1、a、 $f_3 = f_2 + f_2 = 2$

$f_4 = f_3 + f_2 = 3$

when $n=1$, $f_1 = 1 = f_3 - 1$

when $n=2$, $f_1 + f_2 = 2 = f_4 - 1$

we assume $f_1 + f_2 + \cdots + f_k = f_{k+2} - 1$ is true

then $f_1 + f_2 + \cdots f_k + f_{k+1} = f_{k+2} - 1 + f_{k+1}$

$\Rightarrow f_1 + f_2 + \cdots + f_{k+1} = f_{(k+1)+2} - 1$

$\therefore$ $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

b、 $f_3 = f_1 + f_2 = a+b$

$f_4 = f_3 + f_2 = a+2b$

when $n=1$, $f_1 = a = f_3 - b$

when $n=2$, $f_1 + f_2 = a+b = f_4 - b$

Assume $f_1 + f_2 + \cdots + f_k = f_{k+2} - b$ is true

Then $f_1 + f_2 + \cdots + f_k + f_{k+1} = f_{k+2} - b + f_{k+1}$

$\Rightarrow f_1 + f_2 + \cdots + f_{k+1} = f_{(k+1)+2} - b$

$\therefore$ $f_1 + f_2 + \cdots + f_n = f_{n+2} - b$

2\. a. $f(x) = \frac{1}{1-ax}$

proof: when $n=0$, $a_n = a^0 = 1$ $(a \neq 0)$

when $n \geq 1$, $a_n - a a_{n-1} = 0$

$\therefore f(x) - axf(x) = a_0 + (a_1 - aa_0)x + (a_2 - aa_1)x^2 + \cdots$

$= a_0 = 1$

$\Rightarrow f(x) = \frac{1}{1-ax}$

b. $f(x) = (1+x)^m$

proof: $\because (1+x)^m = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 \cdots \binom{m}{m}x^m$

$\therefore f(x) = a_0 + a_1 x + a_2 x^2 \cdots$

$= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 \cdots + \binom{m}{m}x^m + 0x^{m+1} + 0x^{m+2} \cdots$

$= (1+x)^m$

c. $f(x) = \frac{x}{1-x-x^2}$

proof: when $n=0$, $a_n = 0$

when $n=1$, $a_n = 1$

when $n \geq 2$, $a_n - a_{n-1} - a_{n-2} = 0$

$f(x) - xf(x) - x^2 f(x) = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots$

$$= a_0 + (a_1 - a_0)x$$
$$= x$$

$\therefore f(x) = \dfrac{x}{1-x-x^2}$

3. Let $a = \binom{p}{k} = \dfrac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$

Then $k! \cdot a = p(p-1)(p-2)\cdots(p-k+1)$

$\because p \mid p(p-1)(p-2)\cdots(p-k+1)$

$\therefore p \mid k! \cdot a \Rightarrow p \mid k!$ or $p \mid a$

$\because 0 < k < p$

$\therefore p \nmid k,\ p \nmid (k-1),\ p \nmid (k-2), \cdots, p \nmid 1.$

$\Rightarrow p \nmid k!$

$\Rightarrow p \mid a$ namely $p \mid \binom{p}{k}$

proof $n^p \equiv n \pmod p$:

when $n=0$, $0^p \equiv 0 \pmod p$ is true

Assume $n=k$, $k^p \equiv k \pmod p$ is true.

Then $(k+1)^p = \binom{p}{0} + \binom{p}{1}k + \binom{p}{2}k^2 + \cdots \binom{p}{p}k^p$

$(k+1)^p \equiv 1 + 0 + 0 + \cdots + k \pmod p$

namely $(k+1)^p \equiv k+1 \pmod p$

4. $(1+x)^n(1+x)^n = \left(\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right)\left(\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right)$

$$= \binom{n}{0}\binom{n}{0} + \left[\binom{n}{0}\binom{n}{1}+\binom{n}{1}\binom{n}{0}\right]x + \cdots$$
$$+\left[\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}\right]x^n + \cdots$$

$$(1+x)^{2n} = \binom{2n}{0}+\binom{2n}{1}x^1+\cdots+\binom{2n}{n}x^n+\cdots$$

Comparing coefficient of x^n, we have

$$\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$$

$$\because \binom{n}{m} = \frac{n!}{(n-m)!\,m!} = \frac{n!}{m!\,(n-m)!} = \frac{n!}{(n-(n-m))!\,(n-m)!} = \binom{n}{n-m}$$

$$\therefore \binom{2n}{n} = \sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \sum_{m=0}^{n}\binom{n}{m}^2$$

5. Since $X \equiv 5 \pmod{6}$

then $X = 5 + 6u$

since $X \equiv 3 \pmod{10}$

then $5 + 6u \equiv 3 \pmod{10}$

$\Leftrightarrow$ $6u = -2 \pmod{10}$

$\Leftrightarrow$ $6u = -2 + 10v$

$\Leftrightarrow$ $3u = -1 + 5v$

$\Leftrightarrow$ $3u \equiv -1 \pmod{5}$

$\Leftrightarrow$ $2 \times 3u \equiv 2 \times (-1) \pmod{5}$

$\Leftrightarrow$ $u \equiv -2 \pmod{5}$

$\Leftrightarrow$ $u = -2 + 5w$

Since $x \equiv 8 \pmod{15}$

then $x = 5 + 6u = 5 + 6(-2 + 5w) = -7 + 30w \equiv 8 \pmod{15}$

$\Leftrightarrow$ $30w \equiv 15 \pmod{15}$

$\Leftrightarrow$ $30w \equiv 0 \pmod{15}$, which is always true.

