Differential Topology Notes

Simon Xiang

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Contents

Lecture 1

April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

A lot of definitions, here's the new ones:

Definition 1.1. A **subalgebra** of an algebra A is a linear subspace $A' \subseteq A$ containing 1 such that $a_1' a_2' \in A'$ for all $a_1', a_2' \in A'$. A **2-sided ideal** $I \subseteq A$ is a linear subspace such that $AI = I$ and $IA = I$. A $\mathbb{Z}$**-grading** of an algebra A is a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that $A^{k_1} A^{k_2} \subseteq A^{k_1+k_2}$ for all $k_1, k_2 \in \mathbb{Z}$. If A is a $\mathbb{Z}$-graded algebra and $a \in A^k, k \in \mathbb{Z}^{>0}$, then a is **decomposable** if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \cdots, a_k \in A^1$. If not, a is **indecomposable**.

1.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra $\bigotimes V$, thought of as the "free algebra generated by V".

Definition 1.2. Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map $i \colon V \to A$ such that for all (B, T) of an algebra B and a linear map $T \colon V \to B$ such that φ_T is a homomorphism of algebras.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "T"'] & & A \arrow[dl, dotted, "\exists!"', "\varphi_T"] \\ & B & \end{tikzcd}$$

(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If $(\xi \neq 0) \in V$ and $i(\xi) = 0$, set $B = \mathbb{R} \oplus \mathbb{R}\xi$ and define $\xi^2 = 0$.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "\pi"'] & & A \arrow[dl, dotted, "\varphi"] \\ & \mathbb{R} \otimes \mathbb{R}\xi & \end{tikzcd}$$

Note that $\pi|_{\mathbb{R}\xi} = \mathrm{id}$. But $\xi = \pi(\xi) = \varphi_1(\xi) = 0$, a contradiction. Furthermore, A has a canonical $\mathbb{Z}$-grading. $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$, $T_\lambda \colon V \to V$ is scalar multiplication, $\varphi_\lambda \colon A \to A$ is a homomoprhism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

Definition 1.3. Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map $b \colon V \times V'' \to X$ such that for all (W, B),

$$\begin{tikzcd} V' \times V'' \arrow[rr, "b"] \arrow[dr, "B"'] & & X \arrow[dl, "\exists!"', "T_B"] \\ & W & \end{tikzcd}$$

We denote $X = V' \otimes V''$, and $b(\xi', \xi'') = \xi' \otimes \xi''$, $\xi' \in V', \xi'' \in V''$.

If S' is a basis of V', S'' a basis of V'', then $S' \times S''$ is a basis of $V' \otimes V''$, where

$$S' \times S'' \cong \{\xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S''\}.$$

Note that $\bigotimes$ is "commutative" and "associative" with unit $\mathbb{R}$, so

$$\begin{gathered} \mathbb{R} \otimes V \to V \\ V_1 \otimes V_2 \to V_2 \otimes V_1 \\ (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), \end{gathered}$$

forming what we call a **symmetric monoidal category**. We write $\otimes^1 V = V$, $\otimes^2 V = V \otimes V$, $\otimes^3 V = V \otimes V \otimes V$ and so on. We also write $\otimes^0 V = \mathbb{R}$, and sometimes replace $\otimes^n V$ with $V^{\otimes n}$.

1.2 Existence of tensor algebras

Let V be a vector space, and $A = \bigoplus_{k=0}^{\infty} \otimes^k V$. Let $i : V \hookrightarrow A$ be the inclusion into $\otimes' V = V$.

Claim. (A, i) is a tensor algebra over V.

To see this, note that

$$\underbrace{\xi_1 \otimes \cdots \otimes \xi_k)}_{\in \otimes^k V} \cdot_A \underbrace{\eta_1 \otimes \cdots \otimes \eta_\ell}_{\in \otimes^\ell V} = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that $A = \otimes' V$ is *not* commutative.

1.3 The Exterior Algebra

We want to impose todo:come back

Lecture 2

April 8, 2020

todo:see notes on chapter 21, multivariate analysis

2.1 Exterior algebra of a direct sum

Definition 2.1. Let V be a vector space. An **exterior algebra** (E, j) over V is an algebra E and a linear map $j : V \to E$ satisfying $j(\xi)^2 = 0$ for all $\xi \in V$ such that for all pairs (B, T) consisting of an algebra B and a linear map $T : V \to B$ satisfying $T(\xi)^2 = 0$ for all $\xi \in V$, there exists a unique algebra homomorphism $\varphi : E \to B$ such that $T = \varphi \circ j$.

Let L_1, L_2 be linear, and $\bigwedge^*(L_1 \oplus L_2 = V)$.

