1. (a) Base Case. $f_3 = f_1 + f_2 = 2$. $f_4 = f_2 + f_3 = 3$, $f_5 = f_3 + f_4 = 5$,

$f_1 + f_2 = 2 = f_4 - 1$.

Induction. Assume $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$. for all n,

so $f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$,

then $f_1 + f_2 + \cdots + f_{n-1} + f_n + f_{n+1} = f_{n+1} - 1 + f_n + f_{n+1}$

$= f_n + f_{n+1} + f_{n+1} - 1$

$= f_{n+2} + f_{n+1} - 1$

$= f_{n+3} - 1$ $\qquad P(n) \to P(n+1)$

therefore, $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

(b) Base case. $f_3 = f_1 + f_2 = a + b$, $f_4 = f_2 + f_3 = a + 2b$,

$f_1 + f_2 = a + b = f_4 - b$

Induction. Assume $f_1 + f_2 + \cdots + f_n = f_{n+2} - b$ for any n,

$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - b$

then $f_1 + f_2 + \cdots + f_{n+1} = f_1 + f_2 + \cdots + f_{n-1} + f_n + f_{n+1}$

$= f_{n+1} - b + f_n + f_{n+1}$

$= f_{n+2} - b + f_{n+1}$

$= f_{n+3} - b$, $\qquad P(n) \to P(n+1)$

therefore, $f_1 + f_2 + \cdots + f_n = f_{n+2} - b$

2. (a) Generate function $f(x)$.

$$f(x) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$$
$$= 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$$
$$= \sum_{i=0}^{\infty} (ax)^i$$
$$= \lim_{n\to\infty} \frac{1-(ax)^n}{1-ax}$$
$$= \frac{1}{1-ax} \qquad (\text{if } f(x) \text{ converges, } (ax)^n \to 0 \text{ as } n\to\infty)$$

When $x=1$, $f(1) = 1+a+a^2+\cdots+a^n+\cdots = \sum_{i=0}^{\infty} a^i$

$\therefore \frac{1}{1-ax}$ is the closed form formula of sequence $a_n = a^n$

(b) Generate function $f(x)$

$$f(x) = (1+x)^m \qquad (m\in N^+)$$
$$= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{n}x^n + \cdots \binom{m}{m}x^m$$

When $x=1$, $f(x) = \sum_{i=0}^{m} \binom{m}{i}$

$\therefore f(x) = (1+x)^m$ is closed form formula of sequence $\binom{m}{n}$ $(m\in N^+)$

(c) Generate function $f(x)$

$$f(x) = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n + \cdots$$
$$xf(x) = f_0x + f_1x^2 + f_2x^3 + \cdots$$
$$x^2f(x) = f_0x^2 + f_1x^3 + f_2x^4 + \cdots$$

since $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$,

$\therefore f(x) = x + xf(x) + x^2f(x)$,

$\therefore f(x) = \frac{x}{1-x-x^2}$, When $x=1$, $f(1) = f_0+f_1+f_2+\cdots = \sum_{i=0}^{\infty} i$,

therefore, $f(x) = \frac{x}{1-x-x^2}$ is the closed form formula of Fibonacci sequence.

4.Using the identity

$$(1+x)^n(1+x)^n = (1+x)^{2n}$$

Prove that

$$\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$$

Deduce that

$$\sum_{m=0}^{n}\binom{n}{m}^2 = \binom{2n}{n}$$

4. $(1+x)^n(1+x)^n = \left[1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n\right]$

$\cdot\left[1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n\right]$

So for the term of x^n, we have

$1\cdot\binom{n}{n}x^n+\binom{n}{1}x\cdot\binom{n}{n-1}x^{n-1}+\binom{n}{2}x^2\cdot\binom{n}{n-2}x^{n-2}+\cdots+\binom{n}{n}x^n\cdot 1$

$=\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m}x^n$, its coefficient is $\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m}$

$(1+x)^{2n} = 1+\binom{2n}{1}x+\binom{2n}{2}x^2+\cdots+\binom{2n}{n}x^n+\cdots+\binom{2n}{2n}x^{2n}$

