Differential Topology Notes

Simon Xiang

April 15, 2021

Notes for the Spring 2021 graduate section of Differential Topology (Math 382D) at UT Austin, taught by Dr. Freed. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

A lot of definitions, here's the new ones:

Definition 1.1. A **subalgebra** of an algebra A is a linear subspace $A' \subseteq A$ containing 1 such that $a_1'a_2' \in A'$ for all $a_1', a_2' \in A'$. A **2-sided ideal** $I \subseteq A$ is a linear subspace such that $AI = I$ and $IA = I$. A $\mathbb{Z}$**-grading** of an algebra A is a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that $A^{k_1}A^{k_2} \subseteq A^{k_1+k_2}$ for all $k_1, k_2 \in \mathbb{Z}$. If A is a $\mathbb{Z}$-graded algebra and $a \in A^k, k \in \mathbb{Z}^{>0}$, then a is **decomposable** if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \cdots, a_k \in A^1$. If not, a is **indecomposable**.

1.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra $\bigotimes V$, thought of as the "free algebra generated by V".

Definition 1.2. Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map $i\colon V \to A$ such that for all (B, T) of an algebra B and a linear map $T\colon V \to B$ such that φ_T is a homomorphism of algebras.

$$\begin{tikzcd}
V \arrow[rr, "i"] \arrow[dr, "T"'] & & A \arrow[dl, dotted, "\exists!"', "\varphi_T"] \\
& B &
\end{tikzcd}$$

(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If $(\xi \neq 0) \in V$ and $i(\xi) = 0$, set $B = \mathbb{R} \oplus \mathbb{R}\xi$ and define $\xi^2 = 0$.

$$\begin{tikzcd}
V \arrow[rr, "i"] \arrow[dr, "\pi"'] & & A \arrow[dl, dotted, "\varphi"] \\
& \mathbb{R} \otimes \mathbb{R}\xi &
\end{tikzcd}$$

Note that $\pi|_{\mathbb{R}\xi} = \mathrm{id}$. But $\xi = \pi(\xi) = \varphi_1(\xi) = 0$, a contradiction. Furthermore, A has a canonical $\mathbb{Z}$-grading. $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$, $T_\lambda\colon V \to V$ is scalar multiplication, $\varphi_\lambda\colon A \to A$ is a homomoprhism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

Definition 1.3. Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map $b\colon V \times V'' \to X$ such that for all (W, B),

$$\begin{tikzcd}
V' \times V'' \arrow[rr, "b"] \arrow[dr, "B"'] & & X \arrow[dl, "\exists!"', "T_B"] \\
& W &
\end{tikzcd}$$

We denote $X = V' \otimes V''$, and $b(\xi', \xi'') = \xi' \otimes \xi''$, $\xi' \in V', \xi'' \in V''$.

If S' is a basis of V', S'' a basis of V'', then $S' \times S''$ is a basis of $V' \otimes V''$, where

$$S' \times S'' \cong \{\xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S''\}.$$

Note that $\bigotimes$ is "commutative" and "associative" with unit $\mathbb{R}$, so

$$\begin{gathered}\mathbb{R} \otimes V \to V \\ V_1 \otimes V_2 \to V_2 \otimes V_1 \\ (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),\end{gathered}$$

forming what we call a **symmetric monoidal category**. We write $\otimes^1 V = V$, $\otimes^2 V = V \otimes V$, $\otimes^3 V = V \otimes V \otimes V$ and so on. We also write $\otimes^0 V = \mathbb{R}$, and sometimes replace $\otimes^n V$ with $V^{\otimes n}$.

1.2 Existence of tensor algebras

Let V be a vector space, and $A = \bigoplus_{k=0}^{\infty} \otimes^k V$. Let $i: V \hookrightarrow A$ be the inclusion into $\otimes' V = V$.

Claim. (A, i) is a tensor algebra over V.

To see this, note that

$$\underbrace{\xi_1 \otimes \cdots \otimes \xi_k)}_{\in \otimes^k V} \cdot_A \underbrace{\eta_1 \otimes \cdots \otimes \eta_\ell}_{\in \otimes^\ell V} = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that $A = \otimes' V$ is *not* commutative.

1.3 The Exterior Algebra

We want to impose todo:come back

Lecture 2

April 8, 2020

todo:see notes on chapter 21, multivariate analysis

2.1 Exterior algebra of a direct sum

Definition 2.1. Let V be a vector space. An **exterior algebra** (E, j) over V is an algebra E and a linear map $j: V \to E$ satisfying $j(\xi)^2 = 0$ for all $\xi \in V$ such that for all pairs (B, T) consisting of an algebra B and a linear map $T: V \to B$ satisfying $T(\xi)^2 = 0$ for all $\xi \in V$, there exists a unique algebra homomorphism $\varphi: E \to B$ such that $T = \varphi \circ j$.

Let L_1, L_2 be linear, and $\bigwedge^*(L_1 \oplus L_2 = V)$.

Lecture 3

April 15, 2021

todo:is this lecture 24??

Theorem 3.1. *Let X be a smooth manifold. Then there exists a unique $d: \Omega^*(X) \to \Omega^{*+1}(X)$ satisfying*

(i) Linearity,

(ii) The Liebniz rule,

(iii) $d^2 = 0$*,*

(iv) $d|_{\Omega^0(X)}$ *is the usual differential.*

Proof. Let $\{(U_i, x_i)\}_{i\in I}$ be an open cover of X by charts. Let $\{\rho_i\}_{i\in I}$ be a partition of unity, where $\operatorname{Supp}\rho_i \subseteq U_i$. If $\alpha \in \Omega^*(X)$, then $\alpha = \sum_i \rho_i\alpha_i$, where $\operatorname{supp}(\rho_i\alpha) \subseteq U_i$. Define $d\alpha = \sum_i d(\rho_i\alpha)$, where we compute $x_i(U_i) \subseteq A_i$, $\operatorname{supp} d(\rho_i\alpha)$ (note that d increases support).