Lecture 3

April 15, 2021

todo:is this lecture 24??

Theorem 3.1. *Let X be a smooth manifold. Then there exists a unique $d : \Omega^*(X) \to \Omega^{*+1}(X)$ satisfying*

(i) Linearity,

(ii) *The Liebniz rule,*

(iii) $d^2 = 0$,

(iv) $d|_{\Omega^0(X)}$ *is the usual differential.*

Proof. Let $\{(U_i, x_i)\}_{i\in I}$ be an open cover of X by charts. Let $\{\rho_i\}_{i\in I}$ be a partition of unity, where $\operatorname{Supp}\rho_i \subseteq U_i$. If $\alpha \in \Omega^*(X)$, then $\alpha = \sum_i \rho_i\alpha_i$, where $\operatorname{supp}(\rho_i\alpha) \subseteq U_i$. Define $d\alpha = \sum_i d(\rho_i\alpha)$, where we compute $x_i(U_i) \subseteq A_i$, $\operatorname{supp} d(\rho_i\alpha)$ (note that d increases support).

For this to be a good definition, we need to show that this is well-defined. say $\{(V_a, y_a\}_{a\in A}$ is another atlas, $\{\sigma_a\}_{a\in A}$ a partition of unity. Then

$$\begin{aligned}\sum_i d(\rho_i\alpha) &= \sum_i\sum_a d(\rho_i\sigma_a\alpha)\\ &= \sum_a\sum_i d(\sigma_a\rho_i\alpha)\\ &= \sum_\alpha d(\sigma_a\alpha).\end{aligned}$$

Note that $\operatorname{supp}\rho_i\sigma_a\alpha \subseteq U_i\cap V_a$. Something about d commuting with pullback, the first is defined on $x_i(U_i\cap V_a)$, the second on $y_a(U_i\cap V_a)$, and the final on $y_a(V_a)$. todo:this, plus something about transition maps ⊠

3.1 Orientation

We have all seen Riemann integration on the line, and hopefully you have learned how to integrate in $\mathbb{R}^n$, and perhaps Lebesgue integration. We do not focus on the analytic aspects, but the geometric aspects, which allows us to integrate on manifolds. Unfortunately we do not have a fixed vector space, giving a fixed Lebesgue measure, so we have to start from the beginning. Let's talk about orientation.

Recall that if L is a real line (1-dimensional vector space), then an **orientation** of L is an element of $\pi_0(L\setminus\{0\})$.

Definition 3.1. If V is a finite dimensional real vector space, then an **orientation** of V is an orientation of $\det V$. A **basis** of V is an isomorphism $b\colon \mathbb{R}^n \to V$ if $\dim V = n$.

Remark 3.1. Let $\mathcal{O}(V)$ be the set of bases of V. The group $\mathrm{GL}_n\mathbb{R} = \{g\colon \mathbb{R}^n \xrightarrow{\simeq} \mathbb{R}^n\}$ acts simply transitively on $\mathcal{O}(V)$.[1] This is a right action $\mathrm{GL}_n\mathbb{R}$, or a torsor. Then $\det\colon \mathrm{GL}_n\mathbb{R} \to \mathbb{R}^{\neq 0}$ is an isomorphism on π_0. Introduce $\mathcal{O}(V) \to \det V\setminus\{0\}$, $e_1,\cdots,e_n \mapsto e_1\wedge\cdots\wedge e_n$. An orientation partitions $\mathcal{O}(V)$ into $\mathscr{B}^{\pm}(V)$. If $T\colon V' \to V$, then $\dim V' = \dim V$ if T is an ismorphism. Then $\det T\colon \det V' \to \det V$[2] is an isomorphism, and T is orientation preserving (resp reversing) if $T(O') = 0$ (resp $T(O') \neq O$). (Here O denotes the orientation of a space.)

Definition 3.2. Let V be a finite dimensional real vector space. A nonzero element of $\operatorname{Det} V^*$ is a **volume form**. For $\xi_1,\cdots,\xi_k \in V$, $(\xi_1,\cdots,\xi_k) = \{t^i\xi_i \mid 0 \leq\to i \leq 1\} \subseteq \operatorname{span}\{\xi_i\}$, the vectors are **nondegenerate** if the $\xi_1,\cdots,\xi_k$ are LI iff $\xi_1\wedge\cdots\wedge\xi_k \neq 0$ in $\bigwedge^k V$. If $e_1,\cdots,e_n$ is a basis of V, define

$$\operatorname{vol}(//(e_1,\cdots,e_n) = \|\langle\omega, e_1\wedge\cdots\wedge e_n\rangle\|.$$

