Gromov's Norm and Bounded Cohomology Lecture Notes

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Lecture notes for the Spring 2022 graduate section of a topics course called "Gromov's norm and bounded cohomology" (Math 392C) at UT Austin, taught by Dr. Chen. These notes were taken live in class (and so they may contain many errors). Source files: https://git.simonxiang.xyz/math_notes/files.html

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1 Mostow Rigidity

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Lemma 1.1. *$\partial\widetilde{\varphi}$ preserves $\mathfrak{V}$.*

Why is this true? $\widetilde{\varphi}$ is a lift of φ, a homotopy equivalence between $M \to N$. Since simplicial volume is a homotopy invariant, Gromov's proportionality tells us $\|M\|_1 = \mathrm{vol}(M)/v_n$ is equal to $\|N\|_1 = \mathrm{vol}(N)/v_n$. Recall that v_n is uniquely achieved by the volume of regular ideal simplices.

Proof. Suppose there exists a regular ideal simplex Δ_r such that $\partial\widetilde{\varphi}(V(\Delta_r)) = V(\widetilde{\Delta})$ where $\widetilde{\Delta}$ is not regular. Then

$$\mathrm{vol}\left(\widetilde{\Delta}\right) + 2\varepsilon < \mathrm{vol}(\Delta_r) = v_n.$$

Take an approximation of Δ_r by a finite straight simplex Δ, then $\mathrm{vol}(\Delta') > v_n - \varepsilon$. By our smearing argument, we can represent the fundamental class of M as a summation, where $[M] = \sum \lambda_i c_i, \lambda_i > 0$. Each c_i is the projection of some Δ'. Sending this through th emap φ, this sends $\varphi\colon [M] \to [N] = \sum \lambda_i \mathrm{str}(\varphi_* c_i)$. Now $\mathrm{vol}(\mathrm{str}(\varphi_* c_i))$ differes from $\mathrm{vol}(\widetilde{\Delta})$ by a factor of ε. Computing the volume of N we get a contradiction:

$$\mathrm{vol}(N) = \langle [N], \mathrm{vol}\rangle = \sum \lambda_i \mathrm{vol}(\mathrm{str}(\varphi_* c_i)) < \sum \lambda_i (v_n - \varepsilon) = \left(\sum \lambda_i\right)(v_n - \varepsilon) \le (\|M\|_1 + \delta)(v_n - \varepsilon) = (\|N\|_1 + \delta)(v_n - \varepsilon)$$

ε is fixed as the distance between the volumes. We choose δ very very small, such that $(\|N\|_1 + \delta)(v_n - \varepsilon)$ is approximately $\|N\|_1 (v_n - \varepsilon) < \|N\|_1 \cdot v_n$. If we have a map on the boundary taking the vertex set of any regular ideal simplex to another regular ideal simplex, then this has to be the boundary map of an isometry.

Proposition 1.1. *For $n \ge 3$, $h\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ a homeomorphism such that h preserves $\mathfrak{V}$, then $h = \partial F$ for some $F\colon \mathbb{H}^n \to \mathbb{H}^n$ a hyperbolic isometry.*

Note that any two regular ideal simplices Δ_1, Δ_2 differ by some hyperbolic isometry, since equilateral triangles differ by some Euclidian similary ($\mathrm{stab}(\infty)$). By composing with some $F \in \mathrm{Isom}(\mathbb{H}^n)$, we may assume h fixes the vertices of some regular ideal simplex. The goal is to have $h = \mathrm{id}_{\partial\mathbb{H}^n}$. $\mathrm{Fix}(h)$ contains a dense set in $\partial\mathbb{H}^n$. Then the convex hull $\mathrm{co}(v_0, v_1, v_2, v_3)$ is regular, as well as $\mathrm{co}(v_0, v_1', v_2, v_3)$. The point is that there are exactly two regular simplices containing these points, so they must be the only two regular simplices containing v_0, v_2, v_3. Therefore v_1' must be fixed.

The inversion takes $v_1' \to m_1$ the midpoint of $[v_2, v_3]$, which implies that $m_1 \in \mathrm{Fix}(h)$. Continuing this process gives us the fact that $\mathrm{Fix}(h)$ contains a dense set of $\partial\mathbb{H}^n$.

