Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

Category Theory

Today we talk about abstract nonsense! These notes will follow Evan Chen's Napkin §60 and May's "A Concise Course in Algebraic Topology" §2. Some examples are peppered in from Hatcher §2.3.

1.1 Motivation

Why do we talk about categories? Categories rise from objects (sets, groups, topologies) and maps between them (bijections, isomorphisms, homeomorphisms). Algebraic topology speaks of maps from topologies to groups, which makes maps between categories a suitable tool for us.

Example 1.1. Here are some examples of morphisms between objects:

- A bijective homomorphism between two groups G and H is an isomorphism. What also works is two group homomorphisms $\phi : G \to H$ and $\psi : H \to G$ which are mutual inverses, that is $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.
- Metric (or topological) spaces X and Y are isomorphic if there exists a continuous bijection $f : X \to Y$ such that f^{-1} is also continuous.
- Vector spaces V and W are isomorphic if there is a bijection $T : V \to W$ that's a linear map (aka, T and T^{-1} are linear maps).
- Rings R and S are isomorphic if there is a bijective ring homomorphism ϕ (or two mutually inverse ring homomorphism).

1.2 Categories

Definition 1.1 (Category)**.** A **category** $\mathscr{A}$ consists of

- A class of **objects** , denoted $\mathrm{obj}(\mathscr{A})$.
- For any two objects $A_1, A_2 \in \mathrm{obj}(\mathscr{A})$, a class of **arrows** (also called **morphisms** or **maps** between them). Let's denote the set of arrows by $\mathrm{Hom}_{\mathscr{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \mathrm{obj}(\mathscr{A})$, if $f : A_1 \to A_2$ is an arrow and $g : A_2 \to A_3$ is an arrow, we can compose the two arrows to get $h = g \circ f : A_1 \to A_3$ an arrow, represented in the **commutative diagram** below:

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[dr, dotted, "h"] & A_2 \arrow[d, "g"] \\ & A_3 \end{tikzcd}$$

  The composition operation can be denoted as a function

$$\circ : \mathrm{Hom}_{\mathscr{A}}(A_2, A_3) \times \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathscr{A}}(A_1, A_3)$$

  for any three objects A_1, A_2, A_3. Composition must be associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$. In the diagram above, we say h **factors** through A_2.
- Every object $A \in \mathrm{obj}_{\mathscr{A}}$ has a special **identity arrow** $\mathrm{id}_{\mathscr{A}}$. The identity arrow has the expected properties $\mathrm{id}_{\mathscr{A}} \circ f = f$ and $f \circ \mathrm{id}_{\mathscr{A}} = f$.

Note. We can't use the word "set" to describe the class of objects because of some weird logic thing (there is no set of all sets). But you can think of a class as a set.

From now on, $A \in \mathscr{A}$ is the same as $A \in \operatorname{obj}(\mathscr{A})$. A category is **small** if it has a set of objects, and **locally small** if $\operatorname{Hom}_{\mathscr{A}}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathscr{A}$.

Example 1.2 (Basic Categories)**.** Here are some basic examples of categories:

- We have the category of groups Grp.
  - The objects of Grp are groups.
  - The arrows of Grp are group homomorphisms.
  - The composition of Grp is function composition.
- You can also think of the subcategory of abelian groups AbGrp. We can generalize this to the category of modules over a fixed ring R denoted $R\mathsf{Mod}$, with morphisms the module homomorphisms.
- Describe the category CRing (of commutative rings) in a similar way.
- Consider the category Top of topological spaces, whose arrows are continuous maps between spaces. We can also restrict the spaces to special classes like CW complexes (CellCw), or the maps to homeomorphisms.
- Also consider the category Top_* of topological spaces with a distinguished basepoint, that is, a pair (X, x_0), $x_0 \in X$. Arrows are continuous maps $f : X \to Y$ with $f(x_0) = y_0$.
- Similarly, the category of (possibly infinite-dimensional) vector spaces over a field k Vect_k has linear maps for arrows. There is even a category FDVect_k of finite-dimensional vector spaces.
- Finally, we have a category Set of sets, arrows denote any map between sets. You can restrict the maps to injectections, bijections, and surjections.

Definition 1.2 (Isomorphism)**.** An arrow $A_1 \xrightarrow{f} A_2$ is an **isomorphism** if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \mathrm{id}_{A_2}$ and $g \circ f = \mathrm{id}_{A_1}$. We say A_1 and A_2 are **isomorphic**, denoted $A_1 \cong A_2$.

Remark 1.1. In the category Set, $X \cong Y \iff |X| = |Y|$.

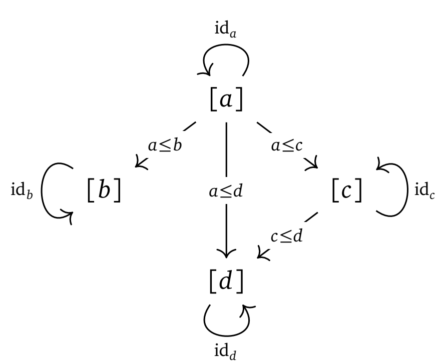
In other fields, we can tell a lot about the structure of an object by looking at maps between them. In category theory, we *only* look arrows, and ignore what the objects themselves are.

Example 1.3 (Posets are Categories)**.** Let $\mathscr{P}$ be a poset. Then we can construct a category P for it as follows:

- The objects of P are elements of $\mathscr{P}$.
- We define the arrows of P as follows:
  - For every object $p \in P$, we add an identity arrow id_p, and
  - For any pair of distinct objects $p \leq q$, we add a single arrow $p \to q$.

  There are no other arrows.
- We compose arrows in the only way possible, examining the order of the first and last object.

Here's a figure depicting the category of a poset $\mathscr{P}$ on four objects $\{a, b, c, d\}$ with $a \leq b$ and $a \leq c \leq d$.



Note that no two distinct objects of a poset are isomorphic.

This shows that categories don't have to refer to just structure preserving maps between sets (these are called "concrete categories".

Example 1.4 (Groups as a category with one object)**.** A group G can be thought of as a category $\mathscr{G}$ with one object $*$, all of whos arrows are isomorphisms.

If the universe were structured differently and kids learned category theory before groups, symmetries transforming X into itself would be a natural extension of categories that transform X into other objects, a special case in which all the maps are invertible. Alas, this is not the right timeline.

Example 1.5. We have the homotopy category hTop whose objects are topological spaces and morphisms are homotopy classes of maps. This uses the fact that composition is well-defined on homotopy classes: $f_0 g_0 \simeq f_1 g_1$ if $f_0 \simeq f_1$ and $g_0 \simeq g_1$.

Example 1.6. Finally, chain complexes are objects of a category Ch_K for K a commutative ring (usually $\mathbb{Z}$), with chain maps as morphisms. This category has many interesting subcategories by restricting the objects, for example we can consider chain complexes whose groups are zero in negative dimensions (or outside a finite range). Or we could talk about exact sequences or short exact sequences, in either case morphisms are chain maps which are commutative diagrams. To go even deeper, there is a category whose objects are short exact sequences of chain complexes and morphisms are the square shaped commutative diagrams. Scary stuff!

Example 1.7 (Deriving Categories)**.** We can make categories from other categories!

(a) Given a category $\mathscr{A}$, we can construct the **opposite category** A^{op}, which is the same as $\mathscr{A}$ but with all the arrows reversed.

(b) Given categories $\mathscr{A}$ and $\mathscr{B}$, we can construct the **product category** $\mathscr{A} \times \mathscr{B}$ as follows: the objects are pairs (A, B) for $A \in \mathscr{A}$, $B \in \mathscr{B}$, and the arrows from (A_1, B_1) to (A_2, B_2) are pairs

$$\left(A_1 \xrightarrow{f} A_2, B_1 \xrightarrow{g} B_2\right).$$

The composition is just pairwise composition, and the identity is the pair of identity functions on A and B.

