CSC3001 Discrete Mathematics: Tutorial 5

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Tough problems for bored students

1. Consider subset of $\{1, 2, ..., n\}$ such that no elements are neighbors. Attach weight $wt(S) = \sum_{a \in S} a$ to such subsets. Consider $p_n(q) = \sum_S q^{wt(S)}$. This is statistical sum or partition function from theoretical physics.

- Find recursion for $p_n(q)$
- Prove that $p(q) = \lim p_n(q)$ exists and compute first terms.
- Prove that $p(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)...(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$

2. There is a rectangular wall with $n \times m$ cells. A girl stacks cubes in lower left corner of the wall (so if some cell is occupied then any cell below and to the left is occupied). How many ways she can stack cubes? What if she stacks in the corner of $m \times n \times k$ room? What about 4D corner?

Exercise 1

Solve recurrence

$$a_{n+1} = 2a_n - a_{n-1}.$$

Solution to Exercise 1

We have

$$a_{n+1} - a_n = a_n - a_{n-1},$$

Hence $b_n = a_{n+1} - a_n$ is a constant sequence. Hence

$$a_{n+1} = a_n + b$$

and $a_n = bn + c$.

Exercise 2

Find generating function for sequence in exercise 1 with $a_0 = 1$, $a_2 = 2$.

Solution to Exercise 2

Let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Then

$$f(x) - 2xf(x) + x^2 f(x) = a_0 + (a_1 - 2a_0)x = 1,$$

hence

$$f(x) = \frac{1}{1 - 2x + x^2}$$

Exercise 3

Let f_n be n-th Fibonacci number, $f_0 = 0$, $f_1 = 1$. Find $f_1 + ... + f_n$.

Solution to Exercise 3

By induction we prove that

$$f_1 + ... + f_n = f_{n+2} - 1.$$

Indeed,

$$f_1 + ... + f_n + f_{n+1} = f_{n+1} + f_{n+2} - 1 = f_{n+3} - 1.$$

Alternative solution: Let $g(x)$ be generating function of Fibonacci numbers. From lectures we know that

$$g(x) = \frac{x}{1 - x - x^2}.$$

But

$$g(x)\frac{x}{1-x} = \frac{1}{1 - x - x^2} - \frac{1}{1-x}.$$

Exercise 4

Compute

$$\gcd(2^{30} - 1, 2^{54} - 1).$$

Solution to Exercise 4

$$2^{54} - 1 = (2^{30} - 1) \cdot 2^{24} + 2^{24} - 1,$$

hence

$$\gcd(2^{30} - 1, 2^{54} - 1) = \gcd(2^{30} - 1, 2^{24} - 1).$$

We notice that this is euclid algorithm on exponents, hence answer is

$$2^{\gcd(30,54)} - 1 = 2^6 - 1.$$

Thank You

Thank you for your attention!