Simulation of the solar system – with an asteroid passing close by Jupiter

Simon Westberg

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1 Introduction

The aim of this project was to simulate our solar system and to study the behavior of an asteroid passing close by Jupiter. Three different numerical integrators were studied in terms of energy conservation – Euler, Euler-Cromer, and velocity Verlet. Two different simulations with an added asteroid were studied – one where the asteroid's mass was realistic, and one were the asteroid was extremely massive.

2 Physics and model

Our solar system consists of one star, the Sun, eight planets, some of which have orbiting moons, and many other smaller objects such as dwarf planets and comets. This project focuses on simulating the Sun together with the orbiting planets.

The solar system was modelled as lying in a flat plane, the xy-plane. This is a suitable approximation since every planet's inclination to the invariable plane, the weighted average of all planetary orbits, is less than 2.2°, except for Mercury who has an inclination of 6.34° [1]. The planets and the Sun were modelled as point masses and no other force than gravity was considered.

The gravitational force acting on an object with mass M from another object with mass m is given by

$$\boldsymbol{F} = G\frac{M \cdot m}{r^2}\hat{\boldsymbol{r}}, \tag{1}$$

where $G = 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$ is the gravitational constant [2], r is the distance between the two objects and $\hat{\boldsymbol{r}}$ is a unit vector pointing in the direction from M to m [3]. Since the gravitational force is a vector quantity, the total force acting on an object with mass M from several objects with masses $m_1, m_2, m_3 \ldots$ is given by the vector sum

$$\boldsymbol{F} = GM\sum_i \frac{m_i}{r_i^2}\hat{\boldsymbol{r}}_i. \tag{2}$$

In this simulation all the forces are in the xy-plane and they can thus be decomposed into an x-component and into a y-component. If θ is the angle from the positive x-axis to the line that intersects the two bodies, see figure 1, then the force can be decomposed as

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y = GM \sum_i \frac{m_i}{r_i^2} (\cos\theta_i \,\hat{\boldsymbol{x}} + \sin\theta_i \,\hat{\boldsymbol{y}}). \tag{3}$$

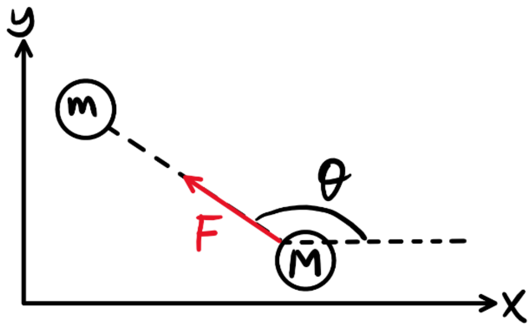


Figure 1: Gravitational force acting between two bodies.

When the total force acting on a body is known, you can determine its acceleration $\boldsymbol{a}$ using Newton's second law of motion,

$$\boldsymbol{F} = m\boldsymbol{a}, \tag{4}$$

or in terms of the x- and y-position,

$$F_x = m\ddot{x}, F_y = m\ddot{y}, \tag{5}$$

where a dot indicates a time derivative.

The mechanical energy of an object with mass m is given by

$$E = \tfrac{1}{2}mv^2 + U = \tfrac{1}{2}m(v_x^2 + v_y^2) + U, \tag{6}$$

where v is the objects velocity, v_x and v_y are the x- and y-component of the velocity, and U is the potential energy of the object. For a two-body system, e.g. the Sun and Earth, the potential energy is given by [3]

$$U = -G\frac{M \cdot m}{r}. \tag{7}$$

For several bodies, the total mechanical energy is given by

$$E = \sum_i \tfrac{1}{2}m_i v_i^2 + \tfrac{1}{2}\sum_{i \neq j} -G\frac{m_i m_j}{r_{ij}}, \tag{8}$$

where r_{ij} is the distance from body i to body j. The factor of $\frac{1}{2}$ before the double sum of the potential energies is there so you do not count the potential energy associated with two bodies twice.

Since the gravitational force is conservative, the total mechanical energy is conserved.

