Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TeX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: https://git.simonxiang.xyz/math_notes/files.html

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Lecture 1

The Fundamental Group

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups. These notes follow Hatcher §1.2, Lee §10, and May §1,§2.7.

1.1 Defining the fundamental group

Let X be a space, we say two paths $f, g: I \to X$ from x to y are **equivalent up to homotopy** if there exists a homotopy $h: I \times I \to X$ such that

$$h(s,0) = f(s),\ h(s,1) = g(s),\ h(0,t) = x,\ h(1,t) = y$$

for all $s, t \in I$. In other words, starting at 0 for the second interval ensures you begin at f, and evaluating at the end takes you to g. Since the starting position of f and g are the same, for the first interval starting at 0 must give x, and similarly for y. Denote the homotopy equivalence class of f as $[f]$: we say that f is a **loop** if $f(0) = f(1)$.

Definition 1.1 (Fundamental group)**.** The **fundamental group** of a space X with a basepoint x denoted $\pi_1(X, x)$ is the set of loops up to homotopy starting and ending at x. For paths f, g we have

$$(g \circ f)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2, \\ g(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Similarly, we define $f^{-1}(t) = f(1-s)$, f transversed the other way around, and the identity is the constant loop, $c_x(t) = x$. Multiplication of equivalence classes is given by $[g][f] = [g \circ f]$, which is well-defined, under this operation $\pi_1(X.x)$ becomes associative and unital.

We can define a change-of-basepoint homomorphism between pointed groups $\gamma[a]: \pi_1(X, x) \to \pi_1(X, y)$ for $x, y \in X$ by $\gamma[a][f] = [a \cdot f \cdot a^{-1}]$, where a is a path from x to y (note that X has to be path-connected!). To see that $\gamma[a]$ is a homomorphism, consider $\gamma[a][fg] = [afga^{-1}] = [afa^{-1}aga^{-1}] = [afa^{-1}][aga^{-1}] = \gamma[a][f] \circ \gamma[a][g]$. Note that $\gamma[b \cdot a] = \gamma[b] \circ \gamma[a]$, then $\gamma[a]$ becomes an isomorphism by $\gamma[a] \circ \gamma[a^{-1}] = \gamma[a^{-1}] \circ \gamma[a] = \mathrm{id}_{\pi_1(X)}$.

We can also define the induced map on the fundamental group of a map between spaces $p: X \to Y$ as $p_*: \pi_1(X, x) \to \pi_1(Y, p(x))$ by $p_*[f] = [p \circ f]$, where f is a loop $I \to X$. This is a homomorphism, and $\mathrm{id}: X \to X$ induces the identity homomorphism. Note that for $q: Y \to Z$, $q_* \circ p_* = (q \circ p)_*$. We would like to say that for $p \simeq q$ homotopic by a homotopy h that $p_* = q_*$, but this isn't quite true because homotopies don't respect basepoints. But h determines a path $a: p(x) \to q(x)$ by $a(t) = h(x, t)$, which helps. Then, we can draw a diagram and show the fundamental group is preserved under homotopy type.

1.2 Fundamental group of the circle

If this is a first introduction to fundamental groups, then our first fundamental group of real interest is $\pi_1(S^1) = \mathbb{Z}$. Before we do this, let's do a quick calculation to show $\pi_1(\mathbb{R}) = 0$. Take the origin as a convenient basepoint. Define $k: \mathbb{R} \times I \to \mathbb{R}$ by $k(s, t) = (1-t)s$. Then k is a homotopy from the identity to the constant map at 0. For a loop $f: I \to \mathbb{R}$ at 0, define a homotopy from this map to a point (via k) by $h(s, t) = k(f(s), t)$. Then f is equivalent to a constant c_0 by the homotopy h.

Now let's talk about circles: we can view S^1 as the circle group (let's denote it U^1), that is, $U^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Multiplication is continuous, so this is a topological group. Take the identity 1 as a convenient basepoint for S^1.

Theorem 1.1. *We have the fundamental group of a circle isomorphic to the integers, that is,*

$$\pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. Define (the covering map) $p: \mathbb{R} \to S^1$ by $p(s) = p(s) = e^{2\pi i s}$, wrapping each interval $[n, n+1]$ around the circle. Now $f_n = p \circ \widetilde{f}_n$, where $\widetilde{f}_n(s) = sn$ is the uniquely lifted path in $\mathbb{R}$. This is true because exponentiation is a homomorphism between multiplication and addition, and the path lifting holds true for arbitrary paths. That is, for any path $f: I \to S^1$ with $f(0) = I$, we have a uniquely lifted path $\widetilde{f}: I \to \mathbb{R}$ such that $\widetilde{f}(0) = 0$ and $p \circ \widetilde{f} = f$. Observe that the preimage of any connected neighborhood in S^1 in $\mathbb{R}$ is a disjoint union of a copy of such nbd in the intervals $(r, r+1), (r+1, r+2), \cdots$ for some $r \in \mathbb{R}$. Since I is compact, we can subdivide I into a finite amount of closed subintervals, so f takes each subinterval into such nbd. Working backward, we thus determine the unique lifting of f by using the fact that each lifting of the subintervals is determined inquely by the lifting of its initial point, then repeat.

Consider $j: \pi_1(S^1, 1) \to \mathbb{Z}$ by $j[f] = \widetilde{f}(1)$, where then endpoint lifts to. Since $p(\widetilde{f}(1)) = 1$, this number is an integer. To show such integer is independent of the choice of f in the homotopy class $[f]$, say we have a homotopy $h: f \simeq g$ through loops at 1, then this lifts uniquely to a homotopy $\widetilde{h}: I \times I \to \mathbb{R}$ such that $\widetilde{h}(0,0) = 0$ and $p \circ \widetilde{h} = h$. This is the homotopy lifting property. Since paths lift uniquely, the paths $c(t) = \widetilde{h}(0, t)$ and $d(t) = \widetilde{h}(1, t)$ determine constant paths since $h(0, t) = h(1, t) = 1$ for all t. Now c is constant at 0, so by the unique lifting property we have

$$\widetilde{f}(s) = \widetilde{h}(s, 0) \quad \text{and} \quad \widetilde{g}(s) = \widetilde{h}(s, 1).$$

Then our second constant path starts at $\widetilde{f}(1)$ and ends at $\widetilde{g}(1)$, thus $j(\widetilde{f}(1))$ is independent of the choice of path in $[f]$.

Since $j[f_n] = n$, $j \circ i: \mathbb{Z} \to \mathbb{Z}$ is the identity, since i takes integers to the path with such winding number, and j sends a path with such winding number back down to the integers. All we have left to check is that j is injective, since the composition of injections results in an injection, which the identity must be (it is a bijection)[1]. Suppose $j[f] = j[g]$, then $\widetilde{f}(1) = \widetilde{g}(1)$. Then $\widetilde{g}^{-1} \cdot \widetilde{f}$ is a loop at 0 in $\mathbb{R}$. Since $\mathbb{R}$ is contractible, $[\widetilde{g}^{-1} \cdot \widetilde{f}] = [c_0]$, thus projecting down yields

$$[g^{-1}][f] = [g^{-1} \cdot f] = [c_1].$$

Then $[f] = [g]$, finishing the proof. ⊠

To summarize: we defined a function from the integers into the fundamental group $\pi_1(S^1)$ by considering the class of loops with winding number n, and defined a complementary function by considering the winding number of any class of loops in S^1. How we showed this function was an injection is by considering the upstairs covering space $\mathbb{R}$, in which loops lift uniquely and preserve homotopy type. The fact that $\mathbb{R}$ covers S^1 was also used in the definition of the complementary function assigning winding numbers, precisely as the endpoint of the lifted loop. We carefully made sure that homotopy type mattered not when assigning winding numbers to functions. Then considering that the endpoint of two lifted loops, such paths being equal meant that they ended in the same place upstairs, and composing one with the others inverse results in a loop in $\mathbb{R}$ centered at zero. But since $\mathbb{R}$ is contractible, this loop becomes zero, and projecting downstairs also makes the composition zero, implying that the two loops are equal. After some function mumbo jumbo, we can conclude that the first map is a bijection, finishing the proof.

We can start with our first real-world application, that is, Brouwers fixed point theorem.

Proposition 1.1. *There is no continuous map $r: D^2 \to S^1$ such that $r \circ i = \mathrm{id}_{S^1}$, where i denotes the inclusion $S^1 \hookrightarrow D^2$.*

Proof. If such an r existed, then

$$\pi_1(S^1, 1) \xhookrightarrow{i_*} \pi_1(D^2, 1) \xrightarrow{r_*} \pi_1(S^1, 1)$$

would compose to the identity, since the identity induces the identity homomorphism on the fundamental groups. Since the identity homomorphism of $\mathbb{Z}$ doesn't factor through the zero group, this cannot happen. ⊠

Brouwer's Fixed-point Theorem. *Any continuous map $f: D^2 \to D^2$ has a fixed point.*

Proof. Suppose f has no fixed point, that is, $f(x) \neq x$ for all x. Define $r(x) \in S^1$ to be the intersection of the ray starting at $f(x)$ and ending at x with the boundary S^1. This is well defined since $f(x) \neq x$ for all x. Note that $r(x) = x$ if $x \in S^1$. Since r is continuous, this contradicts our proposition, since this is a retraction of the disk onto the boundary that, when composed with inclusion, becomes the identity on the circle, since $r(x) = x$ for $x \in S^1$. This finishes the proof. ⊠

[1] Actually, apparently the converse of this is an example of something I missed in discrete math called the **Cantor-Bernstein-Schröeder Theorem**, which states that if we have injections $A \to B$ and $B \to A$, then A and B must be in bijection.

1.3 The van Kampen Theorem (Hatcher)

Let's take a space X and say it's the union of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi\colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is often onto but in general, we can expect Φ to have a nontrivial kernel.

For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both of these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Van Kampen says under fairly broad hypotheses that this determines all of Φ.

Theorem 1.2. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi\colon *_\alpha\,(A_\alpha) \to \pi_1(X)$$

is onto. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N.$$

Example 1.1 (Wedge Sums)**.** I like the visual of the wedge sum but the terminology of the smash product. If only we could keep the `\vee` symbol ($\vee$) and say we "smash the spaces together" at a point.

We define the wedge sum $\bigvee_\alpha X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ as the disjoint union $\amalg_\alpha X_\alpha$ with all the basepoints x_α identified to a single point. If each $x_\alpha$is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Then by van Kampens theorem,

$$\Phi\colon *_\alpha\, \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$$

is an isomorphism, provided each X_α is path-connected, hence also each A_α. Therefore for a wedge sum of circles, $\pi_1(\bigvee_\alpha S^1_\alpha)$ is a free group, the free product of copies of $\mathbb{Z}$.

I know it always helps to see something done somewhere else. For me, the above definition fails to make any sense at all whatsoever. So, let's revisit van Kampens from two more lens: one from the words of Lee (*Introduction to Topological Manifolds*) and another from the categorical perspective.

1.4 The van Kampen Theorem (Lee)

Let's say we have a space X that's made up of the union of two open subsets $U, V \subseteq X$. We know how to compute the fundamental groups of U, V, and $U \cap V$ (each of which is path-connected). Every loop can be written as a product of loops in U or V (visualized as the free product $\pi_1(U) * \pi_1(V)$), but any loop in $U \cap V$ only represents a single element of $\pi_1(X)$, even though it represents two distinct elements of the free product (one in $\pi_1(U)$ and one in $\pi_1(V)$). So $\pi_1(X)$ can be though of as the quotient of this free product modulo some relations from $\pi_1(U \cap V)$ that express this redundancy.

Let's do some setup so we can state a precise version of van Kampens. Let X be a topological space and $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V \neq \emptyset$. Let $q \in U \cap V$. Then the four inclusion maps shown below,

$$\begin{tikzcd}
 & U \arrow[dr, "k"] & \\
U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\
 & V \arrow[ur, "l"'] &
\end{tikzcd}$$

induce fundamental group homomorphisms as shown below.

$$\begin{tikzcd}
& \pi_1(U,q) \arrow[dr, "k_*"] & \\
\pi_1(U \cap V,q) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & \pi_1(X,q) \\
& \pi 1(V,q) \arrow[ur, "l_*"'] &
\end{tikzcd}$$

Now insert the free product $\pi_1(U,q) * \pi_1(V,q)$ in the middle of the diagram and let $\iota_U : \pi_1(U,q) \hookrightarrow \pi_1(U,q) * \pi_1(V,q)$ and $\iota_V : \pi_1(V,q) \hookrightarrow \pi_1(U,q) * \pi_1(V,q)$ be the canonical injections. By the characteristic property (unique induced homomorphisms) of the free product, k_* and l_* induce a homomorphism $\Phi : \pi_1(U,q) * \pi_1(V,q) \to \pi_1(X,q)$ such that the right half of the following diagram commutes:

$$\begin{tikzcd}
& \pi_1(U,q) \arrow[d, dashed, "\iota_U"] \arrow[dr, "k_*"] & \\
\pi_1(U \cap V,q) \arrow[ur, "i_*"] \arrow[r, dotted, "F"] \arrow[dr, "j_*"'] & \pi_1(U,q) * \pi_1(V,q) \arrow[r, "\Phi"] & \pi_1(X,q) \\
& \pi_1(V,q) \arrow[u, dashed, "\iota_V"'] \arrow[ur, "l_*"'] &
\end{tikzcd}$$

Finally, define a map $F : \pi_1(U \cap V,q) \to \pi_1(U,q) * \pi_1(V,q)$ by setting $F(\gamma) = (i_*\gamma)^{-1}(j_*\gamma)$[2]. Let $\overline{F(\pi_1(U \cap V,q))}$ denote the *normal closure*[3] of the image of F in $\pi_1(U,q) * \pi_1(V,q)$.

