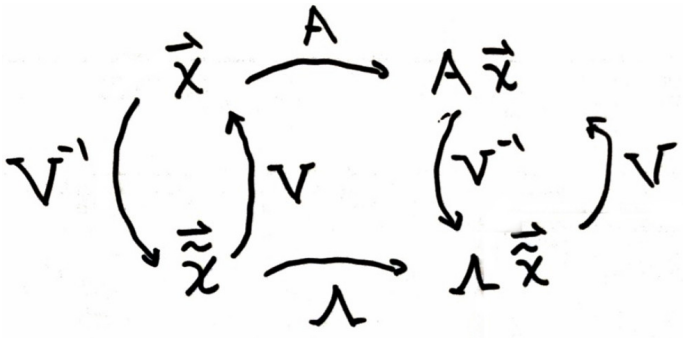

Figure 6.1: Illustration of multiplication actions of A, Λ, V, and V^{-1}.

1. Compute eigenvalues λ_1 and λ_2 of A, as well as their respective eigenvectors $\vec{v}_1$ and $\vec{v}_2$.
2. Construct $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ and define $\vec{\tilde{x}} = V^{-1}x$.
3. Construct $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\tilde{b} = V^{-1}b$. Solve the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} + \tilde{b}u$ with initial condition $\tilde{x}(t_0) = V^{-1}x(t_0)$. (More on this later.)
4. Recover a solution for $\vec{x}$ using $\vec{x} = V\vec{\tilde{x}}$.

6.2 Numerical example from RCRC circuit

Equation 5.11 captured a second-order low-pass filter using

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \tag{6.6}$$

We will solve the differential equation for $\vec{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ using the technique of the previous section.

Eigenvalues and eigenvectors

We will solve for eigenvectors λ as roots of $\det(\lambda I - A)$, the characteristic polynomial of A.

$$\det \begin{bmatrix} \lambda + 5 & -2 \\ -2 & \lambda + 2 \end{bmatrix} = \lambda^2 + 7\lambda + 6 = 0 \tag{6.7}$$

This quadratic equation in the indeterminate λ is called the *characteristic equation* of A. It has the following roots:

$$\lambda_1 = -1; \quad \lambda_2 = -6. \tag{6.8}$$

Next we will solve for an eigenvector belonging to eigenvalue λ_1, by choosing a nonzero vector from the null space of $\lambda_1 I - A$:

$$\lambda_1 I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \tag{6.9}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{6.10}$$

. . . and, *mutatis mutandis*, for λ_2:

$$\lambda_2 I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \tag{6.11}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{6.12}$$

Differential equation in new coordinates

In our example,

$$V = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \text{so} \tag{6.13}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}. \tag{6.14}$$

Our differential equation in $\vec{\tilde{x}}$ will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} + V^{-1}\vec{b}u \tag{6.15}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}\vec{\tilde{x}} + \begin{bmatrix} \frac{3}{5} \\ \frac{6}{5} \end{bmatrix}. \tag{6.16}$$

With $t_0 = 0$, $\vec{\tilde{x}}$ is solved as follows:

$$\vec{\tilde{x}}(t) = \big(\text{homogeneous solution}\big) + \big(\text{particular solution}\big) \tag{6.17}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\vec{\tilde{x}}(0) + \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} & 0 \\ 0 & e^{\lambda_2(t-\tau)} \end{bmatrix}\vec{\tilde{b}}u(\tau)\,\mathrm{d}\tau, \tag{6.18}$$

viz., in individual components,

$$\begin{cases} \tilde{x}_1(t) = e^{\lambda_1 t}\tilde{x}_1(0) + \int_0^t e^{\lambda_1(t-\tau)}\tilde{b}_1 u(\tau)\,\mathrm{d}\tau \\ \tilde{x}_2(t) = e^{\lambda_2 t}\tilde{x}_2(0) + \int_0^t e^{\lambda_2(t-\tau)}\tilde{b}_2 u(\tau)\,\mathrm{d}\tau \end{cases} \tag{6.19}$$

