On the Correlation Properties of Generalized Chirp-Like Sequences and Their Application

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Abstract A generalized chirp-like (GCL) sequence is constructed by modulating a Zadoff-Chu sequence with a unimodular sequence. Under some specific conditions, the magnitude of the cross-correlation between GCL sequences is known to be the same as that of the corresponding Zadoff-Chu sequences. In this paper, we further investigate the conditions under which such a relation holds. Using them, we construct a new class of optimal zero-correlation zone (ZCZ) sequences.

key words: *Generalized chirp-like sequence, Zadoff-Chu sequence, zero-correlation zone sequence.*

1. Introduction

For two positive integers N and r such that $\gcd(N, r) = 1$ and any integer q, a Zadoff-Chu sequence $\{c_r(n)\}$ of period N is defined [1], [2] as

$$c_r(n) = \begin{cases} W_N^{\frac{rn^2}{2}+qrn}, & \text{if } N \text{ is even} \\ W_N^{\frac{rn(n+1)}{2}+qrn}, & \text{if } N \text{ is odd,} \end{cases}$$

where $W_N = \exp(2\pi\sqrt{-1}/N)$. Zadoff-Chu sequences have optimal cross-correlation with respect to the Sarwate bound [3] under some specific condition as well as they have ideal autocorrelation. Recently, Kang *et al.* [4] computed the magnitudes of their cross-correlations completely.

There is a general class of sequences called generalized chirp-like (GCL) sequences [5] which include Zadoff-Chu sequences as a special case. A GCL sequence is constructed by modulating a Zadoff-Chu sequence with a unimodular sequence. Not only do GCL sequences have ideal autocorrelation, but they also have optimal cross-correlation with respect to the Sarwate bound when the corresponding Zadoff-Chu sequences satisfy the optimality condition with respect to the Sarwarte bound. For this reason, they can be used in many applications, for example, as spreading sequences for spread-spectrum multiple-access (SSMA) systems [5], [6] or preambles for synchronization [7], etc. However, there have been few theoretical results on their cross-correlation properties in general cases.

In this paper, we study the cross-correlation of GCL sequences and investigate the conditions under which the magnitude of their cross-correlation becomes the same as that of the corresponding Zadoff-Chu sequences in a general case, i.e., the case that the corresponding Zadoff-Chu sequences are not an optimal pair with respect to the Sarwarte bound. As an application, we construct a new zero correlation zone (ZCZ) sequence set which is optimal with respect to the Tang-Fan-Matsufuji bound [8].

The outline of the paper is as follows. In Section II, we give some preliminaries. In Section III, we then present sufficient conditions under which the magnitude of the cross-correlation between GCL sequences is the same as that of the corresponding Zadoff-Chu sequences. As an application, we give a construction of optimal ZCZ sequences in Section IV. Finally, we give some concluding remarks in Section V.

2. Preliminaries

Throughout the paper, we denote by $\langle x \rangle_y$ the least nonnegative residue of x modulo y for an integer x and a positive integer y and denote by $\delta_K(\cdot)$ the Kronecker delta function defined by

$$\delta_K(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{a(n)\}_{n=0}^{N-1}$ be a complex-valued sequence of period N, i.e., $a(n+N) = a(n)$ for any n. For short notation, it is denoted by $\{a(n)\}$ when there is no confusion. A sequence $\{a(n)\}$ is said to be unimodular if $|a(n)| = 1$ for all n. Let $\{a(n)\}$ and $\{b(n)\}$ be two complex-valued sequences of period N. The periodic cross-correlation $\theta_{a,b}(\tau)$ between $\{a(n)\}$ and $\{b(n)\}$ is defined as

$$\theta_{a,b}(\tau) = \sum_{n=0}^{N-1} a(n) b^*(n+\tau)$$

for an integer $0 \le \tau \le N-1$, where $*$ denotes the complex conjugation and all the operations among the indices are computed modulo N. In particular, if $a(n) = b(n)$ for all $n \in \mathbb{Z}$, $\theta_{a,b}(\tau)$ is called the autocorrelation of $\{a(n)\}$ and is denoted by $\theta_a(\tau)$. Kang *et al.* [4] computed the magnitudes of their cross-correlations completely in the following theorem.

