Assignment 2

Due 23:59, Oct 28

Question 1 [10 marks]

Prove or disprove: There exists some integer k such that $4k + 2$ is the difference between two integers, both of which are perfect squares.

Question 2 [10 marks]

Use a proof by contraposition to show if $x^2 + y^2$ and $3xy$ are both even numbers, then x and y are both even.

Question 3 [10 marks]

Prove or disprove that all checkerboards of these shapes can be completely covered using right triominoes whenever n is a positive integer: (Hint: A right triomino is an L-shaped piece that covers 3 squares) (A proof must be done via induction):

1. 3×2^n.
2. $3^n \times 3^n$.

Question 4 [10 marks]

Find the flaw with the following proof that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k + 1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

Question 5 [10 marks]

Prove or disprove: The product of a nonzero rational number and an irrational number is irrational.

Question 6 [10 marks]

Show that you can select two out of the three real numbers (which can be any arbitrary real number) such that their product is nonnegative.

Question 7 [10 marks]

Let the sequence a_n be defined as $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Prove that

$$a_n < 2^n$$

holds for all $n \in \mathbb{Z}_+$

Question 8 [10 marks]

Let S be a finite set with |S| elements and let P(S) denotes power set of S. Prove by induction that there are $2^{|S|}$ number of elements in P(S).

Question 9 [10 marks]

Suppose that n people ($n \geq 3$) play a round robin tournament. Show that at least one of the following statements must be true:

1. There is a person p who beats everyone else.
2. There are three people p, q, r such that p beats q, q beats r, and r beats p.

Question 10 [10 marks]

Which of these sets are well ordered under the given operator? Give a short justification for your answer, and if it is not, provide a counterexample.

1. $(\mathbb{R}^+ \cup \{0\}, <)$
2. $(\{0, 1, 2, 3\}, >)$
3. $(P, <)$ where $P = \{p \in \mathbb{Z}^+ \mid p \text{ is a prime }\}$