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1.Let f_n be the n-th Fibonacci sequence,$f_{n+2} = f_{n+1} + f_n$

a.$f_1 = f_2 = 1$,prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$
b.$f_1 = a, f_2 = b$,prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - b$

1. a. $f_{n+2} = f_{n+1} + f_n = (f_n + f_{n-1}) + (f_{n-1} + f_{n-2}) = (f_n + f_{n-1} + f_{n-2}) + (f_{n-2} + f_{n-3})$

$= (f_n + f_{n-1} + f_{n-2} + f_{n-3}) + (f_{n-3} + f_{n-4})$

$\cdots \quad \cdots$

$= (f_n + f_{n-1} + \cdots f_3) + (f_3 + f_2)$

$= (f_n + f_{n-1} + \cdots f_2) + (f_2 + f_1)$

$= f_1 + f_2 + \cdots f_n + f_2$

$\because f_2 = 1 \quad \therefore f_{n+2} - f_2 = f_1 + f_2 + \cdots f_n = f_{n+2} - 1$

b. the same as a.

$f_{n+2} = f_1 + f_2 + \cdots f_n + f_2 \quad \because f_2 = b$

$\therefore f_{n+2} - f_2 = f_1 + f_2 + \cdots f_n = f_{n+2} - b$

2.Find and prove closed form formulas for generating functions
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
of the following sequences

(a) $a_n = a^n$,where $a \in \mathbf{R}$;

(b)$a_n = \binom{m}{n}$,where $m \in N$;

(c)$a_n = f_n$, where f_n is the n-th Fibonacci number(assume $f_0 = 0, f_1 = f_2 = 1$)

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2.(a) let $f(x)=\sum_{n=0}^{\infty} a^n x^n$, Then $axf(x)=\sum_{n=0}^{\infty} a^{n+1}x^{n+1}=f(x)-1$

hence $f(x)=\frac{1}{1-ax}$

(b) let $f(x)=\sum_{n=0}^{\infty}\binom{m}{n}x^n$. Note that this sum is, in fact, finite. One of the definitions of binomial coefficients implies that $f(x)=(1+x)^m$

(Base case: $\binom{1}{0}+\binom{1}{1}x=(1+x)^1$

Step: $\sum_{n=0}^{\infty}\binom{m+1}{n}x^n=\sum_{n=0}^{\infty}\left(\binom{m}{n-1}x^n+\binom{m}{n}x^n\right)=x(1+x)^m+(1+x)^m=(1+x)^{m+1}$)

by induction. to prove.

(c) $f(x)=f_0+f_1x+f_2x^2+f_3x^3+\cdots$

$xf(x)=\quad f_0x+f_1x^2+f_2x^3+\cdots$

$x^2f(x)=\quad f_0x^2+f_1x^3+\cdots$

$\because f_n=f_{n-1}+f_{n-2}$

$\therefore f(x)=x+xf(x)+x^2f(x)$

$\therefore f(x)=\frac{x}{1-x-x^2}$

when $x=1$ $f(1)=f_0+f_1+f_2+\cdots=\sum_{i=0}^{\infty}f_i$

$\therefore f(x)=\frac{x}{1-x-x^2}$ is the closed form formula of sequence $a_n=f_n$

3.Using the formlar

$$\binom{n}{m}=\frac{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(n-m\cancel{+1}^{\,-2})}{m\cdot(m-1)\cdot(m-2)\cdot\cdots\cdot2\cdot1}$$

p is a prime number.

① Prove that $\binom{p}{k}$ is divisible by p for $0<k<p$;

② Deduce by induction on n that $n^p\equiv_p n.(_pn$ means $n(mod\,p))$

3.① $\binom{p}{k}=\frac{p\cdot(p-1)\cdot(p-2)\cdots(p-k+1)}{k\cdot(k-1)\cdot(k-2)\cdots2\cdot1}$ we will prove it is an integer.

when $k=1$, $C_p^{k-1}=C_p^0=1$, $C_p^k=C_p^1=p$ it is an integer.

assume, $k>1$, C_p^{k-1} and C_p^k are integers. we will recurse on k.

when $k=k+1$ $C_p^{k+1} = C_{p-1}^{k+1} + C_{p-1}^{k}$, according to n, we know that C_{p-1}^{k} is an Integer

$C_{p-1}^{k+1} = C_{p-2}^{k+1} + C_{p-2}^{k}$ $C_{p-2}^{k+1} = C_{p-3}^{k+1} + C_{p-3}^{k}$ - - - -

$C_{p-i}^{k+1} = C_{p-1-i}^{k+1} + C_{p-1-i}^{k}$

$\left.\begin{matrix} \\ \end{matrix}\right\} C_p^{k+1} = C_{p-1-i}^{k+1} + \sum_i C_{p-1-i}^{k}$

$C_{p-1-(k+1)} = i$

when $i=0$ $C_p^{k+1} = 1 + C_{p-1}^{k}$ $\because C_{p-1-i}^{k+1} = 1$ $\therefore \sum_{i=0} C_{p-1-i}^{k}$ always is an integer.