Proposition 3.1. *If* $e_1',\cdots,e_n'$ *is another basis, and* $e_j' = T_j^i e_i$ *for* $T_i^i \in \mathbb{R}$, *then*

$$\operatorname{vol}//(e_1',\cdots,e_n') = (\det T)\operatorname{vol}//(e_1,\cdots,e_n).$$

Remark 3.2. *Ratios* of volume are defined without a volume form. A k-form $\alpha \in \bigwedge^k V_6*$ induces a notion of volume on all k-dimensional subspaces $W \subseteq V$ such that $\alpha|_W \neq 0$. On $\mathbb{R}^n$ we take $\omega = e^1\wedge\cdots\wedge e^n \in \operatorname{Det}\mathbb{R}^{n*}$.

todo:?? canonical double cover, orientation bundle, homology

Definition 3.3. An orientation of X is a section of $\pi_0^{\mathrm{vert}}(\operatorname{Det} TX\setminus 0) \to X$. A **volume form** on X is a nonvanishing $\omega \in \Omega^n(X)$ if $\dim X = n$.

[1] Apparently in physics, left vs right actions form the idea of passive vs active actions or something like that. This is a right action.

[2] Confused on usage of det and Det

Example 3.1. If $X = S^1$, then we have two double covers up to isomorphism. If $X = \mathbb{RP}^2$, then $D^2 \subseteq \mathbb{A}^2$ todo:something happen, so the orientation double cover has total space S^2, and $\mathbb{RP}^2$ is not orientable.

Definition 3.4. Suppose X is an oriented manifold. A standard chart $(U, x), x : U \to \mathbb{A}^n$ is **oriented** if $\frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p$ is an oriented basis of T_pX for all $p \in U$.

If $(U, x), (V, y)$ are oriented charts, then $\det d(y \circ x^{-1}) > 0$. Look forward to integration.

Lecture 4

April 20, 2021

4.1 Integration on manifolds

To integrate on an interval $[a, b] \subseteq \mathbb{R}$, partition the interval into small intervals I_i, and for $x_i \in I_i$, $f : [a, b] \to \mathbb{R}$ consider

$$\int_a^b f \approx \sum_{I_i} f(x_i) \cdot \text{length } (I_i).$$

For a region $\Omega \subseteq \mathbb{A}^2$, we want to integrate $f : \Omega \to \mathbb{R}^2$. Then break up Ω into regions P_{ij}, and define

$$\int_\Omega f \approx \sum_{i,j} f(p_{ij}) \cdot \text{Area } (P_{ij})$$

To integrate on a 2-manifold Σ, consider $\xi_{ij} \wedge \eta_{ij} \in \bigwedge^2 T_{p_{ij}}\Sigma$ for $\xi_{ij}, \eta_{ij} \in T_{p_{ij}}\Sigma$. Then for $\omega \in \Omega^2(\Sigma)$, to imitate the previous integrals do something like $\sum_{i,j} \omega_{p_{ij}}(\xi_{ij} \wedge \eta_{ij})$. todo:? So we don't actually integrate over 2-forms, we integrate over something called the *density*.

4.2 Change of variables

In dimension

1: Consider $\int_1^2 x^2\, dx = -\int_{-1}^{-2} y^2\, dy = \int_{-2}^{-1} y^2\, dy$, where $\varphi^*x = -y$, $\varphi^*dx = -dy$, $\varphi : y \to x$ is an orientation reversing map. This is integration of a differential form, which you learn in single variable calculus.

2: Now consider regions V, U with respect to variables u, v and x, y. Then $\varphi : V \to U$, and

$$\int_U f = \int_{U'} (\overset{=\varphi^* f}{f \circ \varphi}) |\det \varphi|$$

More intelligently, we have

$$|dx\, dy| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right| |du\, dv|, \quad \int_U f |dx \wedge dy| = \int_{U'} \varphi^*[f |dx \wedge dy|].$$

4.3 Integration in $\mathbb{A}^n$

Suppose $U \subseteq \mathbb{A}^n$ is open, $\Omega_c^0(U)$ denotes the compactly supported smooth functions. Then

$$\int_U : \Omega_c^0(U) \to \mathbb{R}$$

is linear, and satisfies the change of variables: if $\varphi : U' \to U$ is a diffeomorphism, then $\int_U f = \int_{U'} \varphi^* f |\det d\varphi|$. To identify $\Omega_c^0(U)$ with $\Omega_c^n(U)$, identify f with $\omega_f = f\, dx^1 \wedge \cdots \wedge dx^n$. Then $\int_U \omega = \int_{U'} \varphi^*\omega$ if φ is orientation-preserving.