Another important aspect of the planetary orbits is that they are time reversible, meaning that if time is reversed, the planets' orbits still obey the involved laws of physics, e.g. Newton's second law of motion (4) [4].

3 Simulation and method

3.1 Energy conservation

Since the mechanical energy of all the planets and the Sun, equation (8), is conserved, it is important to use a numerical integrator that mostly conserves energy, i.e. a symplectic integrator. Two such integrators were used and compared in the simulation – the Euler-Cromer integrator and the velocity Verlet integrator. To highlight that energy is not conserved by a non-symplectic integrator, the performance of the two symplectic integrators were also compared with the Euler method, a non-symplectic integrator.

The dependence on the time step in terms of energy conservation and computation time was also analyzed for the velocity Verlet method.

3.2 Numerical integrators

Let Δt be a short interval of time. Let x^n be the x-position of a certain body at time $\Delta t \cdot n$, with analogous definitions for the other quantities, y, v_x, v_y, F_x and F_y, where F_x and F_y is given by equations (3) and (5). The position and velocity at time $\Delta t \cdot (n+1)$ can then be calculated using one of the following numerical methods.

3.2.1 Euler method

Given the above definitions, the Euler method can be written as [5]

$$\begin{cases} x^{n+1} = x^n + v_x^n \Delta t \\ y^{n+1} = y^n + v_y^n \Delta t \\ v_x^{n+1} = v_x^n + \dfrac{F_x^n}{m} \Delta t \\ v_y^{n+1} = v_y^n + \dfrac{F_y^n}{m} \Delta t \end{cases} \tag{9}$$

3.2.2 Euler-Cromer

The Euler-Cromer method, also known as symplectic Euler, is similar to the usual Euler method, although you first calculate the new velocities, v_x^{n+1} and v_y^{n+1}, and then use these velocities when calculating the new positions,

$$\begin{cases} v_x^{n+1} = v_x^n + \dfrac{F_x^n}{m} \Delta t \\ v_y^{n+1} = v_y^n + \dfrac{F_y^n}{m} \Delta t \\ x^{n+1} = x^n + v_x^{n+1} \Delta t \\ y^{n+1} = y^n + v_y^{n+1} \Delta t \end{cases} \tag{10}$$

Unlike the Euler method, Euler-Cromer is a symplectic integrator [6].

3.2.3 Velocity Verlet

The velocity Verlet method is a bit more complicated than the former Euler-methods. With velocity Verlet you first calculate the new positions of the bodies according to (11). Then you use these positions to calculate the new forces acting on the bodies, F_x^{n+1} and F_y^{n+1}, and then you determine the new velocities using both the forces at time $\Delta t \cdot n$ and at time $\Delta t \cdot (n+1)$. See equations (11).

$$\begin{cases} x^{n+1} = x^n + v_x^n \Delta t + \dfrac{F_x^n}{2m} \Delta t^2 \\ y^{n+1} = y^n + v_y^n \Delta t + \dfrac{F_y^n}{2m} \Delta t^2 \\ v_x^{n+1} = v_x^n + \frac{1}{2m}[F_x^n + F_x^{n+1}]\Delta t \\ v_y^{n+1} = v_y^n + \frac{1}{2m}[F_y^n + F_y^{n+1}]\Delta t \end{cases} \tag{11}$$

The velocity Verlet method is a symplectic method. It is also time reversible [7], unlike the Euler-Cromer method.

3.3 Initial conditions, units and coordinate system

Since the masses of the planets and the distances between planets are so large in the solar system, the usual SI units are inconvenient. Instead, the astronomical system of units was used in the simulation. Length was thus stated in AU (149 597 870 700 m), time in days (86 400 s), and mass in Earth mass (5.9742×10^{24} kg), $\mathrm{M_E}$ [8]. The astronomical unit of mass is actually solar mass, but Earth mass was more convenient in the simulation. In these units, the gravitational constant is given by $G = 8.8874 \times 10^{-10}\ \mathrm{AU}^3 \cdot \mathrm{M_E^{-1}} \cdot \mathrm{days}^{-2}$.

