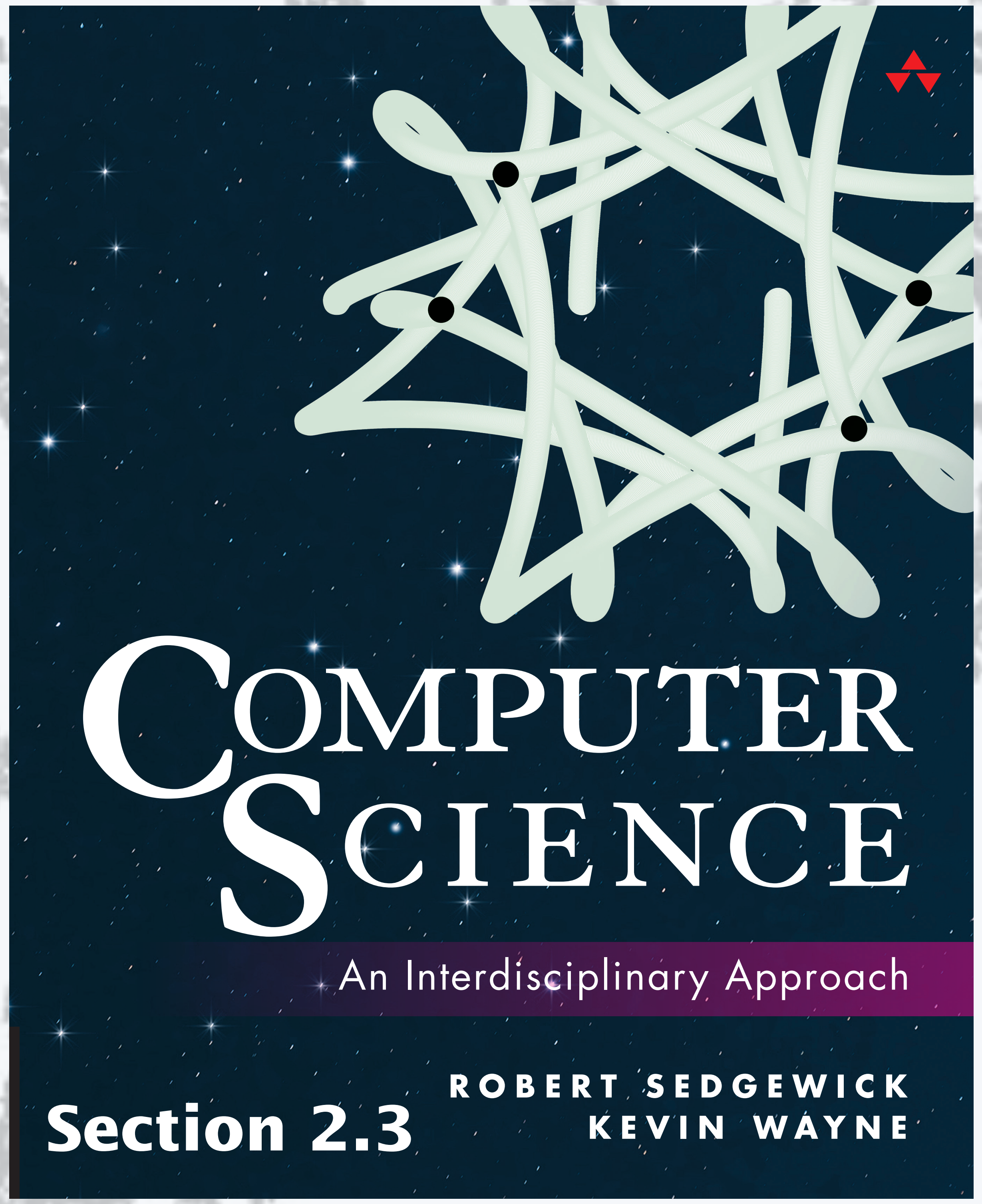
COMPUTER SCIENCE

SEDGEWICK / WAYNE

PART I: PROGRAMMING IN JAVA



http://introcs.cs.princeton.edu

6. Recursion

Midterm Arrangement

- **20 Mar, 10:30–12:00 (Wed)** in total 90 minutes
- **Closed-book exam**
- **No electronic devices (including calculators)** are allowed

- Scope: from 1.Basics to 6.Recursion

- Around 20% choice questions
- Around 50% read the program, give the expected output
- Around 30% short answer questions
- The above are rough numbers, the proportion can be changed.

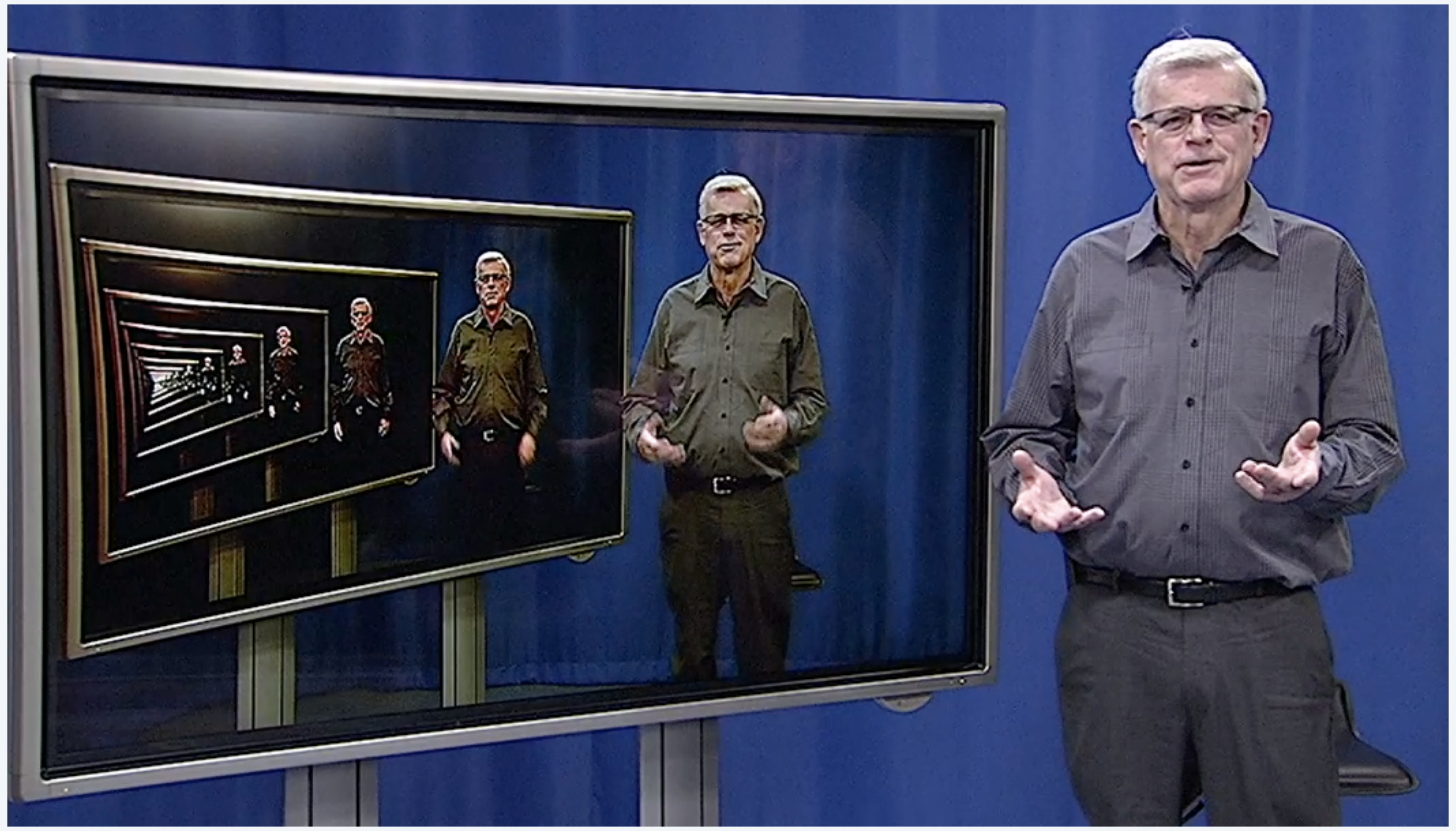
6. Recursion

- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

Overview

Q. What is recursion?

A. When something is specified in terms of *itself*.

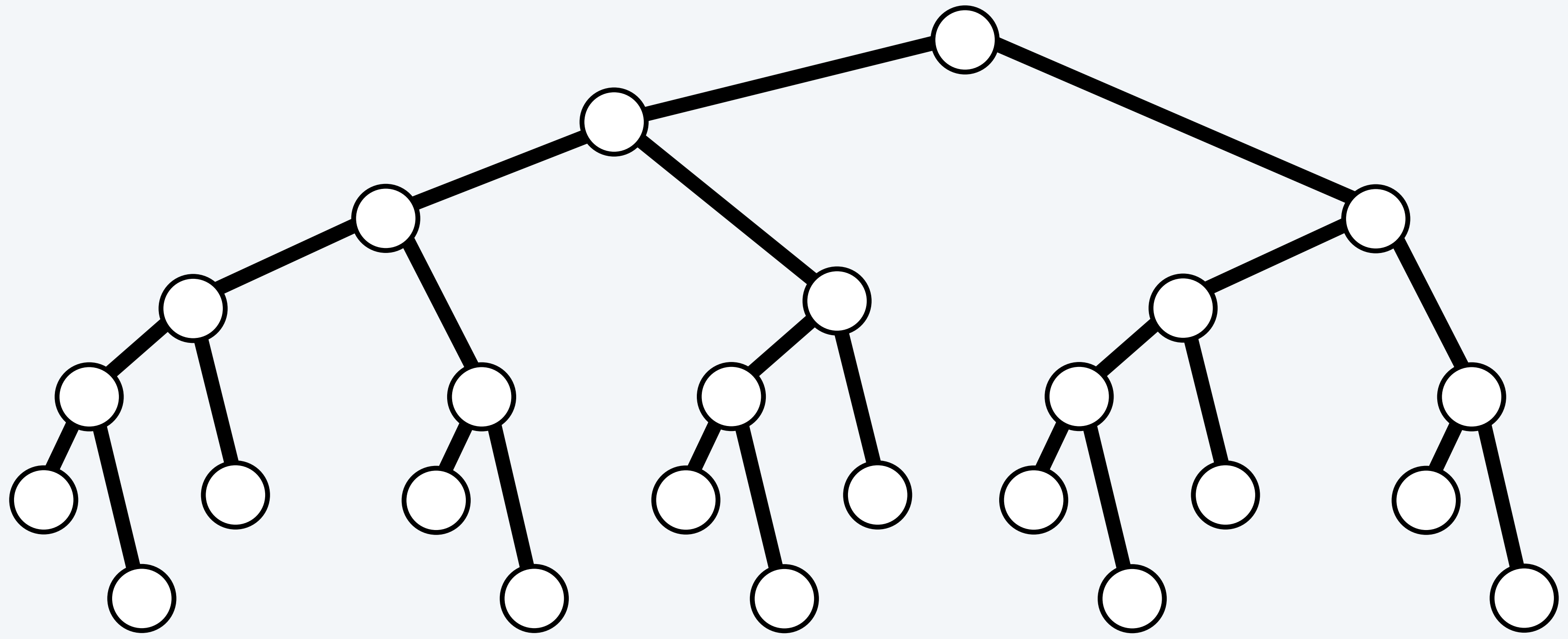


Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Enables reasoning about correctness.
- Gives insight into the nature of computation.

Many computational artifacts are *naturally* self-referential.

- File system with folders containing folders.
- Fractal graphical patterns.
- Divide-and-conquer algorithms (stay tuned).



Example: Convert an integer to binary

Recursive program

To compute a function of a positive integer *N*

- Base case. Return a value for small *N* .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for *N*.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      StdOut.println(convert(N));
   }
}
```

int 0 or 1 automatically converted to String "0" or "1"

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```

Q. How can we be convinced that this method is correct?

A. Use *mathematical induction*.

Mathematical induction (quick review)

To prove a statement involving a positive integer N

- Base case. Prove it for some specific values of N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

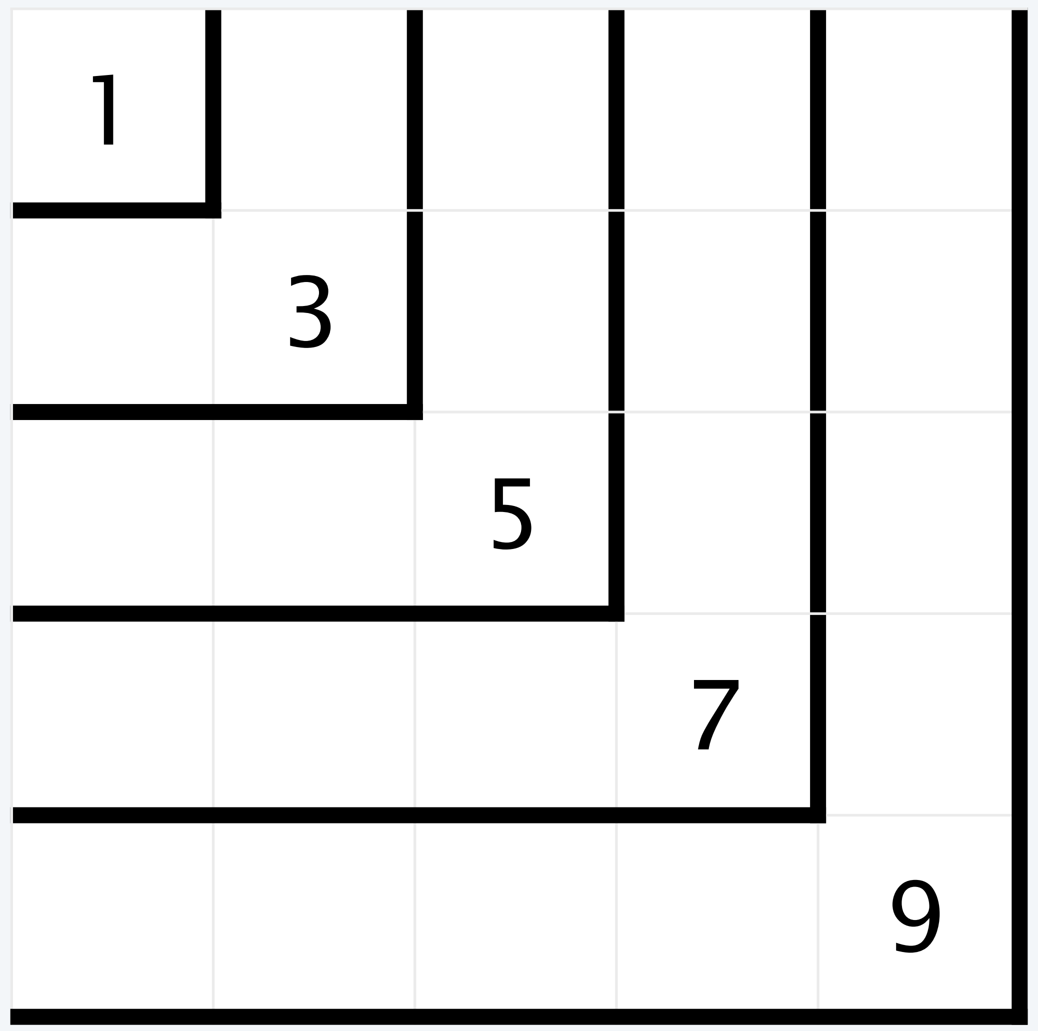
Example

The sum of the first N odd integers is N^2.

Base case. True for $N = 1$.

Induction step. The Nth odd integer is $2N - 1$. Let $T_N = 1 + 3 + 5 + \ldots + (2N - 1)$ be the sum of the first N odd integers.

- Assume that $T_{N-1} = (N - 1)^2$.
- Then $T_N = (N - 1)^2 + (2N - 1) = N^2$.



An alternate proof

Proving a recursive program correct

Recursion

To compute a function of N

- Base case. Return a value for small N .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for N.

Mathematical induction

To prove a statement involving N

- Base case. Prove it for small N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

Recursive program

```
public static String convert(int N)
{
   if (N == 1) return "1";
   return convert(N/2) + (N % 2);
}
```

Correctness proof, by induction

`convert()` computes the binary representation of N

- Base case. Returns "1" for $N = 1$.
- Induction step. Assume that `convert()` works for $N/2$

  1. Correct to append "0" if N is even, since $N = 2(N/2)$.

  2. Correct to append "1" if N is odd since $N = 2(N/2) + 1$.

Mechanics of a function call

System actions when *any* function is called

- *Save environment* (values of all variables and call location).
- *Initialize values* of argument variables.
- *Transfer control* to the function.
- *Restore environment* (and assign return value)
- *Transfer control* back to the calling code.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      System.out.println(convert(N));
   }
}
```

```
convert(26)
   if (N == 1) return "1";
   return  "1101"  + "0";
```

```
convert(13)
   if (N == 1) return "1";
   return  "110"  + "1";
```

```
convert(6)
   if (N == 1) return "1";
   return  "11"  + "0";
```

```
convert(3)
   if (N == 1) return "1";
   return  "1"  + "1";
```

```
convert(1)
   if (N == 1) return "1";
   return convert(0) + "1";
```

```
% java Binary 26
11010
```

Programming with recursion: typical bugs

Missing base case

```
public static double bad(int N)
{
   return bad(N-1) + 1.0/N;
}
```

No convergence guarantee

```
public static double bad(int N)
{
   if (N == 1) return 1.0;
   return bad(1 + N/2) + 1.0/N;
}
```

Try $N = 2$

Both lead to *infinite recursive loops* (bad news).

need to know how to stop them on your computer

Collatz Sequence

Collatz function of *N*.