Theorem 1.3 (Seifert-Van Kampen). *Let X be a topological space. Suppose $U, V \subseteq X$ are open, $U \cup V = X$, and U, V, and $U \cap V$ are path-connected. Then for any $q \in U \cap V$, the homomorphism Φ is surjective, and its kernel is $\overline{F(\pi_1(U \cap V,q))}$. Therefore we have*

$$\pi_1(X,q) \cong \pi_1(U,q) * \pi_1(V,q) \big/ \overline{F(\pi_1(U \cap V,q))}.$$

When the fundamental groups in question are finitely presented, the theorem has a useful reformulation in terms of generators and relations.

Corollary 1.1. *In addition to the hypothesis of van Kampen, assume that the fundamental groups of U, V, and $U \cap V$ have the following finite presentations:*

$$\begin{aligned}
\pi_1(U,q) &\cong \langle \alpha_1, \cdots, \alpha_m \mid \rho_1, \cdots, \rho_r \rangle; \\
\pi_1(V,q) &\cong \langle \beta_1, \cdots, \beta_n \mid \sigma_1, \cdots, \sigma_s \rangle; \\
\pi_1(U \cap V,q) &\cong \langle \gamma_1, \cdots, \gamma_p \mid \tau_1, \cdots, \tau_t \rangle.
\end{aligned}$$

Then $\pi_1(X,q)$ has the presentation

$$\pi_1(X,q) \cong \langle \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \mid \rho_1, \cdots, \rho_r, \sigma_1, \cdots, \sigma_s, u_1 = v_1, \cdots, u_p = v_p \rangle$$

where for each $a = 1, \cdots, p$, u_a is an expression for $i_*\gamma_a \in \pi_1(U,q)$ in terms of the generators $\{\alpha_1, \cdots, \alpha_m\}$, and v_a similarly expresses $j_*\gamma_a \in \pi_1(V,q)$ in terms of $\{\beta_1, \cdots, \beta_n\}$.

1.5 The fundamental groupoid

We backtrack a little to talk about categorical nonsense. This doesn't fit too well with the section on category theory, so it's here. These will follow May §2.5.

[2] F is not a homomorphism.

[3] the *normal closure* of a set means the smallest normal subgroup that contains such set.

We often talk of pointed spaces, but it would to nice to talk about spaces without making such a choice. We define the fundamental groupoid $\prod(X)$ of a space X to be the category whose objects are the points of X and whose morphism $x \to y$ are the equivalence classes of paths from x to y[4]. Then the set of endomorphisms of the object x is the fundamental group $\pi_1(X, x)$.

We say "groupoid" because a group is simply a groupoid with only one object (the class of morphisms or symmetries on an object). However, the category of groupoids has several objects. We also defined groupoids as categories whose morphisms are all isomorphisms. If morphisms are functors, then we have the category Grpd of groupoids. So we can see $\prod$ as a functor $\mathsf{Top}_* \to \mathsf{Grpd}$.

Let's talk about skeletons. We have the skeleton of a category $\mathscr{C}$ denoted by $\mathrm{sk}(\mathscr{C})$. This is a "full" subcategory with one object from each isomorphism class of objects of $\mathscr{C}$, "full" meaning that the morphisms between two objects of $\mathrm{sk}(\mathscr{C})$ are all of the morphisms between these objects in $\mathscr{C}$. The inclusion functor $J\colon \mathrm{sk}(\mathscr{C}) \to \mathscr{C}$ is an equivalence of categories. We can find an inverse functor $F\colon \mathscr{C} \to \mathrm{sk}(\mathscr{C})$ by letting $F(A)$ be the unique object in $\mathrm{sk}(\mathscr{C})$ that is isomorphic to A, choosing an isomorphism $\alpha_A\colon A \to F(A)$, and defining $F(f) = \alpha_B \circ f \circ \alpha_A^{-1}$ for a morphism $f : A \to B$ in $\mathscr{C}$. Choose α to be the identity morphisms if $A \in \mathrm{sk}(\mathscr{C})$, then $FJ = \mathrm{id}_{\mathrm{sk}(\mathscr{C})}$; the α_A specify a natural isomorphism $\alpha\colon \mathrm{id} \to JF$.

A category is connected if any two objects can be connected by a sequence of morphisms. Then a groupoid is connected iff any two of its objects are isomorphic. The group of endomorphisms of any object C is then a skeleton of $\mathscr{C}$, so we can generalize our results about skeletons to give the relationship between a fundamental group and a fundamental groupoid of a path connected space X.

Proposition 1.2. *Let X be a path connected space. Then for each $x \in X$, the inclusion $\pi_1(X, x) \to \prod(X)$ is an equivalence of categories.*

Proof. View $\pi_1(X, x)$ as a groupoid with on object x: then $\pi_1(X, x)$ is a skeleton of $\prod(X)$ and we are done. ⊠

May's presentation and proofs are very concise and elegant. I like this.

Lecture 2

Covering Spaces

Today we talk about covering spaces, another central topic in algebraic topology. The notes will follow various texts, including Hatcher, Lee, and May.

2.1 Some preliminary definitions

Sometimes we need to know what words mean so we can talk about big concepts. These notes will follow May §3. We can talk about the theory of covering spaces on *locally contractible* spaces that are path-connected, that is, spaces with a base of contractible spaces, that is, open sets that are contractible when viewed as a space under the subspace topology. However, to get the full picture, we must talk about *locally path-connected* spaces.

Definition 2.1 (Locally path-connected)**.** A space X is *locally path-connected* if for any $x \in X$ and any neighborhood U of x, there exists a smaller neighborhood V of x, with each of whose points can be connected to x by a path in U. We could also say X has a base consisting of open sets that are path-connected (under the subspace topology).

Note that if X is connected and locally path-connected, then it is path-connected. From now on[5], we assume that spaces are connected and locally path-connected. Let's look at how May defines covering spaces.

[4] Recall Example 1.4 of a group being realized as a category with all its arrows isomorphisms.

[5] By this, we mean any sections following May.

Definition 2.2 (Covering space)**.** A map $p: E \to B$ is a covering (or cover, covering space) if it is onto and if each point $b \in B$ has an open neighborhood V such that each component of $p^{-1}(V)$ is open in E and is mapped homeomorphically onto V by p. We say that a path connected open subset V with this property is a fundamental neighborhood of B. We call E the total space, B the base space, and $F_b = p^{-1}(b)$ a fiber of the covering p.

Some notes: in other texts, we have

- covering $\to$ covering map,
- U is a fundamental neighborhood $\to U$ is evenly covered,
- total space $\to$ covering space,
- base space $\to$??,
- $F_b = p^{-1}(b)$ is a fiber of $p \longrightarrow F_b$ is the preimage of b (points) in the union of sheets of $\widetilde{X}$ over U_b.

Another definition that will come in handy when classifying covering spaces is the notion of something being semilocally simply-connected, that is, given a "hole" (of genus one), we can always find a neighborhood contained in that hole such that the fundamental group induced by the inclusion map is trivial in π_1 of the entire space.

Definition 2.3 (Semilocally simply-connected)**.** A space X is *semilocally simply-connected* if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

We'll define this again when we need it, and talk a little more about what it means for a space to be semilocally simply-connected.

This is kind of out of place, but now we'll state Lebesgue's number lemma. It's useful when dealing with compact metric spaces.

Lemma 2.1 (Lebesgue's number lemma)**.** *If a metric space (X, d) is compact and we have an open cover of X, then there exists a $\delta > 0$ such that every subset of X having a diameter less than δ is contained in some member of the cover. We say δ is the* Lebesgue number *of such cover.*

Proof. If the subcover is trivial then any $\delta > 0$ will suffice. Otherwise, if $\bigcup_{i \in I} A_i$ is a finite subcover, then for $i \in I$, define $C_i := X \setminus A_i$ (note that C_i is nonempty since the subcover is nontrivial). Define a function

$$f : X \to R, \quad x \mapsto \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

Since f is continuous on a compact set, it obtains a minimum δ. The key thing to note is that every x is in some A_i, so by the extreme value theorem $\delta > 0$. To show that this δ is indeed the Lebesgue number of the cover, let $x_0 \in Y$, where $\operatorname{diam}(Y) < \delta$, such that $Y \subseteq B(x_0, \delta)$. Since $f(x_0) \geq \delta$, there exists at least one i such that $d(x_0, C_i) \geq \delta$. But then $B(x_0, \delta) \subseteq A_i$, and so $Y \subseteq A_i$. ⊠

2.2 Covering spaces

These notes will follow Hatcher §1.3.

We've already seen these briefly when we calculated $\pi_1(S^1)$, using the projection $\mathbb{R} \to S^1$ of a helix onto a circle. Covering spaces can be used to calculated fundamental groups of other spaces as well, but the connection runs much deeper than this. We can talk about algebraic aspects of the fundamental group through the geometric language of covering spaces, exemplified in one of the main results in this section: a one to one correspondence between connected covering spaces of a space X and subgroups of $\pi_1(X)$ (spoilers, smh). This is really really similar to Galois theory, where we looked at the towers of field extensions and related them to the subgroup lattice of the Galois group of automorphisms[6].

[6] I actually know this! Thank goodness for an entire semester of algebra to understand an example.

Definition 2.4 (Covering space)**.** A *covering space* of a space X is a space $\widetilde{X}$ together with a map $p: \widetilde{X} \to X$ (we say p is a *covering map*) satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p. Then we say U is *evenly covered* and the disjoint open sets in $\widetilde{X}$ that project homeomorphically to U by p are called *sheets* of $\widetilde{X}$ over U.

If U is connected these sheets are the connected components of $p^{-1}(U)$ so they're uniquely determined by U. If U is not connected, however, the decomposition of U into sheets may not be unique. $p^{-1}(U)$ is allowed to be empty, so p doesn't have to be onto. The number of sheets over U can be given by the cardinality of $p^{-1}(x)$, given $x \in U$. This number is a constant if X is connected.

Example 2.1. A prototypical example (or way to wrap your head around) this section is the helix embedded in $\mathbb{R}^3$: if you think of it projecting on a circle, then $p^{-1}(U)$ is just $\amalg_\alpha U_\alpha$, where each U_α corresponds to the U of a coil or wind of the helix.

Example 2.2. Another example is the helicoid surface $S \subseteq \mathbb{R}^3$ given by $(s\cos 2\pi t, s\sin 2\pi t, t)$ for $(s,t) \in (0,\infty)\times\mathbb{R}$. This projects onto $\mathbb{R}^2 \setminus \{0\}$ via the map $(x,y,z) \mapsto (x,y)$, and defines a covering space $p: S \to \mathbb{R}^2 \setminus \{0\}$ since each point of $\mathbb{R}^2 \setminus \{0\}$ is contained in an open disk U in $\mathbb{R}^2 \setminus \{0\}$ with $p^{-1}(U)$ consisting of countably many disjoint open disks in S projecting homeomorphically onto U. (I can't really see this example...)

Example 2.3. We also have the map $p: S^1 \to S^1$, $p(z) = z^n$ where we view z as a complex number with $|z| = 1$ and n any positive integer[7]. This projection is as described in the footnote, but intersects itself in $n-1$ points (that one can't really imagine as intersections). To see this without the defect, embed S^1 in the boundary torus of a solid torus $S^1 \times D^2$ such that it winds n times monotonically around the S^1 factor without self-intersections, then restrict the projection $S^1 \times D^2 \to S^1 \times \{0\}$ to this embedded circle. What?

We usually restrict our attention to connected covering spaces, as these contain all the interesting examples.

2.3 The covering spaces of $S^1 \vee S^1$ (todo figures)

Covering spaces of $S^1 \vee S^1$ form a rich family that demonstrate the general theory very concretely. For convenience, let $X = S^1 \vee S^1$. View it as a graph with one vertex and two edges, with the edges labeled a and b.

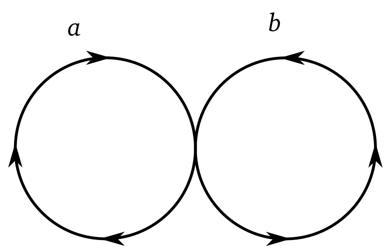

Figure 1: The graph of $S^1 \times S^1$.

Let $\widetilde{X}$ be any other graph with four edges connected to each point, like X at its singular vertex, and that each edge has been assigned an orientation like the ones assigned to each edge of X. That is, for each vertex there are two a-edges and b-edges oriented toward and away from the vertex. Help I can't include figures that are the proper size! Let's call $\widetilde{X}$ a 2-oriented graph.

Given a 2-oriented graph $\widetilde{X}$ we can construct a map $p: \widetilde{X} \to X$ that sends all vertices of $\widetilde{X}$ to the vertex of X, and all edges of $\widetilde{X}$ to the edge of X with the same label. Say p is a homeomorphism on the regions bounded by the edges, and preserves the orientation of the edges. Then p is a covering map. Conversely, every covering space of

[7] Something about the order of z I realized when thinking about this example: z^n means z coils around in S^1 n times. So if z was a fifth root of unity, the covering space would be a circle with five coils projecting onto S^1. Now what if z has infinite order. Can z even have infinite order? I'm not entirely sure...
Edit: elements that are irrational multiples of 2π have infinite order. So does that mean it never winds back to itself? How is this isomorphic to S^1?