For the term of x^n, we have its coefficient $\binom{2n}{n}$

since $(1+x)^n(1+x)^n = (1+x)^{2n}$,

so the coefficient of x^n must equal. $\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$

$\binom{n}{m} = \dfrac{n\cdot(n-1)\cdot\cdots\cdot(n-m+1)}{m(m-1)(m-2)\cdots\cdot 2\cdot 1}$, $\binom{n}{n-m} = \dfrac{n(n-1)\cdot\cdots\cdot(m+1)}{(n-m)\cdot(n-m-1)\cdot\cdots\cdot 2\cdot 1}$

since $n\cdot(n-1)\cdot\cdots\cdot(n-m+1)\cdot(n-m)\cdot(n-m-1)\cdot\cdots\cdot 2\cdot 1$

$= n\cdot(n-1)\cdot\cdots\cdot(m+1)\cdot m\cdot(m-1)(m-2)\cdot\cdots\cdot 2\cdot 1 = n!$

$\therefore \dfrac{n\cdot(n-1)\cdot\cdots\cdot(n-m+1)}{m(m-1)(m-2)\cdot\cdots\cdot 2\cdot 1} = \dfrac{n(n-1)\cdot\cdots\cdot(m+1)}{(n-m)\cdot(n-m-1)\cdot\cdots\cdot 2\cdot 1}$. means $\binom{n}{m}=\binom{n}{n-m}$

$\therefore \sum_{m=0}^{n}\binom{n}{m}^2 = \sum_{m=0}^{n}\binom{n}{m}\cdot\binom{n}{n-m} = \binom{2n}{n}$

5.Find all solutions,if any,solutions to the system

$$x \equiv 5(mod6)$$
$$x \equiv 3(mod10)$$
$$x \equiv 8(mod15)$$

6.Show steps to find;
(a) the greatest common divisor of 1234567 and 7654321.
(b)the greatest common divisor of $2^3 3^5 5^7 7^9 11$ and $2^9 3^7 5^5 7^3 13$

7.Label the first prime number 2 asP_1.Label the second prime number 3 as P_2.Similarly,label the n-th prime number as P_n.Prove that $P_n < 2^{2^n}$ for an arbitrary $n \in N^+$. (Hint:consider $P_1 P_2 P_2 \ldots P_{n-1} + 1$.)

8.In a round-robin tournament,every team plays every other team exactly

5. Assume there are some integers x,

then $x = 6k+5 = 10m+3 = 15n+8$ $(m,n,k \in \mathbb{Z})$

$$\begin{cases} 6k+5 = 10m+3 \\ 10m+3 = 15n+8 \Rightarrow m = \frac{3n+1}{2} \\ 6k+5 = 15n+8 \Rightarrow k = \frac{5n+1}{2} \end{cases}$$