- If *N* is 1, stop.
- If *N* is even, divide by 2.
- If *N* is odd, multiply by 3 and add 1.

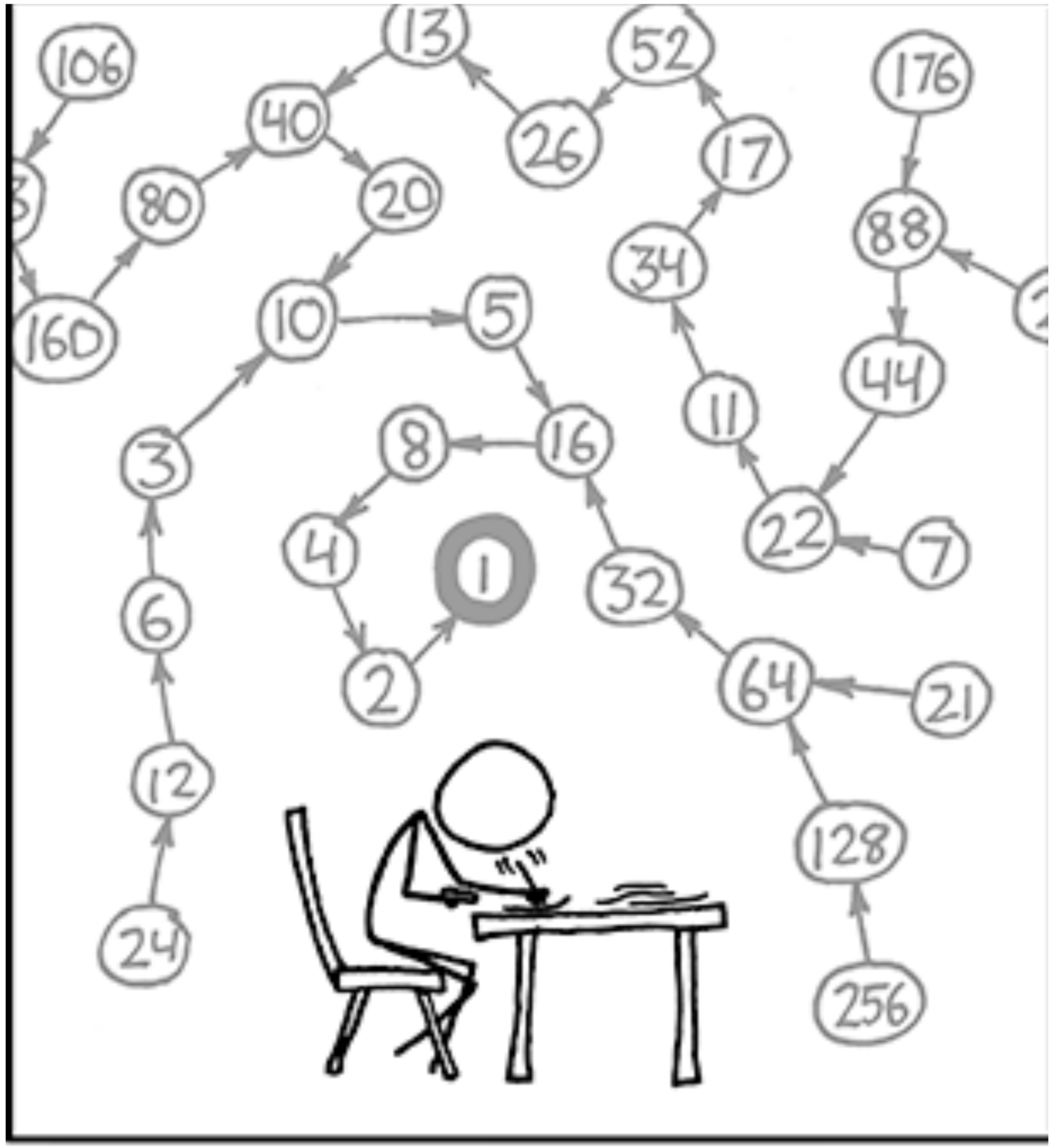
7	22	11	34	17	52	26	13	40	20	...

```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    else if (N % 2 == 0) collatz(N / 2);
    else collatz(3*N + 1);
}
```

```
% java Collatz 7
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

Amazing fact. No one knows whether or not this function terminates for all *N* (!)

Note. We usually ensure termination by only making recursive calls for smaller *N*.



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

Image sources

http://xkcd.com/710/

6. Recursion

- Foundations
- **A classic example**
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

Warmup: subdivisions of a ruler (revisited)

`ruler(n)`: create subdivisions of a ruler to $1/2^n$ inches.

- Return one space for $n = 0$.
- Otherwise, sandwich n between two copies of `ruler(n-1)`.

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

```
public class Ruler
{
   public static String ruler(int n)
   {
      if (n == 0) return " ";
      return ruler(n-1) + n + ruler(n-1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(ruler(n));
   }
}
```

```
% java Ruler 1
 1
% java Ruler 2
 1 2 1
% java Ruler 3
 1 2 1 3 1 2 1
% java Ruler 4
 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
% java Ruler 50
Exception in thread "main"
java.lang.OutOfMemoryError:
Java heap space
```

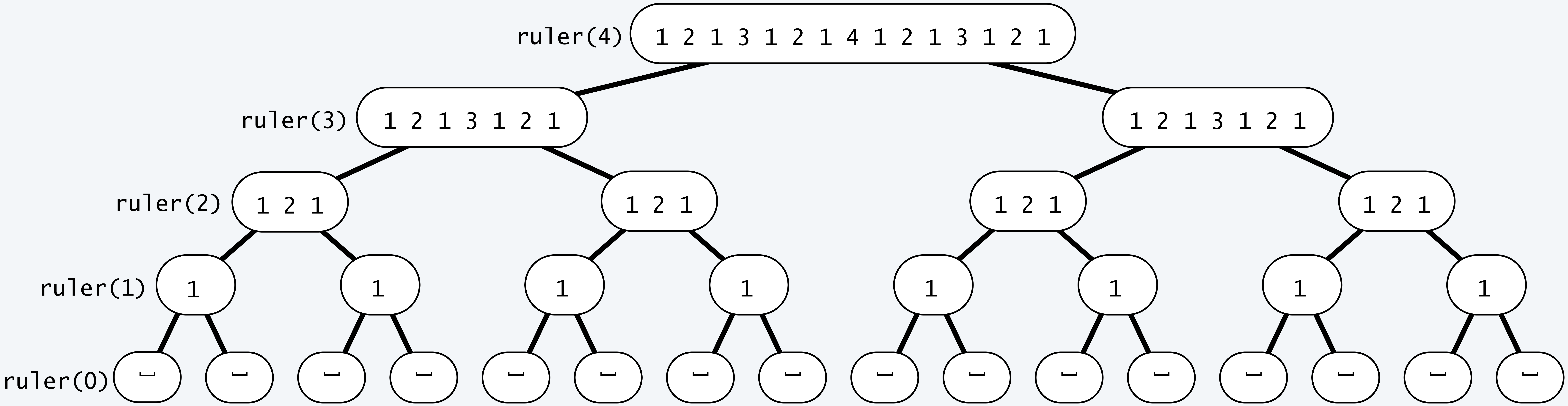
$2^{50} - 1$ integers in output.

Tracing a recursive program

Use a *recursive call tree*

- One node for each recursive call.
- Label node with return value after children are labeled.

```
public static String ruler(int n)
{
   if (n == 0) return " ";
   return ruler(n-1) + n + ruler(n-1);
}
```



Towers of Hanoi puzzle

A legend of uncertain origin

- $n = 64$ discs of differing size; 3 posts; discs on one of the posts from largest to smallest.
- An ancient prophecy has commanded monks to move the discs to another post.
- When the task is completed, *the world will end.*

n = 10

Rules

- Move discs one at a time.
- Never put a larger disc on a smaller disc.

before

after

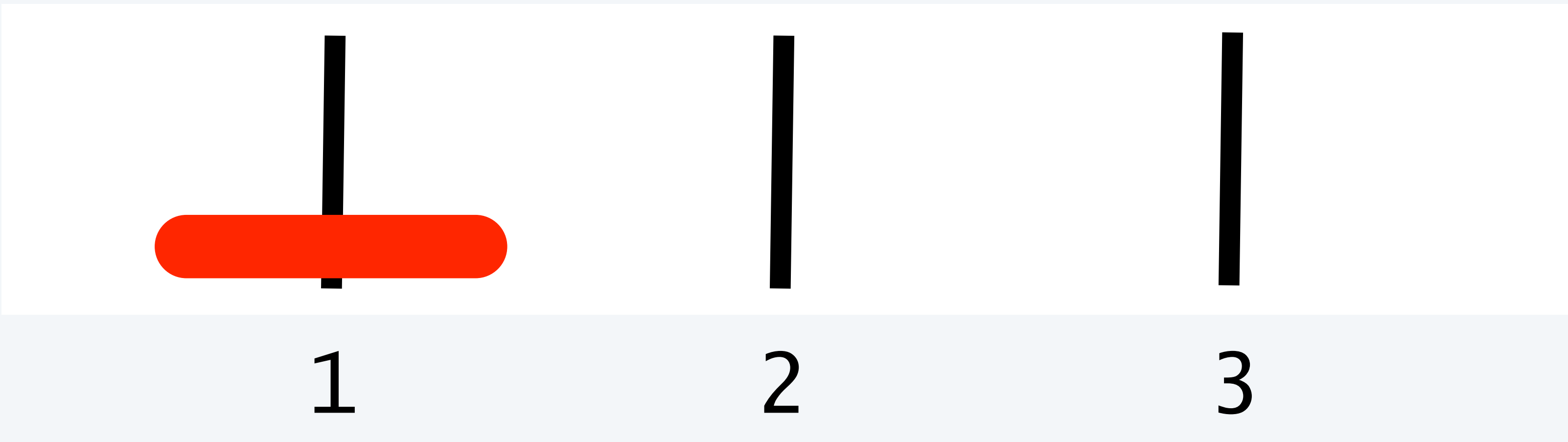
Q. Generate list of instruction for monks ?

Q. When might the world end ?

Towers of Hanoi

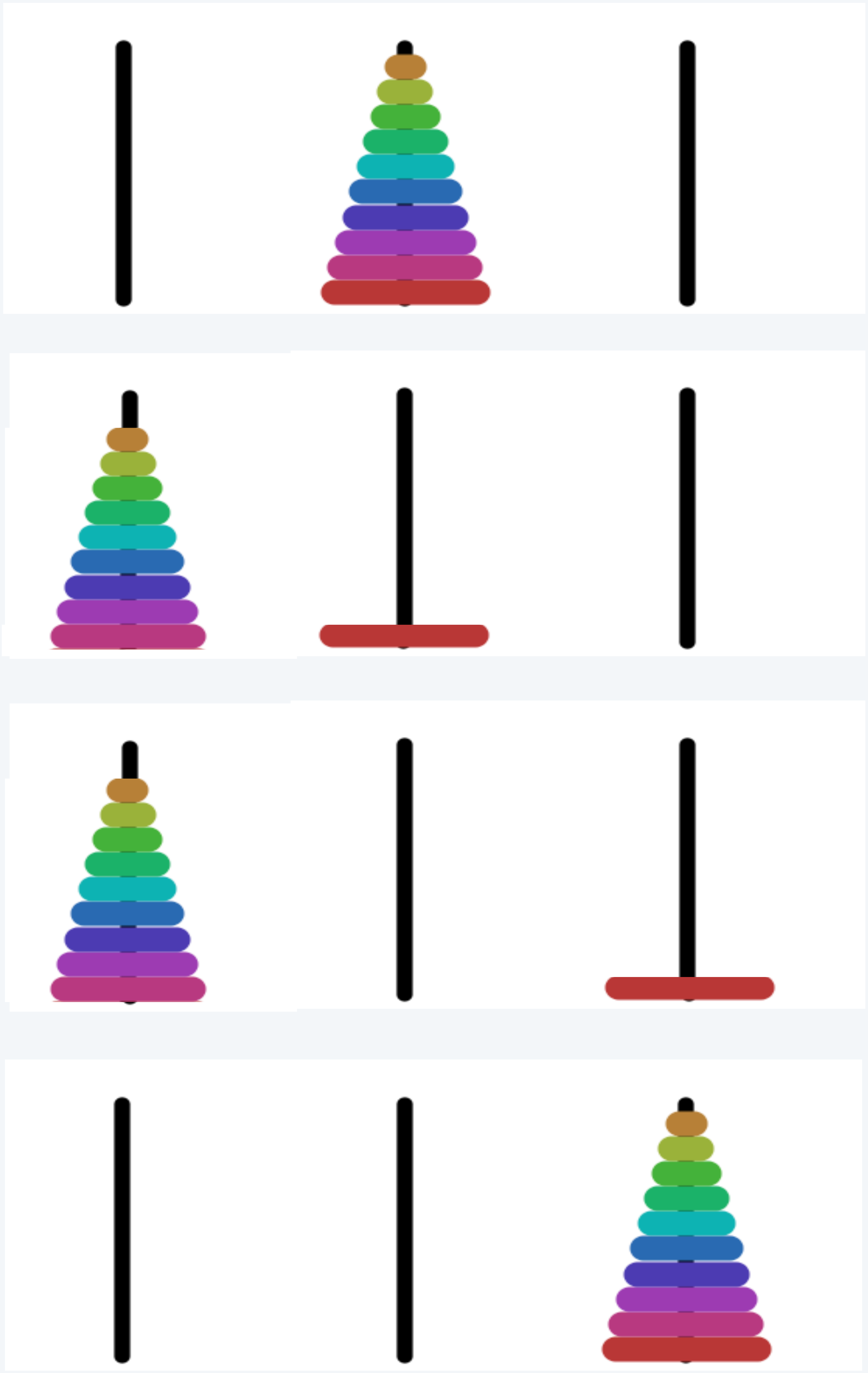
For simple instructions, use cyclic wraparound

- Move *right* means 1 to 2, 2 to 3, or 3 to 1.
- Move *left* means 1 to 3, 3 to 2, or 2 to 1.

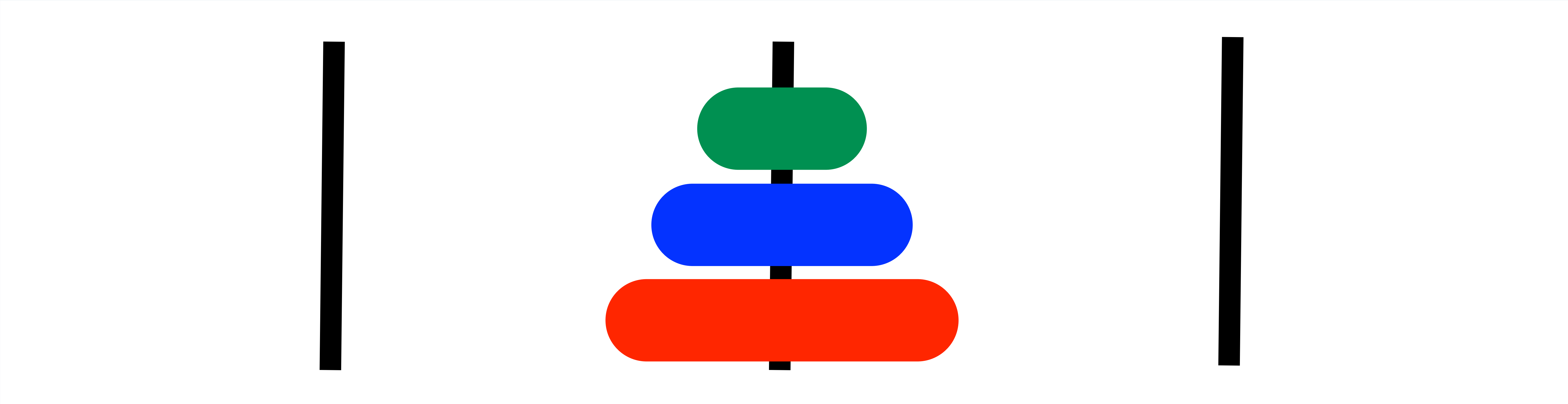


A recursive solution

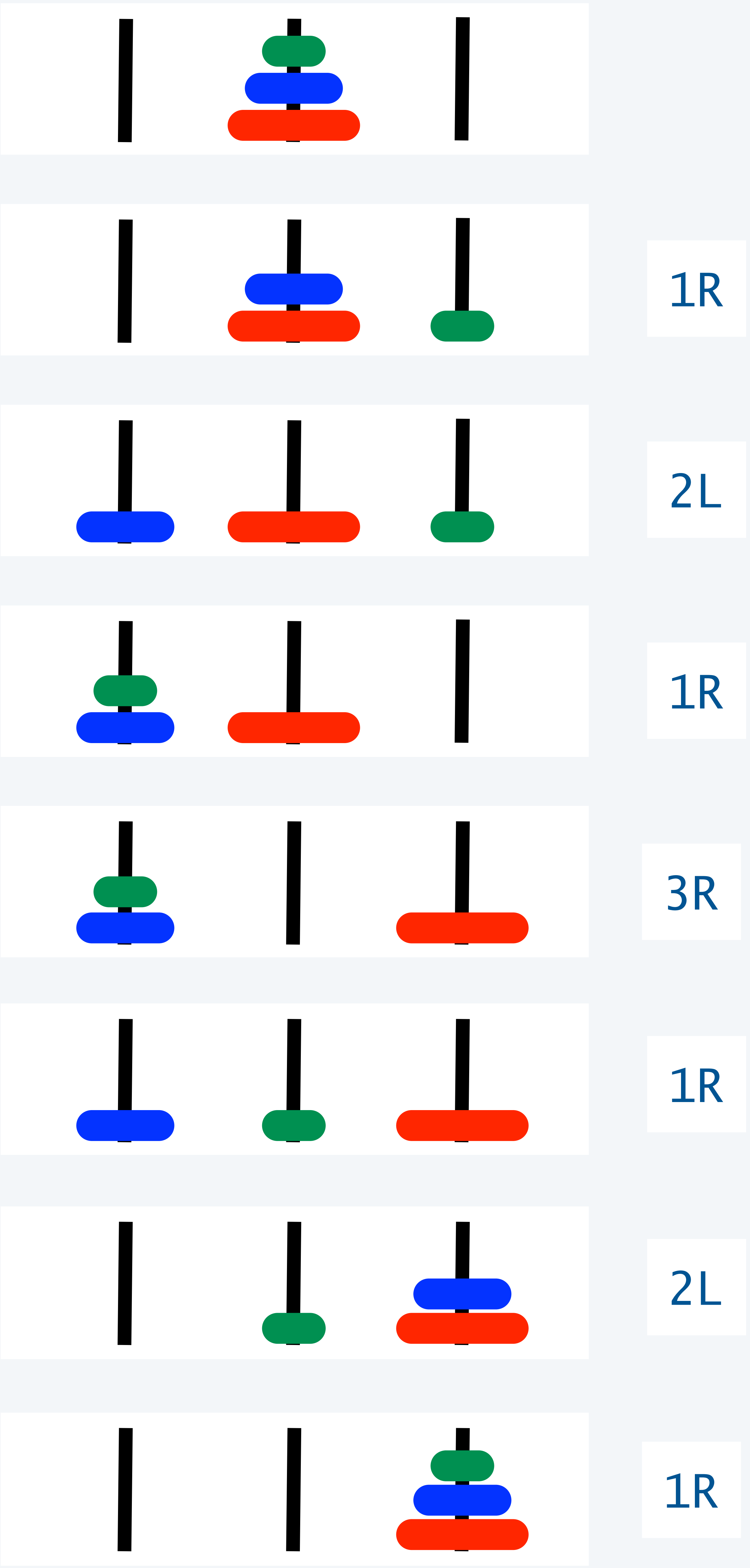
- Move $n - 1$ discs to the left (recursively).
- Move largest disc to the *right*.
- Move $n - 1$ discs to the left (recursively).



Towers of Hanoi solution (n = 3)



1R	2L	1R	3R	1R	2L	1R



Towers of Hanoi: recursive solution

`hanoi(n)`: Print moves for *n* discs.

- Return one space for *n* = 0.
- Otherwise, set `move` to the specified move for disc *n.*
- Then sandwich `move` between two copies of `hanoi(n-1)`.

