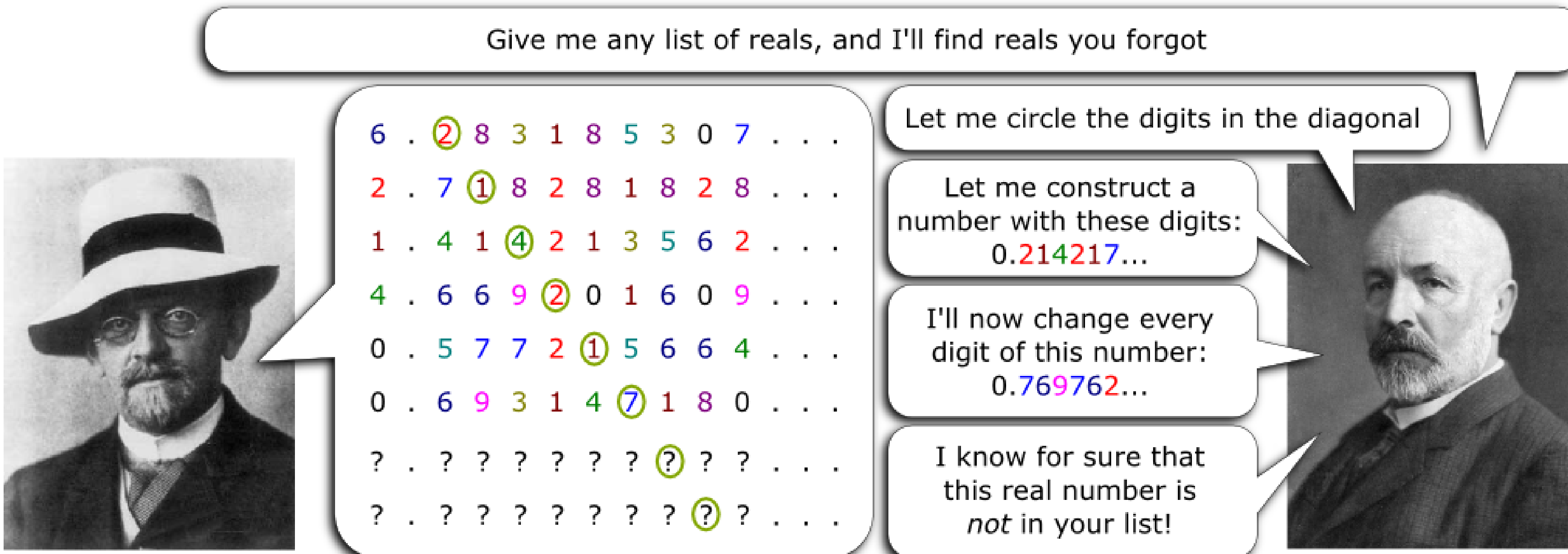
Cardinality (optional)



This Lecture

Last lecture we learn the technique of counting by mapping.

In this lecture we will study how to count infinity, i.e., the cardinality (size of infinity).

We will also discuss some applications of cardinality.

Cardinality

Questions:

- | the set of positive integers | = | the set of integers | ?
- | the set of integers | = | the set of rational numbers | ?
- | the set of integers | = | the set of real numbers | ?

How to compare the sizes for infinite sets?

The cardinality of a set A is a number that measures "the number of elements in A".

e.g. the cardinality of {a,b,c} is 3.
the cardinality of pow({1,…,n})=2^n.

What is the cardinality of an infinite set?

Cardinality

What does it mean by two infinite sets have the same cardinality ?

Recall that
- $f: A \rightarrow B$ is injective $\Rightarrow |A| \leq |B|$
- $f: A \rightarrow B$ is surjective $\Rightarrow |A| \geq |B|$
- $f: A \rightarrow B$ is bijective $\Rightarrow |A| = |B|$

Georg Cantor
1845-1918

Two sets A and B have the **same cardinality** if and only if there is a bijection between A and B.

A set is countable (cardinality $\aleph_0$) if it has the same cardinality as the set of positive integers.

Integers vs Positive Integers

Is the set of integers countable?

Define a bijection between the positive integers and all integers.

1	2	3	4	5	6	7	8	...
0	1	−1	2	−2	3	−3	4	...

$$f(n) = \begin{cases} n/2, \text{ if } n \text{ is even;} \\ -(n-1)/2, \text{ if } n \text{ is odd.} \end{cases}$$

So, the set of integers is countable.

Rational Numbers vs Positive Integers

Question: Is the set of **rational numbers** countable?

The set of “integer pairs” (a,b) is not smaller than the set of rational numbers.

We want to show that the set of “integer pairs” is countable,
by defining a **surjection** from the positive integers to this set.

This would then imply the set of rational numbers is countable.

Rational Numbers vs Positive Integers

(0, 0), (0, 1), (0,-1), (0, 2), (0,-2), (0, 3), (0,-3), . . .

