Modulatable Orthogonal Sequences and their Application to SSMA Systems

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***Abstract*—A set of $(N-1)$ orthogonal sequences of period N^2 is proposed, where N is a natural number. Each orthogonal sequence proposed can be modulated by N complex numbers of absolute value 1, so the modulated sequence is also orthogonal. When N is an odd prime number, the absolute value of the cross-correlation function between any two of the $(N-1)$ orthogonal sequences is constant and satisfies the mathematical lower bound. This property of the cross-correlation function is not changed when each of the two orthogonal sequences is modulated by N complex numbers of absolute value 1. Two spread spectrum multiple access (SSMA) systems using these sequences are proposed. One system is an asynchronous SSMA system, using the proposed sequences unmodulated. The co-channel interference peak between any two channels in this system realizes the mathematical lower bound for an asynchronous SSMA system using a set of orthogonal sequences. The other system is a synchronous SSMA system without co-channel interference which uses the modulated form of the proposed sequences.**

I. Introduction

SPREAD spectrum multiple access (SSMA) systems [1] usually use binary sequences, such as *m*-sequences [2] or Gold's sequences [3]. These systems exhibit co-channel interference, which is caused by cross correlations between pairs of these sequences. Co-channel interferences exceed $1/\sqrt{N}$ in these systems, where N is the SSMA spread ratio.

A periodic sequence whose autocorrelation function is 0 except for the period-multiple-shift terms is called an orthogonal sequence. Frank and Zadaff [4] proposed an orthogonal sequence of period N^2, where N is a natural number. However, it cannot be used for an SSMA system.

Suehiro [5] proposed a set of $(N-1)$ orthogonal sequences of period N which can be used for an asynchronous SSMA system. However, these sequences are not directly modulatable.

We propose a set of $(N-1)$ readily modulatable orthogonal sequences of period N^2, where N is a prime number. The absolute value of the cross-correlation function between any two of these $N-1$ sequences is $1/N$ for every nonzero lag. This realizes the mathematical lower bound for the absolute value peak in the cross-correlation function between two orthogonal sequences.

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Each of our $N-1$ orthogonal sequences can be modulated by N complex numbers of absolute value 1 such that each of the modulated sequences remains orthogonal. The cross-correlation properties already mentioned also are inherited by the modulated sequences.

Section II concerns periodic sequence properties, which were discussed by Suehiro [5]. Especially, the mathematical lower bound of the absolute value peak for the cross-correlation function between two orthogonal sequences is covered.

In Section III we propose a set of modulatable orthogonal sequences and discuss their autocorrelation and cross-correlation properties, with the following results. A modulated orthogonal sequence can be obtained by modulating one of these $N-1$ original orthogonal sequences. The orthogonal sequences modulated from the same original orthogonal sequence are defined to compose a "class" of modulated orthogonal sequences. The resulting $N-1$ classes of modulated orthogonal sequences have the following properties.

1) Each class includes an infinite number of modulated orthogonal sequences of period N^2.

2) The absolute value of the cross-correlation function between any two sequences in different classes is constant and equals the lower bound of the absolute value peak of the cross-correlation function.

3) The cross-correlation function between any two sequences in the same class is 0 for every lag that is not a multiple of N.

In Section IV two SSMA systems using proposed sequences are proposed. One system is an asynchronous SSMA system, using the proposed sequences unmodulated. The co-channel interference peak between any two channels in this system realizes the mathematical lower bound for an asynchronous SSMA system, using a set of orthogonal sequences. The other system is a synchronous SSMA system without co-channel interference which uses the modulated form of the proposed sequences.

II. Periodic Sequence Properties

A. Periodic Sequence Description

Periodic sequences are usually described by writing finite sequences, as $(a_1, a_2, \cdots, a_{n-1}, a_n)$, where n is the period for the periodic sequence. It can also be described as $(a_2, a_3, \cdots, a_n, a_1)$.

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It is convenient to describe a periodic sequence as a normalized cyclic matrix when investigating correlation function properties. Therefore, a periodic sequence of period n will be described as

$$A = \frac{1}{\sqrt{n}} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}. \tag{1}$$

B. Orthogonal Sequences Properties

A periodic sequence of period n is called an orthogonal sequence when the autocorrelation function is zero for every term, except for the 0 (mod n) shift term. When, and only when, A is a unitary matrix, the periodic sequence, described as A, is an orthogonal sequence.

When another periodic sequence of period n is described as

$$B = \frac{1}{\sqrt{n}} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ b_n & b_1 & \cdots & b_{n-1} \\ \vdots & \vdots & & \vdots \\ b_2 & b_3 & \cdots & b_1 \end{bmatrix}, \tag{2}$$

the cross-correlation function between two periodic sequences, represented as A and B, is described as

$$A \cdot {}^t(\overline{B}) \text{ or } AB^*$$

where elements in $\overline{B}$ are complex conjugates of elements in B, and B^* is the adjoint matrix for B. Note that AB^* is a cyclic matrix.