Some categories have things called *initial objects*. For example the empty set $\emptyset$, the trivial group, the empty space, initial element in a poset, etc. More interestingly: the initial object of CRing is the ring $\mathbb{Z}$.

Definition 1.3 (Initial object)**.** An **initial object** of $\mathscr{A}$ is an object $A_{\text{init}} \in \mathscr{A}$ such that for any $A \in \mathscr{A}$ (possibly $A = A_{\text{init}}$), there is exactly one arrow from A_{init} to A.

The yang to this yin is the *terminal object*:

Definition 1.4 (Terminal object)**.** A **terminal object** of $\mathscr{A}$ is an object $A_{\text{final}} \in \mathscr{A}$ such that for any $A \in \mathscr{A}$ (possibly $A = A_{\text{final}}$, there is exactly one arrow from A to A_{final}.

For example, the terminal object of Set is $\{*\}$, Grp is $\{1\}$, CRing is the zero ring, Top is the single point space, and a poset its maximal element (if one exists).

1.3 Products and coproducts

We have a way of uniquely describing objects (up to isomorphism) called the "universal property". For example, in the category Set say we have two sets X, Y, and we want to construct $X \times Y$. How would we do this without talking about the sets themselves, but just the maps between them?

Observation. *A function $A \xrightarrow{f} X \times Y$ amounts to a pair of functions $\left(A \xrightarrow{g} X, A \xrightarrow{h} Y\right)$.*

In other words, we have natural projection maps $\pi_X : X \times Y \twoheadrightarrow X$ and $\pi_Y : X \times Y \twoheadrightarrow Y$:

$$
\begin{tikzcd}
 & X \\
X \times Y \arrow[ur, two heads, "\pi_X"] \arrow[dr, two heads, "\pi_Y"'] & \\
 & Y
\end{tikzcd}
$$

By our observation, we have a bijection between functions $A \xrightarrow{f} X \times Y$ and pairs of functions (g, h), so each pair $A \xrightarrow{g} X$ and $A \xrightarrow{h} Y$ there is a *unique* function $A \xrightarrow{f} X \times Y$. This demonstrates how $X \times Y$ is "universal", since we can build a unique function into $X \times Y$ from pairs of functions to the component spaces, as demonstrated in the following diagram.

$$
\begin{tikzcd}
 & & X \\
A \arrow[rru, two heads, "f"] \arrow[r, dotted, "\exists! f"] \arrow[rrd, two heads, "g"'] & X \times Y \arrow[ur, two heads, "\pi_X"'] \arrow[dr, two heads, "\pi_Y"] & \\
 & & Y
\end{tikzcd}
$$

Figure 1: Diagram for the product of objects in a category.

We can do this for general categories, defining a product.

Definition 1.5 (Product)**.** Let X and Y be objects in a category $\mathscr{A}$. The **product** consists of an object $X \times Y$ and arrows π_X, π_Y to X and Y (thought of as projection), such that for any object A and arrows $A \xrightarrow{g} X$, $A \xrightarrow{h} Y$, there exists a *unique* arrow $A \xrightarrow{f} X \times Y$ such that Figure 1 commutes. Note: usually the product should consist of *both* the object $X \times Y$ and the projections π_X, π_Y, however if the projection maps are understood we often refer to $X \times Y$ as both the object and the product.

Claim. Products do not always exists, consider the category with two objects and no non-identity morphisms. However, when they do, they are unique up to isomorphism. That is, given two products P_1, P_2 of objects X and Y, we can find an isomorphism between them.

Proof. Consider two products P_1, P_2, and their associated projection maps. In Figure 1, if we replace A with P_i we get the following diagram:

$$
\begin{tikzcd}
 & X & \\
P_1 \arrow[ur, two heads, "\pi_X^1"] \arrow[r, "f"] \arrow[dr, two heads, "\pi_Y^1"'] & P_2 \arrow[u, two heads, "\pi_X^2"'] \arrow[r, "g"] \arrow[d, two heads, "\pi_Y^2"] & P_1 \arrow[ul, two heads, "\pi_X^1"'] \arrow[dl, two heads, "\pi_Y^1"] \\
 & Y &
\end{tikzcd}
$$

Since the P_i are products, we have the existence of the unique morphisms f, g such that the diagram commutes, by the universal property. If we just look at the outer square, $g \circ f$ is the unique map that makes this portion of the diagram commute. But id_{P_1} also makes this portion of the diagram commute, so $g \circ f = \mathrm{id}_{P_1}$. Similarly, we can rearrange the diagram such that $f \circ g$ factors through g, and thus $f \circ g = \mathrm{id}_{P_2}$ and therefore f and

g are isomorphisms. For uniqueness, if we have maps $g\colon P_1 \to P_2$ and $h\colon P_2 \to P_1$ satisfying the properties of isomorphism, they must be the unique maps from $P_1 \to P_2$ and vice versa, since the projection arrows define a unique arrow up to isomorphism into the other product. Combined with the fact that g and h make the following diagram commute since they satisfy the isomorphism properties,

$$\begin{tikzcd}
& X & \\
P_1 \arrow[ur, "\pi_X^1"] \arrow[rr, shift left, "g"] \arrow[dr, "\pi_Y^1"'] & & P_2 \arrow[ul, "\pi_X^2"'] \arrow[ll, shift left, "h"] \arrow[dl, "\pi_Y^2"] \\
& Y &
\end{tikzcd}$$

we conclude that such arrows are precisely the arrows f and g induced by the other projections as stated above, and we are done. ⊠

Note. Actually, we've only shown that P_1 and P_2 are isomorphic as objects, and said nothing about the projection maps. Don't worry about it too much, when we say $P_1 \simeq P_2$ we're referring to the objects.

The universal property is nice because we don't have to explicity construct such an object P, we can just say that "such object satisfying the given properties is unique up to isomorphism", and refer to it henceforth without getting our hands dirty and messing with its inner workings. However, that doesn't stop us from giving examples.

Example 1.8 (Examples of products)**.**

(a) In the category Set, the product of sets X, Y is their Cartesian product $X \times Y$.

(b) For Grp, the product of groups G, H is the direct product $G \times H$.

(c) Similarly, in Vect_k the product of spaces V and W is the direct product $V \oplus W$.

(d) In CRing, the product of two rings R, S is the product ring $R \times S$.

(e) Thinking of a poset as a category, the product of two objects (elements) x, y is the *greatest lower bound*; for example,
   - For the poset $(\mathbb{R}, \leq)$, the product is $\min\{x, y\}$.
   - For the poset of subsets (or subgroups, rings, fields etc) the product is $X \cap Y$.
   - For the poset of positive integers ordered by divisibility, the product is $\gcd(x, y)$.

We can also define products of more than one object. For objects $\{X_i \mid i \in I\}$ in a category $\mathscr{A}$, we define a **cone** on the X_i to be an object A with the projection maps. Then the **product** is a cone P satisfying the universal property, that is, given any other cone A we have a unique map $f : A \to P$ making the diagram below commute.

$$\begin{tikzcd}
& & A \arrow[d, dotted, "\exists! f"] \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
& & P \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
X_1 & X_2 & & X_3 & X_4
\end{tikzcd}$$

Definition 1.6 (Coproduct)**.** We can do the dual construction to get the **coproduct**: given objects X and Y, the coproduct is the object $X+Y$ with maps $X \xrightarrow{\iota_X} X+Y$ and $Y \xrightarrow{\iota_Y} X+Y$ (think inclusion) such that for any object A and maps $X \xrightarrow{g} A$, $Y \xrightarrow{h} A$ there is a unique f for which the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\iota_X"'] \arrow[drr, "g"] & & \\
& X+Y \arrow[r, "\exists! f"] & A \\
Y \arrow[ur, "\iota_Y"] \arrow[urr, "h"'] & &
\end{tikzcd}$$

As expected, a coproduct is a universal **cocone**.

Example 1.9 (Examples of coproducts)**.**

(a) In Set, the coproduct of sets X, Y is the disjoint union $X \amalg Y$.

