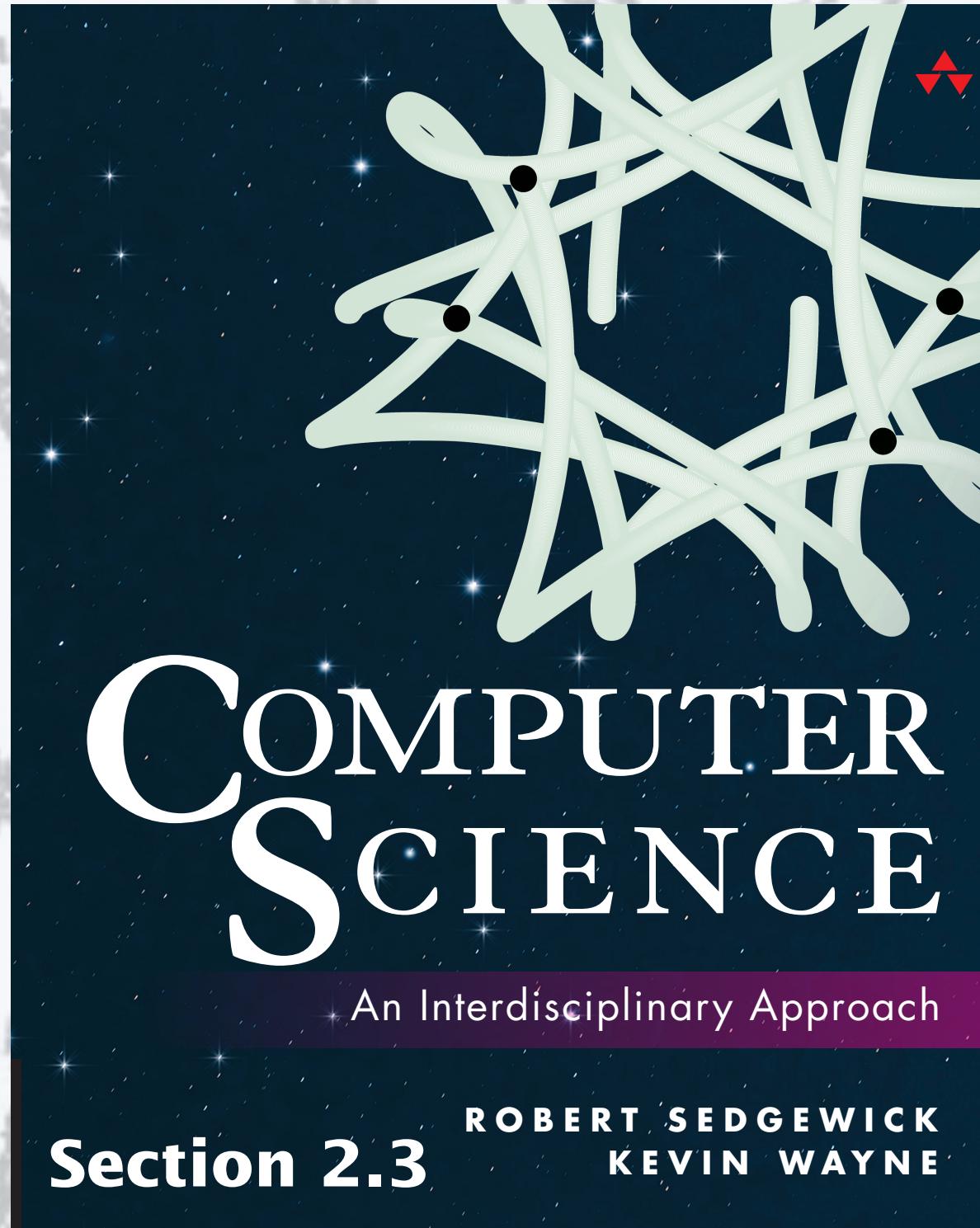


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PART I: PROGRAMMING IN JAVA



<http://introcs.cs.princeton.edu>

## 6. Recursion

## Midterm Arrangement

---

- **20 Mar, 10:30–12:00 (Wed)** in total 90 minutes
- **Closed-book exam**
- **No electronic devices (including calculators)** are allowed
- Scope: from 1.Basics to 6.Recursion
- Around 20% choice questions
- Around 50% read the program, give the expected output
- Around 30% short answer questions
- The above are rough numbers, the proportion can be changed.

## 6. Recursion

- **Foundations**
  - A classic example
  - Recursive graphics
  - Avoiding exponential waste
  - Dynamic programming

# Overview

---

**Q.** What is recursion?

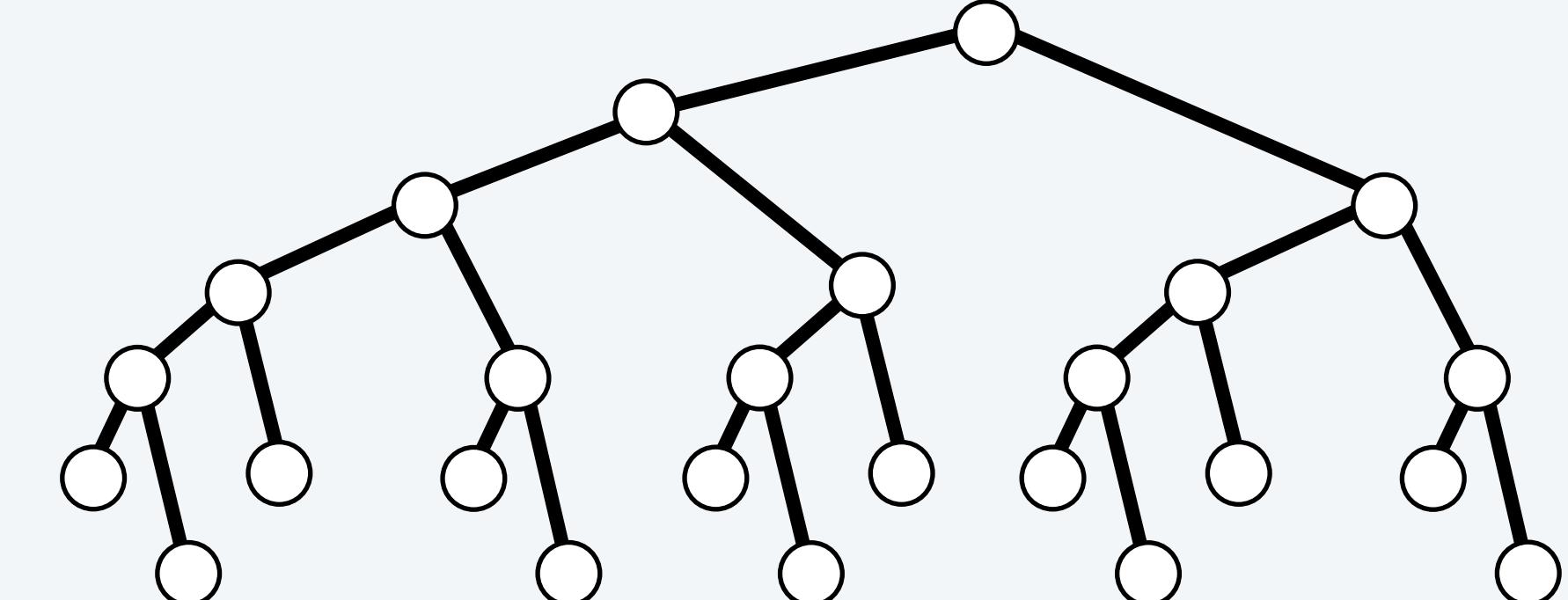
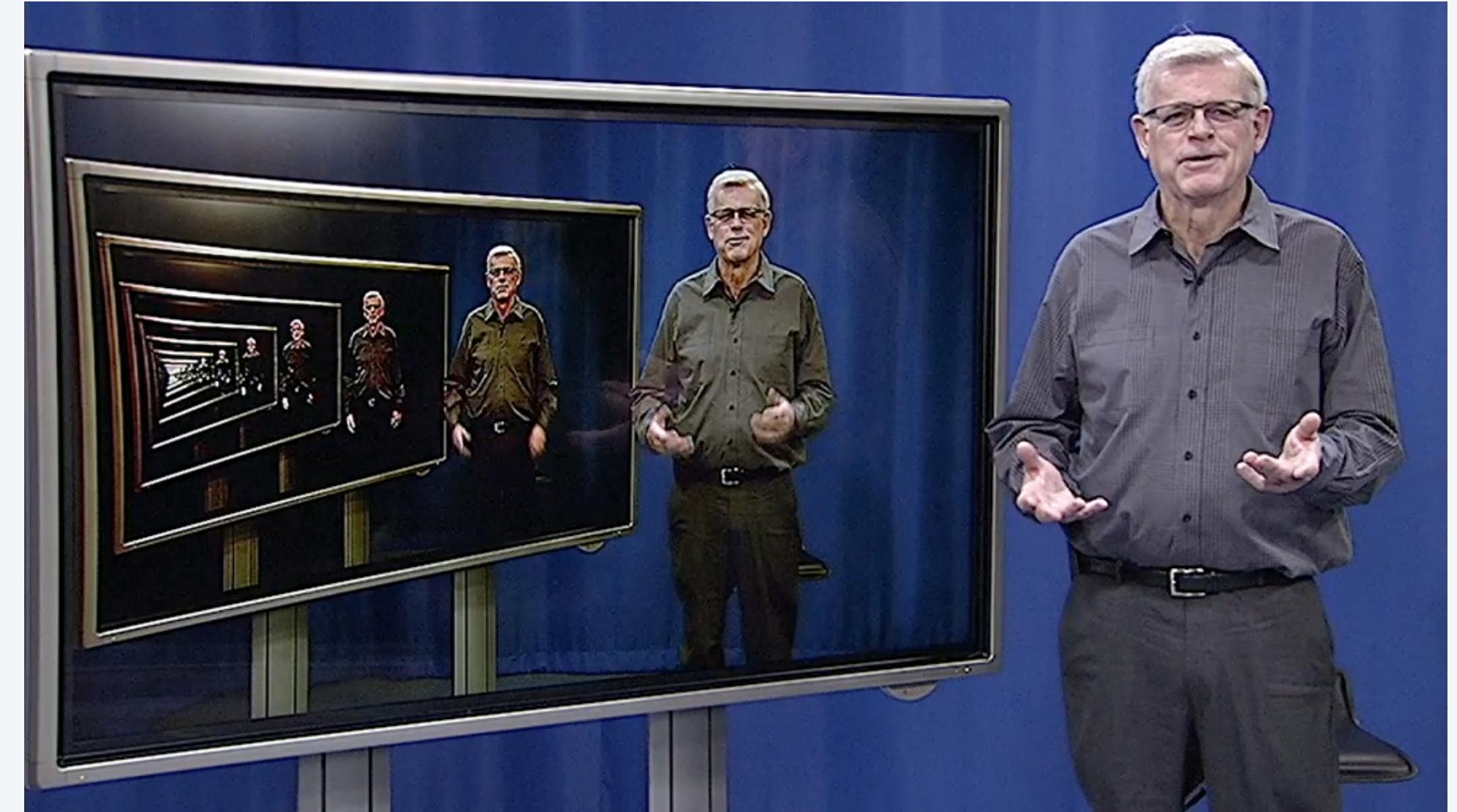
**A.** When something is specified in terms of *itself*.

Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Enables reasoning about correctness.
- Gives insight into the nature of computation.

Many computational artifacts are *naturally* self-referential.

- File system with folders containing folders.
- Fractal graphical patterns.
- Divide-and-conquer algorithms (stay tuned).



# Example: Convert an integer to binary

## Recursive program

To compute a function of a positive integer  $N$

- **Base case.** Return a value for small  $N$ .
- **Reduction step.** Assuming that it works for smaller values of its argument, use the function to compute a return value for  $N$ .

```
public class Binary
{
    public static String convert(int N)
    {
        if (N == 1) return "1";
        return convert(N/2) + (N % 2);  
    }
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        StdOut.println(convert(N));
    }
}
```

int 0 or 1 automatically converted to String "0" or "1"

```
% java Binary 6  
110  
% java Binary 37  
100101  
% java Binary 999999  
11110100001000111111
```

Q. How can we be convinced that this method is correct?

A. Use *mathematical induction*.

# Mathematical induction (quick review)

To prove a statement involving a positive integer  $N$

- **Base case.** Prove it for some specific values of  $N$ .
- **Induction step.** Assuming that the statement is true for all positive integers less than  $N$ , use that fact to prove it for  $N$ .

## Example

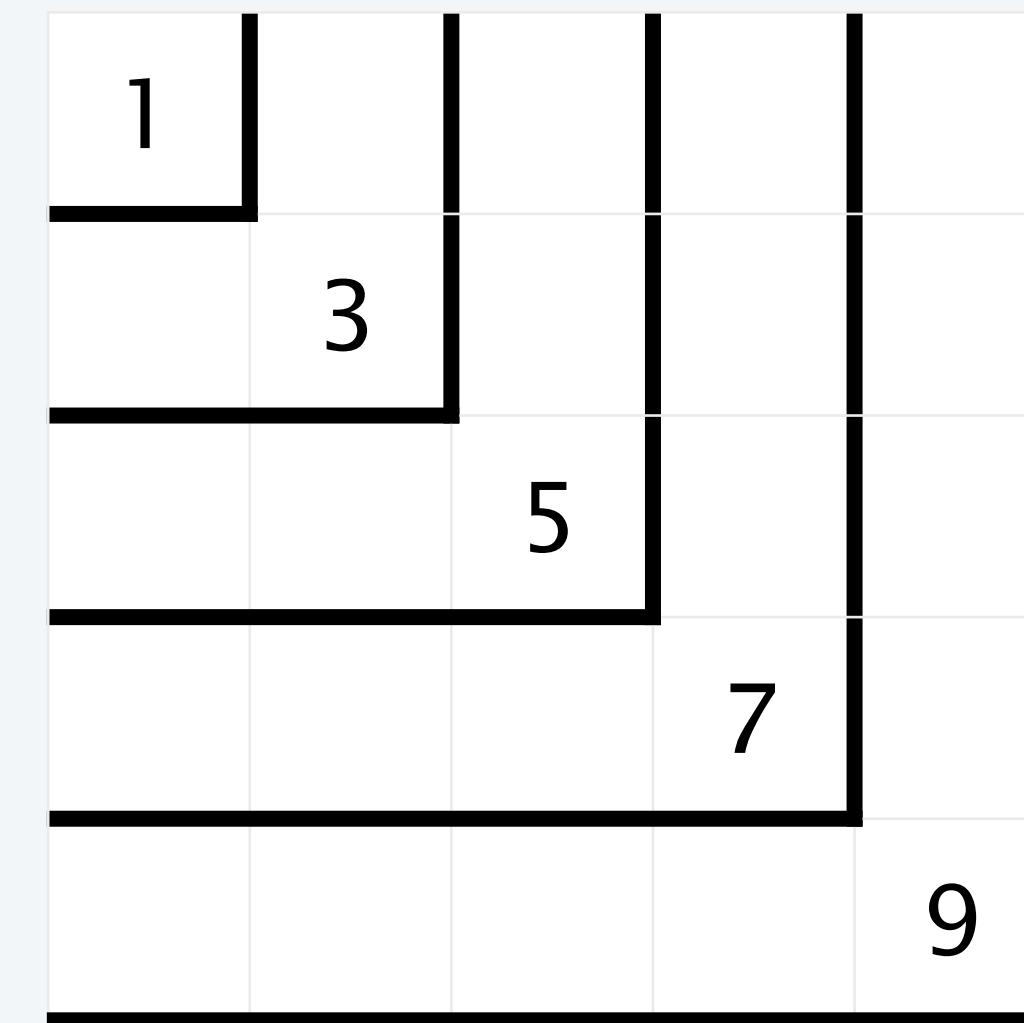
The sum of the first  $N$  odd integers is  $N^2$ .

Base case. True for  $N = 1$ .

Induction step. The  $N$ th odd integer is  $2N - 1$ .

Let  $T_N = 1 + 3 + 5 + \dots + (2N - 1)$  be the sum of the first  $N$  odd integers.

- Assume that  $T_{N-1} = (N-1)^2$ .
- Then  $T_N = (N-1)^2 + (2N-1) = N^2$ .



An alternate proof

# Proving a recursive program correct

## Recursion

To compute a function of  $N$

- **Base case.** Return a value for small  $N$ .
- **Reduction step.** Assuming that it works for smaller values of its argument, use the function to compute a return value for  $N$ .

## Mathematical induction

To prove a statement involving  $N$

- **Base case.** Prove it for small  $N$ .
- **Induction step.** Assuming that the statement is true for all positive integers less than  $N$ , use that fact to prove it for  $N$ .

## Recursive program

```
public static String convert(int N)
{
    if (N == 1) return "1";
    return convert(N/2) + (N % 2);
}
```

## Correctness proof, by induction

**convert()** computes the binary representation of  $N$

- **Base case.** Returns "1" for  $N = 1$ .
- **Induction step.** Assume that convert() works for  $N/2$ 
  1. Correct to append "0" if  $N$  is even, since  $N = 2(N/2)$ .

$N/2$ 

--	--	--	--	--	--

 $N$ 

						0
--	--	--	--	--	--	---

2. Correct to append "1" if  $N$  is odd since  $N = 2(N/2) + 1$ .

$N/2$ 

--	--	--	--	--	--

 $N$ 

						1
--	--	--	--	--	--	---

# Mechanics of a function call

System actions when *any* function is called

- *Save environment* (values of all variables and call location).
- *Initialize values* of argument variables.
- *Transfer control* to the function.
- *Restore environment* (and assign return value)
- *Transfer control* back to the calling code.

```
public class Binary
{
    public static String convert(int N)
    {
        if (N == 1) return "1";
        return convert(N/2) + (N % 2);
    }
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        System.out.println(convert(N));
    }
}
```

```
convert(26)
if (N == 1) return "1";
return "1101" + "0";

convert(13)
if (N == 1) return "1";
return "110" + "1";

convert(6)
if (N == 1) return "1";
return "11" + "0";

convert(3)
if (N == 1) return "1";
return "1" + "1";

convert(1)
if (N == 1) return "1";
return convert(0) + "1";
```

```
% java Binary 26
11010
```

# Programming with recursion: typical bugs

## Missing base case

```
public static double bad(int N)
{
    return bad(N-1) + 1.0/N;
}
```



## No convergence guarantee

```
public static double bad(int N)
{
    if (N == 1) return 1.0;
    return bad(1 + N/2) + 1.0/N;
}
```

Try  $N = 2$



Both lead to *infinite recursive loops* (bad news).



need to know  
how to stop them  
on your computer

# Collatz Sequence

Collatz function of  $N$ .

- If  $N$  is 1, stop.
- If  $N$  is even, divide by 2.
- If  $N$  is odd, multiply by 3 and add 1.



