

Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source code: https://git.simonxiang.xyz/math_notes/file/freshman_year/algebraic_topology/master_homework.tex.html

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§1 August 29, 2020: Homework 1

Hatcher Chapter 0 (p. 18): 1, 3ab, 17,

Hatcher Section 1.1 (p. 38): 3, 6, 7, 16.

§1.1 Question 1

Problem 1. Suppose X, Y are compact Hausdorff spaces and $f: X \rightarrow Y$ is continuous and onto. Define \sim as the equivalence relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

- (a) Prove the quotient space X/\sim is Hausdorff.
- (b) Use this to show that the induced map $X/\sim \rightarrow Y$ is a homeomorphism.
- (c) Show that identifying the ends of the interval gives S^1 .
- (d) Give a cooler example.

Solution. We examine the structure of topologies generated by identifying points together who lie in the same “class” after a map.

- (a) Let $[a], [b]$ be elements (equivalence classes) of the quotient space X/\sim . We want to separate $[a]$ and $[b]$ by open sets: since singletons are closed, we can separate $q^{-1}[a]$ and $q^{-1}[b]$ by open sets in X (where $q: X \rightarrow X/\sim$ is the canonical quotient map), due to the fact that the quotient map is continuous (and so the inverse image of closed sets are closed) and Hausdorff plus compact implies normal. Then their images are disjoint and open in X/\sim as well, since the quotient map is open.
- (b) We claim that f induces a map $g: X/\sim \rightarrow Y$ such that $g \circ q = f$. Then g is bijective (the equivalence classes all identify to the same point in Y), and continuous (by a theorem in Munkres, Corollary 22.3). We claim g is open: let a be open in X/\sim ($q^{-1}(a)$ is open in X), then $g(a) = f(q^{-1}(a))$ is open in the topology induced by f on Y (since $q^{-1}(a)$ is open in X), so g is open. Then open, continuous, and bijective implies homeomorphism, and we are done.
- (c) Identify the endpoints $\{0, 1\}$ in the interval $[0, 1]$: then the quotient space $[0, 1]/\{0, 1\}$ is homeomorphic to S^1 by defining the map $f: [0, 1] \rightarrow S^1$ as

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Then this identifies the endpoints $\{0, 1\}$ together, and points on the interval to points on the unit circle.

- (d) A cooler example (from Munkres): Identify the corners of the edges of the box $X = [0, 1] \times [0, 1]$ by partitioning X into the singletons $\{(x, y)\}$ where $0 < x < 1, 0 < y < 1$, the two point sets $\{(x, 0), (x, 1)\}$ where $0 < x < 1$ and $\{(0, y), (1, y)\}$ where $0 < y < 1$, and the four point set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Then X/\sim is homeomorphic to the torus.

■

§1.2 Problem 1 Chapter 0

Problem. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. Informal idea: take the hole in the torus and stretch it all the way to the boundaries. So the torus becomes two circles in parallel (in three dimensional space) connected by a point, “flatten” the two circles to obtain two circles intersecting at a point.

Formal idea: As seen earlier, we can glue the borders of a unit square to obtain a torus. Take $I = [-1, 1]$, then if we can show that $I^2 \setminus \{0\}$ retracts to ∂I^2 we are done (since ∂I^2 glued together is two circles). Since $I^2 \setminus \{0\}$ is convex, we can define $f: I^2 \setminus \{0\} \rightarrow S^1$ as the unit length $\frac{x}{|x|}$ of any ray from the origin to x , which is a retraction onto S^1 . Restricting f to the boundary ∂I^2 then taking its inverse yields a map $g: S^1 \rightarrow \partial I^2$ from the circle to the boundary of the square: then the composition $g^{-1} \circ f$ is a retraction from the entire square to the circle then to the boundary. Define the homotopy $F: I^2 \setminus \{0\} \times [0, 1] \rightarrow \partial I^2$ as

$$F(x, t) = x(1 - t) + t(g^{-1} \circ f),$$

the desired homotopy from the torus (glued square) onto the two circles connected by a point (∂I^2). ■

§1.3 Problem 3a

Problem. Show that the composition of homotopy equivalences $X \rightarrow Y$ and $Y \rightarrow Z$ is a homotopy equivalence $X \rightarrow Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution. dkjfsklkj ■

§1.4 Problem 3b

Problem. Show that the relation of homotopy among maps $X \rightarrow Y$ is an equivalence relation.

§1.5 Problem 17a

Problem. Show that the mapping cylinder of every map $f: S^1 \rightarrow S^1$ is a CW complex.

§1.6 Problem 17b

Problem. Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

§1.7 Problem 3 Section 1.1

Problem. For a path-connected space X , show that $\pi(X)$ is abelian if and only if all basepoint-change homeomorphisms β_h depend only on the endpoints of the path h .

§1.8 Problem 6

Problem. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \rightarrow (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \rightarrow X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \rightarrow [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

§1.9 Problem 7

Problem. Define $f : S^1 \times I \rightarrow S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on both boundary circles. [Consider what f does to the map $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.

§1.10 Problem 17

Problem. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \rightarrow S^1$.