When both of two periodic sequences, represented as A and B, are orthogonal sequences, A and B are unitary matrices. In this case, the norm for each row of AB^* is 1, because AB^* is a unitary matrix. Therefore, when the absolute values of all AB^* elements are equal to each other, the maximum absolute value is minimized. In other words, when AB^* is a cyclic matrix whose elements have a constant absolute value, the peak of the absolute value is minimum for the cross-correlation function between two orthogonal sequences, described as A and B. The sequences proposed here have this property.

III. Modulatable Orthogonal Sequences

A. Periodic Sequences to be Modulated

This section proposes a class of periodic sequences, as well as a method to modulate these sequences. The next section shows that the proposed sequences and the modulated sequences are orthogonal sequences.

Let N be a natural number. An N-dimensional normalized discrete Fourier transformation (DFT) matrix F_N is defined as

$$F_N = [f_N(i_0, i_1)]$$

$$f_N(i_0, i_1) = \frac{1}{\sqrt{N}} \exp\left(-\frac{2\pi\sqrt{-1}}{N} i_0 i_1\right) \tag{3}$$

where $0 \le i_0 \le N-1$, $0 \le i_1 \le N-1$. Then matrix F_{Nm} is defined as

$$F_{Nm} = [f_{Nm}(i_0, i_1)]$$

$$f_{Nm}(i_0, i_1) = \frac{1}{\sqrt{N}} \exp\left(-\frac{2\pi\sqrt{-1}}{N} m i_0 i_1\right) \tag{4}$$

where m is a natural number.

An N-dimensional diagonal matrix B is introduced as

$$B = [b(i_0, i_1)]$$

where the absolute value $b(i_0, i_0)$ of each diagonal element of B is 1, and other elements are 0. The (i_0, i_1) element of $\overline{F_{Nm}}B$ is

$$\overline{f_{Nm}(i_0, i_1)} b(i_1, i_1) = b(i_1, i_1) \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi\sqrt{-1}}{N} m i_0 i_1\right). \tag{5}$$

A sequence of length N^2 is derived from the $N \times N$ matrix $\overline{F_{Nm}}B$. Let i be

$$i = i_0 N + i_1;$$

then

$$0 \le i \le N^2 - 1.$$

The ith element of a sequence of length N^2 composed by linking N rows of $N \times N$ matrix $\overline{F_{Nm}}B$ is the (i_0, i_1) element of $\overline{F_{Nm}}B$. Let G be a sequence composed of linked rows of $\sqrt{N}\,\overline{F_{Nm}}B$; then

$$G = (g(i))$$

$$\begin{aligned} g(i) &= \sqrt{N}\,\overline{f_{Nm}(i_0, i_1)} b(i_1, i_1) \\ &= b(i_1, i_1) \exp\left(\frac{2\pi\sqrt{-1}}{N} m i_0 i_1\right). \end{aligned} \tag{6}$$

G is modulated by $b(i_1, i_1)$, which are the diagonal elements of B. A periodic sequence of period N^2, obtained by repeating G, is modulated by the diagonal elements of B.

For example, when $N = 3$ and $m = 1$,

$$\overline{F_{31}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3 & W_3^2 \\ 1 & W_3^2 & W_3 \end{bmatrix}$$

where $W_N = \exp(2\pi\sqrt{-1}/N)$. If

$$B = \begin{bmatrix} b_0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{bmatrix},$$

the obtained periodic sequence is $\{b_0, b_1, b_2, b_0, b_1 W_3,$

$b_2W_3^2, b_0, b_1W_3^2, b_2W\}$. If B is a unit matrix, the obtained sequence corresponds to the sequence without modulation.

B. Modulated Sequences Autocorrelation Function

This section shows that the modulated periodic sequences introduced in the last section are orthogonal sequences. As is well-known, cyclic convolution is transformed into multiplication by Fourier transformation. Therefore, a cyclic convolution can be described as

$$AX = F^{-1}\Lambda FX;$$

so

$$A = F^{-1}\Lambda F \tag{7}$$

where A is a cyclic matrix, X is a vector, F is the DFT matrix, and Λ is a diagonal matrix. The diagonal elements of Λ are the multipliers in the Fourier-transformed domain. Multiplying cyclic matrix A from the left describes the cyclic convolution. The sequence, whose elements are diagonal elements of Λ, is transformed into one of the rows in A by DFT.

Let N be a natural number. An N^2-dimensional normalized DFT matrix F_{N^2} is defined as

$$F_{N^2} = [f_{N^2}(i,j)]$$

$$f_{N^2} = \frac{1}{N}\exp\left(-\frac{2\pi\sqrt{-1}}{N^2}ij\right) \tag{8}$$

where

$$0 \le i \le N^2-1 \qquad 0 \le j \le N^2-1.$$

Let Λ be a diagonal matrix as

$$\Lambda = [\lambda(i,j)]$$

$$\lambda(i,j) = \begin{cases} 0, & i \ne j \\ g(i), & i = j \end{cases} \tag{9}$$

where $g(i)$ is defined in (6). Let A be defined as

$$A = F_{N^2}^{-1}\Lambda F_{N^2} \tag{10}$$

and

$$A = [a(i,j)]$$

$$a(i,j) = \sum_{k=0}^{N^2-1} \overline{f_{N^2}(i,k)}\lambda(k,k)f_{N^2}(k,j). \tag{11}$$