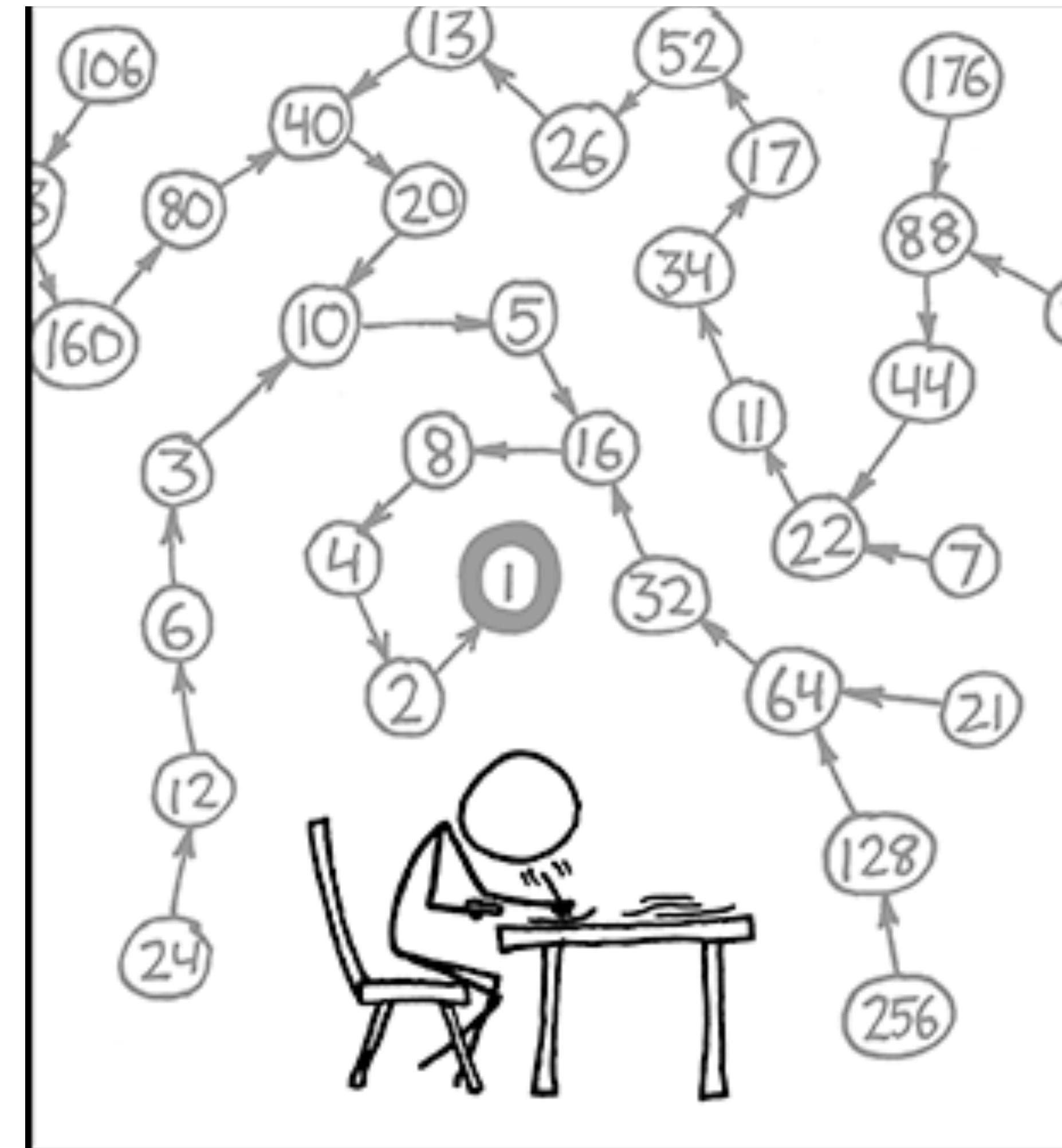
7 22 11 34 17 52 26 13 40 20 ...

```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    else if (N % 2 == 0) collatz(N / 2);
    else collatz(3*N + 1);
}
```

```
% java Collatz 7
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

**Amazing fact.** No one knows whether or not this function terminates for all  $N$  (!)

**Note.** We usually ensure termination by only making recursive calls for smaller  $N$ .



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

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*Image sources*

<http://xkcd.com/710/>

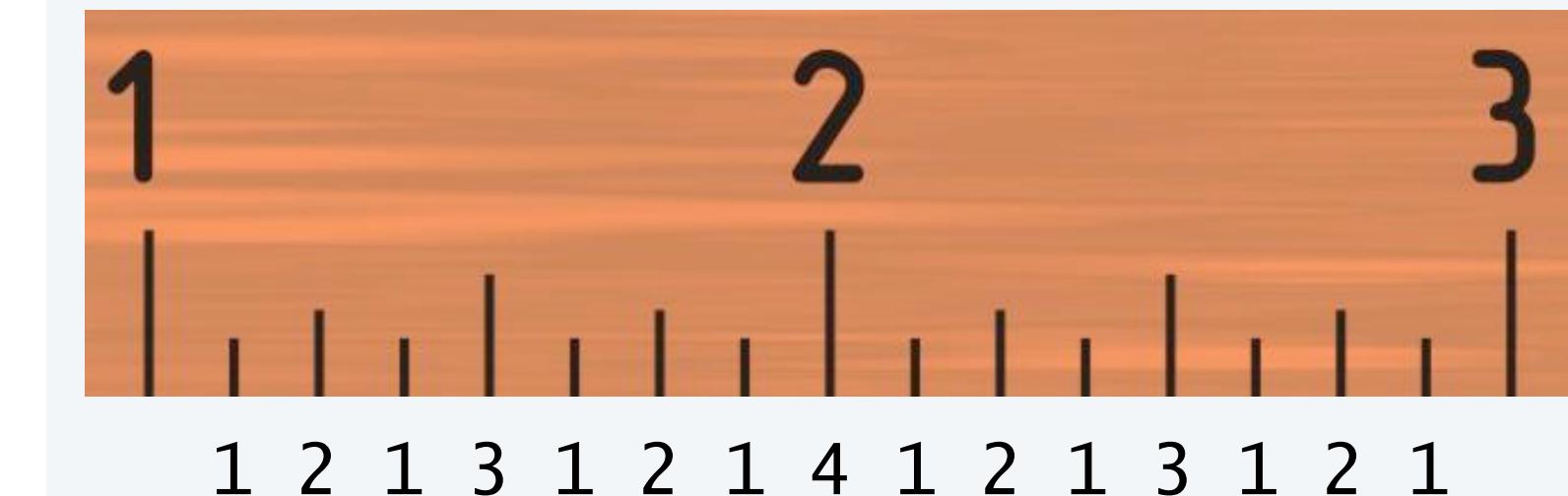
## 6. Recursion

- Foundations
- **A classic example**
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

## Warmup: subdivisions of a ruler (revisited)

ruler( $n$ ): create subdivisions of a ruler to  $1/2^n$  inches.

- Return one space for  $n = 0$ .
- Otherwise, sandwich  $n$  between two copies of ruler( $n-1$ ).



```
public class Ruler
{
    public static String ruler(int n)
    {
        if (n == 0) return " ";
        return ruler(n-1) + n + ruler(n-1);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        StdOut.println(ruler(n));
    }
}
```

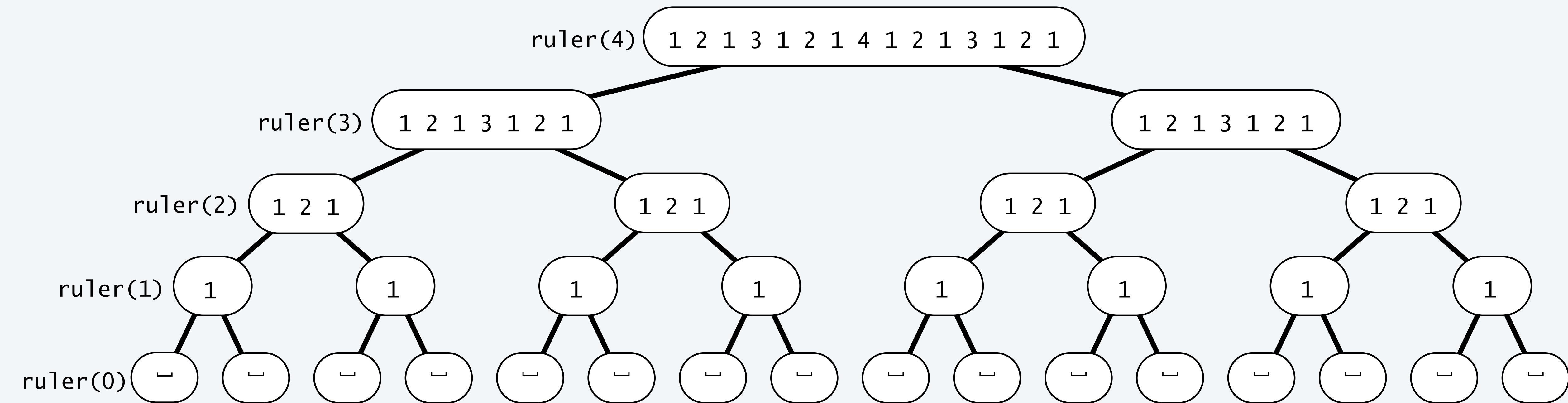
```
% java Ruler 1
1
% java Ruler 2
1 2 1
% java Ruler 3
1 2 1 3 1 2 1
% java Ruler 4
1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
% java Ruler 50
Exception in thread "main"
java.lang.OutOfMemoryError:
Java heap space
```

# Tracing a recursive program

Use a *recursive call tree*

- One node for each recursive call.
- Label node with return value after children are labeled.

```
public static String ruler(int n)
{
    if (n == 0) return " ";
    return ruler(n-1) + n + ruler(n-1);
}
```



# Towers of Hanoi puzzle

## A legend of uncertain origin

- $n = 64$  discs of differing size; 3 posts; discs on one of the posts from largest to smallest.
- An ancient prophecy has commanded monks to move the discs to another post.
- When the task is completed, *the world will end.*

$n = 10$

### Rules

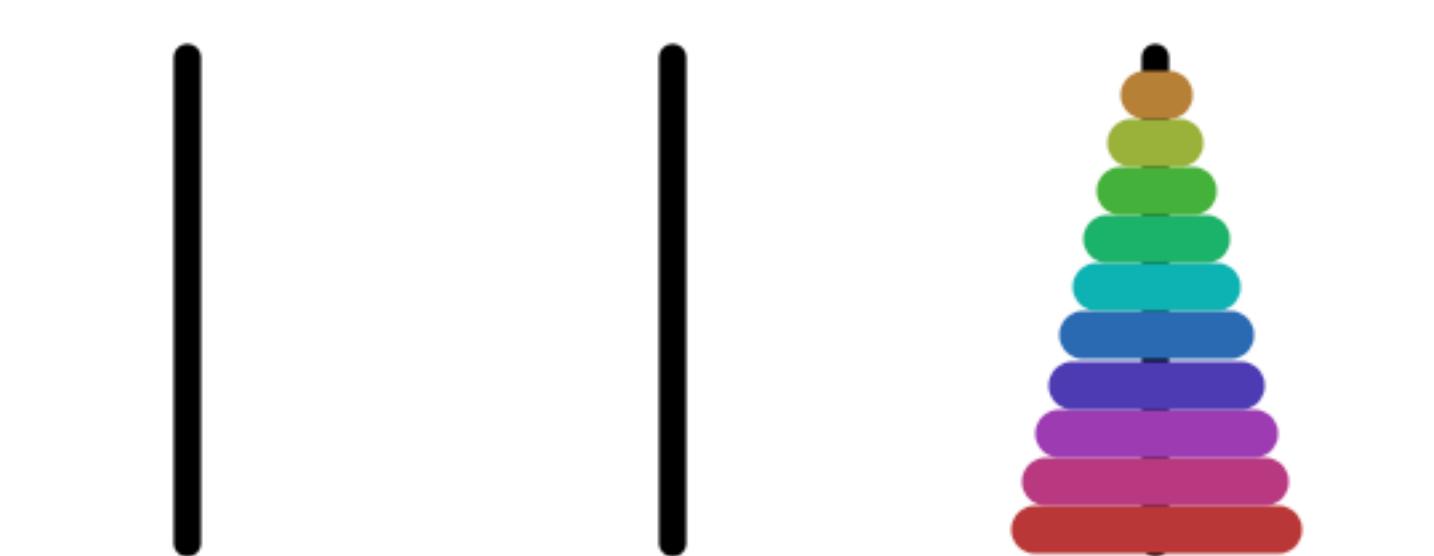
- Move discs one at a time.
- Never put a larger disc on a smaller disc.

before



Q. Generate list of instruction for monks ?

after

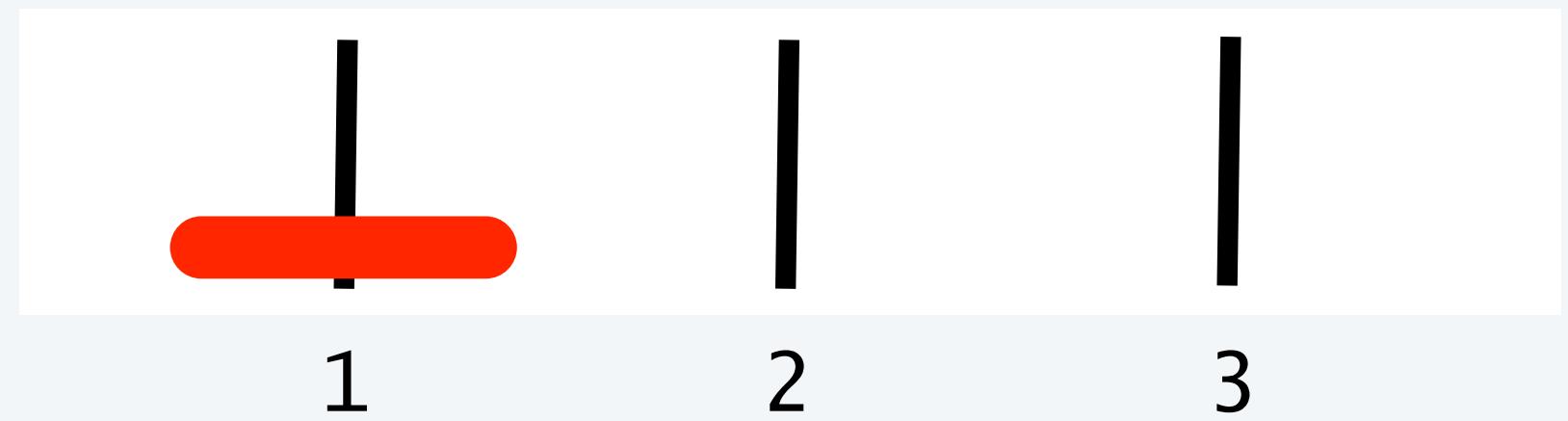


Q. When might the world end ?

# Towers of Hanoi

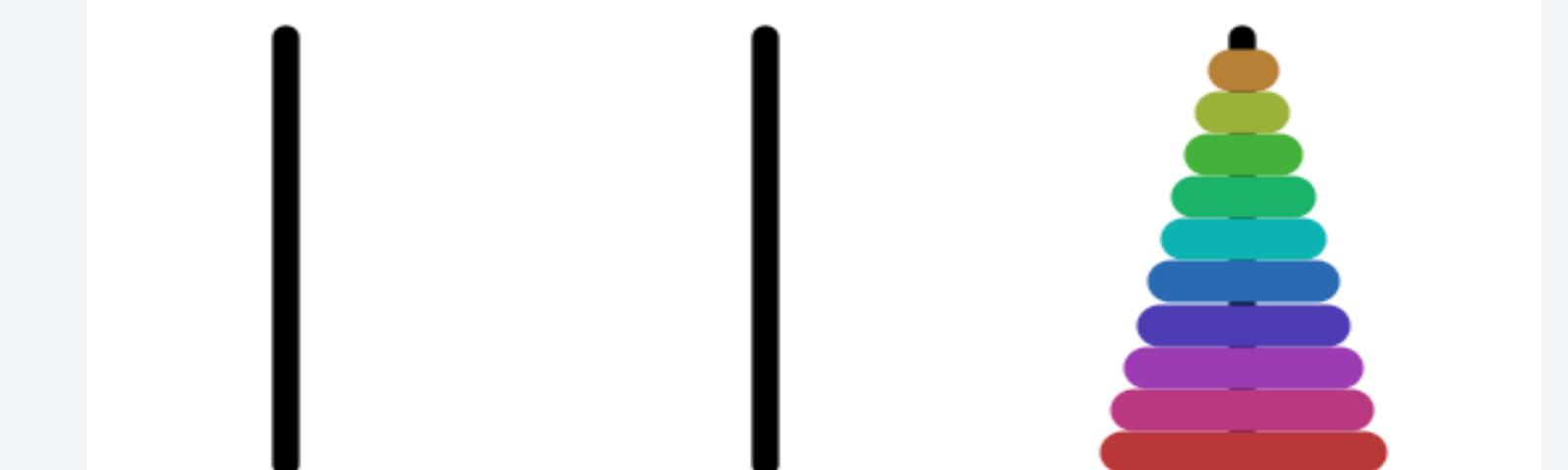
For simple instructions, use cyclic wraparound

- Move *right* means 1 to 2, 2 to 3, or 3 to 1.
- Move *left* means 1 to 3, 3 to 2, or 2 to 1.

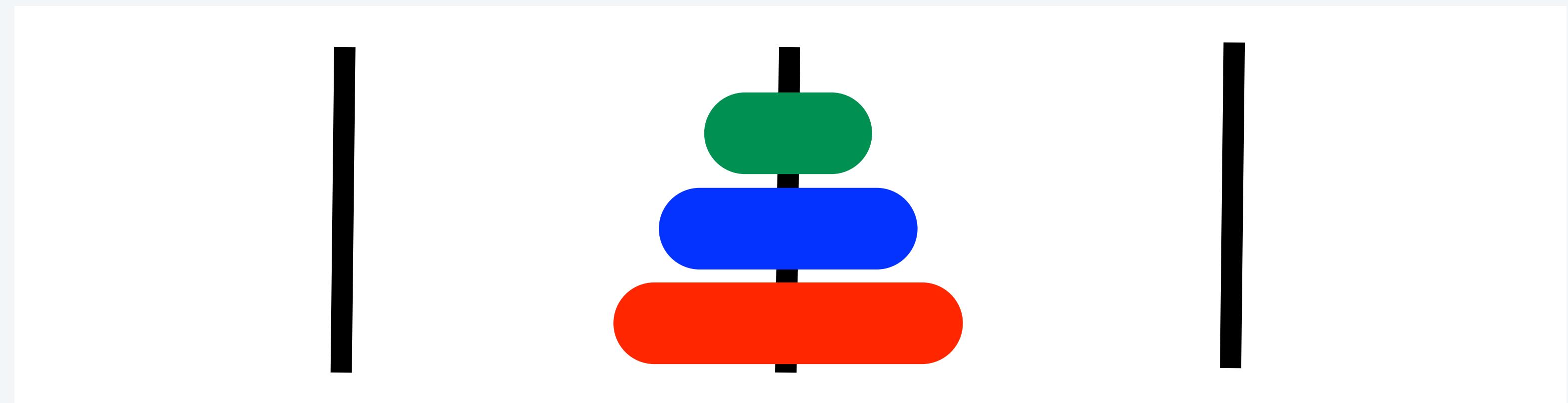


A recursive solution

- Move  $n - 1$  discs to the left (recursively).
- Move largest disc to the *right*.
- Move  $n - 1$  discs to the left (recursively).



# Towers of Hanoi solution (n = 3)



1R

2L

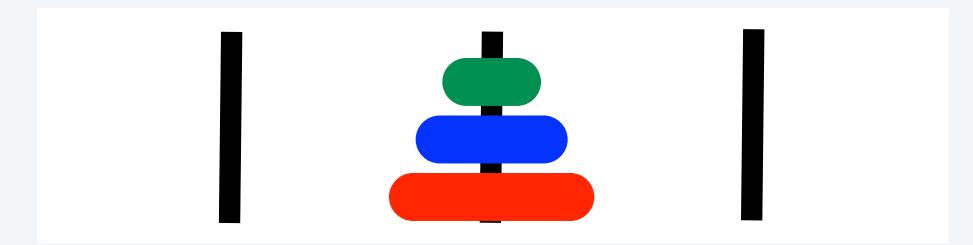
1R

3R

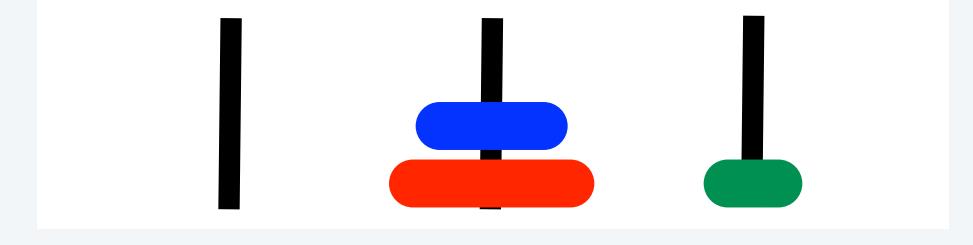
1R

2L

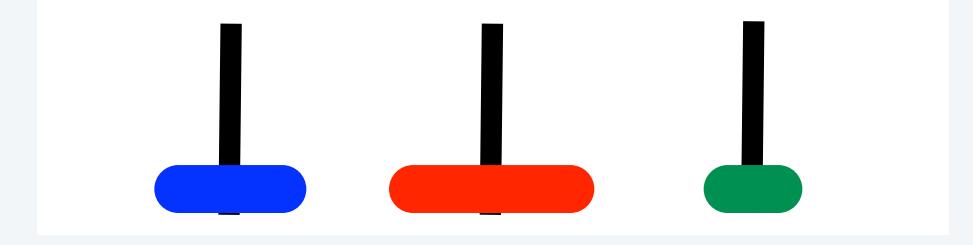
1R



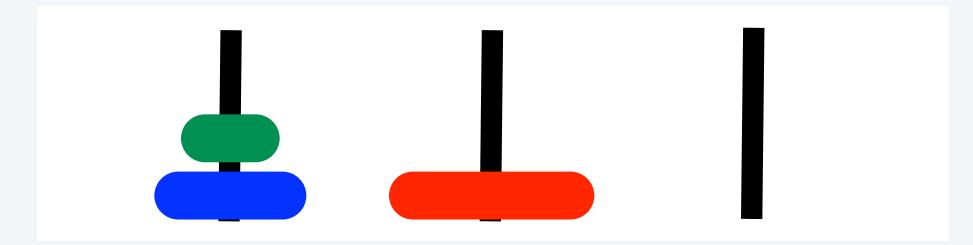
1R



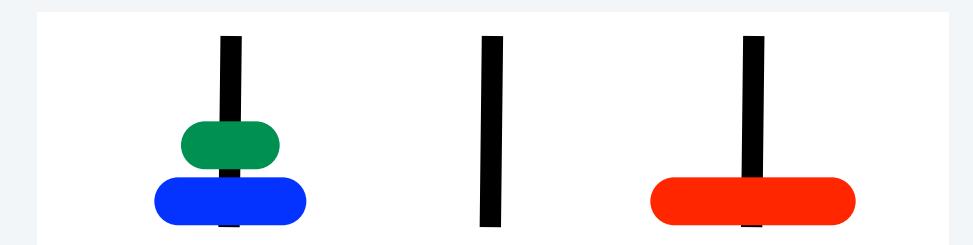
1R



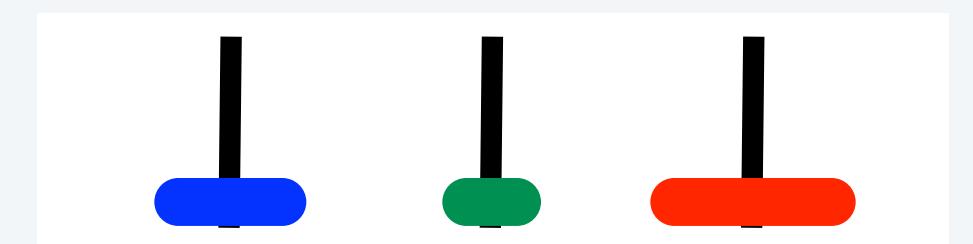
2L



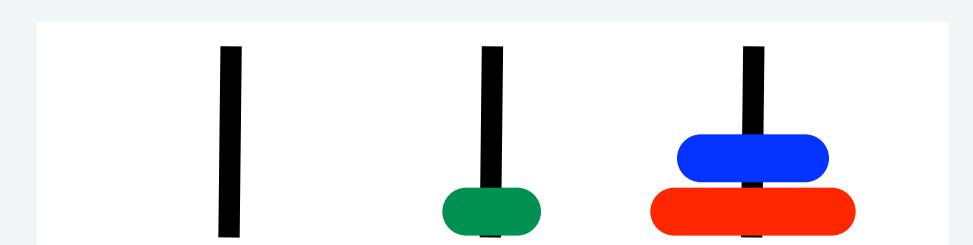
1R



3R



1R



2L



1R

## Towers of Hanoi: recursive solution

hanoi( $n$ ): Print moves for  $n$  discs.

- Return one space for  $n = 0$ .
- Otherwise, set move to the specified move for disc  $n$ .
- Then sandwich move between two copies of hanoi( $n-1$ ).