```
public class Hanoi
{
   public static String hanoi(int n, boolean left)
   {
      if (n == 0) return " ";
      String move;
      if (left) move = n + "L";
      else      move = n + "R";
      return hanoi(n-1, !left) + move + hanoi(n-1, !left);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(hanoi(n, false));
   }
}
```

```
% java Hanoi 3
 1R 2L 1R 3R 1R 2L 1R
```

Recursive call tree for towers of Hanoi

Structure is the *same* as for the ruler function and suggests 3 useful and easy-to-prove facts.

- Each disc always moves in the same direction.
- Moving smaller disc always alternates with a unique legal move.
- Moving n discs requires $2^n - 1$ moves.

Answers for towers of Hanoi

Q. Generate list of instructions for monks ?

A. (Long form). 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L 5L 1L 2R 1L 3L 1L 2R 1L 4R ...

A. (Short form). Alternate "1L" with the only legal move not involving the disc 1.

"L" or "R" depends on whether *n* is odd or even

Q. When might the world end ?

A. Not soon: need $2^{64} - 1$ moves.

Note: Recursive solution has been proven optimal.

moves per second	*end of world*
1	5.84 billion centuries
1 billion	5.84 centuries

CS.6.B.Recursion.Hanoi

6. Recursion

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- Recursive graphics
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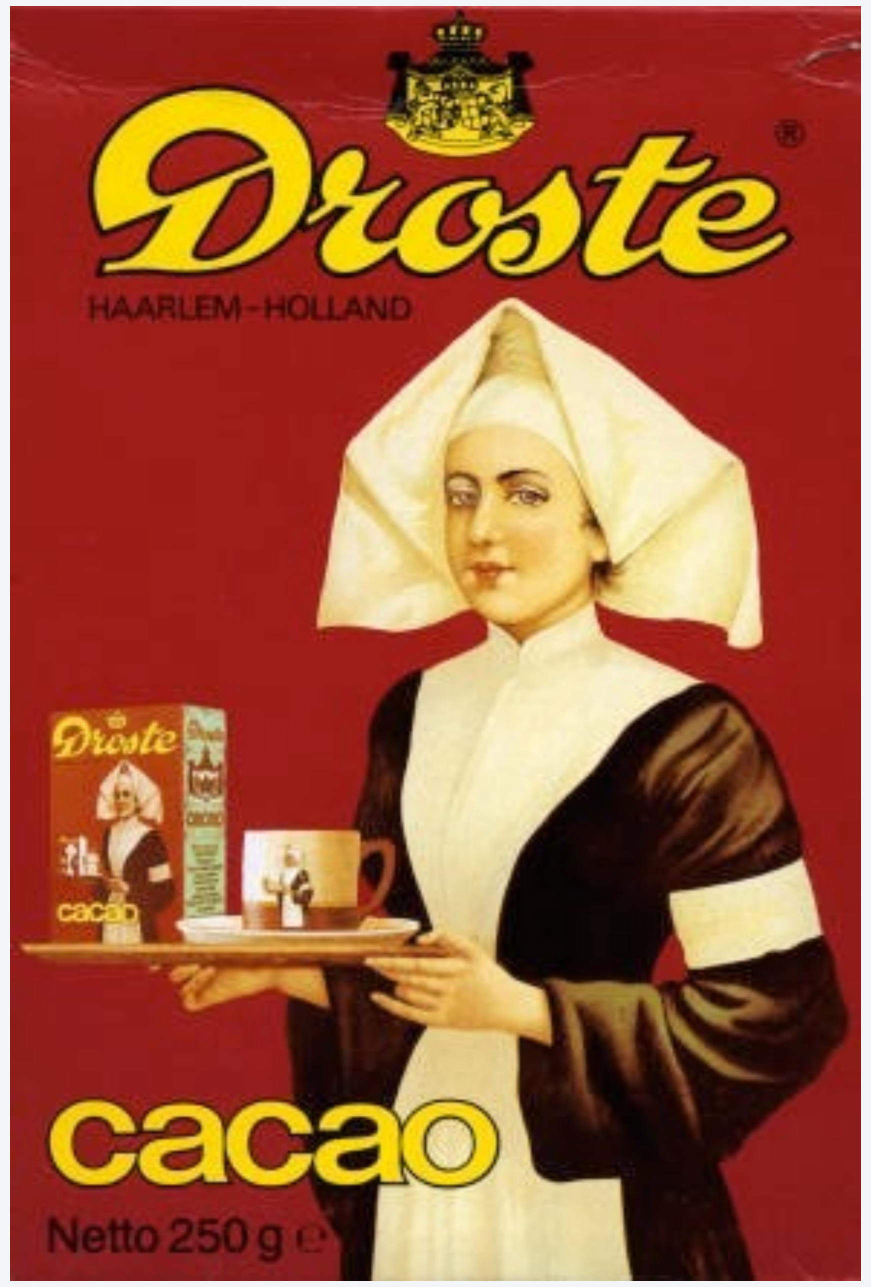
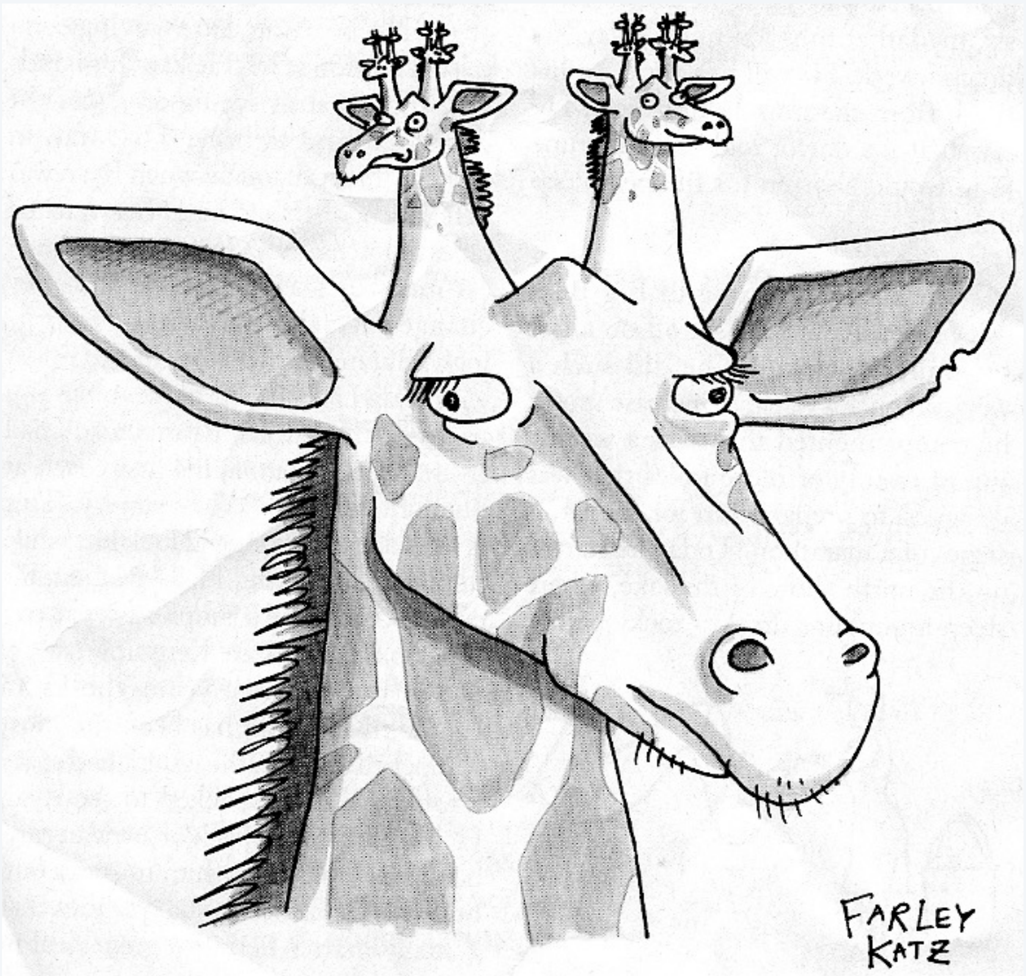