Theorem 1 ([4]). *Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of period N. Then the magnitude of the cross-correlation $\theta_{c_r,c_s}(\tau)$ between $\{c_r(n)\}$ and $\{c_s(n)\}$ is given by*

$$|\theta_{c_r,c_s}(\tau)| = \begin{cases} \sqrt{Ng}\,\delta_K(\tau_2), \\ \qquad \text{if } \langle N \rangle_2 = \langle uv \rangle_2 = 0 \text{ or } \langle N \rangle_2 = 1 \\ \sqrt{Ng}\,\delta_K(\tau_2 - \frac{g}{2}), \\ \qquad \text{if } \langle N \rangle_2 = 0 \text{ and } \langle uv \rangle_2 = 1, \end{cases}$$

where $g = \gcd(N, r-s)$, $u = N/g$, $v = (r-s)/g$ and $\tau_2 = \langle \tau \rangle_g$.

Note that $g = 1$ happens only when $\langle N \rangle_2 = 1$, i.e., N is odd. In this case, the magnitude of the cross-correlation $\theta_{c_r,c_s}(\tau)$ was determined by Popovic [5].

A GCL sequence, which belongs to a general category of modulatable orthogonal sequences [9], is obtained by modulating a Zadoff-Chu sequence with a unimodular sequence.

Definition 2 ([5]). *Let $\{c_r(n)\}_{n=0}^{N-1}$ be a Zadoff-Chu sequence of period N with $N = tm$ for two positive integers t and m such that m is a divisor of t. Let $\{b(n)\}_{n=0}^{m-1}$ be an arbitrary unimodular sequence of period m. A GCL sequence $\{h_{c_r,b}(n)\}_{n=0}^{N-1}$ of period N is defined as*

$$h_{c_r,b}(n) = c_r(n)b(\langle n\rangle_m).$$

In the definition of a GCL sequence, $\{c_r(n)\}$ and $\{b(n)\}$ are called the carrier sequence and the modulation sequence, respectively. From now on, m is assumed to be a divisor of t, unless otherwise specified. The following theorem gives partial information on the correlation of GCL sequences.

Theorem 3 ([5]). *Let $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_s,b_j}(n)\}$ be two GCL sequences of period $N = tm$. Then the magnitude of the cross-correlation $\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)$ between $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_s,b_j}(n)\}$ is given by*

$$|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)| = \begin{cases} N\delta_K(\tau), & \text{if } g = N \text{ and } i = j \\ \sqrt{N}, & \text{if } g = 1, \end{cases}$$

where $g = \gcd(N, r-s)$.

Note that $g = N$ if and only if $r = s$, i.e., the two GCL sequences are obtained from a common Zadoff-Chu sequence.

Definition 4 ([10]). *A set $\{\{x_i(n)\} \,|\, 1 \le i \le M\}$ of M sequences of period N is called an (N, M, Z) ZCZ sequence set if it satisfies*

$$\theta_{x_i,x_j}(\tau) = \begin{cases} N, & \text{for } \tau = 0 \text{ when } i = j \\ 0, & \text{for } 0 < |\tau| < Z \text{ when } i = j \\ 0, & \text{for } 0 \le |\tau| < Z \text{ when } i \ne j, \end{cases}$$

for some positive integer Z called the zero-correlation zone with $2 \le Z \le N$. Moreover, an (N, M, Z) ZCZ sequence set is said to be optimal if $N = MZ$.