```
public class Hanoi
{
    public static String hanoi(int n, boolean left)
    {
        if (n == 0) return " ";
        String move;
        if (left) move = n + "L";
        else      move = n + "R";
        return hanoi(n-1, !left) + move + hanoi(n-1, !left);
    }

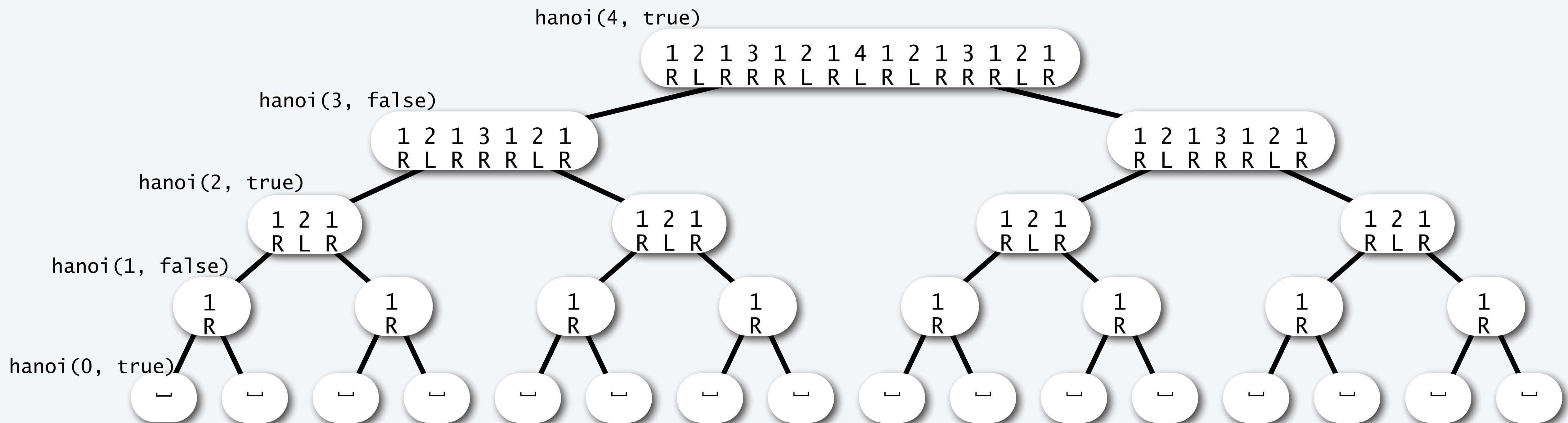
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        StdOut.println(hanoi(n, false));
    }
}
```

