Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: https://git.simonxiang.xyz/math_notes/files.html

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§1 August 26, 2020: Homework 0

§1.1 Question 1

Problem. *Prove that the finite product of manifolds is a manifold.*

Proof. We prove $M \times N$ is a manifold, where M is an m-manifold and N is an n-manifold, which is a sufficient condition for the finite product

$$\prod_{i=1}^{n} M_i$$

to be a manifold for M_i a m_i-manifold, $m_i \in \mathbb{N}$. First, note that the product of two T_2 spaces is T_2. Take τ_1, τ_2 to be topological spaces, and let X be their product. We have two distinct points $(a, b), (c, d)$ in X, which we can separate by open sets $X_1 \times U_2$, $X_1 \times V_2 \in X$ for $X_1 \in \tau_1$, $U_2, V_2 \in \tau_2$ if $a = c$ (which implies $b \neq d$), and $U_1 \times X_2$, $V_1 \times X_2$ for $U_1, V_1 \in \tau_1$, $X_2 \in \tau_2$ if $a \neq c$.

Now let (a, b) be in X, where a is in τ_1 and b is in τ_2. Then there exist $U_1 \in \tau_1$, $U_2 \in \tau_2$ such that $a \in U_1$, $b \in U_2$, and U_1 homeomorphic to $\mathbb{R}^m$, U_2 homeomorphic to $\mathbb{R}^n$. Simply take the open set $U_1 \times U_2 \in X$ containing the point (a, b), and define the homeomorphism $f : M \times N \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) = (g(x), h(y))$, where g and h are the homeomorphisms of U_1 onto $\mathbb{R}^m$ and U_2 onto $\mathbb{R}^n$ respectively. Clearly f is continuous since g and h are continuous, and by the same logic f^{-1} exists and is continuous, and is given by $f^{-1}(x, y) = (g^{-1}(x), h^{-1}(y))$ (whose components are continuous since g and h are homeomorphisms).

Finally, we have $\mathbb{R}^m \times \mathbb{R}^n$ homeomorphic to $\mathbb{R}^{m+n} = \mathbb{R}^{n+m}$, so we conclude the product manifold $M \times N$ is indeed an $n + m$-manifold. ⊠

§1.2 Question 2

Problem. *Prove that a manifold is connected if and only if it is path-connected.*

Proof. First, note that every path-connected space is connected. By way of contradiction, assume that a path-connected topological space (X, τ) is not connected, that is, there exist $U, V \in \tau$ such that $U \cap V = \emptyset$ and $U \cup V = X$.

Recall that an *interval* is a set $I \subset \mathbb{R}$ such that for all $a, b \in I$, $a < x < b$ implies $x \in I$. Furthermore, all intervals are connected (we omit the proof for brevity). Since τ is path-connected, for all $x, y \in X$ there exists a path $f : [a, b] \to X$ such that f is continuous and $f(a) = x$ and $f(b) = y$. Now the image of the path denoted $f([a, b])$ is connected, since the image of a connected set under a continuous function is connected. Choose $x \in U$ and $y \in V$: then the path f cannot connect x and y since $U \cap V = \emptyset$, and $f([a, b])$ must either be fully contained in U or V. Therefore path-connected spaces (and manifolds) are connected, proving the reverse implication.

For the forward implication, let $a \in M$. Consider X, the set of points that are path-connected to a. Note that $a \in X$ so $X \neq \emptyset$ (this is important). We claim X and X^c are open: to see this, let $x \in X$. Then we have an open neighborhood of x homeomorphic to $\mathbb{R}^n$, let us denote its image under the homeomorphism f as $U \subset \mathbb{R}^n$. We can find a convex neighborhood of $f(x)$ denoted $B(f(x), \epsilon) \subset U$ that is path-connected by definition. Since path-connectedness is preserved under a continuous map, the inverse image of the convex neighborhood containing $f(x)$ under the homeomorphism f denoted $f^{-1}(B(f(x), \epsilon))$ is path-connected. Note that $x \in f^{-1}(B(f(x), \epsilon))$, so there exists a path between every point in $f^{-1}(B(f(x), \epsilon))$ and x, therefore $x \in f^{-1}(B(f(x), \epsilon)) \subset X$ and is open. Since for all $x \in X$ we have $x \in X^\circ$, we conclude X is open. A similar argument follows for the fact that X^c is open: simply examine $y \in X^c$ and $B(f(y), \delta)$ instead.

We reach the final stage of this proof. By assumption, our manifold M is connected. This is equivalent to the fact that the only subsets of M that are both open and closed are M and $\emptyset$: if there existed an $A \subset M$ that were both open and closed, then $A \cap A^c = \emptyset$ and $A \cup A^c = M$, contradicting the fact that M is connected. Now we have constructed a path-connected set X that is both open and closed— both X and X^c are open, and $X \neq \emptyset$ as stated earlier in the proof. We conclude that $X = M$, and so the manifold M is path-connected. ⊠

§1.3 Question 3

Problem. *Suppose a finite group G acts on a manifold M. Suppose the action is free, meaning that only the identity element has any fixed points. Then the orbit space M/G is also a manifold. ("Lying in the same G-orbit" is an equivalence relation on M. M/G means the set of equivalence classes. The topology on M induces one on M/G, which is the one you must work with.)*

Proof. Why helpppppp

G acts on M: a map $* : G \times M \to M \ni ex = x \forall x \in M$, $*(g1, g2)x = *(g1 * (g2, x)) \forall g_1, g_2 \in G$, $x \in M$ or alternatively $(g_1 g_2)x = g_1(g_2 x) \forall x \in M$, $g_1, g_2 \in G$. M is a G-set.

Free action: $g \in G \wedge \exists x \in X \ni gx = x \implies g = e$.

Orbit of $x \in X$: $Gx = \{gx \mid g \in G\}$ for some $x \in X$. x $\sim$ y iff $\exists g \in G \ni gx = y \implies$ orbits are equivalence classes under this relation.

Orbit space: set of all equivalence classes (under the same orbit relation), denoted X/G (also called quotient, space of coinvariants).

Induces a topology: Recall the quotient map induces the quotient topology, e.g. $U \in G/M$ open iff $f^{-1}(U) \in G$ open.

Things to consider: G is finite, M is a manifold (locally Euclidian), the action is *free*, WTS: M/G is locally euclidian, T2, etc.

Let G be a finite group acting on a manifold M. Consider the quotient map $p : M \to M/G$, where p maps elements of M to their respective G-orbit in G/M. Then p induces the desired topology on the quotient space M/G (that is, sets of orbits are open if their union is open in M).

First, we show G/M is T_2. Since the action of G is free, we have $\boxtimes$

§2 August 29, 2020: Homework 1

Hatcher Chapter 0 (p. 18): 1, 3ab, 17,
Hatcher Section 1.1 (p. 38): 3, 6, 7, 16.

§2.1 Question 1

Problem 1. *Suppose X, Y are compact Hausdorff spaces and $f: X \to Y$ is continuous and onto. Define $\sim$ as the equivalence relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.*

(a) Prove the quotient space $X/\sim$ is Hausdorff.

(b) Use this to show that the induced map $X/\sim \to Y$ is a homeomorphism.

(c) Show that identifying the ends of the interval gives S^1.

(d) Give a cooler example.

Solution. We examine the structure of topologies generated by identifying points together who lie in the same "class" after a map.

(a) Let $[a], [b]$ be elements (equivalence classes) of the quotient space $X/\sim$. We want to separate $[a]$ and $[b]$ by open sets: since singletons are closed, we can separate $q^{-1}[a]$ and $q^{-1}[b]$ by open sets in X (where $q: X \to X/\sim$ is the canonical quotient map), due to the fact that the quotient map is continuous (and so the inverse image of closed sets are closed) and Hausdorff plus compact implies normal. Then their images are disjoint and open in $X/\sim$ as well, since the quotient map is open.

(b) We claim that f induces a map $g: X/\sim \to Y$ such that $g \circ q = f$. Then g is bijective (the equivalence classes all identify to the same point in Y), and continuous (by a theorem in Munkres, Corollary 22.3).We claim g is open: let a be open in $X/\sim$ ($q^{-1}(a)$ is open in X), then $g(a) = f(q^{-1}(a))$ is open in the topology induced by f on Y (since $q^{-1}(a)$ is open in X), so g is open. Then open, continuous, and bijective implies homeomorphism, and we are done.

(c) Identify the endpoints $\{0, 1\}$ in the interval $[0, 1]$: then the quotient space $[0, 1]/\{0, 1\}$ is homeomorphic to S^1 by defining the map $f: [0, 1] \to S^1$ as

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Then this identifies the endpoints $\{0, 1\}$ together, and points on the interval to points on the unit circle.