The simulation was initialized with a heliocentric coordinate system, i.e. with the Sun initially at the origin. The positions and velocities of the planets in relation to the Sun at 2019-01-09 00.00 (Barycentric Dynamical Time) were used as initial conditions for the numerical integrators, retrieved from [9]. The z-position and z-velocity, v_z, were set to zero.

The masses of the planets and the Sun were retrieved from [10].

3.4 Adding an asteroid

An asteroid was added to the solar system, with initial position and velocity chosen so that it would pass close by Jupiter. The trajectory of the asteroid was then studied. In order to choose realistic values for mass and initial velocity of the asteroid, [11] was studied. An asteroid of very large mass (almost as massive as Jupiter) were also studied.

4 Results and analysis

Since the outer planets in the solar system are so much farther from the Sun than the inner planets, plotting all the orbits in scale at the same time does not look good. Therefore, throughout the results, I will either plot the inner solar system (the Sun together with Mercury, Venus, Earth, and Mars) or the outer solar system (the Sun together with Jupiter, Saturn, Uranus, and Neptune), even though all the planets were still used in the simulation.

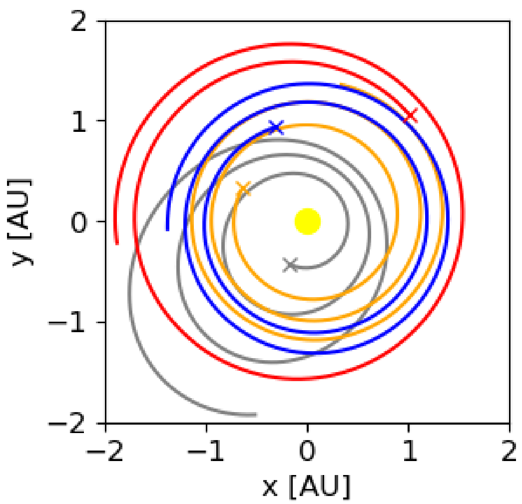
Throughout the results, the following colors were used to represent the planets and the Sun:

Sun – yellow, Mercury – grey, Venus – orange, Earth – blue, Mars – red, Jupiter – gold, Saturn – beige, Uranus – light blue, Neptune – blue.

4.1 Energy conservation

Figure 2 shows the resulting orbits for the inner planets in a simulation using the Euler method (9). A time step of $\Delta t = 1$ day was used, simulating three years. The x's mark the planets' initial positions. As can be seen in the figure, the planets' distances from the Sun increases with time, and each orbit creates a spiral pattern. In figure 3 we can see that the total mechanical energy of the solar system increases with time, and thus the non-symplectic Euler method does not conserve energy, as expected.

Euler method



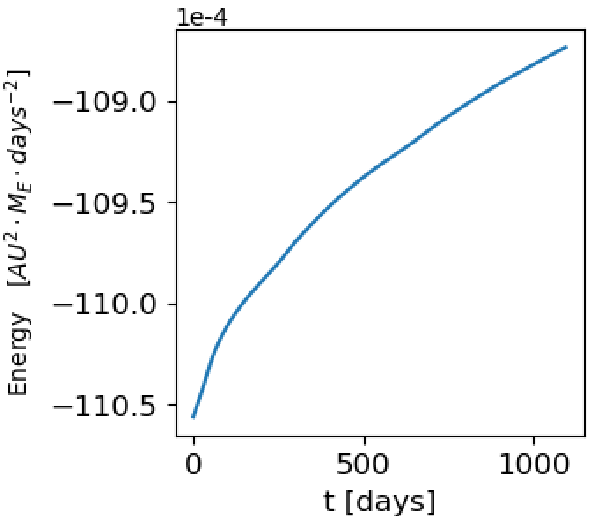


Figure 2: The inner planets' orbits around the Sun with the Euler method, using $\Delta t = 1$ day and simulating three years. The x's mark the planets' initial positions.

Figure 3: The total mechanical energy of the Sun and the planets as a function of time, using the Euler method with $\Delta t = 1$ day.