```
% java Hanoi 3
1R 2L 1R 3R 1R 2L 1R
```

## Recursive call tree for towers of Hanoi

Structure is the *same* as for the ruler function and suggests 3 useful and easy-to-prove facts.

- Each disc always moves in the same direction.
- Moving smaller disc always alternates with a unique legal move.
- Moving  $n$  discs requires  $2^n - 1$  moves.



# Answers for towers of Hanoi

Q. Generate list of instructions for monks ?

A. (Long form). 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L 5L 1L 2R 1L 3L 1L 2R 1L 4R ...

A. (Short form). Alternate "1L" with the only legal move not involving the disc 1.

"L" or "R" depends on whether  $n$  is odd or even

Q. When might the world end ?

A. Not soon: need  $2^{64} - 1$  moves.

Note: Recursive solution has been proven optimal.



<i>moves per second</i>	<i>end of world</i>
1	5.84 billion centuries
1 billion	5.84 centuries

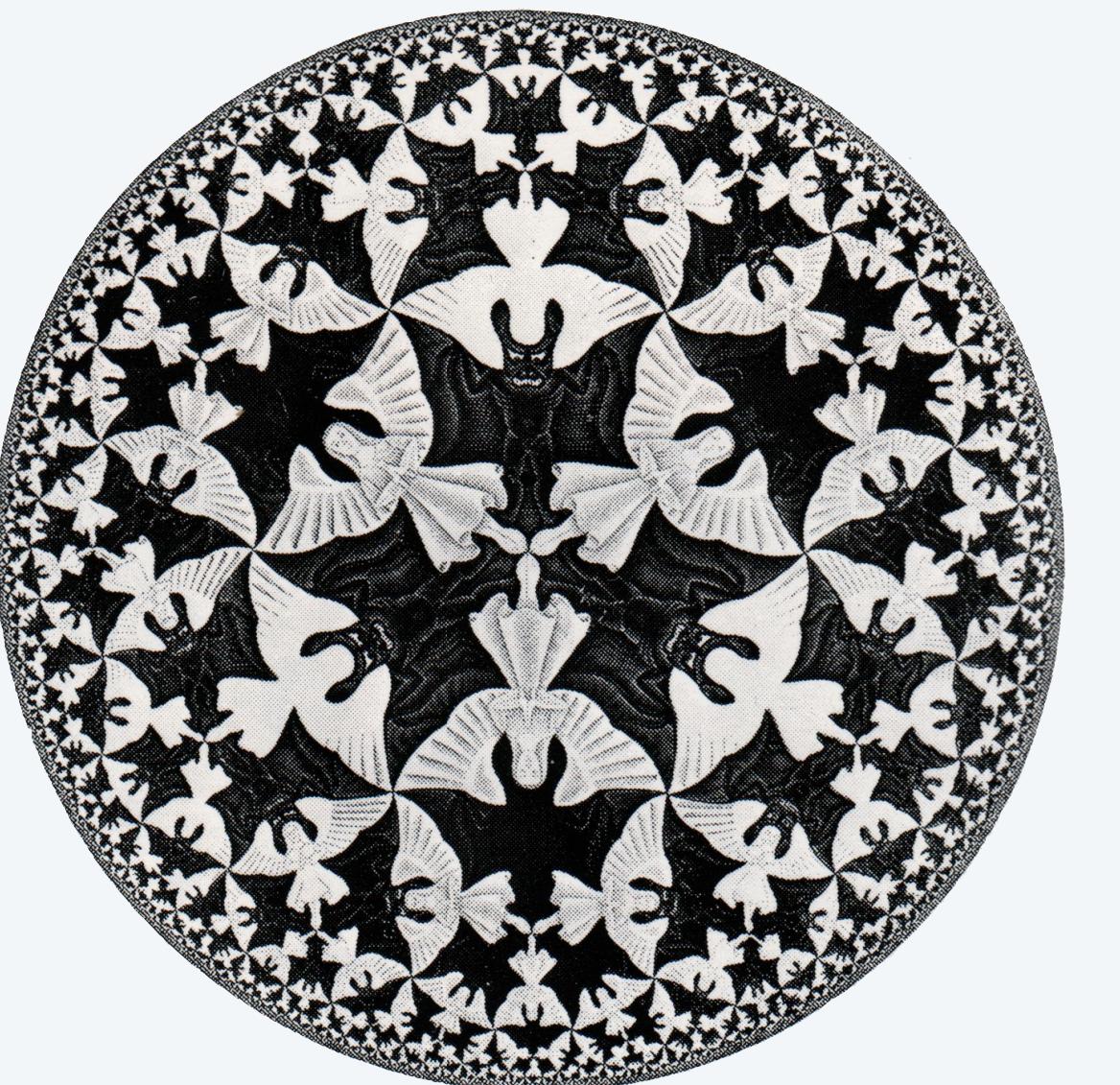
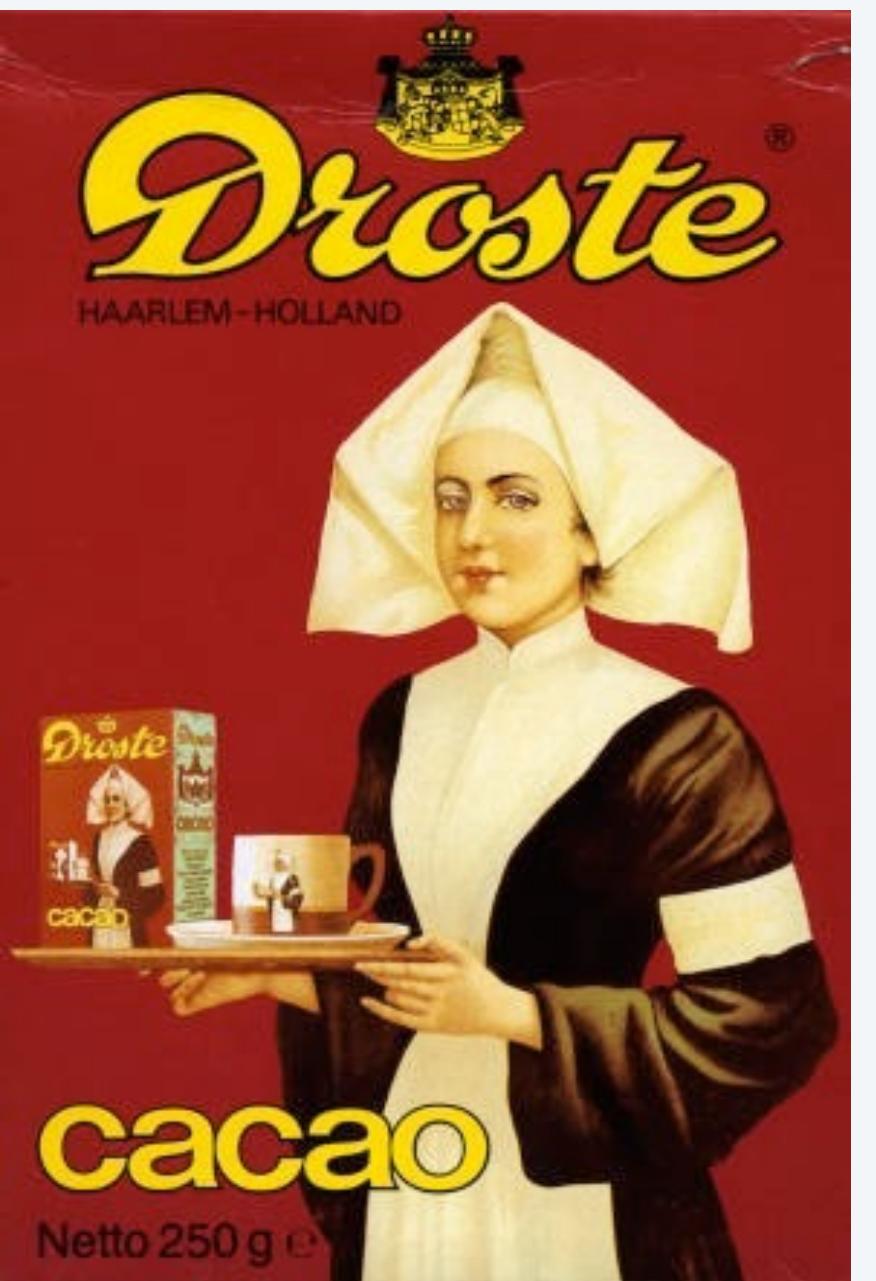


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## 6. Recursion

- Foundations
- A classic example
- **Recursive graphics**
- Avoiding exponential waste
- Dynamic programming

# Recursive graphics in the wild



**WEEKEND Arts FINE ARTS LEISURE**

**The New York Times**

**WEEKEND Arts FINE ARTS LEISURE**

**FRIDAY, DECEMBER 15, 2006**

**Design Life Now** or the Cooper-Hewitt National Design Museum makes this toy from the New York company Kidrobot.

**Fruits of Design, Certified Organic**

It's Triennial time at the Cooper-Hewitt National Design Museum. This means that the former Andrew Carnegie mansion is up to its necks in mostly American design from the 19th century to the present. "Design Life Now," the museum's third National Design Triennial, is a craze affair that illuminates the field but fails to call it to order.

**ROBERTA SMITH** The exhibition has been organized by the Cooper-Hewitt curator Barbara Blomberg, along with Martin McCann and a guest, Brooke Hodge, a curator at the Museum of Contemporary Art, Los Angeles.

Once again, we are asked the question: "What's design?" with the evasive catchall: "What's not?" Covering so many bases so equivalently, it never gets anywhere. A question like "What is good design?" or, more to the point, "What is design good for?" It refers to take sides on the issue of whether design is art and can be appreciated for its own merit or serve a relatively decorative purpose. Still, the show's benefits are many, even if you have to work for them.

The exhibits are a rambling tour of the field, delightful to digest. They cover little-extending innovations, completely frivolous restatements of received ideas (far too many of which trace to the realists) and more varieties of design than can be imagined. Fashion, building materials, furniture, toys, theatrical sets, jewelry and textiles, medical and military hardware, all quality and quantity.

The main point comes across loud and clear: design permeates every aspect of contemporary life. Everything is designed, and nothing is left to chance. And while all of nature's designs are intelligent, whether you go by Darwin or the Bible, the human kind are much

*Continued on Page S1*

**The Gifts to Open Again and Again**

It's Triennial time at the Cooper-Hewitt National Design Museum. This means that the former Andrew Carnegie mansion is up to its necks in mostly American design from the 19th century to the present. "Design Life Now," the museum's third National Design Triennial, is a craze affair that illuminates the field but fails to call it to order.

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*Continued on Page S1*

**Divine and Devotee Meet Across Hinges**

In one of Jorge Luis Borges's best-known short stories, "Pierre Menard, Author of the Quixote," a 20th-century man writes an exact copy of a very famous 17th-century novel. The copy is an enormous, endlessly fascinating family scrapbook for New Yorkers, who can eavesdrop baby pictures of the Platonic philosopher, his wife, and their children, hundreds of pages and thousands of photographs, to the big, grown-up New York of the Lipstick Building, countless Trump projects and the Woolworth Building.

*Continued on Page 40*

**Black, White and Read All Over Over**

When a young Turkish artist named Serkan Ozkaya set out recently to practice his skills as a copier — a scrivener, as he says — his goals were a bit more modest than those of the man in the bathtub copy of Cervantes's 17th-century masterpiece. He simply wanted to draw and see printed a faithful copy of all the type and pictures planned for a broadsheet newspaper.

Probably nothing interests Ozkaya more than the printed page. Creating and reading newspapers is his passion. "I'm a newspaper freak," he says. "I collect them. I can't live without them. I can't sleep without them. I can't eat without them. I can't drink without them. I can't do anything without them."

He may be a bit obsessed, but he's also a genius. His drawings duplicate what Borges's narrator finds to be "infinitely richer" than the original because it contains all manner of new meanings and inflections, wrenching it as it is from its proper time and context.

*Continued on Page 51*

## Divine and Devotee Meet Across Hinges

WASHINGTON — For soothie, did St. Apollonia ASAP. She'll bring relief if a flash. Keep St. Matthew, ex-harpooner, in mind in April; he'll help get your taxes in shape. Everyone knows that a prayer to St. Rita, pray'r, can end a bad marriage.

**HOLLAND** plague, as good as a fire shot, and that lightning with a broadsword, is still a mystery.

**COTTER** Fa's on the job.

**ART REVIEW** Most important, for dire mental confusion, incurable grief, sickness of soul — there's the Virgin. Day and night she's on the toll-free hot line offering gentle attention and prudent advice.

To European Christians half a millennium ago, the Virgin and a raft of familiar saints were the exalted persons in kind of celestial welfare system, available to all believers. And one quick way to remember them was to look at a picture of the kind found in "Prayers and Portraits: Unfolding the Netherlands Dijptych" in the National Gallery of Art.

Probably nothing in Washington art compares to it for perfection. And these pictures, produced by the likes of Jan van Eyck, Rogier van der Weyden and Hugo van der Goes across an area that now encompasses the Netherlands, Belgium, Luxembourg and parts of France. These painters were pictorial magicians, conjuring visual worlds, cosmopolitan abstract art and microscopically realistic, of peerless beauty.

You see all of this in one glance at the 40 double-painted panels, of which 16 are here. Then you learn gradually as you move through the show how diptych art came to be, how it was made and reconstructed over the centuries, with the result that few survive in their intended form.

"Prayers and Portraits" is an attempt to restore some of the lost art. It brings art historians and art critics back to the fold. It brings art historians and art critics back to the fold.

*Continued on Page 44*

**Prayers and Portraits:** Unfolding the Netherlands Dijptych. Two panels of an early 15th-century diptych by Michel Sittow, left, are on view in an exhibition at the National Gallery of Art in Washington through Feb. 18.

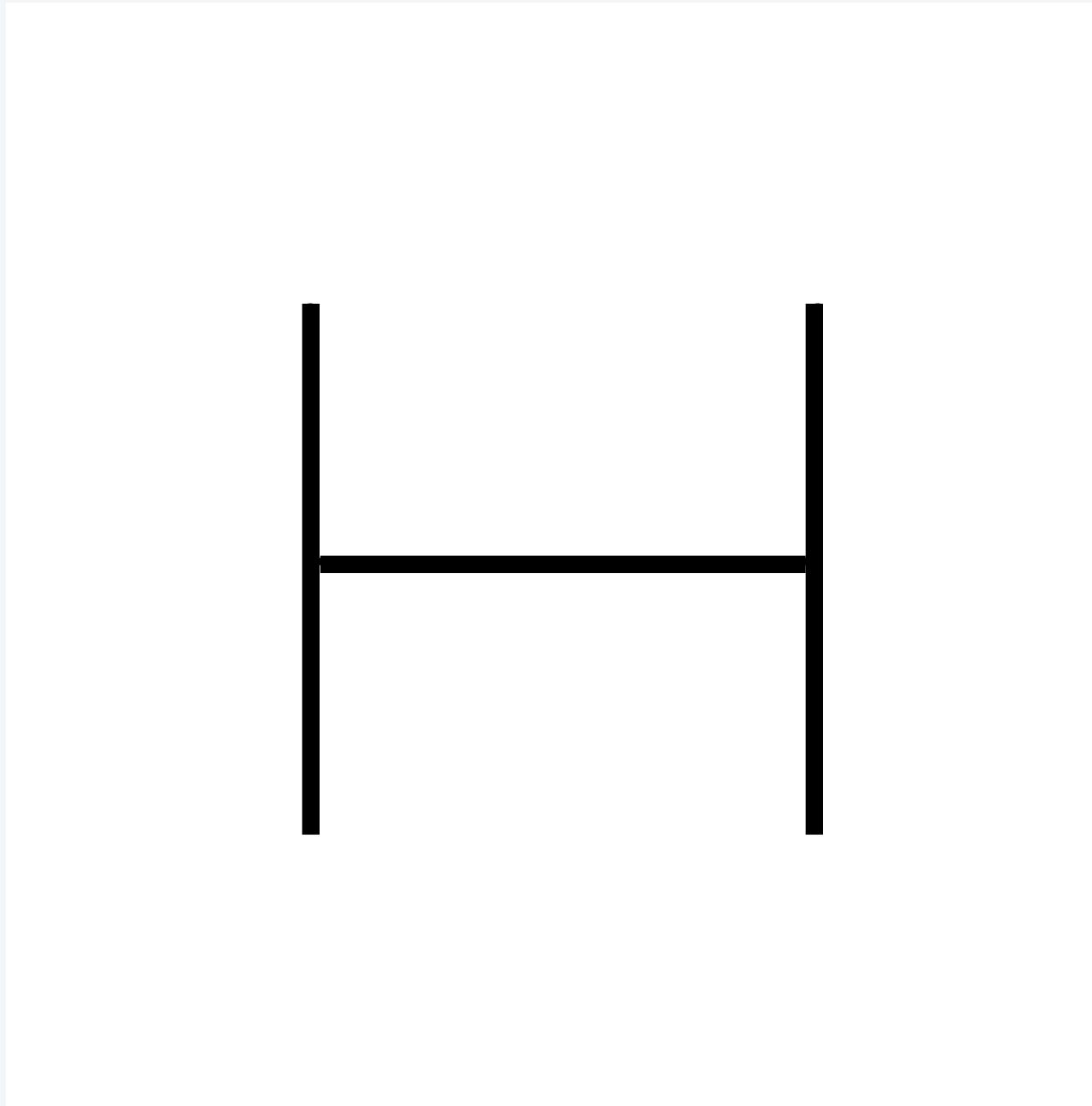


# "Hello, World" of recursive graphics: H-trees

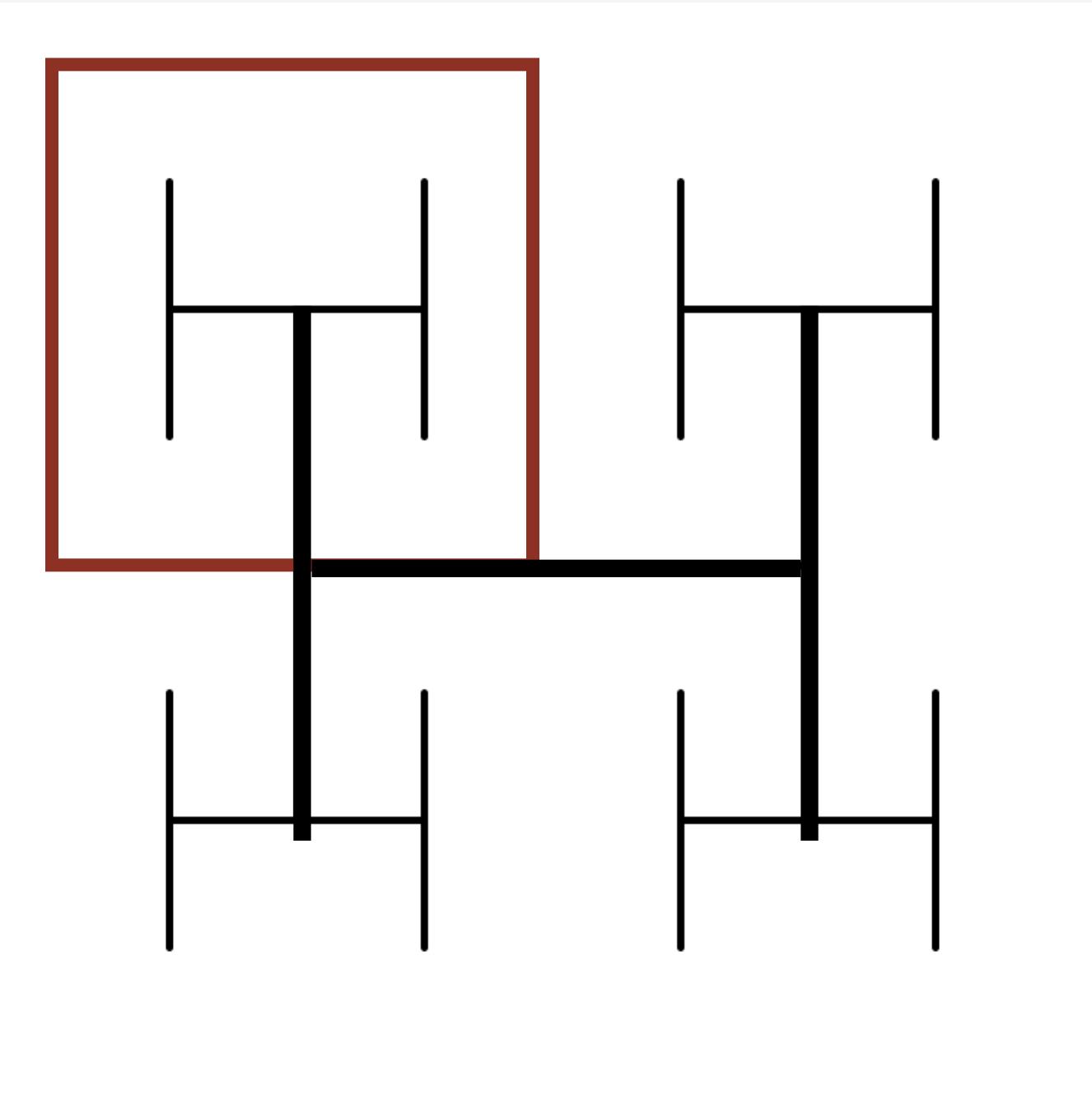
H-tree of order  $n$

- If  $n$  is 0, do nothing.
- Draw an H, centered.
- Draw four H-trees of order  $n - 1$  and half the size, centered at the tips of the H.

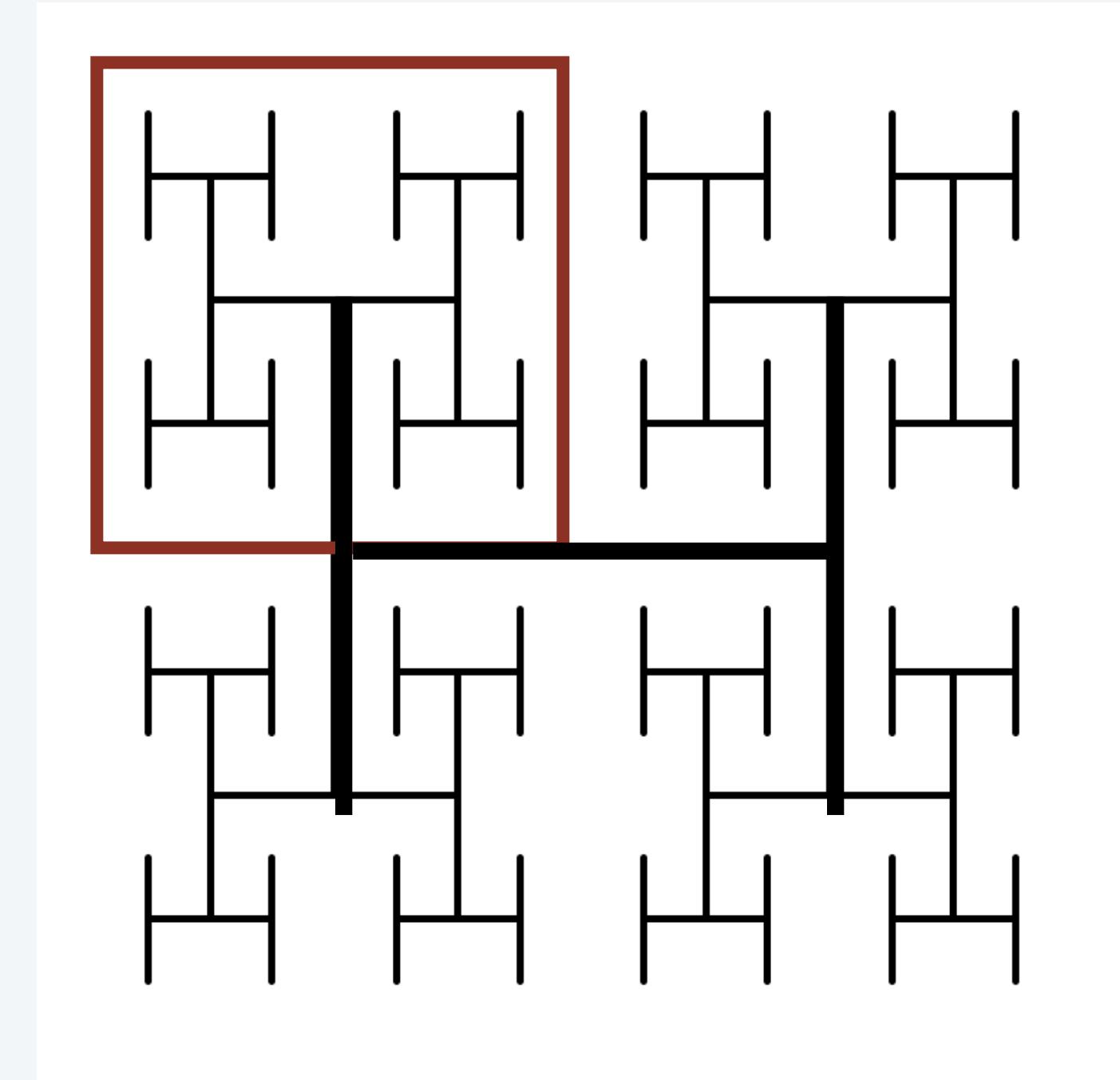
order 1



order 2



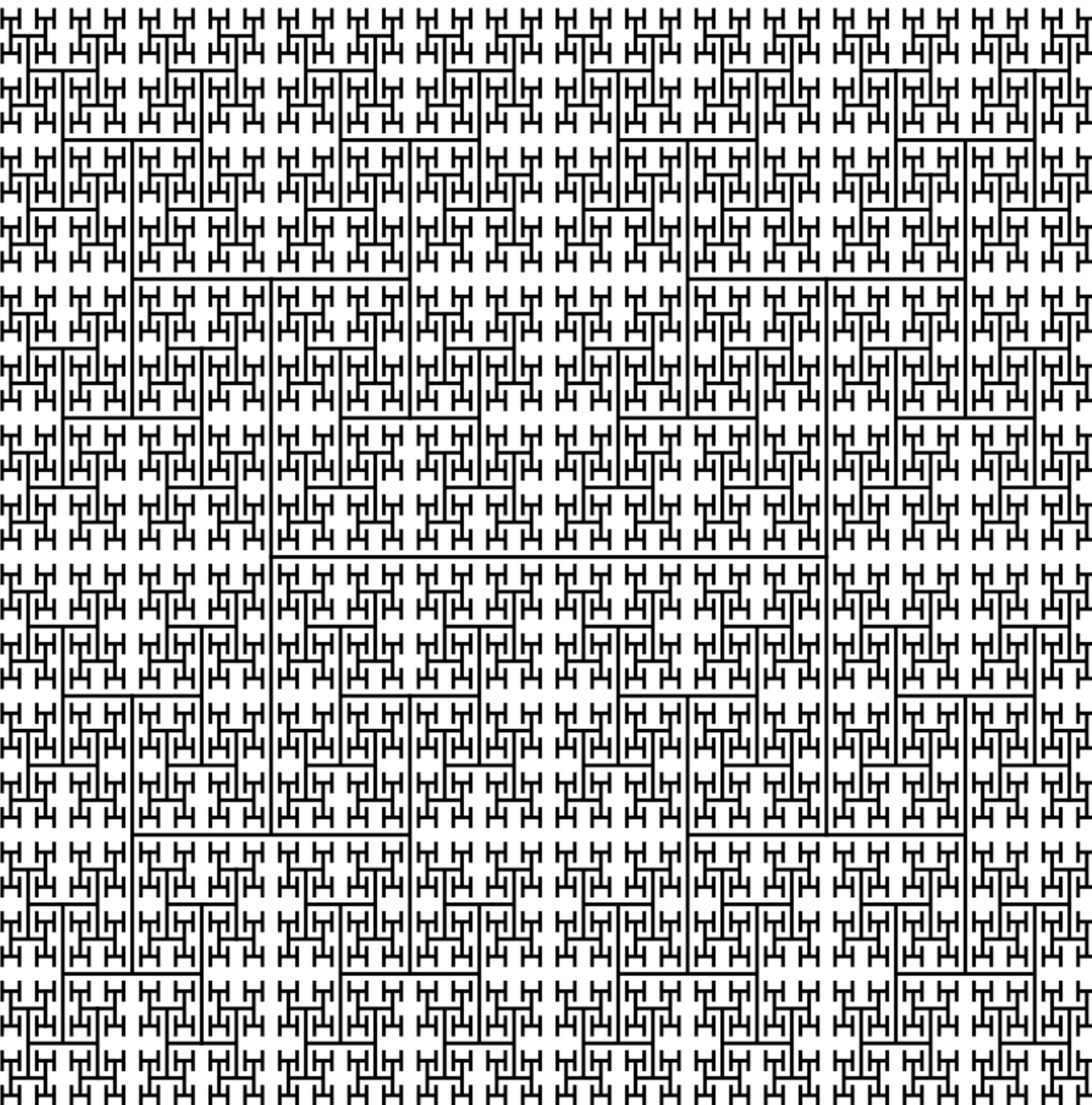
order 3



# H-trees

---

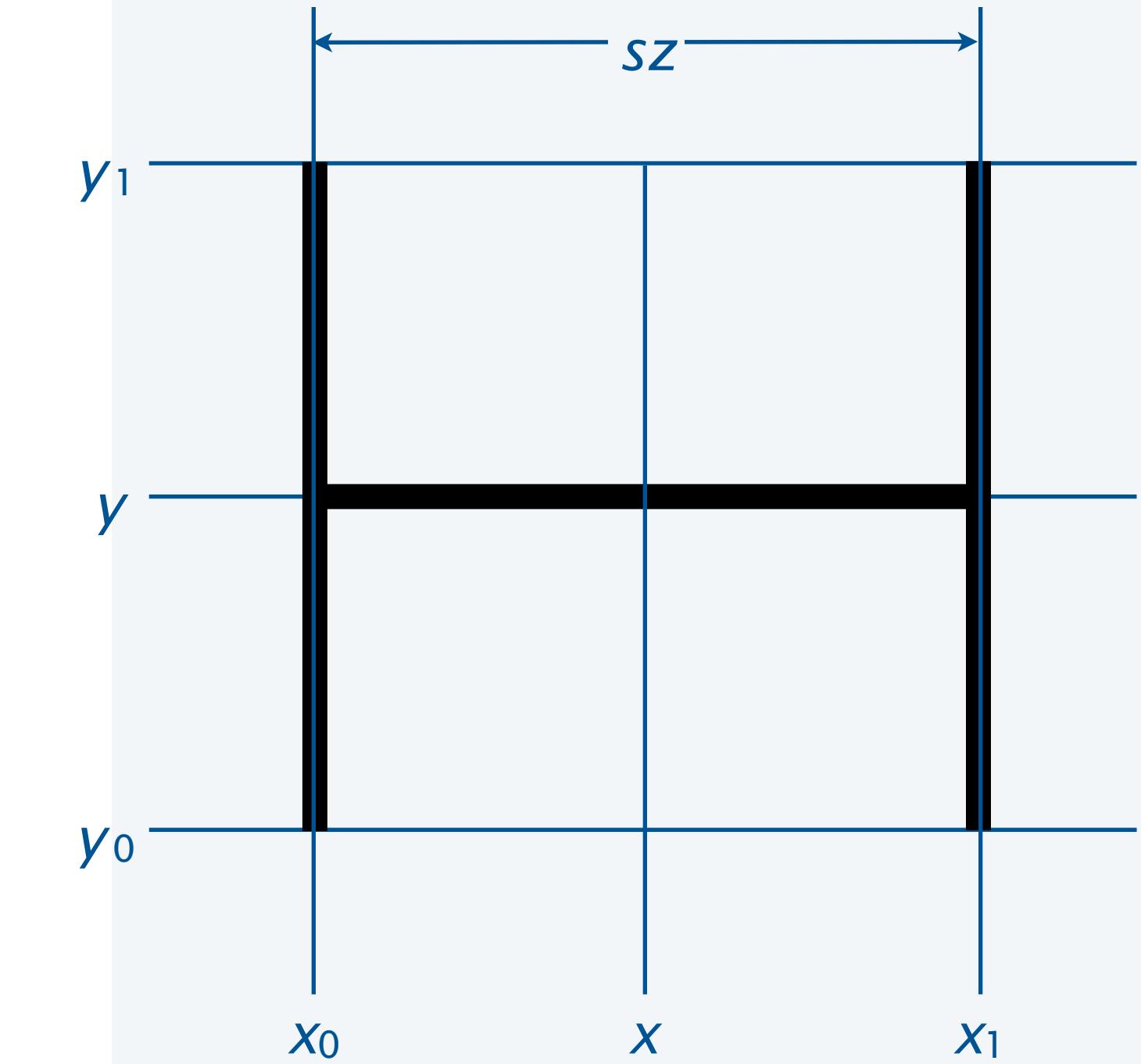
**Application.** Connect  
a large set of regularly  
spaced sites to a  
single source.



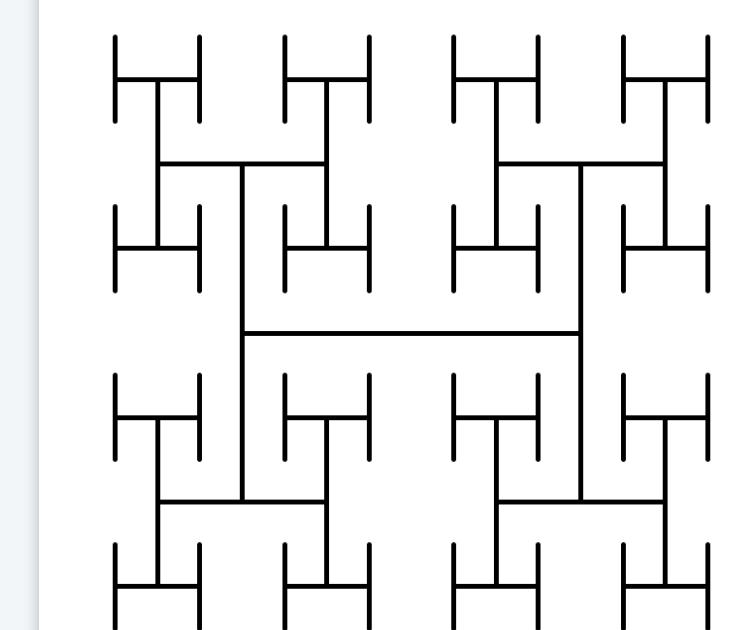
order 6

# Recursive H-tree implementation

