Distributed Synchronization on Weakly Connected Networks

Stojan Denic, Orestis Georgiou, and Umberto Spagnolini

***Abstract*— Distributed synchronization for wireless networks is based on the mutual exchange of the same chirp-signature by nodes. Collisions of these signatures drive the system toward time and (carrier) frequency synchronization using distributed consensus algorithms. This letter investigates the convergence and the asymptotic distortion properties on noisy networks when neighboring clusters of nodes are weakly connected to each other only through a subset of nodes and bridging links. These heavily connected clusters act as macroagents, and the consensus properties of the ensemble depend on the number of bridge links between them. The convergence rate and mean square synchronization deviation are derived as functions of the number of bridge links for different examples of weakly connected noisy networks via the analytic calculation of Laplacian spectra. Our approach facilitates the study of network topology optimization for distributed synchronization.**

***Index Terms*— Algebraic connectivity, consensus methods, distributed inference, distributed synchronization, Laplacian spectrum.**

I. INTRODUCTION

DISTRIBUTED consensus algorithms (DCA) is the key analytical tool to evaluate the properties of the distributed (time) synchronization based on the exchange of pulses [1]. On the same line, chirp-signatures routinely used in radar-systems can be adapted in distributed synchronization to let each node to estimate (carrier) frequency and time error of the synchronization reference of each node with respect to an ensemble of the others. The collision of these signatures originated from nodes in transmission mode, enables the receiving nodes to correct their carrier frequency offset (CFO) and time offset (TO), and convergence to the consensus of CFO and TO. Even if experimentally validated for a single cluster [2], the synchronization properties when two (or more) clusters have few inter-clusters links is still open. In this letter, we consider DCAs where two clusters of nodes are weakly connected via a subset of nodes in each cluster (i.e., through bridge links) subject to additive zero-mean noise from local oscillators' jitter. Fig. 1. shows a reference scenario where two mobile clusters of nodes (e.g., platoons of vehicles), attempt to synchronise over $K \geq 1$ bridging inter-cluster links.

Interest for DCAs within the wireless communication community is growing for replacing centralised approaches related to, *inter alia*, network calibration and localisation [3], spectrum sensing and resource scheduling [4], distributed inference and signal processing [5]. The DCAs have improved robustness to link failures, network scalability, and enable the use of energy efficient and simpler (and less expensive) nodes.

Manuscript received January 10, 2017; revised March 14, 2017; accepted March 26, 2017. Date of publication March 31, 2017; date of current version July 8, 2017. The associate editor coordinating the review of this letter and approving it for publication was D. Darsena. *(Corresponding author: Umberto Spagnolini.)*

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Digital Object Identifier 10.1109/LCOMM.2017.2690284

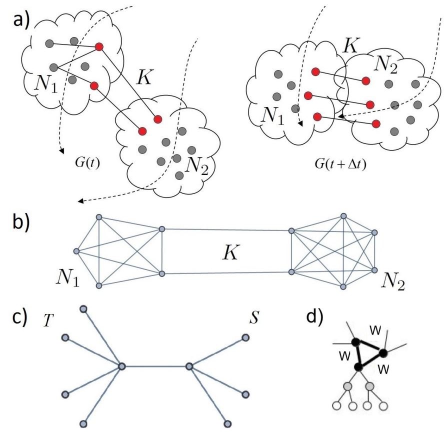

Fig. 1. a) A time dependent graph $G(t)$ consisting of two weakly connected clusters of N_1 and N_2 nodes. Intra-cluster links remain constant, whilst K inter-cluster bridging links can change. Cluster heads are shown in red. b) Two weakly connected complete graphs with $K = 2$ bridging links. c) Two weakly connected star graphs with $T = N_1 - 1 = 4$ and $S = N_2 - 1 = 3$. d) Three weakly connected 2-level binary tree graphs. Only one tree is shown.

Some of the problems in the design of DCAs for wireless networks stem from the stochastic nature of connectivity, and the impact of the additive noise present locally in every node onto the ensemble. Making reference to the influence of CFO/TO jitter onto the network synchronization, the DCA with an additive zero-mean noise has been originally addressed in [6], and recently generalised for a non-zero mean noise in [7]. While the general DCA is well understood, it is desirable to investigate specific examples in search for further insights that globally investigates CFO/TO convergence. As with the general setting, two metrics can be used to quantify the performance of DCAs: i) the convergence rate λ_2 (i.e., the algebraic connectivity of a graph [8]) which was recently studied analytically in [9], and ii) the steady-state mean square deviation (MSD) of the node synchronization δ_{ss} which accounts for the residual fluctuation of the network that can be tolerated. Significantly, the convergence rate and the MSD strongly depend on the network topology which can be described by the graph Laplacian matrix and its spectrum.