X is a graph that inherits a 2-orientation from X. It can be shown that every graph with four edges at each vertex can be 2-oriented: the proof follows from graph theory. We could also generalize this to n-oriented graphs, which are covering spaces of the wedge sum of n circles.

How would we generate a simply-connected covering space of X? Start with the open intervals $(-1,1)$ in $\mathbb{R}^2$ (one per coordinate axis). Then for a fixed λ, $0 < \lambda < 1/2$, say $\lambda = 1/3$, adjoin four open segments of length 2λ to the ends of the previous segments, and shift each back by a distance of λ. These new adjoined segments are perpendicular and bisected by the old ones: continue with four more new segments of distance $2\lambda^2$ at a distance λ^2 to the (now 12) end segments, and so on. Then at the n-th iteration, we would be adding open segments of length λ^{2n-1} at a distance λ^{n-1} from the previous endpoints. Then the union of the segments is a graph (the Cayley graph of the fundamental group of S^1!), with vertices the intersections, labeling horizontal edges a and orienting them to the right, and vertical edges b, orienting them upward.

This covering space is called the *universal cover* of X, because it covers every connected covering space of X.

2.4 More on covering spaces

These notes will follow Lee §11.

The definition of a covering space is the same as Hatcher except: the covering space $\widetilde{X}$ must be connected. Once again, the only interesting covering spaces are connected ones, and so we eliminate the need to frit fret around about details when introducing new theorems and just make sure covering spaces are connected in the definition.

Example 2.4. The exponential quotient map $\varepsilon\colon \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a covering map. Another example: define $E\colon \mathbb{R}^n \to \mathbb{T}^n$ by

$$E(x_1, \cdots, x_n) = (\varepsilon(x_1), \cdots, \varepsilon(x_n)).$$

We will show in an exercise that a product of covering maps is a covering map. So E is a covering map.

Example 2.5. Define a map $\pi\colon S^n \to \mathbb{R}P^n$ (where $n \geq 1$) by sending each point x in the sphere to the line through the origin and x, thought of as a point in $\mathbb{R}P^n$. Then π is a covering map, and the fiber of each point in $\mathbb{R}P^n$ is a pair of antipodal points $\{x, -x\}$.

Lemma 2.2 (Elementary properties of covering maps)**.** *Every covering map is a local homeomorphism, an open map, and a quotient map. An injective covering map is a homeomorphism.*

Proof. Left as an exercise to the reader. ⊠

Proposition 2.1. *For any covering map $p\colon \widetilde{X} \to X$, the cardinality of each fiber $p^{-1}(q)$ is the same for any fiber.*

Proof. If U is any evenly covered open set in X, each sheet in $p^{-1}(U)$ contains exactly one point of each fiber. Then for any $q, q' \in U$, there are one-to-one correspondences

$$p^{-1}(q) \longleftrightarrow \{\text{sheets of}\, p^{-1}(U)\} \longleftrightarrow p^{-1}(q'),$$

which shows that the number of sheets is constant on U. It follows that the set of points $q' \in X$ such that $p^{-1}(q')$ has the same cardinality as $p^{-1}(q)$ is open. Now let $q \in X$, and let A be the set of points in X whose fibers have the same cardinality as $p^{-1}(q)$. Then A is open, and $X \setminus A$ is open since it's a union of open sets (one open set for each cardinality not equal to $p^{-1}(q)$). Since X is connected and nonempty, we have $A = X$. ⊠

If $p\colon \widetilde{X} \to X$ is a covering map, then the cardinality of any fiber is the *number of sheets* of the covering. For example, the n-th power map $(S^1 \to S^1)$ is an n-sheeted covering, $\pi\colon S^n \to \mathbb{R}P^n$ is a two sheeted covering, and $\varepsilon\colon \mathbb{R} \to S^1$ is a countably sheeted covering.

2.5 Lifting properties

Here we'll talk about some important lifting properties, that we discussed when we proved that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. Recall: if $p : \widetilde{X} \to X$ is a covering map and $\varphi : B \to X$ is any continuous map, a *lift* of φ is a continuous map $\widetilde{\varphi} : B \to \widetilde{X}$ such that $p \circ \widetilde{\varphi} = \varphi$. See the commutative diagram below for reference.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ B \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Proposition 2.2 (Unique lifting property). *Let $p : \widetilde{X} \to X$ be a covering map. Suppose B is connected, $\varphi : B \to X$ is continuous, and $\widetilde{\varphi}_1 \widetilde{\varphi}_2 : B \to \widetilde{X}$ are lifts of φ that agree at some point of B. Then $\widetilde{\varphi}_1 \equiv \widetilde{\varphi}_2$, that is, lifts are unique.*

Proof. We show that the set

$$\mathscr{S} = \{b \in B \mid \widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)\}$$

is both open and closed in B, contradicting the connectedness of B if $\mathscr{S}$ is a proper nontrivial subset of B. We conclude that $\mathscr{S}$ must be all of B since $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ agree at a point (so $\mathscr{S}$ is nontrivial) and therefore $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are unique.

Let $b \in \mathscr{S}$ and $U \subset X$ be an evenly covered neighborhood of $\varphi(b)$, and let U_α be the component of p^{-1} containing $\widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)$. On the neighborhood $V = \widetilde{\varphi}_1^{-1}(U_\alpha) \cap \widetilde{\varphi}_2^{-1}(U_\alpha)$ of b, we have $\varphi = p \circ \widetilde{\varphi}_1 = p \circ \widetilde{\varphi}_2$. Since p is 1-1 on U_α, this implies $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$ on V, so $\mathscr{S}$ is open.

OTOH, for $b \notin \mathscr{S}$, if U is an evenly covered neighborhood of $\varphi(b)$, there are disjoint components U_1, U_2 of $p^{-1}(U)$ containing $\widetilde{\varphi}_1(b)$, $\widetilde{\varphi}_2(b)$ such that p is a homeomorphism from each U_i to U. Letting $V = \widetilde{\varphi}_1^{-1}(U_1) \cap \widetilde{\varphi}_2^{-1}(U_2)$, we conclude that $\widetilde{\varphi}_1 \neq \widetilde{\varphi}_2$ on V, so $\mathscr{S}$ is closed. This proof is much easier to follow if you trace everything out with all the inverse relations on the commutative diagram above. ⊠

Proposition 2.3 (Path lifting property). *Let $p : \widetilde{X} \to X$ be a covering map. Suppose $f : I \to X$ is any path, and $\widetilde{q}_0 \in \widetilde{X}$ is any point in the fiber of p over $f(0)$. Then there exists a unique lift $\widetilde{f} : I \to \widetilde{X}$ of f such that $\widetilde{f}(0) = \widetilde{q}_0$.*

Proof. By the Lebesgue number lemma, n can be chosed large enough that p maps each subinterval $[k/n, (k+1)/n]$ into an evenly covered open subset of X. Starting with $\widetilde{f}(0) = \widetilde{q}_0, \widetilde{f}$ is defined inductively by choosing an evenly covered neighborhood U_k containing $f[k/n, (k+1)/n]$, a local section[8] $\sigma_k : U_k \to \widetilde{X}$ such that $\sigma_k(f(k/n)) = \widetilde{f}(k/n)$, and setting $f = \sigma_k \circ f$ on $[k/n, (k+1)/n]$. Because $p \circ \widetilde{f} = (p \circ \sigma_k) \circ f = f$, this is indeed a lift, and it is unique by the unique lifting property. ⊠

Proposition 2.4 (Homotopy lifting property). *Let $p : \widetilde{X} \to X$ be a covering map. Suppose $f_0, f_1 : I \to X$ are path homotopic, and $\widetilde{f}_0, \widetilde{f}_1 : I \to \widetilde{X}$ are lifts of f_0 and f_1 such that $\widetilde{f}_0(0) = \widetilde{f}_1(0)$. Then $\widetilde{f}_0 \sim \widetilde{f}_1$.*

Proof. If $H : f_0 \sim f_1$ is a path homotopy, by the Lebesgue number lemma we can choose n large enough that H maps each square of side $\frac{1}{n}$ into an evenly covered open set. Labeling the squares $S_{ij} = [i/n, (i+1)/n] \times [j/n, (j+1)/n]$, we define a lift $\widetilde{H}$ of H square by square along the bottom row, then the next row, and so on by induction. On each square S_{ij}, set $\widetilde{H} = \sigma \circ H$, for an appopiate local section σ chosen such that the new definition of $\widetilde{H}$ matches the previous one at the corner point $(i/n, j/n)$. Then since two such definitions agree on a line segment (by restricting H to it), they are equal by the unique lifting property.

On the left-hand and right-hand edges of $I \times I$, where $s = 0$ or $s = 1$, $\widetilde{H}$ is a lift of the constant loop and therefore constant. The restriction $\widetilde{H}_0$ to the bottom edge where $t = 0$ is a lift of f_0 starting at $\widetilde{f}_0(0)$, and therefore is equal to $\widetilde{f}_0$, similarly $\widetilde{H}_1 = \widetilde{f}_1$. Therefore $\widetilde{H}$ is the required path homotopy between $\widetilde{f}_0$ and $\widetilde{f}_1$. ⊠

[8] A *local section* of a continuous map is a continuous right inverse defined on some open subset. This exists here by Lee's Lemma 11.7, which shows the existence of local sections of covering maps.

2.6 Connections to the fundamental group

Back to Hatcher §1.3.

Here are some applications of the lifting properties with respect to the fundamental group.

Proposition 2.5. *The map $p_*: \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

Proof. An element of the kernel of p_* is represented by a loop $\widetilde{f}_0: I \to \widetilde{X}$ with a homotopy $f_t: I \to X$ of $f_0 = p\widetilde{f}_0$ to the trivial loop f_1. By the homotopy lifting property, there is a lifted homotopy of loops $\widetilde{f}_t$ starting with $\widetilde{f}_0$ and ending with a constant loop. Basically, since elements of the kernel start with the same point, and there exist unique lifts to them that are nullhomotopic, we conclude the kernel is trivial and p_* is 1-1.

For the second part of the proposition, loops at x_0 lifting to loops at $\widetilde{x}_0$ represent elements of the image of $p_*: \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Conversely, a loop representing an element of the image p_* is homotopic to a loop having such a lift, and by the homotopy lifting property, this loop must also have such a lift. ⊠

Proposition 2.6. *The number of sheets (cardinality of a fiber) of a covering space $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with X and $\widetilde{X}$ path-connected equals the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$.*

Proof. For a loop g in X based at x_0, let $\widetilde{g}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h \cdot g$ with $[h] \in H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ has the lift $\widetilde{h} \cdot \widetilde{g}$ ending at the same point as $\widetilde{g}$ since $\widetilde{h}$ is a loop ($\widetilde{h}$ denotes the same lift as $\widetilde{g}$, just of h instead). All this is saying is that you can lift a product of loops by a product of loops, and we're choosing one loop to the in the image subgroup of p_*. Then we can define a function Φ from cosets $H[g]$ to $p^{-1}(x_0)$ by sending $H[g]$ to $\widetilde{g}(1)$. $H[g]$ denotes $h \cdot g$, where $h \in H$, the coset of g. If you think about it, these are cosets since we just vary g: and so the number of cosets is the index of the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$. Now we just have to show Φ is a bijection to complete the proof.

Φ is onto by the path-connectedness of $\widetilde{X}$, since $\widetilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ by a path $\widetilde{g}$ projecting to a loop g at x_0. To show Φ is 1-1, note that $\Phi(H[g_1]) = \Phi(H[g_2])$ implies that $g_1 \cdot \overline{g_2}$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ and hence $H[g_1] = H[g_2]$. ⊠

Question: for a continuous map $\varphi: Y \to X$, does φ admit a lift $\widetilde{\varphi}$ to a covering space $\widetilde{X}$ of X? The lifting criterion can help us out.

Theorem 2.1 (Lifting criterion)**.** *Suppose we are given a covering space $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and a map $f: (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\widetilde{f}: (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. If a lift $\widetilde{f}$ exists, then $p\widetilde{f} = f$, so $f_* = p_*\widetilde{f}_*$ ⊠

2.7 Classification of covering spaces (todo split it up)

How can we catch all the covering spaces? This whole topic deals closely with its analogue in algebra, Galois theory, with a 1-1 correspondence between connected covering spaces of X (towers of field extensions) and subgroups of $\pi_1(X)$ (subgroups of $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$). This comes from the function that assigns each covering space $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$.

Definition 2.5 (Semilocally simply-connected)**.** A space X is semilocally simply-connected if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Basically, the fundamental group of U is trivial *inside* the fundamental group of X, that is, loops in $\pi_1(U, x)$ are nullhomotopic in X (not necessarily U, if that were the case, U would be locally simply connected). Intuitively, there are lower bounds on the size of holes (genus-wise): if theres a hole, you can find a neighborhood smaller than it so that loops are still trivial. For example, take the Hawaiian earring: loops here are very very small, and at the base every neighborhood will contain a hole, so it's not semilocally simply-connected (a "bad" space)[9].