(d) A cooler example (from Munkres): Identify the corners of the edges of the box $X = [0, 1] \times [0, 1]$ by partitioning X into the singletons $\{(x, y)\}$ where $0 < x < 1$, $0 < y < 1$, the two point sets $\{(x, 0), (x, 1)\}$ where $0 < x < 1$ and $\{(0, y), (1, y)\}$ where $0 < y < 1$, and the four point set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Then $X/\sim$ is homeomorphic to the torus.

■

§2.2 Problem 1 Chapter 0

Problem. *Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.*

Solution. Informal idea: take the hole in the torus and stretch it all the way to the boundaries. So the torus becomes two cirles in parallel (in three dimensional space) connected by a point, "flatten" the two circles to obtian two circles intersecting at a point.

Formal idea: As seen earlier, we can glue the borders of a unit square to obtain a torus. Take $I = [-1, 1]$, then if we can show that $I^2 \setminus \{0\}$ retracts to ∂I^2 we are done (since ∂I^2 glued together is two circles). Since $I^2 \setminus \{0\}$ is convex, we can define $f: I^2 \setminus \{0\} \to S^1$ as the unit length $\frac{x}{|x|}$ of any ray from the origin to x, which is a retraction onto S^1. Restricting f to the boundary ∂I^2 then taking its inverse yields a map $g: S^1 \to \partial I^2$ from the circle to the boundary of the square: then the composition $g^{-1} \circ f$ is a retraction from the entire square to the circle then to the boundary. Define the homotopy $F: I^2 \setminus \{0\} \times [0, 1] \to \partial I^2$ as

$$F(x, t) = x(1 - t) + t(g^{-1} \circ f),$$

the desired homotopy from the torus (glued square) onto the two circles connected by a point (∂I^2). ■

§2.3 Problem 3a

Problem. *Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.*

Solution. Recall that two spaces X and Y are homotopy equivalent if there exists a pair of continuous maps $f\colon X \to Y$, $g\colon Y \to X$ such that $gf = \iota_X$, $fg = \iota_Y$. Clearly the relation is reflexive (consider $f\colon X \to X$ and f^{-1}) and symmetric (use the same pair of maps f and g to show that Y and X are homotopy equivalence). We show the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$, fulfilling the transitive requirement. Let $f_1\colon X \to Y$, $g_1\colon Y \to X$ be the maps on X, Y, and $f_2\colon Y \to Z$, $g_2\colon Z \to Y$ the maps on Y, Z. Let $f_3\colon X \to Z$ (and $g_2\colon Z \to X$) be defined by $f_3 = f_2 \circ f_1$ ($g_3 = g_1 \circ g_2$). Then $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1 = g_1 \circ \iota_Y \circ f_1 = g_1 \circ f_1 = \iota_X$. Similarly, $f_3 \circ g_3 = f_2 \circ f_1 \circ g_1 \circ g_2 = f_2 \circ \iota_Y \circ g_2 = f_2 \circ g_2 = \iota_Z$, and we are done. ■

§2.4 Problem 3b

Problem. *Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.*

Solution. Clearly a map $f\colon X \to Y$ is homotopic to itself (take a constant homotopy $f_t = f$ for all t). If $f \simeq g$, then define a homotopy g_t from g to f as $g_t = f_{1-t}$ where f_t denotes the original homotopy. Then $g_0 = g$ and $g_1 = f$ (connecting g and f) so g_t is a homotopy between g and f. Finally, let maps f and g be homotopic, along with maps g and h. We want to find a homotopy from f to h: If f_t is the homotopy connecting f and g and g_t is the homotopy connecting g and h, define the homotopy h_t on the interval as

$$h_t = \begin{cases} f_{2t} & \text{if } t \in [0, 0.5], \\ g_{2t-1} & \text{if } t \in [0.5, 1]. \end{cases}$$

The homotopy agrees with itself at $t = 0.5$ since $f_{2 \cdot 0.5} = f_1 = g$, and $g_{2 \cdot 0.5 - 1} = g_0 = g$. Furthermore, $h_0 = f_0 = f$, and $h_1 = g_1 = h$, so h is a homotopy between f and h, and we are done. ■

§2.5 Problem 17a

Problem. *Show that the mapping cylinder of every map $f\colon S^1 \to S^1$ is a CW complex.*

Solution. This is the generalization of the idea that if we have a cellular map between two CW complexes, then the mapping cylinder is also a CW complex. To do this, note the product of two CW complexes is also a CW complex: so $S^1 \times I$ is a CW complex. Then we define cells in the mapping cylinder as the cells of $S^1 \times I$ and S^1. ■

§2.6 Problem 17b

Problem. *Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.*

Solution. Both the annulus and the Möbius band retract to S^1. So construct a 2-dimensional CW complex by pasting points to S^1 to the 2-cells $S^1 \times I$ and the Möbius band, then identifying the annulus and the Möbius band by S^1. So this CW has both the annulus and the Möbius band as deformation retracts. ■

§2.7 Problem 3 Section 1.1

Problem. *For a path-connected space X, show that $\pi(X)$ is abelian if and only if all basepoint-change homeomorphisms β_h depend only on the endpoints of the path h.*

Solution. (Not attempted). ■

§2.8 Problem 6

Problem. *We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map Φ : $\pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.*

Solution. (Not attempted). ■

§2.9 Problem 7

Problem. *Define $f: S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on both boundary circles. [Consider what f does to the map $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$].*

Solution. (didn't finish). ■

§2.10 Problem 16

Problem. *Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.*

Solution. (didn't finish) Take the family of retractions that map the first circle to itself (identity) and wrap the second circle around the first n times, then if $n \in \mathbb{N}$ this is an infinite family of retractions. (Not sure how to formalize this or show they're nonhomotopic). ■

§3 September 5, 2020: Homework 2

Hatcher Chapter 0 (p. 18): 9, 20,
Hatcher Section 1.1 (p. 38): 17, 18, 20,
Hatcher Section 1.2 (p. 52): 2, 4.

§3.1 Problem 1

Problem. *An n-dimensional* manifold with boundary *means a Hausdorff space M, such that every $x \in M$ has a neighborhood U such that the pair (U, x) is homeomorphic to either $(\mathbb{R}^n, 0)$ or $(\mathbb{R}^{n-1} \times [0, \infty), 0)$, where in both cases 0 means $(0, \ldots, 0)$. We call x an interior or boundary point according to which of these holds. Note that this is* not *the usual use of "interior" and "boundary" from point-set topology. The set of boundary points is written ∂M.*

Assume that ∂M is compact. Prove that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence.

You may use without proof the fact that no point can be both an interior and a boundary point. Also, the 2-dimensional case is enough to give a complete understanding. Finally, a hint: chain together a sequence of homotopies, being careful that the result makes sense and is continuous.

Remarks: informally, I think of $M \setminus \partial M$ as a sort of deformation-retract of M. But it is easy to see that if $\partial M \neq \emptyset$ then M does not actually deformation retract to $M \setminus \partial M$. Also, without the extra hypotheses, the only solution I know uses something you probably have not seen: topological dimension, which lets you build an open cover with good overlap properties.

Solution. (Thank you for changing the problem to assume ∂M is compact!) We want to show the inclusion $M \setminus \partial M \hookrightarrow M$ is a homotopy equivalence. Let $\mathcal{G}_\alpha$ be an open cover of ∂M. Then we can reduce $\mathcal{G}_\alpha$ to a finite subcover $\bigcup_{i=1}^n G_i$ of ∂M. Consider the inclusion $M \setminus \partial M \hookrightarrow (M \setminus \partial M) \cup G_1$, we'll denote this as ι. We'll also denote the set $(M \setminus \partial M) \cup G_1$ as MG_1 from now on (sorry for the confusing notation, but I couldn't think of anything better). We can find a quotient map

$$f \colon MG_1 \to MG_1/(MG_1 \cap \partial M)$$

by identifying all $x \in MG_1 \cap \partial M$ with an $x_0 \in M \setminus \partial M \subseteq MG_1$ (which exists since G_1 is part of a cover of ∂M). Then the quotient space is homeomorphic to $M \setminus \partial M$. Furthermore, the homotopies $H_1 \colon M \setminus \partial M \times I \to M \setminus \partial M$, $H_2 \colon MG_1 \times I \to MG_1$ given by $H_1(x, t) = (1 - t)(f \circ \iota)(x) + tx$, $H_2(x, t) = (1 - t)(\iota \circ f)(x) + tx$ satisfy the conditions for homotopy equialence, that is,

$$\begin{aligned} H_1(x, 0) &= (f \circ \iota)(x), & H_2(x, 0) &= (\iota \circ f)(x), \\ H_1(x, 1) &= \mathrm{id}_{M \setminus \partial M}, & H_2(x, 1) &= \mathrm{id}_{MG_1}. \end{aligned}$$

So the inclusion $M \setminus \partial M \hookrightarrow MG_1$ is a homotopy equivalence.

