CSC3001 Discrete Mathematics

Homework 3

Deadline: 23:59, Sunday, July 17, 2022

The details should be provided, and you can refer to any theorem in the lecture notes without proof. Otherwise, please provide a proof or cite the reference for that.

1. Does there exist a graph G with no loops and no parallel edges such that all its vertices have different degrees, i.e. $deg(v) \neq deg(u)$ when $v \neq u$?

2. Find the number of perfect matchings in K_{2n}.

3. How many edges do the following graphs have:
   (a) P_n - a path through n vertices;
   (b) C_n - a cycle through n vertices;
   (c) K_n - a complete graph on n vertices;
   (d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other.

4. Consider a graph $G_{n,m}$ of a rectangular grid with n vertices on horizontal lines and m vertices on vertical lines, $n, m > 2$. What is the smallest number of colors to properly color vertices of this graph?

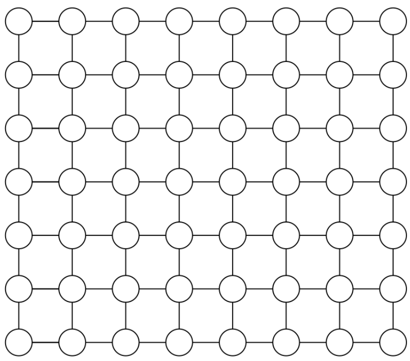


Figure 1: $G_{8,7}$

5. Let $G_{n,m}$ be the same as in question above, $n, m > 2$. What is the smallest number of colors to properly color edges of this graph?

6. Show that any tree can be vertex-colored in 2 colors.

7. Let T_n be the number of binary trees with n vertices - that is, trees that have a root node r and where every node has at most two child nodes, labeled L (left child) and R (right child).

   Argue that deleting a root node r from binary tree of size $n+1$ will break it into two binary trees with sizes k and $n-k$ for some k.

   Conclude that we have a recursion

$$T_{n+1} = T_0 T_n + T_1 T_{n-1} + \ldots + T_{n-1} T_1 + T_n T_0.$$

8. Prove that if $T_0 = 1$ and $T_{n+1} = T_0 T_n + T_1 T_{n-1} + \ldots + T_{n-1} T_1 + T_n T_0$, then $T_n = C_n$ - n-th Catalan number.

9. Count the number of ways to color a tree with n vertices in t colors. Compute the number of ways to color path graph P_5 in 11 colors.

   Hint: start from some vertex, name it 1. It has t choices for its color. Now let's grow our tree by adding new vertices, they will have $t-1$ choices for color.

10. Count the number of ways to color a complete graph K_n with n vertices in t colors with $t > n$. Denote this number as $P(K_n, t)$. Compute exponential generating function

$$g_{K_n}(x) = \sum_{t=1}^{\infty} \frac{P(K_n, t)}{n!} x^t$$

   Can you derive formula for g_{K_n} ~~without computing $P(K_n, t)$ in advance~~?

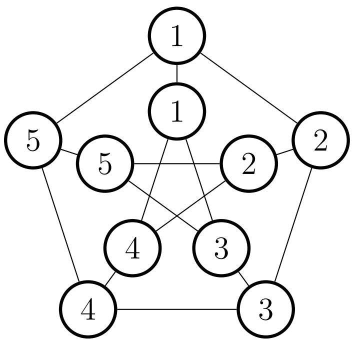
11. For which positive integers n does K_n have an
    (a) Eulerian cycle.
    (b) Eulerian path.

12. Let G be a simple graph with n vertices. Show that if G has more than $\frac{(n-1)(n-2)}{2}$ edges, then G must be connected.

13. Let T be a tree with 21 vertices such that $\deg(v) \in \{1, 3, 5, 6\}$ for every vertex of T. If T has 15 leaves and one vertex of degree 6, how many vertices with degree 5 are in T?

14. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.
    (a) Model the possible marriages on the island using a bipartite graph.
    (b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.
    (c) Is the matching you found in part (b) a maximum matching, that is, a matching with the largest number of edges?

15. Find the chromatic number of the Petersen graph, shown as:



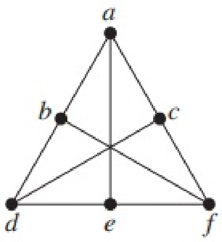
Prove your answer by exhibiting a k-coloring and showing that fewer colors will not be sufficient.

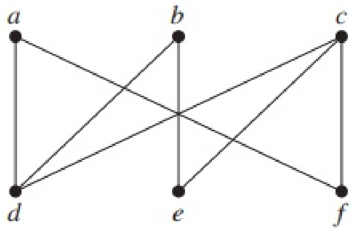
16. If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no cycles of length three, then $e \leq 2v - 4$.

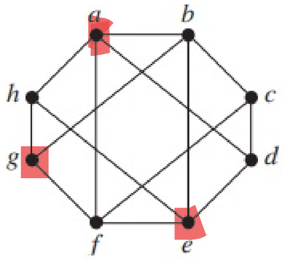
17. The complement of a simple graph $G = (V, E)$ is given by $G^c = (V, E^c)$, where $E^c = V \times V - E$, i.e., the complement has the same vertex set and an edge is in E^c if and only if it is not in E. A graph G is said to be self-complenmentray if G is isomorphic to G^c. Show that a self-complementary graph must have either $4m$ or $4m + 1$ vertices, $m \in \mathbb{N}$. Hint: consider the sum of edges.

18. Show that if G is a simple graph with at least 11 vertices, then either G or G^c, the complement of G (defined in the last problem), is nonplanar. Hint: use $m \leq 3n - 6$ in the lecture.

19. Given the Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5. Determine whether the following graph are planar. If so, draw it so that no edges cross.







20. Can you arrange the numbers 1, $2, \cdots$, 9 along a circle, so that the sum of two neighbors are never divisible by 3, 5, or 7?