Recursive graphics in the wild






WEEKEND Arts FINE ARTS LEISURE

E41

FRIDAY, DECEMBER 15, 2006

The New York Times

Fruits of Design, Certified Organic

ROBERTA SMITH

ART REVIEW

The Gifts to Open Again and Again

WILLIAM GRIMES

BOOKS

Black, White and Read All Over Over

By RANDY KENNEDY

Divine and Devotee Meet Across Hinges

HOLLAND COTTER

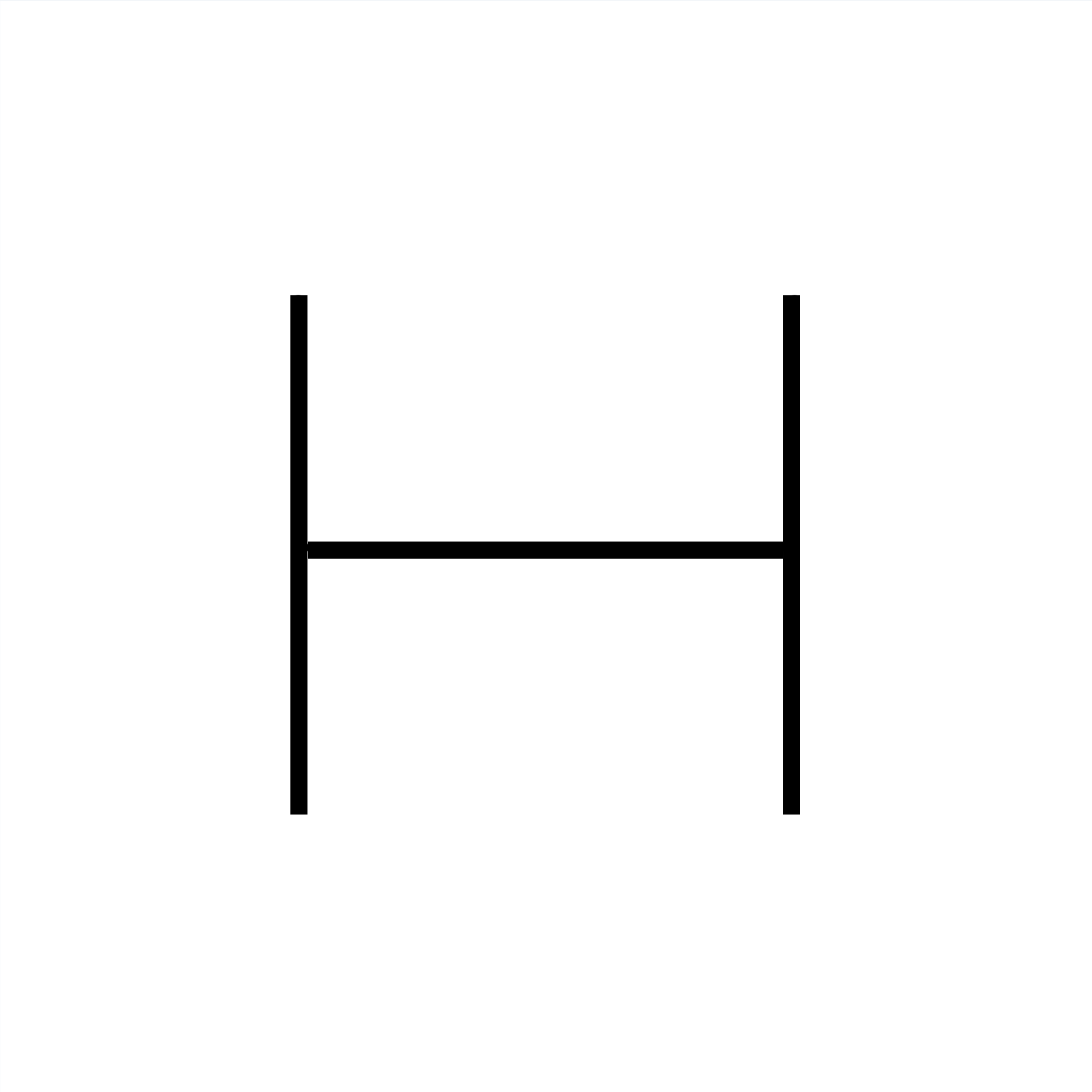
ART REVIEW

"Hello, World" of recursive graphics: H-trees

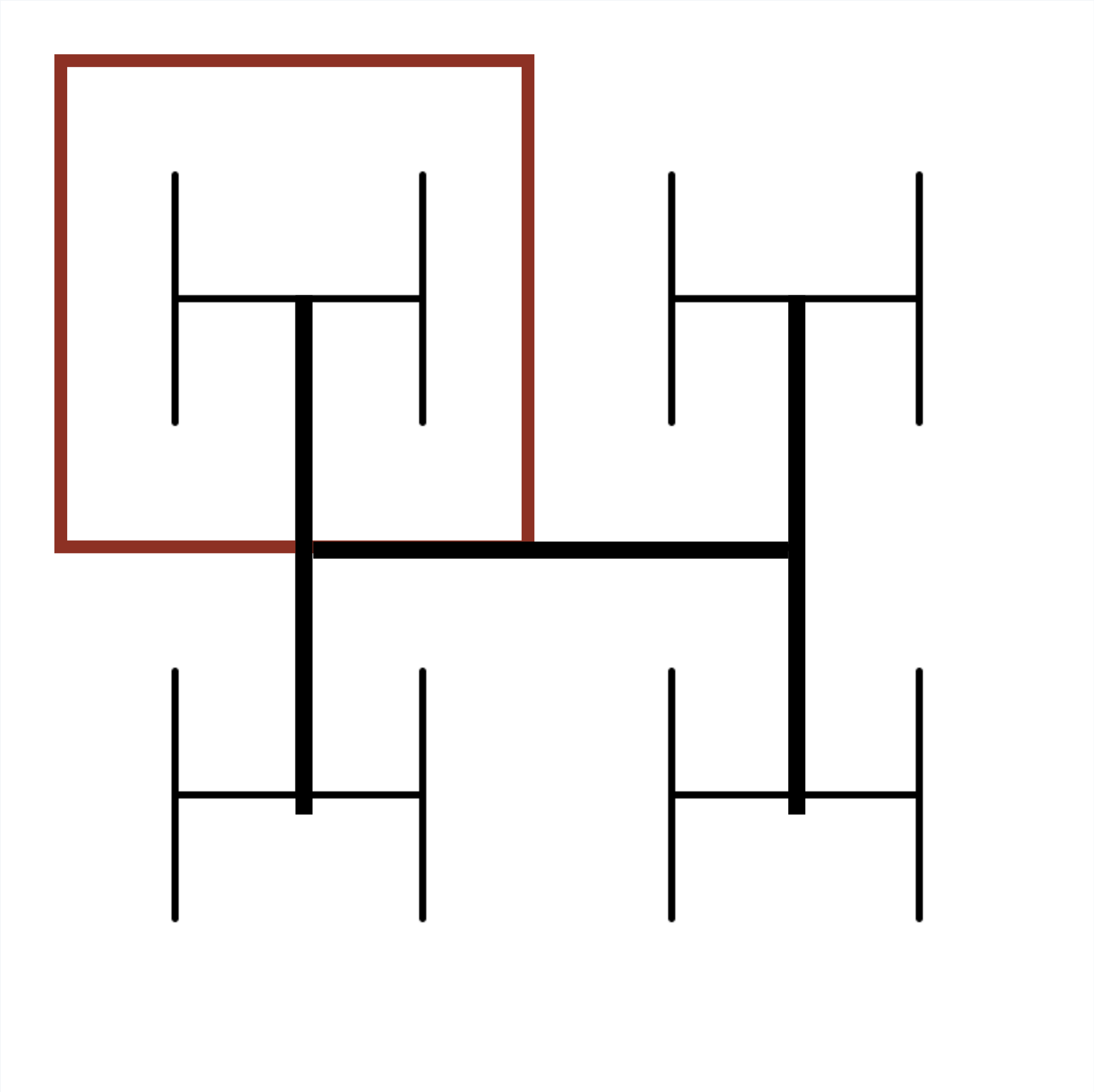
H-tree of order *n*

- If *n* is 0, do nothing.
- Draw an H, centered.
- Draw four H-trees of order *n* –1 and half the size, centered at the tips of the H.

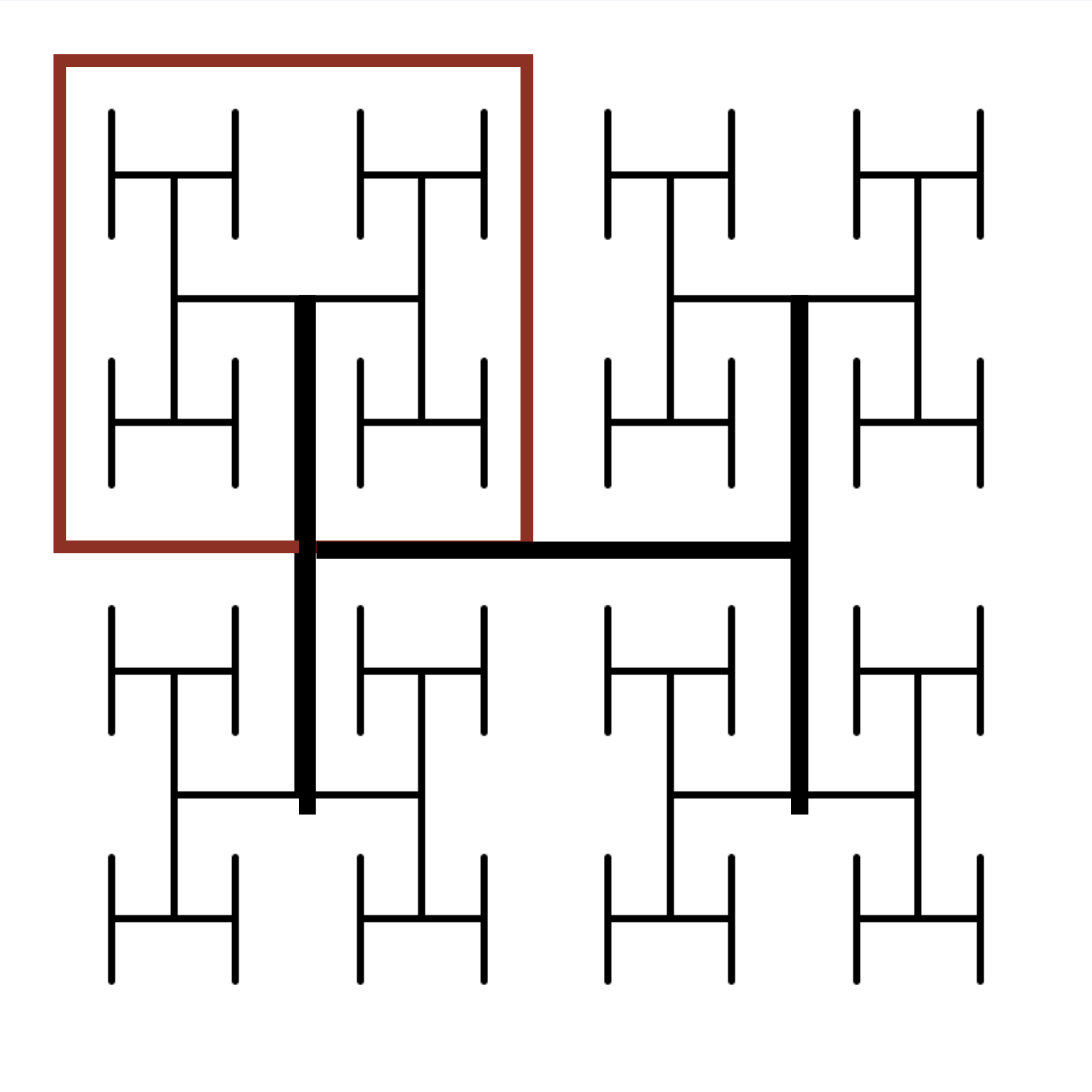
order 1



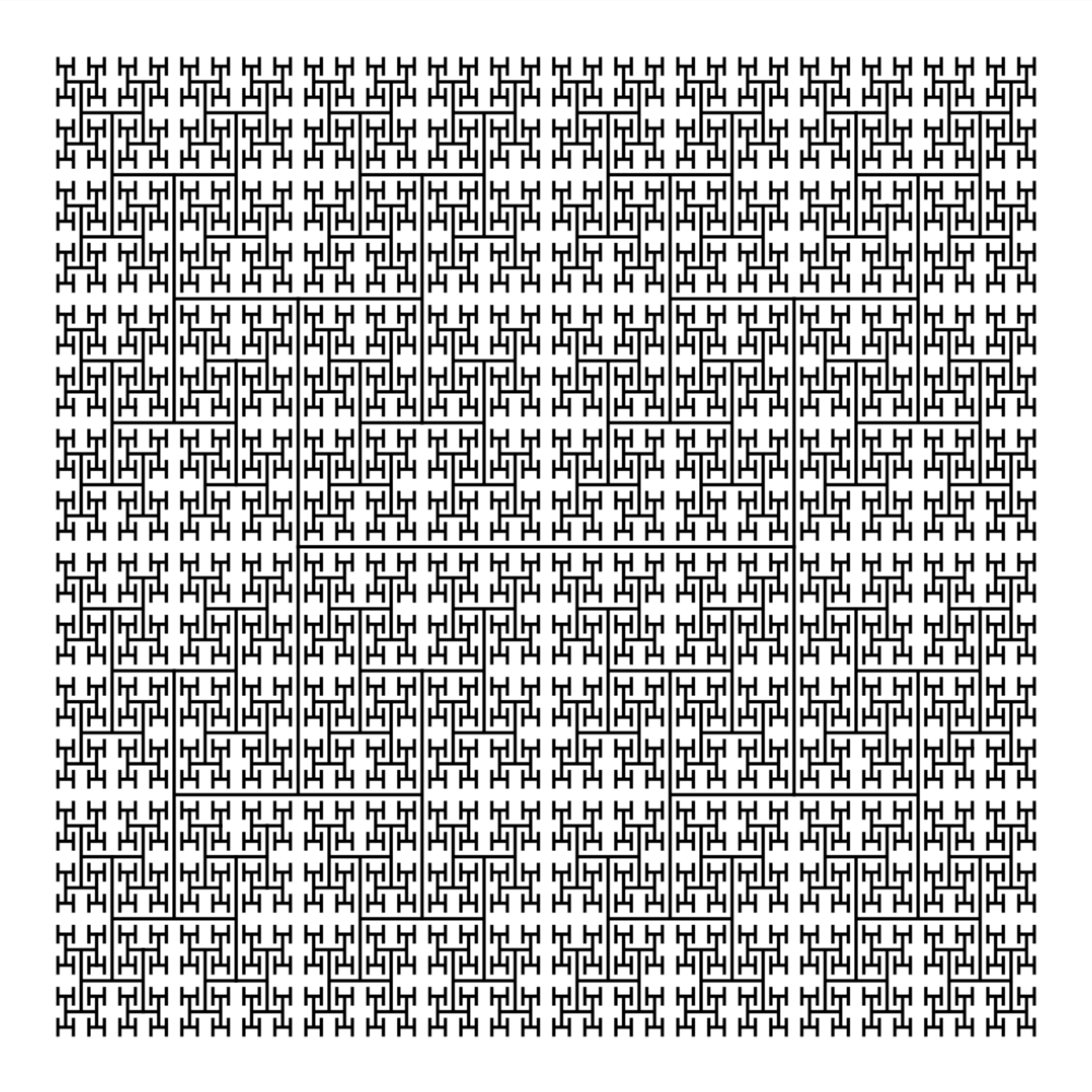
order 2



order 3



H-trees



order 6

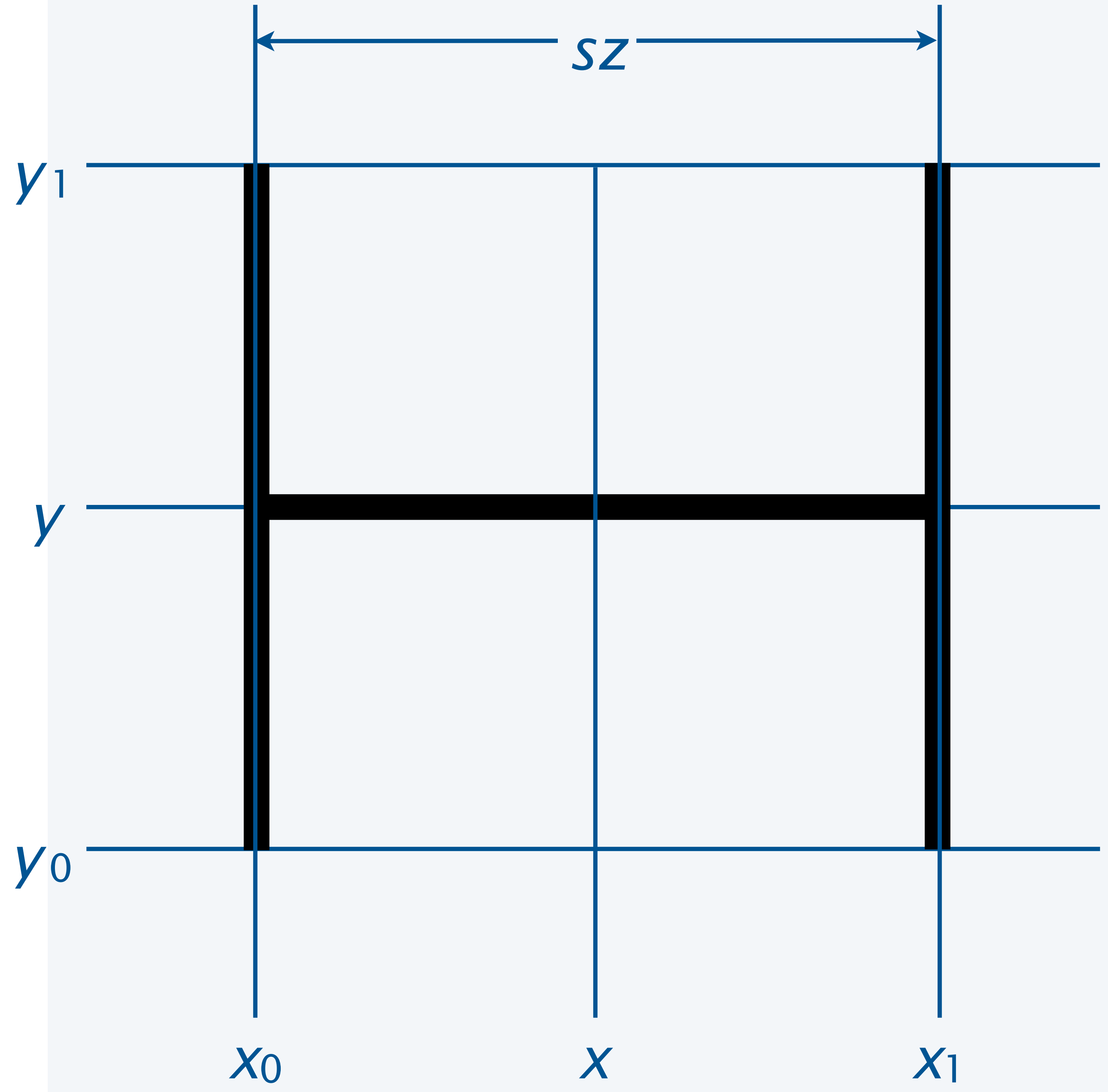
Application. Connect a large set of regularly spaced sites to a single source.

Recursive H-tree implementation

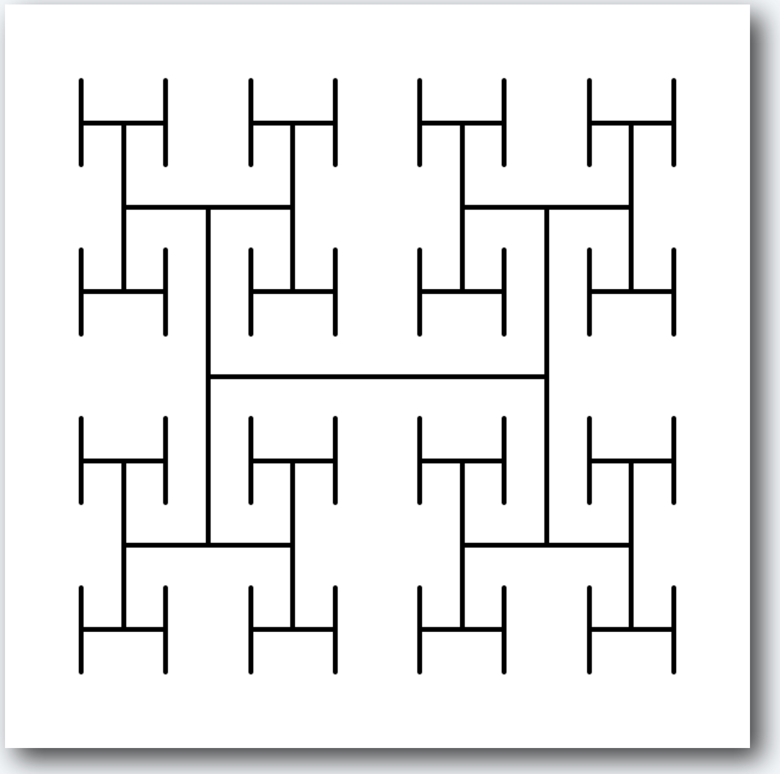
```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

draw the H, centered on (*x*, *y*)

draw four half-size H-trees



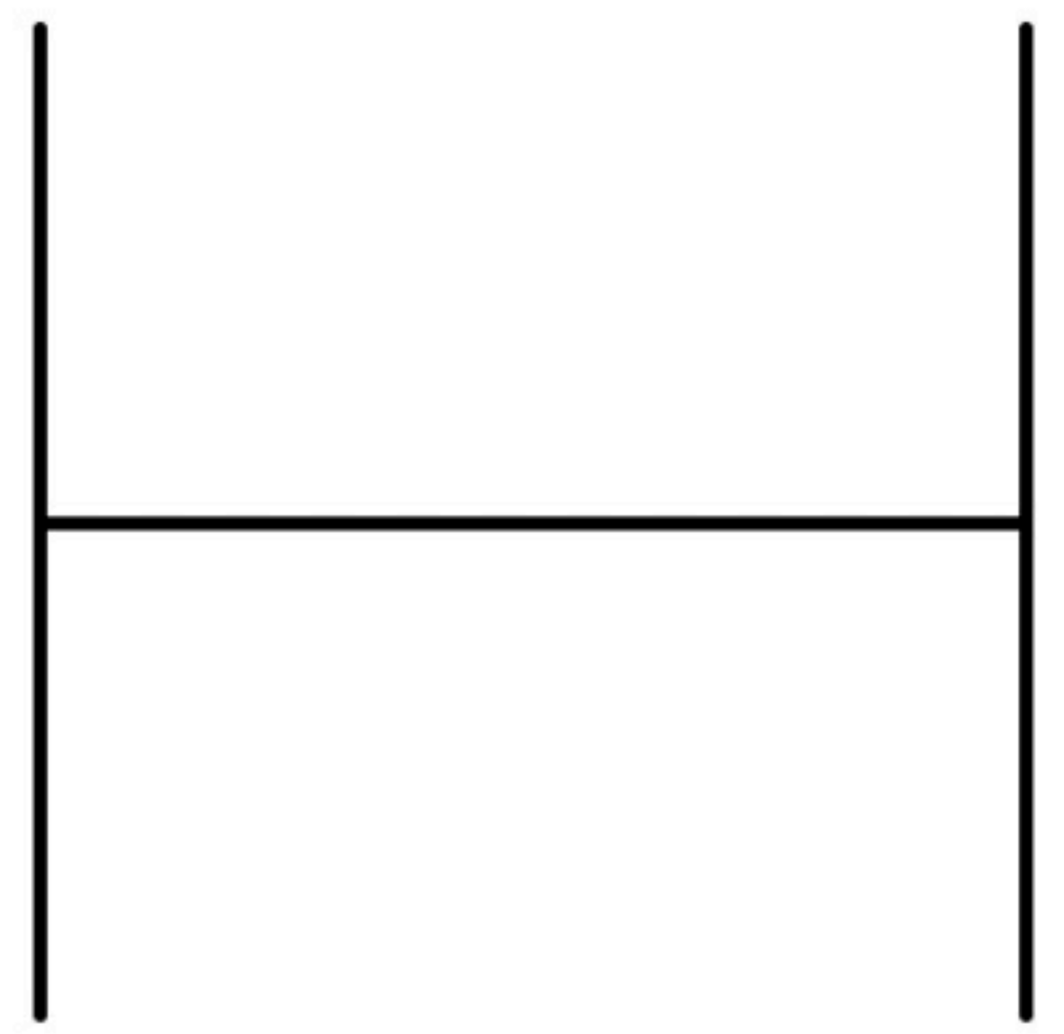

```
% java Htree 3
```



Deluxe H-tree implementation

```
public class HtreeDeluxe
{
   public static void draw(int n, double sz,
                                double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      StdAudio.play(PlayThatNote.note(n, .25*n));
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