Proposition 2.7. *If X is a path-connected, locally path-connected, and semilocally simply-connected space, then for every subgroup H of $\pi_1(X, x_0)$, there is a covering space $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$.*

We'll prove Proposition 2.7 much later, let's talk about it first. If $H = 1$, then $\widetilde{X}$ is simply-connected, so $\widetilde{X}$ is the universal cover of X. Now given an evenly covered open set U, any loop in U will lift to a sheet in $\widetilde{X}$, which implies it's nullhomotopic in $\widetilde{X}$, and therefore nullhomotopic in X (we don't know if it's nullhomotopic in U), we can see this just by projecting the loop with p. This implies that if $U \overset{\iota}{\hookrightarrow} X$ denotes the inclusion of U in X, then $\iota_*(\pi_1(U)) = 1$ in $\pi_1(X)$. So X must be semilocally simply-connected. The following claim shows why we need these claims for X to be a "nice" space.

Claim. If X is path-connected, locally path-connected, and semilocally simply-connected, then there exists a universal cover of X.

Proof. We prove this by directly constructing a universal cover of X through the fundamental groupoid. First assume that X has a universal cover $\widetilde{X} \xrightarrow{p} X$. Let $\widetilde{x}_0 \in \widetilde{X}$. Then for some other $\widetilde{x} \in \widetilde{X}$, there is a unique path homotopy class of paths from $\widetilde{x}_0$ to $\widetilde{x}$. So points in $\widetilde{X}$ are in a 1-1 correspondence of path homotopy classes of paths starting at $\widetilde{x}_0$. But by the path lifting property, these are all homotopic.

Let's turn this around and define the universal cover of X by its path homotopy classes, that is, let

$$\widetilde{X} := \{[f] \in \Pi_1(X) \mid f(0) = x_0\},$$

where $\Pi_1(X)$ denotes the fundamental groupoid of X. The covering is given by $p : \widetilde{X} \to X$, $[f] \mapsto f(1)$. We want to define a topology on $\widetilde{X}$ that makes p continuous and a covering map. To do this, we define a basis $\mathscr{B}$ and check to see if the inverse image of open sets in the *basis* are continuous. Albin, 24 min lecutre 8 unfinished ⊠

Now that we've proved that for every subgroup we have a covering space, the next question is how many covering spaces per subgroup? We have two covering spaces $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are *equivalent* if there is a homeomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$, or such that the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

If so, it's easy to see that this is an equivalence relation.

Theorem 2.2. *The covering spaces $(\widetilde{X}_i, \widetilde{x}_i) \xrightarrow{p_i} (X, x_0)$, where $i \in \{1, 2\}$ and X is path-connected, locally path-connected are equivalent if and only if $p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

So it turns out the answer to the question above is just one.

Proof. One direction is easy: look at the diagram of induced fundamental groups, and notice that the homeomorphism induces an isomorphism on the subgroups of $\pi_1(X)$. The other direction is more interesting. Let $H_1 = p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$ and $H_2 = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. Since $H_1 \subseteq H_2$ and p_2 is a covering map and X is path-connected and locally path-connected, there exists a lift of $\widetilde{p}_1$ to a map $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ by the lifting criterion, making

[9] Does anyone reading this know of a space that's path-connected but not locally path-connected? I know of many counterexamples for the converse, but without a counterexample to the implication I don't see why local path-connectedness is a necessary condition on top of path-connectedness.

the diagram commute. Similarly, $H_2 \subseteq H_1$, so there's a lift of p_2 to a map $g\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ making the appropiate diagram commute. In particular, we have

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "g\circ f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_1, \widetilde{x}_1) \arrow[dl, "p_1"] \\
& (X, x_0) &
\end{tikzcd}$$

Since the identity is also a lift of p_1 to $(\widetilde{X}_1, \widetilde{x}_1)$, by uniqueness of lifts we have $g \circ f$ equal to the identity, that is, $g \circ f = \mathrm{id}_{\widetilde{X}_1}$. Similarly, we have $f \circ g = \mathrm{id}_{\widetilde{X}_2}$. So f is a homeomorphism. [10] ⊠

Now for the theorem we all came here for.

Theorem 2.3. *Let X be a path-connected, locally path-connected, and semilocally simply-connected space. Then there is a bijection between the coverings $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ up to equivalence and the subgroups of $\pi_1(X, x_0)$. This bijection is given by $p \mapsto p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, we also have a 1-1 correspondence between the non pointed covering spaces $\widetilde{X} \xrightarrow{p} X$ and the conjugacy classes of subgroups, given by the same map $p \mapsto [p_*(\pi_1(\widetilde{X}))]$.*

It's important that we have a covering space and choice of basepoint: if we change the basepoint, we might not necessarily give the same group. Changing the basepoint gives a conjugacy isomorphisms between fundamental groups. This conjugacy isomorphism might give rise to different subgroups, conjugating by some element of the group possibly gives a different subgroup. Hence the second part of the theorem.

It turns out there's an equivalence of posets between covers of a space (X, x_0) (for X a "nice" space) and subgroups of $\pi_1(X, x_0)$, known as the Galois correspondence. The partial order is given by defining two elements to be comparable if one is a cover of another.

2.8 Actions on the fibers

If $p\colon \widetilde{X} \to X$ a cover, $\alpha \in \pi_1(X, x_0)$, define $L_\alpha \in \mathrm{Sym}(p^{-1}(x_0))$ by $L_\alpha \widetilde{x} = \widetilde{\alpha}(0)$, where $\widetilde{\alpha}$ is the lift of x to a path ending at $\widetilde{x}$. We have $L_{\alpha\beta} = L_\alpha \circ L_\beta$, since $L_{\alpha\beta}(\widetilde{x}) = \widetilde{\alpha\beta}(0) = L_\alpha(\widetilde{\beta}(0)) = L_\alpha(L_\beta(\widetilde{x}))$. This is why we defined $L_\alpha(\widetilde{x})$ starting at the left endpoint 0. Albin lecture 9, 36 minutes

Lecture 3

Homology

The big boy has arrived. These notes will follow Hatcher §2.1.

Remark 3.1. This is something I heard even before I enrolled in this course. The homotopy groups are easy to define, but impossible to compute and work with. The homology groups take a lot of work to define, but the resulting groups are much nicer and easier to work with.

The fundamental group is a cool tool when dealing with low-dimensional spaces (the pride and joy of UT Austin), but it doesn't do well with higher dimensional spaces, for example, it can't distinguish between the n-spheres S^n for $n \geq 2$. We can get rid of this limitation by considering the higher homotopy groups $\pi_n(X)$, which are defined in terms of maps from the n-dimensional cube I^n and homotopies $I^n \times I \to X$ of such maps. Cool things about higher homotopy groups: for X a CW complex, $\pi_n(X)$ only depends on the $(n+1)$-skeleton, and $\pi_i(S^n) = 0$ for $i < n$ and

[10] I don't understand where f came from: how can we guarantee its existence?

$\mathbb{Z}$ for $i = n$, as expected. However, the drawback is that they're extremely difficult to compute in general— take the "simple" task of computing $\pi_i(S^n)$ for $i > n$.

Enter the homology groups $H_n(X)$. Similar to $\pi_n(X)$, $H_n(X)$ for X a CW complex depends only on the $(n+1)$-skeleton, and for the spheres $H_i(S^n) \simeq \pi_i(S^n)$ for $1 \le i \le n$, but the homology groups have the advantage in that $H_i(S^n) = 0$ for $i > n$. However, everything has a price. How exactly do we define these so called homology groups? We start by motivating, then doing simplicial homology, before moving onto singular homology. Most efficient method for computing homology groups is called cellular homology. We'll also talk about Mayer-Vietoris sequences, the analogue of the van Kampens for the fundamental group.

Something interesting about homology: most of the time we only use the basic properties of homology, not the definition itself. So we could almost invoke an axiomatic approach, which will happen soon. We could also skip the algebra and talk about geometry, but then Dr. Brand would be unhappy (and so would I), so we'll approach it with a mix of the two (talk about intuition first then state the axioms later).

3.1 The big idea of homology

Issues with homotopy groups: things get really wacky because S^2 has no cells of dimension greater than 2, but some (infinitely many) of the higher homotopy groups $\pi_n(S^2)$ are nontrivial. $\langle god\,shattering\,star\,noises \rangle$ However, homology groups are (directly) related to cell structures, in that you can regard them as an algebraization of how cells of dimension n attach to cells of dimension $n-1$.

Imagine a circle with two antipodal points x and y, with four arrows a, b, c, d drawn in the direction from x to y, which we'll denote by X_1.

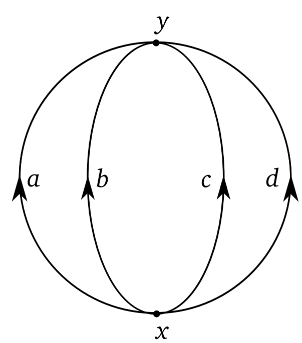

Figure 2: The graph X_1, consisting of two vertices and four edges.

Usually loops are nonabelian, so suppose we abelianize the loops. That is, the loops ab^{-1} and $b^{-1}a$ are "the same circle" (but with a different starting point), so we'll just say they're equal. Formally (not really), rechoosing the basepoint just permutes the letters cyclically, so by abelianizing we can cast off our silly worries about the basepoint. So we make the transition from loops (chosen basepoint) $\longrightarrow$ cycles (no chosen basepoint).

Now we abelian, and all the cool abelian groups use additive notation. So a cycle looks something like $a-b+c-d$ now, a linear combination of edges with integer coefficients. We'll call these linear combinations **chains** of edges. We can decompose these into cycles by several ways, eg $(a-c)+(b-d) = (a-d)+(b-c)$, so it's better just to say cycles are any LC of edges st at least one decomposition make geometric sense. When is a chain a cycle? Cycles are distinguished by the fact that they enter and exit a vertex the same amount of times. So for an arbitrary chain $ka + lb + mc + nd$, it enters y about $k+l+m+n$ times (one for each thing) and enters x (or leaves it) $-k-l-m-n$ times. So if we want $ka+lb+mc+nd$ to be a cycle, we just need to require $k+l+m+n = 0$.

To generalize this, let C_1 be the free abelian group with a basis set $\{a, b, c, d\}$ (edges), and C_0 be the free abelian group with basis $\{x, y\}$ (vertices). Elements of C_1 are chains of edges, and elements of C_0 are linear combinations of vertices. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element to $y - x$, then $\partial(ka+lb+mc+nd) = (k+l+m+n)y - (k+l+m+n)x$, so cycles are precisely $\ker\partial$. It can be seen that $a-b$, $b-c$, and $c-d$ form a basis for $\ker\partial$, so every cycle in X_1 is a unique linear combination of these three elts. Basically, X_1 has three "holes", the three gaps in between the four edges.

Now let's attach a 2-cell to X_1 to get X_2, as seen below.

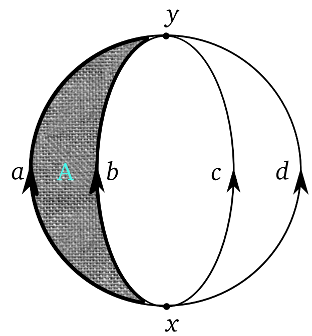

Figure 3: X_1 with a 2-cell attached, denoted X_2. Have you ever seen a 2-cell that looks like cloth?

The 2-cell is attached along the cycle $a-b$, forming the 2-skeleton X_2. Now the cycle is trivial (homotopically), which suggest we form a quotient by factoring out the subgroup generated by $a-b$. For example, $a-c$ and $b-c$ are now equivalent, since they're homotopic in X_2. Algebraically, we define a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the infinite cyclic group generated by A, and $\partial_2(A) = a - b$. ∂_1 is the boundary homomorphism, defined earlier. We are interested in $\ker \partial_1 / \operatorname{im} \partial_2$, that is, the 1-dimensional cycles modulo the boundaries (multiples of $a-b$). Remember, factor groups collapse everything we don't like to the identity. This quotient group is the **homology group** $H_1(X_2)$. If we were to talk about X_1, since it has no 2-cells C_2 is simply zero, so $H_1(X_1) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1$, which is free abelian on three generators. $H_1(X_2)$ is free abelian on two generators ($b-c$ and $c-d$), which expresses the geometric observation that there are two holes remaining after filling one of them in with the 2-cell A.

Let's go farther. Add another 2-cell to the pre-existing 2-cell A, to get the 3-complex X_3.

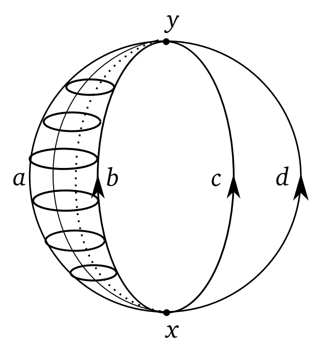

Figure 4: The 3-complex X_3, formed by attaching a 2-cell to X_2.

This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B, and the boundary homomorphism $\partial_2 \colon C_2 \to C_1$ sends A, B to $a-b$. $H_1(X_3) = \ker \partial_1 / \operatorname{im} \partial_2 = H_1(X_2)$, but now ∂_2 has a nontrivial kernel (the infinite cyclic group generated by $A-B$). We view $A-B$ as a 2d cycle generating $H_2(X_3) = \ker \partial_2 \simeq \mathbb{Z}$. The second homology detects the 2d "hole" in X_3.

Unfortunately the diagrams will have to stop now, but let's go even farther and make the complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere by A and B, creating a chain group C_3 generated by C. The boundary homomorphism $\partial_3 \colon C_3 \to C_2$ that sends C to $A-B$ should be seen as the boundary of C, similar to how $a-b$ is the boundary of A. Now we have a sequence of boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and $H_2(X_4) = \ker \partial_2 / \operatorname{im} \partial_3$ is now trivial. $H_3(X_4) = \ker \partial_3 = 0$, note that $H_1(X_4) = H_1(X_3) \simeq \mathbb{Z} \times \mathbb{Z}$, so this is the only homology group of X_4 that isn't trivial.