(b) For Grp, the coproduct of groups G, H is the free product $G * H$. In AbGrp, this is the direct *sum* $G \oplus H$: it has the same structure as the direct product in the finite case, but is the dual construction in the categorical sense. To make sense of this, consider the direct product as having morphisms from every component to itself, while the direct sum has morphisms from itself to every component, which is why the components must be zero for all but finitely many in this case.

(c) The same holds for Vect_k, that is, the coproduct of two spaces V, W is the direct sum $V \oplus W$. The notions of direct sum and product yet again coincide in the finite case: this is an example of a **biproduct**, which is both a product and a coproduct. In preadditive categories (AbGrp with extra structure), biproducts exists for a finite collection of objects.

(d) In a poset, coproducts are the least upper bounds.

1.4 Monomorphisms and epimorphisms

[1] Injectivity and surjectivity don't really make sense when talking about categories, because morphisms need not be functions. Here's the correct categorical notion:

Definition 1.7 (Monomorphisms)**.** A map $X \xrightarrow{f} Y$ is a **monomorphism** (or monic) if for any commutative diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

we must have $g = h$. In other words, $f \circ g = f \circ h$ implies that $g = h$.

In a concrete category, injective implies monic: what the heck even is a concrete category? Anyway, consider $f \circ g = f \circ h$, so $f(g(a)) = f(h(a)$ for all $a \in A$: but since f is injective, this implies that $g(a) = h(a)$, and so $g = h$ and f is a monomorphism. Similarly, the composition of two monomorphisms is also a monomorphism: let f, g be monomorphisms. Then $(f \circ g) \circ \alpha = (f \circ g) \circ \alpha' \implies f \circ (g \circ \alpha) = f \circ (g \circ \alpha')$ by associativity of arrows. Since f is a monomorphism, $g \circ \alpha = g \circ \alpha'$, but since g is also a monomorphism, $\alpha = \alpha'$ and we are done. In most but

[1] Here Evan uses the terminology "*monic*" and "*epic*", but I've noticed no one else really does that, so I'm replacing each instance with "*monomorphism*" and "*epimorphism*".

not all situations, the converse of the definition also holds. For example, in $\mathsf{Set}, \mathsf{Grp}$, and CRing, monic implies injective.

There are many categories with a "free" object that you can think of as elements. For example, an element of a set is a function $1 \to S$, and an element of a ring is a function $\mathbb{Z}[x] \to R$, etc. In all these categories, the definition of monomorphims literally say that "f is injective on $\mathrm{Hom}_{\mathscr{A}}(A, X)$". However, there is a standard counterexample involving the category of "divisible" abelian groups $\mathsf{DivAbGrp}$ and the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

Definition 1.8 (Epimorphisms). A map $X \xrightarrow{f} Y$ is an **epimorphism** (or epic) if for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \end{tikzcd}$$

we must have $g = h$. In other words, $g \circ f = h \circ f \implies g = h$.

This is like surjectivity, but a little farther off. Furthermore, the correspondence failure rate is a little higher.

Example 1.10 (Epimorphisms that aren't onto).

(a) In CRing, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism that isn't onto. If two homomorphisms agree on an integer, they agree everywhere since we can extend linearly.

(b) In the category of *Hausdorff* topological spaces Haus, a map is an epimorphism iff it has a dense image (for example $\mathbb{Q} \hookrightarrow \mathbb{R}$).

Basically, things fail when $f : X \to Y$ can be determined by just some of the points (some subset) of X.

1.5 Functors

Example 1.11 (Basic Functors). Here are some basic examples of functors:

- Given an algebraic structure (group, field, vector space) we can take its underlying set S: this is a functor from $\mathsf{Grp} \to \mathsf{Set}$ (or whatever you want to start with).
- If we have a set S, if we consider the vector space with basis S we get a functor $\mathsf{Set} \to \mathsf{Vect}$.
- Taking the power set of a set S gives a functor $\mathsf{Set} \to \mathsf{Set}$.
- Given a locally small category $\mathscr{A}$, we can take a pair of objects (A_1, A_2) and obtain a set $\mathrm{Hom}_{\mathscr{A}}(A_1, A_2)$. This turns out to be a functor $\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathsf{Set}$.

Finally, the most important examples (WRT this course):

- In algebraic topology, we build groups like $H_n(X)$, $\pi_1(X)$ associated to topological spaces. All these group constructions are functors $\mathsf{Top} \to \mathsf{Grp}$.

Definition 1.9 (Functors). Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **functor** F takes every object of $\mathscr{A}$ to an object of $\mathscr{B}$. In addition, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. Refer to the commutative diagram:

$$\begin{tikzcd} & A_1 \arrow[d, "f"'] & B_1 = F(A_1) \arrow[d, "F(f)"] & \\ \mathscr{A} \ni & {} \arrow[r, dotted, "F"] & {} & \in \mathscr{B} \\ & A_2 & B_2 = F(A_2) & \end{tikzcd}$$

Functors also satisfy the following requirements:

- Identity arrows get sent to identity arrows, that is, for each identity arrow id_A, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

- Functors respect composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3$ are arrows in $\mathscr{A}$, then $F(g \circ f) = F(g) \circ F(f)$.

More precisely, these are **covariant** functors. A **contravariant** functor F reverses the direction of arrows, so that F sends $f : A_1 \to A_2$ to $F(f) \colon F(A_2) \to F(A_1)$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ instead. A category $\mathscr{A}$ has an opposite category $\mathscr{A}^{\mathrm{op}}$ with the same objects and with $\mathscr{A}^{\mathrm{op}}(A_1, A_2) = \mathscr{A}(A_2, A_1)$. A contravariant functor $F \colon \mathscr{A} \to \mathscr{B}$ is just a covariant functor $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

Example 1.12. A classical example of functors is the dual vector space functor. For K a field, V a K-vector space the dual vector space functor assigns to V the dual vector space $F(V) = V^*$ of linear maps $V \to K$, and to each linear transformation $f : V \to W$ the dual map $F(f) = f^* \colon W^* \to V^*$. Note that this functor is contravariant.

Example 1.13. We have already talked about **free** and **forgetful** functors in Example 1.3: the forgetful functors are functors from spaces to sets (the underlying set of a group) and free functors are from sets to spaces (the basis set forming a vector space).

- Another example of a forgetful functor is a functor $\mathsf{CRing} \to \mathsf{Grp}$ by sending a ring R to its abelian group $(R, +)$.

- Another example of a free functor is a functor $\mathsf{Set} \to \mathsf{Grp}$ by taking the free group generated by a set S (who would have known this is free?)

Here is a cool example: functors preserve isomorphism. If two groups are isomorphic, then they must have the same cardinality. In the language of category theory, this can be expressed as such: if $G \cong H$ in Grp and $U \colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor, then $U(G) \cong U(H)$. We can generalize this to *any* functor and category!

Theorem 1.1. *If $A_1 \cong A_2$ are isomorphic objects in $\mathscr{A}$ and $F \colon \mathscr{A} \to \mathscr{B}$ is a functor then*

$$F(A_1) \cong F(A_2).$$

Proof. Let's go diagram chasing!

$$\begin{tikzcd}
\mathscr{A} \ni & A_1 \arrow[d, shift right, "f"'] \arrow[rr, dotted, "F"] & & B_1 = F(A_1) \arrow[d, shift right, "F(f)"'] & \in \mathscr{B} \\
& A_2 \arrow[u, shift right, "g"'] & & B_2 = F(A_2) \arrow[u, shift right, "F(g)"'] &
\end{tikzcd}$$

The main idea of the proof follows from the fact that functors preserve composition and the identity map. ⊠

This is very very useful for us (people who are doing algebraic topology) because functors will preserve isomorphism between spaces (we get that homotopic spaces have isomorphic fundamental groups).

Example 1.14 (Functors in algebraic topology)**.** As expected, functors show up all the time in algebraic topology. Here are some of the constructions we have studied so far that are functors:

- The act of assigning a fundamental group to a space is a functor $\pi_1 \colon \mathsf{Top}_* \to \mathsf{Grp}$.