3. Sufficient Conditions on the Correlation of GCL Sequences

Theorem 3 tells that when two GCL sequences $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_s,b_j}(n)\}$ are constructed from two different Zadoff-Chu sequences $\{c_r(n)\}$ and $\{c_s(n)\}$ such that $g = 1$, the modulation sequences do not give any effect on the magnitude of the cross-correlation of GCL sequences as well as when $g = N$ and $i = j$. It is easily checked that $|\theta_{c_r,c_s}(\tau)|$ in Theorem 1 becomes equal to $|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|$ in Theorem 3 when $g = 1$ or $g = N$ and $i = j$. Hence, our goal is to further investigate sufficient conditions under which $|\theta_{c_r,c_s}(\tau)| = |\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|$ for any τ and g.

Theorem 5. *Let $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_s,b_j}(n)\}$ be two GCL sequences of period $N = tm$ where m is an odd prime. Let $g = \gcd(N, r-s)$, $u = N/g$, $v = (r-s)/g$ and $\tau_2 = \langle\tau\rangle_g$. Then the magnitude of the cross-correlation $\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)$ between $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_s,b_j}(n)\}$ is the same as $|\theta_{c_r,c_s}(\tau)|$, i.e.,*

$$|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)| = \begin{cases} \sqrt{Ng}\,\delta_K(\tau_2), \\ \quad \text{if } \langle N\rangle_2 = \langle uv\rangle_2 = 0 \\ \quad \text{or } \langle N\rangle_2 = 1 \\ \sqrt{Ng}\,\delta_K(\tau_2 - \frac{g}{2}), \\ \quad \text{if } \langle N\rangle_2 = 0 \\ \quad \text{and } \langle uv\rangle_2 = 1, \end{cases}$$

if either (a) or (b) is satisfied:

(a) $\langle u\rangle_{m^2} = 0$;

(b) $u = m$ and each modulation sequence is a linear polyphase sequence of the form

$$b_i(n) = W_m^{\phi_i n} \quad \text{and} \quad b_j(n) = W_m^{\phi_j n}$$

for integers ϕ_i and ϕ_j.

Proof. We consider only the case that $\langle N\rangle_2 = \langle uv\rangle_2 = 0$ or $\langle N\rangle_2 = 1$, since the other case can be similarly proved. It suffices to show that we get

$$|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|^2 = N\sum_{k=0}^{g-1} W_u^{\frac{vku(ku+\langle N\rangle_2)}{2} - \frac{sku\tau}{g}}$$

if either (a) or (b) is satisfied, because the Proof of Theorem 1 in [4] starts with the relation

$$|\theta_{c_r,c_s}(\tau)|^2 = N\sum_{k=0}^{g-1} W_u^{\frac{vku(ku+\langle N\rangle_2)}{2} - \frac{sku\tau}{g}}.$$

After some manipulation, we have

$$\begin{aligned} &|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|^2 \\ &= \sum_{n_1=0}^{N-1} W_N^{\frac{rn_1^2}{2}+qrn_1} W_N^{-\frac{s(n_1+\tau)^2}{2}-qs(n_1+\tau)} \\ &\qquad b_i(\langle n_1\rangle_m) b_j^*(\langle n_1+\tau\rangle_m) \\ &\quad \sum_{n_2=0}^{N-1} W_N^{-\frac{rn_2^2}{2}-qrn_2} W_N^{\frac{s(n_2+\tau)^2}{2}+qs(n_2+\tau)} \\ &\qquad b_i^*(\langle n_2\rangle_m) b_j(\langle n_2+\tau\rangle_m) \\ &= \sum_{n_1=0}^{N-1} W_u^{\frac{vn_1^2}{2}-\frac{sn_1\tau}{g}+gvn_1} \sum_{n_2=0}^{N-1} W_u^{vn_1n_2} \\ &\qquad b_i^*(\langle n_2\rangle_m) b_j(\langle n_2+\tau\rangle_m) \\ &\qquad b_i(\langle n_2+n_1\rangle_m) b_j^*(\langle n_2+n_1+\tau\rangle_m). \end{aligned}$$