```
public class Htree
{
    public static void draw(int n, double sz, double x, double y)
    {
        if (n == 0) return;
        double x0 = x - sz/2, x1 = x + sz/2;
        double y0 = y - sz/2, y1 = y + sz/2;
        StdDraw.line(x0, y, x1, y);
        StdDraw.line(x0, y0, x0, y1); ← draw the H,
        StdDraw.line(x1, y0, x1, y1); centered on (x, y)
        draw(n-1, sz/2, x0, y0);
        draw(n-1, sz/2, x0, y1); ← draw four
        draw(n-1, sz/2, x1, y0); half-size H-trees
        draw(n-1, sz/2, x1, y1);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        draw(n, .5, .5, .5);
    }
}
```



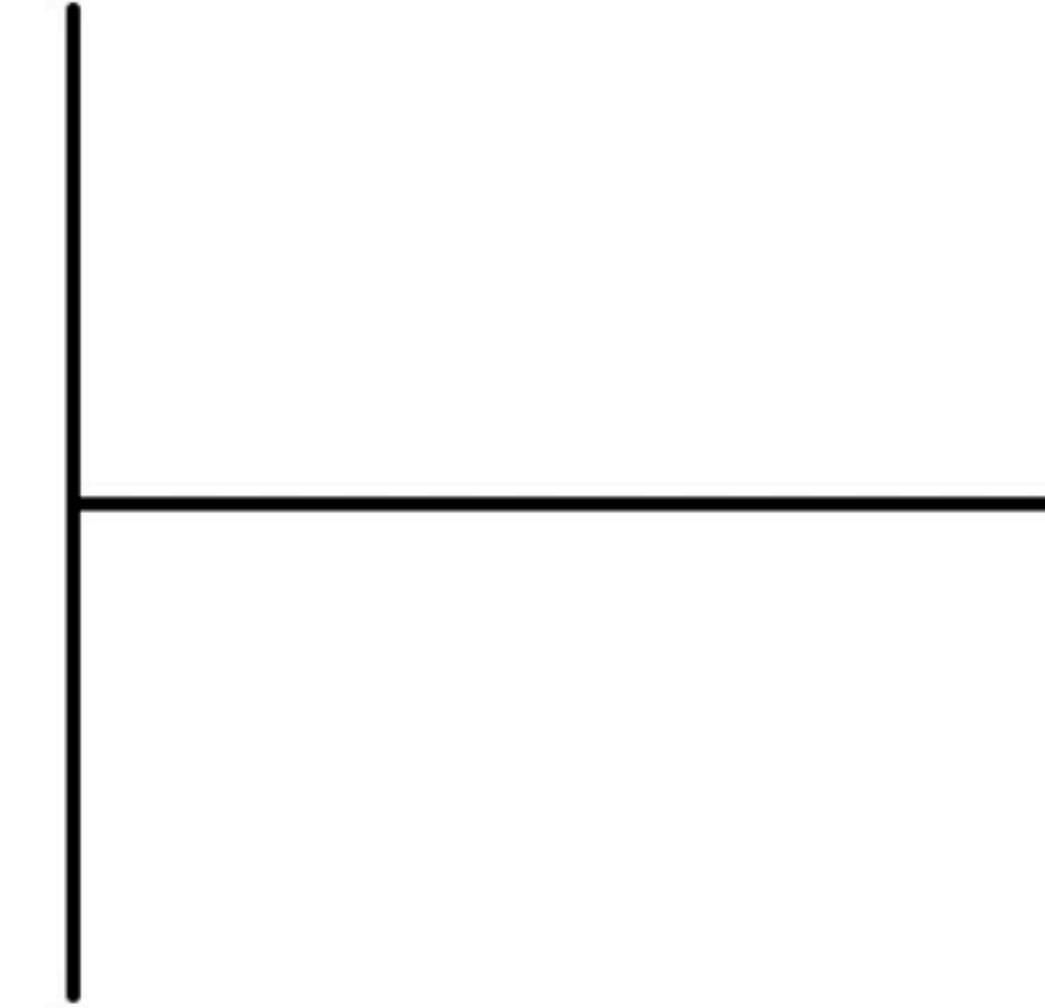
% java Htree 3



# Deluxe H-tree implementation

```
public class HtreeDeluxe
{
    public static void draw(int n, double sz,
                           double x, double y)
    {
        if (n == 0) return;
        double x0 = x - sz/2, x1 = x + sz/2;
        double y0 = y - sz/2, y1 = y + sz/2;
        StdDraw.line(x0, y, x1, y);
        StdDraw.line(x0, y0, x0, y1);
        StdDraw.line(x1, y0, x1, y1);
        StdAudio.play(PlayThatNote.note(n, .25*n));
        draw(n-1, sz/2, x0, y0);
        draw(n-1, sz/2, x0, y1);
        draw(n-1, sz/2, x1, y0);
        draw(n-1, sz/2, x1, y1);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        draw(n, .5, .5, .5);
    }
}
```

```
% java HtreeDeluxe 4
```



# Fractional Brownian motion

---

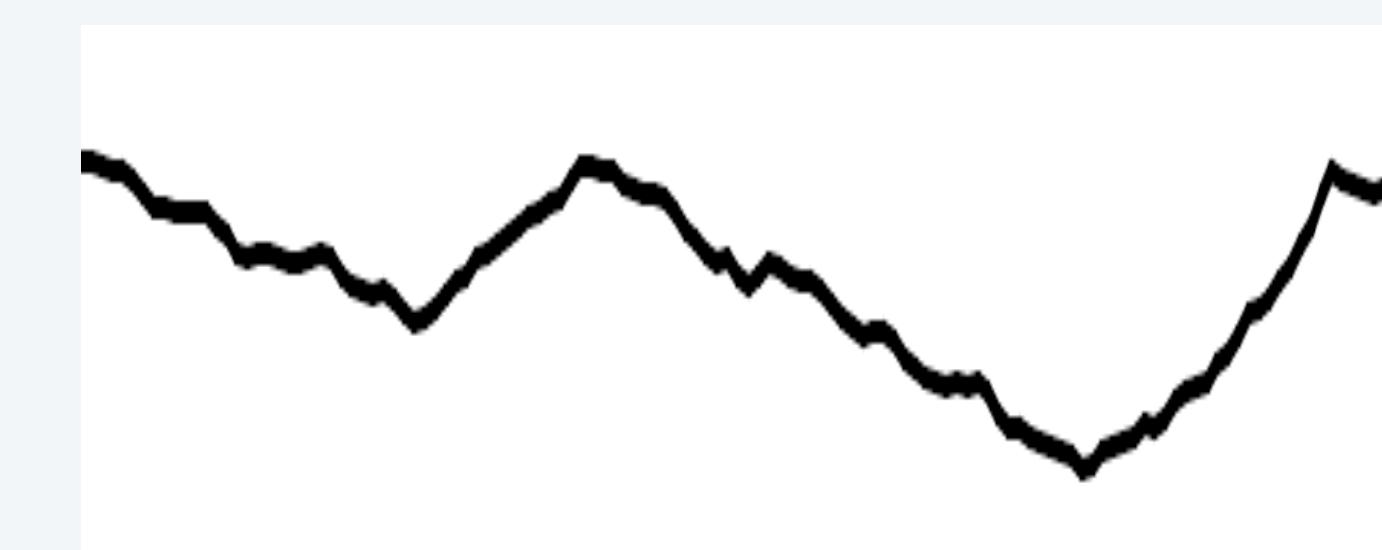
A process that models many phenomenon.

- Price of stocks.
  - Dispersion of fluids.
  - Rugged shapes of mountains and clouds.
  - Shape of nerve membranes.
- ...

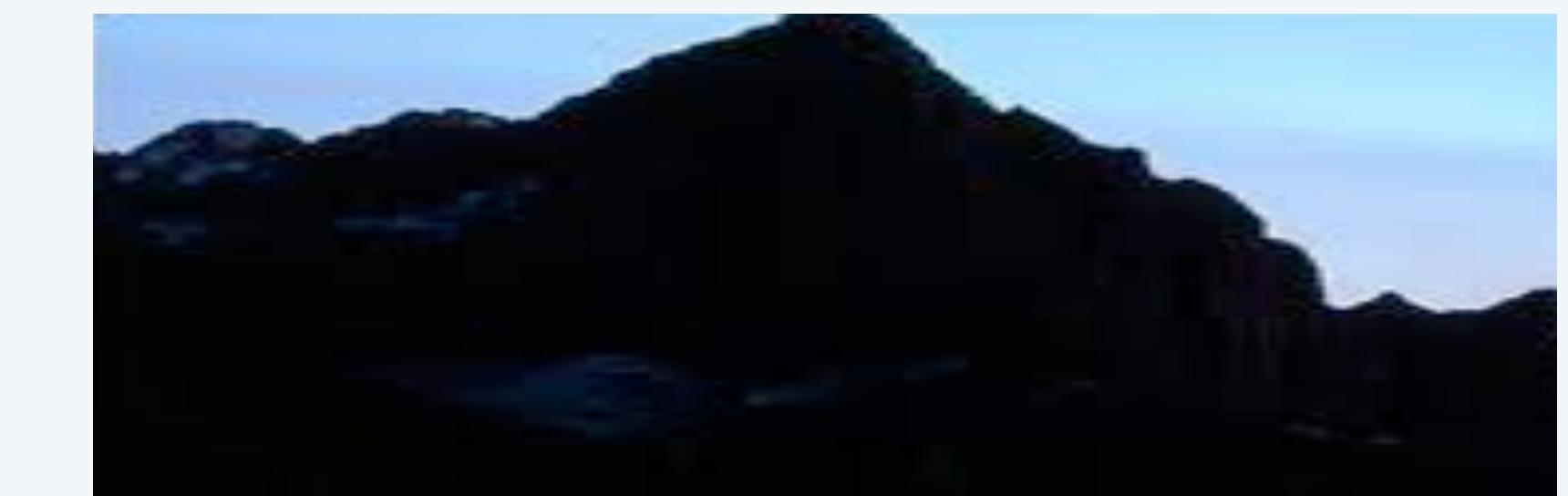
Price of an actual stock



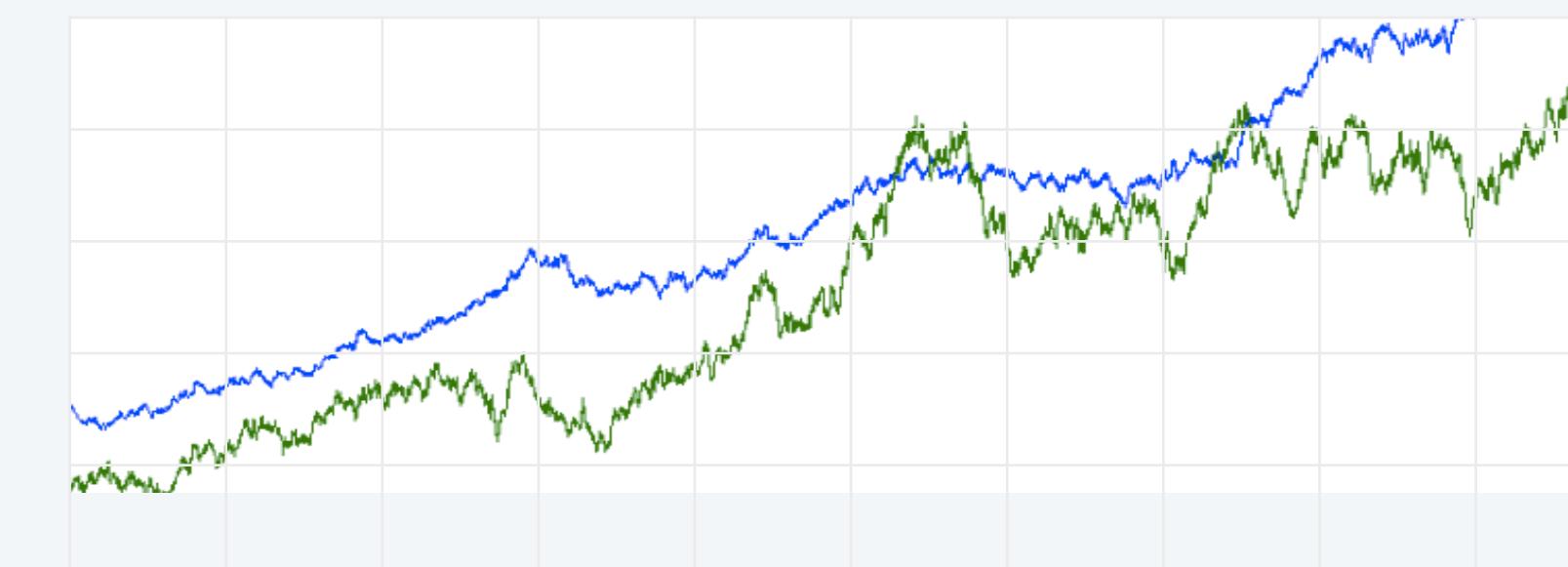
Brownian bridge model



An actual mountain



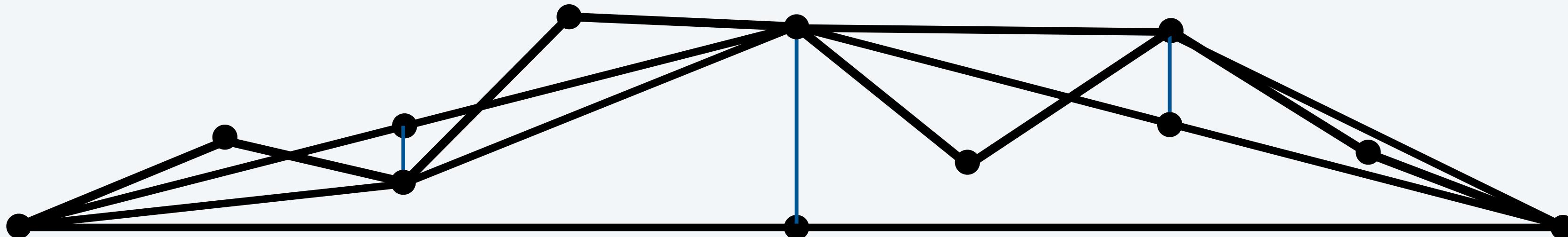
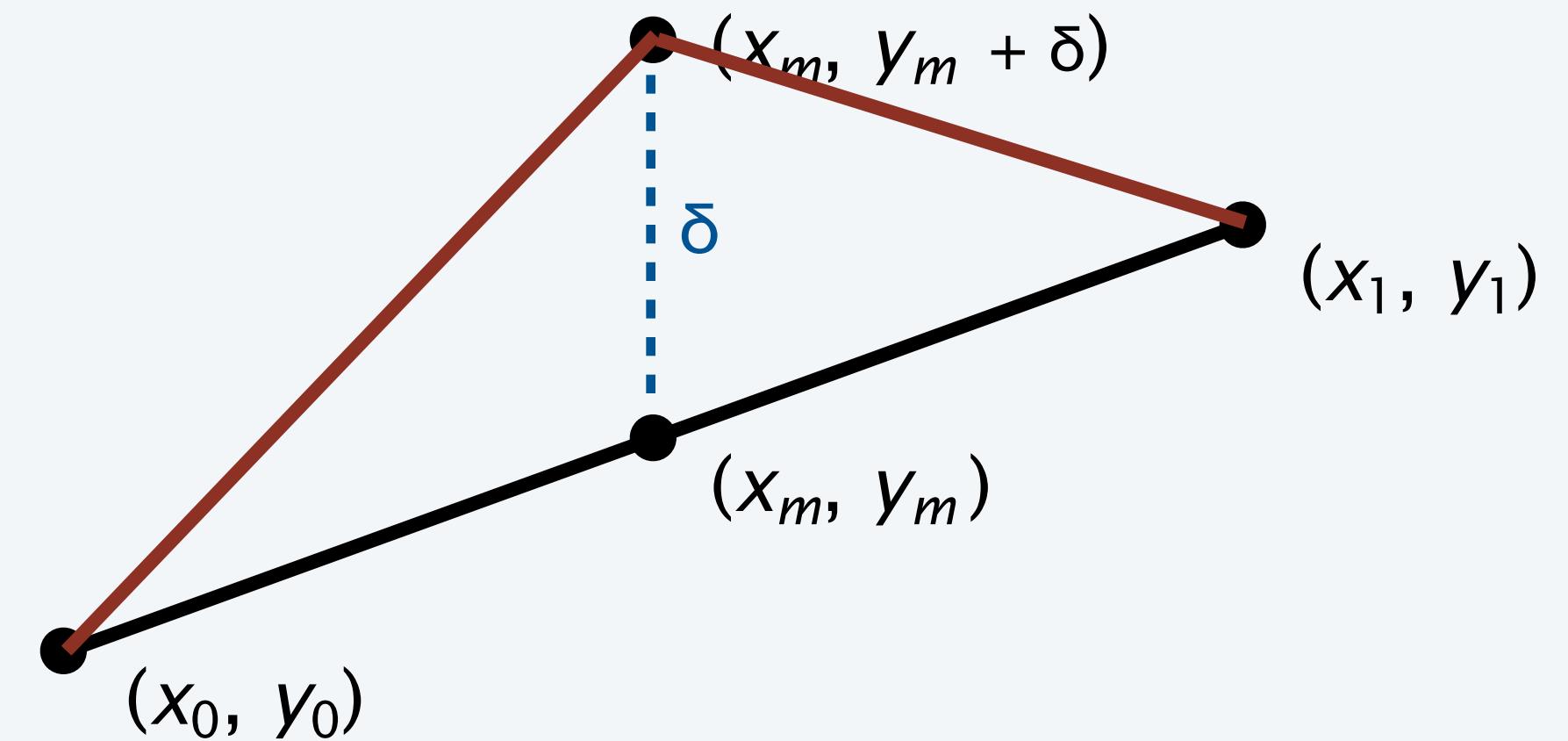
Black-Scholes model (two different parameters)



# Fractional Brownian motion simulation

## Midpoint displacement method

- Consider a line segment from  $(x_0, y_0)$  to  $(x_1, y_1)$ .
- If sufficiently short draw it *and return*
- Divide the line segment in half, at  $(x_m, y_m)$ .
- Choose  $\delta$  at random *from Gaussian distribution*.
- Add  $\delta$  to  $y_m$ .
- Recur on the left and right line segments.



# Brownian motion implementation