Now we generalize this to show MG_i has the same homotopy type as $MG_i \cup G_{i+1}$. Consider the inclusion $\iota \colon MG_i \hookrightarrow MG_i \cup G_{i+1}$. We find a quotient map

$$f \colon MG_i \cup MG_{i+1} \to MG_i \cup MG_{i+1}/\left((MG_i \cup MG_{i+1}) \setminus MG_i\right)$$

by identifying all $x \in MG_i \cup G_{i+1}$ with an $x_0 \in (MG_i \cup MG_{i+1}) \cap \partial M$ (which exists since the G_i are part of a cover of ∂M). Then the quotient space

$$MG_i \cup MG_{i+1}/\left((MG_i \cup MG_{i+1}) \setminus MG_i\right)$$

is homeomorphic to MG_i. Furthermore, the homotopies $H_1 \colon MG_i \times I \to MG_i$, $H_2 \colon MG_i \cup MG_{i+1} \times I \to MG_i \cup MG_{i+1}$ given by $H_1(x, t) = (1 - t)(f \circ \iota)(x) + tx$, $H_2(x, t) = (1 - t)(\iota \circ f)(x) + tx$ satisfy the conditions for homotopy equialence, that is,

$$\begin{aligned} H_1(x, 0) &= (f \circ \iota)(x), & H_2(x, 0) &= (\iota \circ f)(x), \\ H_1(x, 1) &= \mathrm{id}_{MG_i}, & H_2(x, 1) &= \mathrm{id}_{MG_i \cup MG_{i+1}}. \end{aligned}$$

So the subsequent inclusion maps $MG_i \hookrightarrow MG_i \cup MG_{i+1}$ are all inclusion maps, and preserve homotopy type. Let $i = 1$, then $M \setminus \partial M$ has the same homotopy type as $\bigcup_{i=1}^n (M \setminus \partial M) \cup G_i$, but all the G_i form an open cover of ∂M. So

$$\bigcup_{i=1}^n G_i = \partial M \implies \bigcup_{i=1}^n (M \setminus \partial M) \cup G_i = M \setminus \partial M \cup \partial M = M,$$

and we are done. ■

§3.2 Problem 2

Problem (A "bad" group action). *Let $X = \mathbb{R}^2 \setminus \{0\}$ where 0 is the origin. Let G be the group of homeomorphisms of X generated by the transformation $(x, y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.*

(a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).

(b) Prove that G's action on X satisfies the hypothesis of the theorem from class about $\pi_1(X/G) \cong G$, namely: every $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for every $g \in G \setminus \{1\}$.

(c) Prove that Y is a manifold, except for the fact that it is not *Hausdorff.*

(When working on a theorem involving a group action, if I wonder whether some hypothesis can be omitted, checking it for this single example usually reveals the answer.)

Solution. (The condition for an orbit to be discrete comes from the Wikipedia page for a discrete group.)

(a) Let $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, $G(x, y)$ be the orbit of (x, y). We want to show that the singleton containing the identity $\{(x, y)\}$ is open, a sufficient condition for the orbit to be discrete. We know the next two subsets of the orbit "closest" to $\{(x, y)\}$ are the singletons $\{(2x, y/2)\}$ and $\{(x/2, 2y)\}$, generated by the given homeomorphism and its inverse. Let $\varepsilon = 1/4 \min\{x, y\}$. So take an open set $B((x, y), \varepsilon)$ around (x, y): this doesn't intersect the other two sets, and
$$B((x, y), \varepsilon) \cap \{(x, y)\} = \{(x, y)\}.$$
Therefore $\{(x, y)\}$ is open, and orbits in this group action are discrete.

(b) The neighborhood $B((x, y), \varepsilon)$ from the previous part does the trick: the minimum possible distance from one singleton subset to another is $\min\{x, y\}/2$, so taking $\varepsilon = 1/4 \min\{x, y\}$ ensures that two open $g(B((x, y), \varepsilon))$'s won't intersect (given $g \in G, g \neq 1$). This fulfills the condition from class.

(c) (Not entirely sure about this one): We want to show that Y is a manifold, but *not* Hausdorff. G acts freely (fixing only identity), smoothly, and properly (not sure how to show these two conditions, but I *think* they're true), so since $\mathbb{R}^2 \setminus \{0\}$ is a manifold, the quotient space Y is a manifold. Forgot how to formalize this, but to show Y is not Hausdorff, take the orbits of a rational and irrational "right next to each other" (by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$). The rational orbit contains entirely rational points but the irrational orbit contains entirely irrational points (since rational times irrational is irrational) so these two orbits are distinct. Then any open set containing the rational orbit (and therefore base point) must also contain the irrational base point by the definition of open sets in $\mathbb{R}$, so we have found two points that can't be separated by open sets, and we are done.

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§3.3 Problem 9 Chapter 0

Problem. *Show that a retract of a contractible space is contractible.*

Solution. Let A be a retract of a contractible space X. Then there exists a homotopy of X onto a point, and a retract of X onto A: denote this retract with f, and the homotopy encoding the contraction as $H : X \times I \to X$, where $f|_A = \text{id}\, A$, $H(x, 0) = \text{id}_X$, $H(x, 1) = \{x_0\}$. Consider the homotopy $H' = f \circ H|_A$ from $A \times I \to A$. Then this homotopy is continuous since f and H are continuous and shows that id_A is nullhomotopic since $H'(x, 0) = \text{id}_A$, $H'(x, 1) = \{x_0\}$. Therefore A is contractible. ■

§3.4 Problem 20

Problem. *Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.*

Solution. We can contract the intersecting disk of the Klein bottle to itself, so the resulting structure resembles S^2/S^0 (the sphere with two points identified), which is homotopy equivalent to $S^1 \vee S^2$ by Example 0.8. Counting the boundary of the intersecting disk itself, this forms another S^1 identified with the rest of the bottle (now $S^1 \vee S^2$) at a point. Therefore the self-intersecting Klein bottle in $\mathbb{R}^3$ has the homotopy type of $S^1 \vee S^1 \vee S^2$. ■

§3.5 Problem 17 Section 1.1

Problem. *Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$ (whoops, attempted this one last week).*

Solution. Informal idea: Take the first S^1 and twist it like a pretzel n times. Then fold these loops onto the second S^1, a retraction. This is an infinite family of nonhomotopic retractions.

Formal idea: We can express $S^1 \vee S^1$ as a unit circles centered at $(-2, 0)$ and the origin (the left and right unit circles, respectively) wedged together at the point $(-1, 0)$. Then we have an infinite family of retractions $R = \{r_n \mid n \in \mathbb{Z}\}$, where each r_n is defined as
$$r_n \colon (\cos\theta - 2, \sin\theta) \mapsto (e^{ni\theta}).$$

(Is there a way to write the left side in terms of e? My algebra is lacking). These retractions are the identity on the right circle and are non-homotopic because they correspond to different elements of the fundamental group $\pi_1(S^1) = \mathbb{Z}$. ■

§3.6 Problem 18

Problem. *Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:*

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

Solution. Let X be a space obtained from a path connected subspace A by attaching a cell e^n with $n \geq 2$. We know $\pi_1(e^n)$ is trivial (Proposition 1.14). Let $e^n = A_\alpha$ and $A = A_\beta$ in Lemma 1.15. Then $X = A_\alpha \cup A_\beta$. We also have $A_\alpha \cap A_\beta = e^n \cap A = \partial e^n$ which is path connected— let $x_0 \in \partial e^n$. Applying Lemma 1.15, every loop in X at x_0 is homotopic to a product of loops, but since $\pi_1(e^n)$ is trivial every loop in e^n is nullhomotopic. Therefore every loop in X is homotopic to a loop in A, and so the inclusion map induces a surjection on $\pi_1(X)$, and we are done.

(a) We want to show the wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$. We attach the 1-cell S^1 to the 2-cell S^2: then by the thing we proved earlier, we have a surjection $\pi_1(S^1) \to \pi_1(S^1 \vee S^2)$ induced by the inclusion map $S^1 \hookrightarrow S^1 \vee S^2$ (let's denote this map ι. Also note that S^1 is path-connected). We use a nice theorem from group theory: apply the Fundamental Homomorphism Theorem to get
$$\pi_1(S^1 \vee S^2) \simeq \pi_1(S^1)/\ker \iota.$$
Since $\pi_1(S^1) = \mathbb{Z}$, we have $\pi_1(S^1 \vee S^2)$ isomorphic to a quotient group of Z. Furthermore, $\pi_1(S^1 \vee S^2)$ is infinite because each loop winds around n times for n an integer (see problem 17), so $\pi_1(S^1 \vee S^2) \simeq \mathbb{Z}$.

