CSC3001 Discrete Mathematics

Homework 2

Deadline: 23:59, Friday, Nov 4, 2022

The details should be provided, and you can refer to any theorem in the lecture notes without proof. Otherwise, please provide a proof or cite the reference for that.

1. (a) Let f_n be the n-th Fibonacci number, i.e. $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$. Prove that
$$f_1 + f_2 + ... + f_n = f_{n+2} - 1.$$
(b) Let g_n satisfy the same recurrence as Fibonacci sequence, but have different initial values: $g_1 = a, g_2 = b$, $g_{n+2} = g_{n+1} + g_n$. Prove that
$$g_1 + g_2 + ... + g_n = g_{n+2} - b.$$

2. Find and prove closed-form formulas for generating functions
$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots.$$
of the following sequences
   (a) $a_n = a^n$, where $a \in \mathbb{R}$;
   (b) $a_n = \binom{m}{n}$, where $m \in \mathbb{N}$;
   (c) $a_n = f_n$, where f_n is the n-th Fibonacci number (assume $f_0 = 0, f_1 = f_2 = 1$).

3. Using the formula
$$\binom{n}{m} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-m+1)}{m \cdot (m-1) \cdot \cdots \cdot 2 \cdot 1}$$
Let p be prime. Prove that $\binom{p}{k}$ is divisible by p for $0 < k < p$. Deduce by induction on n that $n^p \equiv n \pmod{p}$.

4. Using the identity
$$(1+x)^n(1+x)^n = (1+x)^{2n}$$
prove that
$$\sum_{m=0}^{n} \binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}.$$

Deduce that

$$\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}.$$

5. Find all solutions, if any, solutions to the system

$$x \equiv 5 \pmod 6$$
$$x \equiv 3 \pmod{10}$$
$$x \equiv 8 \pmod{15}.$$

6. Show steps to find
   (a) the greatest common divisor of 1234567 and 7654321.
   (b) the greatest common divisor of $2^3 3^5 5^7 7^9 11$ and $2^9 3^7 5^5 7^3 13$.

7. Label the first prime number 2 as P_1. Label the second prime number 3 as P_2. Similarly, label the n-th prime number as P_n. Prove that $P_n < 2^{2^n}$ for an arbitrary $n \in \mathbb{N}^+$. Hint: consider $P_1 P_2 \cdots P_{n-1} + 1$.

8. In a round-robin tournament, every team plays every other team exactly once and each match has a winner and a loser. We say that the team $p_1, p_2, \cdots, p_m$ form a cycle if p_1 beats p_2, p_2 beats p_3, $\cdots$, p_{m-1} beats p_m, and p_m beats p_1. Show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle of three of these players. Hint: Use the well-ordering principle.