```
public class Brownian
{
    public static void
    curve(double x0, double y0, double x1, double y1,
          double var, double s)
    {
        if (x1 - x0 < .01)
        { StdDraw.line(x0, y0, x1, y1); return; }
        double xm = (x0 + x1) / 2;
        double ym = (y0 + y1) / 2;
        double stddev = Math.sqrt(var);
        double delta = StdRandom.gaussian(0, stddev);
        curve(x0, y0, xm, ym+delta, var/s, s);
        curve(xm, ym+delta, x1, y1, var/s, s);
    }

    public static void main(String[] args)
    {
        double hurst = Double.parseDouble(args[0]);
        double s = Math.pow(2, 2*hurst); ← control parameter
        curve(0, .5, 1.0, .5, .01, s);
    }
}
```

% java Brownian 1



% java Brownian .125

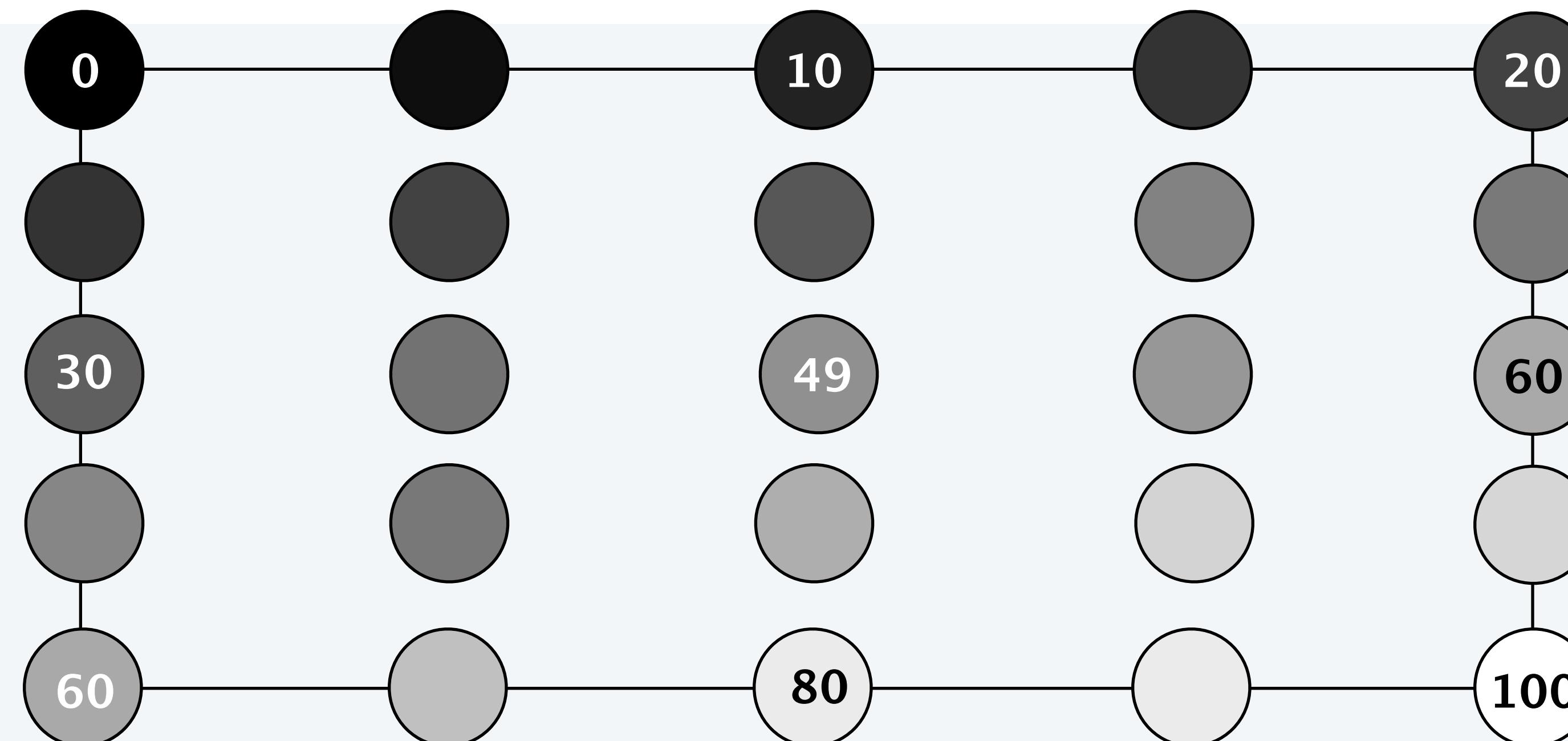


# A 2D Brownian model: plasma clouds

## Midpoint displacement method

- Consider a rectangle centered at  $(x, y)$  with pixels at the four corners.
- If the rectangle is small, do nothing.
- Color the midpoints of each side the average of the endpoint colors.
- Choose  $\delta$  at random *from Gaussian distribution*.
- Color the center pixel the average of the four corner colors *plus*  $\delta$
- Recurse on the four quadrants.

Books site code actually  
draws a rectangle to  
avoid artifacts



# A Brownian cloud

# A Brownian landscape



*Image sources*

[http://en.wikipedia.org/wiki/Droste\\_effect#mediaviewer/File:Droste.jpg](http://en.wikipedia.org/wiki/Droste_effect#mediaviewer/File:Droste.jpg)

<http://www.mcescher.com/gallery/most-popular/circle-limit-iv/>

<http://www.megamonalisa.com/recursion/>

<http://fractalfoundation.org/0FC/FractalGiraffe.png>

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## 6. Recursion

- Foundations
- A classic example
- Recursive graphics
- **Avoiding exponential waste**
- Dynamic programming

# Fibonacci numbers

Let  $F_n = F_{n-1} + F_{n-2}$  for  $n > 1$  with  $F_0 = 0$  and  $F_1 = 1$ .

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
$F_n$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	...



Models many natural phenomena and is widely found in art and architecture.

Leonardo Fibonacci  
c. 1170 – c. 1250

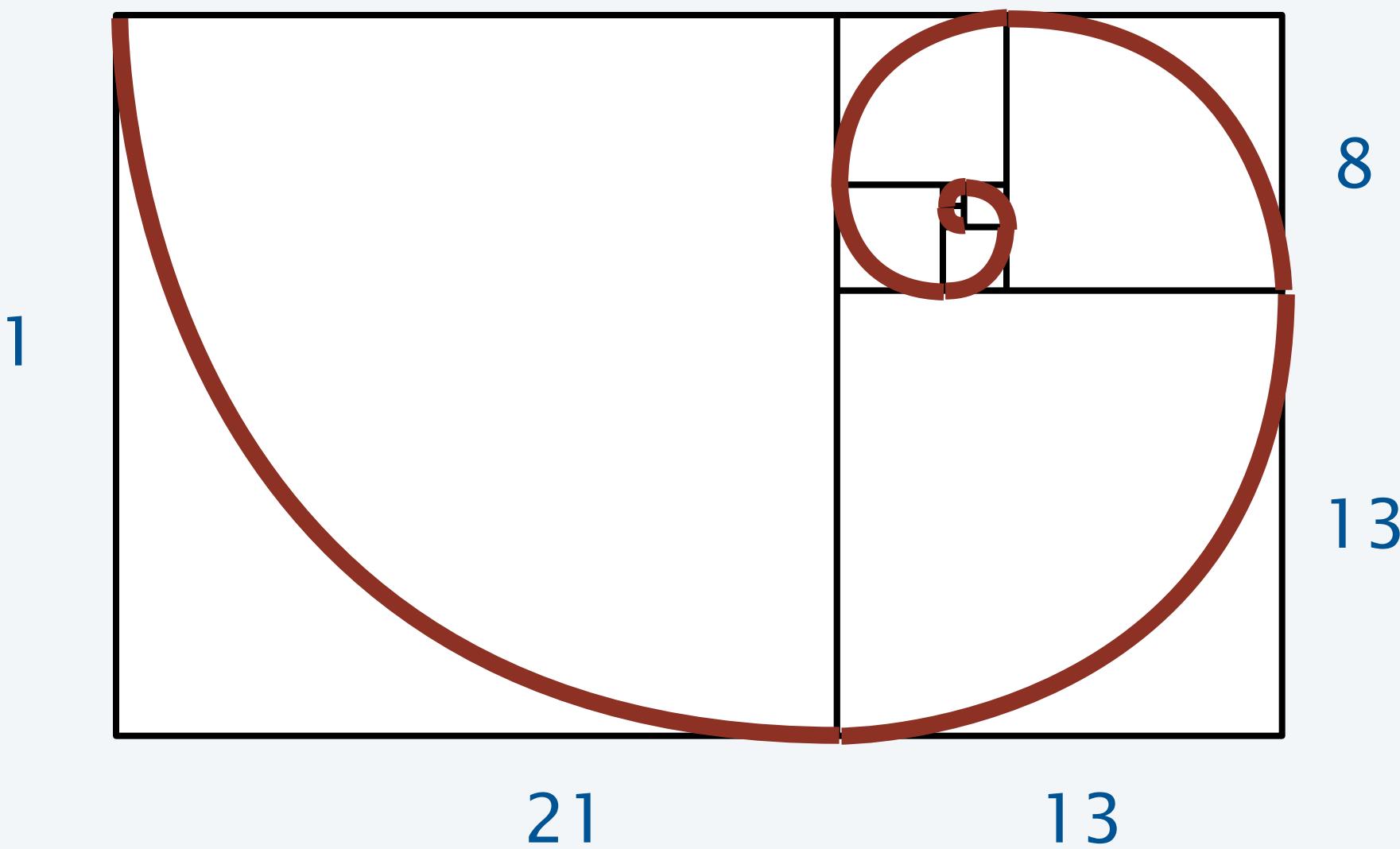
## Examples.

- Model for reproducing rabbits.
- Nautilus shell.
- Mona Lisa.
- ...

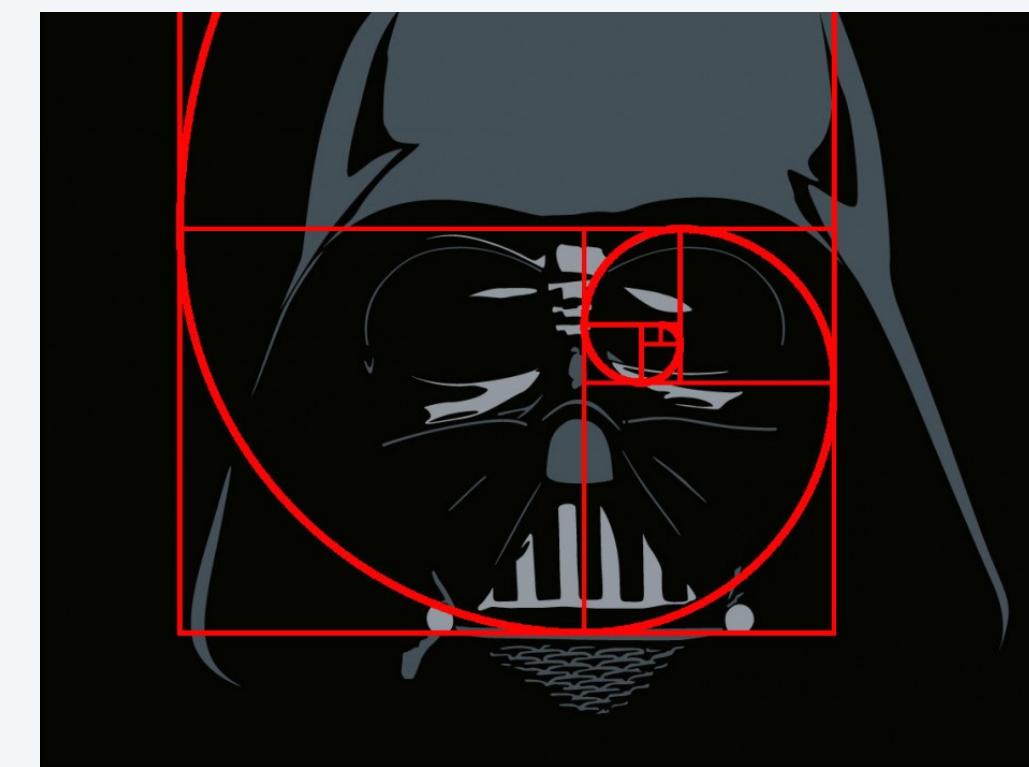
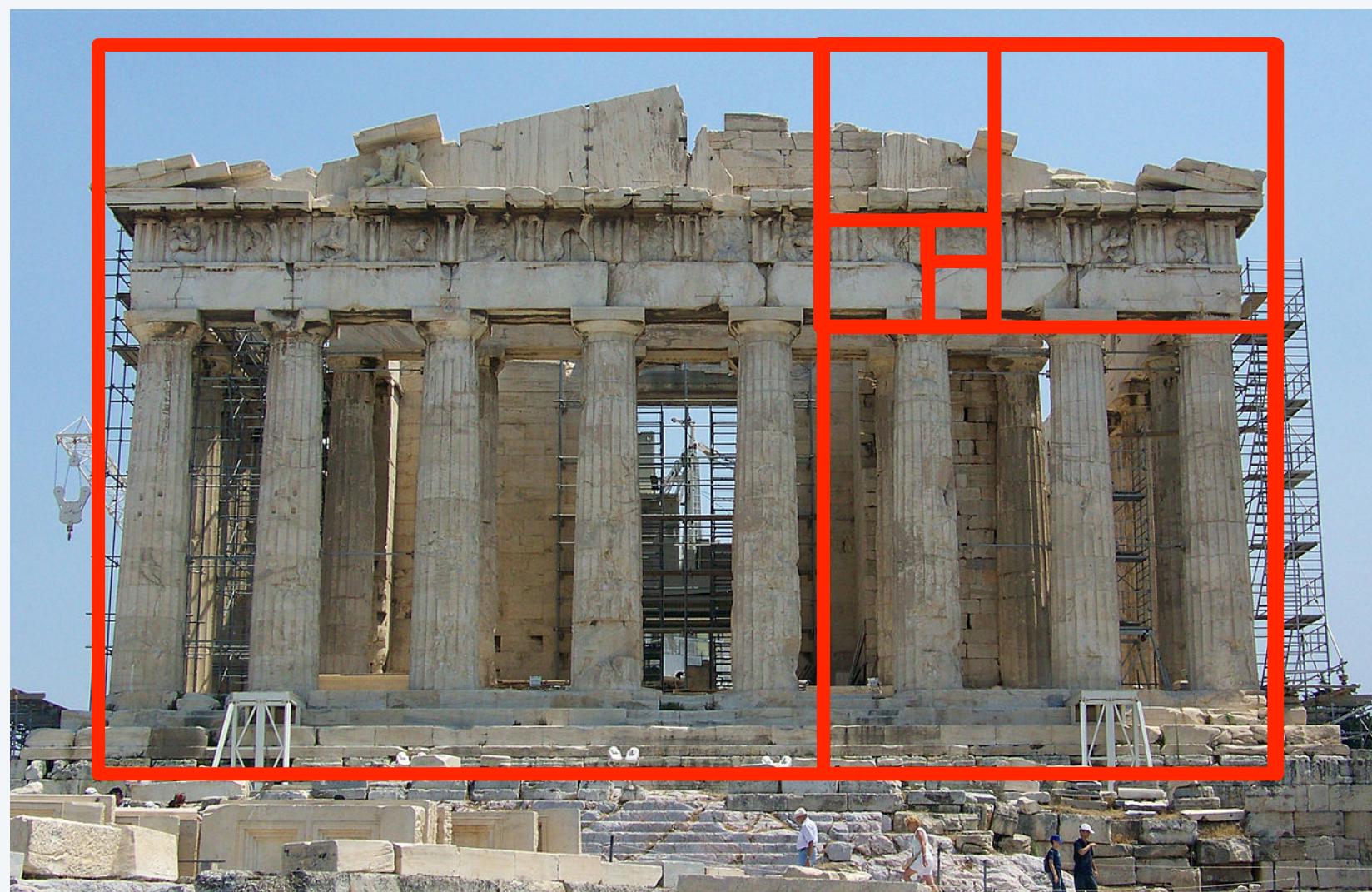
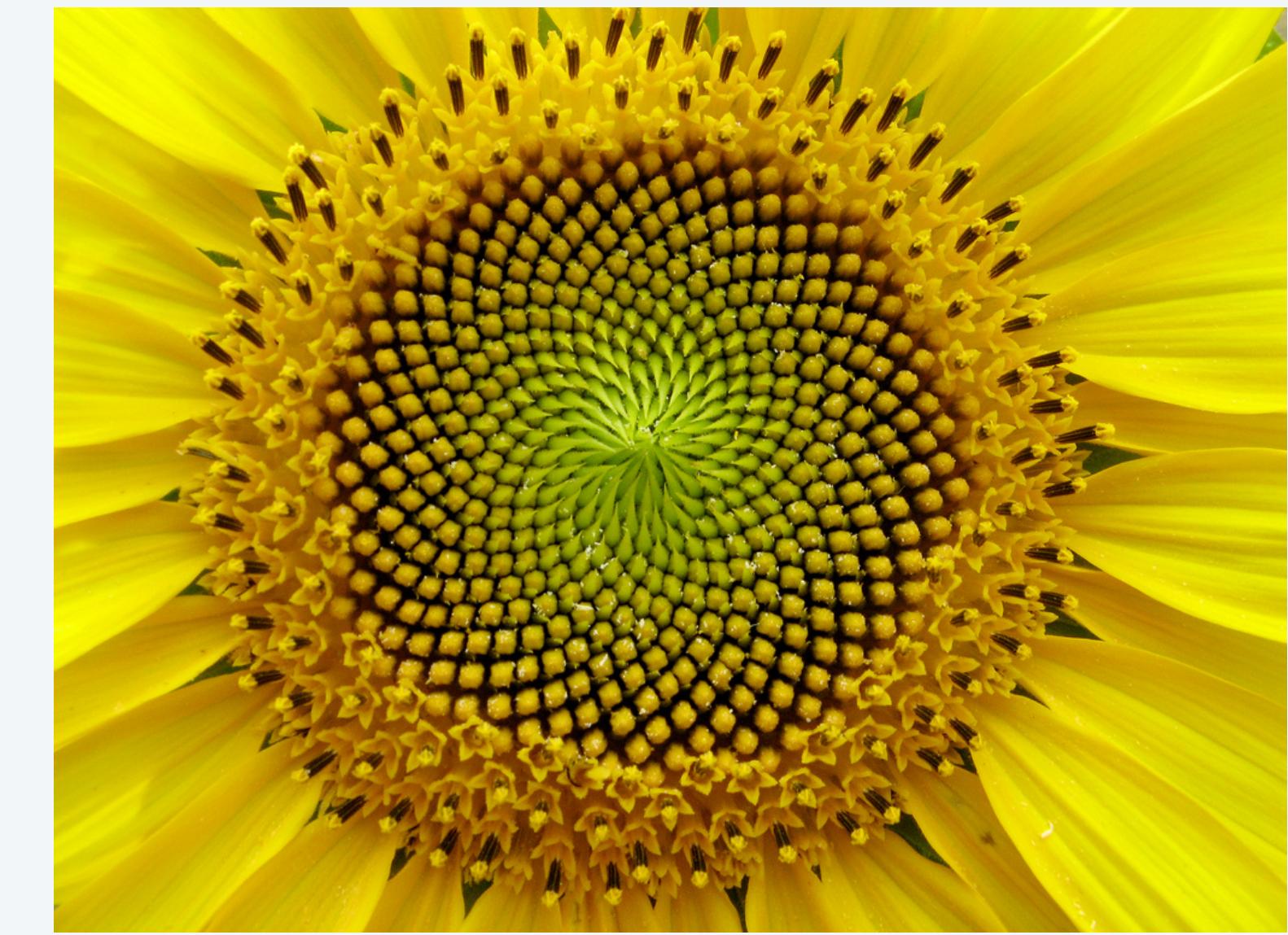
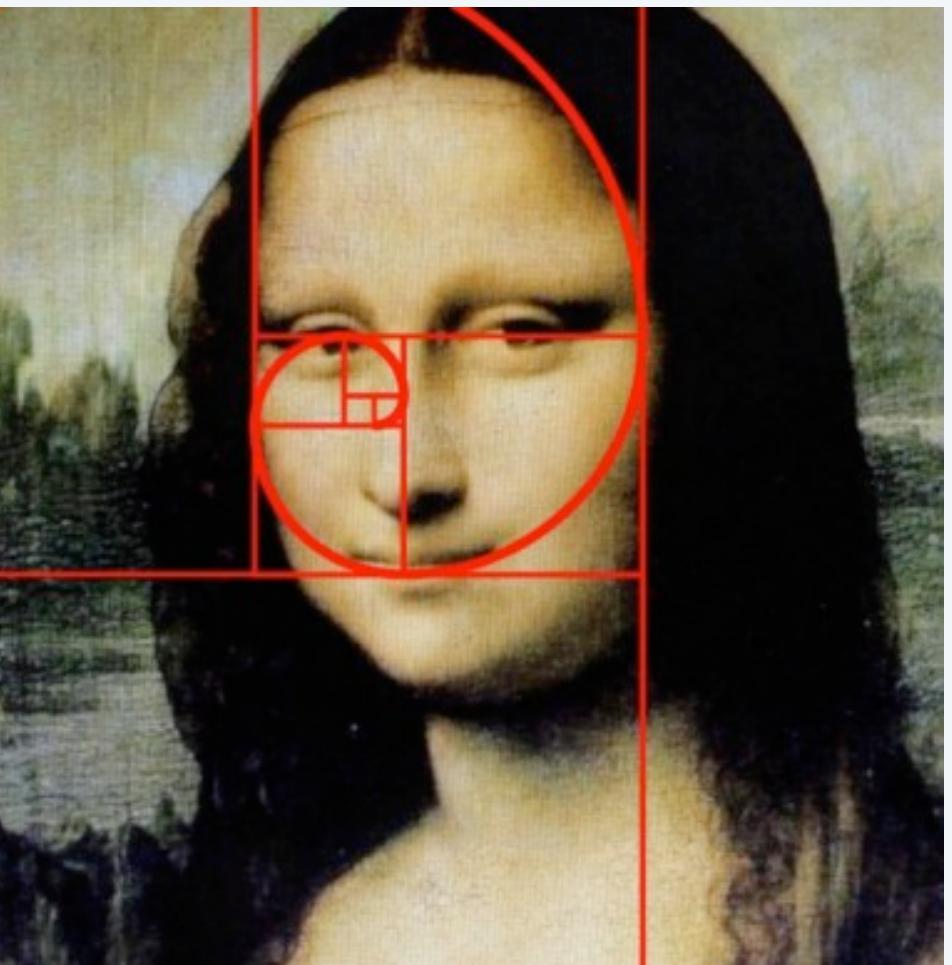
## Facts (known for centuries).

- $F_n / F_{n-1} \rightarrow \Phi = 1.618\dots$  as  $n \rightarrow \infty$
- $F_n$  is the closest integer to  $\Phi^n/\sqrt{5}$

golden ratio  $F_n / F_{n-1}$



# Fibonacci numbers and the golden ratio in the wild



1	1	2	3	5	8	13	21
1	1	2	1	1	2	1	
1	3	3	1	1	3	3	1
1	4	6	4	1	1	4	6
1	5	10	10	5	1	1	5
1	6	15	20	15	6	1	1
1	7	21	35	35	21	7	1

# Computing Fibonacci numbers

Q. [Curious individual.] What is the exact value of  $F_{60}$  ?

A. [Novice programmer.] Just a second. I'll write a recursive program to compute it.