```
% java HtreeDeluxe 4
```



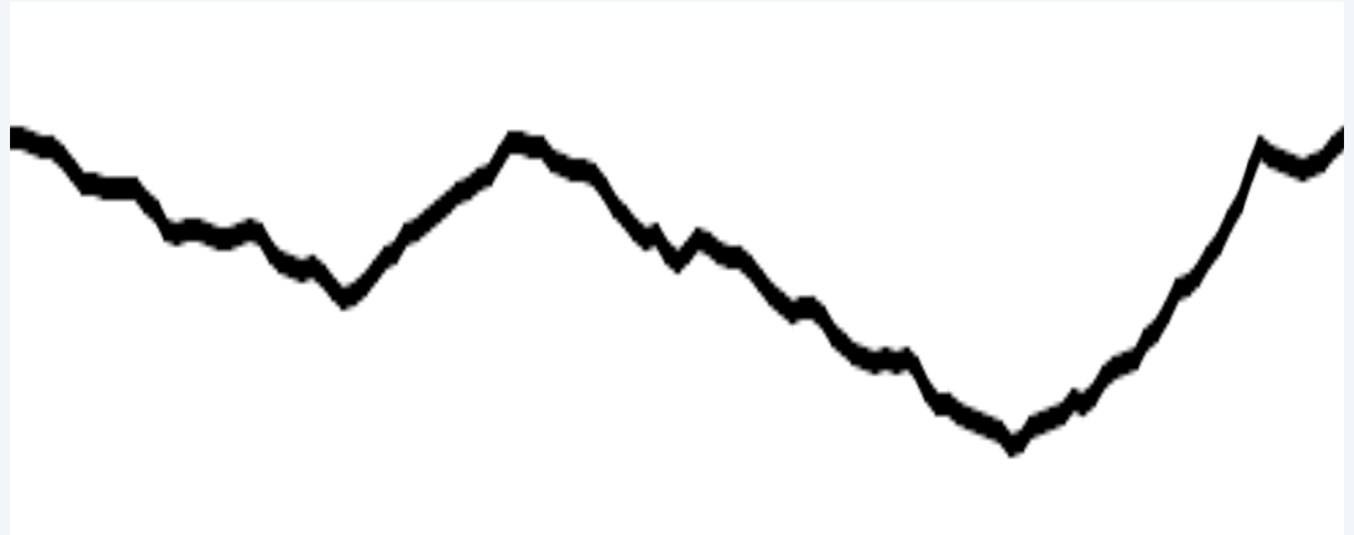
Fractional Brownian motion

A process that models many phenomenon.

- Price of stocks.
- Dispersion of fluids.
- Rugged shapes of mountains and clouds.
- Shape of nerve membranes.

. . .

Brownian bridge model



An actual mountain

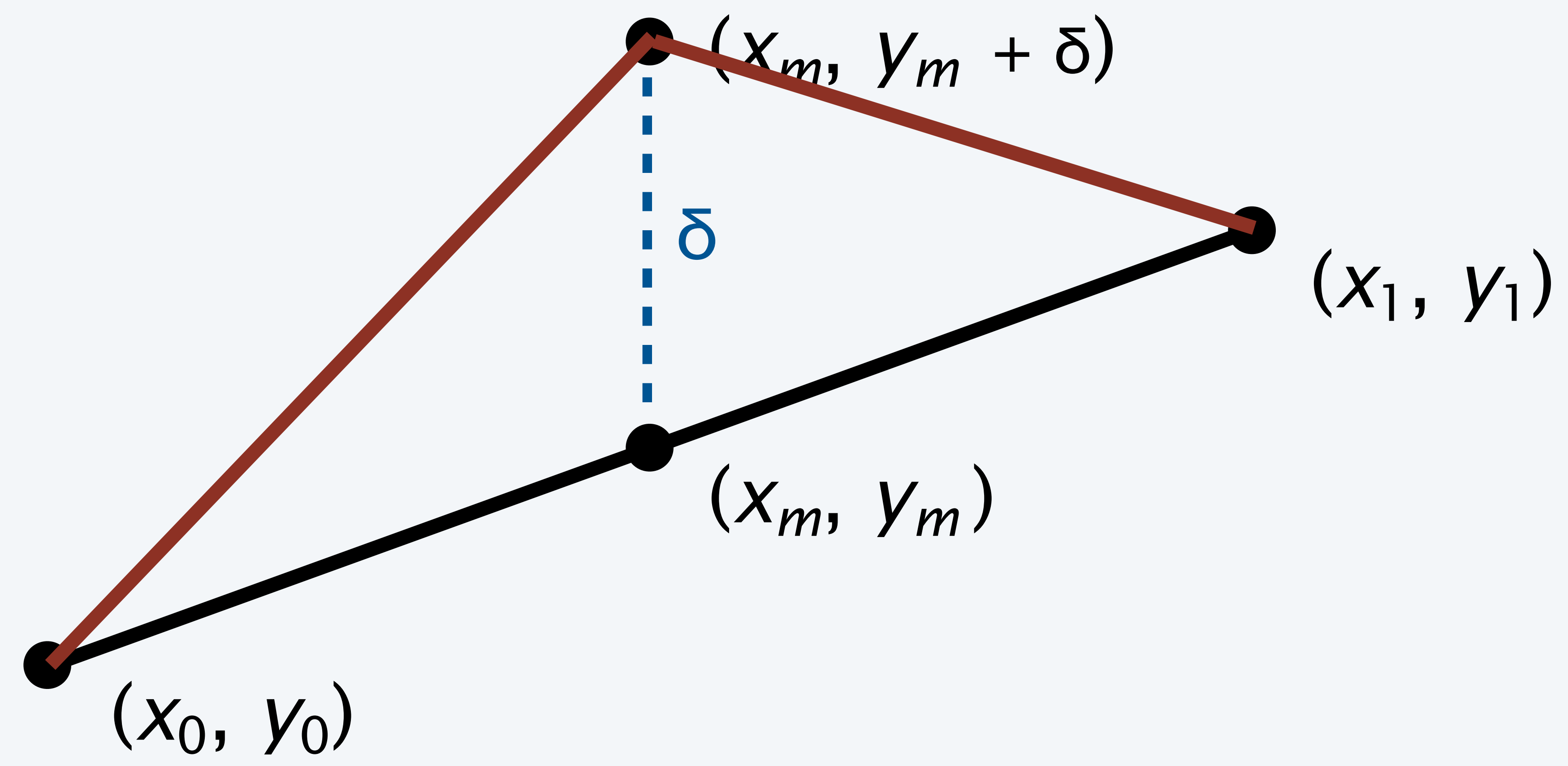
Price of an actual stock

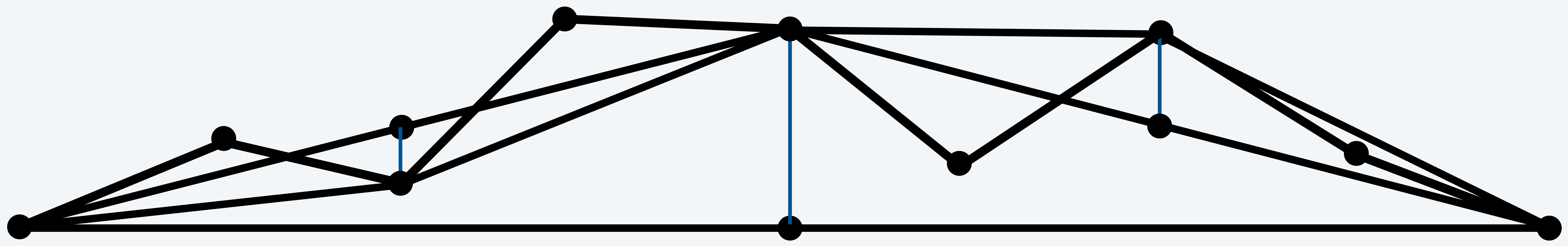
Black-Scholes model (two different parameters)

Fractional Brownian motion simulation

Midpoint displacement method

- Consider a line segment from (x_0, y_0) to (x_1, y_1).
- If sufficiently short draw it *and return*
- Divide the line segment in half, at (x_m, y_m).
- Choose δ at random *from Gaussian distribution*.
- Add δ to y_m.
- Recur on the left and right line segments.



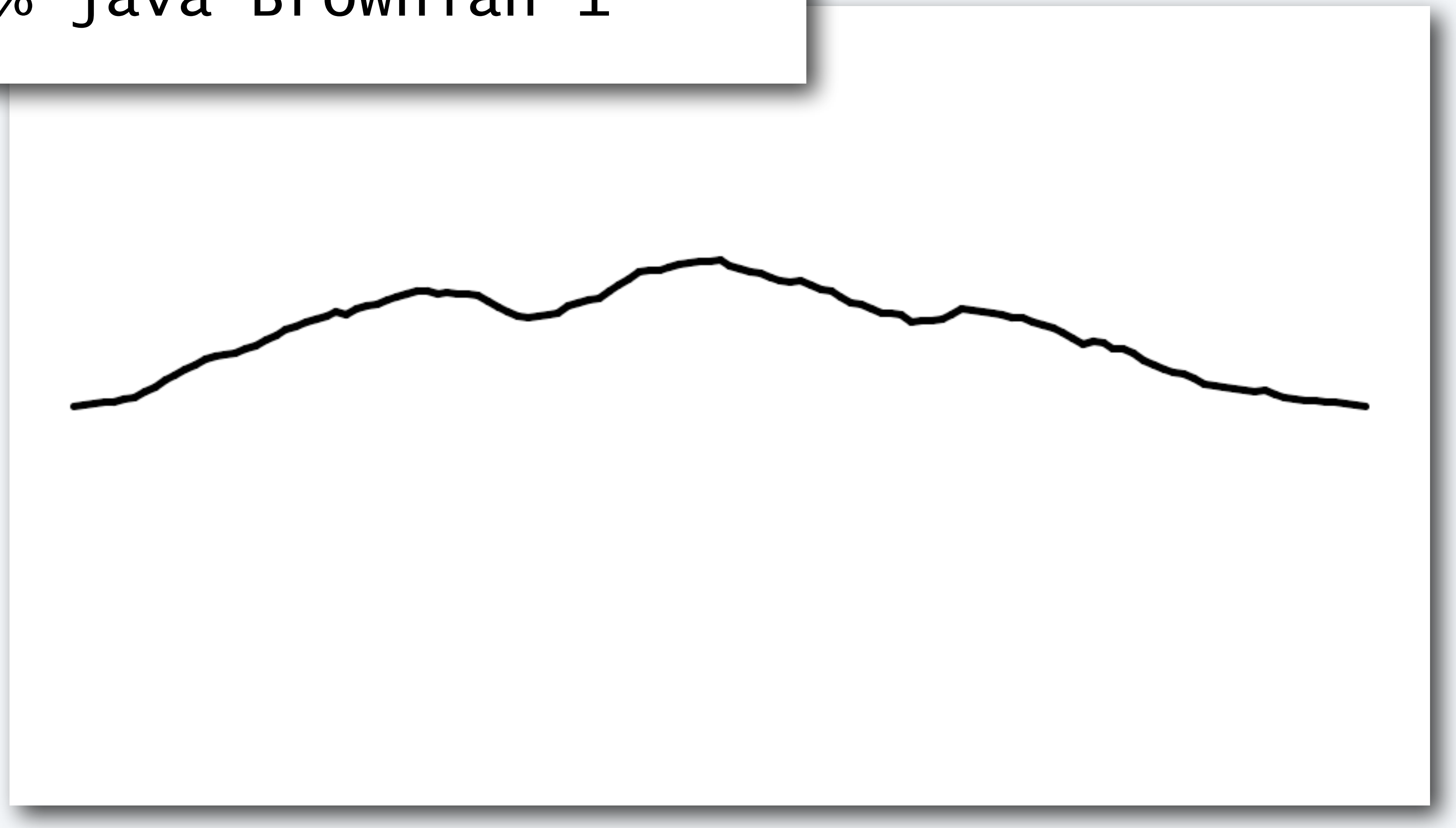


Brownian motion implementation

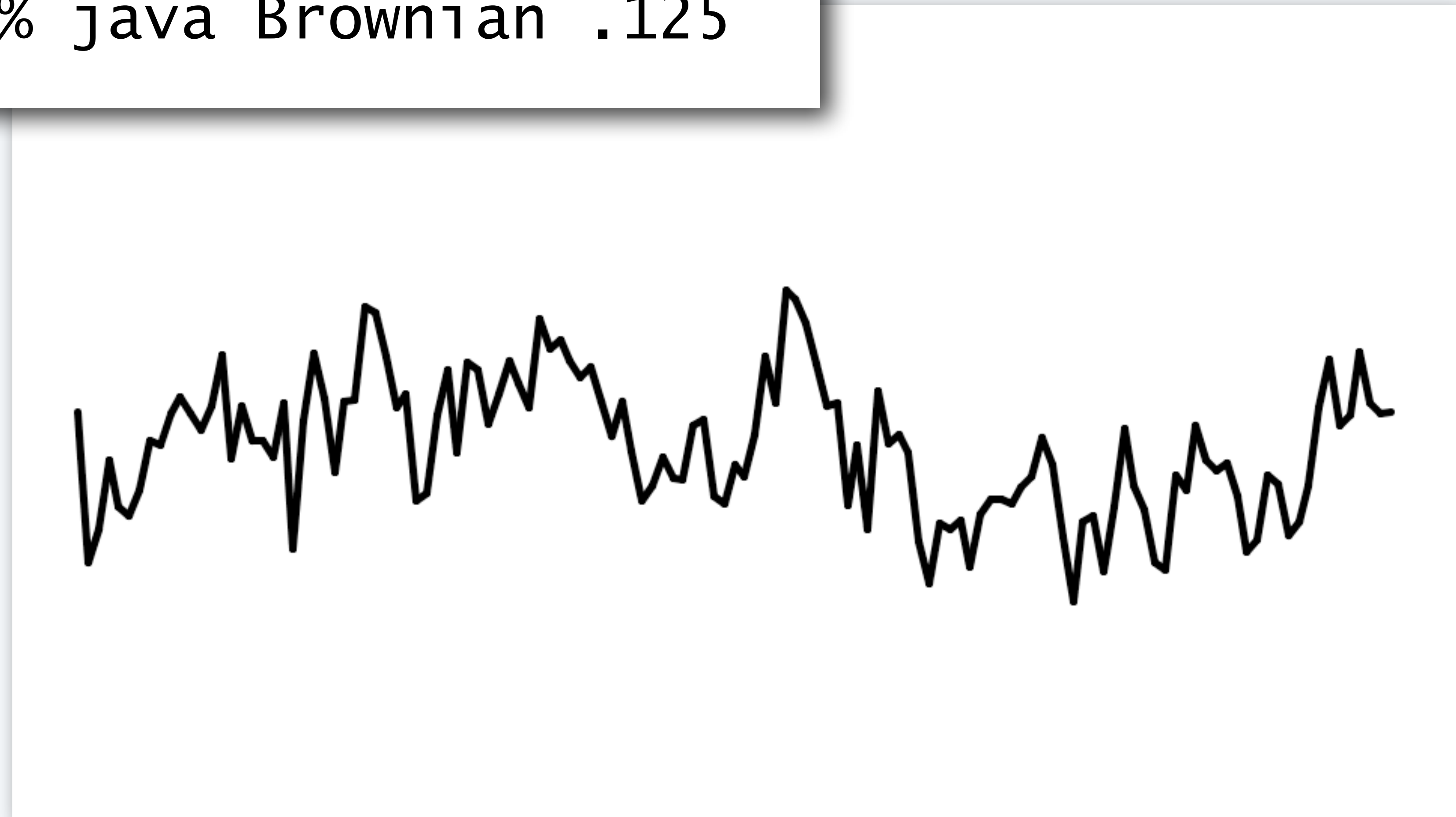
```
public class Brownian
{
   public static void
   curve(double x0, double y0, double x1, double y1,
                                   double var, double s)
   {
      if (x1 - x0 < .01)
      {  StdDraw.line(x0, y0, x1, y1); return;  }
      double xm = (x0 + x1) / 2;
      double ym = (y0 + y1) / 2;
      double stddev = Math.sqrt(var);
      double delta = StdRandom.gaussian(0, stddev);
      curve(x0, y0, xm, ym+delta, var/s, s);
      curve(xm, ym+delta, x1, y1, var/s, s);
   }

   public static void main(String[] args)
   {
      double hurst = Double.parseDouble(args[0]);
      double s = Math.pow(2, 2*hurst);
      curve(0, .5, 1.0, .5, .01, s);
   }
}
```

control parameter (see textbook)

