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THE FOURIER TRANSFORM OF RECTANGULAR FUNCTION

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INTRODUCTION

Fourier analysis, when applied to a continuous periodic signal in the time domain, yields a series of discrete frequency components in the frequency domain. However, fourier transform is relevant to power quality.

By allowing the integration period T to extend to infinity, the spacing between, the harmonic frequencies tends to zero and the function X(f_n) of equation (1.5.4) becomes a continuous and infinite function of frequency such that

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{1.6.1}$$

The expression for the time domain function x(t), which is also continuous and infinite, in terms of $X(f)$ is then

$$x(t) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}df \tag{1.6.2}$$

of equation (1.5.7). $X(f)$ is known as the spectral density function of x(t).

Equations (1.6.1) and (1.6.2) from the Fourier transform pair. Equation (1.6.1) is referred to as the 'forward transform' and equation (1.6.2) as the 'reverse' or 'inverse transform'.

In general $X(f)$ is complex and can be written as

$$X(f) = Re\, X(f) + jIm\, X(f) \tag{1.6.3}$$

The part of $X(f)$ is obtained from

$$\text{Re}\, X(f) = \frac{1}{2}[X(f) + X(-f)] \tag{1.6.4}$$

$= \int_{-\infty}^{\infty} x(t) \cos 2\pi ft \, dt.$

Similarly, for the imaginary part of $X(f) = \frac{1}{2} j[X(f) - X(-f)]$

$= \int_{-\infty}^{\infty} x(t) \, sin \, 2\pi ft \, dt.$ (1.6.5)

The amplitude spectrum of the frequency signal is obtained from

$| \, X(f)| = [(Re \, X(F))^2 + (Im \, X(f)^2)]^{½}$ (1.6.6)

The phase spectrum is

$\emptyset(f) = tan^{-1} [\frac{Im \, X(f)}{Re \, X \, (f)}].$ (1.6.7)

Using equations (1.6.3)-(1.6.7) the inverse Fourier transform can be expressed in terms of the magnitude and phase spectra components:

$x(t) = \int_{-\infty}^{\infty} |X(f)| \cos[2\pi ft - \emptyset(f)] \, df$ (1.6.8)

As an example let us consider a rectangular function such as Figure 1.9, defined by

$$x(t) = \begin{Bmatrix} K \; for \; |t| \leq T/2 \\ o \; for \; |t| > T/2; \end{Bmatrix}$$

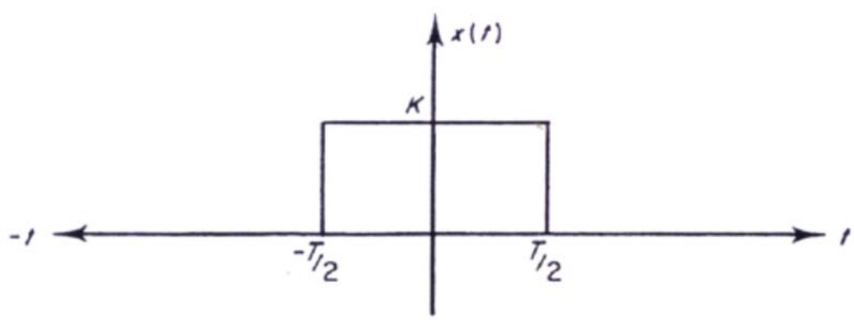

Figure 1.9. Rectangular function.

$$x(f) = KT[\frac{\sin(\pi fT)}{\pi fT}]$$

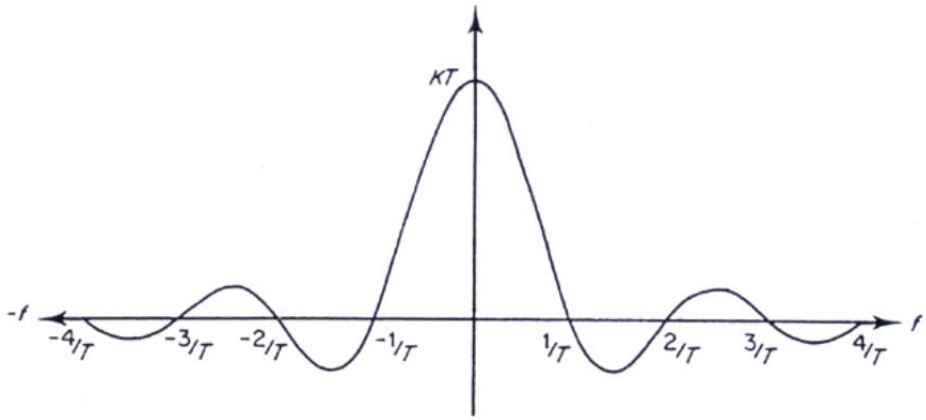

Figure 1.10. The sine function, sin $\frac{\pi f t}{\pi f t}$.