(b) X is obtained from the 1-skeleton X^1 (which is path-connected since X is path-connected) by attaching the e^n cells (for $n \geq 2$). Then it follows from the thing above that the inclusion $X^1 \hookrightarrow X$ induces a surjection $\pi_1(X^1) \to \pi_1(X^2)$. Each subsequent inclusion induces another surjection $\pi_1(X^i) \to \pi_1(X^{i+1})$, so chaining the inclusions together yields a surjection from $\pi_1(X^1) \to \pi_1(X)$.

■

§3.7 Problem 20

Problem. *Suppose $f_t \colon X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]*

Solution. Let $f_t(x_0) = h$. We have f_0, f_1 equal to id_X, so f_{0*}, f_{1*} are also the identity map. By Lemma 1.19, composing $\varphi_{1*} = f_{1*}$ (the identity) and β_h gives $\varphi_{0*} = f_{0*}$ (also the identity), which implies β_h is also the identity map. But β_h is the change-of-basepoint map, defined as $\beta_h[f] = [hf\bar{h}]$. So if β_h is the identity,
$$\beta_h[f] = f = [hf\bar{h}] \implies [fh] = [hf]$$
for all f. Recall the center of the fundental group is defined as
$$Z[\pi_1(X, x_0)] = \{[h] \in \pi_1(X, x_0) \mid \forall [f] \in \pi_1(X, x_0), [hf] = [fh]\}.$$
Then $h = f_t(x_0)$ is in the center of $\pi_1(X, x_0)$. ■

§3.8 Problem 2 Section 1.2

Problem. *Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.*

Solution. Intuition: A bunch of simply connected open sets linked together, although some two maybe not appear to be connected, the third will connected with another "structure", all the way down. So you can draw paths through the whole thing/retract it all to a point.

Formal idea: We show that $\pi_1(X)$ is trivial by Van Kampen's theorem, proving that X is simply-connected (since X is path-connected). We want to find a basepoint to use for Van Kampen's, then the fact that there is a basepoint in all the X_n's ($\bigcap_{i=n} X_i \neq \emptyset$) can be realized by a sort of "reverse" induction: If $\bigcap_{i=1}^{n-1} X_i \neq \emptyset$, then $\bigcap_{i=1}^{n} X_i = \emptyset$, this would contradict our hypothesis by choosing $i = n-2, j = n-1, k = n$ (since the intersect with X_k would be empty). We can repeat this process all the way down to three sets, which is nonempty by assumption. Apply Van Kampen's theorem to get
$$\pi_1(X) \simeq *_i \pi_1(X_i)/N.$$
But all the $\pi_1(X_i)$'s are trivial, so $\pi_1(X)$ must be trivial, and we are done.

■

§3.9 Problem 4

Problem. *Let $X \subseteq \mathbb{R}^3$ be the finite union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.*

Solution. Informal idea: Retract space minus a line to the square with two sides glued together, then compress this into a circle. So when $n = 1$, $\pi_1(\mathbb{R}^3 \setminus X) = \mathbb{Z}$. Now if we take two points, this deformation retracts onto S^1 minus four points (like cutting a cross through $\mathbb{R}^3$ given $n = 2$). The fundamental group of this spaces is the free group on $2n - 1$ generators.

(somewhat) Formal idea: We have $\pi_1(\mathbb{R}^3 \setminus X) = \mathbb{Z}$ (or the F_1, the free group on one generator) when $n = 1$. We can retract $\mathbb{R}^3 \setminus X$ onto S^1 minus $2n$ points for n the number of lines through the origin (let's call this set $S^1 - 2n$). We want to find $\pi_1(S^1 - 2n)$: then the problem will be solved, since $\mathbb{R}^3 \setminus X$ deformation retracts onto $S^1 - 2n$. (Not sure how to do this formally): Take one of the holes in the sphere and use it to "flatten out" the sphere onto the plane. Then $\pi_1(\text{new space}) = *\mathbb{Z}^{2n-1}$ or F_{2n-1}, the free group on $2n - 1$ generators since we took $2n$ holes and got ride of one of them to "flatten" the sphere onto the plane, resulting in $2n - 1$ holes (so $\pi_1(\text{new space}) = F_{2n-1}$). Therefore since $\mathbb{R}^3 \setminus X$ deformation retracts onto this new space, $\pi_1(\mathbb{R}^3 \setminus X) = F_{2n-1}$. ■

§4 September 14, 2020: Homework 3

Hatcher Section 1.2 (p. 52): 1, 10, 14, 16, 21,
Hatcher Section 1.3 (p. 79): 30,
Hatcher Section 1.A (p. 86): 5.

§4.1 Problem 1 Section 1.2

Problem. *Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.*

Solution. Assume the center of the free product $G * H$ of nontrivial groups is nontrivial, that is, there exists a $z \in Z(G * H)$ such that $zw = wz$ for all $w \in G * H$. WLOG, take a nontrivial reduced word $w \in G * H$ (we can do this because G and H are nontrivial) that ends in $h \in H$. If z ends in $g \in G$, we are done, since zw will end in h while wz will end in g, contradicting the fact that z lies in $Z(G * H)$. If z ends in h, then if the ending letter of w is of the form h^n while the ending letter of z is of the form h^m, zw will end in h^n while wz will end in h^{n+m}, implying that $zw \neq wz$, a contradiction.

Next, we'll show the only elements of $G * H$ of finite-order are the conjugates of finite-order elements of G and H. We'll do this proof by cases.

Case 1: WLOG, $w \in G * H$ starts with $g \in G$ and ends with $h \in H$. Clearly this doesn't terminate as $(g_i \cdots h_i) \cdot (g_i \cdots h_i)$ can't reduce down at $h_i \cdot g_i$, so this product will just keep growing longer with each multiplication.

Case 2: WLOG, $w \in G * H$ starts and ends with g, but the ending term isn't the inverse of the beginning term. That is, $w = g^n \cdots g^m$, but $n + m \neq 0$. So $w^2 = (g^n \cdots g^m) \cdot (g^n \cdots g^m) = g^n \cdots g^{n+m} \cdots g^m$. Since g^{n+m} can't simplify, none of the other terms can, so this product will just keep growing longer to infinity like the previous one.

Case 3: WLOG, $w \in G * H$ starts with g and ends with g^{-1}, but is not the conjugate of some element. In the previous cases, the possibility of w being a conjugate wasn't even there, but now we can consider it (in the next case). For now, assume it isn't: then $(g \cdots g^{-1}) \cdot (g \cdots g^{-1})$ will reduce to $g \cdots 1 \cdots g$, but the middle dots won't reduce because w isn't the conjugate of some element. (Note that w starting with g and ending with g^{-1} is just a special case of w being a conjugate of some element with the $\omega \in G * H$ set to g. This time, the WLOG also includes if w starts with g^{-1} and ends with g).