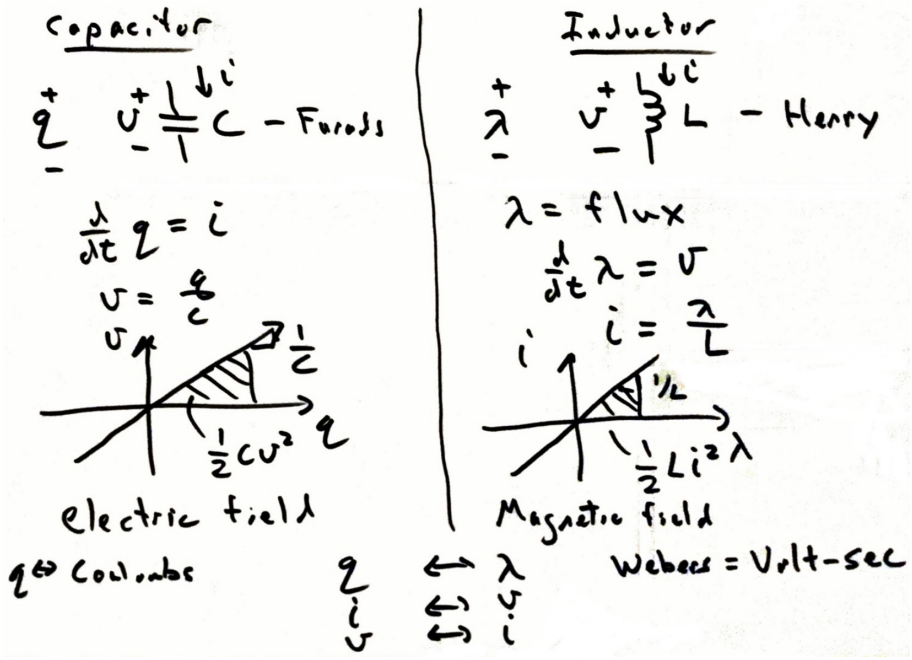

Figure 6.2: Parallels between capacitors and inductors.

Solution in original coordinates

A solution for $\vec{x}(t)$ may be reconstituted from eigenbasis-aligned coordinates using the following equation:

$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}. \tag{6.20}$$

6.3 Introduction to inductors

Inductors are a branch element that are analogous to capacitors. Figure 6.2 compares them with capacitors, and the parallels are repeated below.

$$q = \text{charge (Coulomb)} \qquad \lambda = \text{flux (Weber = Volt-second)} \tag{6.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q = i \qquad \frac{\mathrm{d}}{\mathrm{d}t} \lambda = v \tag{6.22}$$

$$v = \frac{q}{C} \qquad i = \frac{\lambda}{L} \tag{6.23}$$

$$E_C = \frac{1}{2} C v^2 \qquad E_L = \frac{1}{2} L i^2 \tag{6.24}$$

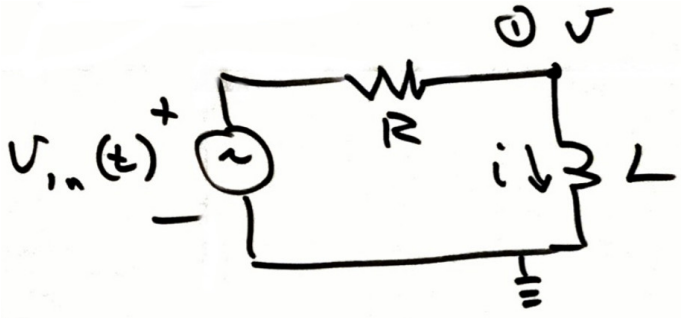

Figure 6.3: RL circuit, which is similar to an RC circuit (cf. Figure 4.2).

6.4 Example: RL circuit

Figure 6.3 shows a circuit with a time-varying voltage source, a resistor, and an inductor. KCL at the marked upper right node yields

$$\frac{v - v_{\text{in}}}{R} + i = 0. \tag{6.25}$$

In addition, from the current-voltage relationship of an inductor,

$$L \frac{\mathrm{d}}{\mathrm{d}t} i = v. \tag{6.26}$$

Eliminating v and isolating $\dfrac{\mathrm{d}}{\mathrm{d}t} i$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} i = -\frac{R}{L} i + \frac{v_{\text{in}}}{L}. \tag{6.27}$$

The state variable for an inductor is i, and this differential equation may be solved the same way we solved RC circuits.