- The singular chain complex functor $F \colon \mathsf{Top} \to \mathsf{Ch}_{\mathbb{Z}}$ assigns to a space X the chain complex of singular chains in X and to each map $f : X \to Y$ the induced chain map.

- The algebraic homology functor $F \colon \mathsf{Ch}_{\mathbb{Z}} \to \mathsf{AbGrp}$[2] assigns to a chain complex its sequence of homology groups, and chain maps the induced homomorphisms on homology.

[2] Not really, it's actually the category of sequences of abelian groups, but I wasn't sure how to denote that.

- Composing the previous functors, we have a functor $F\colon \mathsf{Top} \to \mathsf{AbGrp}$ assigning to each space its singular homology groups.
- There is a functor assigning pairs of spaces (X, A) to the associated LES of homology groups. In the domain category, morphisms are maps between pairs, and in the target category morphisms are commutative diagrams of maps between exact sequences.
- The previous functor is a composition of the functor from pairs of spaces to $\mathsf{Ch}_{\mathbb{Z}}$ restricted to short exact sequences, and a functor from the aforementioned restriction on $\mathsf{Ch}_{\mathbb{Z}}$ to the LES of homology groups.
- Finally, in the next section we will study the contravariant version of homology, called *cohomology*.

Note. As a meme (or not really, but it's still funny), we can construct the category Cat whose objects are categories and arrows are functors.

1.6 Homotopy categories and homotopy equivalence

Let Top_* be the category of pointed topological spaces. Then the fundamental group gives a functor $\mathsf{Top}_* \to \mathsf{Grp}$. When we have a suitable relation of homotopy between maps in a category $\mathscr{C}$, we define the homotopy category $\mathsf{Ho}(\mathscr{C})$ to be the category sharing the same objects as $\mathscr{C}$, but morphisms the homotopy classes of maps. On Top_*, we require homotopies to map basepoint to basepoint, and we get the homotopy category hTop_* of pointed spaces.

Homotopy equivalences in $\mathscr{C}$ are isomorphisms in $\mathsf{Ho}(\mathscr{C})$. More concretely, recall that a map $f : X \to Y$ is a homotopy equivalence if there is a map $g\colon Y \to X$ such that both $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In the language of category theory, we can obtain the analogous notion of a pointed homotopy equivalence. Functors carry isomorphisms to isomorphisms, so then the pointed homotopy equivalence will induce an isomorphism of fundamental groups. This also holds, but less obviously, for the category of non pointed homotopy equivalences.

Theorem 1.2. *If $f : X \to Y$ is a homotopy equivalence, then*

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is an isomorphism for all $x \in X$.

Proof. Let $g : Y \to X$ be a homotopy inverse of f. By our homotopy invariance diagram, we see that the composites

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x))$$

and

$$\pi_1(Y, y) \xrightarrow{g_*} \pi_1(X, g(y)) \xrightarrow{f_*} \pi_1(Y, (f \circ g)(y))$$

are isomorphisms determined by paths between basepoints given by chosen homotopies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Then in each displayed composite, the first map is a monomorphism and the second is an epimorphism. Taking $y = f(x)$ in the second composite, we see that the second map in the first composite is an isomorphism. Therefore so is the first map, and we are done. ⊠

A space X is said to be contractible if it is homotopy equivalent to a point.

Corollary 1.1. *The fundamental group of a contractible space is zero.*

1.7 Natural transformations

We talked about maps between objects which led to categories, and then maps between categories which lead to functors. Now let's talk about maps between functors, the natural transformation: this is actually not too strange (recall the homotopy, a "deformation" from a map to another map).

In this case, we also want to pull a map (functor) F to another map G by composing a bunch of arrows in the target space $\mathscr{B}$.

Definition 1.10 (Natural Transformations)**.** Let $F, G\colon \mathscr{A} \to \mathscr{B}$ be two functors. A **natural transformation** $\alpha\colon F \to G$ denoted

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathscr{B} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

consists of, for each $A \in \mathscr{A}$ an arrow $\alpha_A \in \operatorname{Hom}_{\mathscr{B}}(F(A), G(A))$, which is called the component of α at A. Pictorially, it looks like this:

$$\begin{tikzcd} & F(A) \in \mathscr{B} \arrow[dd, "\alpha_A"] \\ \mathscr{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\ & G(A) \in \mathscr{B} \end{tikzcd}$$

The α_A are subject to the "naturality" requirement such that for any $A_1 \xrightarrow{f} A_2$, the following diagram commutes:

$$\begin{tikzcd} F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\ G(A_1) \arrow[r, "G(f)"'] & G(A_2) \end{tikzcd}$$

The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (like in a homotopy from f to g the point $f(t)$ gets deformed to the point $g(t)$ continuously). Think of f representing the homotopy and the basepoints being $F(A_1), G(A_1)$ to $F(A_2), G(A_2)$.

Natural transformations can be composed. Take two natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$. Consider the following commutative diagram:

$$\begin{tikzcd} & F(A) \arrow[d, "\alpha_A"] \\ \mathscr{A} \ni A \arrow[ur, dotted, "F"] \arrow[r, dotted, "G"] \arrow[dr, dotted, "H"'] & G(A) \arrow[d, "\beta_A"] \\ & H(A) \end{tikzcd}$$

We can also construct inverses: suppose α is a natural transformation such that α_A is an isomorphism for each A. Then we construct an inverse arrow β_A in the following way:

$$\begin{tikzcd} & F(A) \in \mathscr{B} \arrow[dd, shift right, "\alpha_A"'] \\ \mathscr{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\ & G(A) \in \mathscr{B} \arrow[uu, shift right, "\beta_A"'] \end{tikzcd}$$

We say α is a **natural isomorphism**. Then $F(A) \cong G(A)$ *naturally* in A (and β is an isomorphism too!) We write $F \cong G$ to show that the functors are naturally isomorphic.

Example 1.15. If $F\colon \mathsf{Set} \to \mathsf{Grp}$ is the free functor that sends a set to the free group on such set and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor sending a free group to its generating set, then we have a natural inclusion of $S \hookrightarrow UF(S)$. The functors F and U are left and right adjoint to each other, in the sense that we have a natural isomorphism

$$\mathsf{Grp}(F(S), A) \cong \mathsf{Set}(S, U(A))$$

for a set S and an abelian group A. This expresses the "universal property" of free objects: a map of sets $S \to U(A)$ extends uniquely to a homomorphism of groups $F(S) \to A$.

Definition 1.11. Two categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if there are functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$ and natural isomorphisms $FG \to \mathrm{Id}$ and $GF \to \mathrm{Id}$, where the Id are the respective identity functors.

Example 1.16 (Natural transformations in algebraic topology)**.** As expected, these also show up in algebraic topology.

- Consider the boundary maps $H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)$ in singular homology, or any homology theory really.
- The change-of-coefficient homomorphisms $H_n(X; G_1) \to H_n(X; G_2)$ induced by a homomorphism $G_1 \to G_2$ are also natural transformations.

1.8 The Yoneda lemma (todo)

Definition 1.12 (The functor category)**.** The **functor category** of two categories $\mathscr{A}$ and $\mathscr{B}$, denoted $[\mathscr{A}, \mathscr{B}]$ is defined as follows:

- The objects of $[\mathscr{A}, \mathscr{B}]$ are (covariant) functors $F\colon \mathscr{A} \to \mathscr{B}$, and
- The morphisms are natural transformations $\alpha\colon F \to G$.

todo

1.9 (todo) equalizers, limits, colimits, pullback, pushout, kernesl and cokernesl, additive and abelian category, homological algebra

Lecture 2

Free Groups and Group Theory

Not to be confused with free **abelian** groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39,40 and Hatcher §1.2.

I've decided to expand this section to include any miscellaneous group theory that I may not have covered/forgot. What texts they follow will probably be at the beginning of each subsection.

2.1 Words and Reduced Words

Let A_i be a set of elements (not necessarily finite). We say A is an **alphabet** and think of the $a_i \in A$ as **letters**. Symbols of the form a_i^n are **syllables** and **words** are a finite string of syllables. We denote the **empty word** 1 as the word with no syllables.