```
public class FibonacciR
{
    public static long F(int n)
    {
        if (n == 0) return 0;
        if (n == 1) return 1;
        return F(n-1) + F(n-2);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        StdOut.println(F(n));
    }
}
```

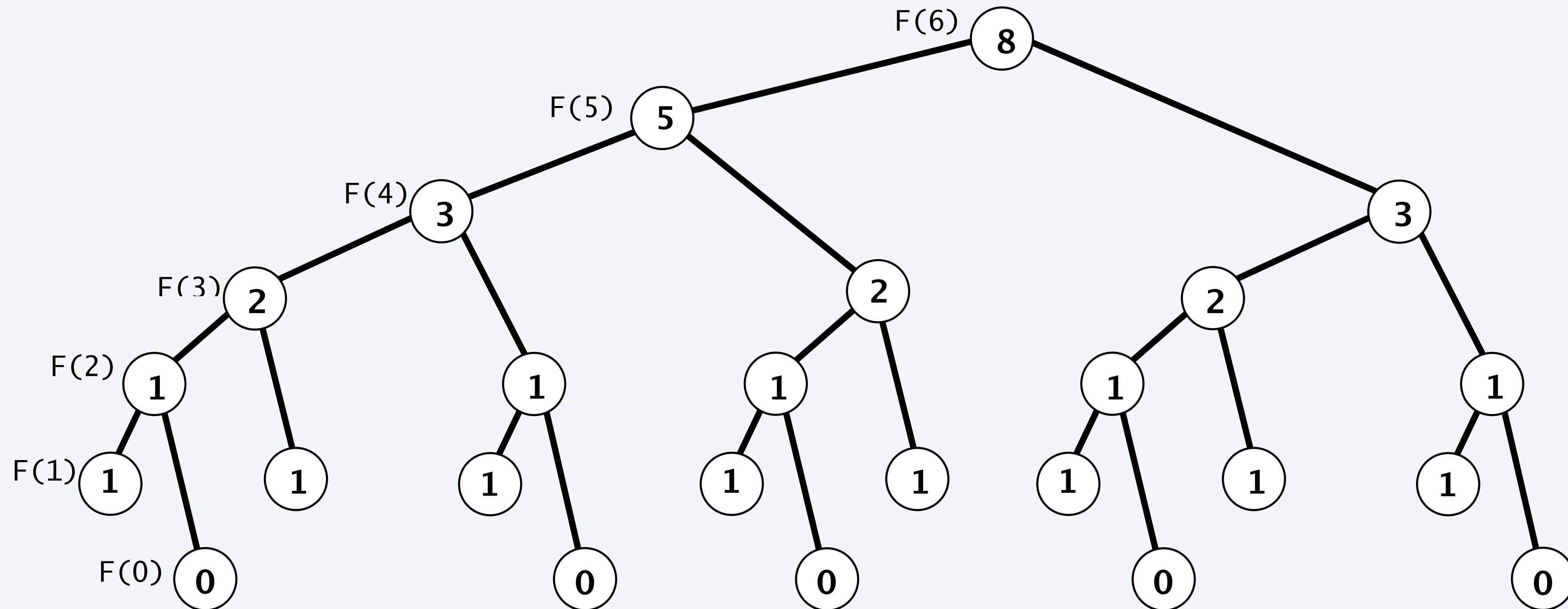
```
% java FibonacciR 5
5
% java FibonacciR 6
8
% java FibonacciR 10
55
% java FibonacciR 12
144
% java FibonacciR 50
12586269025
% java FibonacciR 60
```

takes a few minutes  
Hmmm. Why is that?

Is something wrong with my computer?

## Recursive call tree for Fibonacci numbers

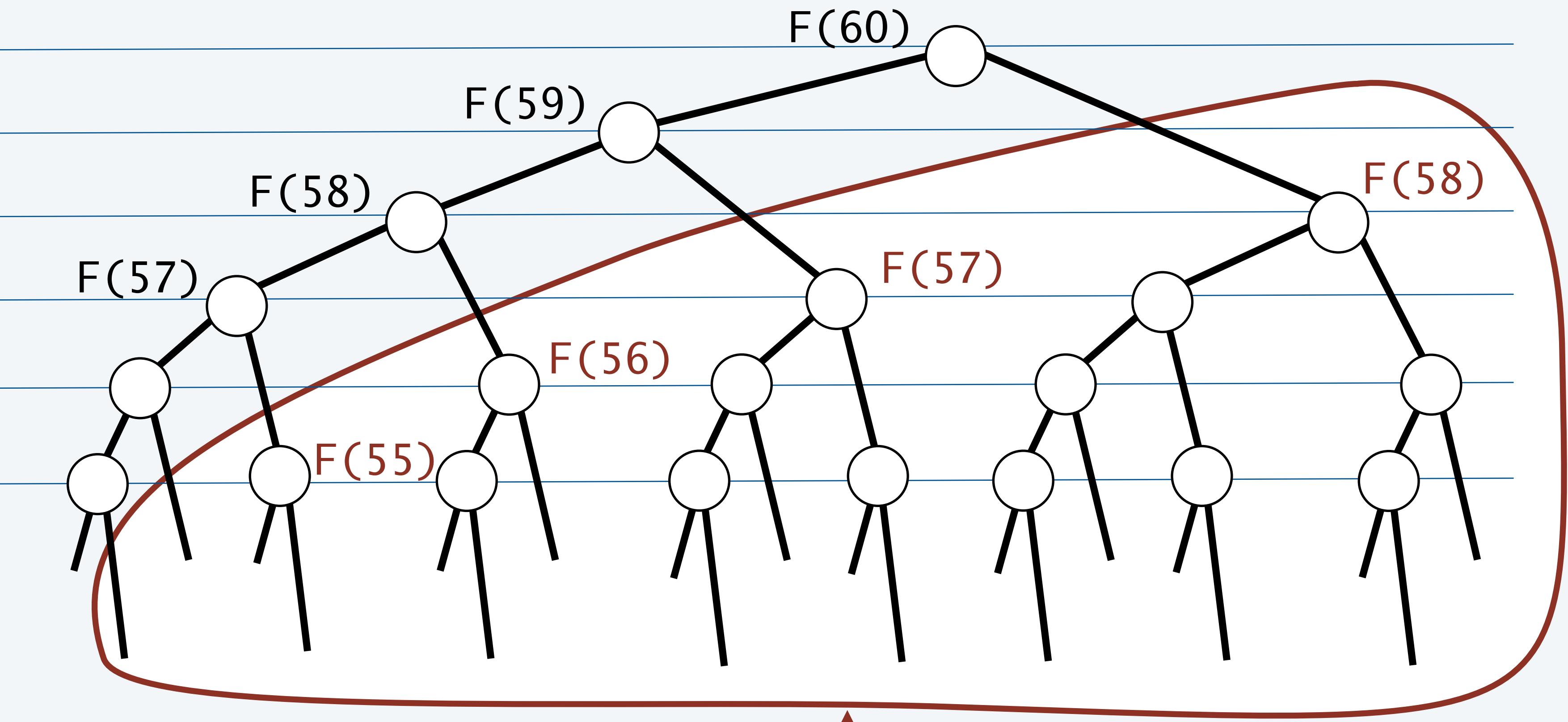
---



# Exponential waste

Let  $C_n$  be the number of times  $F(n)$  is called when computing  $F(60)$ .

$n$	$C_n$	
60	1	$F_1$
59	1	$F_2$
58	2	$F_3$
57	3	$F_4$
56	5	$F_5$
55	8	$F_6$
...	...	
0	$>2.5 \times 10^{12}$	$F_{61}$

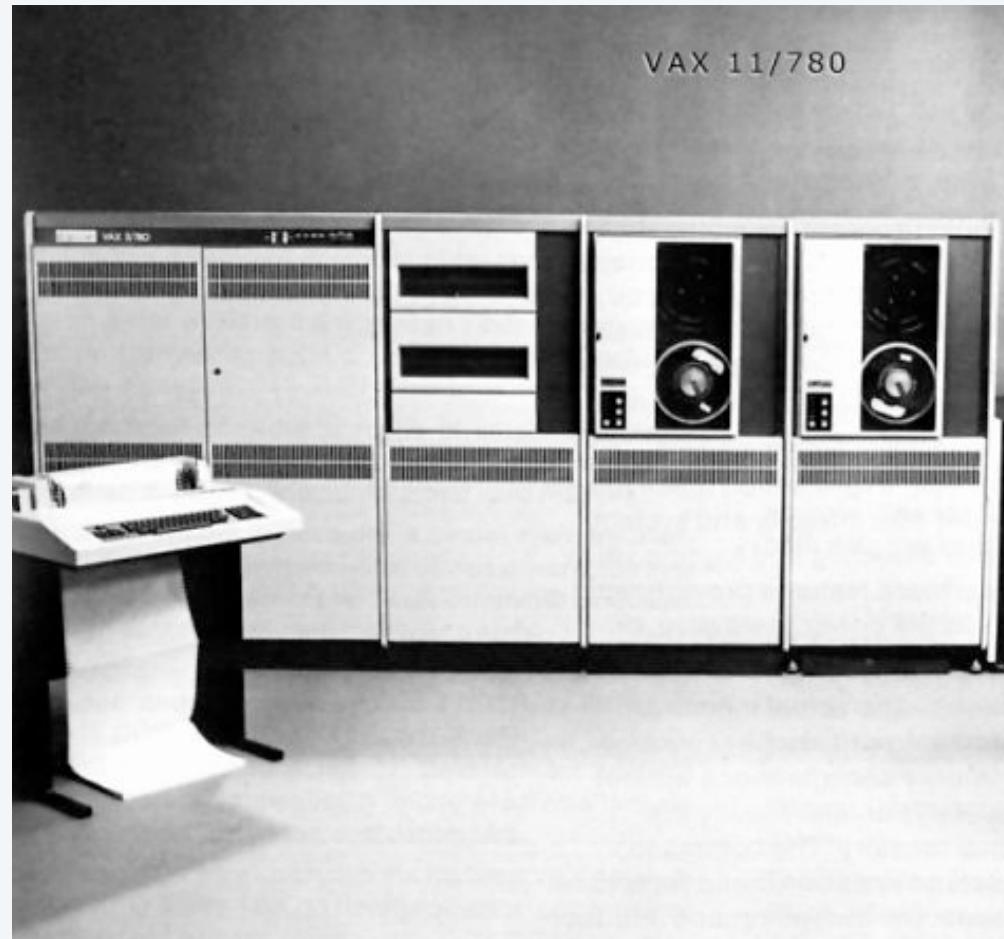


*Exponentially wasteful to recompute all these values.*  
(trillions of calls on  $F(0)$ , not to mention calls on  $F(1), F(2), \dots$ )

# Exponential waste dwarfs progress in technology

If you engage in exponential waste, you *will not* be able to solve a large problem.

1970s



VAX 11/780

$n$	<i>time to compute <math>F_n</math></i>
30	minutes
40	hours
50	weeks
60	years
70	centuries
80	millenia

2010s: 10,000+ times faster

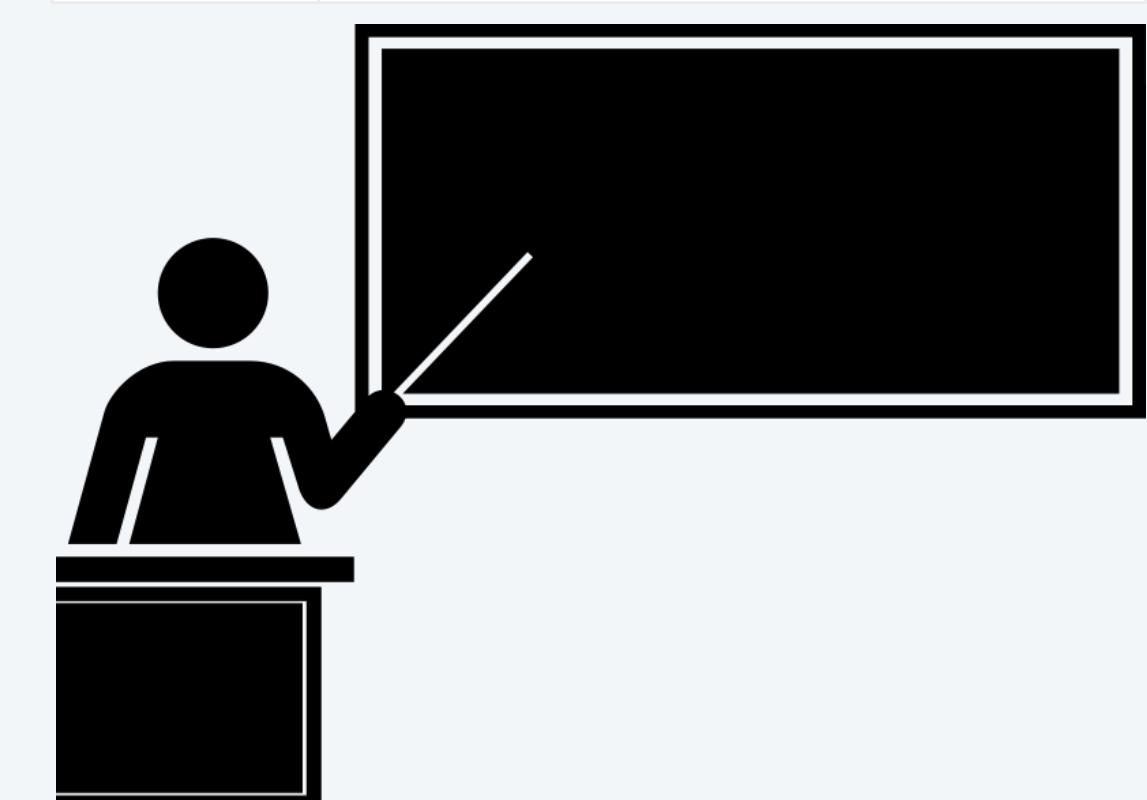


Macbook Air

$n$	<i>time to compute <math>F_n</math></i>
50	minutes
60	hours
70	weeks
80	years
90	centuries
100	millenia

1970s: "That program won't compute  $F_{60}$  before you graduate! "

2010s: "That program won't compute  $F_{80}$  before you graduate! "



# Avoiding exponential waste

## Memoization

- Maintain an array `memo[]` to remember all computed values.
- If value known, just return it.
- Otherwise, compute it, remember it, and then return it.

