

# An introduction to de Rham cohomology

How algebra and calculus relate to topology

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April 26, 2021

## Question

Does there exist a function that is the gradient of some other function? More precisely, when does  $F: U \rightarrow \mathbb{R}^2$  for some open  $U \subseteq \mathbb{R}^2$  satisfy

$$\frac{\partial F}{\partial x} = f_1, \quad \frac{\partial F}{\partial y} = f_2 \quad \text{for some } f = (f_1, f_2)?$$

(You could also think of this question as asking when vector fields have potential.)

# Motivation

## Question

Does there exist a function that is the gradient of some other function? More precisely, when does  $F: U \rightarrow \mathbb{R}^2$  for some open  $U \subseteq \mathbb{R}^2$  satisfy

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(You could also think of this question as asking when vector fields have potential.)

## Answer

It depends on the topology of  $U$ !

# Some vector calculus

Note that  $\frac{\partial F}{\partial x} = f_1$ ,  $\frac{\partial F}{\partial y} = f_2$  implies  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ . Is this condition sufficient to show  $F$  is the gradient of some other function?

## Example

The vector field  $F(x, y) = (y, x)$  satisfies  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = 1$ .  $F$  is also the gradient of  $f = xy$ .

## Example

However, consider  $F(x, y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ . You can show this cannot be the gradient of some function by integrating around the circle, resulting in two different values.

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## Proposition

The condition  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$  is sufficient if  $U$  looks like a ball (convex).

# Sneak peek of de Rham cohomology

## Definition

If  $C^\infty(U, \mathbb{R}^k)$  denotes the set of smooth functions  $U \rightarrow \mathbb{R}^k$  define the maps  $C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R})$ , where  $\text{grad}(\phi) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$ ,  $\text{curl}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_2}{\partial x}$ .

## Proposition

The curl of the gradient is zero, or  $\text{curl} \circ \text{grad} = 0$ .

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## Proposition

The curl of the gradient is zero, or  $\text{curl} \circ \text{grad} = 0$ .

This implies that  $\text{im}(\text{grad}) \subseteq \ker(\text{curl})$ , so we can form the *quotient space*  $H^1(U) := \ker(\text{curl}) / \text{im}(\text{grad})$ .

## Definition

The space  $H^1(U)$  is the **1st de Rham cohomology group** of  $U$ .

Now the aforementioned proposition is equivalent to saying  $H^1(U) = 0$  whenever  $U$  is convex. **todo: image and gradient**

# Divergence, gradient, curl

Let's move to three dimensions!

## Definition

Define  $C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U, \mathbb{R})$ ,

$$\text{grad}: f \mapsto \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\text{curl}: (f_1, f_2, f_3) \mapsto \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right),$$

$$\text{div}: (f_1, f_2, f_3) \mapsto \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

## Proposition

$$\text{div} \circ \text{curl} = 0.$$



# More on de Rham cohomology

## Definition

Since the composition  $\text{div} \circ \text{curl}$  is zero, we can also form the **2nd de Rham cohomology group**  $H^2(U) := \ker(\text{div}) / \text{im}(\text{curl})$ . To fit with the theme, define  $H^0(U) = \ker(\text{grad})$ .

It turns out de Rham cohomology measures the amount of “holes” in a space. Since  $\mathbb{R}^3$  is “completely solid”, there should be no nontrivial de Rham cohomology. The following theorem demonstrates this:

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## Theorem

*For a convex open set  $U \subseteq \mathbb{R}^3$ , we have  $H^0(U) = \mathbb{R}$ ,  $H^1(U) = 0$ , and  $H^2(U) = 0$ .*

Intuitively,  $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$  since this space has a nontrivial topology. But how do we show this?

# A commutative diagram, because why not

The relationships between divergence, gradient, and curl, and the 0th, 1st, and 2nd de Rham cohomology groups of a convex open subset of  $\mathbb{R}^3$  can be summarized in the following commutative diagram:

$$\begin{array}{ccccccc} \ker(\text{grad}) & & \ker(\text{curl})/\text{im}(\text{grad}) & & \ker(\text{div})/\text{im}(\text{curl}) & & H^3(U) \\ H^0(U) & \longrightarrow & H^1(U) = 0 & \longrightarrow & H^2(U) = 0 & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ C^\infty(U, \mathbb{R}) & \xrightarrow{\text{grad}} & C^\infty(U, \mathbb{R}^3) & \xrightarrow{\text{curl}} & C^\infty(U, \mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(U, \mathbb{R}) \\ & & \text{exact} & & \text{exact} & & \end{array}$$

# The de Rham complex

Let's shift gears to a more abstract setting.