If you instead use the Euler-Cromer method (10) with a time step of $\Delta t = 1$ day and simulating three years, you get the orbits and energy shown in figures 4 and 5, respectively. As we can see in figure 4, the planets' orbits are relatively constant. The energy in figure 5 shows an oscillatory behavior in time, though it is hard to determine the long-term behavior of the energy from the figure. In order to do this, a 20-year simulation was conducted with the Euler-Cromer method using the same time step as before. The resulting energy can be seen in figure 6. We can see that in addition to the small oscillations in time, the oscillating energy also shows an overall oscillating behavior over longer periods. The mechanical energy of the solar system thus seems to be at least periodically conserved.

Euler-Cromer method

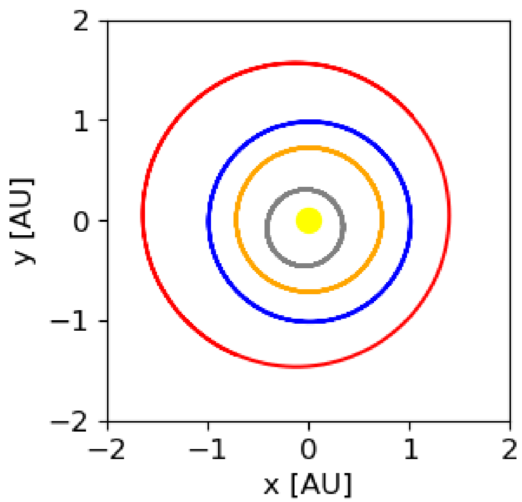


Figure 4: The inner planets' orbits around the Sun with the Euler-Cromer method, using $\Delta t = 1$ day and simulating three years.

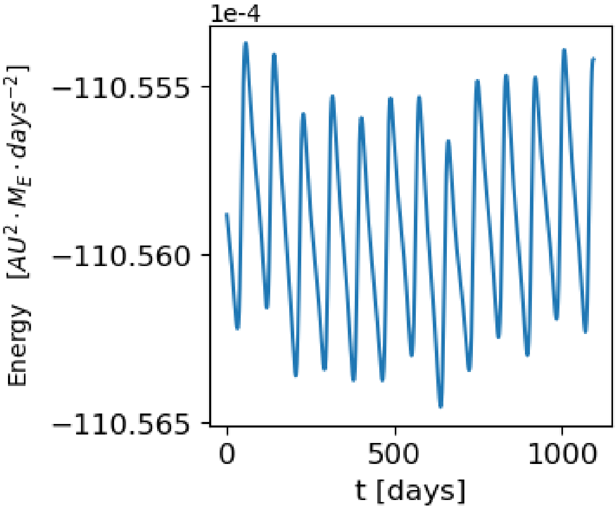


Figure 5: The total mechanical energy of the Sun and the planets as a function of time, using the Euler-Cromer method with $\Delta t = 1$ day.

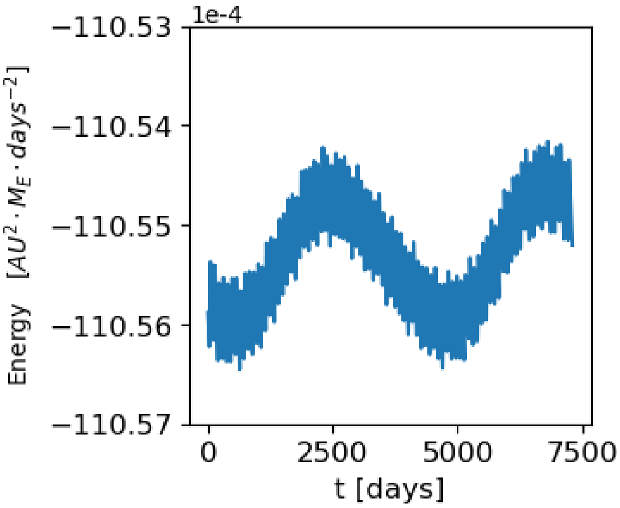


Figure 6: Long-term behavior of the mechanical energy of the solar system, using the Euler-Cromer method with $\Delta t = 1$ day.