$\because p$ is a prime number, $0<k<p$

$\therefore p$ has not factors except 1, and p.

② $n^p \equiv n \pmod p$: Bcase: when $n=1$ $1 \equiv 1 \pmod p$ always true.

assume, $n^p \equiv n \pmod p$

now, to prove $(1+n)^p \equiv (n+1) \pmod p$

$(1+n)^p = C_p^0 n^0 + C_p^1 n^1 + C_p^2 n^2 + \cdots C_p^p n^p$

$\because p$ is a prime number $\therefore C_p^1 n^1 + C_p^2 n^2 + \cdots C_p^{p-1} n^{p-1} \equiv 0 \pmod p$

$C_p^p n^p = n^p \equiv n \pmod p$ $C_p^0 n^0 = 1$ $1 \equiv 1 \pmod p$

$\therefore (1+n)^p = (1+0+0+\cdots+n) \pmod p = (n+1) \pmod p$

4.Using the identity

$$(1+x)^n(1+x)^n = (1+x)^{2n}$$

① Prove that

$$\sum_{m=0}^{n} \binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$$

② Deduce that

$$\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}$$

4.① $(1+x)^n(1+x)^n$

$= [C_n^0 \cdot 1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n]$

$\cdot [C_n^0 \cdot 1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n]$

we will obtain the coefficient of x^n

so $C_n^0 \cdot 1 \cdot C_n^n \cdot x^n + C_n^1 \cdot x \cdot C_n^{n-1} \cdot x^{n-1} + \cdots$

$\cdots + C_n^k \cdot x^k \cdot C_n^{n-k} \cdot x^{n-k} + \cdots$

$= \sum_k^n C_n^k \cdot C_n^{n-k} \cdot x^n$

$\sum_k^n C_n^k C_n^{n-k} = \sum_{m=0}^{n} \binom{n}{m}\binom{n}{n-m}$

For $(1+x)^{2n}$, we obtain the coefficient of x^n : $C_{2n}^n x^n$ $\therefore C_{2n}^n = \sum_{m=0}^{n} \binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$

② First, we will prove $\binom{n}{m} = \binom{n}{n-m}$

$\because C_n^m = \dfrac{n\cdot(n-1)(n-2)\cdots(n-m+1)}{m\cdot(m-1)\cdots 2\cdot 1}$ $\dfrac{n\cdot(n-1)\cdots(m+1)}{(n-m)(n-m-1)\cdots 2\cdot 1} = C_n^{n-m}$

Mulitiply thely numerator and denominator by each other.

we will find they are both $n!$. so $C_n^m = C_n^{n-m} \Rightarrow \binom{n}{m} = \binom{n}{n-m}$

Second, according to ①, so $\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \sum_{m=0}^{n}\binom{n}{m}\binom{n}{m} = \sum_{m=0}^{n}\binom{n}{m}^2 = \binom{2n}{n}$

5.Find all solutions,if any,solutions to the system

$$x \equiv 5(mod6) \text{ ①}$$
$$x \equiv 3(mod10) \text{ ②}$$
$$x \equiv 8(mod15) \text{ ③}$$

5. ① $x \equiv 5 \pmod 6$ let $x = 6k+5$

② $6k+5 \equiv 3 \pmod{10}$ $6k \equiv (-2) \pmod{10}$ $3k \equiv (-1) \pmod 5$

$k \equiv (-2) \pmod 5$ $k = 5v-2$ $\therefore x = 30v-7$

③ $30v-7 \equiv 8 \pmod{15}$ $30v \equiv 15 \pmod{15}$ This equation is always true

so we conclude that $x = 30v+23$ or $x \equiv 23 \pmod{30}$

6.Show steps to find;
(a) the greatest common divisor of 1234567 and 7654321.
(b)the greatest common divisor of $2^3 3^5 5^7 7^9 11$ and $2^9 3^7 5^5 7^3 13$

(a) $7654321 = 1234567 \times 6 + 246919$ $GCD(1234567, 7654321)$

$1234567 = 246919 \times 4 + 246891$ $= GCD(1234567, 246919)$

$246919 = 246891 \times 1 + 19$ $= GCD(246919, 246891)$

$246891 = 19 \times 12994 + 5$ $= GCD(246891, 19)$

$19 = 5 \times 3 + 4$ $= GCD(19, 5)$

$5 = 4 \times 1 + 1$ $\quad = GCD(5,4)$

$4 = 1 \times 4 + 0$ $\quad = GCD(4,1) = GCD(1,0)$

the greatest common divisor is 1.