```
% java Brownian 1
```



```
% java Brownian .125
```

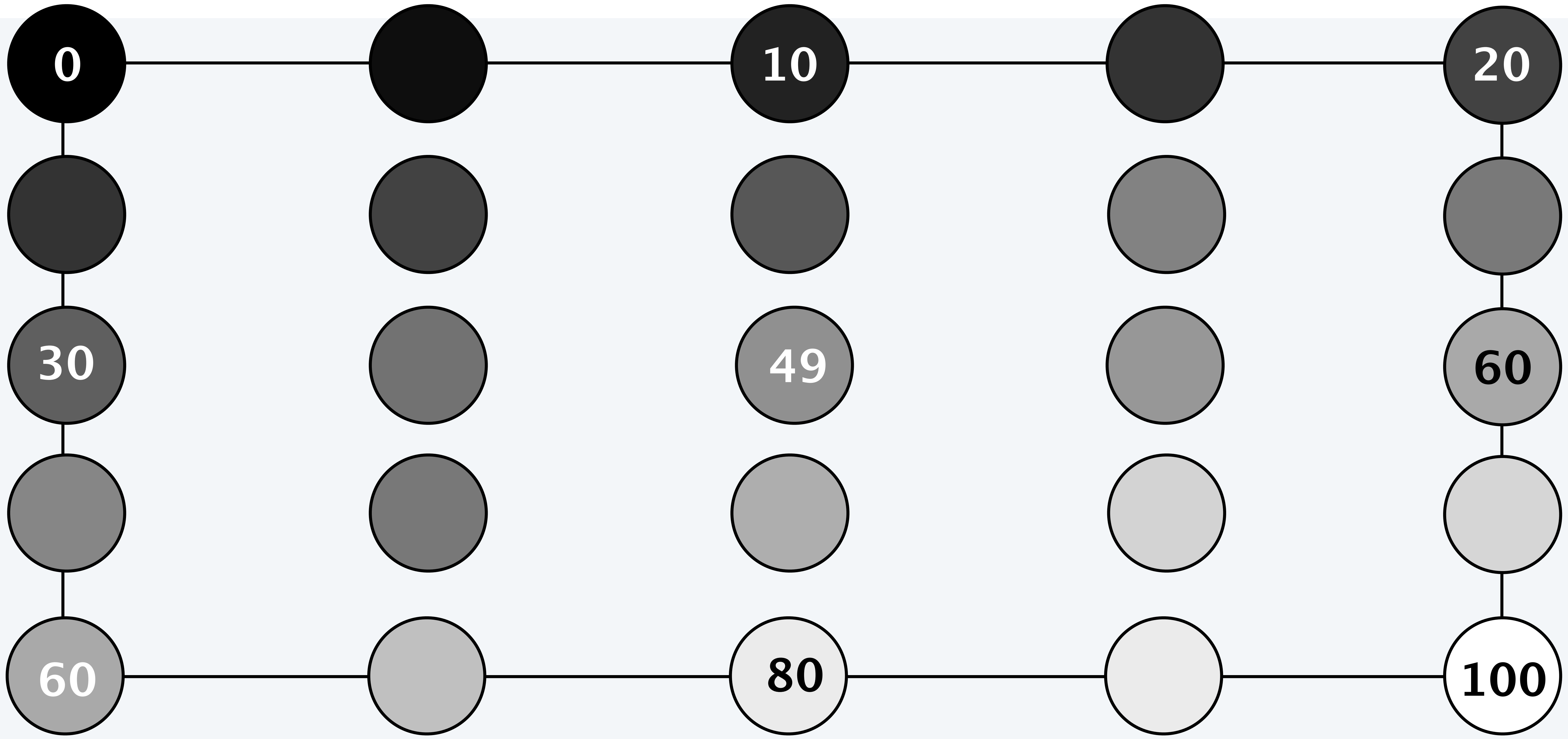


A 2D Brownian model: plasma clouds

Midpoint displacement method

- Consider a rectangle centered at (*x*, *y*) with pixels at the four corners.
- If the rectangle is small, do nothing.
- Color the midpoints of each side the average of the endpoint colors.
- Choose δ at random *from Gaussian distribution*.
- Color the center pixel the average of the four corner colors *plus* δ
- Recurse on the four quadrants.

Booksite code actually draws a rectangle to avoid artifacts



A Brownian cloud

A Brownian landscape

Image sources

http://en.wikipedia.org/wiki/Droste_effect#mediaviewer/File:Droste.jpg

http://www.mcescher.com/gallery/most-popular/circle-limit-iv/

http://www.megamonalisa.com/recursion/

http://fractalfoundation.org/OFC/FractalGiraffe.png

http://www.nytimes.com/2006/12/15/arts/design/15serk.html?pagewanted=all&_r=0

http://www.geocities.com/aaron_torpy/gallery.htm

6. Recursion

- Foundations
- A classic example
- Recursive graphics
- **Avoiding exponential waste**
- Dynamic programming

Fibonacci numbers

Let $F_n = F_{n-1} + F_{n-2}$ for $n > 1$ with $F_0 = 0$ and $F_1 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	…

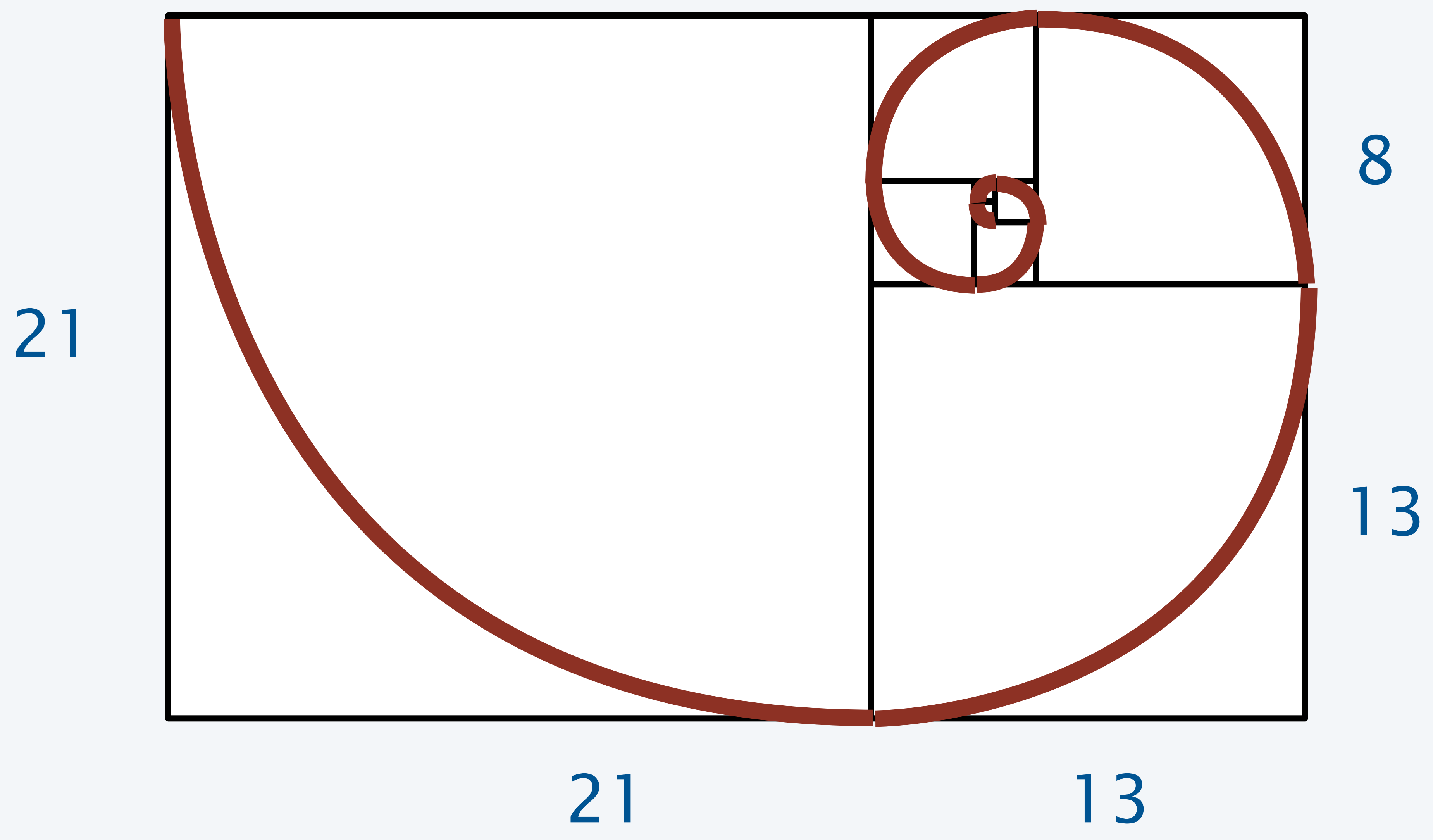
Leonardo Fibonacci
c. 1170 – c. 1250

Models many natural phenomena and is widely found in art and architecture.

Examples.

- Model for reproducing rabbits.
- Nautilus shell.
- Mona Lisa.
- …

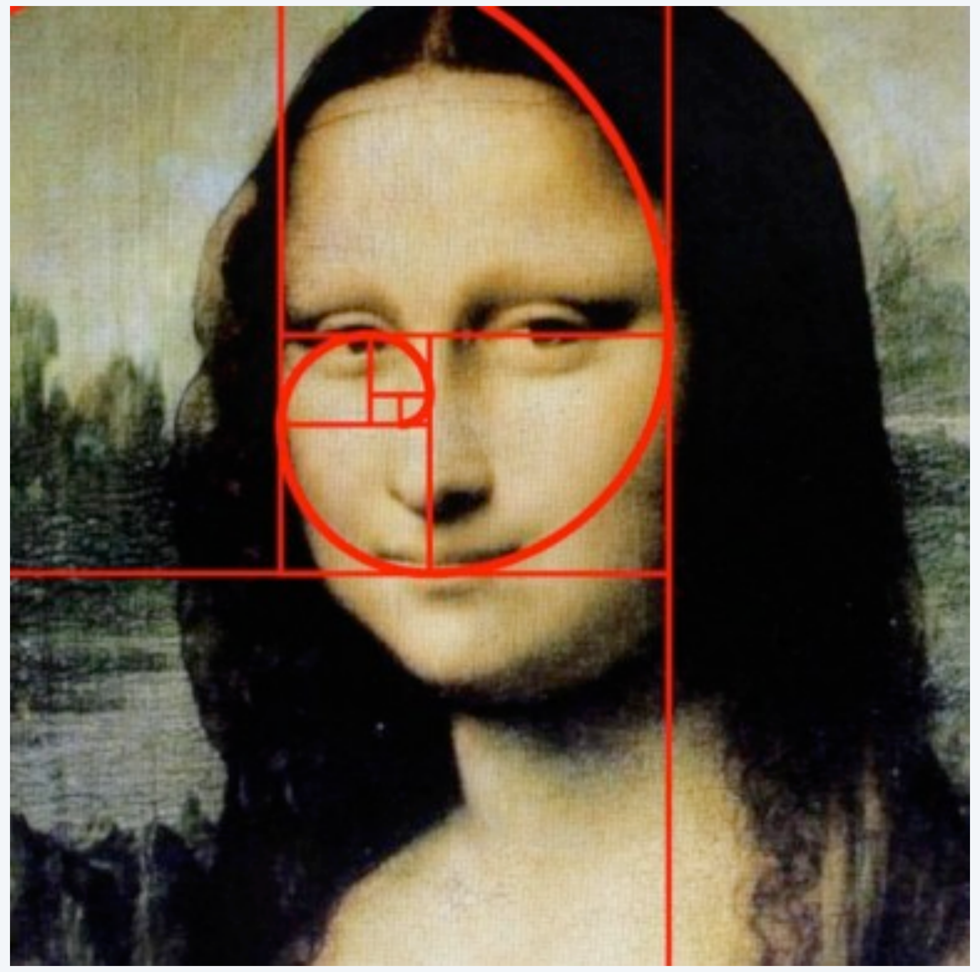
golden ratio $F_n \,/\, F_{n-1}$

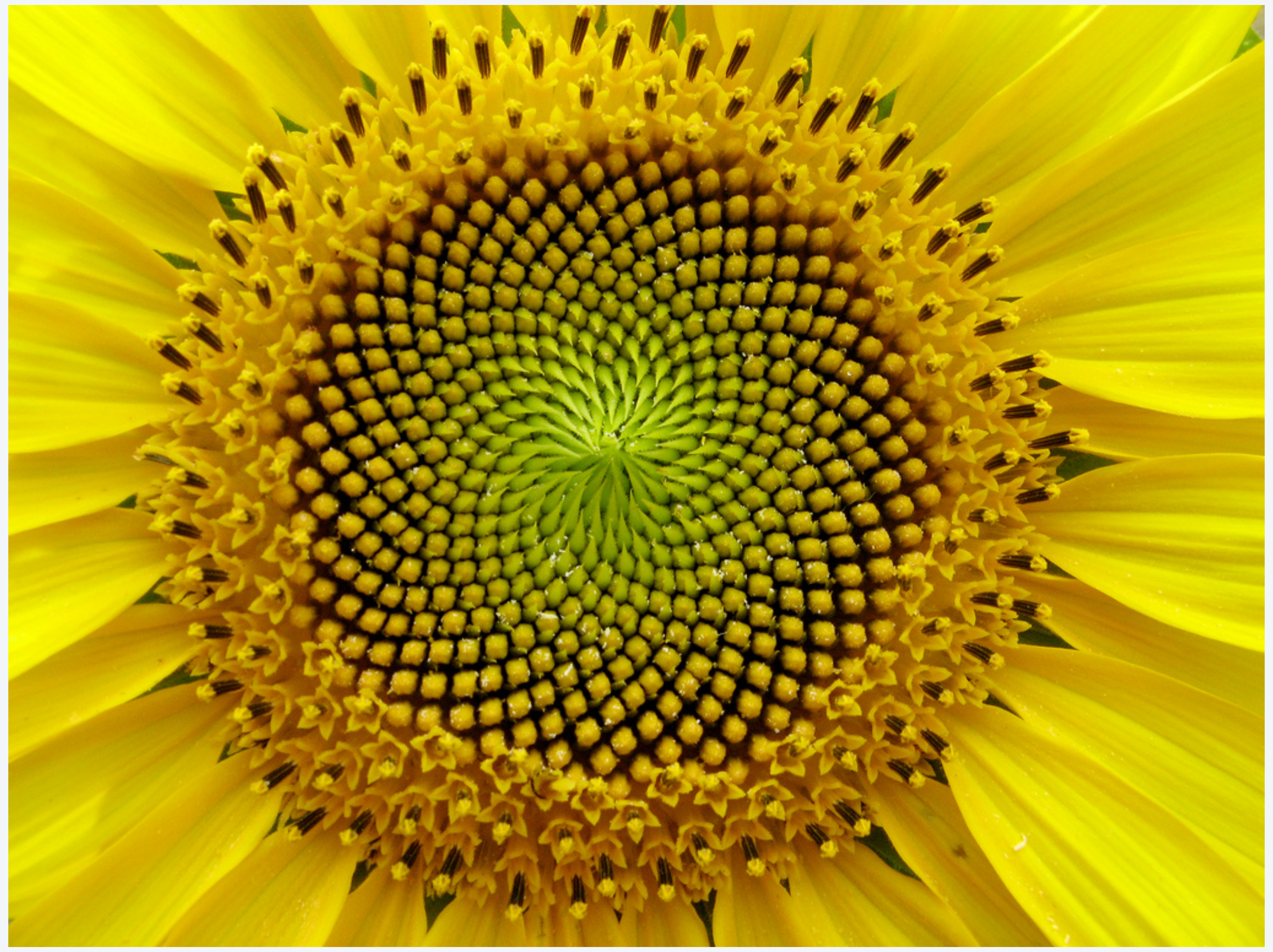


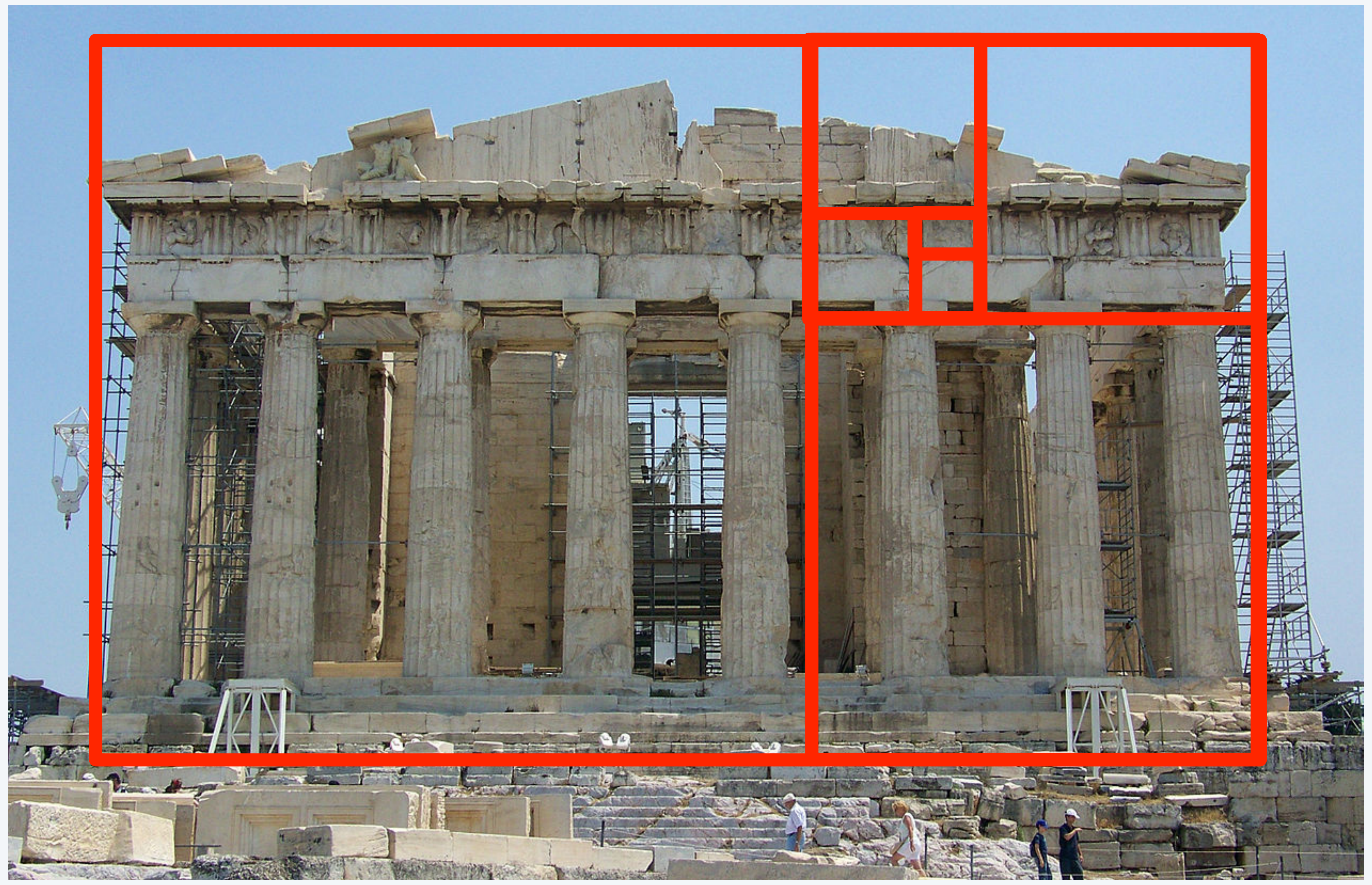
Facts (known for centuries).

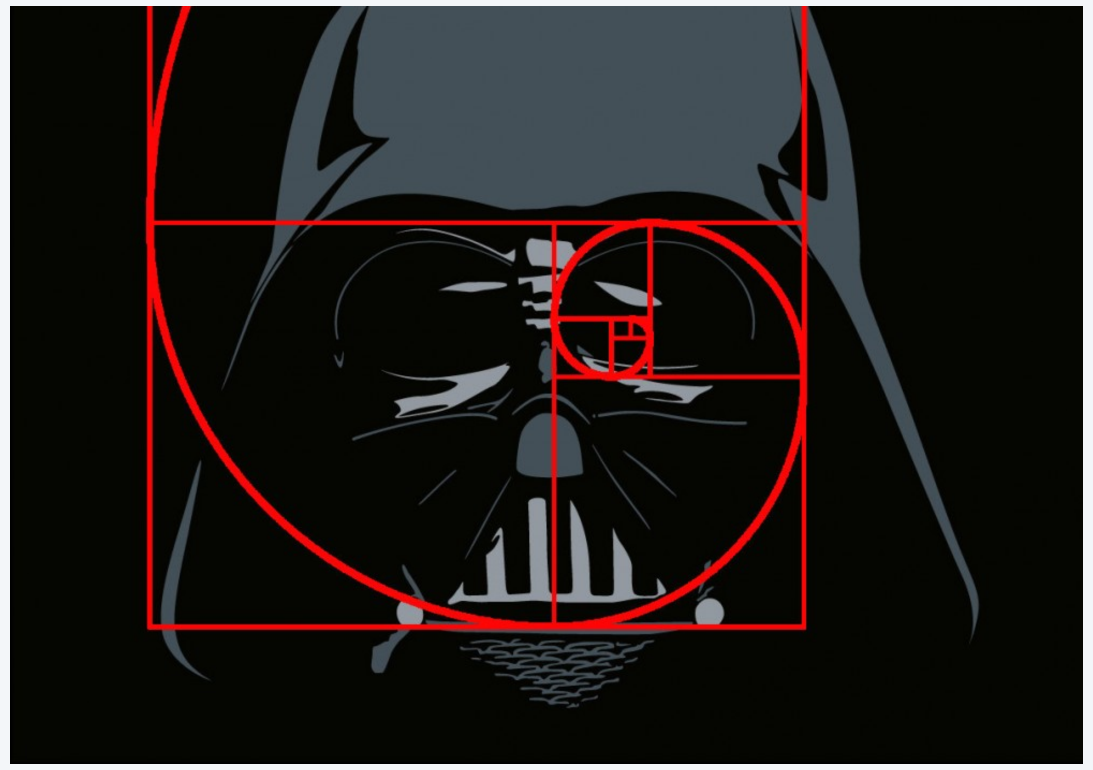
- $F_n \,/\, F_{n-1} \to \Phi = 1.618\ldots$ as $n \to \infty$
- F_n is the closest integer to $\Phi^n/\sqrt{5}$

Fibonacci numbers and the golden ratio in the wild









