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Problem 1

Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

1. let a bipartite simple graph with x vertices and $(v-x)$ vertices.

the maximum egdes is $x \cdot (v-x) = e_{max}$

when $x = \frac{v}{2}$, $(e_{max})_{max} = \frac{v^2}{2} - \frac{v^2}{4} = \frac{v^2}{4}$

$\therefore e \leq \frac{v^2}{4}$

Problem 2

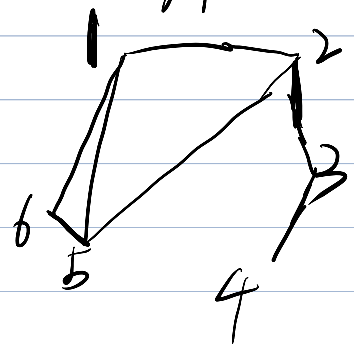
Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.

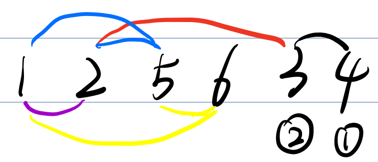
Suppose six new radio stations are to be set up in a currently unpopulated (by radio stations) region. The distances among stations are recorded in the table below. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

Table 1: Distances in miles among stations

	1	2	3	4	5	6
1	——	85	175	200	50	100
2	85	——	125	175	100	160
3	175	125	——	100	200	250
4	200	175	100	——	210	220
5	50	100	200	210	——	100
6	100	160	250	220	100	——

Q2: we creat this graph accordy to (vertices is stations, edges is within 150 miles)





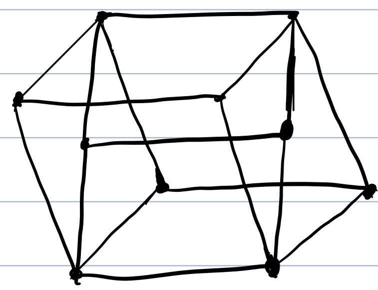
we will colour this graph. 1 -> 1 colours (black) ← 4

2 -> 1 colours (green) ← 6 ← 3

5 -> 1 colours (red)

The minumum color is three.

Q3 (a)



(b) Base case: G_1 is a bipartite graph.

Induction: Assume G_n is bipartite. then we have two groups of vertices which has no inside-group edges. Let them be V_1, V_2

For G_{n+1}, there is a copy of V_1, V_2, Let them be V_3, V_4

V_3, V_4 preserved between V_1-V_2 in V_3-V_4 and some edges added between V_1-V_3 and V_2-V_4 for joining correspondly vertices

Since there are no edges with V_1, V_2, V_3, V_4 and no edges within V_1 and V_4, V_2 and V_3.

So we have two new biparite groups formed by $V_1 V_4$, V_2, V_3 combination

Therefore, G_{n+1} is a bipartite graph.

By induction, G_n is bipartite for all $n \geq 1$

Q4 $\binom{m+n}{r} = C_{m+n}^{r}$

$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$: pick k elements from n elements and pick r-k elements from m elements.

when k be any integer from 0 to r, the number of ways means pick (r-k+k) element from (m+n) element

C_{m+n}^{r} is the same as it. $\therefore$ LHS = RHS

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Q5

Proof. RHS is number of ways picking 5 balls from $(n+3)$ balls

$LHS = \binom{2}{2}\binom{n}{2}\binom{1}{1} + \binom{3}{2}\binom{n-1}{2}\binom{1}{1} + \cdots + \binom{n}{2}\binom{2}{2}\binom{1}{1}$

means pick 2 from k balls $(2 \le k \le n)$ pick another 2 from $(n+2-k)$ balls and the last 1 ball combining number of ways with all possible k. equals to pickiy 5 balls from $(n+3)$ balls.

$\therefore LHS = RHS$

Q6 since $0 \le x_1 \le 3$, $0 \le x_2 \le 4$, $0 \le x_3 \le 6$

1. $x_1 = 0$ $\quad x_2 + x_3 = 11$ $\quad x_{2max} + x_{3max} \le 10 < 11$ impossible.
2. $x_1 = 1$ $\quad x_2 + x_3 = 10$ $\quad x_{2max} + x_{3max} \le 10$. possible

   $x_1 = 1,\ x_2 = 4,\ x_3 = 6$
3. $x_1 = 2$ $\quad x_2 + x_3 = 9$ $\quad x_{2max} + x_{3max} \le 10 > 9$

   ① $x_1 = 2$ $\quad x_2 = 4$ $\quad x_3 = 5$

   ② $x_1 = 2$ $\quad x_2 = 3$ $\quad x_3 = 6$
4. $x_1 = 3$ $\quad x_2 + x_3 = 8$ $\quad 10 > 8$

   ① $x_1 = 3$ $\quad x_2 = 4$ $\quad x_3 = 4$

   ② $x_1 = 3$ $\quad x_2 = 3$ $\quad x_3 = 5$

   ③ $x_1 = 3$ $\quad x_2 = 2$ $\quad x_3 = 6$

$\therefore$ Totally. It has 6 solutions

Q7 Proof. we divide the set into n-classes $\{1,2\}\{3,4\}\cdots\{2n-1,2n\}$

By the pigeonhole principle, given $n+1$ elements, at least two of them will be in the sam class. $\{2k-1, 2k\}$ $(1 \le k \le n)$ but $2k-1$ and $2k$ are relatively prime, because their difference is 1.

Q8

(a) For abnormal 0-1 sequence. we have k terms as small as possible. In this sequence $(a_1, a_2 -- a_k)$, a_k is equal to 1. and for the first $(n-1)$ terms, the number of 1s is equal to the number of 0s ("0"="1"+1) and find that the number of 0s is equal to the number of (1s +1)

So, In this sequence the number of 1s is equal to the number of 0s.

If we let $0 \to 1$, $1 \to 0$, we find the number of 0s is equal to the number of (1s+2), ∴ It has $(m+1)$ 0s and $(m-1)$ 1s

According to this, we can conclude that for the first k terms, 0s are more than 1s. we can get an abnormal sequence by transforming 0 and 1. This transforming is inversible, there is a bijection between abnormal and normal. So the number of abnormal sequences a_n with $2m$ terms equals that of sequences a_n of which $(m+1)$ terms are 0s and $(m-1)$ terms are 1s.

(b) Number of sequence with m 1s and m 0s

$$= \binom{2m}{m} = \binom{8}{4} = \frac{8\times7\times6\times5}{4\times3\times2\times1} = 70$$

Number of abnormal sequence $= \binom{2m}{m+1} = \binom{8}{5} = 56$

∴ Number of normal sequence $= 70 - 56 = 14$