Motivated by the above, and focusing on weakly-clustered networks (as in Fig. 1.), we: i) explicitly compute for the first time the spectra of weakly connected star and complete graphs, as a function of number of bridge links K between the two clusters, for any number of nodes N_1 and N_2 in each of the clusters; ii) describe a method for calculating the spectrum of multi-clustered graphs; and iii) define optimisation problems in terms of the MSD and the number of bridge links, which can be easily solved via the derived expressions.

II. DCA ON WEAKLY CONNECTED NETWORKS

Consider a wireless communications network shown in Fig. 1b), consisting of two clusters which have N_1 and N_2 nodes, respectively. Between the two clusters, $K \leq \min(N_1, N_2)$ bridge links can be established. The nodes synchronize with their neighbours within the clusters and over the K bridges. Here, we have abstracted the specifics of the wireless devices and communication protocol, and focus on the underlying network graph topology represented by an undirected graph having $N_T = N_1 + N_2$ nodes and K bridges $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N_T\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The adjacency matrix $\mathbf{A}$ describes the connectivity of $\mathcal{G}$ with entries a_{ij} equal to 1 if $(i, j) \in \mathcal{E}$, and 0 otherwise. From $\mathbf{A}$, we can obtain the degree matrix $\mathbf{D}$, a diagonal $N_T \times N_T$ matrix with entries d_{ii} equal to the degree of node i. The Laplacian matrix is then given by $\mathbf{L}_K = \mathbf{D} - \mathbf{A}$.

Further, each node $i \in \mathcal{V}$ of $\mathcal{G}$ is associated to a local state variable $x_i \in \mathbb{R}$ that represents either the CFO or the TO as the analysis is general for both. A DCA is an iterative process which instructs a node to update x_i according to a rule which depends on the states of its neighbours as well as on its own. The update for the node $i \in \mathcal{V}$ can thus be defined as

$$x_i(n+1) = x_i(n) + \alpha \sum_{j=1}^{N_T} a_{ij}(x_j(n) - x_i(n)) + v_i(n) \qquad (1)$$

where $\alpha \ll 1$ is a constant, $n \in \mathbb{N}^+$, and the $\{v_i(n)\}$ are independent and identically distributed random variables, with zero mean and variance σ^2, representing the additive CFO or TO noise due to the local oscillators. All N_T updating rules given by (1) can be expressed in a compact matrix form

$$\mathbf{x}(n+1) = \mathbf{P}_K \mathbf{x}(n) + \mathbf{v}(n) \qquad (2)$$

where $\mathbf{P}_K = \mathbf{I} - \alpha \mathbf{L}_K$ is the Perron matrix, $\mathbf{x}(n) = [x_1(n), x_2(n) \ldots, x_{N_T}(n)]^{\mathrm{T}}$ is the state vector, and $\mathbf{v}(n) = [v_1(n), v_2(n) \ldots, v_{N_T}(n)]^{\mathrm{T}}$ is the noise vector.

A consensus problem, expressed by a function $\chi : \mathbb{R}^{N_T} \to \mathbb{R}$, is a mapping from all initial states $\mathbf{x}(0)$ to $\chi(\mathbf{x}(0)) \in \mathbb{R}$ [10]. The unique solution of the consensus problem exists if and only if (2) converges to an asymptotically stable equilibrium $\mathbf{x}(n) \to \mathbf{x}^* = x^* \mathbf{1}$ as $n \to \infty$, where $\mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{N_T}$, and $x^* = \sum_{i=1}^{N_T} x_i^* / N_T$ recalls that the synchronization is reached to the average from all the nodes. In the presence of noise, there is no unique solution x^*, but rather DCA convergence towards the current mean of all nodes.

