CSC3001 Discrete Mathematics

Homework 4

Deadline: 2022.07.27

1. Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.

2. Let a_n denote the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two white squares are adjacent. Find a_n.

3. Codewords from the alphabet {0, 1, 2, 3} are called legitimate if they have an even number of 0's. How many legitimate codewords are there, of length k?

4. A chessboard has five rows and five columns. A rook can attack in its row and in its column. So 'placing rooks' means placing points such that in each row and column has at most one rook. How many ways are there to place five (identical) rooks on a 5×5 chessboard, with no rooks occupying places $(1, 1), (2, 2), (3, 3), (4, 4)$? Note: the forbidden squares are four of the five diagonal squares.

5. The Schröder numbers S_n are related to the Catalan numbers as follows: The alphabet consists of $\{\uparrow, \downarrow, \rightarrow\rightarrow\}$. A Schröder word is the empty string or a word w characterized by

   P1 $\#(\uparrow) = \#(\downarrow)$

   P2 In any word consisting of the first k letters of w, $\#(\uparrow) \geq \#(\downarrow)(k = 1, ..., l(w))$

   The length of a Schröder word w is given by $l(w) = \#(\uparrow) + \#(\downarrow) + 2\#(\rightarrow\rightarrow)$. For example, the possible Schröder words of length 4 are illustrated below:

   

   The Schöder number S_n is the number of Schröder words of length $2n$. One defines $S_0 := 1$.

(a) Give a recursive definition of a Schröder word and from it derive the recurrence relation

$$S_n = S_{n-1} + \sum_{i+j=n-1} S_i S_j, \quad n \geq 1$$

(You do not have to prove that the recursive definition coincides with that of P1 and P2 above).

(b) Find the closed form of the generating function $S(x) = \sum_{n=0}^{\infty} S_n x^n$

6. You are getting 10 ice cream sandwiches for 10 students. There are 4 types: Mint, Chocolate, Resse's and Plain. If there are only 2 Mint ice cream sandwiches and only 3 Plain (and plenty of the other two), how many different ways could you select the ice cream sandwiches for students?

7. Prove that at a cocktail party with ten or more people, there are either three mutual acquaintances or four mutual strangers. (This is a special case of Ramsey theory, i.e., prove $R(3,4) \leq 10$. In fact, you can also prove a more stricter condition $R(3,4) = 9$ if you are interested.)

8. Prove that if every point on a line is painted cardinal or white, there exists three points of the same color such that one is the midpoint of the line segment formed by the other two.

9. A stressed-out computer science professor consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days exactly 100 espressos were consumed.

10. Give a combinatorial proof that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

[Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

11. Let $n > 0$ be a natural number. Prove the following identities

(a)

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + (-1)^n \binom{n}{n} = 0.$$

(b)

$$\binom{n}{0} - 2\binom{n}{1} + 2^2\binom{n}{2} - 2^3\binom{n}{3} + ... + (-1)^n 2^n \binom{n}{n} = (-1)^n.$$

(c)

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + ... = 2^{n-1}.$$

12. Let
$$a_n = \sum_{k=0}^{n} \binom{n-k}{k}.$$
Prove that $a_n = f_{n+1}$ - Fibonacci number. Where $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$.

13. Let
$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k}.$$
Find closed form solution for a_n.

14. Find the number of paths from point $(0,0)$ to point (m,n) that use only steps $(0,1)$ and $(1,0)$. Using it and considering points $(m-k,k)$ prove that for $m \leq n$ we have
$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{m-k} = \binom{n+m}{m}.$$

15. Using
$$(1-x)^n(1+x)^n = (1-x^2)^n$$
prove that
$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{2m-k}(-1)^k = (-1)^m\binom{n}{m}.$$

16. Prove that the number of binary words of length n with exactly m substrings of the form 01 is equal to $\binom{n+1}{2m+1}$. Hint: extend a string by 0 at the end and 1 at the beginning, then put $2m+1$ “bit switches” in gaps.

17. Count the number of triples (x,y,z) from $\{1,2,...,n+1\}$ with $z > \max(x,y)$.

18. Obtain combinatorial identities for binomial coefficients using the following generating function identities

$$\sum_{n=0}^{\infty} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}}.$$

$$\frac{1}{(1-x)^n} \frac{1}{(1-x)^m} = \frac{1}{(1-x)^{n+m}}.$$

19. Obtain combinatorial identities for Fibonacci numbers and binomial coefficients using the following generating function identities with

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2},$$

$$\sum_{n=0}^{\infty} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}}.$$

(a)

$$\frac{x}{1-x-x^2} \frac{x}{1-x} = \frac{1}{1-x-x^2} - \frac{1}{1-x},$$

(b)

$$\frac{1}{1-x-x^2} \frac{x^2}{(1-x)^3} = \frac{1}{1-x-x^2} \frac{1}{(1-x)^2} - \frac{1}{(1-x)^3}.$$

20. How many pairs (A, B) of non-empty subsets of $\{1, 2, ..., n\}$ satisfy $A \cap B = \emptyset$?