CSC3001 Discrete Mathematics: Tutorial 2

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September 13, 2022

Suggested extra Materials

- Popular math: 3Blue1Brown channel, in particular, Circle Division Problem, Hanoi Tower, Hamming Code, Olympiad Counting, etc.
- Advanced CS: Introduction to Algorithms, Cormen, Leiserson, Rivest, Stein.
- Advanced Problem-Solving: Problem Solving Strategies, Arthur Engel.
- Advanced math: Concrete Mathematics, Graham, Knuth, Patashnik.
- Advanced math: Enumerative Combinatorics, Volume 1, Richard Stanley.

Tough problem for bored students: For which a, b, n you can fill an $a \times b$ rectangular grid with $1 \times n$ rectangles?

Rules of the game

- Ask any questions about slides / blackboard. No need to raise your hand.
- More questions = better

Recap: What we need before Discrete Mathematics

Simplifying to the extreme, Discrete Mathematics is concerned with a question:

Given some finite set A, what is $|A|$?

To start answering such questions, we need a bit of machinery:

- Set Theory
- Logic
- First-order Logic

Recap: Set Theory

In set theory we simply simply have objects $a, b, c, ..., x, y, z$ and a single relation $\in$. E.g.

$$-1 \in \mathbb{Z}$$

$$13 \in \mathbb{P} \quad \text{(primes)}$$

$$\{1, 2\} \in P(\{1, 2, 3, 4\})$$

Recap: Basic Logic

In logic we have statements $A, B, C, D, \ldots$ and logical operations

$$A \vee B, \quad A \implies B, \quad A \wedge B, \quad \neg A.$$

In basic logic it is assumed that every statement can only take two values:

true or false

equivalently,

yes or no

equivalently,

1 or 0

Recap: Basic Logic

The process of deriving the value of logical statement (e.g. $A \implies (B \implies A)$) under some assumptions (e.g. B is false) is called *logical deduction*. In Basic Logic we can derive rules from truth tables. E.g. we can derive *Modus Ponens* (A is true, $A \implies B$ is true, hence B is true) from truth table for

$$A \implies B$$

Exercise 0

Compute a truth-table for $(\neg A) \implies B$

Recap: Propositional Logic

- Assume now that our logical statements $A, B, C, ...$ depend on some finite number of parameters $x_1, x_2, ..., x_n$. Such statements (terms) are written as

$$A(x_1, ..., x_n),$$

- For example $x_1 = x_2$ is a statement with variables x_1, x_2.
- Logical truth (in meaning of basic logic) can now be analyzed for different values of $x_1, x_2, ..., x_n$.
- Term can be made into a logical statement by adding *predicates* $\forall, \exists$.
- For example,

$$\forall x (x^2 \geq x),$$

$$\exists x (x^3 = 3x).$$

$$\forall y \exists x (x + y = 0) \implies (xy < 0)$$

Exercise 1

Combining Set theory and Basic Logic we can define operations

$$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$$

$$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$$

$$x \in B \setminus A \Leftrightarrow (x \in B) \wedge (x \notin A)$$

Prove that

$$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$$

Exercise 2

Write a logical statement "there exist exactly 3 solutions of the equation $P(x) = 0$"

Solution to Exercise 2

$$\exists x_1, \exists x_2 \exists x_3 (x_1 \neq x_2) \wedge (x_2 \neq x_3) \wedge (x_1 \neq x_3) \wedge$$

$$\wedge (P(x_1) = 0) \wedge (P(x_2) = 0) \wedge (P(x_3) = 0) \wedge$$

$$\wedge (\forall y (y \neq x_1 \wedge y \neq x_2 \wedge y \neq x_3) \implies (P(y) \neq 0))$$

In real arguments

Of course, during real proofs we do not write such long logical statements explicitly. Instead, we re-use some common equivalences, notations, shorthands, E.g.

- proof by contradiction, $\neg\neg A \Leftrightarrow A$
- disproof by example, $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$

Exercise 3

Prove / disprove the following statements

$$\forall x \in \mathbb{R}\ (x < 0) \implies (1 - x > 0)$$

$$A \implies (B \implies A)$$

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (xy = 1)$$

Solution to Exercise 3

Thank You

Thank you for your attention!