Figures 7 and 8 show the resulting orbits and energy from a three-year simulation using the velocity Verlet method (11), with a time step of $\Delta t = 1$ day. As can be seen in figure 7, the planetary orbits seem to be relatively constant. In figure 8 we can see that the energy also is relatively constant, although with small, periodic, variations. The same behavior can be seen when plotting the energy over longer time periods.

Velocity Verlet method

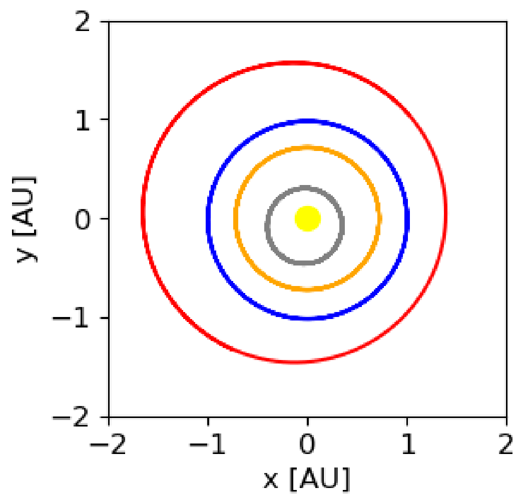


Figure 7: The inner planets' orbits around the Sun with the velocity Verlet method, using $\Delta t = 1$ day and simulating three years.

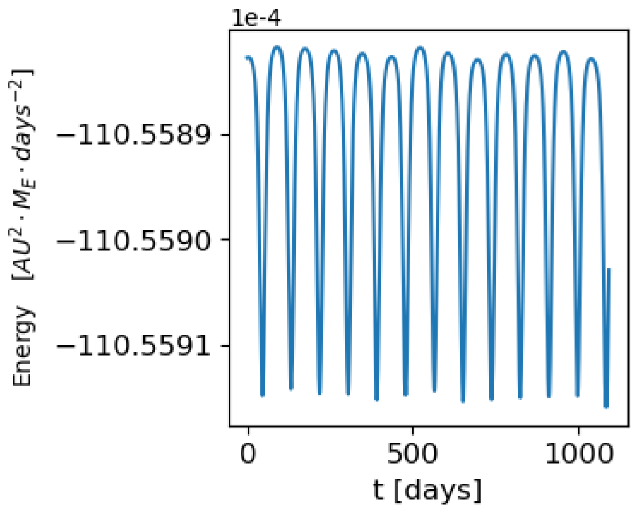


Figure 8: The total mechanical energy of the Sun and the planets as a function of time, using the velocity Verlet method with $\Delta t = 1$ day.

The results presented in figures 2-8 were as expected. The two symplectic integrators – Euler-Cromer and velocity Verlet – almost conserved energy, at least periodically, but the non-symplectic integrator – Euler – did not conserve energy.

From figures 6 and 8, the energy variation with the Euler-Cromer method and the velocity Verlet method can be determined as approximately 0.02 % and 0.0003 %, respectively. In terms of energy conservation, then, the velocity Verlet method is a more suitable choice of integrator. Since the velocity Verlet method also is time reversible, it is a better choice than Euler-Cromer for planetary motion. However, Euler-Cromer had a shorter computation time when simulating the orbits, since you only need to calculate the forces on the planets once, instead of twice as with the velocity Verlet method.

Table 1 shows the energy variation and computation time for a one-year simulation of the solar system, using the velocity Verlet method, for a few different time steps. The energy variation was determined as

$$\frac{|E_{max} - E_{min}|}{|E_{max}|}, \tag{12}$$

where E_{max} and E_{min} is the maximum and minimum value of the energy.

As can be seen in the table, the energy variation gets smaller and smaller as the time step decreases. The energy variation when using a time step of $\Delta t = 0.1$ days is approximately 4000 times smaller than when using a time step of $\Delta t = 10$ days. However, we can also see that a smaller time step leads to a longer computation time, and for a time step of $\Delta t = 0.1$ days the computation time was several minutes.