Now to conclude the proof. By our proposition, $\partial\widetilde{\varphi} = \partial F$ which is π_1-equivariant, $F \in \mathrm{Isom}(\mathbb{H}^n)$, which implies F is π_1-equivariant. So F induces an isometry $f\colon M \to N$. We also have an equivariant homotopy H between $\widetilde{\varphi}$ and F. The map $\{H(x,t) \mid 0 \le t \le 1\}$ (the geodesic from $\widetilde{\varphi}(x)$ to $F(x)$), which means H induces a homotopy between φ and f. But f is an isometry, and we are done. This concludes Mostow rigidity. ⊠

2 Consequences of Mostow rigidity

Recall we have a homotopy equivalence of manifolds $\varphi\colon M \to N$ and an π_1-equivariant map $\widetilde{\varphi}(\widetilde{M} = \mathbb{H}^n) \to (\widetilde{N} = \mathbb{H}^n)$ with boundary $\partial\widetilde{\varphi}\colon \partial\mathbb{H}^n \to \mathbb{H}^n$ a self-homeomorphism. Then this boundary map $\partial\widetilde{\varphi}$ must be ∂F for some isometry F, which implies $f\colon M \to N$ is an isometry $f \simeq \varphi$. This can be proved in different ways:

(1) $\partial\varphi$ is a quasi-conformal ergordicity means constant distortion. Here $\pi_1 M$ is ergodic on $\partial\mathbb{H}^n$, which implies $\partial\widetilde{\varphi} = \partial F$ is conformal.

(2) We can view $M = \mathbb{H}^n/\Gamma$, where $\Gamma \cong \pi_1 M$ acting on $\mathbb{H}^n$ by isometry. Then $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$, which is a discrete subgroup (lattice acting by isometries). This is a so called **cocompact lattice**. Furthermore since the quotient is a manifold this lattice is also torsion free. Let $M = \mathbb{H}^n/\Gamma_1, N = \mathbb{H}^n/\Gamma_2$. Any $\varphi\colon \Gamma_1 \xrightarrow{\cong} \Gamma_2$ is realized by conjugation in $F \in \mathrm{Isom}(\mathbb{H}^n)$, $\varphi(\gamma) = F\gamma F^{-1}$. The reason for this is basically lifting of universal covers.

$$\begin{tikzcd} \mathbb{H}^n \arrow[r, "\tilde{\varphi}=F"] \arrow[d] & \mathbb{H}^n \arrow[d] \\ \mathbb{H}^n/\Gamma_1 = M \arrow[r, "\varphi"] & \mathbb{H}^n/\Gamma_2 = N \end{tikzcd}$$

Here φ is a homotopy equivalence, but Mostow rigidity tells us that we can choose φ to be an isometry. Then for $\gamma \in \Gamma_1$, this leads to a diagram

$$\begin{tikzcd} \mathbb{H}^n \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathbb{H}^n \arrow[d, "\partial(\gamma)=F\gamma F^{-1}"] \\ \mathbb{H}^n \arrow[r, "F"'] & \mathbb{H}^n \end{tikzcd}$$

There are higher analogues for Lie groups, one of them being Margul's super rigidity. It essentially says a linear representation of a higher dimensional lattice comes from a representation of the underlying Lie group.

Some consequences.

Theorem 2.1. *For $M = \mathbb{H}^n/\Gamma$, the following groups are isomorphic and finite if $n \geq 3$.*

(1) $\mathrm{Isom}(M)$.

(2) N_Γ/Γ *(the normalizer of Γ), where $N_\Gamma = \{F \in \mathrm{Isom}(\mathbb{H}^n) \mid F\Gamma F^{-1} = \Gamma\}$. In other words, this is the largest subgroup of the ambient group such that Γ is a normal subgroup.*

(3) $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$.

(4) $\mathrm{MCG}(M) = \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$.

Proof. ($\mathrm{Isom}(M) \cong N_\Gamma/\Gamma$): This is a standard process that doesn't have much to do with Mostow rigidity. For $f \in \mathrm{Isom}$ acting on M, $\Gamma\colon \mathbb{H}^n \to M$, we can lift f to the universal cover F which implies $F\Gamma F^{-1} = \Gamma$. Conversely, if f is a normalizer, then $\gamma \mapsto F\gamma F^{-1}$ is an isometry $\Gamma \cong \Gamma$. Here $\Gamma = \ker(N_\Gamma \to \mathrm{Isom}(M))$, F a life of id iff $F \in \Gamma$ (the deck group). This is a general argument, we could even replace $\mathbb{H}^n$ with some other geometric group.

($N_\Gamma/\Gamma \cong \mathrm{Out}(\Gamma)$): One direction is clear. For $F \in N_\Gamma$, $\varphi_\Gamma\colon \Gamma \to \Gamma, \gamma \mapsto F\gamma F^{-1}$ if $F \in \Gamma$ implies $\partial_F \in \mathrm{Inn}(F)$.

$$\begin{tikzcd} N_\Gamma \arrow[r, "F\mapsto\varphi_F"] \arrow[dr, "h"'] & \mathrm{Aut}(\Gamma) \arrow[d] \\ & \mathrm{Out}(\Gamma) \end{tikzcd}$$