Example 2.1. Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to a_i^{m+1} (**elementary contractions**) or replacing a_i^0 by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a **reduced word**.

Example 2.2. The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

2.2 Free Groups

Denote the set of all reduced words from our alphabet A as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words w_1 and w_2 in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of w_2 onto w_1.

Example 2.3. If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet A. Man, the weather outside today is nice.

Definition 2.1 (Free Group)**.** The group $F[A]$ described above is the **free group generated by** A.

Sometimes we have a group G and a generating set $A = \{a_i \mid \in I\}$, and we want to know whether or not G is *free* on $\{a_i\}$, that is, G is the free group generated by $\{a_i\}$.

Definition 2.2 (Free Generators)**.** If G is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi : G \to F[A]$ such that $\phi(a_i) = a_i$, then G is **free on** A, and the a_i are **free generators of** G. A group is **free** if it is free on some nonempty set A.

Oh you'll be free... free indeed...

Example 2.4. $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on n letters" as opposed to the "free group on n generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

Example 2.5. $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

Theorem 2.1. *If G is free on A and B, then A and B have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

Proof. Refer "to the literature". ⊠

Definition 2.3 (Rank)**.** If G is free on A, then the number of letters in A is the **rank of the free group** G.

Theorem 2.2. *Two free groups are isomorphic if and only if they have the same rank.*

Proof. Immediate. ⊠

Theorem 2.3. *A nontrivial proper subgroup of a free group is free.*

Proof. Back "to the literature", says Fraleigh. This can be proving with the theory of covering spaces in algebraic topology. ⊠

Example 2.6. Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

2.3 Homomorphisms of Free Groups

Theorem 2.4. *Let G be generated by $A = \{a_i \mid\in I\}$ and let G' be any group. If a_i' for $i \in I$ are any elements in G' not necessarily distinct, then there is at most one homomorphism $\phi : G \to G'$ such that $\phi(a_i) = a_i$'. If G is free on A, then there is exactly one such homomorphism.*

Proof. Let ϕ be a homomorphism from G into G'such that $\phi(a_i) = a_i$'. Then any $x \in G$ can be written as a finite product of the generators a_i, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the a_i not necessarily distinct. Since ϕ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i$'.

Now suppose that G is free on A, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi : G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in G, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ⊠

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

Corollary 2.1. *Every group G' is a homomorphic image of a free group G.*

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid\in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 2.4 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i$'. Clearly the image of G under ψ is all of G'. ⊠

Only the free group on one letter is abelian.

2.4 Free Products of Groups

Definition 2.4 (Free Products)**.** As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of w_1 and the beginning of w_2 belong to the same G_α? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of a and b are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, \ldots$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$ extends uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under φ.

Example 2.7. For a free product $G*H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G*H \to G \times H$.

2.5 Group Presentations

Apparently, I never took group theory. Let's talk about group presentations!

Motivation: form a group by giving generators and having them follow certain relations. We want the group as free (free indeed) as possible on these generators.

Example 2.8. Suppose G has generators x and y and is **free except for the relation** $xy = yx$, or $xyx^{-1}y^{-1} = 1$. This makes sure G is abelian, and so G is isomorphic to $F[\{x,y\}]$ modulo its commutator subgroup, the smallest normal subgroup containing $xyx^{-1}y^{-1}$. This is because any normal subgroup containing $xyx^{-1}y^{-1}$ gives rise to an abelian factor group and thus contians the commutator subgroup (by a previous theorem).

This example illustrates what we want: let $F[A]$ be a free group and we want a new group as free as possible, with certain equations satisfied. We can always write these equations with the RHS equal to 1, so we consider the equations to be $r_i = 1$ for $i \in I$, where $r_i \in F[A]$. If $r_i = 1$, then

$$x(r_i^n)x^{-1} = 1$$

for any $x \in F[A]$, $n \in \mathbb{Z}$. Any product of elements equal to 1 again equals 1, so any finite product of the form

$$\prod_j x_j \left(r_{i_j}^{n_j}\right) x_j^{-1}$$

where r_{i_j} need not be distinct equals 1 in the new group. It can be seen that the set of all these finite products is a normal subgroup R of $F[A]$. Then any group that looks like $F[A]$ given $r_i = 1$ also has $r = 1$ for all $r \in R$. But $F[A]/R$ looks like $F[A]$, except that R has been collapsed to form the identity 1. Hence the group we are after is (at least isomorphic to) $F[A]/R$. We can view this group as described by the generating set A and the set $\{r_i \mid i \in I\}$, abbreviated $\{r_i\}$.

Definition 2.5 (Group Presentations)**.** Let A be a set and $\{r_i\} \subseteq F[A]$. Let R be the least normal subgroup of $F[A]$ containing the r_i. An isomorphism ϕ of $F[A]/R$ onto a group G is a **presentation of** G. The sets A and $\{r_i\}$ give a **group presentation**. The set A is the set of **generators for the presentation** and each r_i is a **relator**. Each $r \in R$ is a **consequence of** $\{r_i\}$. An equation $r_i = 1$ is a **relation**. A **finite presentation** is one in which both A and $\{r_i\}$ are finite sets.

Refer back to Example 2.1: $\{x,y\}$ is our set of generators and $xyx^{-1}y^{-1}$ is the only relator. The equation $xyx^{-1}y^{-1} = 1$ or $xy = yx$ is a relation— this was an example of a finite presentation.

2.6 Free Abelian Groups (todo)

Finally, another missing part of group theory has come back to bite me at a crucial moment. Today we talk about free **abelian** groups, needed to define chain groups in homology. Some notation stuff (since we're dealing with abelian groups): for an abelian group G, 0 is the identity, + is the operation, and

$$\left.\begin{aligned} na &= \underbrace{a + \cdots + a}_{n \text{ times}} \\ -na &= \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}} \end{aligned}\right\} \text{for } n \in \mathbb{Z}^+ \text{ and } a \in G.$$

We also have $0a = 0$, where the LHS $0 \in \mathbb{Z}$, and the RHS $0 \in G$.

Note. $\{(1,0),(0,1)\}$ is a generating set for $\mathbb{Z} \times \mathbb{Z}$, since $(n,m) = n(1,0) = m(0,1)$ for any $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. Note that each element can be *uniquely* expressed in such form, that is, n, m are unique. Linear algebra much?

Theorem 2.5. *Let $X \subseteq G$ for G a nonzero abelian group. Then TFAE:*

1. *Every nonzero $a \in G$ can be expressed* uniquely *(up to order of summands) in the form $a = n_1x_1 + n_2x_2 + \cdots + n_rx_r$ for $n_i \neq 0 \in \mathbb{Z}$ and distinct $x_i \in X$.*
2. *X generates G, and $n_1x_1 + n_2x_2 + \cdots + n_rx_r = 0$ for $n_i \in \mathbb{Z}$ and distinct $x_i \in X \iff n_1 = n_2 = \cdots = n_r = 0$.*

This is looking an awful lot like linear algebra. Inb4 $\mathbb{Z}$-modules?

Proof. Suppose Condition 1 holds, then $G \neq \{0\} \implies X \neq \{0\}$. To show $0 \notin X$, if $x_i = 0$ and $x_j = 0$, then $x_j = x_i + x_j$ contradicting uniqueness of the expression of x_j. Condition 1 implies that X generates G, and $n_1x_1 + n_2x_2 + \cdots + n_rx_r = 0$ if $n_1 = n_2 = \cdots n_r = 0$. Suppose $n_1x_2 + n_2x_2 + \cdots + n_rx_r = 0$ for some $n_i \neq 0$: if we drop the zero terms and relabel, we can assume all $n_i \neq 0$. Basically, we've created something from nothing (which is equal to nothing), from which we can derive our contradiction. Then

$$x_1 = x_1 + (n_1x_1 + x_2x_2 + \cdots + n_rx_r) = (n_1 + 1)x_1 + n_2x_2 + \cdots n_rx_r,$$

two ways of writing some nonzero x_1, contradicting uniqueness. So every n_i must equal zero, which implies Condition 2.