4.4 Globalizing integration

Now we want to globalize.

Theorem 4.1. *Let X be an oriented manifold. Then there exists a unique linear map*

$$\int_X : \Omega_c^n(X) \to \mathbb{R}$$

such that if $(U; x^1, \cdots, x^n)$ is an oriented standard chart and $\omega \in \Omega_c^n(U)$, then

$$\int_X \omega = \int_{X(U)} (x^{-1})^* \omega$$

Proof. Let $\{(U_i, x_i)\}_{i \in I}$ is an atlas of *oriented* charts, and $\{\rho_i\}_{i \in I}$ be a subordinate partition of unity. Then for $\omega \in \Omega_c^n(X)$, let $\omega = \sum_{i \in I} \rho_i \omega$, where $\operatorname{supp}(\rho_i \omega) \subseteq U_i$. Define

$$\int_X \omega = \sum_{i \in I} \int_{x_i(U)} (x^{-1})^* (\rho_i \omega).$$

If $\{(V_a, y_a)\}_{a \in A}$ is an oriented atlas, $\{\sigma_a\}_{a \in A}$ a partition of unity, then this is equal to

$$\begin{aligned}
&= \sum_i \sum_a \int_{x_i(U_i \cap V_a)} (x_i^{-1})^* (\rho_i \sigma_a \omega) \\
&= \sum_a \sum_i \int_{y_a(U_i \cap V_a)} (y_a^{-1})^* (\sigma_a \rho_i \omega) \\
&= \sum_a \int_{y_a(V_a)} (y_a^{-1})^* (\sigma_a \omega).
\end{aligned}$$

⊠

Example 4.1. Let's work through an example to see how we actually calculate integrals. Let $\varphi : (0, \pi) \times (0, 2\pi) \to S^2 \subseteq \mathbb{A}^3_{x,y,z}$, $\phi, \theta \mapsto \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi$. Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$. Then

$$\begin{aligned}
x &= \sin\phi\cos\theta, & dx &= \cos\phi\cos\theta\,d\phi - \sin\phi\sin\theta\,d\theta, \\
y &= \sin\phi\sin\theta, & dy &= \cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta, \\
z &= \cos\phi, & dz &= -\sin\phi\,d\phi.
\end{aligned}$$

Then

$$\begin{aligned}
\omega &= x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \\
&= (\sin\phi\cos\theta)(\cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta) \wedge (-\sin\phi\,d\phi) \\
&= \sin\phi\,d\phi \wedge d\theta.
\end{aligned}$$

todo:finish

Some properties of the integral: for an oppositely oriented manifold $-X$,

$$\int_{-X} \omega = -\int_X \omega,$$

and if $\varphi : X' \to X$ is an oriented diffeomorphism $\omega \in \Omega_c^n(X)$,

$$\int_{X'} \varphi^* \omega = \int_X \omega.$$

4.5 Stoke's theorem and boundary orientations

Let

$$0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$$

be a short exact sequence of finite dimensional real vector spaces. We know

- $\dim V = \dim V'' + \dim V'$,
- $\det V \xleftarrow{\cong} \det V'' \otimes \det V'$.

Say $e_1', \cdots, e_k'$ is a basis of V', $e_1'', \cdots, e_\ell''$ is a basis of V'', and $\widetilde{e}_1'', \cdots, \widetilde{e}_\ell''$ be vectors in V such that $j(\widetilde{e}_\alpha'') = e_\alpha''$. Then $\widetilde{e}_1'', \cdots, \widetilde{e}_\ell''$, $i(e_1'), \cdots, i(e_k')$ is a basis of V. Slogan: quotient before sub.

Stokes' Theorem. *Let X^n be an oriented manifold with boundary, and $i\colon \partial X \hookrightarrow X$. Fix $\omega \in \Omega_c^{n-1}(X)$. Then*

$$\int_X d\omega = \int_{\partial X} i^*\omega.$$

Example 4.2 (The fundamental theorem of calculus)**.** Let $X = [a,b] \subseteq \mathbb{R}$, $\partial X = \{a,b\}$. Then $\omega = f, f : [a,b] \to \mathbb{R}$, and $d\omega = df = f'(x)dx$. Then

$$\int_{[a,b]} f = f(a) - f(b).$$

todo:not sure

Example 4.3. Let $\partial D^3 = S^2$. Then

$$\begin{aligned}\int_{S^2} \omega = \int_{D^3} d\omega &= \int_{D^3} 3dx \wedge dy \wedge dz \\ &= 3\,\mathrm{vol}(D^3) \\ &= 3 \cdot \frac{4}{3}\pi\end{aligned}$$