Now $b(i_1, i_1)$ is defined as

$$b(i_1,i_1) = \exp\left\{\frac{2\pi\sqrt{-1}}{N^2}c(i_1)\right\} \tag{12}$$

where $c(i_1)$ denotes the phase element for $b(i_1, i_1)$. Then the (i, j) element of A is

$$\begin{aligned} a(i,j) &= \sum_{k=0}^{N^2-1}\frac{1}{N}\left\{\exp\left(\frac{2\pi\sqrt{-1}}{N^2}ik\right)\right\}\cdot b(k_1,k_1) \\ &\quad\cdot\left\{\exp\left(\frac{2\pi\sqrt{-1}}{N^2}Nmk_0k_1\right)\right\} \\ &\quad\cdot\frac{1}{N}\left\{\exp\left(-\frac{2\pi\sqrt{-1}}{N^2}kj\right)\right\} \\ &= \frac{1}{N^2}\sum_{k=0}^{N^2-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k(i-j) \right. \\ &\quad \left. + Nmk_0k_1 + c(k_1))\right\} \\ &= \frac{1}{N^2}\sum_{k_1=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_1(i-j)+(c(k_1))\right\} \\ &\quad\cdot\sum_{k_0=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_0N(i-j)+Nmk_0k_1)\right\} \end{aligned} \tag{13}$$

where

$$k = k_0N + k_1$$

$$0 \le k_0 \le N-1$$

$$0 \le k_1 \le N-1.$$

Introducing $P(i, j)$,

$$a(i,j) = \frac{1}{N^2}\sum_{k_1=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_1(i-j) + c(k_1))\right\}P(i,j) \tag{14}$$

where

$$\begin{aligned} P(i,j) &= \sum_{k_0=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N}k_0(i-j+mk_1)\right\} \\ &= \begin{cases} 0, & i-j+mk_1 \ne 0 \pmod N \\ N, & i-j+mk_1 = 0 \pmod N. \end{cases} \end{aligned} \tag{15}$$

Since i and j can be regarded as residue classes of modulo N^2, $(j-i)$ can be described as

$$j - i = (j-i)_0N + (j-i)_1 \tag{16}$$

where

$$0 \le (j-i)_0 \le N-1 \qquad 0 \le (j-i)_1 \le N-1.$$

When $i - j + mk_1 = 0$ (modulo N), the condition under which k_1 can be decided uniquely is when m and N are relatively prime. When m and N are relatively prime,

$$\begin{aligned} a(i,j) &= \frac{1}{N}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(\frac{j-i}{m}(\mathrm{mod}\,N)\right.\right.\\ &\qquad \left.\left.\cdot(i-j)(\mathrm{mod}\,N^2)+c\left(\frac{j-i}{m}(\mathrm{mod}\,N)\right)\right)\right\}\\ &= \frac{1}{N}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(c\left(\frac{(j-i)_1}{m}\right)\right.\right.\\ &\qquad \left.\left.-\frac{N}{m}(j-i)_0(j-i)_1-\frac{1}{m}((j-i)_1)^2\right)\right\} \end{aligned} \tag{17}$$

where $c(\cdot)$ is defined in (12).

This means A is a cyclic matrix since $a(i,j)$ depends on only $(j-i)$(modulo N^2). Besides, A is a unitary matrix since

$$A = F_{N^2}^{-1}\Lambda F_{N^2}$$

as defined in (10). A cyclic matrix A represents a polyphase periodic sequence since the absolute values of all its elements $a(i,j)$ are equal. The periodic sequence represented by A is also an orthogonal sequence because A is a unitary matrix.

Introducing l, $c'(l_1)$, and m' as

$$l = j-i = l_0N + l_1 \tag{18}$$

$$c'(l_1) = \left\{c\left(\frac{(j-i)_1}{m}\right)-\frac{1}{m}(\mathrm{mod}\,N)((j-i)_1)^2\right\} \tag{19}$$

$$m' = -\frac{1}{m}(\mathrm{mod}\,N), \tag{20}$$

$a(i,j)$ can be written as

$$\begin{aligned} g'(l) &= a(i,j)\\ &= \frac{1}{N}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(c'(l_1)+Nm'l_0l_1)\right\}. \end{aligned} \tag{21}$$

In the notation in (6) and (12) in the previous section,

$$g(i) = \exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(c(i_1)+Nmi_0i_1)\right\}. \tag{22}$$

Equations (21) and (22) have the same structure. Therefore, the cyclic matrix corresponding to a periodic sequence obtained by repeating G is a cyclic orthogonal matrix. The periodic sequence represented by this matrix is an orthogonal sequence. Note that $g(i) = \lambda(i,i)$.