Let $n_1 = a_1m + a_2$, where $0 \le a_1 \le t-1$ and $0 \le a_2 \le m-1$. Then

$$\begin{aligned} &|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|^2 \\ &= \sum_{a_1=0}^{t-1}\sum_{a_2=0}^{m-1} W_u^{\frac{v(a_1m+a_2)^2}{2}-\frac{s(a_1m+a_2)\tau}{g}+gv(a_1m+a_2)} \\ &\quad \sum_{n_2=0}^{N-1} W_u^{v(a_1m+a_2)n_2} b_i^*(\langle n_2\rangle_m) b_j(\langle n_2+\tau\rangle_m) \\ &\qquad b_i(\langle n_2+a_2\rangle_m) b_j^*(\langle n_2+a_2+\tau\rangle_m). \end{aligned} \tag{1}$$

For our convenience, define $B_{\tau,m,u,v}(a_1m+a_2)$ as

$$\begin{aligned} &B_{\tau,m,u,v}(a_1m+a_2) \\ &\triangleq \sum_{n_2=0}^{N-1} W_u^{v(a_1m+a_2)n_2} b_i^*(\langle n_2\rangle_m) b_j(\langle n_2+\tau\rangle_m) \\ &\quad b_i(\langle n_2+a_2\rangle_m) b_j^*(\langle n_2+a_2+\tau\rangle_m) \end{aligned}$$

and let $n_2 = l_1m + l_2$, where $0 \le l_1 \le t-1$ and $0 \le l_2 \le m-1$. Then we get

$$\begin{aligned} &B_{\tau,m,u,v}(a_1m+a_2) \\ &= \sum_{l_1=0}^{t-1} W_u^{v(a_1m+a_2)l_1m} \\ &\quad \sum_{l_2=0}^{m-1} W_u^{v(a_1m+a_2)l_2} b_i^*(l_2) b_i(\langle l_2+a_2\rangle_m) \\ &\qquad b_j(\langle l_2+\tau\rangle_m) b_j^*(\langle l_2+a_2+\tau\rangle_m). \end{aligned}$$

One of the key facts we need for the proof is that

$$B_{\tau,m,u,v}(a_1m+a_2) = \begin{cases} N, & \text{if } \langle a_1m\rangle_u = 0 \\ & \text{and } a_2 = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

if either (a) or (b) is satisfied. The detailed proof of (2) is omitted here due to the limit of the space. Using (2), (1) leads to

$$\begin{aligned} &|\theta_{h_{c_r,b_i},h_{c_s,b_j}}(\tau)|^2 \\ &= N \sum_{a_1=0}^{t-1} W_u^{\frac{v(a_1m)^2}{2} - \frac{s(a_1m)\tau}{g} + gv(a_1m)} \\ &= N \sum_{k=0}^{g-1} W_u^{\frac{v(ku)^2}{2} - \frac{sku\tau}{g}} \end{aligned} \tag{3}$$

if either (a) or (b) is satisfied, where the last equality comes from the fact that

$$\langle a_1m\rangle_u = 0 \ \text{ for } \ 0 \le a_1 \le t-1$$

if and only if

$$a_1 = k\frac{u}{m} \ \text{ for } \ 0 \le k \le g-1.$$

Hence, we get the desired result. □

4. New Optimal ZCZ Sequences from GCL Sequences

Popovic and Mauritz [11] constructed optimal ZCZ sequences which are the GCL sequences obtained by modulating a single Zadoff-Chu sequence with unimodular orthogonal sequences. The construction are based on the following fact:

Fact 6. *[11]: Let $\{h_{c_r,b_i}(n)\}$ and $\{h_{c_r,b_j}(n)\}$ be two GCL sequences of period $N = tm$ constructed by modulating a Zadoff-Chu sequence $\{c_r(n)\}$ with modulation sequences $\{b_i(n)\}$ and $\{b_j(n)\}$, respectively. Then we have*

(a) $|\theta_{h_{c_r,b_i},h_{c_r,b_j}}(\tau)| = 0$ if $\langle\tau\rangle_t \ne 0$;

(b) $|\theta_{h_{c_r,b_i},h_{c_r,b_j}}(0)| = 0$ if $\{b_i(n)\}$ and $\{b_j(n)\}$ are orthogonal to each other.