i.e., the function is continuous over all t but is zero outside the limits (- T /2, T/2).
Its Fourier transform is

$$X(f) = \int_{-T/2}^{T/2} x(t) e^{-j2\pi f t} dt$$

$$= \int_{-T/2}^{T/2} K e^{-j2\pi f t} dt$$
$$= \frac{-K}{\pi f} \frac{1}{2j} \left[e^{-j\pi f T} - e^{jxfT} - e^{j\pi f T} \right]. \quad (1.6.9)$$

and using the identity

$$\sin \emptyset = \frac{1}{2j} (e^{j\emptyset} - e^{-j\emptyset})$$

yields the following expression for the Fourier transform:

$$X(f) = \frac{K}{\pi f} \sin(\pi f T) = KT \left[\frac{\sin(\pi f T)}{\pi f T} \right]. \quad (1.6.10)$$

The term in brackets, the sin x/x or sinc function, is shown in Figure 1.10.

While the function is continuous, it has zero value at the points f =n/T for n = ± 1, $\pm$ 2, …, and the side lobes decrease in magnitude as 1/T. This should be compared to the fourier series of a periodic square wave which has discrete frequencies and odd harmonics. The interval 1/T is the effective bandwidth of the signal.

The power in the signal, obtained by squaring this function, is shown in Figure 1.11. Most of the power of the signal is contained within the frequency range – 1 /T < f > 1/T. The total power is the area under the curve of Figure 1.11, i.e.,

$$P = K^2 T^2 \int_{-\infty}^{\infty} \frac{sin^2(\pi f T)}{(\pi f T)^2} df. \quad (1.6.11)$$

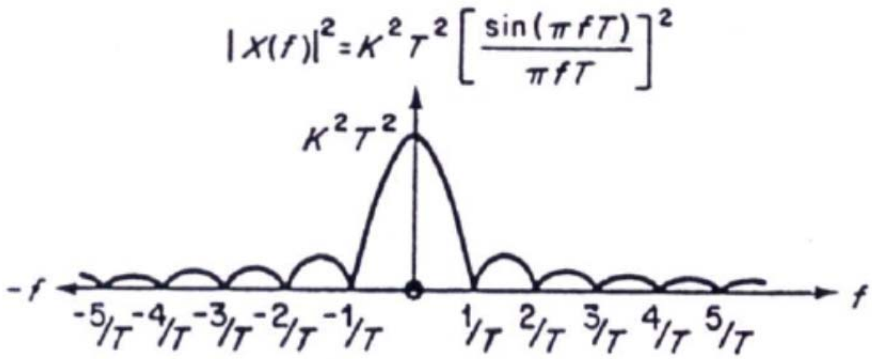

Figure 1.11. Power spectrum of a function.

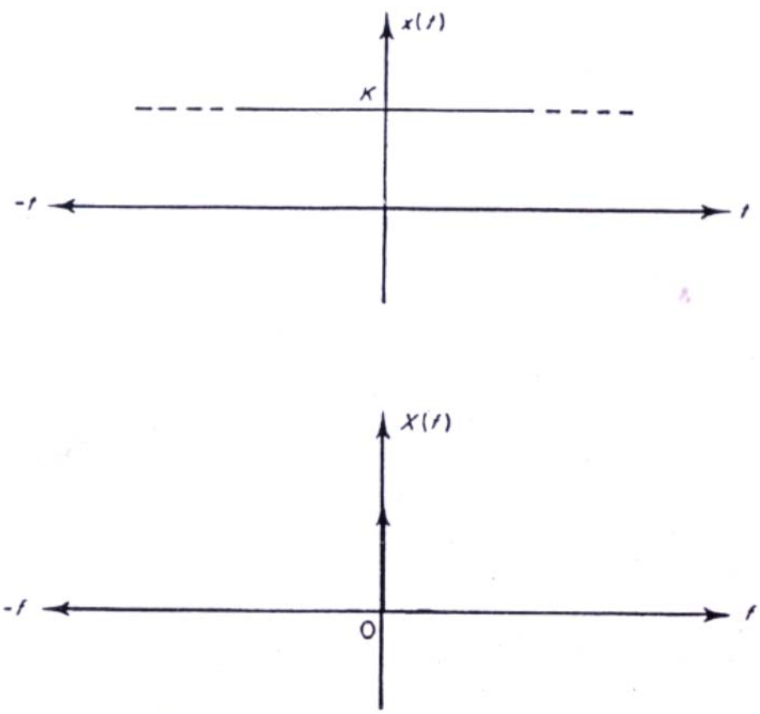

Figure 1.12. Infinite d.c. time domain signal and its frequency spectrum.

and since the value of the integral is 1/T, the total power is

$$P = K^2T^2(\frac{1}{T}) \tag{1.6.12}$$

i.e., the magnitude of the signal squared multiplied by the effective bandwidth.