For example, when $N=3$, $m=1$, $b(0,0)=W_9^0$, $b(1,1)=W_9^1$, and $b(2,2)=W_9^3$, $G=(g(i))$ defined in (6) becomes

$$\begin{aligned} G &= (g(i))\\ &= (1, W_9^1, W_9^3, 1, W_9^4, 1, 1, W_9^7, W_9^6). \end{aligned}$$

Matrix A is calculated as follows:

$$A = F_9^{-1}\begin{bmatrix} 1 & & & & & & & & \\ & W_9^1 & & & & & & 0 & \\ & & W_9^3 & & & & & & \\ & & & 1 & & & & & \\ & & & & W_9^4 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & 0 & & & & & & W_9^7 & \\ & & & & & & & & W_9^6 \end{bmatrix} F_9$$

$$= \frac{1}{3}\begin{bmatrix} W_9^0 & W_9^0 & W_9^8 & W_9^0 & W_9^6 & W_9^2 & W_9^0 & W_9^3 & W_9^5\\ W_9^5 & W_9^0 & W_9^0 & W_9^8 & W_9^0 & W_9^6 & W_9^2 & W_9^0 & W_9^3\\ W_9^3 & W_9^5 & W_9^0 & W_9^0 & W_9^8 & W_9^0 & W_9^6 & W_9^2 & W_9^0\\ W_9^0 & W_9^3 & W_9^5 & W_9^0 & W_9^0 & W_9^8 & W_9^0 & W_9^6 & W_9^2\\ W_9^2 & W_9^0 & W_9^3 & W_9^5 & W_9^0 & W_9^0 & W_9^8 & W_9^0 & W_9^6\\ W_9^6 & W_9^2 & W_9^0 & W_9^3 & W_9^5 & W_9^0 & W_9^0 & W_9^8 & W_9^0\\ W_9^0 & W_9^6 & W_9^2 & W_9^0 & W_9^3 & W_9^5 & W_9^0 & W_9^0 & W_9^8\\ W_9^8 & W_9^0 & W_9^6 & W_9^2 & W_9^0 & W_9^3 & W_9^5 & W_9^0 & W_9^0\\ W_9^0 & W_9^8 & W_9^0 & W_9^6 & W_9^2 & W_9^0 & W_9^3 & W_9^5 & W_9^0 \end{bmatrix}.$$

According to (12), (18), (19), and (20), we obtain

$$\begin{aligned} &c(0) = 0\ (\mathrm{mod}\,9)\\ &c(1) = 1\ (\mathrm{mod}\,9)\\ &c(2) = 3\ (\mathrm{mod}\,9)\\ &\quad m' = -\frac{1}{m}\ (\mathrm{mod}\,N) = 2\\ &c'(0) = 0\\ &c'(1) = 0\\ &c'(2) = 8\\ &g'(l) = \frac{1}{3}\exp\left\{\frac{2\pi\sqrt{-1}}{9}(c'(l_1)+3\cdot 2l_0l_1)\right\}\\ &\quad G' = (g'(l))\\ &\qquad = (1,1,W_9^8,1,W_9^6,W_9^2,1,W_9^3,W_9^5). \end{aligned}$$

It is shown that $(1/3)G'$ equals the first row in the obtained cyclic matrix A.

C. Cross-Correlation Function between Modulated Orthogonal Sequences in Different Classes

This section discusses properties of the cross-correlation function between the modulated orthogonal sequences introduced previously. Two sequences, G_{I} and G_{II} of length N^2, are introduced by the modulation method already described. G_{I} and G_{II} are described as

$$G_{\mathrm{I}} = g_{\mathrm{I}}(i)$$

$$G_{\mathrm{II}} = g_{\mathrm{II}}(i)$$

$$g_{\mathrm{I}}(i) = b_{\mathrm{I}}(i_1, i_1)\exp\left(\frac{2\pi\sqrt{-1}}{N} m_{\mathrm{I}} i_0 i_1\right) \tag{23}$$

$$g_{\mathrm{II}}(i) = b_{\mathrm{II}}(i_1, i_1)\exp\left(\frac{2\pi\sqrt{-1}}{N} m_{\mathrm{II}} i_0 i_1\right). \tag{24}$$

Two diagonal matrices Λ_{I} and Λ_{II} are introduced as

$$\Lambda_{\mathrm{I}} = [\lambda_{\mathrm{I}}(i,j)]$$

$$\Lambda_{\mathrm{II}} = [\lambda_{\mathrm{II}}(i,j)]$$

$$\lambda_{\mathrm{I}}(i,j) = \begin{cases} 0, & i \neq j \\ g_{\mathrm{I}}(i), & i = j \end{cases} \tag{25}$$

$$\lambda_{\mathrm{II}}(i,j) = \begin{cases} 0, & i \neq j \\ g_{\mathrm{II}}(i), & i = j. \end{cases} \tag{26}$$

A_{I} and A_{II} are defined as

$$A_{\mathrm{I}} = F_{N^2}^{-1}\Lambda_{\mathrm{I}} F_{N^2} \tag{27}$$

$$A_{\mathrm{II}} = F_{N^2}^{-1}\Lambda_{\mathrm{II}} F_{N^2}. \tag{28}$$

A_{I} and A_{II} are cyclic orthogonal matrices described in the last section. Now $A_{\mathrm{I\,II}}$ is defined as

$$A_{\mathrm{I\,II}} = [a_{\mathrm{I\,II}}(i,j)]$$

$$= A_{\mathrm{I}} A_{\mathrm{II}}^{*}. \tag{29}$$

$A_{\mathrm{I\,II}}$ represents the cross-correlation function between two sequences respresented by A_{I} and A_{II}:

$$A_{\mathrm{I\,II}} = F_{N^2}^{-1}\Lambda_{\mathrm{I}} F_{N^2} F_{N^2}^{-1}\Lambda_{\mathrm{II}}^{*} F_{N^2}$$

$$= F_{N^2}^{-1}\Lambda_{\mathrm{I}}\Lambda_{\mathrm{II}}^{*} F_{N^2}. \tag{30}$$