Δt [days]	Energy variation [%]	Computation time [s]
10	$110 \cdot 10^{-4}$	0.35
5	$56 \cdot 10^{-4}$	0.81
2	$12 \cdot 10^{-4}$	3.5
1	$3.0 \cdot 10^{-4}$	11
0.5	$0.75 \cdot 10^{-4}$	44
0.1	$0.030 \cdot 10^{-4}$	880

Table 1: Dependence on the time step in terms of energy variation and computation time with the velocity Verlet method for a one-year simulation.

In order to both have a rather small energy variation, and a reasonable computation time, a time step of $\Delta \mathrm{t} = 1$ day is an acceptable choice. If a simulation of a much longer time than one year is needed, then the computation time will obviously increase, and thus a larger time step may be necessary, at the price of reduced accuracy.

Energy variation is of course only one way to measure the performance of the method. To conduct a more comprehensive study of the time step dependence and performance of the integrator, conservation of total linear- and angular momentum [3] could also be examined.

4.2 Animation of the solar system

Figures 9 and 10 show snapshots of the inner and outer solar system, respectively, from an animation of a simulation of the solar system. The velocity Verlet method was used in the simulation, with a time step of $\Delta t = 1$ day. Note that the sizes of the planets and the Sun are not to scale. The snapshots were taken $359\ days$ into the simulation.

As we can see in figure 9, the Earth (blue) is just about to complete one orbit around the Sun, which is expected, since it takes Earth approximately 365 days – one year – to revolve around the Sun.

In section 6, Attachments, there is a link to a movie showing the simulation.

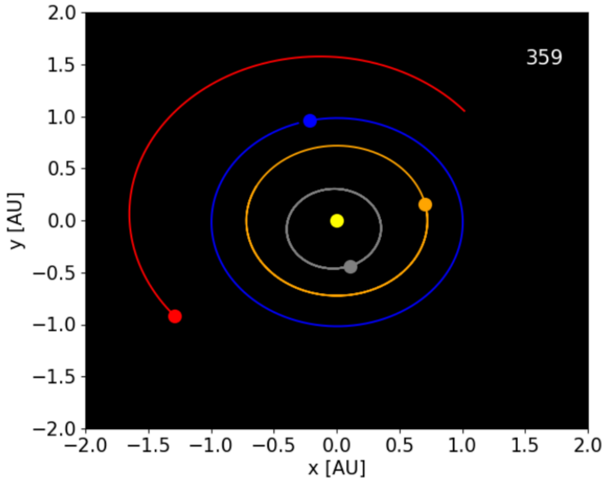


Figure 9: A snapshot of an animation of the inner planets' orbits around the Sun after 359 days, with the velocity Verlet method, using $\Delta t = 1$ day. The sizes of the planets and the Sun are not to scale.

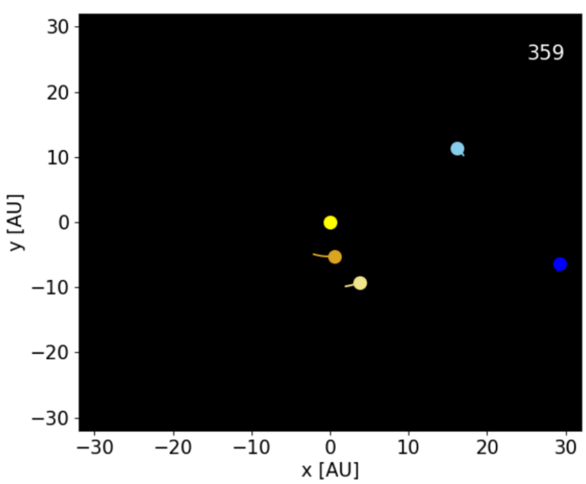


Figure 10: A snapshot of an animation of the outer planets' orbits around the Sun after 359 days, with the velocity Verlet method, using $\Delta t = 1$ day. The sizes of the planets and the Sun are not to scale.

4.3 Adding an asteroid

Throughout this section, all simulations were conducted with the velocity Verlet method, with a time step of $\Delta t = 0.5$ days.