In a multicast-based scenario, all nodes are synchronised in time via an external refernce signal (e.g., GPS or LTE) and each node transmits at regular time instants in order to achieve carrier frequency synchronization [10]. Here, the $\mathbf{L}_K$ and $\mathbf{P}_K$ matrices are symmetric ($\mathbf{P}_K^T = \mathbf{P}_K$), and $\mathbf{P}_K \mathbf{1} = \mathbf{1}$, meaning that $\mathbf{1} = [1, 1, \ldots, 1]^T$, is an eigenvector with eigenvalue 1. The eigenvalues of $\mathbf{L}_K$ ($\{\lambda_i : 1 \le i \le N_T\}$) and $\mathbf{P}_K$ ($\{\mu_i : 1 \le i \le N_T\}$) can be ordered in an increasing order, $0 = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{N_T}$, and $\mu_1 \le \mu_2 \le \ldots \le \mu_{N_T} = 1$, respectively. Note that the eigenvalues of $\mathbf{P}_K$ and $\mathbf{L}_K$ are related through $\mu_{N_T+1-i} = 1 - \alpha \lambda_i$. The performance of DCAs can be characterised by these eigenvalues, as discussed below.

1) No Noise ($\mathbf{v}(n) = 0$): When there is no noise, the consensus value $x^* = \mathbf{w}^T \mathbf{x}(0)$, where $\mathbf{w}$ is given by the dominant left eigenvector of $\mathbf{P}_K$, normalised with respect to the 1-norm, i.e., $\| \mathbf{w} \|_1 = 1$. The convergence towards x^* is exponential with n, with a rate proportional to λ_2. To see this, one can define a difference vector $\Delta(n) = \mathbf{x}(n) - \mathbf{x}^*$ at iteration n such that $\| \Delta(n) \| = \| \mathbf{P}_K^n \Delta(0) \| \le \mu_{N_T-1}^n(K) \| \Delta(0) \| \approx e^{-n\alpha\lambda_2(K)} \| \Delta(0) \|$. Hence, the second smallest eigenvalue $\lambda_2(K)$ of $\mathbf{L}_K$, also referred to as the algebraic connectivity [8], controls the convergence rate of the DCA, under certain convergence conditions, e.g., that $0 \le \alpha < \frac{2}{\lambda_{N_T}}$ [10]. Note that the eigenvalues λ_i of $\mathbf{L}_K$ are functions of K, emphasising their dependence on the number of bridge links.

2) Zero-Mean Noise ($\mathbf{v}(n) \neq 0$): Additive noise induces a zero-mean error in the average of node states defined by $\mathbf{1}^{\mathrm{T}} \mathbf{x}(n) / N_T$, which has a variance that increases linearly with n [6]. Thus, there is no unique solution x^*. Instead, convergence is observed in the mean of the DCA (i.e., mean distance between the current state $\mathbf{x}(n)$ and the current mean $\mathbf{1}^{\mathrm{T}} \mathbf{x}(n) / N_T$) motivating the study of the MSD, defined by $\delta(n) = \mathbb{E}\Big[\sum_{i=1}^{N_T} \Big(x_i(n) - \mathbf{1}^{\mathrm{T}} \mathbf{x}(n) / N_T \Big)^2 \Big]$, where the expectation $\mathbb{E}[\cdot]$ is taken over $\mathbf{v}$. In [6], it was proven that $\delta(n)$ converges to a finite steady state value

$$\delta_{ss} = \lim_{n \to \infty} \delta(n) = \sum_{i=2}^{N_T} \frac{\sigma^2}{1 - (1 - \alpha \lambda_i(K))^2} \qquad (3)$$

which explicitly depends on the spectrum of the Laplacian. Note that if $\sigma = 0$ there is no noise nor MSD and hence the DCA converges to the unique solution x^*.

The above framework follows through in a broadcast-based consensus algorithm as well but with $\alpha \to \alpha / N_T$, making the convergence towards consensus slower (in the noiseless case) and the MSD larger (in the zero-mean noisy case).

III. Connectivity of Weakly Connected Graphs

In the following sections, the spectra of several different weakly connected networks will be explicitly computed, which will enable us to analyse and further understand the MSD and the rate of convergence of DCAs as a function of K.