Therefore,

$$a_{\mathrm{I\,II}}(i,j) = \sum_{k=1}^{N^2-1} \overline{f_{N^2}(i,k)}\lambda_{\mathrm{I}}(k,k)\overline{\lambda_{\mathrm{II}}(k,k)} f_{N^2}(k,j). \tag{31}$$

c_{I}, c_{II}, k_0, and k_1 are introduced as

$$b_{\mathrm{I}}(i_1, i_1) = \exp\left\{\frac{2\pi\sqrt{-1}}{N^2} c_{\mathrm{I}}(i_1)\right\} \tag{32}$$

$$b_{\mathrm{II}}(i_1, i_1) = \exp\left\{\frac{2\pi\sqrt{-1}}{N^2} c_{\mathrm{II}}(i_1)\right\} \tag{33}$$

$$k = k_0 N + k_1$$

$$0 \le k_0 \le N-1$$

$$0 \le k_1 \le N-1.$$

Then,

$$\begin{aligned} a_{\mathrm{I\,II}}(i,j) &= \frac{1}{N^2}\sum_{k=0}^{N^2-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k(i-j) \right. \\ &\quad \left. + Nk_0k_1(m_{\mathrm{I}} - m_{\mathrm{II}}) + c_{\mathrm{I}}(k_1) - c_{\mathrm{II}}(k_1))\right\} \\ &= \frac{1}{N^2}\sum_{k_1=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_1(i-j) \right. \\ &\quad \left. + c_{\mathrm{I}}(k_1) - c_{\mathrm{II}}(k_1))\right\} \\ &\quad \cdot \sum_{k_0=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_0N(i-j) \right. \\ &\quad \left. + Nk_0k_1(m_{\mathrm{I}} - m_{\mathrm{II}}))\right\}. \end{aligned} \tag{34}$$

$P_{\mathrm{I\,II}}(i,j)$ is introduced as

$$\begin{aligned} a_{\mathrm{I\,II}}(i,j) &= \frac{1}{N^2}\sum_{k_1=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}(k_1(i-j) \right. \\ &\quad \left. + c_{\mathrm{I}}(k_1) - c_{\mathrm{II}}(k_1))\right\}\cdot P_{\mathrm{I\,II}}(i,j). \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} P_{\mathrm{I\,II}}(i,j) &= \sum_{k_0=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N}k_0(i-j+k_1(m_{\mathrm{I}} - m_{\mathrm{II}}))\right\} \\ &= \begin{cases} 0, & i-j+k_1(m_{\mathrm{I}} - m_{\mathrm{II}}) \neq 0 \pmod N \\ N, & i-j+k_1(m_{\mathrm{I}} - m_{\mathrm{II}}) = 0 \pmod N. \end{cases} \end{aligned} \tag{36}$$

Similar to $a(i,j)$, when $(m_{\mathrm{I}} - m_{\mathrm{II}})$ and N are relatively prime with regard to each other,

$$\begin{aligned} a_{\mathrm{I,II}}(i,j) &= \frac{1}{N}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(\frac{j-i}{m_{\mathrm{I}} - m_{\mathrm{II}}} \pmod N\right)(i-j) \right. \\ &\quad \left. + c\left(\frac{j-i}{m_{\mathrm{I}} - m_{\mathrm{II}}} \pmod N\right)\right\}, \end{aligned} \tag{37}$$

where $c(\cdot) = c_{\mathrm{I}}(\cdot) - c_{\mathrm{II}}(\cdot)$. In this condition, $A_{\mathrm{I\,II}}$ is a cyclic unitary matrix, and the absolute values of all elements of $A_{\mathrm{I\,II}}$ are the same. Therefore, when $(m_{\mathrm{I}} - m_{\mathrm{II}})$ and N are relatively prime, the absolute value of the cross-correlation function between two periodic sequences, represented by A_{I} and A_{II}, is constant, and its peak is the lowest, as described in Section II.

For example, if $N = 3$, $m_{\mathrm{I}} = 1$, $m_{\mathrm{II}} = 2$, $b_{\mathrm{I}}(0,0) = W_9^0$, $b_{\mathrm{I}}(1,1) = W_9^1$, $b_{\mathrm{I}}(2,2) = W_9^3$, $b_{\mathrm{II}}(0,0) = W_9^1$, $b_{\mathrm{II}}(1,1) = W_9^3$, and $b_{\mathrm{II}}(2,2) = W_9^6$, then

$$G_{\mathrm{I}} = (1, W_9^1, W_9^3, 1, W_9^4, 1, 1, W_9^7, W_9^6)$$

$$G_{\mathrm{II}} = (W_9^1, W_9^3, W_9^6, W_9^1, 1, 1, W_9^1, W_9^6, W_9^3).$$

According to (27) and (28), two cyclic matrices A_{I} and A_{II}

are obtained. The first row in cyclic matrix A_{I} is

$$\frac{1}{3}\left(1,1,W_9^8,1,W_9^6,W_9^2,1,W_9^3,W_9^5\right),$$

and the first row in cyclic matrix A_{II} is

$$\frac{1}{3}\left(W_9^1,W_9^4,W_9^1,W_9^1,W_9^7,W_9^7,W_9^1,W_9^1,W_9^4\right).$$