Figure 11 shows the inner solar system together with Jupiter and an asteroid, 314 days after the simulation started. The asteroids initial position and initial velocity was selected so that it would pass close by Jupiter. Its initial velocity was set to $20\,000$ m/s, or 0.012 AU/day, pointing straight upward (in the y-direction). The mass of the asteroid was set to 10^{-6} Earth masses, which is approximately $3 \cdot 10^{-9}$ Jupiter masses.

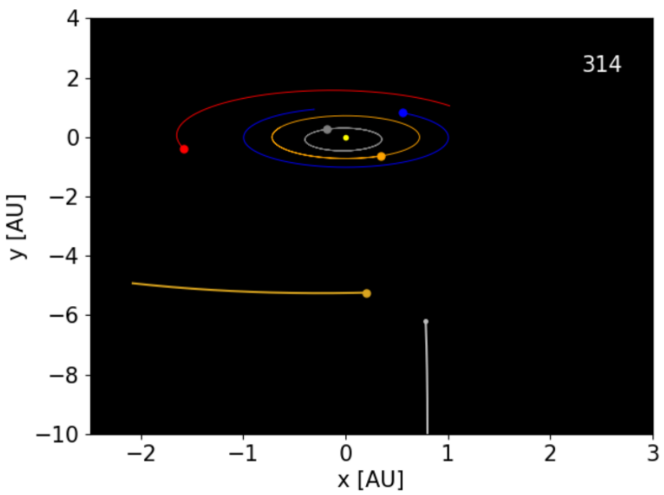


Figure 11: The inner solar system together with Jupiter and an asteroid. Sizes of bodies not to scale. The number in the corner indicates how many days that have passed since the simulation started.

Figures 12 and 13 show how the asteroids trajectory has developed at two subsequent times in the simulation. At the closest point, the asteroid was approximately 0.006 AU from Jupiter. We can see that the asteroid's trajectory changes significantly when passing by Jupiter, as a consequence of the gravitational force from Jupiter. In figure 13 we can see that the Sun's gravitational field has attracted the asteroid, again changing its trajectory.

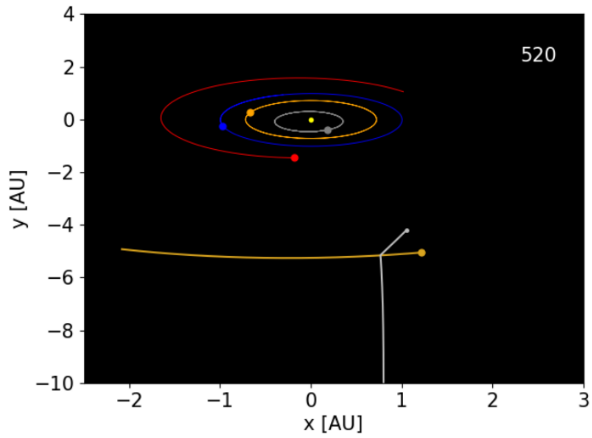


Figure 12: The inner solar system together with Jupiter and an asteroid. We can see that the asteroids trajectory has changed significantly.

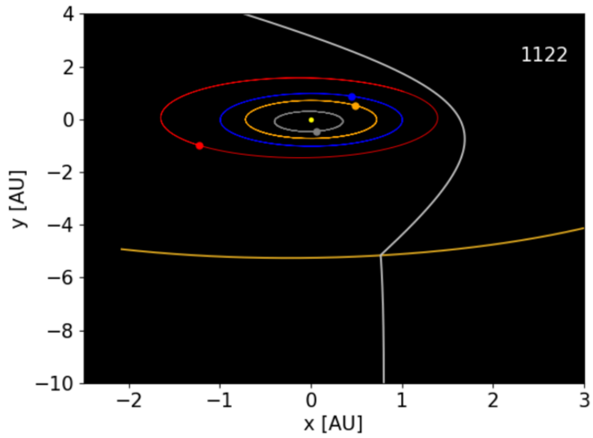


Figure 13: The inner solar system together with Jupiter and an asteroid. The asteroid is attracted by the Sun's gravitational field.