Note that the ZCZ sequences in [11] are GCL sequences constructed from one common Zadoff-Chu sequence. Since we know the cross-correlation properties among GCL sequences constructed from different Zadoff-Chu sequences under some circumstances, it is possible to apply them to construction of ZCZ sequences. As in [11], the modulation sequences in our construction are also obtained from the rows of the $m \times m$ discrete Fourier transform (DFT) matrix $\mathbf{D}$ defined by

$$D_{ij} = W_m^{(i-1)(j-1)}, \ 1 \le i,j \le m.$$

Construction. *Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of even period $N = tm$ for an odd prime m, where r and s are chosen so that $\langle uv\rangle_2 = 1$ and $u = m$ (or equivalently, $g = t$). Let $\{\{b_i(n)\} \,|\, 1 \le i \le m\}$ be the set of m sequences of period m, where $\{b_i(n)\}_{n=0}^{m-1}$ is the ith row of the $m \times m$ DFT matrix. Let $\{X_i(n)\}_{n=0}^{N-1}$, $1 \le i \le 2m$, be a GCL sequence of period N defined by*

$$X_i(n) = \begin{cases} c_r(n)b_i(\langle n\rangle_m), & \text{if } 1 \le i \le m \\ c_s(n)b_{i-m}(\langle n\rangle_m), & \text{if } m+1 \le i \le 2m. \end{cases}$$

Construct the set $\mathcal{X}$ as

$$\mathcal{X} = \{\{X_i(n)\} | 1 \le i \le 2m\}.$$

Theorem 7. *The set $\mathcal{X}$ in the above construction is an optimal $(tm, 2m, t/2)$ ZCZ sequence set.*

Proof. The calculation of $|\theta_{X_i,X_j}(\tau)|$ is divided into three cases. Without loss of generality, we assume that $i < j$ when $i \ne j$.

Case i) $i = j$: In this case, we have

$$|\theta_{X_i,X_i}(\tau)| = \delta_K(\tau)$$

by Theorem 3.

Case ii) $1 \le i,j \le m$ or $m+1 \le i,j \le 2m$: In this case, $|\theta_{X_i,X_j}(\tau)| = 0$ when $1 \le \tau < t$ from (a) of Fact 6. In addition, since all the rows of the DFT matrix are orthogonal to each other, we have $|\theta_{X_i,X_j}(0)| = 0$ from (b) of Fact 6. Therefore,

$$|\theta_{X_i,X_j}(\tau)| = 0, \ \ 0 \le \tau < t.$$

Case iii) $1 \le i \le m$ and $m+1 \le j \le 2m$: In this case, $|\theta_{X_i,X_j}(\tau)| = |\theta_{c_r,c_s}(\tau)|$ by Theorem 5. Since $\langle N\rangle_2 = 0$, $\langle uv\rangle_2 = 1$ and $g = t$ by the assumption of the construction,

$$|\theta_{X_i,X_j}(\tau)| = t\sqrt{m}\,\delta_K(\tau_2 - t/2)$$

where $\tau_2 = \langle \tau \rangle_t$. Hence, we get

$$|\theta_{X_i,X_j}(\tau)| = 0, \ \ 0 \leq \tau < t/2.$$

Combining Case i), Case ii) and Case iii), the proof is completed. □

5. Conclusions

Motivated by a question on whether the effect of the modulation sequences on the cross-correlation among GCL sequences is eliminated or not in a general case, we found some sufficient conditions under which the magnitude of the cross-correlation between two GCL sequences becomes the same as that of the corresponding Zadoff-Chu sequences. We applied the result to construction of new optimal ZCZ sequences. The conditions found in this paper may not be necessary, so finding a sufficient and necessary condition may be an interesting problem.

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