$A_{\mathrm{I\,II}}=A_{\mathrm{I}}A_{\mathrm{II}}^*$ is obtained as another cyclic matrix. The first row in cyclic matrix $A_{\mathrm{I\,II}}$ is

$$\frac{1}{3}\left(W_9^8,W_9^4,W_9^5,W_9^8,W_9^7,W_9^2,W_9^8,W_9^1,W_9^8\right).$$

In this example,

$$c(0)=c_{\mathrm{I}}(0)-c_{\mathrm{II}}(0)\ (\mathrm{mod}\,9)=8$$

$$c(1)=c_{\mathrm{I}}(1)-c_{\mathrm{II}}(1)\ (\mathrm{mod}\,9)=7$$

$$c(2)=c_{\mathrm{I}}(2)-c_{\mathrm{II}}(2)\ (\mathrm{mod}\,9)=6.$$

According to (37), $a_{\mathrm{I\,II}}(i,j)$ is obtained as follows:

$j-i$	$\frac{j-i}{m_{\mathrm{I}}-m_{\mathrm{II}}}$ (modulo N)	$c\left(\frac{j-i}{m_{\mathrm{I}}-m_{\mathrm{II}}}\text{ (modulo } N)\right)$	$a_{\mathrm{I\,II}}(i,j)$
0	0	8	$\frac{1}{3}W_9^8$
1	2	6	$\frac{1}{3}W_4^4$
2	1	7	$\frac{1}{3}W_9^5$
3	0	8	$\frac{1}{3}W_9^8$
4	2	6	$\frac{1}{3}W_9^7$
5	1	7	$\frac{1}{3}W_9^2$
6	0	8	$\frac{1}{3}W_9^8$
7	2	6	$\frac{1}{3}W_9^1$
8	1	7	$\frac{1}{3}W_9^8$

It is shown that matrix $A_{\mathrm{I\,II}}$, obtained by product A_{I} and A_{II}^*, is composed of $a_{\mathrm{I\,II}}(i,j)$, obtained by (37).

D. Cross-Correlation Function between Modulated Orthogonal Sequences in the Same Class

Next, the cross-correlation function under another condition is discussed. When N is a prime number and

$$m_{\mathrm{I}}-m_{\mathrm{II}}=0\ (\mathrm{mod}\,N),$$

it follows that

$$i-j+k_1(m_{\mathrm{I}}-m_{\mathrm{II}})=i-j\ (\mathrm{mod}\,N).$$

Therefore,

$$P_{\mathrm{I\,II}}(i,j)=\begin{cases}0, & i-j\neq 0\ (\mathrm{mod}\,N)\\ N, & i-j=0\ (\mathrm{mod}\,N).\end{cases}\tag{38}$$

Accordingly, when

$$i-j\neq 0\ (\mathrm{mod}\,N)$$
$$a_{\mathrm{I\,II}}(i,j)=0$$

and when

$$i-j=0\ (\mathrm{mod}\,N)$$

$$a_{\mathrm{I\,II}}(i,j)=\frac{1}{N}\sum_{k_1=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N}\left(k_1\cdot\frac{i-j}{N}+\frac{1}{N}\left(c_{\mathrm{I}}(k_1)-c_{\mathrm{II}}(k_1)\right)\right)\right\}\tag{39}$$

when

$$i-j=0\ (\mathrm{mod}\,N),$$

$(i-j)/N$ can be regarded as a residue class of modulo N because $(i-j)$ can be regarded as a residue class of modulo N^2. Therefore, when

$$i-j=0\ (\mathrm{mod}\,N),$$

a residue class l is defined as

$$l=\frac{i-j}{N},\tag{40}$$

and a sequence $f_{\mathrm{I\,II}}(l)$ of length N is defined as

$$f_{\mathrm{I\,II}}(l)=a_{\mathrm{I\,II}}(i,j)\tag{41}$$

and

$$a_{\mathrm{I\,II}}(i,j)=\frac{1}{N}\sum_{k_1=0}^{N-1}\left\{\exp\left(\frac{2\pi\sqrt{-1}}{N}k_1 l\right)\right\}\cdot\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(c_{\mathrm{I}}(k_1)-c_{\mathrm{II}}(k_1)\right)\right\}.\tag{42}$$

This equality shows that

$$\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(c_{\mathrm{I}}(i_1)-c_{\mathrm{II}}(i_1)\right)\right\}=b_{\mathrm{I}}(i_1,i_1)\overline{b_{\mathrm{II}}(i_1,i_1)}$$

is transformed into $f_{\mathrm{I\,II}}(l)$ by N-dimensional inverse DFT. Therefore, when $(m_{\mathrm{I}}-m_{\mathrm{II}})$ and N are not relatively prime and N is a prime number, $A_{\mathrm{I\,II}}$ is a cyclic orthogonal matrix, and the elements of $A_{\mathrm{I\,II}}$ follow.