Let's go the other way around. Let $a \in G$, since X generates G we have $a = n_1x_1 + n_2x_2 + \cdots + n_rx_r$. Using the standard technique for proving uniqueness, assume another expression for a, that is, $a = n_1x_1 + m_2x_2 + \cdots + m_rx_r$. So

$$0 = (n_1 - m_1)x_1 + (n_2 - m_2)x_2 + \cdots + (n_r - m_r)x_r,$$

and by Condition two, we have $n_i = m_i$ for $1 \leq i \leq r$. We conclude the coefficients must be unique, finishing the proof. ⊠

Definition 2.6 (Free abelian group)**.** A **free abelian** group is a group with a generating set X that satistifies the conditions stated in Theorem 2.5. We say X is a **basis** for the group.

Example 2.9. The group $\mathbb{Z} \times \mathbb{Z}$ is free abelian with $\{(1,0),(0,1)\}$ a basis. We also have a basis for the free abelian group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ as $\{(1,0,0),(0,1,0),(0,0,1)\}$. So finite direct products of $\mathbb{Z}$ are free abelian groups. However, $\mathbb{Z}_n$ is not free abelian, for $nx = 0$ for all $x \in \mathbb{Z}_n$, and since $n \neq 0$ this contradicts Condition 2 of being a free abelian group.

Suppose G is free abelian with basis $X = \{x_1, x_2, \cdots, x_r\}$. If $a \in G$ and $a \neq 0$, then a has a unique expression of the form

$$a = n_1x_1 + n_2x_2 + \cdots + n_rx_r \quad \text{for } n_i \in \mathbb{Z}.$$

Note that while in our definition of a free abelian group, we required all the coefficients to be nonzero, since we only used some elements of the basis to uniquely define the expression (the basis might have been infinite). However, in this definition we cover all the basis, so we can allow some terms to be zero, since our basis is finite. Define

$$\phi : G \to \overbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}^{r \text{ factors}}$$

by $\phi(a) = (n_1, n_2, \cdots, n_r)$ and $\phi(0) = (0, 0, \cdots, 0)$. This leads us to the following result, which we'll state as a theorem.

Theorem 2.6. *If G is a nonzero free abelian group with rank*[3] *r, then G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r factors.*

Proof. All we have to do is check that $\phi : G \to \overbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}^{r \text{ factors}}$, $a \mapsto (n_1, n_2, \cdots, n_r)$, $0 \mapsto (0, 0, \cdots, 0)$ is an isomorphism. Let $(n_1, n_2, \cdots, n_r)$ be an arbitrary element of the r products of $\mathbb{Z}$. Since we can write $a \in G$ as $a = n_1x_1 + n_2x_2 + \cdots + n_rx_r$ uniquely (since our basis is finite, see the above expression), simply take $\phi(a)$ which maps onto $(n_1, n_2, \cdots, n_r)$. We know that such a exists with the desired coefficients since G is generated by $X = \{x_i\}_{i \in I}$. Note that this covers the cases where some terms of the $\mathbb{Z}$ product are zero since we relaxed

[3] The **rank** of G free abelian is the number of elements in a basis for G (since they all have the same number of elements by this theorem).

the nonzero coefficient requiremeent since our basis is finite. This also hits zero by definition, and also by noting that Condition 2 exists. Therefore ϕ is onto. To show ϕ is 1-1, let $a, b \in G$, where $a = n_1x_1 + \cdots n_rx_r$, $b = m_1x_1 + \cdots + m_sx_s$. If $\phi(a) = \phi(b)$, then $n_1 = m_1, n_2 = m_2, \cdots, n_r = m_s$. So $r = s$, and a and b have the same amount of terms and coefficients in their *unique* expression, so they must be equal. So ϕ is 1-1.

Finally, we want to show that ϕ is a homomorphism. Take $a, b \in G$ with the representations as in the previous paragraph. Then WLOG assume that $r < s$, so $a + b = (n_1 + m_1)x_1 + (n_2 + m_2)x_2 + \cdots (n_r + m_r)x_r + m_{r+1}x_{r+1} + \cdots + m_sx_s$. Then $\phi(a+b) = (n_1+m_1, n_2+m_2, \cdots, n_r+m_r, m_{r+1}, \cdots, m_s)$. Now $\phi(a) = (n_1, \cdots, n_r)$ and $\phi(b) = (m_1, \cdots, m_s)$, so $\phi(a)+\phi(b) = (n_1+m_1, \cdots, n_r+m_r, \cdots, m_s)$. Clearly the two are equal, and so ϕ is a homomorphism. Therefore ϕ is a bijective homomorphism and therefore an isomorphism, and we are done. ⊠

Theorem 2.7. *Let $G \neq \{0\}$ be a free abelian group with a finite basis. Then every basis of G is finite, and all bases of G have the same number of elements.*

Proof. Say $\{x_1, x_2, \cdots, x_r\}$ is a basis for G. Then by Theorem 2.6, $G \simeq \overbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \mathbb{Z}}^{r \text{ factors}}$. Define the group $2G := \{2g \mid g \in G\}$, which clearly a normal subgroup of G. Then

$$G/2G \simeq (\overbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}^{r \text{ factors}})/(\overbrace{2\mathbb{Z} \times 2\mathbb{Z} \times \cdots \times 2\mathbb{Z}}^{r \text{ factors}}) \simeq \overbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \mathbb{Z}_2}^{r \text{ factors}}.$$

So $|G/2G| = 2^r$, which implies that the number of elements in any finite basis is $r = \log_2 |G/2G|$. Therefore any two finite bases have the same amount of elements. To show that G cannot also have an infinite basis, let Y be an arbitrary basis for G and $\{y_1, y_2, \cdots, y_s\}$ be distinct elements of Y. Define H as the subgroup of G generated by $\{y_1, y_2, \cdots, y_s\}$ and K the subgroup generated by the rest of Y. Then $G \simeq H \times K$, since this covers everything generated by elements of Y, so

$$G/2G \simeq (H \times K)/(2H \times 2K) \simeq (H/2H) \times (K/2K).$$

Since $|H/2H| = 2^s$, it can be seen that $|G/2G| \geq 2^s$. Since $|G/2G| = 2^r$, it must be that $s \leq r$, however this implies that Y cannot be infinite since we could simply take $s > r$, and we are done. ⊠

TODO: proof of the fundamental theorem?

2.7 Semidirect products and Commutators(todo)

I had an existential crisis when Dr. Allcock said to simply observe that one group is a semidirect of another by such and such group. These notes will follow Dummit and Foote §5.

The direct product is what you think it is: the set of n-tuples with the group operation done componentwise.

Definition 2.7 (Commutators)**.** Let G be a group and $x, y \in G$. Let A, B be nonempty subsets of G. Then

1. Define $[x, y] = x^{-1}y^{-1}xy$ as the **commutator** of x and y.

2. Define $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$, the group generated[4] by commutators of elements from A and B.

3. Define $G' = [G : G] = \langle [x, y] \mid x, y \in G \rangle$, the subgroup of G generated by commutators of elements from G, called the **commutator subgroup** of G.

The commutator of x and y is 1 iff x and y commute, hence the name.

Proposition 2.1. *The factor group G/G' is abelian. Furthermore, G/G' is the largest abelian quotient of G in the sense that if $H \trianglelefteq G$ and G/H is abelian, then $G' \leq H$. Conversely, if $G' \leq H$, then $H \trianglelefteq G$ and G/H is abelian.*

[4] Generators and presentations are starting to blur the line for me...