You can pretty much see where this is going. For a cell complex X, we have chain groups $C_n(X)$ free abelian with basis the n-cells of X, with boundary homomorphisms $\partial_n \colon C_n(X) \to C_{n-1}(X)$, by which we define the homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. So what's the problem? It's how to define ∂_n in general— for $n = 1$ this is easy, it's the vertex head minus the one at the tail. For $n = 2$, it still isn't hard per say, if the cell is attached on a loop of edges, just take the cycle of edges, keeping in mind orientation. This is much tricker for higher dimension cells, even with restrictions to polyhedral cells and nice attaching maps we still have to worry about orientation and stuff.

So what do we do? Use triangles, of course. We can subdivide arbitary polyhedra into certain special types of polyhedra called simplices (what we talked about in class day 1), so there isn't any loss of generality (but there is a loss of efficiency). This gives rise to our more basic **simplicial homology**, which deals with cell complexes from simplices. However, we are still quite limited in what we can do.

So, what do we really do this time? Make things less simple, and make your life difficult by considering the collection of all possible continuous maps of simplices into a space X (wow). The chain groups $C_n(X)$ are tremendously large, but the quotients $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, the **singular homology groups**, are much smaller and easier to work with[11]. For example, in the examples above the singular homology groups coincide with the ones computed from cellular chains. Furthermore (as we will see later), singular homology lets us define these nice cellular homology groups for *all* cell complexes, which solves the issue of how to define boundary maps for cellular chains.

3.2 The structure of Δ-complexes

I have a feeling we're gonna be typing a lot of `\Delta`'s. So basically, the only thing cool kids talk about is singular homology, but it's kinda complicated so we gotta talk about the inferior version for those who have the brain capacity of a literal ape[12], simplicial homology, first. We talk about simplicial homology in the domain of Δ-complexes. Take the standard fundamental polygons with orientation for $\mathbb{T}^2$, $\mathbb{R}P^2$, and the Klein bottle K. Cut the squares in half with a diagonal to get two triangles, from here we can get the original shape by identifying in pairs. We can do this with any n-gon, decomposing it into $n-2$ base triangles. So we can make any closed surface from triangles, furthermore, we could also make a larger class of spaces that aren't surfaces by allowing more than two edges to be glued together at the same time.

The idea of a Δ-complex is to generalize these constructions to n-dimensions. The n-dimensional triangle is the n-simplex, the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that don't lie in a hyperplane of dimension less than n, where by "hyperplane" we mean the set of solutions to a system of linear equations. We could also just say that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are LI. The v_i are **vertices** of the simplex, and the simplex itself is $[v_0, \cdots, v_n]$.

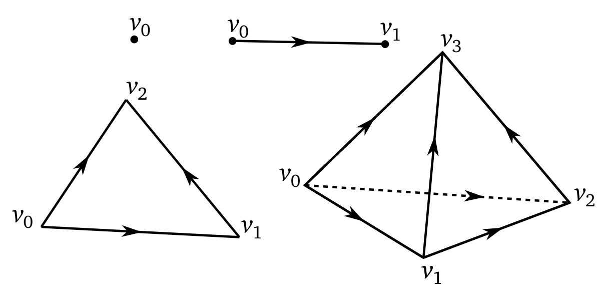

Figure 5: The 0-simplex to the 3-simplex, respectively (with ordered vertices and oriented edges).

For example, we have the standard n-simplex given by

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\},$$

whose vertices are the unit vectors along the coordinate axes. Think of this as taking the unit vectors, and drawing a triangle from each of their endpoints. This works because the difference vectors are LI. For homology, orientation

[11] For reasonably "nice" spaces X, of course.

[12] The book simply says "primitive" version, so I used my imagination a little bit.

of vertices is really important, so n-simplex really means n-simplex with an ordering on its vertices. Ordering the vertices will determine an orientation on its subscripts, as can be seen in Figure 5. This also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other simplex $[v_0, \cdots, v_n]$ that preserves the order of the vertices, given by

$$(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i.$$

We say the coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i \in [v_0, \cdots, v_n]$. Deleting a vertex of a n-simplex yields something that spans an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$. We'll adopt the following convention: *The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.* That sounds reasonable enough. We say the union of all faces of Δ^n is the **boundary** of Δ^n, written $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is equal to $\Delta^n \setminus \partial\Delta^n$, the interior of Δ^n.

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is onto, and each point of X is in the image of exactly one restriction $\sigma_\alpha | \mathring{\Delta}^n$.

2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear order-preserving homeomorphism.

3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

A consequence of (3) is that X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha : \Delta^n \to X$, the quotient space obtained by idenfying each face of a Δ^n_α with the Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face in question. You can think of this as basically cell complexes, attaching 0-simplices (cells) to 1-simplices and 2-simplices, and so on.

In general, we can make Δ-complexes from collections of disjoint simplices by identifying various subspaces spanned by subsets of the vertices, with identifications performed by the canonical order-preserving linear homeomorphisms. Note that if we think of a Δ-complex X as a quotient space of disjoint simplices, then X must be Hausdorff. Each restriction $\sigma_\alpha | \mathring{\Delta}^n$ is a homeomorphism onto its image by condition (3), which is an open simplex in X. Then these open simplices are the cells e^n_α of a CW complex structure on X with the σ_α's as characteristic maps (we won't use this fact yet).

3.3 Simplicial homology

Goal: define simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Formally, we can write elements of $\Delta_n(X)$ as finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$, called **n-chains.** We could also write $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e^n_α, with image the closure of e^n_α. Such a sum can be thought of as a finite collection, or 'chain', of n-simplices in X.

Take a look at $\partial[v_0, v_1] = [v_1] - [v_0]$, $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$, and $\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$. Naïvely, one might assume the boundary of an n-simplex to be the sum of the faces delete a point, denoted by $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ where v_i is the vertex to be deleted. However, note the signs to take orientations into account, it just happens that they work out based on the position of v_i. So we have

$$\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots v_n].$$

Keeping this in mind, let's define a **boundary homomorphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ for X a general Δ-complex by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \ \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Lemma 3.1. *The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.*

Proof. Note that

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i}(-1)^i(-1)^j\ \sigma|[v_0,\cdots,\hat{v}_j,\cdots,\hat{v}_i,\cdots,v_n] + \sum_{j>i}(-1)^i(-1)^{j-1}\ \sigma|[v_0,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_n].$$

Then after switching i and j in the second term, it becomes the negative of the first. Alternate proof from Dr. Allcock: note that $\partial\sigma := \sum_{i=0}^n(-1)^i\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,v_n]$. Then

$$\partial\partial\sigma = \sum_{i=0}^n(-1)^i\partial\left(\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,v_n]\right),$$

which distributes because C_{n-1} is free on {singular $(n-1)$-simplex}. So defining any function {singular $(n-1)$-simplex} $\to C_{n-2}$ extends to a $\mathbb{Z}$-linear map $C_{n-1}\to C_{n-2}$. Then

$$\partial\partial\sigma = \sum_{i=0}^n(-1)^i\left(\sum_{j=0}^{i-1}\sigma\circ[v_0,\cdots,\hat{v}_j,\cdots,\hat{v}_i,\cdots,v_n](-1)^j + \sum_{j=i+1}^{n}\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_n](-1)^{j-1}\right),$$

which is equal to zero by cancellation[13]. ⊠

What we have here is a sequence of homomorphisms of abelian groups

$$\cdots\to C_{n+1}\xrightarrow{\partial_{n+1}}C_n\xrightarrow{\partial_n}C_{n-1}\to\cdots\to C_1\xrightarrow{\partial_1}C_0\xrightarrow{\partial_0}0$$

with $\partial_n\partial_{n+1}=0$ for all n. This is called a **chain complex**. Note that we've extended the sequence to 0, with $\partial_0=0$. The equation $\partial_n\partial_{n+1}=0$ is equivalent to the inclusion $\operatorname{im}\partial_{n+1}\subseteq\ker\partial_n$, so we can define the **nth homology group** of the chain complex as $H_n=\ker\partial_n/\operatorname{im}\partial_{n+1}$. Elements of $\ker\partial_n$ are called **cycles** and elements of $\operatorname{im}\partial_{n+1}$ are called boundaries. Cosets of $\operatorname{im}\partial_{n+1}$ in H_n are called **homology classes**. Two cycles representing the same homology class are said to be **homologous**, that is, their difference is a boundary. When $C_n=\Delta_n(X)$, the homology group $\ker\partial_n/\operatorname{im}\partial_{n+1}$ will be denoted by $H_n^\Delta(X)$ and called the **$\mathbf{n^{th}}$ simplicial homology group** of X.

3.4 Homological algebra

We'll take this section to digress a bit and talk about some homological algebra. These notes will follow May §12.

Let R be a commutative ring: the main example is $R=\mathbb{Z}$. A **chain complex** over R is a sequence of R-modules

$$\cdots\to X_{i+1}\xrightarrow{d_{i+1}}X_i\xrightarrow{d_i}X_{i-1}\to\cdots$$

such that $d_i\circ d_{i+1}=0$ for all i (abbreviated $d=d_i$). A **cochain complex** over R is an analogous sequence

$$\cdots\to Y^{i-1}\xrightarrow{d^{i-1}}Y^i\xrightarrow{d^i}Y^{i+1}\to\cdots$$

with $d^i\circ d^{i+1}=0$. Usually $X_i=0$ for $i<0$ and $Y^i=0$ for $i<0$ (or else $\{X_i,d_i\}\to\{X^{-i},d^{-i}\}$, making chain and cochain complexes equivalent). An element of the kernel of d_i is a **cycle** and an element of the image of d_{i+1} is a **boundary**. This makes a lot more sense if you picture the boundary map d_i as removing a vertex to get an $n-1$ simplex each time. We say two cycles are **homologous** if their difference is a boundary, and write $B_i(X)\subseteq Z_i(X)\subseteq X_i$ for the submodules of boundaries and cycles, respectively. Then we can define the **ith homology group** $H_i(X)$ as the quotient module $Z_i(X)/B_i(X)$, and write $H_*(X)$ for the sequence of R-modules $H_i(X)$. To get things straight, we've defined things the following way:

$$Z_i(X) = \text{ cycles } := \ker d_i\subseteq X_i$$
$$B_i(X) = \text{ boundaries } := \operatorname{im} d_{i+1}\subseteq X_i.$$

[13] The proof from Dr. Allcock was for singular homology, but the idea is the same.

A **chain map** $f : X \to X'$ of chain complexes is a sequence of maps of R-modules $f_i : X_i \to X'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$ for all i. That is, the following diagram commutes for all i[14]:

$$\begin{tikzcd} \cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r, "d_i"] \arrow[d, "f_i"] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\ \cdots \arrow[r] & X'_{i+1} \arrow[r, "d'_{i+1}"] & X'_i \arrow[r, "d'_i"] & X'_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

It follows that $f_i(B_i(X)) \subseteq B_i(X')$ and $f_i(Z_i(X)) \subseteq Z_i(X')$. Therefore we have that f induces a map of R-modules $f_* = H_i(f) \colon H_i(X) \to H_i(X')$. A **chain homotopy** $s \colon f \simeq g$ between chain maps $f, g \colon X \to X'$ is a sequence of homomorphisms $s_i : X_i \to X'_{i+1}$ such that

$$d'_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

for all i. Chain homotopy is an equivalence relation (this was an exercise) since if $t \colon g \simeq h$, then $s + t = \{s_i + t_i\}$ is a chain homotopy $f \simeq h$.

Lemma 3.2. *Chain homotopic maps induce the same homomorphism of homology groups.*

Proof. Let $s \colon f \simeq g$, $f, g \colon X \to X'$. If $x \in Z_i(X)$, then $f_i(x) - g_i(x) = d'_{i+1} s_i(x)$ such that $f_i(x)$ and $g_i(x)$ are homologous. ⊠

A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of modules is **exact** if $\operatorname{im} f = \ker g$. If $M' = 0$, then g is a monomorphism; if $M'' = 0$, then f is an epimorphism. We proved this as an exercise! A longer sequence is exact if it is exact at each position. A **short exact sequence** of chain complexes is a sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

that is exact in each degree. Here 0 denotes that chain complex that is the 0 module in each degree.

Proposition 3.1. *A short exact sequence of chain complexes naturally gives rise to a LES of R-modules*

$$\cdots \to H_q(X') \xrightarrow{f} H_q(X) \xrightarrow{g_*} H_q(X'') \xrightarrow{\partial} H_{q-1}(X') \to \cdots .$$

Proof. Let $[x]$ denote the homology class of a cycle x. Define the "connecting homomorphism" $\partial \colon H_q(X'') \to H_{q-1}(X')$ by $\partial[x''] = [x']$, where $f(x') = d(x)$ for some x such that $g(x) = x''$. There exists such an x because g is an epimorphism, and x' exists because $gd(x) = dg(x) = 0$. Use a "diagram chase" to verify that ∂ is well defined and the sequence is exact. Naturality means that a commutative diagram of short exact sequences of chain complexes gives rise to a commutative diagram of long exact sequences of R-modules. The big idea is the naturality of the connecting homomorphism, which is left as an exercise to the reader. ⊠

3.5 Singular homology

These notes will follow Massey §2 and the rest of Hatcher §2.1.

[14] May's diagram showed much less, but I feel this illustrates the idea much better: it also makes following around the chain homotopy homomorphisms easier.