The elements of $i-j\neq 0$ (modulo N) are 0, and the elements of $i-j=0$ (modulo N), which are N elements in each row, are given from $b_{\mathrm{I}}(i_1,i_1)\overline{b_{\mathrm{II}}(i_1,i_1)}$ by N-dimensional inverse DFT. This cyclic orthogonal matrix $A_{\mathrm{I\,II}}$ represents the cross-correlation function between two sequences in the same class, which are represented by A_{I} and A_{II}.

E. Properties of Modulated Orthogonal Sequences Whose Period is a Second Power of a Prime Number

This section describes the properties of modulated orthogonal sequences, each of which has a period that is a second power of a prime number. These properties are introduced from the discussion in previous sections.

Let N be a prime number, and let m be a number relatively prime with regard to N. The number m is included in one of $(N-1)$ residue classes of modulo N. These $(N-1)$ residue classes characterize the matrices $\overline{F_{Nm}}$. When B is a unit matrix, a sequence composed of linked rows of $N\overline{F_{Nm}}B$ is an original (unmodulated) orthogonal sequence. Therefore, there are $(N-1)$ original orthogonal sequences to be modulated.

A modulated orthogonal sequence can be obtained by modulating one of these $(N-1)$ original orthogonal sequences. This modulation is achieved with the diagonal elements of B. The orthogonal sequences modulated from the same original orthogonal sequence are defined to compose a "class" of modulated orthogonal sequences. In other words, modulated orthogonal sequences of the same m compose a class. There are $(N-1)$ classes of modulated orthogonal sequences. These modulated orthogonal sequences have the following properties.

1) Each class includes an infinite number of modulated orthogonal sequences of period N^2.

2) The absolute value of the cross-correlation function between any two sequences which are included in different classes is constant. This absolute value satisfies the mathematical lower bound of the absolute value peak for cross-correlation function.

3) The cross-correlation function between any two sequences which are included in the same class is 0 for every term, except for N-multiple shift terms. The N-multiple shift terms are obtained by inverse DFT from the sequence composed of ratios between corresponding diagonal elements of each B. In other words, when the two sequences are modulated by

$$\{b_{\mathrm{I}}(0,0), b_{\mathrm{I}}(1,1),\cdots, b_{\mathrm{I}}(N-1,N-1)\}$$

and

$$\{b_{\mathrm{II}}(0,0), b_{\mathrm{II}}(1,1),\cdots, b_{\mathrm{II}}(N-1,N-1)\},$$

the N-multiple shift terms of cross-correlation function are obtained by inverse DFT from the sequence

$$\{b_{\mathrm{I}}(0,0)/b_{\mathrm{II}}(0,0), b_{\mathrm{I}}(1,1)/b_{\mathrm{II}}(1,1),\cdots, \cdot b_{\mathrm{I}}(N-1,N-1)/b_{\mathrm{II}}(N-1,N-1)\}.$$

IV. Application to SSMA Systems

A. Synchronous SSMA System Using the Modulatable Orthogonal Sequences

This section proposes a synchronous spread spectrum multiple access system without co-channel interferences, based on modulation of orthogonal sequences. Let N be a prime number. Now $(N-1)$ original orthogonal sequences, proposed previously, are assigned to $(N-1)$ channels. Each transmitter possesses a device to modulate the assigned original orthogonal sequence with diagonal elements of B. The diagonal elements of B, which are complex numbers of absolute value 1, carry the information to be communicated. Each transmitter also possesses another device to transmit the modulated orthogonal sequence. Each receiver possesses a filter matched to the assigned original orthogonal sequence. Each receiver also possesses an N-dimensional inverse DFT device. This inverse DFT device transforms the N-multiple elements of the matched filter's outputs into the diagonal elements of B.

Each receiver receives not only the signal transmitted by the corresponding transmitter, but also signals transmitted by other transmitters as well. Therefore, co-channel interferences should be discussed at this point.

At first it is assumed that the SSMA system is an asynchronous system. In this case, noncorresponding signals are added to the corresponding signal asynchronously.

Equation (37) represents the cross-correlation function between two periodic sequences represented by A_{I} and A_{II}. Now it is assumed that A_{I} represents the noncorresponding modulated sequence and that A_{II} represents the corresponding original sequence.