Here $\ker(h) \supseteq \Gamma$. We need to show that h is surjective (Mostow rigidity) and $\ker h = \Gamma$. Pick $\varphi \in \mathrm{Aut}(\Gamma)$, then $\varphi_F = \varphi$ up to conjugation (different lifts differ by conjugation). Suppose $\varphi_F = \varphi_\gamma$ for $\gamma \in P, F \in N_\Gamma$. Then $\varphi F = \mathrm{id}, FgF^{-1} = g$ for all $g \in F$. This implies F commutes with all $g \in \Gamma$, and $F = \mathrm{id}$. Here $\mathrm{Fix}(g) = \mathrm{Fix}(FgF^{-1}) = F(\mathrm{Fix}(G))$, attracting/repelling implies F fixes g^+ and g^-. Conjugate g by $\gamma \in P$, then $g^+ \mapsto \gamma(g^+)$ (fixed by F) by varying γ.

$$\begin{tikzcd} \mathrm{Isom}(M) \arrow[r] \arrow[dr, hook, "\cong"] \arrow[ddr, "\cong"'] & \mathrm{Homeo}(M) \arrow[d] \\ & \mathrm{MCG}(M) \arrow[d] \\ & \mathrm{Out}(M) \end{tikzcd}$$

Mostow rigidity tells us the red map is surjective, which implies the blue map is an isomorphism. ⊠

Next time we will explain that the isometry group of a closed hyperbolic manifold is finite. Then $\mathrm{Isom}(M)$ acts on M acting on FM (compact), so orbits are discrete, and subsequently finite. Then we need to show the stabilizers are trivial, which is true because isometries cannot fix the frame (otherwise it is trivial).

3 ok

4 ok

Last day of class today wow. Last time we talked about the bounded Euler class associated to a group action on the circle. One is the that the bounded Euler class is trivial iff the action has a global fixed point. The idea is that we can use that point to lift the action. The other thing is that this Euler class captures the action completely up to semiconjugal equivalence.

Example 4.1. Consider $G = \mathbb{Z}$. Pick $f \in T = \mathrm{Homeo}^+(S^1)$. Let $\rho : \mathbb{Z} \to T$, then $\rho(n) = f^n$. This gives us an action. Something nice is that $\mathbb{Z}$ is amenable, so $H_b^2(\mathbb{Z};\mathbb{R}) = 0$ as with all amenable groups. We mentioned earlier there is an exact sequence relating these;

$$0 \to \mathrm{Hom}(\mathbb{Z}, S^1) \to H_b^2(\mathbb{Z};\mathbb{Z}) \to H_b^2(\mathbb{Z};\mathbb{R}) = 0$$

Therefore $\mathrm{Hom}(\mathbb{Z}, S^1) \simeq H_b^2(\mathbb{Z};\mathbb{Z})$, and our Euler class $\mathrm{eu}_b^{\mathbb{Z}}(\rho)$ lives here, so it should correspond to some homomorphism into the circle. It turns out that $\varphi(n) = n \cdot \mathrm{rot}(f) = \mathrm{rot}(f^n)$. Going back to the context of Ghys' theorem, we start with an arbitrary homomorphism and try to understand its action. It says it corresponds to its rotation number, but the rotation number corresponds to another rigid action, giving an Euler class, and this map tells you that this Euler class is equal to the original Euler class, and Ghys' theorem says they are the same action up to semi-conjugacy. In summary, $\mathrm{eu}_b^{\mathbb{Z}}(\rho') = \delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho)$, and by Ghys we have $\rho' \sim \rho$.

Proposition 4.1. *An action ρ of a group G on S^1 has the bounded Euler class with real coefficients $\mathrm{eu}_b^{\mathbb{R}}(\rho) = 0$ iff the action is semiconjugate to an action by rigid rotations. In this case, the rotation number $\mathrm{rot}_\rho : G \to S^1$, $g \mapsto \mathrm{rot}(\rho(g))$ is a homomorphism.*

Proof. We have

$$0 \to \mathrm{Hom}(G, S^1) \to H_b^2(G;\mathbb{Z}) \to H_b^2(G;\mathbb{R})$$

with $H_b^2(G;\mathbb{Z}) \ni \mathrm{eu}_b^{\mathbb{Z}}(\rho) \mapsto \mathrm{eu}_b^{\mathbb{R}}(\rho) = 0$. Then there exists a φ iff $\delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho)$. So $\varphi : G \to S^1$ acting on S^1 by rigid rotations, leading to an action of ρ' at G on S^1 by rigid rotations. Recall the snake lemma: nvm let's skip that. We check that $\delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho')$. By definition this is equal to $\mathrm{eu}_b^{\mathbb{Z}}(\rho)$, and applying Ghys' theorem these two are related by semiconjugacy, explaining the first part.