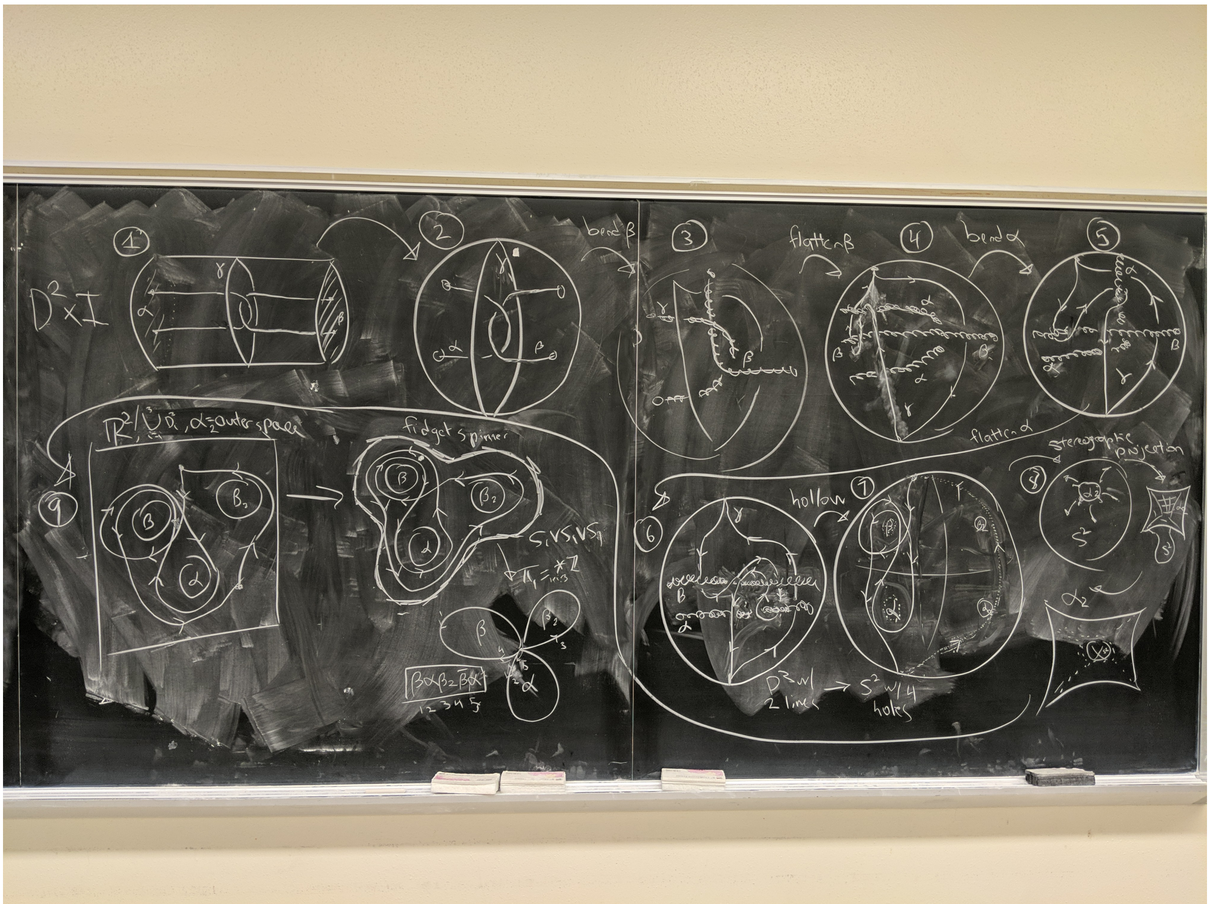
Case 4: $w \in G * H$ is the conjugate of some element of $G * H$. We'll show that the only element that allows w to terminate are elements of finite-order in G or H. We have $w = \omega a \omega^{-1}$ for $\omega, a \in G * H$. Each multiplication of w gives $w^2 = (\omega a \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^2 \omega^{-1})$, $w^2 = (\omega a^2 \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^3 \omega^{-1})$, and so on. Therefore $w^n = \omega a^n \omega^{-1}$. If a is a word (as in a is not just an element of G or H), it must be the conjugate of another word, which must be the conjugate of another word, and so on. We can't repeat this forever, so a can't be a word. Now assume a is an element of G or H. Then if a has infinite order in either of these groups, we have no n satisfying $a^n = 1$, which subsequently means $w = \omega a^n \omega^{-1}$ will never terminate. If a has finite order in G or H, then there exists an n such that $a^n = 1$. Then at that n, $w^n = \omega a^n \omega^{-1} = \omega \cdot 1 \cdot \omega^{-1} = 1$. Therefore w has order n if and only if w is the conjugate of some element of finite order in G or H, and we are done. ■

§4.2 Problem 10

Problem. *Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.*

Hint (from Dr. Allcock): this can be done directly with Van Kampen's, but it becomes easier if you manipulate $(D^2 \times I) \setminus (\alpha \cup \beta)$ first, being careful not to change the homotopy type, and carrying along the loop γ.

Solution. See the figure below: if the image is not clear/zoomed in enough, I'll be happy to email some more to you (don't want to clutter my document too much).



We start off with $D^2 \times I$ minus two "ropes", then we deform it into a sphere, and undo the knots to get a sphere with two lines missing, taking much care to keep track of where γ is. (We denote the ropes α and β with a bunch of swirls at this stage to differentiate it from the loop γ, and because the classroom had no colored chalk). Then, we hollow out the sphere $D^3 \setminus \{\text{two lines}\}$ to get S^2 with four holes, still keeping track of γ. Then we use a technique similar to the homework last week (lines through the origin) of taking one of the four holes and blowing it up to get the plane with three holes in it. We contract that to the fidget spinner, and denote the holes with β, β_2, and α. This has the homotopy type of $S^1 \vee S^1 \vee S^1$, and so π_1 of this space $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The loop γ corresponds to a loop around $\beta, \alpha, \beta_2, \overline{\beta}$, and α in that order. So $\pi_1(D^2 \times I \setminus \{\alpha \cap \beta\}) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, and since γ corresponds to the loop $\beta\alpha\beta_2\overline{\beta}\alpha \neq 1$, we have γ not nullhomotopic in this space. ■