1
1
2
3
5
8
13
21

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Computing Fibonacci numbers

Q. [Curious individual.] What is the exact value of F_{60} ?

A. [Novice programmer.] Just a second. I'll write a recursive program to compute it.

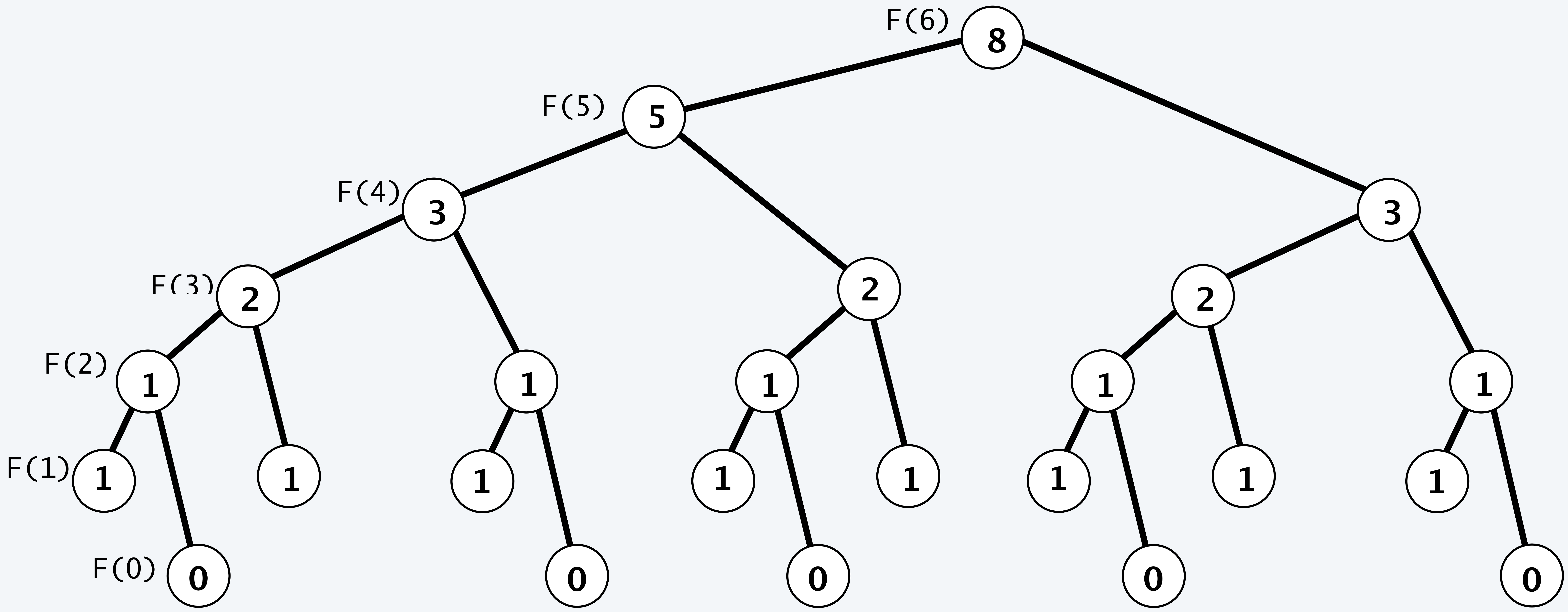
```
public class FibonacciR
{
   public static long F(int n)
   {
      if (n == 0) return 0;
      if (n == 1) return 1;
      return F(n-1) + F(n-2);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciR 5
5
% java FibonacciR 6
8
% java FibonacciR 10
55
% java FibonacciR 12
144
% java FibonacciR 50
12586269025
% java FibonacciR 60
```

takes a few minutes
Hmmm. Why is that?

Is something wrong with my computer?

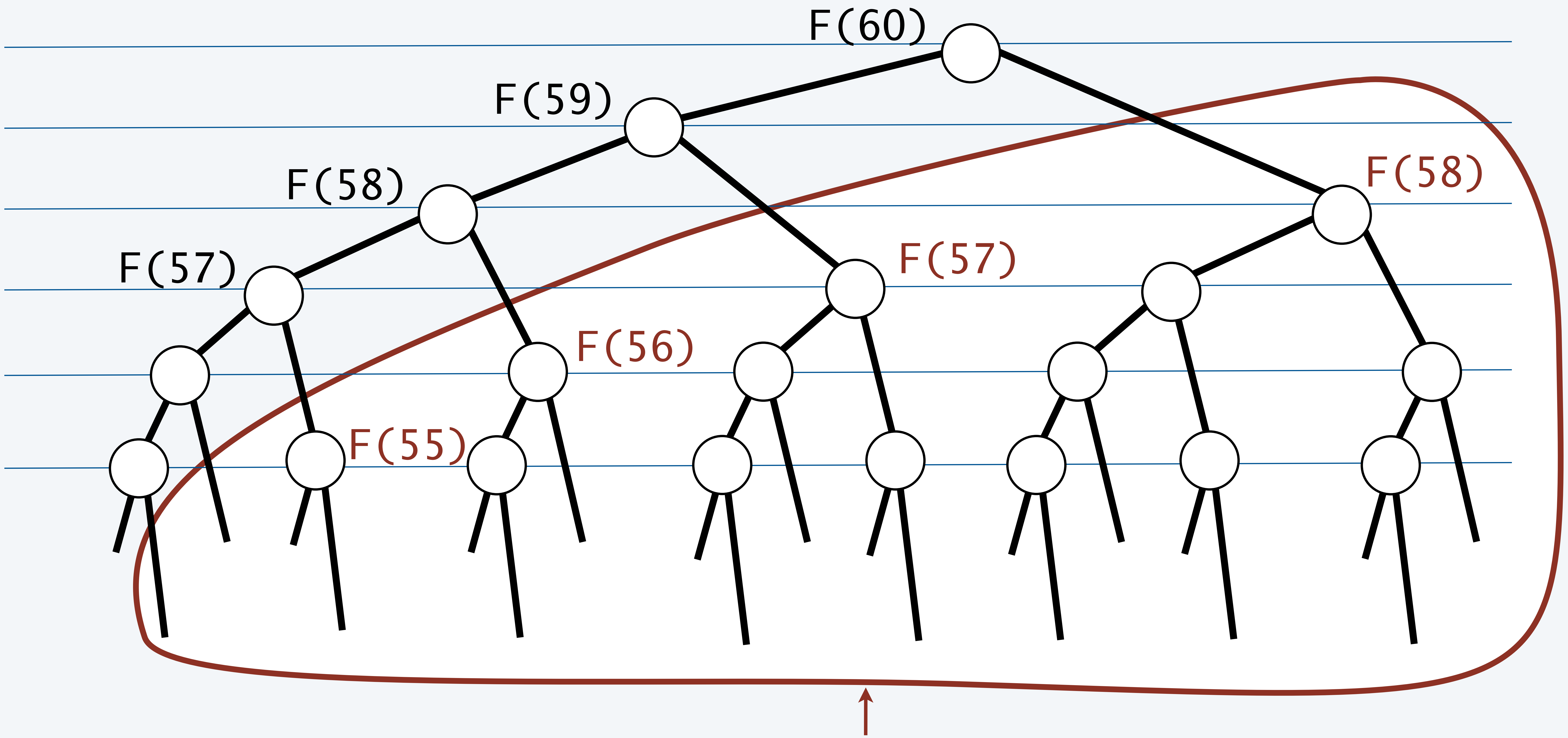
Recursive call tree for Fibonacci numbers



Exponential waste

Let C_n be the number of times F(n) is called when computing F(60).

n	C_n	
60	1	F_1
59	1	F_2
58	2	F_3
57	3	F_4
56	5	F_5
55	8	F_6
	…	…
0	$>2.5\times10^{12}$	F_{61}

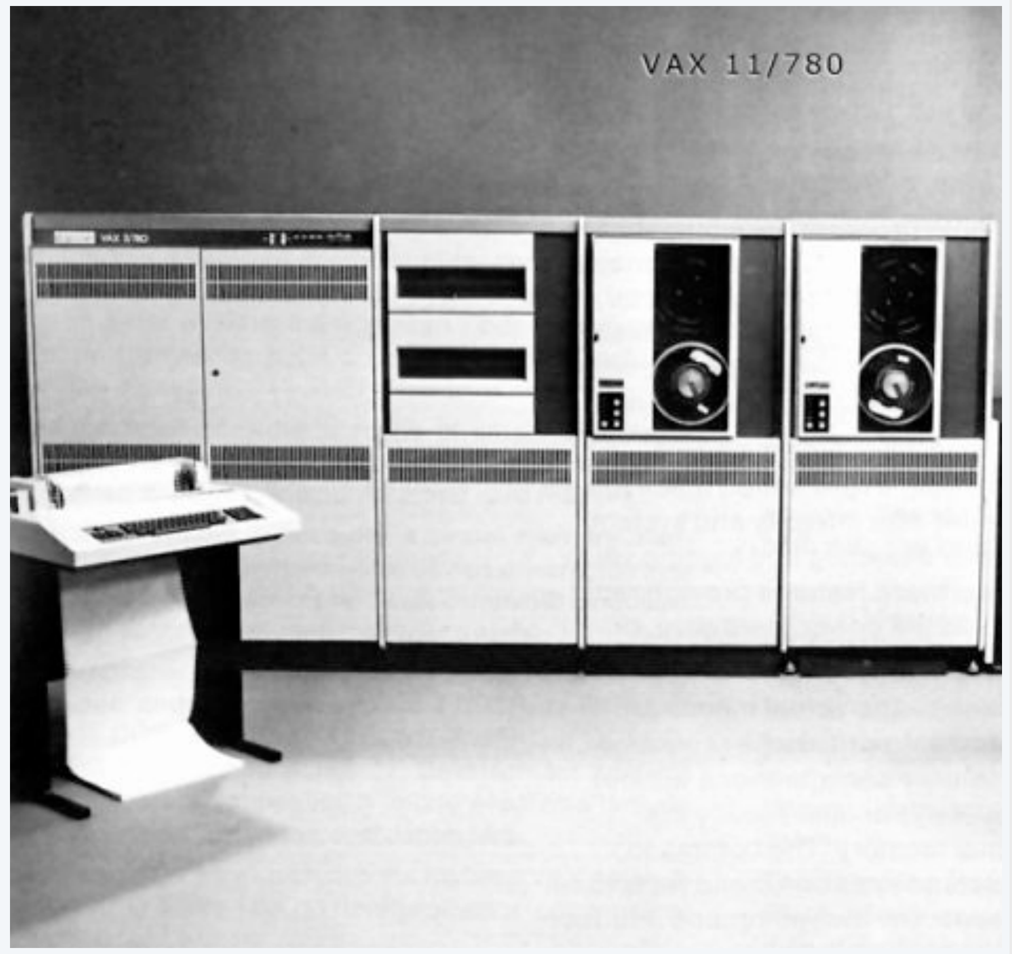


Exponentially wasteful to recompute all these values.
(trillions of calls on F(0), not to mention calls on F(1), F(2),...)

Exponential waste dwarfs progress in technology

If you engage in exponential waste, you *will not* be able to solve a large problem.

1970s



VAX 11/780

n	*time to compute* F_n
30	minutes
40	hours
50	weeks
60	years
70	centuries
80	millenia

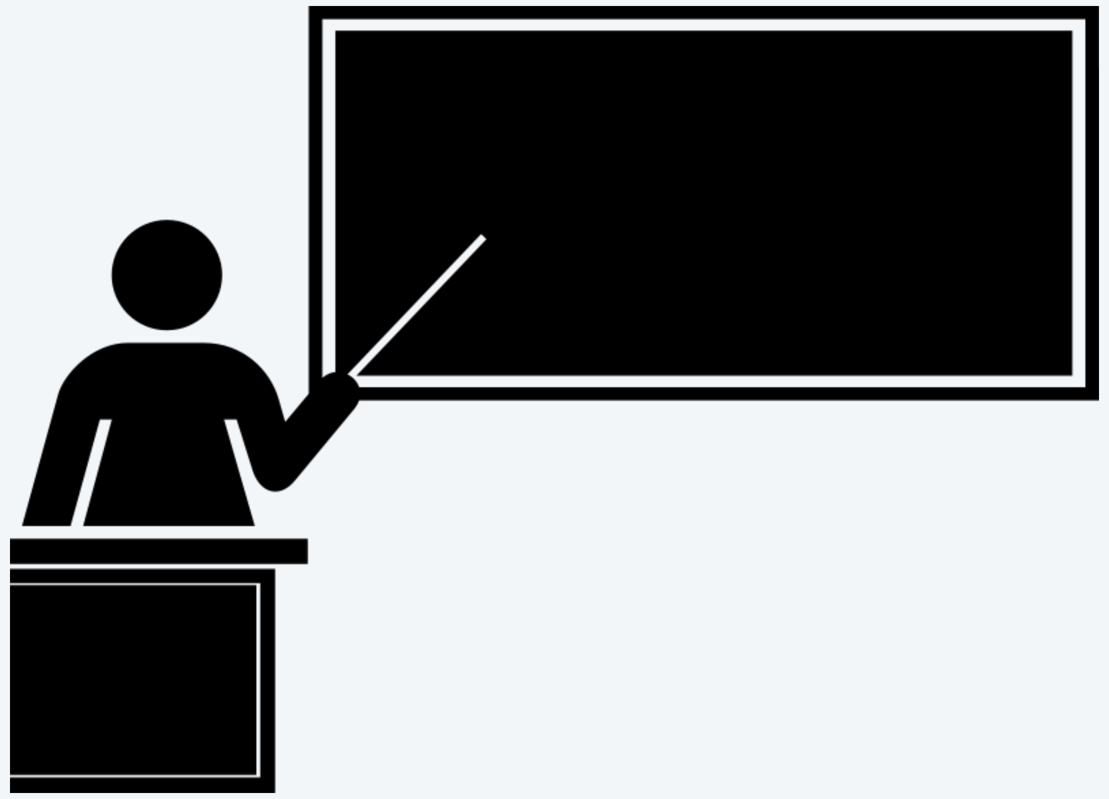
2010s: 10,000+ times faster


Macbook Air

n	*time to compute* F_n
50	minutes
60	hours
70	weeks
80	years
90	centuries
100	millenia

1970s: "That program won't compute F_{60} before you graduate! "

2010s: "That program won't compute F_{80} before you graduate! "



Avoiding exponential waste

Memoization

- Maintain an array memo[] to remember all computed values.
- If value known, just return it.
- Otherwise, compute it, remember it, and then return it.