Figures 14 and 15 show Jupiter together with an asteroid, during a different simulation. This time the asteroid's mass was set (unrealistically) to 300 Earth masses, which is approximately the same mass as Jupiter. At the closest point, the asteroid was approximately 0.002 AU from Jupiter. We can see in figure 15 that this time not only the asteroid's trajectory changed, but also Jupiter's trajectory.

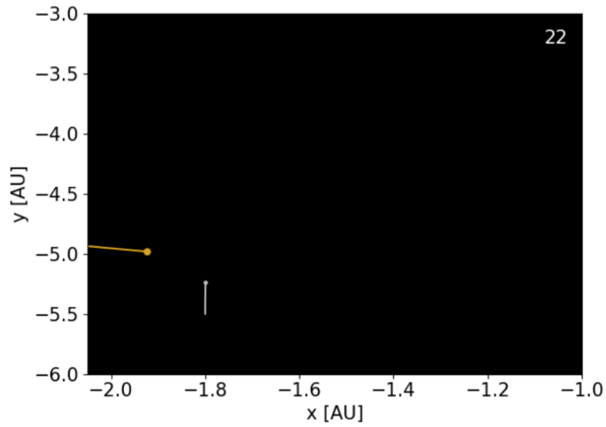


Figure 14: Jupiter together with a very heavy asteroid, 22 *days* into the simulation.

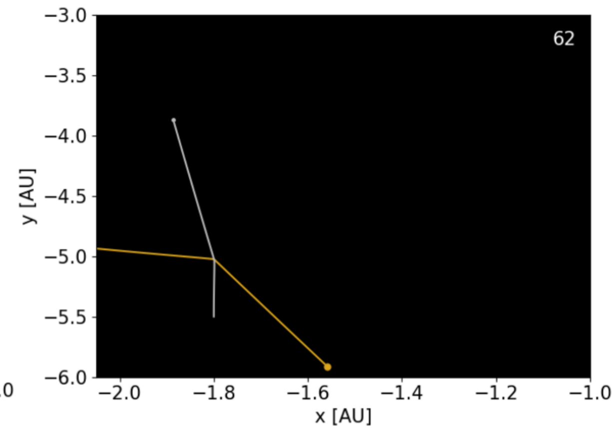


Figure 15: Jupiter together with a very heavy asteroid, 62 *days* into the simulation. We can see that both the asteroid's trajectory and Jupiter's trajectory has changed significantly.

In section 6, Attachments, there are links to videos showing the two simulation.

5 Discussion and conclusions

In section 4.1, Energy conservation, we saw that the velocity Verlet method yielded better results than the Euler-Cromer method, in terms of energy conservation, and that the Euler method did not conserve energy at all. Since the velocity Verlet method also is time reversible, it is a suitable choice as numerical integrator when simulating the solar system. When it comes to choosing a good time step, it depends on what kind of accuracy you need and what kind of computation time is acceptable. For the purposes of this project, a time step of $\Delta t = 1$ day was an acceptable choice. To further examine the performance of the numerical integrators, conservation of total linear- and angular momentum could also have been studied.

To reach an even better performance of the simulation, an adaptive time step could have been implemented, i.e. a time step that gets smaller during periods of rapid change, and larger when the planets' positions and velocities are not changing much. You could also compare the simulated positions and velocities of the planets (after a certain time) for different time steps and then use a small enough time step such that the difference in positions and velocities is below some specified value.

When studying the trajectory of an asteroid in section 4.3, we could see that the trajectory changed considerably when the asteroid passed close by Jupiter. When the asteroid had a small mass, no significant change was seen in Jupiter's trajectory, but when the asteroid had a mass similar to Jupiter, a change in Jupiter's trajectory could also be seen.

6 Attachments

Animation of the solar system: https://streamable.com/e6bcc

Animation of the inner solar system, Jupiter, and an asteroid: https://streamable.com/vl472

Animation of a very massive asteroid and Jupiter: https://streamable.com/x3ojr

Note that the sizes of the planets and the Sun are not to scale.

7 References

[1] https://en.wikipedia.org/wiki/Invariable_plane, retrieved 2019-01-12.

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