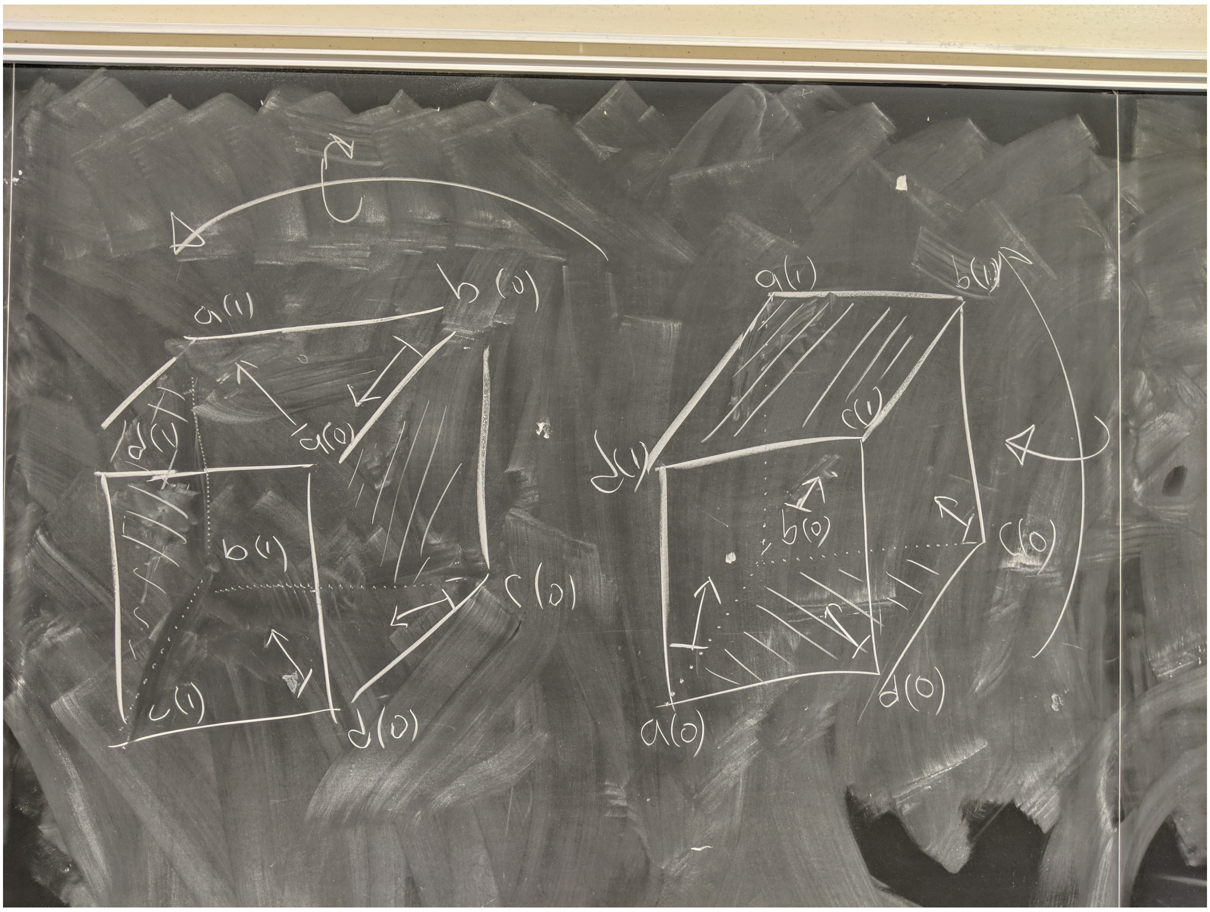
§4.3 Problem 14

Problem. *Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.*

Solution. Recall a group presentation for the quaternion group is given by

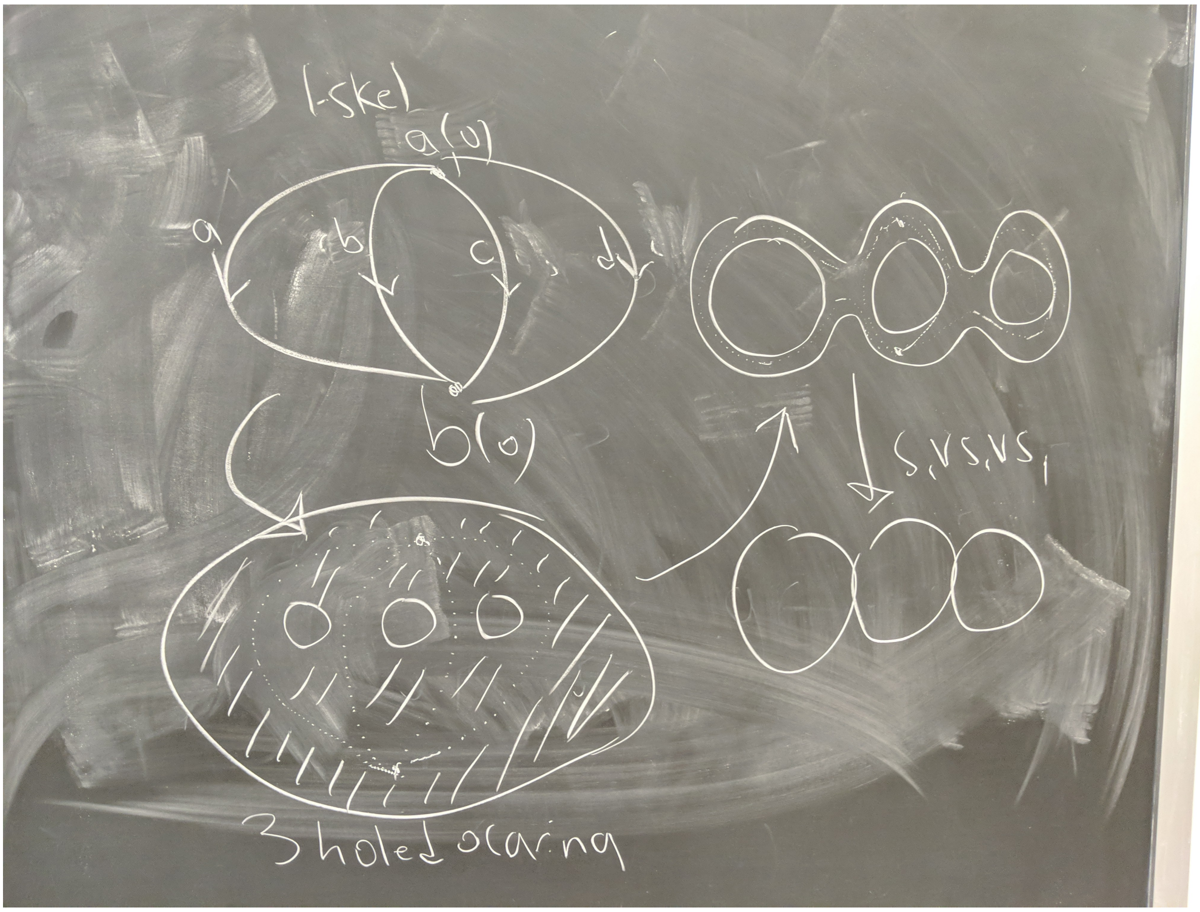
$$\langle -1, i, j, k \mid i^2 = j^2 = k^2 = -1, (-1)^2 = 1 \rangle.$$

Refer to the figure to see what's being identified to what: in the left side, I identified the right face with the left (0 and 1 just denoting what goes where, probably should have used subscripts) and in the right side of the figure I identified the bottom face with the top: together they describe the identification space of I^3.



The cube I^3 has the cell complex structure of eight 0-cells, twelve 1-cells, six 2-cells, and one 3-cell. Applying the first identification will identity eight 0-cells with four, and the second will identify four with two. Similarly, for 1-cells we identify twelve 1-cells onto eight (by mashing four together) and eight onto four (by mashing another four together). Finally, the identification will identify two 2-cells with each other, bringing us from four 2-cells to 3 2-cells. So $I^3/\sim$ has the desired cell complex structure.

We examine the 1-skeleton: See the figure to see why the resultant structure and $S^1 \vee S^1 \vee S^1$ are homotopy equivalent.



Therefore $\pi_1(X^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. We have the loops $a \cdot \overline{b}, a \cdot \overline{c}, a \cdot \overline{d}$. Let i, j, and k denote the homotopy classes of these loops (to make it feel more like the quaternions).

We have three 2-cells attached, being f_1, f_2, and f_3. It can be seen that $f_1 = [a\overline{b}c\overline{d}]$, $f_2 = [c\overline{b}d\overline{a}]$, $f_3 = [d\overline{b}\overline{a}c]$ by following the lines. We can perform some calculations to rewrite these in a form we like. We have

- $f_1 = [a\overline{b}c\overline{d}] = [a\overline{b}c\overline{a}a\overline{d}] = [ij^{-1}k]$.
- $f_2 = [c\overline{b}d\overline{a}] = [c\overline{a}a\overline{b}d\overline{a}] = [j^{-1}ik^{-1}]$.
- $f_3 = [d\overline{b}\overline{a}c] = [d\overline{a}a\overline{b}\overline{a}c] = k^{-1}ij$.

Then by Proposition 1.26, we have

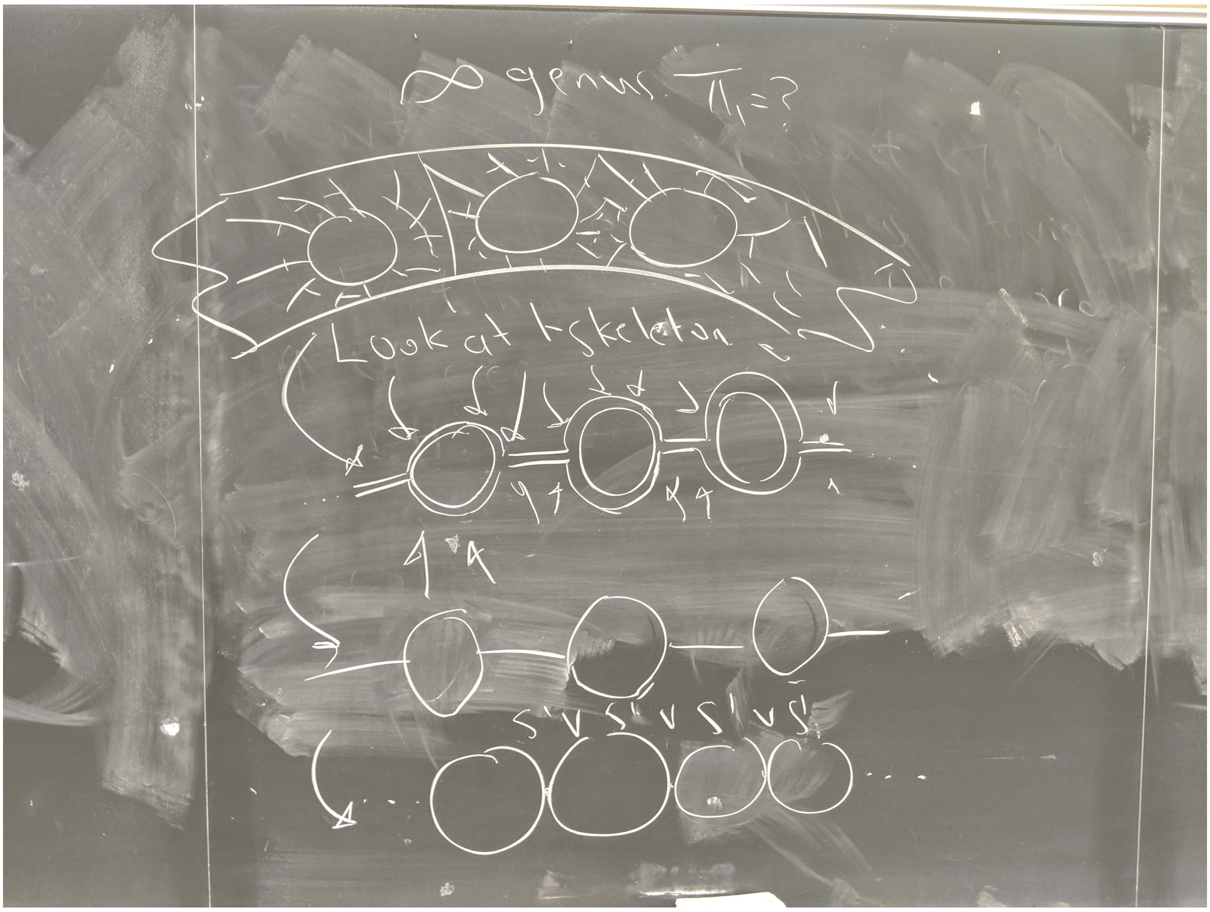
$$\pi_1(X) \simeq \pi_1(X_2) \simeq \langle i, j, k \mid ij^{-1}k = j^{-1}ik^{-1} = k^{-1}ij = 1 \rangle.$$

This is just the quaternion group in disguise: we have $ki = j, i = kj, ij = k$. So $i^2 = (ik)j = j^2, j^2 = k(ij) = k^2$. Therefore $i^2 = k^2 = j^2$, denote this element as -1. Finally, all we have to do is show that $(-1)^2 = 1$ (by the group presentation given at the beginning). We have $(-1)^2 = i^2j^2 = i(ij)j = (k^{-1}k)ikj = k^{-1}(ki)kj = k^{-1}j(kj) = k^{-1}(ji) = k^{-1}k = 1$, and we are done. ■

§4.4 Problem 16

Problem. *Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.*

Solution. I don't have a way to explicitly describe the deformation retraction (since I don't have a way to describe the surface), but this is an image I drew showing that the surface has the same homotopy type as an infinite wedge of S^1's, denoted $S^1 \vee S^1 \vee \cdots$.