## Definition

Define  $\Omega^*$  to be the algebra generated by  $dx_1, \dots, dx_n$  with the relations

$$\begin{cases} (dx_i)^2 = 0, \\ dx_i dx_j = -dx_j dx_i, \quad i \neq j. \end{cases}$$

Then a **differential form** is an element of  $\Omega^*(U)$ , formally defined as  $C^\infty(U, \mathbb{R}) \otimes_{\mathbb{R}} \Omega^*$ , where  $U \subseteq \mathbb{R}^n$  is open. This algebra has a natural grading  $\Omega^*(U) = \bigoplus_{q=0}^n \Omega^q(U)$ .

## Example

Concretely, a form  $\omega \in \Omega^q(U)$  can be written uniquely as  $\sum f_I dx_I$ , where  $I$  is a strictly increasing sequence of length  $q < n$ .

# The exterior derivative

## Definition

Define a *differential operator*  $d: \Omega^q(U) \rightarrow \Omega^{q+1}(U)$  by the following properties:

- i if  $f \in \Omega^0(U)$ , then  $df = \sum \frac{\partial f}{\partial x^i} dx^i$ ,
- ii if  $\omega = \sum f_I dx_I$ , then  $d\omega = \sum df_I dx_I$ .

This operator is called **exterior differentiation**.

## Definition

The algebra  $\Omega^*(U)$  paired with the differential operator  $d$  is called the **de Rham complex** on  $U$ .

## Example

A return to grounded reality. Let  $\omega = x dy \in \Omega^1(\mathbb{R})$  be a 1-form. Then  $d\omega = \left( \frac{\partial x}{\partial x} dx + \frac{\partial x}{\partial y} dy \right) dy = 1 dx dy + 0 dy dy = dx dy$ . We will see more exterior derivative calculations very soon.

# How the exterior derivative generalizes calculus

In  $\mathbb{R}^3$ ,  $\Omega^0(\mathbb{R}^3)$  and  $\Omega^3(\mathbb{R}^3)$  are one dimensional and  $\Omega^1(\mathbb{R}^3)$  and  $\Omega^2(\mathbb{R}^3)$  are both three dimensional. So identify

$$\begin{array}{ccccc} \{\text{functions}\} & \simeq & \{\text{0-forms}\} & \simeq & \{\text{3-forms}\}, \\ f & & f & & f \, dx \, dy \, dz \\ \{\text{vector fields}\} & \simeq & \{\text{1-forms}\} & \simeq & \{\text{2-forms}\} \\ X=(f_1, f_2, f_3) & & f_1 dx + f_2 dy + f_3 dz & & f_1 dy \, dz - f_2 dx \, dz + f_3 dx \, dy \end{array} .$$

Furthermore, taking the exterior derivative of a 0-form (function gives gradient), the derivative of a 1-form is curl, and the derivative of a 2-form is divergence. We compute one illuminating example, namely that  $d(\text{2-form}) = d(f_1 dy \, dz - f_2 dx \, dz + f_3 dx \, dy)$  is divergence.

### Example

Consider a 2-form defined by  $f_1 dy dz - f_2 dx dz + f_3 dx dy$ . Then

$$\begin{aligned} d(\text{2-form}) &= \left( \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz \right) dy dz + \cdots \\ &= \frac{\partial f_1}{\partial x} dx dy dz + 0 + 0 + \cdots \\ &= \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx dy dz. \end{aligned}$$

This is precisely divergence. Similarly, we also have

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \text{grad}, \\ d(\text{1-form}) &= \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy + \cdots = \text{curl}. \end{aligned}$$

So the exterior derivative generalizes all the previous notions of derivatives from calculus!

# The boundary of the boundary is zero

## Proposition

$d^2 = 0$ , where  $d$  denotes the exterior derivative.



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## Proof.

$$d^2 f = d \left( \sum_i \frac{\partial f}{\partial x_i} dx_i \right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0$$

since mixed partials commute, while the  $dx_j dx_i$  “anti-commute” (the property that  $dx_j dx_i = -dx_i dx_j$ ). ⊠

This generalizes the previous ideas  $\text{curl} \circ \text{grad} = 0$ ,  $\text{div} \circ \text{curl} = 0$ , which allowed us to define  $H^1(U)$  and  $H^2(U)$ . Since  $d$  is defined in all dimensions, we can define a more general de Rham cohomology group!

# Closed and exact forms

## Definition

If  $d\omega = 0$ , then  $\omega$  is a **closed** form, while if  $\omega = d\tau$  for some form  $\tau$ , we say  $\omega$  is an **exact** form. Precisely,  $\ker d$  consists of all the closed forms, while  $\operatorname{im} d$  are the exact forms.

Since  $d^2 = 0$ ,  $\operatorname{im} d \subseteq \ker d$  trivially. Now the question at the beginning of the talk reduces to “can we find a nontrivial closed form on  $U$ ”? The generalized de Rham cohomology measures to what extent we can do this, by collapsing the trivial solutions to zero.