Let j be

$$j = j_0N + j_1, \qquad 0 \le j_0 \le N-1, \qquad 0 \le j_1 \le N-1.$$

Equation (37) is rewritten as follows:

$$a_{\mathrm{I\,II}}(i, j_0N + j_1) = \frac{1}{N}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\cdot\left(\frac{j_1 - i}{m_{\mathrm{I}} - m_{\mathrm{II}}}(\mathrm{mod}\, N)(i - j_0N - j_1) + c\left(\frac{j_1 - i}{m_{\mathrm{I}} - m_{\mathrm{II}}}(\mathrm{mod}\, N)\right)\right)\right\}. \tag{43}$$

In the ith row there are N elements of $j_1 = 0$. Let D_i be a sequence of length N whose elements are the $j_1 = 0$ elements for the ith row in $A_{\mathrm{I\,II}}$. Therefore,

$$D_i = \{d_i(j_0)\}$$
$$d_i(j_0) = a_{\mathrm{I\,II}}(i, j_0N). \tag{44}$$

This means that when a signal represented by A_{I} is input to the filter matched for A_{II} at sync-shift i, the sampled output is D_i. The sampled output should be transformed by the inverse DFT device as follows:

$$F_N^{-1}D_i(k) = \frac{1}{N\sqrt{N}}\exp\left\{\frac{2\pi\sqrt{-1}}{N^2}\left(\frac{-i}{m_{\mathrm{I}} - m_{\mathrm{II}}}(\mathrm{mod}\, N)i + c\left(\frac{-i}{m_{\mathrm{I}} - m_{\mathrm{II}}}(\mathrm{mod}\, N)\right)\right)\right\} \cdot \sum_{j_0=0}^{N-1}\exp\left\{\frac{2\pi\sqrt{-1}}{N}\left(j_0k + \frac{j_0 i}{m_{\mathrm{I}} - m_{\mathrm{II}}}\right)\right\}. \tag{45}$$

Since N is prime, when

$$k \neq -\frac{i}{m_{\mathrm{I}} - m_{\mathrm{II}}} \pmod N,$$

$$F_N^{-1} D_i(k) = 0, \tag{46}$$

and when

$$k = -\frac{i}{m_{\mathrm{I}} - m_{\mathrm{II}}} \pmod N,$$

$$F_N^{-1} D_i(k) = \frac{1}{\sqrt{N}} \exp\left\{ \frac{2\pi\sqrt{-1}}{N^2} \left(k_i + c(k)\right) \right\}. \tag{47}$$

This means that after the inverse DFT, co-channel interferences gather into one element of

$$k = -\frac{i}{m_{\mathrm{I}} - m_{\mathrm{II}}} \pmod N.$$

Now it is assumed that all channels are synchronous, as $i = 0$ for any combination of m_{I} and m_{II}. In this assumption, all co-channel interferences from any other channels gather into the first element in the output of the inverse DFT device. If the transmitter uses the $(N-1)$ diagonal element, except for the first diagonal element in the diagonal matrix B, to carry information, the information can be carried without co-channel interferences. This is because co-channel interferences gather into the first element; it does not carry information.

In this synchronous SSMA system, each of $(N-1)$ modulatable orthogonal sequences of period N^2 carries $(N-1)$ polyphase data. The $(N-1)/N$ of noise energy is banished at the sampling process. On the other hand, all of the signal energy remains in the demodulated signal. If the noise property is not white Gaussian (for example, pulse noise), the proposed system works more stably than the usual synchronous SSMA systems because many noise terms are summed up into pseudo-Gaussian noise.

B. Asynchronous SSMA System Using Unmodulated Orthogonal Sequences

Letting N be prime, we can obtain $(N-1)$ unmodulated orthogonal sequences of period N^2. The peak absolute value of the cross-correlation function between any two in these sequences is N, which equals the mathematical lower bound. These $(N-1)$ sequences can be used for an asynchronous SSMA system. This system is with lowest co-channel interferences, as an asynchronous SSMA system, based on orthogonal sequences.

C. Signal Generator and Matched Filter

This section presents the signal generator for the modulated orthogonal sequence and the filter matched to the original orthogonal sequence. Fig. 1 shows the signal generator for the modulated orthogonal sequence, where $b_0, b_1, \cdots, b_{N-1}$ represent $b(0,0)$, $b(1,1), \cdots, b(N-1, N-1)$. Multipliers $b_0, b_1, \cdots, b_{N-1}$ are variable and modulate the output sequence. The input signal is made of repeated inpulses. Each interval between two impulses is $N^2\tau$. Fig. 2 shows the filter matched to the original orthogonal sequence.

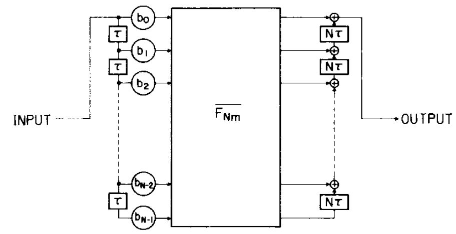

Fig. 1. Signal generator for modulated orthogonal sequence of period N^2.

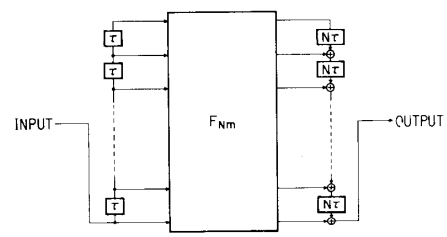

Fig. 2. Filter matched to original orthogonal sequence.

V. Conclusion

We have proposed a family of polyphase orthogonal sequences which remain orthogonal when modulated. They also have the property that the absolute value of their cross-correlation function is constant.

The orthogonal sequences were applied in two categories of SSMA systems. One category includes a synchronous SSMA system without co-channel interferences which works more stably than a usual synchronous SSMA system. The other is an asynchronous SSMA system with low co-channel interferences that uses unmodulated orthogonal sequences.

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