Note that even though the circles are side by side, we can slide them together and glue at a single point. So by Hatcher Example 1.21, the fundamental group of this surface of infinite genus is just

$$\pi_1(\text{surface}) = \pi_1(S^1 \vee S^1 \vee \cdots) = *_\alpha \pi_1(S^1) = \mathbb{Z} * \mathbb{Z} * \cdots .$$

Then π_1 is just the free group on an infinite number of generators (one for each $\mathbb{Z}$), and we are done (although a free product may not always be free, a free product of free groups is, and $\mathbb{Z} \simeq F_1$). ■

§4.5 Problem 21

Problem. *Show that the join $X * Y$ of two nonempty spaces X and Y is simply-connected if X is path-connected.*

Hint (from Dr. Allcock): If you are not comfortable with the join of spaces then wrap your mind around the following examples in order:

1. *Join of two points*
2. *Join of a point and an interval*
3. *Join of a point and a circle*
4. *Join of 2 copies of the interval*
5. *Join of a circle and an interval*
6. *Join of two circles (doesn't embed in $\mathbb{R}^3$, but still understandable).*

That might be enough: if not, work out examples using the figure 8 *or S^2.*

Solution. First we show the join $X * Y$ of two spaces is path-connected. Let $(x_0, 0, y_0) \in X * Y = X \times I \times Y$ (I like putting I in the middle because it captures the idea of one space joining with the interval, and on the other side pops out the second space. The definition is from ncatlab). We have a path $\alpha\colon [0,1] \to X$, $\alpha(0) = x_0, \alpha(1) = x$ since X is path-connected. Then we have a path $f(\alpha(t), it, y)$ for $i \in I$ connecting any (x, i, y) to $(x_0, 0, y_0)$ (this works for y because at $t = 0$ just let $y = y_0$, then see what happens to y naturally and that will be a path, because of the construction of $X * Y$).

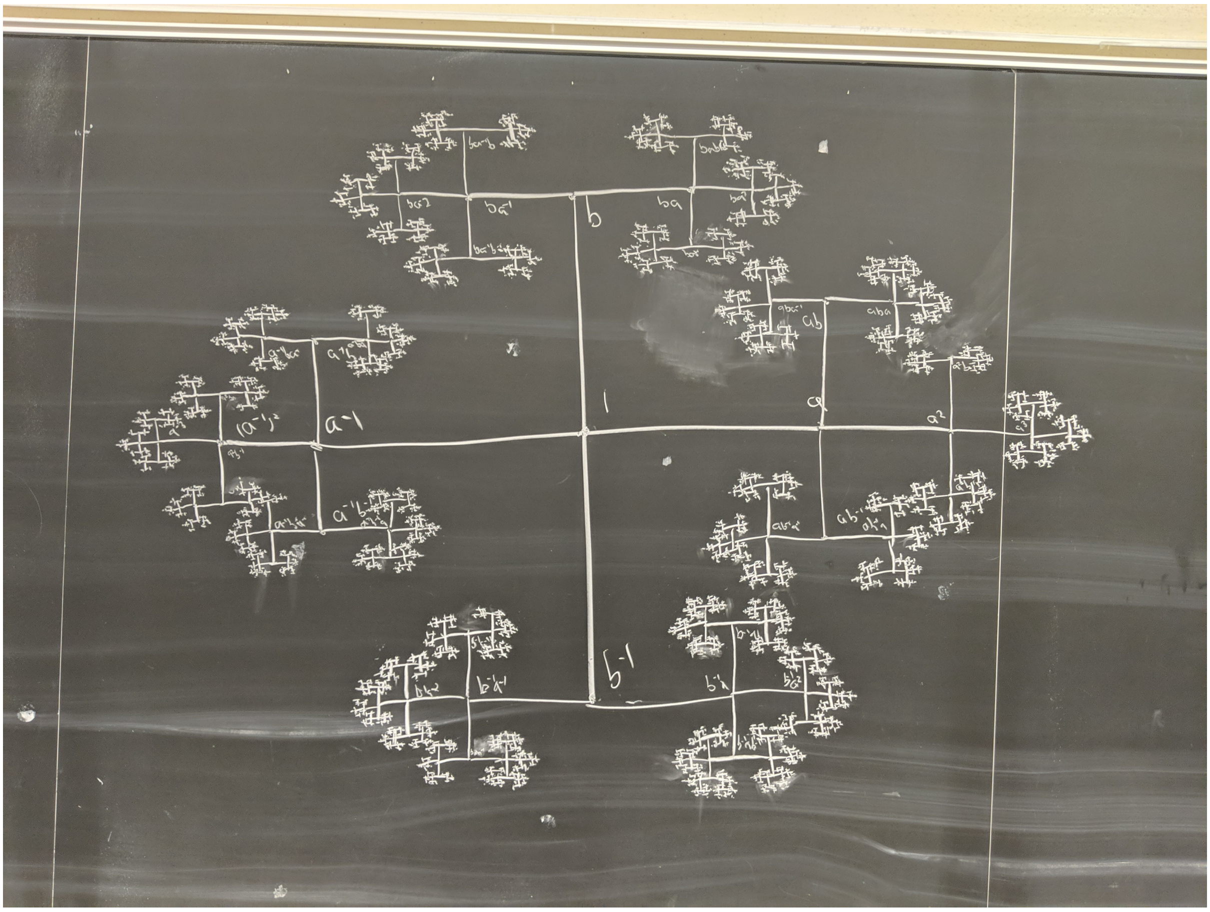
Assume Y is path-connected. Now we split up $X * Y$ into two spaces A and B, one where I becomes $[0, \frac{2}{3})$ and another where I becomes $(\frac{1}{3}, 0]$ (this is just a natural way to split the space in half with a nonempty intersection): these subsets are both open. Then A deformation retracts onto X, B onto Y, $A \cap B = X \times (\frac{1}{3}, \frac{2}{3}) \times Y$ onto $X \times Y$. Then since $X \times Y$ is path-connected by assumption, $\pi_1(A \cap B) \cong \pi_1(X) \times \pi_1(Y)$, $\pi_1(A) \cong \pi_1(X)$, $\pi_1(B) \cong \pi_1(Y)$. Then the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$ induce projections $\pi_1(X) \times \pi_1(Y) \to \pi_1(X)$ and $\pi_1(X) \times \pi_1(Y) \to \pi_1(Y)$, respectively. Then the normal subgroup described by van Kampens is simply the free product $\pi_1(X) * \pi_1(Y)$, since it's free on ab^{-1} for $a \in \pi_1(X)$, $b \in \pi_1(Y)$. So $\pi_1(X * Y) \simeq \pi_1(X) * \pi(Y)/N = \pi_1(X) * \pi_1(Y) \implies \pi_1(X * Y) = 1$. (Man, the conflicting notation for free product and topological join is getting confusing about now).