Let's define $H_0(X)$ as such: let $Z_0(X) = C_0(X)$ and $H_0(X) = Z_0(X)/B_0(X) = C_0(X)/B_0(X)$. Anoter way we could do this is is defining $C_n(X) = \{0\}$ for $n < 0$, then defining $\partial_n \colon C_n(X) \to C_{n-1}(X)$ in the only possible way for $n \leq 0$ (i.e., $\partial_n = 0$ for $n \leq 0$), and finally defining $Z_n(X) = \ker \partial_0$. In general, we could define $Z_n(X) = \ker \partial_n$for all integers n, $B_n(X) = \partial_{n-1}(C_{n+1}(X)) \subseteq Z_n(X)$, and $H_n(X) = Z_n(X)/B_n(X)$ for all n, with $H_n(X) = \{0\}$ for $n < 0$.

Now let's define (not really, we'll ignore the definition) the reduced 0-dimensional homology group $\widetilde{H}_0(X)$. Let's define a homomorphism $\varepsilon \colon C_0(X) \to \mathbb{Z}$, often called the *augmentation*, made by the typical barycentric coordinate sum $\varepsilon \colon \sum_i n_i \sigma_i \mapsto \sum_i n_i$. Then $\varepsilon \circ \partial_1 = 0$: to do this, show that $\varepsilon(\partial_1(T)) = 0$ for some 1-cube (not hard)[15]. Then we can define $\widetilde{Z}_0(X) = \ker \varepsilon$, and

$$\widetilde{H}_0(X) = \widetilde{Z}_0(X)/B_0(X).$$

We say that $\widetilde{H}_0(X)$ is the **reduced 0-dimensional homology group** of X. To avoid weird stuff happening, assume $X \neq \emptyset$. It's often convenient to set $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$.

JK, back to Hatcher. Some examples of simplicial homology:

Example 3.1. Let $X = S^1$, with one vertex v and an edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map ∂_1 is zero since $\partial e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \geq 2$ since there are no simplices in these dimensions. Therefore

$$H_n^{\Delta}(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

Example 3.2. Let $X = \mathbb{T}$, the torus with a Δ-complex structure having one vertex, three edges a, b, and c,, and two 2-simplices U and L. Since $\partial_1 = 0$, $H_0^{\Delta}(\mathbb{T}) \simeq \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(\mathbb{T})$, it follows that $H_1^{\Delta}(\mathbb{T}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^{\Delta}(\mathbb{T})$ is equal to $\ker \partial_2$, which is infinite cyclic generated by $U - L$. So

$$H_n^{\Delta}(\mathbb{T}) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z} & \text{for } n = 0, 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

Let's talk about **singular homology**. A **singular n-simplex** in a space X is just a map $\sigma \colon \Delta^n \to X$. The word 'singular' is used to imply that the map doesn't have to be nice (look like a simplex) but can have weird 'singularities'. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$, called **n-chains** (more precisely, singular n-chains) are finite formal sums $\sum_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i \colon \Delta^n \to X$. A boundary map $\partial_n \colon C_n(X) \to C_{n-1}(X)$ is defined by the same formula as before:

$$\partial_n(\sigma) = \sum_i (-1)^i \big| [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Then $\sigma | [v_0, \cdots, \hat{v}_i, \cdots v_n]$ is a map $\Delta^{n-1} \to X$, that is , a singular $(n-1)$-simplex. We also have $\partial_n \partial_{n+1} = 0$ (more concisely $\partial^2 = 0$), so we define the singular homology group $H_n(x) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. Singular chain groups tend to be really large (often uncountable), but modding out makes the homology groups easier to work with.

Proposition 3.2. *For a space X, there is an isomorphism $H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$, where X_α denotes the path-components of X.*

Proof. Since a singular simplex always has a path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. This is preserved by the boundary maps ∂_n and similarly $\ker \partial_n$ and $\operatorname{im} \partial_{n+1}$. ⊠

Proposition 3.3. *If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

[15] I'm glossing over formal stuff because everywhere else uses triangles instead of cubes. I just want results!

Proof. We have $H_0(X)/\operatorname{im}\partial_1$ since $\partial_0 = 0$. Define a homomorphism $\varepsilon\colon C_0(X) \to \mathbb{Z}$ by $\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$. This is onto if $X \neq \emptyset$: we claim that $\ker\varepsilon = \operatorname{im}\partial_1$ if X is path-connected, and hence ε induces an isomoprhism $H_0(X) \approx \mathbb{Z}$. To see that this is true, observe that $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$ since for a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have $\varepsilon\partial_1(\sigma) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$. To show that $\ker\varepsilon \subseteq \operatorname{im}\partial_1$, suppose $\varepsilon\left(\sum_i n_i\sigma_i\right) = 0$, so $\sum_i n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i\colon I \to X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i\colon [v_0, v_1] \to X$, then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence $\partial\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$ since $\sum_i n_i = 0$. So $\sum_i n_i\sigma_i$ is a boundary, which shows that $\ker\varepsilon \subseteq \partial_1$. ⊠

Proposition 3.4. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof. In this case there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_i (-1)^i\sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. So we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternately isomorphisms and trivial maps, except for the last $\mathbb{Z}$. So the homology groups of this complex are trivial for every group besides $H_0 \simeq \mathbb{Z}$. ⊠

Sometimes weird stuff happens with $H_0(X)$, as can be seen in Proposition 3.4. To avoid this, we can talk about the **reduced homology groups** $\widetilde{H}_n(X)$, defined to be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the same one as in our earlier proposition[16]. Since $\varepsilon\partial_1 = 0$, ε vanishes on $\operatorname{im}\partial_1$ adn hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. Obviously $H_n(X) \simeq \widetilde{H}_n(X)$ for $n > 0$.

3.6 Exact sequences

Definition 3.1 (Exact sequences)**.** A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be **exact** if $\ker\alpha_n = \operatorname{im}_{n+1}$ for each n.

The inclusions $\operatorname{im}\alpha_{n+1} \subseteq \ker\alpha_n$ are equivalent to $\alpha_n\alpha_{n+1} = 0$, so the sequence is a chain complex, and the opposite inclusions $\ker\alpha_n \subseteq \operatorname{im}\alpha_{n+1}$ say that the homology groups of this chain complex are trivial. We can express a number of basic algebraic concepts in terms of exact sequences, for example:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker\alpha = 0$, i.e., α is injective.

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{im}\alpha = B$, i.e., α is surjective.

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism, by (i) and (ii).

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective, and $\ker\beta = \operatorname{im}\alpha$, so β induces an isomorphism $C \simeq B/\operatorname{im}\alpha$. This can be written as $C \simeq B/A$ if we think of α as an inclusion of A as a subgroup of B.

An exact sequence $0 \to A \to B \to C \to 0$ as in (iv) is called a **short exact sequence**. These turn out to be the perfect tool for stuff, in particular, relating the homolofy groups of a space, a subspace, and the associated quotient space.

Theorem 3.1. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0,$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

[16] My clever references aren't working??

Proof. Basically, construct ∂. The idea is that an element $x \in \widetilde{H}_n(X/A)$ can be represented by a chain α in X with $\partial\alpha$ a cycle in A whose homology class is $\partial x \in \widetilde{H}_{n-1}(A)$. The full proof will come later. Pairs of spaces (X,A) that satisfy the hypothesis of the theorem will be called **good pairs**[17]. ⊠

Corollary 3.1. *$\widetilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0$ for $i \neq n$.*

Proof. For $n > 0$ take the good pair $(X,A) = (D^n, S^{n-1})$ so $X/A = S^n$. Since D^n is contractible the terms $\widetilde{H}_i(D^n)$ in the LES for this pair are zero. Then by the exactness of the sequence the maps $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\widetilde{H}_0(S^n) = 0$. Then our result follows by induction on n, in which the base case of S^0 holds by Proposition 3.2 and Proposition 3.4. ⊠

Lemma 3.3. *Every continuous map $h\colon D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.*

Proof. This was actually an earlier theorem in Hatcher. As you can see, this will lead into Brouwer's fixed point theorem. Suppose that $h(x) \neq x$ for all $x \in D^2$. Then we can define a map $r\colon D^2 \to S^1$ by letting $r(x)$ be the point of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2. Now r is continuous, furthermore, $r(x) = x$ if $x \in S^1$. So r is a retraction of D^2 onto S^1, but no such retraction exists: let f_0 be a loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop, for example $f_t(s) = (1-t)f_0(s) + tx_0$ for x_0 the basepoint of f_0. Since the retraction r is the identity on S^1, the composition rf_t is a homotopy in S^1 from $rf_0 = f_0$ to the constant loop at x_0: but this contradicts the fact that $\pi_1(S^1)$ is nonzero. ⊠

Corollary 3.2 (Brouwer's fixed point theorem)**.** *∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.*

Proof. If $r\colon D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for $i\colon \partial D^n \to D^n$ the inclusion map. The composition $\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\widetilde{H}_{n_1}(\partial D^n) \simeq \mathbb{Z}$. But i_* and r_* are both 0 since $\widetilde{H}_{n-1}(D^n) = 0$, and we have a contradiction. For the fixed point portion, just replace π_1 with H_n in Lemma 3.3 and we're good. ⊠

3.7 Relative homology (todo)

Sometimes ignoring things makes things easier, for example arithmetic modulo n (ignoring multiples of n). Relative homology is another example: in this case, we ignore all singular chains in a subspace of a given space.

Given a space X and a subspace $A \subseteq X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$, thus chains in A are trivial in $C_n(X,A)$. Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$. Then we have a sequence of boundary maps

$$\cdots \to C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \to \cdots$$

The relation $\partial^2 = 0$ holds since it held before (then holds for quotients).

Definition 3.2 (Relative homology groups)**.** Given the chain complex above, the homology groups $\ker\partial / \operatorname{im}\partial$ of the chain complex are the **relative homology groups** $H_n(X,A)$. We can see the following:

- Elements of $H_n(X,A)$ are represented by **relative cycles**: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle is trivial in $H_n(X,A)$ iff it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

These properties make precise the intuitive idea that $H_n(X,A)$ is 'homology of X modulo A'.

Goal: show that the relative homology groups $H_n(X,A)$ for any pair (X,A) fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

To do this, we'll go on our first diagram chase. Consider the diagram

[17] *We're a good pair, you and I...*

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

where i is the inclusion map and j is the quotient map. If we let n vary and draw the short exact sequences vertically instead of horizontally, we have a large commutative diagram like the one below, where the columns are exact and the rows are chain complexes denoted by A, B, and C.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A diagram like this is called a **short exact sequence of chain complexes**. We'll show that this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where $H_n(A)$ denotes the homology group $\ker \partial / \operatorname{im} \partial$ at A_n in the chain complex, $H_n(B)$ and $H_n(C)$ similarly defined. To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, let $c \in C_n$ be a cycle. Then since j is onto, $c = j(b)$ for some $b \in B_n$. Then $\partial b \in B_{n-1}$ is also in $\ker j$ since $j(\partial b) = \partial j(b) = \partial c = 0$.

3.8 Homology with coefficients (todo)

3.9 Degrees of maps $S^n \to S^n$ (todo)

3.10 Cellular homology

Following Pierre Albin lecture 19 and Hatcher for more technical things. Recall that if X is a Δ-complex then $H_*^\Delta(X) \simeq H_*(X)$, and that $H_*^\Delta(X)$ is easy to compute and $H_*(X)$ is easy to prove theorems about. In an ideal world, we would like a similar equivalence for when X is a CW complex since they're much more applicable, but we ran into an issue when figuring out how to define the boundary maps. What we're going to do is defined a chain complex $C_n^{\mathrm{CW}}(X)$, and we want it to be free abelian on the n-cells of X.

Lemma 3.4. *If X is a CW complex, then:*

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.

(c) The map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof. Statement (a) follows immediately from the fact that (X^n, X^{n-1}) is a good pair and X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X (it does!). Next consider the following part of the LES of the pair (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

If $k \neq n$ the last term is zero by (a) so the middle map is surjective, while if $k \neq n-1$ then the first term is zero so the middle map is injective. Now look at the inclusion-induced homomorphisms:

$$H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1})$$

It follows that all of these maps are isomorphisms, except that the map to $H_k(X^k)$ may not be surjective and the map from $H_k(X^k)$ may not be injective. Then the first part the sequence gives (b) since $H_k(X^0) = 0$ when $k > 0$. The last part gives (c) when X is finite-dimensional. The proof when X is infinite-dimensional requires a little more work. ⊠

Let X be a CW complex. What we want is a boundary map $C^{\mathrm{CW}}_{n+1}(X) \xrightarrow{\partial^{\mathrm{CW}}_{n+1}} C^{\mathrm{CW}}_n(X)$. By Lemma 3.4, we have

$$\begin{tikzcd}
& C^{\mathrm{CW}}_{n+1}(X) \arrow[rr, dash, "\partial^{\mathrm{CW}}_{n+1}"] \arrow[d, equal] & & C^{\mathrm{CW}}_n(X) \arrow[d, equal] \\
& H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] & H_n(X^n) \arrow[r, "\partial_n"] & H_n(X^n, X^{n-1})
\end{tikzcd}$$

The equalities are from Lemma 3.4, and the boundary maps between homology groups are from the LES of the good pair (X^n, X^{n-1}). Then this naturally extends to the diagram shown in Figure 6.

$$\begin{tikzcd}
& & & & 0 & & \\
& 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
& & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "\partial^{\mathrm{CW}}_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "\partial^{\mathrm{CW}}_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
& & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
& & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Figure 6: The diagram for cellular homology.

