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1. Use the Euclidean algorithm to calculate gcd(102, 70). Use the extended Euclidean algorithm to write gcd(663, 234) as an integer linear combination of 663 and 234.

$102 = 70 \times 1 + 32$ $\quad 70 = 32 \times 2 + 6$ $\quad 32 = 5 \times 6 + 2$ $\quad 6 = 3 \times 2 + 0$

$GCD(102, 70) = GCD(70, 32) = GCD(32, 6) = GCD(6, 2) = GCD(2, 0)$

$gcd(102, 70) = 2$

$663 = 234 \times 2 + 195$ $\qquad 195 = 663 - 2 \times 234 = a - 2b$

$234 = 195 \times 1 + 39$ $\qquad 39 = 234 - 195 = b - (a - 2b) = 3b - a$

$195 = 39 \times 5 + 0$

$gcd = 39$ $\qquad 39 = 234 \times 3 - 663$

2. Prove that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

$\because 10 \equiv 1 \pmod 3$ $\therefore$ we assume the decimal representation of n be $d_k d_{k-1} d_{k-2} \cdots d_0$

This means that $n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots d_1 10 + d_0$

Note tha $d_i 10^i \bmod 3 \equiv (d_i \bmod 3)(10^i \bmod 3) \bmod 3$

$\equiv (d_i \bmod 3) \underbrace{(10 \bmod 3)(10 \bmod 3) \cdots\cdots}_{i \text{ terms}} \bmod 3$

$\equiv d_i \bmod 3$

$\therefore n \pmod 3 \equiv (d_k 10^k + d_{k-1} 10^{k-1} + \cdots d_1 \cdot 10 + d_0) \pmod 3$

$\equiv (d_k + d_{k-1} + \cdots d_1 + d_0) \pmod 3$

$\because 3 \mid (d_k + d_{k-1} + \cdots d_1 + d_0)$

$\therefore 0 \equiv (d_k + d_{k-1} + \cdots d_1 + d_0) \pmod 3$

3. Prove that all numbers in the sequence

$$1007, 10017, 100117, 1001117, \ldots$$

are divisible by 53.

let $a_1 = 1007$ $\quad a_2 = 10017$ $\quad a_3 = 100117 \cdots$ $\quad a_n = 100\overbrace{11\cdots}^{(n-1)\text{ terms}}7$

for $k \geq 2$ $a_k - 10 \times a_{k-1} = 1\underbrace{0011\cdots}_{(k-1)\text{ terms}}7 - 10 \times 1\underbrace{0011\cdots}_{(k-2)\text{ terms}}7 = -53$

$a_k = 10 \times a_{k-1} - 53$

$\because a_1 = 1007 = 53 \times 19$ $a_k = 10 \times a_{k-1} - 53$ when $k \geq 2$, recursion on n

$\therefore$ $a_1\, a_2 \cdots a_n$ are divisible by 53.

4. A robot walks around a two-dimensional grid. It starts out at $(2, 0)$ and is allowed to take four different types of steps as:

   (a) $(+2, -1)$
   (b) $(+1, -2)$
   (c) $(+1, +4)$
   (d) $(-3, 0)$

   Prove that this robot can never reach $(0, -1)$.

let (a) walk a steps, (b) walk b steps, (c) walk c steps, (d) walk d steps.

assume ~~the~~ robot can reach $(0,-1)$ $\begin{cases} 2+2a+b+c-3d=0 \\ 0-a-2b+4c=-1 \end{cases}$

$a = 4c-2b+1$ $4+8c-4b+b+c-3d=0$ $4-3b+9c-3d=0$

$4 = 3(b-3c+d)$ $\because b, c, d$ are integers $\therefore 4$ is not a multiple of 3.

$\therefore$ the hypothesis is not true.

this robot can never reach $(0,-1)$.

5. NIM is a famous game in which two players take turns removing items from a pile of n items. For every turn, the player can remove one, two, or three items at a time. The player removing the last item loses. Prove that if each player plays the best strategy possible, the first player wins if $n \not\equiv 1 \pmod 4$ and the second player wins if $n \equiv 1 \pmod 4$. (For your interest, refer to the general NIM game at this link).

when $n=1$, the second player wins.

when $1 < n \leq 4$ the first player wins.

when $n > 4$,

if $n \equiv 1 \pmod 4$, no matter how many the first player takes, let take a items the second player will take $(4-a)$ items, so when n items are removed to 1 item.

The second player must wins, the first player must remove the last one.

if $n \not\equiv 1 \pmod 4 \Rightarrow n \equiv 0 \pmod 4$, $n \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$

no matter how many the first player takes, (let take a items), the second player takes $(4-a)$ items, the last items must be $1 + (1$ or 2 or $3)$ items, so the first player must wins

6. Find all solutions, if any, to the system:

$$\begin{cases} x \equiv 5 \pmod{6} \\ x \equiv 3 \pmod{10} \\ x \equiv 8 \pmod{15}. \end{cases}$$

$x = 15u + 8$ $\quad 15u + 8 \equiv 3 \pmod{10}$ $\quad 15u \equiv 5 \pmod{10}$

$3u \equiv 1 \pmod{2}$ $\quad u \equiv 1 \pmod{2}$ $\quad u = 2v + 1$

$x = 30v + 23$ $\quad 30v + 23 \equiv 5 \pmod{6}$ $\quad 30v + 18 \equiv 0 \pmod{6}$

$30v = -18 \pmod{6}$ this equation is always true.

$\therefore x = 30v + 23$

$x \equiv 23 \pmod{30}$

7. Show with the help of Fermat's little theorem that if n is a positive integer, then $42 | n^7 - n$.

$42 = 2 \times 3 \times 7$

$n^7 - n = n \cdot (n^6 - 1)$ $\begin{cases} \text{when } 7 \nmid n;\ n^6 \equiv 1 \pmod{7}; \\ \text{when } 7 | n;\ 7 | n^7 - n \end{cases}$

$n^7 - n = (n^2 - 1)(n^5 + n^3 + n)$ $\begin{cases} \text{when } 3 | n: \quad 3 | n^7 - n \\ \text{when } 3 \nmid n \quad n^2 \equiv 1 \pmod{3};\ 3 | n^7 - n \end{cases}$

$n^7 - n$ $\begin{cases} \text{when } n \text{ is even}, \quad n^2 - n \text{ is even} \quad 2 | n^2 - n \\ \text{when } n \text{ is odd}, \quad n^2 - n \text{ is even} \quad 2 | n^2 - n \end{cases}$

$\because 2, 3, 7$ coprime.

$\therefore 42 | n^7 - n$