Proof. Let $xG', yG' \in G/G'$. Since the commutator $[x, y] \in G'$ collapses to zero, we have

$$\begin{aligned}(xG')(yG') &= (xy)G' \\ &= (yx[x,y])G' \\ &= (yx)G' \\ &= (yG')(xG').\end{aligned}$$

So G/G' is abelian. Now supposed $H \trianglelefteq G$ and G/H is abelian. Then for all $x, y \in G$ we have $(xH)(yH) = (yH)(xH)$, so

$$\begin{aligned}1H &= (xH)(xH)^{-1}(yH)(yH)^{-1} \\ &= (xH)^{-1}(yH)^{-1}(xH)(yH) \\ &= (x^{-1}y^{-1}xy)H \\ &= [x,y] \in H,\end{aligned}$$

which implies $[x, y] \in H$ for all $x, y \in G$. So $G' \leq H$. Conversely, if $G' \leq H$, then every subgroup of G/G' is normal, in particular, $H/G' \trianglelefteq G/G'$. We have $H \trianglelefteq G$ by the Lattice Isomorphism Theorem, and by the Third Isomorphism Theorem, we have

$$G/H \cong (G/G')/(H/G').$$

Since G/H is isomorphic to a quotient of the abelian group G/G', G/H must be abelian. ⊠

Why does this work? We mod out by the stuff we don't like: in this case, all the commutators collapse to the identity, so elements in the quotient group commute.

Recall that for subgroups $H, K > G$, we define the group $HK = \{hk \mid h \in H, k \in K\}$.

Theorem 2.8. *Suppose G is a group with subgroups H, K such that*

(1) H and K are normal in G, and

(2) $H \cap K = 1$.

Then $HK \simeq H \times K$.

Proof. Now HK is a subgroup of G iff $HK = KH$, which holds since H and K are normal in G. Let $h \in H$ and $k \in K$, since $H \trianglelefteq G$ we have $k^{-1}hk \in H$, so $h^{-1}(k^{-1}hk) \in H$. Similarly, $(h^{-1}k^{-1}h)k \in K$. Now $H \cap K = 1$, so $h^{-1}k^{-1}hk = 1$, or $hk = kh$, so every element of H commutes with every element of K. We can uniquely write each element of HK as a product hk (this was an exercise) with $h \in H$ and $k \in K$, so the map $\varphi\colon HK \to H \times K$, $hk \mapsto (h,k)$ is well defined. To show that φ is a homomorphism let $h_1, h_2 \in H$, $k_1, k_2 \in K$. Since h_2 and k_1 commute, we have $(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2)$, where the latter is a unique way of writing $(h_1k_1)(h_2k_2)$ in the form hk for $h \in H$ and $k \in K$. So

$$\begin{aligned}\varphi((h_1k_1)(h_2k_2)) &= \varphi(h_1h_2k_1k_2) \\ &= (h_1h_2, k_1k_2) \\ &= (h_1,k_1)(h_2,k_2) = \varphi(h_1k_2)\varphi(h_2k_2),\end{aligned}$$

and so φ is a homomorphism. We have φ a surjection since the representation of elements of HK as products hk is unique, and injectivity follows from the fact that $H \cap K = 1$ (and therefore so does $\ker\varphi$), and we are done. ⊠

Definition 2.8. If G is a group and H and K are normal subgroups of G with $H \cap K = 1$, we say that HK is the **internal direct product** of H and K. We will call $H \times K$ the **external direct product** of H and K. The different between the two is purely notational.

In short, if G is a group with $H, K \trianglelefteq G$ such that $H \cap K = 1$ and every element $g = hk$ uniquely (equivalent to the fact that $G = HK$ when $H \cap K = 1$), then $G \simeq H \times K$. The direct product has H and K as subgroups by projection, and thus is uniquely defined by the universal property in category theory.

Now onto semidirect products, a generalization of the direct product by relaxing the normality requirements. This allows us to build “larger” groups from H and K such that G has subgroups isomorphic to H and K, like direct products. Since one of the subgroups need not be normal, we can form a nonabelian group from an abelian groups.

Theorem 2.9. *Let H and K be groups and let φ be a homomorphism from K into $\operatorname{Aut}(H)$. Let $\cdot$ denote the (left) action of K on H determined by φ, and G be the set of ordered pairs (h,k) with $h \in H$ and $k \in H$. Define the following multiplication on G:*

$$(h_1,k_1)(h_2,k_2) = (h_1 k_1 \cdot h_2, k_1 k_2).$$

(1) This turns G into a group of order $|H||K|$.

(2) G contains H and K as subgroups by the projections onto $\{(h,1) \mid h \in H\}$ and $\{(1,k \mid k \in K\}$. If we identify H and K with their projections, we have

(3) $H \trianglelefteq G$

(4) $H \cap K = 1$,

(5) For all $h \in H$ and $k \in K$, $khk^{-1} = k \cdot h = \varphi(k)(h)$.

TODO finish semidirects

Lecture 3

Common Topological Structures

We’ll take this section to digress a little bit and explore some examples of our favorite spaces that we work with a lot in topology.

3.1 Manifolds (todo)

3.2 Cell complexes

The big idea is this: we can build a lot of our favorite topological spaces by starting with “removable parts” from each dimension, then glueing them together via something called a “boundary map”. Naturally, we start from dimension zero (points), add cells from the 1st dimension (arcs), then 2-cells (surfaces), and so on.

For example, we can construct the torus $\mathbb{T} = S^1 \times S^1$ by identifying opposite edges of a square. In general, an orientable (worry about what this means later) surface M_g of genus g can be constructed from a polygon with $4g$ sides by identifying pairs of edges. The $4g$ edges of the polygon becomes a union of $2g$ circles intersecting at a point. The interior of the polygon can be thought of as a **2-cell**, and the union of circles being obtained by attaching $2g$ open arcs, or **1-cells**.

Definition 3.1 (n-cell)**.** An **n-cell** e^n is defined as a space homeomorphic to the open disk $D^n \setminus \partial D^n$, where $\partial D^n = S^{n-1}$. Note that D^0 and e^0 are simply points since $\mathbb{R}^0 = \{0\}$.

Definition 3.2 (CW complexes)**.** We call the spaces defined by attaching n-cells by the name **cell complex** or **CW complex**. When it is clear that we’re referring to cell or CW complexes, sometimes we will drop the beginning word and just refer to these spaces as “complexes”. Here is a natural way to generalize the construction of such complexes:

(1) Start with a discrete set of points X^0, or 0-cells.

(2) Inductively, form the **n-skeleton** X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha \colon S^{n-1} \to X^{n-1}$. Note! These maps are supposed to be hard to define! So X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.

(3) We can either stop this process at a finite step with $X = X^n$ for some $n < \infty$, or continue on forever, with $X = \bigcup_n X^n$. In the infinite-dimensional case, X is given the weak topology: A set $A \subseteq X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the **dimension** of X, or the maximum number of cells of X.

Example 3.1. To better understand what is happening in step (2) of building a CW complex, let us examine the first few cases.

(1) Consider $n = 1$. We start with a 0-skeleton consisting of just discrete points. Then we attach 1-cells homeomorphic to the open interval $(0,1)$ via maps $\varphi_\alpha \colon S^0 \to X^0$, which are maps from the two endpoints of an interval into a discrete set of points. So X^1 is the quotient space of the disjoint union $X^0 \amalg_\alpha D^1_\alpha$ of X^0 with a collection of 1-disks D^1_α (just points in X^0 plus α number of intervals that don't touch the points) under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^1_\alpha = S^0 \simeq \{0,1\}$. In this case, the identifications are just ways of glueing endpoints of the interval $[0,1]$. So as a set, $X^1 = X^0 \amalg_\alpha e^1_\alpha$ where e^1_α is homeomorphic to the interval $(0,1)$.

All this is saying is start out with points, and add some closed intervals (but make sure they're disjoint). Then, identify endpoints of these intervals together with maps φ_α by considering $x \sim \varphi_\alpha(x)$. Post-identification, we have the 1-skeleton equal to a bunch of points with disjoint open intervals. The reason they're disjoint is so points from X^0 can't go in the middle of the interval $(0,1)$, but *can* be attached to the endpoints $0,1$ still preserving disjointedness. Basically, $[0,1] \cup \{0,1\}$ isn't a disjoint union, but $(0,1) \amalg \{0,1\}$ is, and they look the same. Note that the 1-skeleton X^1 is just a **graph** as we call it in algebraic topology, with vertices the 0-cells and edges the 1-cells.