It should be observed that the greater the duration of the signal in the time domain (i.e., as T increases) then the narrower the bandwidth becomes. In the limit, the Fourier transform of a square wave function where T $\rightarrow\infty$ is an impulse function in the frequency domain at f = 0 as shown in Figure1.12.

The Fourier transform pair is such that if x(t) $\leftrightarrow$X (f) then X (t) $\leftrightarrow$x(-f), where $\leftrightarrow$denotes the Fourier transformation. Thus a rectangular function in the frequency domain, representing a band-limited signal, transforms to a sinc function in the time domain.

FINITENESS OF ENERGY AND POWER [4]

A signal x(t) is said to be finite in energy, W, if

$$W = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \tag{1.6.13}$$

and this is a sufficient condition for the existence of the Fourier transform of the signal.

Periodic signals, which are considered to extend to infinity in time, theoretically have infinite energy. But since the existence of a Fourier series requires that the signal be finite in power, P, the following condition must be met:

$$P = \lim_{T \to \infty} \frac{1}{2T} |x(t)|^2 dt < \infty \tag{1.6.14}$$

Thus the modification of the Fourier transform for periodc signals results in a power spectrum, defined by

$$S(f) = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{T} x(t) e^{-j2\pi f t} dt \right|^2 \tag{1.6.15}$$

CONVOLUTION [4-5]

Consider Figure 1.13 in which a continuous signal g (t) is applied to a system with an impulse response of h(t). If the signal g(t) is represented by a series of impulses [$g(t_\pi)$]. Each separated by a time interval Δt. These then become the input to the system. Each of these impulses will produce a response from the system in the form of its impules response scaled according to the amplitude of the impulse $g(t_\pi)$. The output of the system at any instant t_o is the sum of the individual impulse responses at that time. This can be written as

$$y(t_o) = \sum_{\pi=-\infty}^{\infty} y_\pi(t_o) = \sum_{\pi=-\infty}^{\infty} g(t_\pi) h(t_o - t_\pi) \tag{1.6.16}$$

$$y(t) = \sum_{\pi=-\infty}^{\infty} g(t_\pi) h(t - t_\pi). \tag{1.6.17}$$

which in the limit as t $\rightarrow$0 tends to

$$y(t) = \int_{-x}^{x} g(\tau)h(t-\tau)\, d\tau \tag{1.6.18}$$

This is the convolution integral and is conventionally written as

Y(t)=g(t)*h(t). (1.6.19)

using ‘*’ to mean ‘convolved with’.

The Fourier transform has the effect of transforming a convolution in the time domin into a multiplication in the frequency domain.

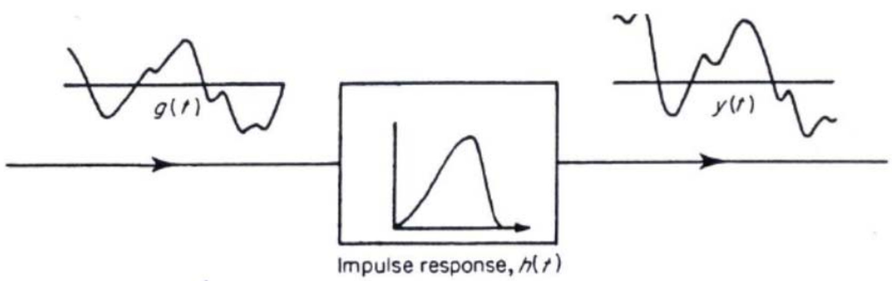

Figure 1.13. Convolution by a circuit of impulse response h(t).

More generally.

Let the forward trandforms og g(t), h(t) and y(t)be

$$G(f) = \int_{-x}^{x} g(t)e^{-j2\pi ft}dt. \tag{1.6.20}$$

$$H(f) = \int_{-x}^{x} h(t)e^{-j2\pi ft}dt. \tag{1.6.21}$$

and

$$y(f) = \int_{-x}^{x} y(t)e^{-j2\pi ft}dt. \tag{1.6.22}$$

as before

$$y(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(t)h(t-\tau)dt. \tag{1.6.23}$$

Using equations (1.6.22) and (1.6.23).

$$y(f) = \int_{-\infty}^{\infty}\left[\int_{-x}^{x} g(\tau)h(t-\tau)dt\right]e^{-j2\pi ft}dt; \tag{1.6.24}$$

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