A. Symmetric Complete Graphs

It was shown in [9] that two clusters of communicating nodes can be modelled by two complete graphs, each with $N = N_T/2 \ge 2$ vertices, linked together through $K \in \{1, \ldots, N\}$ independent bridging links (see Fig. 1. b)), i.e., at most one bridging link per node. The spectrum of corresponding $N_T \times N_T$ block Laplacian matrix $\mathbf{L}_K = \begin{pmatrix} \mathbf{L}_\star + \mathbf{J}^K & -\mathbf{J}^K \\ -\mathbf{J}^K & \mathbf{L}_\star + \mathbf{J}^K \end{pmatrix}$ was computed via the identity: $\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \equiv \det(\mathbf{A} - \mathbf{B}) \det(\mathbf{A} + \mathbf{B})$, where $\mathbf{L}_\star = N \mathbf{I}_N - \mathbf{1}\mathbf{1}^{\mathrm{T}}$ is the Laplacian matrix of a complete graph, $\mathbf{I}_N$ is the identity matrix of size N, and $\mathbf{J}^K$ is an $N \times N$ matrix with K ones along a main diagonal corresponding to the K bridging links. Thus, the characteristic polynomial $p_{\mathbf{L}_K}(\lambda)$ can be explicitly computed yielding

$$\begin{aligned} p_{\mathbf{L}_K}(\lambda) = &(-1)^N (\lambda - N - 2)^{K-1} (\lambda - N)^{N-K-1} \\ &\times (2K + \lambda(\lambda - N - 2))(-1)^N (\lambda - N)^{N-1} \lambda \end{aligned} \qquad (4)$$

The eigenvalues follow from the characteristic polynomial of $p_{\mathbf{L}_K}(\lambda)$: $\{0, N, N+2, \frac{1}{2}\Big(N + 2 \pm \sqrt{4 - 8K + 4N + N^2}\Big)\}$, with multiplicities $\{1, 2N - K - 2, K - 1, 1, 1\}$, respectively. Note that $\lambda_2 = \frac{1}{2}\Big(N + 2 - \sqrt{4 - 8K + 4N + N^2}\Big) = \frac{2K}{N} - O(N^{-2})$ where we have Taylor expanded for large N indicating that $\lambda_2 \to 0$ and $\delta_{ss} \to \infty$ as $N \to \infty$, as expected.

B. Asymmetric Star Graphs

The approach developed in [9] can be extended to the asymmetric case with $N_1 \neq N_2$. To illustrate the computation, a simple case of two star graphs with unequal number of nodes, connected by a single link is considered first (see Fig. 1. c)). Assume that the number of leaf nodes in the first star is $S = N_1 - 1$ and in the second $T = N_2 - 1$ and that $1 \leq S \leq T$, such that $N_T = S + T + 2$. The Laplacian matrix can be represented by four blocks, $\mathbf{L}_1 = \begin{pmatrix} \mathbf{L}_{\star,T} & -\mathbf{J}^1 \\ (-\mathbf{J}^1)^{\mathrm{T}} & \mathbf{L}_{\star,S} \end{pmatrix}$ where $\mathbf{J}^1$ is an $(T+1)\times(S+1)$ matrix, with all zero entries except for its top leftmost entry which is equal to 1, and where $\mathbf{L}_{\star,X}$ is an $(X+1)\times(X+1)$ symmetric matrix with $X \in \{S, T\}$ composed of ones in its first row and first column except at the diagonal entry, and zeros elsewhere. Because the blocks in $\mathbf{L}_1$ have different dimensions, to obtain the characteristic polynomial $p_{\mathbf{L}_1}(\lambda) = \det(\mathbf{L}_1 - \lambda \mathbf{I}_{N_T})$, we exploit the identity: $\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} \end{pmatrix} \equiv \det(\mathbf{A}) \det(\mathbf{D} - \mathbf{B}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{B})$. Here, $\mathbf{A} = \mathbf{L}_{\star,T} - \lambda \mathbf{I}_{T+1}$, $\mathbf{B} = -\mathbf{J}^1$ and $\mathbf{D} = \mathbf{L}_{\star,S} - \lambda \mathbf{I}_{S+1}$. The matrices $\mathbf{A}$ and $\mathbf{D}$ are square, of different dimensions, and $\mathbf{A}$ has to be invertible. It can be shown that $\det(\mathbf{A}) = (1-\lambda)^{T-1}(\lambda^2 - \lambda(T+2) + 1)$. The problem of calculating $p_{\mathbf{L}_1}(\lambda)$ is further mitigated by realising that $\mathbf{B}$ has only one entry different from zero. Therefore, we need only derive the $[\mathbf{A}^{-1}]_{11}$ entry of $\mathbf{A}^{-1}$ which is equal to $(1-\lambda)^T / \det \mathbf{A}$. Hence, $\det(\mathbf{D} - \mathbf{B}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{B})$ assumes the following form $(1-\lambda)^S \phi(\lambda) - S(1-\lambda)^{S-1}$ where we have defined $\phi(\lambda) = S + 1 - \lambda - \frac{(1-\lambda)^T}{\det \mathbf{A}}$. Therefore, we arrive at the characteristic polynomial $\det(\mathbf{L}_1 - \lambda \mathbf{I}_{2N_T}) = \lambda(1-\lambda)^{N_T - 4}(\lambda^3 - (N_T+2)\lambda^2 + (2N_T + ST + 1)\lambda - N_T)$. This result is known in the literature [11], however the proof is different from the one provided here, which lends itself to the natural generalisation involving arbitrary K bridging links.