For the second statement, for rigid rotations the rotation number is exactly the angle of rotation. So $\mathrm{rot}_\rho(g) = \mathrm{rot}(\rho(g)) = \mathrm{rot}(\rho'(g)) = \varphi(g)$. So $\mathrm{rot}_p = \varphi$, which is a homomorphism. ⊠

Note that the ordinary Euler class $\mathrm{eu}(\rho) \in H^2(\mathbb{Z};\mathbb{Z}) = H^2(S^1;\mathbb{Z}) = 0$. This reflects the fact that we can life $f \in T$ to some $\widetilde{f} \in \hat{T}$. The point is that the ordinary Euler class carries no information when we restrict to $\mathbb{Z}$ classes, but the *bounded* Euler class still carries data about the rotation.

Theorem 4.1 (Hirsch-Thurston)**.** *If G is amenable, then any action of G on S^1 is semi-conjugate to an action by rigid rotations. In particular, the rotation number is a homomorphism.*

Proposition 4.2. *Any finite subgroup of $T = \mathrm{Homeo}^+(S^1)$ is cyclic.*

Proof. Let $G \subseteq T$ be finite. We want to deduce that this is cyclic. Finite implies amenable, so we have a homomorphism $\mathrm{rot}\colon G \to S^1$. If G is finite, this homomorphism has to be injective. We have $\mathrm{rot}(G) = 0$ iff g has a fixed point on S^1. If $g \neq \mathrm{id}$, the only case where g has a fixed point is that if it fixed everything. So g acts by translation on complementary intervals. Then $G \simeq \mathrm{rot}(G) \subseteq S^1$ implies $\mathrm{rot}(G) \simeq G$ is cyclic. ⊠

Corollary 4.1. *A group G cannot act faithfully on S^1 if it has a finite subgroup that is not cyclic.*

Example 4.2. The mapping class group of a surface $\mathrm{Mod}(S)$ cannot act faithfully on S^1. The reason is that these always contain a Klein-4 group $K \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. Realize the Klein-4 group as a group of rotations in $\mathbb{R}^3$. We can rotate around each axis by π, so they are order two rotations. We can actually write these as matrices:

$$a = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Then $ab = c, a^2 = b^2 = c^2 = \mathrm{id}$. This tells us that $\{\mathrm{id}, a, b, c\} \simeq K$, where a, b are the two generators. What does this have to do with surfaces and the mapping class group? Embed $S \hookrightarrow \mathbb{R}^3$ "symmetrically" such that K embeds into the homeomorphism group $\mathrm{Homeo}^+(S)$, and inject K into the mapping class group $\mathrm{Mod}(S)$.

In contrast, there is another mapping class group of homeomorphisms preserving a basepoint $\mathrm{Mod}(S, p)$ acts faithfully on S^1. Consider $S \setminus p$ and put a hyperbolic structure on it. Then we have S^1 formed from unique rays coming out of this cusp. Another way to see this is that $\mathrm{Mod}(S, p)$ acts on $\pi_1(S, p)$ by isomorphisms, which is quasi-isomorphic to $\mathbb{H}^2$. So $\mathrm{Mod}(S, p)$ acts on $\partial \pi_1(S, p) \simeq \partial \mathbb{H}^2 = S^1$.

Consider another cocycle representing $2\mathrm{eu}_b^{\mathbb{Z}}$. We can talk about an oriented triple $\mathrm{Or}(x, y, z)$, remove x and this unwraps into an interval $[x, \cdots, y, \cdots, z, \cdots, x]$ (positive orientation). Then for G acting on S^1, take $g, h, k \in G, x \in G$. This defines a function on triples of elements of G given by

$$\mathrm{Or}(g, h, k) = \mathrm{Or}(\rho(g)x, \rho(h)x, \rho(k)x)$$

which is G invariant. Then $\mathrm{Or}(\ell g, \ell h, \ell k) = \mathrm{Or}(g, h, k)$, turning out to be a cocycle. This defines a 2-cocycle in homogeneous coordinates, representing an $[\mathrm{Or}] \in H_b^2(G; \mathbb{Z})$.

Theorem 4.2 (Thurston)**.** *This 2-cocycle describes twice the Euler class, or* $[\mathrm{Or}] = 2 \cdot \mathrm{eu}_b^{\mathbb{Z}}(p)$.

Corollary 4.2. $\|\mathrm{eu}_b^{\mathbb{R}}\|_\infty \leq 1/2$*, leading to Milnor-Wood.*

Theorem 4.3 (Ghys)**.** *If G is countable, $\alpha \in H_b^2(G; \mathbb{Z})$, α can be represented by a cocycle with values in* $\{0, 1\}$ *iff* $\alpha = \mathrm{eu}_b^{\mathbb{Z}}(\rho), \rho : G \to S^1$.

Proof. One direction is by definition, the other is by Ghys. The idea is that the choice of $\{0, 1\}$ determines whether the triple has positive or negative orientation, and we can use this to get a "circular order" on G. This leads to an action of G on S^1. ⊠