For this to be a good definition, we need to show that this is well-defined. say $\{(V_a, y_a\}_{a\in A}$ is another atlas, $\{\sigma_a\}_{a\in A}$ a partition of unity. Then

$$\begin{aligned}\sum_i d(\rho_i\alpha) &= \sum_i\sum_a d(\rho_i\sigma_a\alpha)\\ &= \sum_a\sum_i d(\sigma_a\rho_i\alpha)\\ &= \sum_\alpha d(\sigma_a\alpha).\end{aligned}$$

Note that $\operatorname{supp}\rho_i\sigma_a\alpha \subseteq U_i \cap V_a$. Something about d commuting with pullback, the first is defined on $x_i(U_i\cap V_a)$, the second on $y_a(U_i \cap V_a)$, and the final on $y_a(V_a)$. todo:this, plus something about transition maps ⊠

3.1 Orientation

We have all seen Riemann integration on the line, and hopefully you have learned how to integrate in $\mathbb{R}^n$, and perhaps Lebesgue integration. We do not focus on the analytic aspects, but the geometric aspects, which allows us to integrate on manifolds. Unfortunately we do not have a fixed vector space, giving a fixed Lebesgue measure, so we have to start from the beginning. Let's talk about orientation.

Recall that if L is a real line (1-dimensional vector space), then an **orientation** of L is an element of $\pi_0(L\setminus\{0\})$.

Definition 3.1. If V is a finite dimensional real vector space, then an **orientation** of V is an orientation of $\det V$. A **basis** of V is an isomorphism $b\colon \mathbb{R}^n \to V$ if $\dim V = n$.

Remark 3.1. Let $\mathcal{O}(V)$ be the set of bases of V. The group $\mathrm{GL}_n\mathbb{R} = \{g\colon \mathbb{R}^n \xrightarrow{\simeq} \mathbb{R}^n\}$ acts simply transitively on $\mathcal{O}(V)$.[1] This is a right action $\mathrm{GL}_n\mathbb{R}$, or a torsor. Then $\det\colon \mathrm{GL}_n\mathbb{R} \to \mathbb{R}^{\neq 0}$ is an isomorphism on π_0. Introduce $\mathcal{O}(V) \to \det V\setminus\{0\}$, $e_1,\cdots,e_n \mapsto e_1\wedge\cdots\wedge e_n$. An orientation partitions $\mathcal{O}(V)$ into $\mathcal{B}^{\pm}(V)$. If $T\colon V' \to V$, then $\dim V' = \dim V$ if T is an ismorphism. Then $\det T\colon \det V' \to \det V$[2] is an isomorphism, and T is orientation preserving (resp reversing) if $T(O') = 0$ (resp $T(O') \neq O$). (Here O denotes the orientation of a space.)

Definition 3.2. Let V be a finite dimensional real vector space. A nonzero element of $\operatorname{Det} V^*$ is a **volume form**. For $\xi_1,\cdots,\xi_k \in V$, $(\xi_1,\cdots,\xi_k) = \{t^i\xi_i \mid 0 \leq\to i \leq 1\} \subseteq \operatorname{span}\{\xi_i\}$, the vectors are **nondegenerate** if the $\xi_1,\cdots,\xi_k$ are LI iff $\xi_1\wedge\cdots\wedge\xi_k \neq 0$ in $\bigwedge^k V$. If $e_1,\cdots,e_n$ is a basis of V, define

$$\operatorname{vol}(//(e_1,\cdots,e_n) = \|\langle\omega, e_1\wedge\cdots\wedge e_n\rangle\|.$$

Proposition 3.1. *If* $e_1',\cdots,e_n'$ *is another basis, and* $e_j' = T^i_j e_i$ *for* $T^i_i \in \mathbb{R}$*, then*

$$\operatorname{vol}//(e_1',\cdots,e_n') = (\det T)\operatorname{vol}//(e_1,\cdots,e_n).$$

Remark 3.2. *Ratios* of volume are defined without a volume form. A k-form $\alpha \in \bigwedge^k V_6*$ induces a notion of volume on all k-dimensional subspaces $W \subseteq V$ such that $\alpha|_W \neq 0$. On $\mathbb{R}^n$ we take $\omega = e^1\wedge\cdots\wedge e^n \in \operatorname{Det}\mathbb{R}^{n*}$.

todo:?? canonical double cover, orientation bundle, homology

Definition 3.3. An orientation of X is a section of $\pi_0^{\mathrm{vert}}(\operatorname{Det} TX\setminus 0) \to X$. A **volume form** on X is a nonvanishing $\omega \in \Omega^n(X)$ if $\dim X = n$.

[1] Apparently in physics, left vs right actions form the idea of passive vs active actions or something like that. This is a right action.

[2] Confused on usage of det and Det

Example 3.1. If $X = S^1$, then we have two double covers up to isomorphism. If $X = \mathbb{R}\mathrm{P}^2$, then $D^2 \subseteq \mathbb{A}^2$ todo:something happen, so the orientation double cover has total space S^2, and $\mathbb{R}\mathrm{P}^2$ is not orientable.

Definition 3.4. Suppose X is an oriented manifold. A standard chart (U, x), $x : U \to \mathbb{A}^n$ is **oriented** if $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p$ is an oriented basis of $T_p X$ for all $p \in U$.

If $(U, x), (V, y)$ are oriented charts, then $\det d(y \circ x^{-1}) > 0$. Look forward to integration.