Tutorial 3: Methods of proofs, Induction

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Tough problem for bored students

We start with $0 < x_0 < y_0$.

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_{n+1} y_n}.$$

Find $\lim x_n, \lim y_n$.

Recap: Methods of Proofs

There are endless possibilities for how to construct mathematical proofs. Most common ones used in practice:

- Proof by contradiction $\neg\neg A = A$
- Proof by cases $(A \implies B) \wedge (\neg A \implies B) \implies B$
- Proof by contrapositive $(\neg B \implies \neg A) \iff (A \implies B)$
- Proof by induction $P(1) \wedge (\forall n P(n) \implies P(n+1)) \implies \forall n P(n)$
- Proof by direct construction $P(a) \implies \exists x P(x)$ ...

Exercise 1

Prove that $k(k+1)(k+2)$ is always divisible by 6.

Exercise 1: Solution

First we notice that it is enough to prove that it is always divisible by 3 and 2. Then we prove each of these facts by cases.
If k is odd, then $k+1$ is even and the product is even. If k is even then product is even.
If $k = 3m+1$, then $k+2$ is divisible by 3. If $k = 3m+2$, then $k+1$ is divisible by 4. If $k = 3m$, then k is divisible by 3.

Shorter proof without cases:

Numbers $k, k+1, k+2$ are 3 sequential integers, among them there must be at least one even and at least one divisible by 3.

Exercise 2

Let a, b be real numbers. If $a \neq b$, then $a^2 + b^2 > 2ab$.

Solution to Exercise 2

By contrapositive, if $a^2 + b^2 \leq 2ab$, then $(a - b)^2 \leq 0$, then $(a - b) = 0$, i.e. $a = b$.

Exercise 3

Prove that $1^3 + 2^3 + ... + n^3$ is always a perfect square.
Direct proof? By contradiction? Cases n is even or n is odd? How can we prove it? Seems even a specific statement like

$$1^3 + 2^3 + ... + 5^3 + 6^3 \text{ is a square}$$

is not obvious.

Recap: Mathematical Induction

Recall what is mathematical induction. We want to prove some statement $\forall n > 0 \;\; P(n)$.

Sometimes it is easier to prove $P(1)$ and prove that for all $n > 1$ we can derive $P(n) \rightarrow P(n+1)$. It would follow that:

1. $P(1)$ is true
2. hence $P(2)$ is true
3. hence $P(3)$ is true
4. ...

Essentially, when proving $P(n)$, we got ourselves a (possibly) helpful statement $P(n-1)$.

Strong induction: why only restrict ourselves to usage of $P(n-1)$? Let's make use of all statements we know: $P(1), P(2), ..., P(n-1)$. Strong induction is most useful when some object X_n is decomposed into smaller objects $X_{k_1}, X_{k_2}, ..., X_{k_m}$.

Solution to Exercise 3

Let us try to notice the pattern:

$$1^3 = 1^2$$

$$1^3 + 2^3 = 3^2$$

$$1^3 + 2^3 + 3^3 = 6^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = 10^2$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2$$

Conjecture: $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$
Induction base: $n = 1$ was verified.
Induction step: $(1 + 2 + ... + n)^2 + (n + 1)^3 = n^2(n + 1)^2/4 + (n + 1)^3 =$
$= (n + 1)^2(n^2 + 4n + 4)/4 = (n + 1)^2(n + 2)^2/4 = (1 + ... + (n + 1))^2$

Exercise 4: summation

Prove that

$$1 \cdot 2 + 2 \cdot 3 + ... + (n-1) \cdot n = (n-1)n(n+1)/3.$$

Solution to Exercise 4

Base: $n = 1$ gives $0 = 0$. Step: Assume

$$1 \cdot 2 + 2 \cdot 3 + ... + (n-1) \cdot n = (n-1)n(n+1)/3,$$

then

$$1 \cdot 2 + 2 \cdot 3 + ... + (n-1) \cdot n + n \cdot (n+1) =$$

$$= (n-1)n(n+1)/3 + n \cdot (n+1) = (n-1+3)n(n+1)/3 = n(n+1)(n+2)/3.$$

Problem

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} = ???$$

Comment on problem

This finite sum has no good closed form expression. a_n converges to some number,

$$1.644934066848...$$

nobody could guess the answer. It became known as Basel Problem and remained unsolved until Leonard Euler found that

$$\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} + ... = \frac{\pi^2}{6}.$$

Although he was not quite rigorous by modern standards, it was a very big achievement. Modern days we have tools like Euler-Maclaurin summation formula and specialized software for finding closed form finite and infinite sums (see book "A=B").

Josephus Problem

There are N people standing in a circle and numbered 1 to N clock-wise. The game goes as follows.

- Player 1 starts and eliminates player directly to the left of him.
- We search clockwise (to the left) for the first non-eliminated player and make him eliminate player directly to the left of him.
- these moves are repeated until one player is left. He is declared the winner of the game.

For example, let's say there are 4 players. Initially: $4, 3, 2, 1$. After first step we get the position $4, 3, 1$. Next move is done by player 3 (he is first to the left of 1). He eliminates player 4

$$3, 1.$$

Next move is done by player 1 (he is first to the left of 3). He eliminates player 3

$$1.$$

For $N = 4$ the winner is 1.

Exercise 5: hard guessing and hard induction

You play this game with N players. You can choose your position in circle. Which position you should choose to win?

Hint

Let's look how the games go for small N.

- $N = 1$. The winner is player 1 (no moves).
- $N = 2$. $2, \underline{1} \to 1$. The winner is player 1.
- $N = 3$. $3, 2, \underline{1} \to \underline{3}, 1 \to 3$. The winner is player 3.
- $N = 4$. We have seen. The winner is player 1.
- $N = 5$. $5, 4, 3, 2, \underline{1} \to 5, 4, \underline{3}, 1 \to \underline{5}, 3, 1 \to 5, \underline{3} \to 3$. The winner is player 3.
- $N = 6$. $6, 5, 4, 3, 2, \underline{1} \to 6, 5, 4, \underline{3}, 1 \to 6, \underline{5}, 3, 1 \to 5, 3, \underline{1} \to \underline{5}, 1 \to 5$. The winner is player 5.
- $N = 7$. $7, 6, 5, 4, 3, 2, \underline{1} \to 7, 6, 5, 4, \underline{3}, 1 \to 7, 6, \underline{5}, 3, 1 \to \underline{7}, 5, 3, 1 \to 7, 5, \underline{3} \to \underline{7}, 3 \to 7$. The winner is player 7.
- $N = 8$. Let me skip some steps for you
  $8, 7, 6, 5, 4, 3, 2, \underline{1} \to 7, 5, 3, \underline{1} \to 5, \underline{1} \to 1$. The winner is player 1.

Solution to Exercise 5

Answer: Let $N = 2^m + r$, where $r < 2^m$. Then the winner is $2r + 1$.
Proof: By induction we can prove that for $N = 2^m + 0$ the winner is 1 (each circle of eliminations shrinks the number of players by 2 and leaves 1 as a first-mover).
Let's reduce the problem from $N = 2^m + r$ to $N = 2^m + 0$. Indeed, consider the game state after first r eliminations. Next move is done by player $2r + 1$. Total number of players alive is $2^m + r - r = 2^m$. We know that for power of 2, the winner is the one who does next move. This is player number $2r + 1$.

Question: what if only every k-th move eliminates players?

Thank You

Thank you for your attention!