C. Asymmetric Weighted Star Graphs

Weighted graphs allow adjacency matrix to have non-integer (non-negative) entries. The weights therefore manifest themselves in the form of a weighted average in (1). In the case that the weight is a positive integer greater than 1, the weighted graph can be interpreted as a multi-graph, i.e., permitting for more than one edge between two connected nodes.

We build on the previous results by considering an asymmetric star graph as in Fig. 1. c) but with the bridging link having a weight $W > 0$. The approach is almost identical as in Sec. III-B, i.e., treating the Laplacian matrix as a block matrix, and evoking the relevant determinant identity to calculate $\det(\mathbf{A}) = (1-\lambda)^{T-1}(\lambda^2 - \lambda(T+W+1) + W)$, while $\det(\mathbf{D} - \mathbf{B}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{B})$ is of the same form as before but with $\phi(\lambda) = S + W - \lambda - \frac{(1-\lambda)^T}{\det \mathbf{A}} W^2$. Note that for $S = T$, we have $\lambda_2 = \frac{1}{4}\left(N + 4W - \sqrt{16W(W-2) + 8NW + N^2}\right)$ which is identical to symmetric complete graphs when $W = K = 1$.

D. Asymmetric Complete Graphs

Using a similar approach as in Section III-B and induction, for asymmetric complete graphs connected by $K \in [1, \min(N_1, N2)]$ links (see Fig. 1. b)), we derive that $\det(\mathbf{A}) = -(\lambda - (N_1+1))^{K-1}(\lambda - N_1)^{N_1 - K - 1}(K - (N_1+1)\lambda + \lambda^2)$ while $\det(\mathbf{D} - \mathbf{B}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{B})$ is shown at the top of the next page (5). Multiplying $\det(\mathbf{A})$ by (5) gives the characteristic polynomial of weakly connected asymmetric complete graphs which can be solved using numerical methods.