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## Definition

The  $q$ -th **de Rham cohomology** of  $U$  is the space

$$H_{DR}^q(U) = \{\text{closed } q\text{-forms}\} / \{\text{exact } q\text{-forms}\}.$$

## Example

Let us compute the cohomology of  $\mathbb{R}^1$ .  $\ker d$  in  $\Omega^0(\mathbb{R}^1)$  consists of constant functions, so  $H^0(\mathbb{R}^1) = \mathbb{R}$ . Every 1-form  $\omega = g(x)dx$  is exact, since  $d \int_0^x g(u) du = \omega$ ; this implies  $H^1(\mathbb{R}^1) = 0$ , since we mod out by the entire space. Succinctly, we have

$$H^q(\mathbb{R}^1) = \begin{cases} \mathbb{R}, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$$

More generally, it is true that

$$H^*(\mathbb{R}^n) \begin{cases} \mathbb{R} & \text{in dimension 0,} \\ 0 & \text{otherwise.} \end{cases}$$

This result is called the *Poincaré lemma*.

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We could theoretically show that  $H^*(\mathbb{R}^2 \setminus \{0\})$  is nontrivial by constructing a cohomology class and modding out, but this is a pain. Isn't there an easier way to compute these groups?

# The Mayer-Vietoris sequence

A map between spaces induces a map on forms, formally stated below:

## Remark

Note that a smooth map  $f: X \rightarrow Y$  induces a pullback  $f^*: \Omega^0(Y) \rightarrow \Omega^0(X)$ ,  $g \mapsto g \circ f$ , which naturally extends to a pullback on forms  $f^*: \Omega^*(X) \rightarrow \Omega^*(Y)$

$$f^* \left( \sum g_I dy_{i_1} \cdots dy_{i_q} \right) = \sum (g_I \circ f) d(y_{i_1} \circ f) \cdots d(y_{i_q} \circ f)$$

which commutes with  $d$ . Furthermore, assigning the complexes  $\Omega^*$  to a sequence of maps is a **contravariant functor**.

Suppose our space  $X = U \cup V$  for  $U, V$  open. Then we have a sequence of inclusions

$$X \leftarrow U \amalg V \begin{matrix} \xleftarrow{\partial_0} \\ \xleftarrow{\partial_1} \end{matrix} U \cap V$$

where  $U \amalg V$  is the (set-theoretic) disjoint union, and  $\partial_0, \partial_1$  denote inclusions in  $V, U$  respectively.

# The Mayer-Vietoris sequence

Applying the contravariant functor  $\Omega^*$  to the sequence of inclusions gives

$$\Omega^*(X) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \begin{matrix} \xrightarrow{\partial_0^*} \\ \xleftarrow{\partial_1^*} \end{matrix} \Omega^*(U \cap V),$$

Take the difference of the projections to get the **Mayer-Vietoris sequence**

$$0 \longrightarrow \Omega^*(X) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \begin{matrix} \xrightarrow{\quad \mapsto \quad} \\ (\omega, \tau) \end{matrix} \Omega^*(U \cap V) \xrightarrow{\tau - \omega} 0,$$

which turns out to be exact. This induces a long exact sequence on cohomology

$$\cdots \rightarrow H^q(X) \rightarrow H^q(U) \oplus H^q(V) \rightarrow H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(X) \rightarrow \cdots$$

So how do we actually use this weird algebraic construction?

# The de Rham cohomology of the circle

## Example

For  $X = \mathbb{R}^2 \setminus \{0\}$ , we can cover it with two open sets  $U, V$ , whose intersection  $U \cap V$  is just two solid chunks of  $\mathbb{R}^2$ . So

$$\begin{aligned} H^0(U) \oplus H^0(V) &= H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}, \\ H^1(U) &= H^1(V) = H^1(U \cap V) = 0. \end{aligned}$$

Clearly  $H^2$  and above of  $X, U, V$ , etc are all zero. Our long exact sequence from Mayer-Vietoris becomes

$$H^0(X) \rightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1(X) \rightarrow 0 \rightarrow \dots$$

Since  $\delta: (\omega, \tau) \mapsto (\tau - \omega, \tau - \omega)$ ,  $\text{im } \delta$  is 1-dimensional, and so is  $\ker \delta$ . Then by the first isomorphism theorem,  $H^0(X)/0 \cong \ker \delta = \mathbb{R}$ , and  $H^1(X) \cong \mathbb{R} \oplus \mathbb{R} / (\ker d^* = \text{im } \delta) = \mathbb{R}$ . This shows that  $\mathbb{R}^2 \setminus \{0\}$  has a nontrivial first cohomology, like expected!



# Thank you!

Thank you for listening to my talk! These slides and detailed notes can be found on my website: <https://simonxiang.xyz/math>