and $k, m, n \in \mathbb{Z}$, so $2 \mid 3n+1$, $2 \mid 5n+1$,

therefore, $3n$, $5n$ are both odd, then n is odd,

set $n = 2p+1$. $(p \in \mathbb{Z})$, then $m = \frac{3n+1}{2} = 3p+2$,

$k = \frac{5n+1}{2} = 5p+3$,

$x = 6k+5 = 30p+23$, $(p \in \mathbb{Z})$

$x \equiv 30p+23 \equiv 23 \equiv 5 \pmod 6$

$x \equiv 30p+23 \equiv 23 \equiv 3 \pmod{30}$

$x \equiv 30p+23 \equiv 23 \equiv 8 \pmod{15}$

therefore, for $p \in \mathbb{Z}$, $x = 30p+23$. this is solutions

6. (a) $\gcd(1234567, 7654321)$

$7654321 = 6 \times 1234567 + 246919$

$= \gcd(1234567, 246919)$

$1234567 = 4 \times 246919 + 246891$

$= \gcd(246919, 246891)$

$246919 = 1 \times 246891 + 28$

$= \gcd(246891, 28)$

$246891 = 8817 \times 28 + 15$

$= \gcd(28, 15)$

$28 = 1 \times 15 + 13$

$= \gcd(15, 13)$

$= 1$

(b) $\gcd(2^3 3^5 5^7 7^9 11, 2^9 3^7 5^5 7^3 13)$

$= \gcd(2^3 3^5 5^5 7^3 \cdot 5^2 7^6 11, 2^3 3^5 5^5 7^3 \cdot 2^6 3^2 13)$

$= 2^3 3^5 5^5 7^3 \cdot \gcd(5^2 7^6 11, 2^6 3^2 13)$

$= 2^3 3^5 5^5 7^3 \cdot 1$

$= 10418625000$

$\gcd(5^2 7^6 11, 2^6 3^2 13) = 1$
since they are in prime production form and the 2 numbers have no common divisor of primes

sume $p_n < 2^{2^n}$ for n, then $p_{n+1} \le p_1 p_2 \cdots p_n + 1$

$< 2^{2^1} \cdot 2^{2^2} \cdot \cdots \cdot 2^{2^n} + 1$

7. First we prove 'Every integer >1 is a product of primes'

Assume this is true, set $n \in \mathbb{Z}^+$,

if n is a prime, then it's done

if n is not a prime, let $n = k \cdot m$,

where $k = p_1 p_2 \cdots p_k$ (p_i, q_i are all primes here)

$m = q_1 q_2 \cdots q_m$

(we have assumed every integer >1 is a product of primes, which means k, m are product of primes)

then $n = p_1 p_2 \cdots p_k \cdot q_1 q_2 \cdots q_m$, also a product of primes

so it is proved. Every integer >1 is a product of primes.

Set $Q = p_1 p_2 p_3 \cdots \cdot p_{n-1} + 1$,

for any $p_i \in \{p_1, p_2, \cdots, p_{n-1}\}$, $p_i \mid p_1 p_2 p_3 \cdots \cdot p_{n-1}$,

set $p_1 p_2 \cdots \cdot p_{n-1} = k_i p_i$ ($k_i \in \mathbb{Z}^+$), then $Q = k_i p_i + 1$,

if $p_i \mid Q$, $Q = m_i p_i$ ($m_i \in \mathbb{Z}^+$), then $(m_i - k_i) p_i = 1$,

$m_i - k_i$ should also be in $\mathbb{Z}^+$, but $p_i \nmid 1$. contradict

so $p_i \nmid Q$, means Q is not a product of any primes in $\{p_1, p_2, \cdots p_{n-1}\}$,

and since Q is a product of primes,

so there exist $p_N \mid Q$, where $N \ge n$, $N \in \mathbb{Z}^+$.

so $p_n \le p_N \le Q$.

means $p_n \le p_1 p_2 \cdots \cdot p_{n-1} + 1$, for any $n \in \mathbb{Z}^+$

Base case. $P_1 = 2 < 2^{2^1}$,

Assume $P_n < 2^{2^n}$ for n, then $P_{n+1} \le P_1 P_2 \cdots P_n + 1$

$$< 2^{2^1} \cdot 2^{2^2} \cdot \cdots \cdot 2^{2^n} + 1$$

$$= 2^{2^1 + 2^2 + \cdots + 2^n} + 1$$

$$= 2^{2^{n+1} - 2} + 1$$

$$\le 2^{2^{n+1}-2} + 2^{2^{n+1}-2} \quad (n \ge 1)$$

$$= 2^{2^{n+1}-1}$$

$$< 2^{2^{n+1}}$$

$\therefore P_{n+1} < 2^{2^{n+1}}$ if $P_n < 2^{2^n}$,

therefore by induction,

$P_n < 2^{2^n}$.

8.In a round-robin tournament,every team plays every other team exactly once and each match has a winner and a loser.We say that the team $p_1, p_2, \ldots, p_m$form a cycle if p_1 beats p_2,p_2 beats p_3,and p_m beats p_1.Show that if there is a cycle of length $m(m > 3)$among the players in a round-robin tournament,there must be a cycle off three of these players.(Hint:Use well-ordering principle.)

8.