So $X = -7 + 30w$ or $X \equiv -7 \pmod{30}$

$[\, X \equiv 23 \pmod{30} \,]$

6.a. $\gcd(1234567, 7654321)$

$= \gcd(1234567, 7654321 - 6 \times 1234567)$

$= \gcd(1234567, 246919)$

$= \gcd(1234567 - 4 \times 246919, 246919)$

$= \gcd(246891, 246919)$

$= \gcd(246891, 246919 - 246891)$

$= \gcd(246891, 28)$

$= \gcd(246891 - 8817 \times 28, 28)$

$= \gcd(15, 28)$

$= \gcd(15, 28 - 15)$

$= \gcd(15, 13)$

$= \gcd(15 - 13, 13)$

$= \gcd(2, 13)$

$= \gcd(2, 13 - 6 \times 2)$

$= \gcd(2, 1)$

$= 1$

b. $\gcd(2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9 \cdot 11, 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 13)$

The two number have been factorized.

So we only need to pick the same part.

$\gcd(2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9 \cdot 11, 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 13) = 2^3 \cdot 3^5 \cdot 5^5 \cdot 7^3$

7. Firstly, we proof $P_n \subseteq P_1 P_2 \cdots P_{n-1} + 1$

(i) If $P_1 P_2 \cdots P_{n-1} + 1$ is prime number P_k:

since $P_1 P_2 \cdots P_{n-2} > 1$

then $P_1 P_2 \cdots P_{n-2} P_{n-1} > P_{n-1}$

$\Rightarrow P_1 P_2 \cdots P_{n-1} + 1 > P_{n-1}$

$\Rightarrow P_k > P_{n-1}$

$\Rightarrow k > n-1$

$\Rightarrow k \geqslant n$

$\Rightarrow P_k \geqslant P_n$

$\Rightarrow P_1 P_2 \cdots P_{n-1} + 1 \geqslant P_n$

(ii) If $P_1 P_2 \cdots P_{n-1} + 1$ isn't prime number:

Since $P_1 P_2 \cdots P_{n-1} + 1$ isn't prime number

then $\exists P_k$, $P_k \mid P_1 P_2 \cdots P_{n-1} + 1$

since For $i = 1, 2, 3, \cdots, n-1$,

$P_1 P_2 \cdots P_{n-1} + 1 \equiv 1 \pmod{P_i}$

then $P_i \nmid P_1 P_1 \cdots P_{n-1} + 1$

$\Rightarrow k \neq 1, 2, 3 \cdots n-1$

$\Rightarrow k \geqslant n \Rightarrow P_k \geqslant P_n$

since $P_k \mid P_1 P_2 \cdots P_{n-1} + 1$

then $P_k \le P_1 P_2 \cdots P_{n-1} + 1$

Since $P_n \le P_k$

then $P_n \le P_1 P_2 \cdots P_{n-1} + 1$

Secondly, we proof $P_1 P_2 \cdots P_{n-1} + 1 < 2^{2^n}$

by induction.

when $n=2$, $P_1 + 1 = 3 < 16 = 2^{2^2}$

when $n=k$, $P_1 P_2 \cdots P_{k-1} + 1 < 2^{2^k}$

$$\Rightarrow (P_1 P_2 \cdots P_{k-1} + 1)(P_1 P_2 \cdots P_{k-1} + 1) < (2^{2^k})^2$$

$$\Rightarrow P_k (P_1 P_2 \cdots P_{k-1} + 1) \le (P_1 P_2 \cdots P_{k-1} + 1)(P_1 P_2 \cdots P_{k-1} + 1) < 2^{2^{k+1}}$$

$$\Rightarrow P_1 P_2 \cdots P_k + 1 < P_1 P_2 \cdots P_k + P_k = P_k (P_1 P_2 \cdots P_{k-1} + 1) < 2^{2^{k+1}}$$

So $P_1 P_2 \cdots P_{n-1} + 1 < 2^{2^n}$

In all. $P_n \le P_1 P_2 \cdots P_{n-1} + 1 < 2^{2^n}$

namely $P_n < 2^{2^n}$

8、We use $q_1, q_2, \cdots q_m$ to denote a cycle of n

and q_1 beats q_2, q_2 beats q_3, $\cdots$, q_m beats q_1

Check the results of q_1 plays $q_3, q_4 \cdots q_{m-1}$

(i) "$\exists\, q_i \in \{q_3, q_4, \cdots q_{m-1}\}$, q_i beats q_1." is true.

We take the smallest i as min ($min \geq 3$).

Then q_{min} beats q_1

q_1 beats q_{min-1}

We know q_{min-1} beats q_{min}

So q_{min}, q_1, q_{min-1} form a cycle of 3.

C(ii) "$\exists\, q_i \in \{q_3, q_4, \cdots q_{m-1}\}$, q_i beats q_1." is false.

namely, $\forall\, q_i \in \{q_3, q_4 \cdots q_{m-1}\}$, q_1 beats q_i.

Then q_1 beats q_{m-1}

We know q_{m-1} beats q_m

q_m beats q_1

So q_1, q_{m-1}, q_m form a cycle of 3.

There must be a cycle of 3.