Now let Y be a union of path components Y_i. Let A be the portion of $X * Y$ from $[0, \frac{1}{3})$, and $C_\alpha = A \cap (X * Y_i)$ open sets covering $X * Y$. Each $\cap_\alpha C_\alpha$ is path-connected since the intersections just deform onto X path-connected, and so each C_α is simply-connected. Therefore $X * Y$ is simply-connected by van Kampen's. ∎

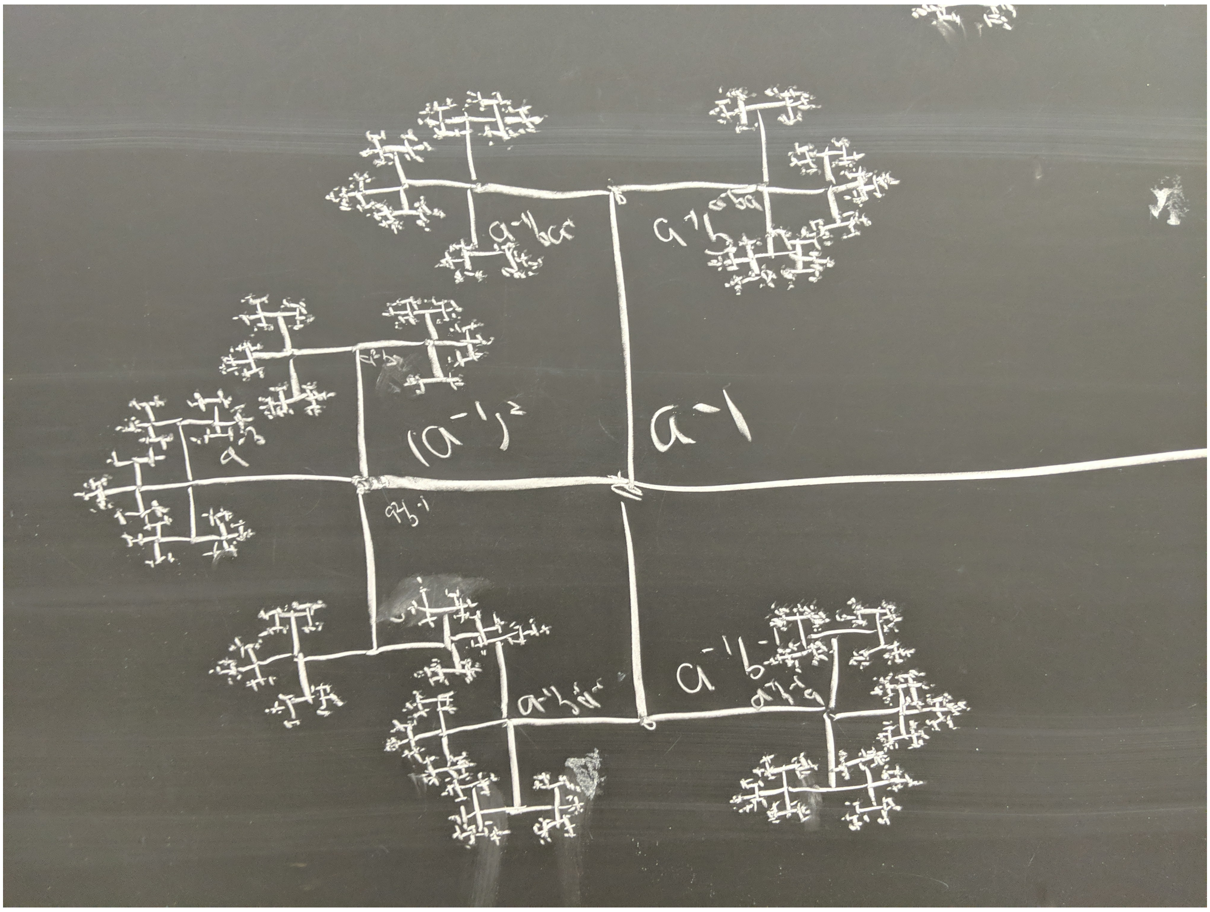
§4.6 Problem 30 Section 1.3

Problem. *Draw the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.*

Solution. Below is the Cayley graph: we start with 1 in the center, and multiplication by b, a, b^{-1}, a^{-1} is denoted by a line to the north, east, south, and west respectively. Only the first four iterations of points are labeled for brevity. It looks like a fractal/tree, similar to the Cayley graph of F_2, with the top half of every other iteration missing because $b^2 = (b^{-1})^2 = 1$.



Here's a zoomed in pictures of the most detailed node:

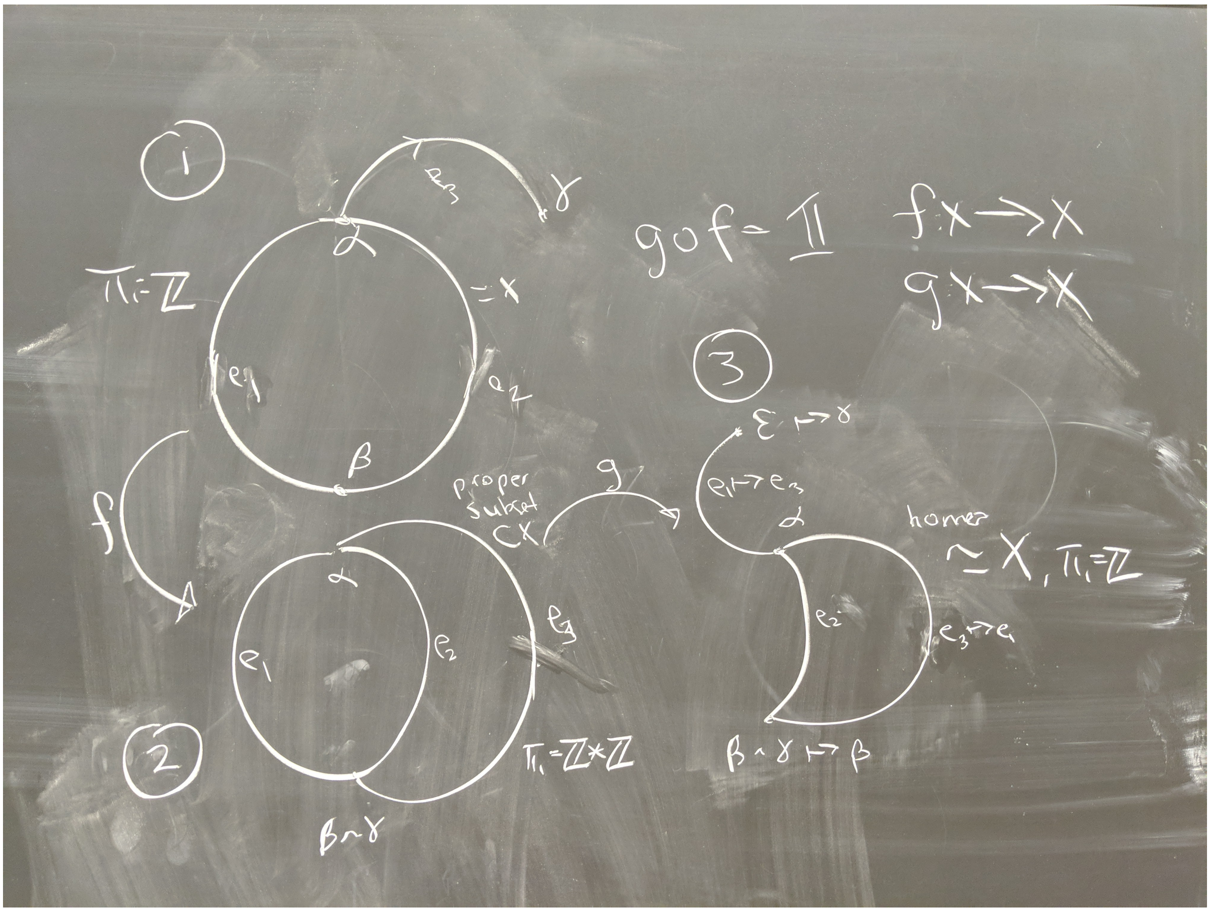


(This took a lot more time than I thought it would...) ■

§4.7 Problem 5 Section 1.A

Problem. *Construct a connected graph X and maps $f, g\colon X \to X$ such that $fg = \mathbb{1}$ but f and g do not induce isomorphisms on π_1. [Note that $f_* g_* = \mathbb{1}$ implies that f_* is surjective and g_* is injective.]*

Solution. We construct a graph X that looks like a cherry with three 0-cells α, β, γ and three 1-cells e_1, e_2, e_3 arranged as shown in the figure.



It is clear that X is connected, and $\pi_1(X) = \mathbb{Z}$. $f\colon X \to X$ identifies β with γ, so the stem of the cherry (e_3) becomes a new circle. Note that f is onto, but not 1-1. The image of f denoted $f[X]$ is a proper subset of X, and its fundamental group $\pi_1(f[X])$is equal to $\mathbb{Z} * \mathbb{Z}$. $g\colon X \to X$ takes e_1 and rips it out to make it the new stem. More precisely, g takes e_1 and identifies its endpoint with the top point, let's call it ε. g is 1-1, but not onto. Although the labels are now different, we can easily remap them to show that $g[f[X]]$ is homeomorphic to X. So $g \circ f = \mathbb{1}$, but f doesn't induce an isomorphism (since $\pi_1(f[X]) = \mathbb{Z} * \mathbb{Z}$) and neither does g (since $g[X]$ becomes two detached stems, implying that $\pi_1(g[X])$ is trivial), so we are done.

■