(1, 0), (1, 1), (1,-1), (1, 2), (1,-2), (1, 3), (1,-3), . . .

(-1, 0),(-1, 1),(-1,-1),(-1, 2),(-1,-2),(-1, 3), (-1,-3), . . .

(2, 0), (2, 1), (2,-1), (2, 2), (2,-2), (2, 3), (2,-3), . . .

(-2, 0),(-2, 1),(-2,-1),(-2, 2),(-2,-2), (-2, 3),(-2,-3), . . .

If you map the set of positive integers to the top row first, then you will not be able to reach the second row.

The trick is to visit the rational numbers diagonal by diagonal.

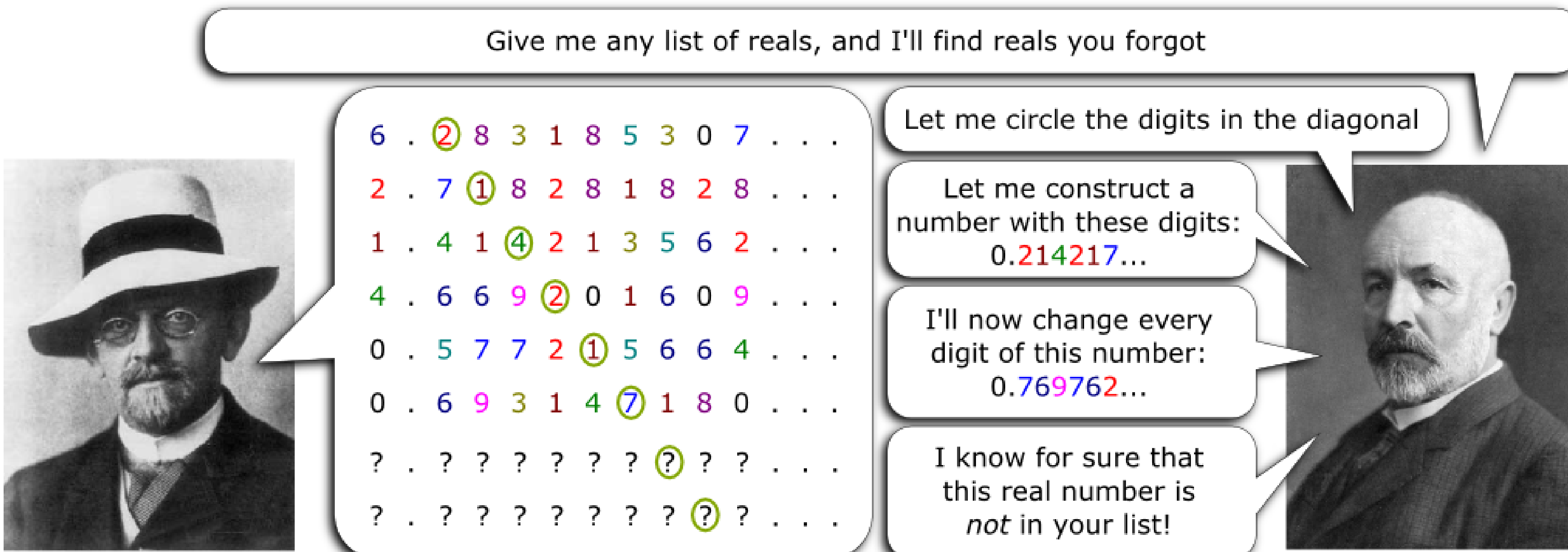
Each diagonal is finite, so every pair will be visited in finite steps.

Therefore, we find a **surjection** from the set of positive integers, to the set of "integer pairs", and so the set of rational numbers is countable.

Real Numbers vs Positive Integers

Question: Is the set of **real numbers** countable?

Theorem. No surjection mapping positive integers to real numbers.



So real numbers are **uncountable**.

Diagonal Argument

The argument we just see is called Cantor's diagonal argument, and has many applications (e.g. Russell's paradox).

In particular, countable and uncountable are different cardinalities.

More surprisingly, there are infinitely many cardinalities (sizes of infinity).

Theorem. For any set S, $|S| \neq |\,\mathrm{pow}(S)\,|$.

Proof (by contradiction).

Suppose $f: S \to \mathrm{Pow}(S)$ is bijective. Consider $T = \{x \in S \mid x \notin f(x)\}$.

Then $T = f(y)$ for some $y \in S$.

- $y \in T \quad \Rightarrow y \notin f(y) = T$ A contradiction!
- $y \notin T = f(y) \quad \Rightarrow y \in T$ A contradiction!

Cardinality and Computability

The set of all computer programs in a given computer language is countable.

The set of all functions is uncountable.

There must exist a non-computable function!

Quick Summary

Cardinality is an alternative word for “size” of an infinite set.

By constructing a bijection we can check if two given infinite sets have equal cardinality.

Diagonal argument is a useful method for checking whether a set is countable or not.