(b) $GCD(2^9 3^7 5^5 7^3 13,\ 2^3 3^5 5^7 7^9 11)$

$= GCD(2^6 3^2 13, 5^2 7^6 11) \times 2^3 3^5 5^5 7^3$

$\because$ $2^6 3^2 13$ and $5^2 7^6 11$ have not the common divisor

$\therefore$ the greatest common divisor is $2^3 3^5 5^5 7^3$

7.Label the first prime number 2 asP_1.Label the second prime number 3 as P_2.Similarly,label the n-th prime number as P_n.Prove that $P_n < 2^{2^n}$ for an arbitrary $n \in N^+$. (Hint:consider $P_1 P_2 P_2 \ldots P_{n-1} + 1$.)

First, we prove $\forall x \in Z^+$ $(x>1)$ is a product of primes

Assume this is true, set $n \in Z^+$

if n is a prime, then it's done

if n is not a prime, let $n = k \cdot m$ where $\begin{cases} k = P_1 P_2 \cdots P_k \\ m = q_1 q_2 \cdots q_m \end{cases}$

$P_1, P_2 \cdots P_k$ and $q_1, q_2 \cdots q_m$ are all primes here.

then $n = P_1 \cdot P_2 \cdot P_3 \cdots P_k \cdot q_1 \cdot q_2 \cdots q_m$, also a product of primes

so it is proved

Second, $P_1 = 2$ $\quad P_2 = 3$ $\quad P_3 = 5$ $\quad P_4 = 7 \cdots$

$P_3 < P_1 \cdot P_2 + 1$ $\quad P_4 < P_1 P_2 \cdot P_3 + 1$ $\cdots$

$P_n < P_1 \cdot P_2 \cdots P_{n-1} + 1$ $\quad$ Base: $n=1$ $\quad P_1 = 2 < 2^2$

let $f_n = P_1 \cdot P_2 \cdots P_n$ $\quad f_1 = P_1 = 2$ $\quad f_2 = P_1 P_2 = 2 \cdot 3 = 6$

$f_1 < 2^2$

assume $P_n < 2^{2^n}$, we consider about P_{n+1}

$P_{n+1} \le P_1 P_2 P_3 \cdots P_n + 1$ $\quad P_1 \cdot P_2 \cdots P_n < 2^{2^1} \cdot 2^{2^2} \cdots 2^{2^n} = 2^{2^1+2^2+\cdots 2^n}$

$2^1 + 2^2 + \cdots 2^n = \frac{2 \cdot (1-2^n)}{1-2} = 2^{n+1} - 2$

$\therefore P_{n+1} < 2^{2^{n+1}-2} + 1$, we will compare $2^{2^{n+1}-2}+1$ with $2^{2^{n+1}}$

$2^{2^{n+1}-2} + 1 < 2^{2^{n+1}}$ for every $n \in N^+$

$\therefore P_{n+1} < 2^{2^{n+1}}$ Through the recursion, we prove that $P_n < 2^{2^n}$

8.In a round-robin tournament,every team plays every other team exactly once and each match has a winner and a loser.We say that the team $p_1, p_2, \ldots, p_m$ form a cycle if p_1 beats p_2,p_2 beats p_3,and p_m beats p_1.Show that if there is a cycle of length $m(m > 3)$among the players in a round-robin tournament,there must be a cycle ~~off~~ of three of these players.(Hint:Use well-ordering principle.)

8. we assume we have a minimum player teams of a cycle, which have a lengeh of m $(m>3)$

assume we have k teams. $P_1, P_2, \cdots P_k$ $(k \ge m)$

first of all, we know that P_1 beats P_2, P_2 beats P_3 $\cdots$ P_{k-1} beats P_k, P_k beats P_1

secondly, we take out P_1, P_{k-1}, P_k

case 1: (P_1 beats P_{k-1}) $P_1 P_{k-1} P_k$ are a circle of three (contradicts)

case 2: (P_{k-1} beats P_1) we take out $P_1 P_{k-2} P_{k-1}$

In case 2: we have case 3, case 4. we take out $P_1 P_{k-2} P_{k-1}$

case 3: (P_1 beats P_{k-2}). $P_1 P_{k-2} P_{k-1}$ are a circle of three (contradicts)

case 4: (P_{k-2} beats P_1), $P_1 P_{k-3} P_{k-2}$

Through case 2 and case 4, we will find the base.

base: when $k=4$ $P_1 P_2 P_3 P_4$

If P_1 beats P_3, $P_1 P_3 P_4$ is a circle (contradicts)

If P_3 beats P_1. we will consider $P_1 . P_2 P_3$. it contradicts.

so, we find that the base is contrary to the hypothetical proposition.

we will recurse on k.

then, we assume case 1 is contrary to the hypothetical proposition.

when $k=k+1$, we take $P_1 P_k P_{k+1}$

if P_1 beats P_k, $P_1 P_k P_{k+1}$ is a circle.

it is contrary to the hypothetical proposition.

if P_k beats P_1, it is the case 1.

it is contrary to the hypothetical proposition.

so no matter k ($k \geq 4$) is what, it always has a cycle of length (=3)

it goes against the hypothetical proposition.

so. there must be a cycle of three of these teams