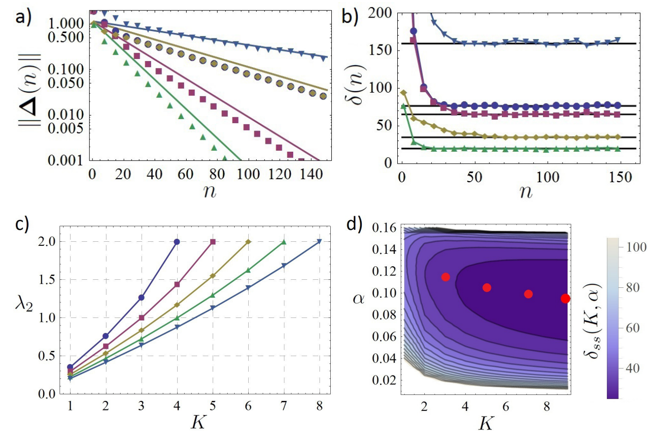

Fig. 2. a) Noiseless ($\sigma^2 = 0$) convergence to steady state x^* as a function of n. b) Mean square displacement of the noisy ($\sigma^2 = 1$) DCA as a function of n. Straight lines correspond to $e^{-n\alpha\lambda_2(K)}$, and δ_{ss}, respectively. Markers are obtained through numerical Monte Carlo simulations. In all cases $\alpha = 0.1$, $N_1 = 6$, and $N_2 = 8$. The blue circles and purple squares are for weakly connected star graphs with $K = 1$ bridging link of weight $W = 1$ and $W = 3$, respectively. The yellow diamonds and green triangles are for weakly connected complete graphs with $K = 1$ and $K = 3$ bridging links, respectively. Finally the light blue inverted triangles are for the broadcast based consensus of weakly connected complete graphs with $K = N_1$ bridging links. c) Plot of the convergence rate λ_2 as a function of the bridging links K for $N_1 = N_2 = 4, \ldots 8$, from left to right. d) Contour plot of the asymptotic MSD δ_{ss} as a function of K and $\alpha < \frac{2}{\lambda_{N_T}}$. The red dots highlight the minimum values of δ_{ss} at different K using $N_1 = 8$ and $N_2 = 11$.

E. Multi-Cluster Graphs

The two-cluster setup can be generalized to the case where m clusters of nodes are linked together via inter-cluster bridging links. For simplicity we assume that all clusters are identical and each cluster has a unique cluster-head node. The adjacency matrix of this type of networks is by the Kronecker product notation $\mathbf{A} = \mathbf{I}_m \otimes \mathbf{A}_- + \mathbf{A}_c \otimes \mathbf{J}^1$, where $\mathbf{A}_-$ is the adjacency matrix of each cluster, and $\mathbf{A}_c$ is the $m \times m$ inter-cluster weighted connectivity matrix. For example this could correspond to chains, stars, or rings of cluster-heads. The Laplacian is therefore easily constructed. By means of an example we consider three clusters, each cluster consisting of a binary tree with two levels (i.e, 7 nodes per cluster) as shown in Fig. 1. d). The three cluster-heads are connected to each other via weighted links with a weight W such that the spectrum is given by the determinant of $\mathbf{L}_3 - \lambda \mathbf{I} = \mathbf{I}_3 \otimes \begin{pmatrix} \mathbf{R} & \mathbf{Y} \\ \mathbf{Y}^{\mathrm{T}} & \mathbf{Q} \end{pmatrix} + (\mathbf{J}_3 - \mathbf{I}_3) \otimes \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where $\mathbf{L}_3$ is the 3-cluster Laplacian matrix, $\mathbf{J}_3$ is a 3×3 all-ones matrix, $\mathbf{R}$ captures the connectivity of a cluster-head and its 2 child nodes, $\mathbf{Q}$ and $\mathbf{Y}$ the leaf node connectivity, and $\mathbf{K}$ the inter-cluster connectivity ($\mathbf{K}$ has all entries equal to zero, except for $[\mathbf{K}]_{11} = W$). The determinant of the leading block principal minor of block order 2 of the previous determinant, $\mathbf{C}_2$, consisting of $\mathbf{R}$, $\mathbf{Y}$, $\mathbf{Y}^{\mathrm{T}}$ and $\mathbf{Q}$ can be computed using the block matrix determinant (BMD) identity where the inverse of $\mathbf{Q}$ is substituted in. To compute a determinant of a leading block principal minor of order 3, $\mathbf{C}_3$ (consisting of $\mathbf{C}_2$, $[\mathbf{K}\ \mathbf{0}]$, $[\mathbf{K}\ \mathbf{0}]^{\mathrm{T}}$ and $\mathbf{R}$) requires the inverse of $\mathbf{R}$. To make progress, from BMD formula we notice that this is similar to calculating the determinant of $\mathbf{C}_2$ with a correction factor $[\mathbf{K}\ \mathbf{0}]\mathbf{R}^{-1}[\mathbf{K}\ \mathbf{0}]^{\mathrm{T}}$.

$$\det(\mathbf{D}-\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}) = (-1)^{N_1-N_2+1}\frac{(-N_2+\lambda)^{N_2-K-1}\lambda[(N_1+1)(N_2+1)-1-(N_1+N_2+2)\lambda+\lambda^2]^{K-1}}{(-(N_1+1)+\lambda)^{K-1}(K-(N_1+1)\lambda+\lambda^2)} \times[-(N_1+N_2)K+((N_1+1)(N_2+1)+2K-1)\lambda-(N_1+N_2+2)\lambda^2+\lambda^3] \quad (5)$$

Thus, to compute the final determinant one needs to iterate and calculate the determinants of $\mathbf{C}_k$ which is analogue to finding the determinant of $\mathbf{C}_{k-1}$, including a correction factor which depends on an inverse matrix, either $\mathbf{R}$ or $\mathbf{Q}$. We have therefore described an iterative process of calculating the determinant of a multi-clustered graph. When, the network structure is irregular, one can employ a general method for determinant computation [12].

