Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Proof. We set one group with k vertices, (similar for $k > v-k$)
another group $(v-k)$ vertices. assume $k \leq v-k \Rightarrow k \leq \frac{1}{2}v$

then the graph with most edges is when all vertices in k vertices connect with every vertex in group of $(v-k)$ vertices. where $e_m = k(v-k)$

$$\therefore e \leq e_m = k(v-k) \leq \frac{1}{2}v\left(\frac{1}{2}v\right) = \frac{v^2}{4}$$

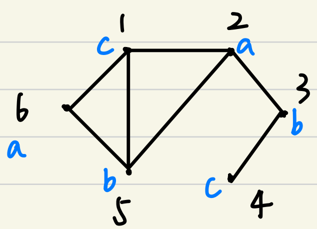
Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.

Suppose six new radio stations are to be set up in a currently unpopulated (by radio stations) region. The distances among stations are recorded in the table below. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

Table 1: Distances in miles among stations

	1	2	3	4	5	6
1	——	85	175	200	50	100
2	85	——	125	175	100	160
3	175	125	——	100	200	250
4	200	175	100	——	210	220
5	50	100	200	210	——	100
6	100	160	250	220	100	——

We construct a graph with 6 vertices representing 6 cities and make an edge if their distance within 150 miles.

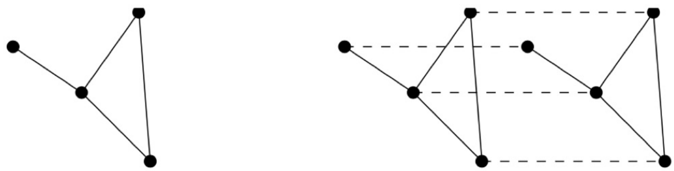


There is a complete graph K_3 of city 1、5、6, so $\chi(G) \geq \chi(K_3) = 3$.

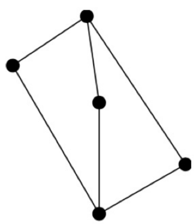
For these 3 frequencies, assume them be a、b、c, set city 2 be a. city 3 be b, city 4 be c, then they will not interfere with each other.

Therefore, they need at least 3 different frequencies and can be arranged as above.

The double of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right. Some edges in the graph on the right are dashed to clarify its structure.

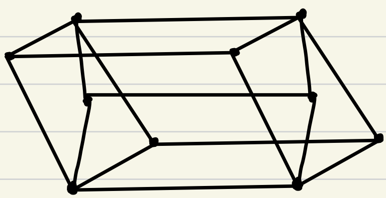


(a) Draw the double of the graph shown below.



(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

(a)



(b) Base case: G_1 is a bipartite graph

Induction: Assume G_n is bipartite,
then there are two groups of vertices which has
no inside-group edges. set them be group V_1, V_2.

Then for G_{n+1}, there is a copy of V_1, V_2, we call them
V_3, V_4. the edges preserved between $V_1 - V_2$
in $V_3 - V_4$, and some edges added between $V_1 - V_3$ and
$V_2 - V_4$ for joining corresponding vertices.

Since there are no edges with V_1, V_2, V_3, V_4,
and no edges within V_1 and V_4, V_2 and V_3,
so 2 new bipartite groups formed by V_1, V_4, V_2, V_3 combination
Therefore, G_{n+1} is a bipartite graph
By induction, G_n is bipartite for all $n \geq 1$.

Let m, n, and r be nonnegative integers with $r \leq m$ and $r \leq n$. Prove the following formula by a combinatorial proof.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Proof. LHS $= \binom{m+n}{r}$ means number of ways to pick r elements from $(m+n)$ elements.

These can be obtain by the below progress:

① pick k elements from n elements

② pick the rest $(r-k)$ elements from m elements

Since k can be any integer from 0 to r, and number of ways can be combined by number of choosing ways with all possible k, which equals to RHS.

$\therefore$ LHS = RHS. $\quad \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$

Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}=\binom{n+3}{5}.$$

Proof. RHS is number of ways picking 5 balls from $(n+3)$ balls.

LHS $= \binom{2}{2}\binom{n}{2}\binom{1}{1}+\binom{3}{2}\binom{n-1}{2}\binom{1}{1}+\cdots+\binom{n}{2}\binom{2}{2}\binom{1}{1}$

means pick 2 from k balls $(2 \le k \le n)$

pick another 2 from $(n+2-k)$ balls,

and last 1 from last 1 ball

combining number of ways with

all possible k,

equals to picking 5 balls from $(n+3)$ balls.

$\therefore$ LHS = RHS.

Find the number of solutions of the equation $x_1 + x_2 + x_3 = 11$, where x_1, x_2, x_3 are non-negative integers with $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$.

Since $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 6$,

we can try enumerate from x_1.

x_1 has only 4 possible cases. 0, 1, 2, 3

1. $x_1 = 0$. $x_2 + x_3 = 11$, since $x_2 \leq 4$, $x_3 \leq 6$, $x_2 + x_3 \leq 10 < 11$,

   impossible.

2. $x_1 = 1$. $x_2 + x_3 = 10$, and $x_2 + x_3 \leq 4 + 6 = 10$,

   so $x_2 + x_3 = 10$ if and only if $x_2 = 4$, $x_3 = 6$, one solution

3. $x_1 = 2$. $x_2 + x_3 = 9$. since $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 6$,

   then $x_3 \geq 5$, $x_2 \geq 3$. $\Rightarrow 3 \leq x_2 \leq 4$, $5 \leq x_3 \leq 6$

   ① $x_2 = 3$, $x_3 = 6$ ② $x_2 = 4$, $x_3 = 5$.

   two solutions

4. $x_1 = 3$. $x_2 + x_3 = 8$. since $0 \leq x_2 \leq 4$, then $x_3 \geq 4$

   since $0 \leq x_3 \leq 6$, then $x_2 \geq 2$

   $\Rightarrow 2 \leq x_2 \leq 4$, $4 \leq x_3 \leq 6$

   ① $x_2 = 2$, $x_3 = 6$ ② $x_2 = 3$, $x_3 = 5$ ③ $x_2 = 4$, $x_3 = 4$

   three solutions

$\therefore$ Totally $1 + 2 + 3 = 6$, six solutions.