```
public class FibonacciM
{
   static long[] memo = new long[100];
   public static long F(int n)
   {
      if (n == 0) return 0;
      if (n == 1) return 1;
      if (memo[n] == 0)
         memo[n] = F(n-1) + F(n-2);
      return memo[n];
   }
   public static void main(String[] args)
   {
      int n = Integer.parseIn
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciM 50
12586269025
% java FibonacciM 60
1548008755920
% java FibonacciM 80
23416728348467685
```

Simple example of *dynamic programming* (next).

Image sources

http://en.wikipedia.org/wiki/Fibonacci

http://www.inspirationgreen.com/fibonacci-sequence-in-nature.html

http://www.goldenmeancalipers.com/wp-content/uploads/2011/08/mona_spiral-1000x570.jpg

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6. Recursion

- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- **Dynamic programming**

An alternative to recursion that avoids recomputation

Dynamic programming.

- Build computation from the "*bottom up*".
- Solve small subproblems *and save solutions*.
- Use those solutions to build bigger solutions.

Richard Bellman
1920-1984

Fibonacci numbers

```
public class Fibonacci
{
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      long[] F =   new long[n+1];
      F[0] = 0; F[1] = 1;
      for (int i = 2; i <= n; i++)
         F[i] = F[i-1] + F[i-2];
      StdOut.println(F[n]);
   }
}
```

```
% java Fibonacci 50
12586269025
% java Fibonacci 60
1548008755920
% java Fibonacci 80
23416728348467685
```

Key advantage over recursive solution. Each subproblem is addressed only *once*.

DP example: Longest common subsequence

Def. A *subsequence* of a string s is any string formed by deleting characters from s.

Ex 1.

```
s = ggcaccacg
    cac          ggcaccacg
    gcaacg       ggcaccacg
    ggcaacg      ggcaccacg
    ggcacacg     ggcaccacg
    ...
```

[2^n subsequences in a string of length n]

Ex 2.

```
t = acggcggatacg
    gacg         acggcggatacg
    ggggg        acggcggatacg
    cggcgg       acggcggatacg
    ggcaacg      acggcggatacg
    ggggaacg     acggcggatacg
    ...
```

longest common subsequence

Def. The *LCS* of s and t is the longest string that is a subsequence of both.

Goal. Efficient algorithm to compute the LCS and/or its length ← numerous scientific applications

Longest common subsequence

Goal. Efficient algorithm to compute the *length* of the LCS of two strings `s` and `t`.

Approach. Keep track of the length of the LCS of `s[i..M)` and `t[j..N)` in `opt[i, j]`

Three cases:

- i = M or j = N
  `opt[i][j] = 0`
- `s[i] = t[j]`
  `opt[i][j] = opt[i+1][j+1] + 1`
- `otherwise`
  `opt[i][j] = max(opt[i][j+1], opt[i+1][j])`

Ex: i = 6, j = 7

```
s[6..9)  = acg
t[7..12) = atacg
LCS(cg, tacg) = cg
LCS(acg, atacg) = acg
```

Ex: i = 6, j = 4

```
s[6..9)  = acg
t[4..12) = cggatacg
LCS(acg, ggatacg) = acg
LCS(cg, cggatacg) = cg
LCS(acg, cggatacg) = acg
```

LCS example

		0	1	2	3	4	5	6	7	8	9	10	11	12
		a	c	g	g	c	g	g	a	t	a	c	g	
0	g	7	7	7	6	6	6	5	4	3	3	2	1	0
1	g	6	6	6	6	5	5	5	4	3	3	2	1	0
2	c	6	5	5	5	5	4	4	4	3	3	2	1	0
3	a	6	5	4	4	4	4	4	4	3	3	2	1	0
4	c	5	5	4	4	4	3	3	3	3	3	2	1	0
5	c	4	4	4	4	4	3	3	3	3	3	2	1	0
6	a	3	3	3	3	3	3	3	3	3	3	2	1	0
7	c	2	2	2	2	2	2	2	2	2	2	2	1	0
8	g	1	1	1	1	1	1	1	1	1	1	1	1	0
9		0	0	0	0	0	0	0	0	0	0	0	0	0

Three cases:

- i = M or j = N
  `opt[i][j] = 0`
- `s[i] = t[j]`
  `opt[i][j] = opt[i+1][j+1] + 1`
- otherwise
  `opt[i][j] = max(opt[i][j+1], opt[i+1][j])`

`opt[i][j] = max(opt[i, j+1], opt[i+1][j])`

`opt[i][j] = opt[i+1, j+1] + 1`

LCS length implementation

```
public class LCS
{
    public static void main(String[] args)
    {
        String s = args[0];
        String t = args[1];
        int M = s.length();
        int N = t.length();

        int[][] opt = new int[M+1][N+1];

        for (int i = M-1; i >= 0; i--)
            for (int j = N-1; j >= 0; j--)
                if (s.charAt(i) == t.charAt(j))
                    opt[i][j] = opt[i+1][j+1] + 1;
                else
                    opt[i][j] = Math.max(opt[i+1][j], opt[i][j+1]);

        System.out.println(opt[0][0]);
    }
}
```

```
% java LCS ggcaccacg acggcggatacg
7
```

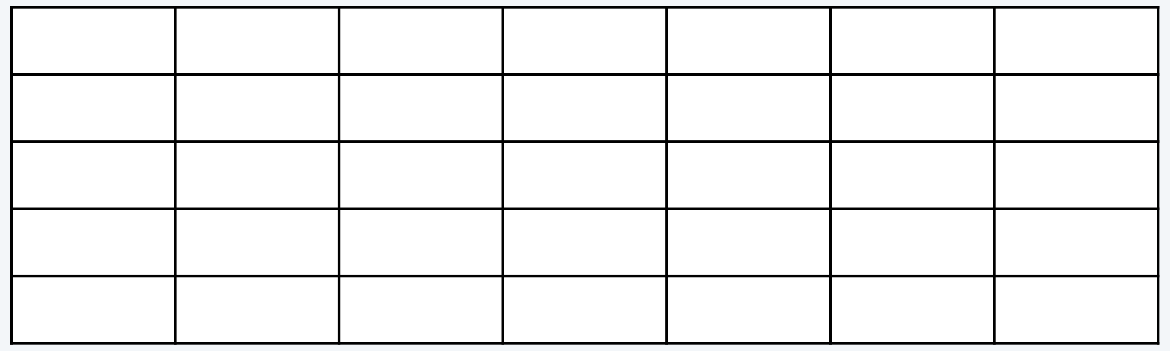
Exercise. Add code to print LCS itself (see `LCS.java` on booksite for solution).

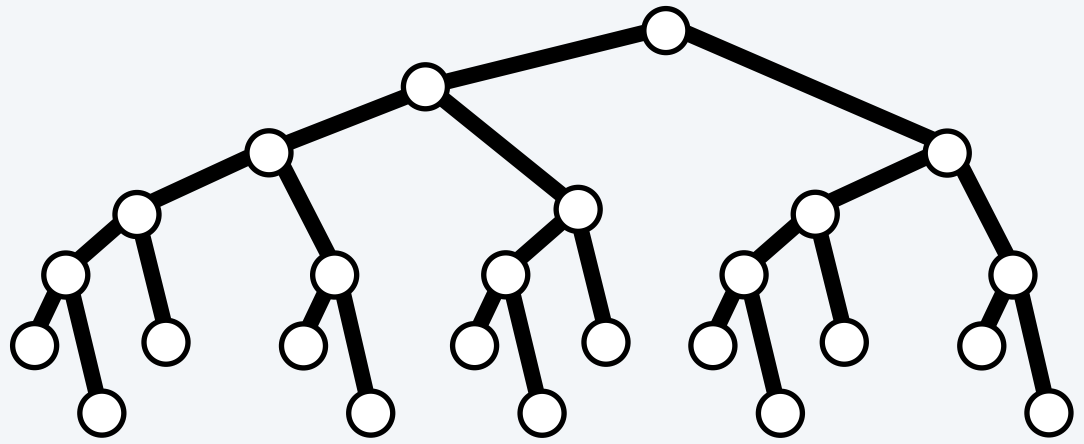
Dynamic programming and recursion

Broadly useful approaches to solving problems by combining solutions to smaller subproblems.

Why learn DP and recursion?

- Represent a new mode of thinking.
- Provide powerful programming paradigms.
- Give insight into the nature of computation.
- Successfully used for decades.





	recursion	*dynamic programming*
advantages	Decomposition often obvious. Easy to reason about correctness.	Avoids exponential waste. Often simpler than memoization.
pitfalls	Potential for exponential waste. Decomposition may not be simple.	Uses significant space. Not suited for real-valued arguments. Challenging to determine order of computation

Image sources

http://upload.wikimedia.org/wikipedia/en/7/7a/Richard_Ernest_Bellman.jpg

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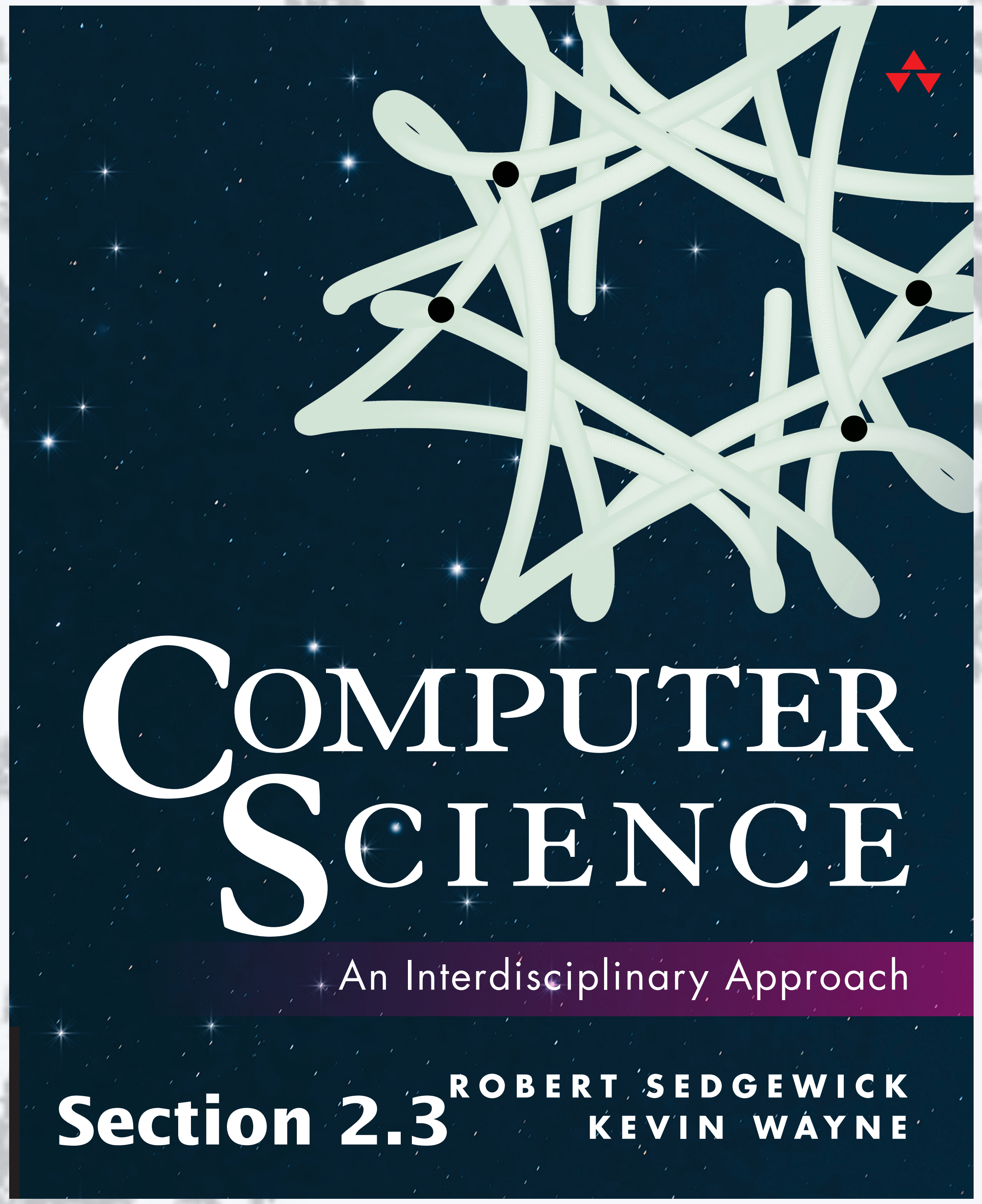
COMPUTER SCIENCE

SEDGEWICK / WAYNE

PART I: PROGRAMMING IN JAVA



http://introcs.cs.princeton.edu

6. Recursion