In this diagram, $\partial^{\mathrm{CW}}_{n+1}$ and ∂^{CW}_n are defined as the compositions $j_n \partial_{n+1}$ and $j_{n-1}\partial_n$, which are just 'relativizations' of the boundary maps ∂_{n+1} and ∂_n. The composition $\partial^{\mathrm{CW}}_n \partial^{\mathrm{CW}}_{n+1}$ contains two successive maps in one of the exact sequences, hence is zero (since image maps onto kernel maps onto zero by exactness). The horizonal row in the diagram is a chain complex, called the **cellular chain complex** of X, since $H_n(X^n, X^{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n-cells of X. The resulting homology groups are called the **cellular homology groups** of X. We temporarily denote them $H^{\mathrm{CW}}_n(X)$.

Theorem 3.2. *$H^{CW}_n(X) \simeq H_n(X)$.*

Proof. We can identify $H_n(X)$ with $H_n(X^n)/\operatorname{im}\partial_{n+1}$ by a simple application of the FHT and exactness. Since j_n is injective, it maps $\operatorname{im}\partial_{n+1}$ isomorphically onto $\operatorname{im}(j_n\partial_{n+1}) = \operatorname{im}\partial^{\mathrm{CW}}_{n+1}$ and $H_n(X^n)$ isomorphically onto $\operatorname{im} j_n = \ker\partial_n$. Since j_{n-1} is injective, $\ker\partial_n = \ker\partial^{\mathrm{CW}}_n$. So j_n induces an isomorphism of the quotient $H_n(X^n)/\operatorname{im}\partial_{n+1} \simeq H_n(X)$ onto $\ker\partial^{\mathrm{CW}}_n / \operatorname{im}\partial^{\mathrm{CW}}_{n+1} = H^{\mathrm{CW}}_n(X)$. ⊠

Some immediate applications:

(i) $H_n(X) = 0$ if X is a CW complex with no 0-cells.

(ii) More generally, if X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements. For since $H_n(X^n, X^{n-1})$ is free abelian on k generators, the subgroup $\ker \partial_n^{\mathrm{CW}}$ must be generated by at most k elements, hence also the quotient $\ker \partial_n^{\mathrm{CW}} / \operatorname{im} \partial_{n+1}^{\mathrm{CW}}$.

(iii) If X is a CW complex having no two of its cells in adjacet dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X. This is because the cellular boundary maps ∂_n^{CW} are automatically zero in this case.

Example 3.3. For $\mathbb{C}\mathrm{P}^n$ having a CW structure with one cell of each even dimension $2k \le 2n$, we have

$$H_i(\mathbb{C}\mathrm{P}^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \cdots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Another example is $S^n \times S^n$ with $n > 1$, using the product CW structure consisting of a 0-cell, two n-cells, and a $2n$-cell.

Proposition 3.5 (Cellular boundary formula). *We have*

$$\partial_n^{CW}(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.

Here we identify the cells e_α^n and e_β^{n-1} with generators of the corresponding summands of the cellular chain groups. The summation in the formula contains only finitely many terms since the attaching map of e_α^n has compact image, so this image meets only finitely many cells e_β^{n-1}. From now on, we'll denote ∂_n^{CW} by d_n.

TODO commutative diagram and justification for cellular boundary formula

Example 3.4. Let M_g be the closed orientable surface of genus g with its usual CW structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

As observed above, d_1 must be 0 since there is only one 0-cell. Also, d_2 is 0 because each a_i or b_i appears with its inverse in $[a_1, b_1] \cdots [a_g, b_g]$, so the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps. Since d_1 and d_2 are both zero, the homology groups of M_g are the same as the cellular chain groups, namely, $\mathbb{Z}$ in dimensions 0 and 2, and $\mathbb{Z}^{2g}$ in dimension 1.

3.11 Axioms for homology

Let's take a formal viewpoint at some properties that all homology theories share.

Definition 3.3. A (reduced) **homology theory** assigns a sequence of abelian groups $\widetilde{h}_n(X)$ and a sequence of homomorphisms $f_* \colon \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ to each nonempty CW complex X and each map $f : X \to Y$ between chain complexes. These groups and homomorphisms satisfy $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$, and the following axioms:

(1) If f is homotopic to g, that is $f \simeq g : X \to Y$, then $f_* = g_* \colon \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$.

(2) There are boundary homomorphisms $\partial : \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ defined for each CW pair (X, A),[18] fitting into an exact sequence

$$\cdots \xrightarrow{\partial} \widetilde{h}_n(A) \xrightarrow{i_*} \widetilde{h}_n(X) \xrightarrow{q_*} \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

[18] A **CW pair** (X, A) is just a CW complex X equipped with a subcomplex inclusion $A \hookrightarrow X$.

where i is the inclusion and q is the quotient map. Furthermore the boundary maps are natural (as in natural transformation): For $f:(X,A)\to(Y,B)$ inducing a quotient map $\overline{f}: X/A\to Y/B$, there are commutative diagrams

$$\begin{tikzcd} \widetilde{h}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"] & \widetilde{h}_{n-1}(A) \arrow[d, "f_*"] \\ \widetilde{h}_n(Y/B) \arrow[r, "\partial"] & \widetilde{h}_{n-1}(B) \end{tikzcd}$$

(3) For a wedge sum $X=\bigvee_\alpha X_\alpha$ with inclusions $i_\alpha: X_\alpha\hookrightarrow X$, the direct sum map $\bigoplus_\alpha i_{\alpha*}: \bigoplus_\alpha \widetilde{h}_n(X_\alpha)\to\widetilde{h}_n(X)$ is an isomorphism for all n.

Some notes on this new definition for homology. Negative values of n are allowed, in our standard singular homology theory they became zero by definition but there exist some interesting homology theories with nontrivial groups in negative dimensions. Also, the third axioms follows from the first two in the case of a finite wedge sum, but not an infinite one.

You can also give axioms for unreduced homology theories, suppose we have relative groups $h_n(X,A)$, define the absolute groups by $h_n(X)=h_n(X,\emptyset)$. Then axiom (2) splits into two, one about long exact sequences with natural boundary maps, and one about excision (eg $h_n(X,A)\approx h_n(X/A, A/A)$ for CW pairs). In axiom (3) replace wedge sum with disjoint union. The axioms are essentially the same as the ones proposed seventy years ago in [Eilenberg & Steenrod 1952], besides the fact that (3) was omitted (to focus on finite CW complexes). There was also an additional axiom called the *dimension axiom* specifying that the groups $h_n(\text{point})$ are zero for $n\neq 0$. At the time there were no interesting homology theories for which the dimension axiom doesn't hold, but now we have *bordism* in which bordism groups of a point are nonzero in infinitely many dimensions.

Reduced and unreduced homology theories are essentially equivalent. We can get a reduced theory $\widetilde{h}$ from an unreduced theory h by setting $\widetilde{h}_n(X)$ equal to the kernel of the canonical map $h_n(X)\to h_n(\text{point})$. For the other direction, set $h_n(X)=\widetilde{h}_n(X\amalg\text{point})$. You can show that these two transformation are each others inverses. We have $h_n(X)\approx\widetilde{h}_n(X)\oplus h_n(x_0)$ for any point $x_0\in X$, since the LES of the pair (X,x_0) splits via the retraction of X onto x_0. Also note that $\widetilde{h}_n(x_0)=0$ for all n, just look at the LES of the pair (x_0,x_0). TODO more stuff on coefficients

We can also get Mayer-Vietoris sequences from the axioms. For a CW complex $X=A\cup B$ with A,B subcomplexes, the inclusion $(B,A\cap B)\hookrightarrow(X,A)$ induces a commutative diagram of exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & h_{n+1}(B,A\cap B) \arrow[r] \arrow[d, "\approx"] & h_n(A\cap B) \arrow[r] \arrow[d] & h_n(B) \arrow[r] \arrow[d] & h_n(B,A\cap B) \arrow[r] \arrow[d, "\approx"] & \cdots \\ \cdots \arrow[r] & h_{n+1}(X,A) \arrow[r] & h_n(A) \arrow[r] & h_n(X) \arrow[r] & h_n(X,A) \arrow[r] & \cdots \end{tikzcd}$$

The vertical maps are isomorphisms since $B/(A\cap B)=X/A$. Then a diagram like this with every third vertical map an isomorphism gives rise to a LES with the remaining nonisomoprhic terms, which looks like

$$\cdots\to h_n(A\cap B)\xrightarrow{\varphi} h_n(A)\oplus h_n(B)\xrightarrow{\psi} h_n(X)\xrightarrow{\partial} h_{n-1}(A\cap B)\to\cdots$$

(This is left as an exercise to the reader.)

Lecture 4

Homotopy theory

Here comes a long block of Hatcher exposition, read if interested, skip if not.

We have met the first homotopy group already, the fundamental group $\pi_1(X)$. The higher dimensional analogues $\pi_n(X)$ are the *homotopy groups*, which have some similarities to the homology groups: $\pi_n(X)$ is abelian for $n \geq 2$, and there are relative homotopy groups fitting into a LES similar to homology. However, neither Seifert-van Kampen's nor excision holds, making the homotopy groups much harder to compute.

However, these groups are still important: one reason is *Whitehead's theorem*, which states that a map between CW complexes inducing isomorphisms on the homotopy groups is a homotopy equivalence. However the stronger statement that if two complexes have isomorphic homotopy groups then they're homotopy equivalent is false usually, aside from the case where we only have one nontrivial homotopy group— these spaces are called *Eilenberg-MacLane spaces*.

Another more direct connection between homology and homotopy is the *Hurewicz theorem*, which says that the first nonzero homotopy group $\pi_n(X)$ of a simply-connected space X is isomorphic to the first nonzero homology group $\widetilde{H}_n(X)$. Though excision doesn't always hold, in some important special cases it does for a range of dimensions. This leads to the idea of *stable homotopy groups*, the beginning of stable homotopy theory. If you figure out how to compute the stable homotopy groups of spheres, you can pick up your Fields medal at the door.

We'll also talk a little about fiber bundles which somewhat generalize the idea of covering spaces for higher homotopy groups, purely to lead toward fibrations. These allow us to describe how the homotopy type of a CW complex is inductively built up from its homotopy groups by forming 'twisted products' of Eilenberg-MacLane spaces, which is the notion of a *Postnikov tower*.

4.1 Whitehead's theorem and relative homotopy groups

Let I^n be the n-cube, and the boundary ∂I^n be the subspace of points with at least one coordinate equal to 0 or 1.

Definition 4.1 (Higher homotopy groups). For a pointed space X, x_0, define the **n-th homotopy group** $\pi_n(X, x_0)$ as the set of homotopy classes $f : (I^n, \partial I^n) \to (X, x_0)$ where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. If f and g are two maps from the n-cube into (X, x_0), then we define the composition as

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n) & s_1 \in [1/2, 1]. \end{cases}$$

If we use $[f]$ to denote the homotopy classes of f (rel ∂), then $[f]+[g]=[f+g]$. We have $(f+g) \in \pi_n(X, x_0)$ since $\partial I^n \to 0$, and thus we have given $\pi_n(X, x_0)$ a group structure. Visualizing composition in terms of spheres, we crush the equatorial S^{n-1} to a point yielding a wedge of two S^n's, and from here the picture is the same as with cubes. Associativity is most naturally proven by the cubical picture.

Theorem 4.1. *$\pi_{n\geq 2}$ is commutative.*

Proof. Consider $f+g$ as the composition of two maps of spheres, with the equator glued to the basepoint. We will show this is the same as $g+f$. The homotopy takes place in the domain S^n (slogan "work in the domain"), which is just a rotation of the sphere until the equator returns to itself, exchanging the top and bottom hemispheres. This defines a homotopy $S^n \times I \to S^n$, so following the rotation, we end up with $g+f$. The simplest way to state this for higher dimensions is by suspending the spheres, but the key point is that this *doesn't* work for $n=1$, since you can't homotope a circle with a basepoint to exchange the top and bottom. ⊠

This extends to π_0 by letting I^0 be a point and ∂I^0 be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X. It turns out that π_1 has a lot of complications since every type of group is possible. The complications for **higher homotopy groups** (for $n \geq 2$) are totally different, in particular, $\pi_{n\geq 2}(X, x_0)$ is always abelian. Although we formalize things with cubes, most visualizations use balls and spheres, with the correspondence $(I^n, \partial I^n) \cong (D^n, S^{n-1})$.

Example 4.1. The basic example is that $\pi_1(X, x_0)$ is the set of maps from the interval into (X, x_0) up to homotopy rel ∂I, which is just the fundamental group. For $\pi_2(X, x_0)$, this is defined as the set of maps from the square to (X, x_0), where the edges of the square $\mapsto x_0$. This can be visualizes as a small square near x_0 with a 2-cell behind it, the reason why it's drawn offset is to distinguish between squares and spheres (more precisely, to indicate that the domain is I^2).

Often we think of π_n as the set of homotopy classes of maps $(S^n, \text{point}) \to (X, x_0)$ or classes of maps $(D^n, \partial D^n) \to (X, x_0)$. These are the same sets, but the formal definition by cubes is often better.