F. Numerical Simulations

We perform numerical simulations for the weakly connecetd graphs consisting of two clusters. Fig. 2 shows numerical simulations of the convergence to consensus for the noiseless case, and also the MSD for the zero-mean noise case. The analytical results derived using the exact spectra of $\mathbf{L}_K$ are compared to the simulations and show very good agreement.

IV. Consensus With Optimal Bridging

Fig. 2. c) shows the dependence of the DCA convergence rate λ_2 on $N_T = N_1 + N_2$ and K for symmetric ($N_1 = N_2$) complete graphs as studied in Sec. III. It is clear that λ_2 increases monotonically with K, and decreases monotonically with N_T. Fig. 2. d) shows the dependence of the asymptotic MSD δ_{ss} on K and α. It is seen that δ_{ss} is a convex function of $\alpha < \frac{2}{\lambda_{N_T}}$ [6] and a decreasing function of $K \leq \min(N_1, N_2)$. This is mostly because λ_2 (the dominant contribution to δ_{ss}) always increases with K. The minimum of δ_{ss} is therefore at $K = \min(N_1, N_2)$ and typically around $\alpha \approx \frac{1}{\lambda_{N_T}}$, although this may vary. The fact that both, the convergence rate and the asymptotic MSD depend on K, enables us to introduce several optimisation problems:

1) Least-Mean-Square Synchronization With Jitter: Find the number of bridging links K and step-size α that yield the smallest asymptotic MSD

$$(K^*, \alpha^*) = \underset{(K,\alpha)}{\arg\min}\ \delta_{ss}(K, \alpha) \quad (6)$$

Note that from Fig. 2. d) it is clear that $K^* = \min(N_1, N_2)$, i.e., stronger inter-cluster connections always improve MSD.

2) Resource Constrained Synchronization: Similar to the above problem, but also requiring that $0 \leq g(K) \leq g_{\max}$, where $g(K)$ is an increasing cost function for using K bridge links and $g_{\max} > 0$ a maximum allowed cost. To exemplify, the $g(K)$ could be the bandwidth B for inter-cluster synchronization, e.g., $g(K) = KB$, or the time-slots allocated to the synchronization. Thus, an optimal solution determines an overall bandwidth required for some given algorithm operation.

3) Performance Targeted: Find the number of bridging links K and step-size α that minimise cost

$$(K^*, \alpha^*) = \underset{(K,\alpha)}{\arg\min}\ g(K) \text{ s.t. } \quad \delta_{ss}(K) \leq \delta_{\max} \quad (7)$$

where $\delta_{\max}$ represent a maximum allowed MSD.

4) Coupled Problem: In resource constrained problems it is likely that the step-size $\alpha = f_1(K)$ such that the optimisation problem is over a single integer variable $K^* = \underset{K}{\arg\min}\ \delta_{ss}(K, f_1(K))$ or similarly, the noise variance $\sigma^2 = f_2(K)$ such that $\delta_{ss}(K, \alpha)$ becomes a convex function

$$(K^*, \alpha^*) = \underset{(K,\alpha)}{\arg\min}\ \sum_{i=2}^{N_T} \frac{f_2(K)}{1-(1-\alpha\lambda_i(K))^2} \quad (8)$$

5) Multi-Cluster: More interesting topology optimisation problems can be considered in the multi-clustered case if resource constraints restrict the form of $\mathbf{A}_c$. For example, the optimal α and $\mathbf{A}_c$ can be set with $\|\mathbf{A}_c\|_1 < f(m)$.

V. Conclusion and Discussion

This letter considered DCAs for distributed CFO/TO synchronization in weakly connected wireless networks with a stochastic perturbation representing the jitter of each individual oscillator. Differently from extensive numerical solutions, we analytically derive the convergence rate λ_2 and the MSD δ_{ss} for specific examples of balanced and unbalanced complete and star networks, which are functions of network topology, here characterised by the number of inter-cluster bridging links K. Our *meso-scale* approach of treating clusters as blocks in the Laplacian matrix of the network, facilitates the analytic calculation of its spectrum, and in turn the explicit dependence of these two measures on K. Inter-cluster topology optimisation was also addressed subject to various resource constraints.

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