```
public class FibonacciM
{
    static long[] memo = new long[100];
    public static long F(int n)
    {
        if (n == 0) return 0;
        if (n == 1) return 1;
        if (memo[n] == 0)
            memo[n] = F(n-1) + F(n-2);
        return memo[n];
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        StdOut.println(F(n));
    }
}
```

```
% java FibonacciM 50
12586269025
% java FibonacciM 60
1548008755920
% java FibonacciM 80
23416728348467685
```

Simple example of *dynamic programming* (next).

*Image sources*

<http://en.wikipedia.org/wiki/Fibonacci>

<http://www.inspirationgreen.com/fibonacci-sequence-in-nature.html>

[http://www.goldenmeancalipers.com/wp-content/uploads/2011/08/mona\\_spiral-1000x570.jpg](http://www.goldenmeancalipers.com/wp-content/uploads/2011/08/mona_spiral-1000x570.jpg)

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[http://en.wikipedia.org/wiki/Ancient\\_Greek\\_architecture#mediaviewer/  
File:Parthenon-uncorrected.jpg](http://en.wikipedia.org/wiki/Ancient_Greek_architecture#mediaviewer/File:Parthenon-uncorrected.jpg)

<https://openclipart.org/detail/184691/teaching-by-ousia-184691>

## 6. Recursion

- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

# An alternative to recursion that avoids recomputation

## Dynamic programming.

- Build computation from the "*bottom up*".
- Solve small subproblems *and save solutions*.
- Use those solutions to build bigger solutions.



Richard Bellman  
1920-1984

### Fibonacci numbers

```
public class Fibonacci
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        long[] F = new long[n+1];
        F[0] = 0; F[1] = 1;
        for (int i = 2; i <= n; i++)
            F[i] = F[i-1] + F[i-2];
        StdOut.println(F[n]);
    }
}
```

```
% java Fibonacci 50
12586269025
% java Fibonacci 60
1548008755920
% java Fibonacci 80
23416728348467685
```

Key advantage over recursive solution. Each subproblem is addressed only *once*.

## DP example: Longest common subsequence

Def. A *subsequence* of a string  $s$  is any string formed by deleting characters from  $s$ .

Ex 1.  $s = \text{ggcaccacg}$

cac

gcaacg

ggcaacg

ggcacacg

...

ggcaccacg

ggcaccacg

ggcaccacg

ggcaccacg

[ $2^n$  subsequences in a string of length  $n$ ]

Ex 2.  $t = \text{acggcgatacg}$

gacg

ggggg

cggcg

ggcaacg

gggaacg

...

acggcgatacg

acggcgatacg

acggcgatacg

acggcgatacg

acggcgatacg

*longest common subsequence*

Def. The *LCS* of  $s$  and  $t$  is the longest string that is a subsequence of both.

Goal. Efficient algorithm to compute the LCS and/or its length

← numerous scientific applications

# Longest common subsequence

**Goal.** Efficient algorithm to compute the *length* of the LCS of two strings  $s$  and  $t$ .

**Approach.** Keep track of the length of the LCS of  $s[i..M]$  and  $t[j..N]$  in  $\text{opt}[i, j]$

Three cases:

- $i = M$  or  $j = N$

$$\text{opt}[i][j] = 0$$

- $s[i] = t[j]$

$$\text{opt}[i][j] = \text{opt}[i+1][j+1] + 1$$

- otherwise

$$\text{opt}[i][j] = \max(\text{opt}[i][j+1], \text{opt}[i+1][j])$$

Ex:  $i = 6, j = 7$

$$s[6..9] = acg$$

$$t[7..12] = atacg$$

$$\text{LCS}(cg, tacg) = cg$$

$$\text{LCS}(acg, atacg) = acg$$

Ex:  $i = 6, j = 4$

$$s[6..9] = acg$$

$$t[4..12] = cgatacg$$

$$\text{LCS}(acg, ggatacg) = acg$$

$$\text{LCS}(cg, cgatacg) = cg$$

$$\text{LCS}(acg, cgatacg) = acg$$

# LCS example

	0	1	2	3	4	5	6	7	8	9	10	11	12		
	a	c	g	g	c	g	g	a	t	a	c	g		Three cases:	
0	g	7	7	7	6	6	6	5	4	3	3	2	1	0	• $i = M \text{ or } j = N$ $\text{opt}[i][j] = 0$
1	g	6	6	6	6	5	5	5	4	3	3	2	1	0	• $s[i] = t[j]$ $\text{opt}[i][j] = \text{opt}[i+1][j+1] + 1$
2	c	6	5	5	5	4	4	4	3	3	3	2	1	0	• otherwise $\text{opt}[i][j] = \max(\text{opt}[i][j+1], \text{opt}[i+1][j])$
3	a	6	5	4	4	4	4	4	3	3	3	2	1	0	
4	c	5	5	4	4	4	3	3	3	3	3	2	1	0	
5	c	4	4	4	4	4	3	3	3	3	3	2	1	0	$\text{opt}[i][j] = \max(\text{opt}[i, j+1], \text{opt}[i+1][j])$
6	a	3	3	3	3	3	3	3	3	3	3	2	1	0	$\text{opt}[i][j] = \text{opt}[i+1, j+1] + 1$
7	c	2	2	2	2	2	2	2	2	2	2	1	0		
8	g	1	1	1	1	1	1	1	1	1	1	1	1	0	
9		0	0	0	0	0	0	0	0	0	0	0	0	0	

Diagram illustrating the computation of the optimal value at  $\text{opt}[7][8]$  (highlighted with a red circle). The value 2 is obtained by summing the values from  $\text{opt}[7][7]$  and  $\text{opt}[8][8]$ , indicated by blue arrows.

# LCS length implementation

```
public class LCS
{
    public static void main(String[] args)
    {
        String s = args[0];
        String t = args[1];
        int M = s.length();
        int N = t.length();

        int[][] opt = new int[M+1][N+1];

        for (int i = M-1; i >= 0; i--)
            for (int j = N-1; j >= 0; j--)
                if (s.charAt(i) == t.charAt(j))
                    opt[i][j] = opt[i+1][j+1] + 1;
                else
                    opt[i][j] = Math.max(opt[i+1][j], opt[i][j+1]);

        System.out.println(opt[0][0]);
    }
}
```

```
% java LCS ggcaccacg acggcggatacg
7
```

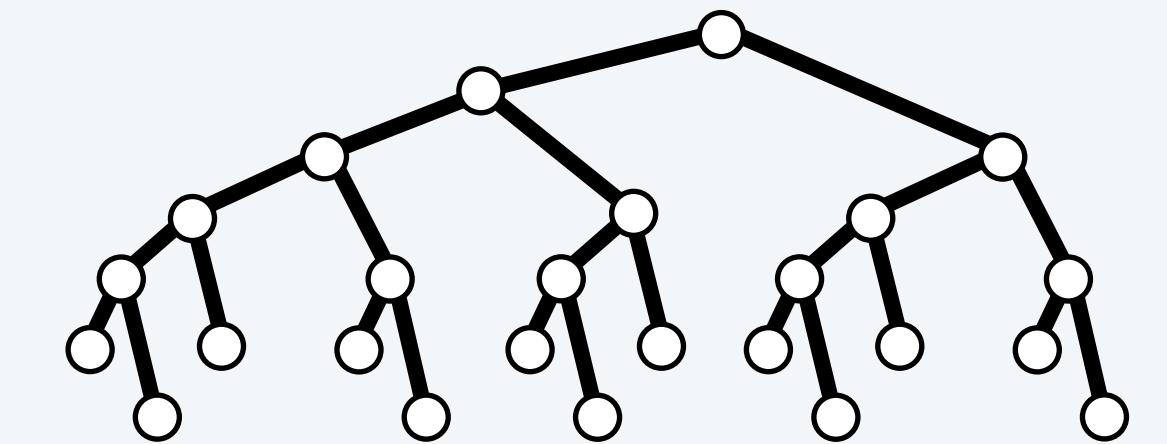
**Exercise.** Add code to print LCS itself (see `LCS.java` on booksite for solution).

# Dynamic programming and recursion

*Broadly useful* approaches to solving problems by combining solutions to smaller subproblems.

## Why learn DP and recursion?

- Represent a new mode of thinking.
- Provide powerful programming paradigms.
- Give insight into the nature of computation.
- Successfully used for decades.

	<i>recursion</i>	<i>dynamic programming</i>
<i>advantages</i>	Decomposition often obvious. Easy to reason about correctness.	Avoids exponential waste. Often simpler than memoization.
<i>pitfalls</i>	Potential for exponential waste. Decomposition may not be simple.	Uses significant space. Not suited for real-valued arguments. Challenging to determine order of computation

*Image sources*

[http://upload.wikimedia.org/wikipedia/en/7/7a/Richard\\_Ernest\\_Bellman.jpg](http://upload.wikimedia.org/wikipedia/en/7/7a/Richard_Ernest_Bellman.jpg)

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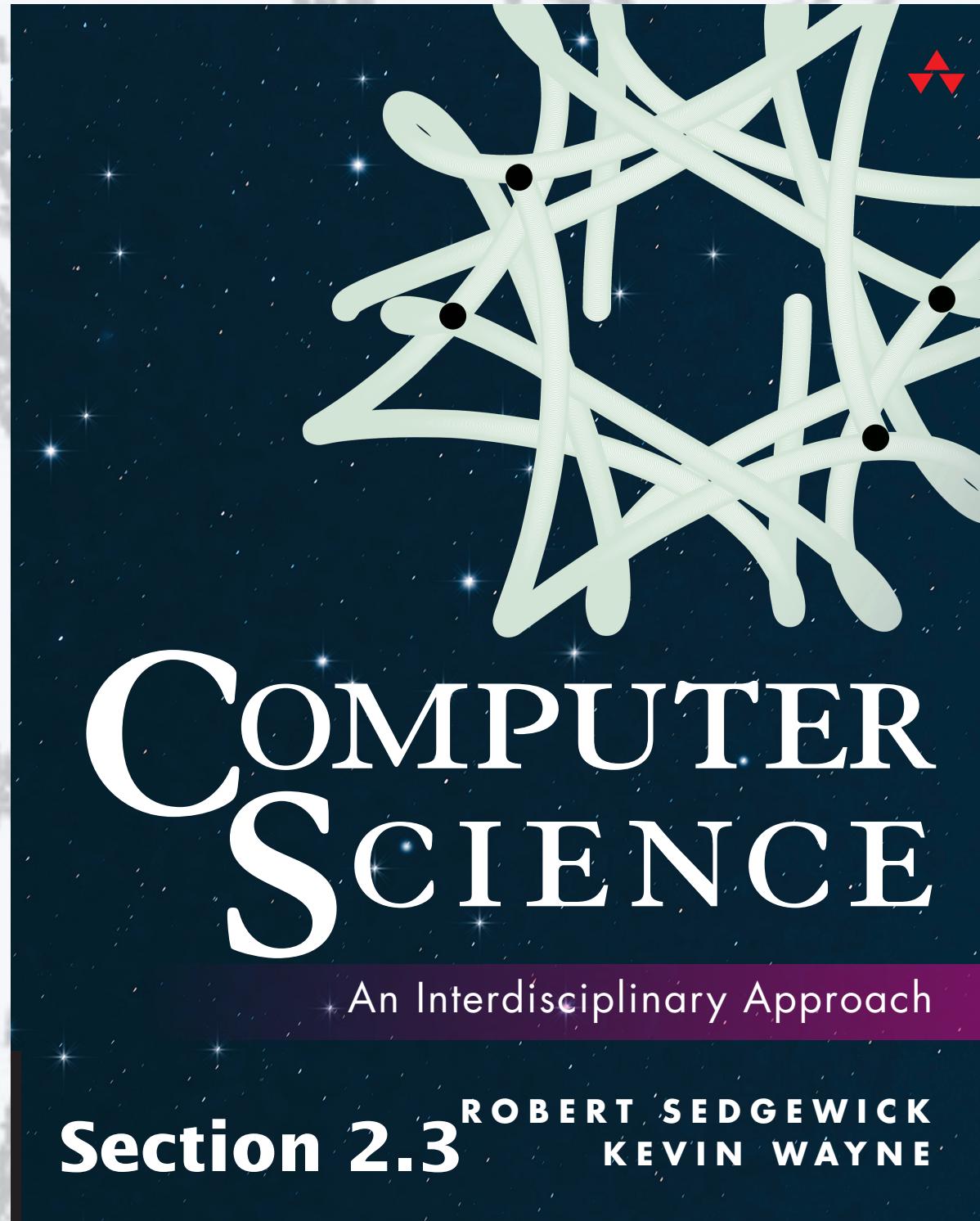
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PART I: PROGRAMMING IN JAVA



## 6. Recursion

<http://introcs.cs.princeton.edu>