Whitehead's Theorem. *If $f : X \to Y$ for X, Y connected cell complexes induces isomorphisms on each π_n, then f is a homotopy equivalence.*

Note. This does *not* say that if $\pi_i(X) \simeq \pi_i(Y)$ for all i, then X and Y are homotopy equivalent! This might happen in a way such that there is no specific function inducing isomorphisms on the homotopy groups. However Whitehead's theorem is still good, it says that any topological map that is an algebraic isomorphism is also a "topological isomorphism" (homotopy equivalence).

If $f : X \to Y$ sends $x_0 \in Y$ to $y_0 \in Y$, then $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is defined as follows: for all $\alpha : (I^n, \partial I^n) \to (X, x_0)$, $f_*(x)$ is the composition $(I^n, \partial I^n) \xrightarrow{\alpha} (X, x_0) \xrightarrow{f} (Y, y_0)$. This formulates things as the level of maps of spaces. We must check that this preserves the equivalence relation of homotopy rel ∂I, which isn't hard. This is exactly the same as for π_1. The omission of basepoint is also similar as for π_1, the point being that if we have a space X with a path γ from $x_0 \to x_1$, then we have an induced isomorphism $\pi_n(X, x_0) \simeq \pi_n(X, x_1)$. In particular, the isomorphism type of π_n is independent of basepoint, so whether $f : (X, x_0) \to (Y, y_0)$ is an isomorphism on pi_n is independent of choices of x_0, y_0, assuming that X, Y are path connected. How do we define this? You can view it as stretching out the n-sphere from one point to another via the map γ.

Note. The isomorphism between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ depends on the choice of γ, just like π_1. For example, take a cube minus a vertical pillar in the middle. We can take γ to loop around the pillar from the left or from the right to connect to points on opposite ends. These two elements of $\pi_2(X, x_0)$ are different, since moving one loop to the other requires moving the basepoint which isn't allowed.

Definition 4.2 (Relative homotopy groups)**.** We will define for $x_0 \in A \subseteq X$, the **relative homotopy groups** $\pi_n(X, A, x_0)$ (note π_1 is not actually a group, just a set with a distinguished point) as the homotopy classes of maps $I^n \to X$ such that

(i) $\overset{I^{n-1}\times\{0\}\subseteq I^{n-1}\times I}{\left(I^{n-1} \subseteq I^n\right)} \to A$,

(ii) all other facets of I^n map to x_0.

If we replace (i) by the stronger condition $I^{n-1} \to x_0$, then we get $\pi_n(X, x_0)$. So it's like $\pi_n(X, x_0)$ except that one facet of I^n can be moved around inside A.

These satisfy a long exact sequence as such:

$$\cdots \to \pi_{n+1(X,A)} \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \longrightarrow \pi_n(X,A) \longrightarrow \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \cdots \to \pi_1(X,A).$$

Since the i_* are isomorphisms by Whitehead's thm (where i_* is induced by the inclusion $A \hookrightarrow X$), then $\pi_n(X,A) = 0$ for all n. We also have a geometric fact (the compression criterion): an element of $\pi_n(X,A)$ is 0, say as a map $(D^n, S^{n-1}, s_0) \xrightarrow{f} (X, A, x_0) \iff f$ is homotopic rel S^{n-1} to a map $D^n \to A$. The image is a cube $X = I^3$, and A the bottom face. Then a "bulge" (element of $\pi_2(X,A)$) can be flattened to the disk in A, so it's zero in $\pi_2(X,A)$. Repeating this process is like building a deformation retraction of X to A.

The relative homotopy groups are groups for $n \geq 2$, it's the same construction as for ordinary π_n except that one coordinate on I^n is reserveed for the definition of relateive π_n (the last coordinate). What goes wrong for π_1? One facet being able to move around in π_1 means you get paths you can't combine.

Theorem 4.2 (Long exact sequence)**.** *If $x_0 \in A \subseteq X$, then the following sequence is exact (suppressing basepoints):*

$$\pi_{n+1}(X,A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \to \cdots \to \pi_1(X,A)$$

where the i_* are induced by inclusion, and $j : \pi_n(X) \to \pi_n(X,A)$ is the obvious map. That is, a map $(I^n, \partial I^n) \to (X, x_0)$ where $(I^n, I^{n-1}) \to (X,A)$, and $(I^n$, rest of $\partial I^n) \to x_0$. The boundary maps $\partial : \pi_n(X,A) \to \pi_{n-1}(A)$ are also "obvious": if $f : I^n \to X$ represents an element of $\pi_n(X,A)$, then $f|_{I^{n-1}} \to A$ sends $\partial I^{n-1} \to x_0$, ie it represents an element of $\pi_{n-1}(A)$, which is precisely ∂f.

Proof. Is it time for a diagram chase? We'll break this up into part (B), (C), (D), there is no part (A). (B) will be about $\beta \in \pi_n(A)$, (C) about $\gamma \in \pi_n(X)$, and (D) about $\delta \in \pi_n(X,A)$.

$$\pi_{n+1}(X,A) \xrightarrow{\partial} \underset{\beta\in}{\pi_n(A)} \xrightarrow{i_*} \underset{\gamma\in}{\pi_n(X)} \xrightarrow{j} \underset{\delta\in}{\pi_n(X,A)} \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \to \cdots \to \pi_1(X,A)$$

Part B: Suppose $\beta \in \pi_n(A)$. We must show that

(i) $ji_*\beta = 0$. Applying j means one facet of I^n is allowed to move within A. We must show that if it allows our facet of I^n to move within A, then we can shrink β to a constant map. Shrink I^n onto $\partial I^n \setminus I^{n-1}$ (Hatcher calls this space J^{n-1}) rel $\partial I^n \setminus I^{n-1}$. So β_t sends the endpoints of the bottom edge of the cube to itself, $t = 0 \implies \beta_0 = \beta_t$, t small $\implies \beta_t \approx$ the pinched cube, when t is large then $\beta_t \approx$ a shrunken pinched cube, and $t = 1 \implies \beta_1$ is constant to x_0. Slogan: β itself is a nullhomotopy of $\partial\beta$.

(ii) If $\beta \in \pi_n(A)$ and $i_*\beta = 0$, then $\beta = \partial\alpha$ for $\alpha \in \pi_{n+1}(X,A)$. In this case we have to find an α st $\partial\alpha = \beta$, the image of α leaving A describes a map $I^{n+1} \to I^{n+1}/J^n = B^n \to X$ restricting to β on I^n. The fact that you can draw something that bounds β is precisely finding an element that bounds β.

Part C: We want to show the following:

(i) If $\gamma \in \pi_n(X)$, then $\partial j(\gamma) = 0$. If $\gamma \in \pi_n(X)$, $\gamma : I^n \to X$, sending $\partial I^n \to x_0$: we want to show that $\partial j(\gamma) = 0$. $j(\gamma)$ is the same picture, ie represented by the same map, the only difference is the homotopy equivalence relation in $\pi_n(X,A)$ is more relaxed than in $\pi_n(X)$. $\partial j(\gamma) = \gamma|_{I^{n-1} \subseteq I_n} =$ the constant map at x_0, or it's already been nullhomotoped.

(ii) If $j(\gamma) = 0$, then $\gamma = i_*\beta$ for some $\beta \in \pi_n(A)$, ie suppose γ can be nullhomotoped with I^{n-1} always mapping into A, with nested $\partial I^n \to x_0$. In this homotopy, I^{n-1} starts constant at x_0 and ends at the constant map x_0, and always maps ∂I^{n-1} to x_0. This homotopy (restricted to A) can be regarded as $I^{n-1} \times I \to A$, sending $\partial(I^{n-1} \times I)$ to x_0, ie this homotopy is an element of $\pi_n(A)$, which represents $\gamma \in \pi_n(X)$.

Part D: We want to show the following:

(i) If $\delta \in \pi_n(X,A)$, then $i_*\partial(\delta) = 0$. We have to show that $\partial\delta$ is nullhomotopic in X, that is, it bounds a ball in X, which it just does.

(ii) If $\partial\delta = 0$ then $\delta = j(\gamma)$ for some $\gamma \in \pi_n(X)$. What does it mean for $\partial\delta = 0$? It means that $\partial\delta$ bounds a disk in A, not just X. We seek a $\gamma \in \pi_n(X)$ with γ homotopic to δ by $\operatorname{im}(X,A)$. Regard E is a nullhomotopy of $\partial\gamma$ in A, extend this to a homotopy of δ in X. (Homotopy extension property: the homotopy is already defined on ∂I^n extends to I^n). After this alteration, we have the same element $\pi_n(X,A)$ since all we changed was moving I^{n-1} around in A. And now, all of $\partial I^n \to x_0$, ie is actually an element of $\pi_n(X)$. Take that to be γ.

With that we are done. ⊠

4.2 The Hureiwicz theorem

Hatcher states a relative version, but here is an absolute version first.

Hureiwicz Theorem. *If X is $(n-1)$-connected ($\pi_i(X) = 0$ for all $i < n$), then $H_n(X) \simeq H_n(X)$.*

For example, $\pi_n(S^n) \simeq \mathbb{Z}$ since S^n is $(n-1)$-connected.

Corollary 4.1. *Suppose a CW complex X is simply connected, and $H_i(X) = 0$ for all $i > 0$. Then X is contractible.*

Proof. Induction shows $\pi_n(X) \simeq H_n(X) \simeq 0$ for all $n > 0$ (knowing this for $n-1$) implies that X is $(n-1)$-connected. By the Hureiwicz theorem, $\pi_n(X) \simeq H_n(X) = 0$ by hypothesis. Then consider (point) $\hookrightarrow X$, inducing isomorphisms $\pi_n(\text{point}) \to \pi_n(X)$ for all n. So by Whitehead's theorem, point $\hookrightarrow X$ is a homotopy equivalence, ie X is contractible. ⊠

Note. $\pi_n(S^n) \simeq \mathbb{Z}$ isn't actually an application of the Hureiwicz theorem, because you need this very fact to prove Hureiwicz. To prove $\pi_n(S^n) \simeq \mathbb{Z}$, we need excision for homotopy groups.

The Hureiwicz Map. Let $h_n : \pi_n(X) \to H_n(X)$ for all spaces X. If $(f : S^n \to X) \in \pi_n(X)$, then f_* sends $H\_n(S^n) \to H_n(X)$. So you take a generator of $H_n(S^n) \simeq \mathbb{Z}$ (fixed once and for all). Then $h_n(f) = f_*(\alpha)$.

Back to the Whitehead theorem. Recall the long exact homotopy sequence, which tell us that isomorphisms on all π_n's implies that $\pi_n(X, A, x_0) = 0$ for all n. Caution: the LES refers to $x_0 \in A \subseteq X$, but we don't have a definition of $\pi_n(X, A, x_0)$ when $A \subsetneq X$, which means we can't apply this directly to Whitehead's theorem. We use a trick called the mapping cylinder, which justifies the slogan "every map is an inclusion".

Definition 4.3 (Mapping Cylinder). Given a map $f : A \to X$, consider $(A \times I) \amalg X$ under the relation $A \times \{1\} \sim f(A) \subseteq X$, or $(a, 1) \sim f(a) \in X$. This is the **mapping cylinder** of f, denoted M_f.

Now A *does* include into M_f as a left end of $A \times I$ by $a \mapsto (a, 0) \in M_f$, and M_f deformation retracts to X. So the meaning of the slogan is given a map $f : A \to X$, there exists an inclusion $A \hookrightarrow M_f \simeq X$, so the composition is homotopic to f. For Whitehead's theorem, suppose that $A \hookrightarrow X$, so we can talk about $\pi_n(X, A, x_0)$, which we know to be trivial. To geometrically interpret this, every map $(D^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map $D^n \to A$, we can think of "the disk compressing to A". Now we prove Whitehead's theorem.

Proof of Whitehead's theorem. Suppose $A \subseteq X$ is a CW subcomplex, where $A \hookrightarrow X$ induces homotopies on π_n, ie $\pi_n(X, A) = 0$ for all n. We construct deformation retractions of X to A one cell at a time, where the domain of the homotopy will be $X \times [0, \infty]$. Suppose after time n that $X^{(n)} \cup A$ has been deformation retracted to A. By homotopy extension, we can extend this to a homotopy of X in itself whose restriction to $X^{(n)} \cup A$ is this deformation retration. On the time interval $[n, n+1]$, we will deformation retract all the $(n+1)$-cells of X into A. The fact that we can do this is essentially the compression criterion. Whoops, had to write essay, couldn't pay attention. ⊠

A small technicality: The mapping cylinder M_f to turn $A \xrightarrow{f} X$ into an inclusion need not be a CW complex. If f happens to be cellular (every n-cell of A maps into the n-skeleton of X), then M_f *is* a CW complex, and the proof works. What could go wrong? Take $X = I^2$ to be the square, $A = I^2$ with a CW structure mapping into X by some random map, then $(A \times I) \amalg X / \sim$ isn't a CW complex. An edge *should* be a 1-cell of M_f, but it isn't because the boundary contains a random point of X not in X^0. To resolve this, we use cellular approximation, which turns out to be enough to finish the proof of Whitehead's theorem. Any map f of a CW complex is homotopic to a cellular map f', this $M_f \simeq M_{f'}$ homotopy equivalent allows us to replace it with something homotopy equivalent, and $M_{f'}$ actually *is* a CW complex. Another consequence of cellular approximation is that attaching n-cells doesn't affect $\pi_0, \pi_1, \cdots, \pi_{n-2}$, eg if an $S^{n-2} \to X$ is contractible in X, then there exists a suitable map $S^{n-2} \times I \to X$. This new map can be homotoped into X^{n-1}, because $S^{n-2} \times I$ is $(n-1)$-dimensional.