(2) Now consider $n = 2$. We start with the 1-skeleton X^1 which looks like a graph with vertices and edges. Then we attach 2-cells homeomorphic to the open ball $B(0,1) \subseteq D^2$, which is a disk in $\mathbb{R}^2$ minus its boundary, S^1. These look like sheets of paper cut in a circle, with an open boundary. We attach these sheets via maps $\varphi_\alpha \colon S^1 \to X^1$, which map from a boundary circle into a graph. Then X^2 is the quotient space of the disjoint union $X^1 \amalg_\alpha D^2_\alpha$ with the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial D^2_\alpha = S^1$. This time, we consider a graph with a bunch of closed sheets, then attach the sheets based off where the boundary circle gets glued onto the graph (realized by φ_α). So as a set, $X^2 = X^1 \amalg_\alpha e^2_\alpha$, which means that the 2-skeleton X^2 looks like a graph plus open sheets, glued together at the boundaries. You could also think of attaching 2-cells as "filling in" regions/faces of the graph enclosed by edges.

(3) For $n = 3$, we start with X^2, a graph with sheets glued in circles along the edges. Then we form the 3-skeleton X^3 by attaching 3-cells (imagine these as "filling" cells, like how a cat will conform to the shape its container, sans the container)[5]. These cats are attached via maps $\varphi_\alpha \colon S^2 \to X^2$, which take their containers S^2 (a real-life ball minus its interior, looks like a shell) into the 2-skeleton. Then consider the 2-skeleton together with the space-filling cats and their containers, denoted by $X^2 \amalg_\alpha D^3_\alpha$, and identify $x \sim \varphi_\alpha(x)$ with $x \in \partial D^3_\alpha = S^2$, which is just saying where the containers (or empty shells) get glued onto the 2-skeleton. Then, the cat will conform to fit its space, and thus we have attached our 3-cells, and we express this by saying $X^3 = X^2 \amalg_\alpha e^3_\alpha$.

At this point, you should have a good idea of where this is going, although it does get more difficult to visualize in higher dimensions, which is why we have the handy topological construction we stated before moving on to cases.

Definition 3.3 (Euler characteristic)**.** The **Euler characteristic** $\chi(X)$ of a finite cell complex is the number of even-dimensional cells minus the odd-dimensional cells. Formally, we define

$$\chi(X) = \sum_{n=0}^{k} (-1)^n c_n,$$

where c_n is the number of n-cells of X and k is the dimension of X. For $k = 2$, we have $\chi(X) =$ 0-cells $-$1-cells $+$2-cells, which if you think of 0-cells as vertices, 1-cells as edges, and 2-cells as faces, is our familiar formula $V - E + F = 2$ for convex polyhedron, so it can be seen that the Euler characteristic is a generalization of Euler's formula. We will later use homology to show that $\chi(X)$ depends only on the homotopy type of X, which is why $V - E + F = 2$ holds for all convex polyhedra.

[5] For a less exciting but more pedagogically effective metaphor, replace all instances of "cat" with the appropiate form of the word "liquid".

Example 3.2. The sphere S^n has a cell complex structure consisting of one 0-cell and one n-cell, with the n-cell attached by the constant map $S^{n-1} \to e^0$. You can think of this as wrapping up the "boundary" of the n-plane down to a single point, and closing it up (the best concrete example is $\mathbb{R}^2$ plus a point). This is formally expressed by the quotient space $D^n/\partial D^n$.

Definition 3.4. Each cell e^n_α in a cell complex X has a **characteristic map** $\phi_\alpha \colon D^n_\alpha \to X$ extending the attaching map φ_α, and is a homeomorphism from the interior of D^n_α onto e^n_α. We define φ_α as the composition $D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \to X^n \hookrightarrow X$, where the map $X^{n-1} \amalg_\alpha D^n_\alpha$ is the quotient map that builds X^n from X^{n-1}.

Example 3.3. For examples of the characteristic map, considering the canonical CW structure on S^n, a characteristic map for the lone n-cell is the quotient map $D^n \to S^n$ collapsing D^n to a point. For $\mathbb{R}\mathrm{P}^n$ (see next section), a characteristic map for e^i is the quotient map $D^i \to \mathbb{R}\mathrm{P}^i \subseteq \mathbb{R}\mathrm{P}^n$ identifying antipodal points of $\partial D^i = S^{i-1}$, and similarly for $\mathbb{C}\mathrm{P}^n$.

Definition 3.5 (Subcomplexes and CW pairs)**.** A **subcomplex** of a cell complex X is a closed subspace $A \subseteq X$ consisting of a union of cells of X. Since A is closed, the characteristic map of each cell in A has image contained in A, and thus the image of each attaching map is also in A, so A is also a cell complex. A pair (X,A) consisting of a cell complex X and a subcomplex A is called a **CW pair**.

Example 3.4. Each n-skeleton X^n of a cell complex X is a subcomplex.

Example 3.5. With the canonical structure on S^n, we have natural inclusions $S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$, but these aren't subcomplexes since S^n only has two cells. However, if we put a different CW structure on S^n by attaching two hemispheres represented by k-cells onto the equatorial S^{k-1} to obtain S^k, then each subspace S^k of S^n is a subcomplex. The k-cells are precisely the two remaining components of $S^k \setminus S^{k-1}$. Then the infinite sphere $S^\infty = \bigcup_{n=0}^\infty S^n$ also becomes a cell complex.

In the examples of cell complexes we've given so far, the closure of each cell is a subcomplex, and the same goes for any collection of cells. However, this doesn't have to hold in general: consider the canonical structure of S^1, then if we attach a 2-cell by a map $\varphi \colon S^1 \to S^1$ where $\operatorname{im}\varphi$ is a nontrivial arc in S^1, then the closure of the 2-cell is not a subcomplex since it only contains part of the 1-cell.

3.3 Operations on CW complexes

We can do a lot of things to cell complexes. Here are some of them.

Products. If X, Y are complexes, then $X \times Y$ has a CW structure with cells $e^m_\alpha \times e^n_\beta$ where e^m_α ranges over the cells of X and e^n_β the cells of Y. An example is the torus $\mathbb{T} = S^1 \times S^1$ with the standard cell structure on S^1. For general complexes X and Y there is a minor point-set topological issue, that is, the topology of $X \times Y$ as a complex is sometimes finer than the product topology, however the topologies coincide if either X or Y has finitely many cells or both X and Y have countably many cells. This kind of triviality rarely causes real issues though.

Quotients. If (X,A) is a CW pair, then the quotient space X/A takes on a natural CW structure from X, consisting of the cells of $X \setminus A$ plus a 0-cell representing the image of A, or the point it gets identified to. For a cell e^n_α of $X \setminus A$ attached by $\varphi_\alpha \colon S^{n-1} \to X^{n-1}$, the attaching map for such cell in the quotient X/A is the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$. For example, given any structure on S^{n-1} we can build D^n by attaching an n-cell, so the quotient D^n/S^{n-1} is S^n with the usual structure, since the old structure all gets identified to a point.

Suspension. Given a space X, the **suspension** SX is the quotient of $X \times I$ with the identifications of collapsion $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. The example that makes the suspension somewhat clear is $X = S^1$: $S^1 \times I$ is the hollow cylinder, and identifying $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to points "closes up" the ends to make a two-sided hollow top. We can regard SX as the union of two copies of the **cone** CX, defined as $CX := (X \times I)/(X \times \{0\})$. If X is a CW complex, then so are CX and SX as quotients of the product with I, where I has the standard structure of two 0-cells and a 1-cell. Don't tune out just yet, suspension becomes important later in algebraic topology. It's hard to give relevant things as of now, but suspension is a functor that sends maps $f : X \to Y$ to $Sf : SX \to SY$ where Sf is the quotient map of $f \times \mathbb{1} \colon X \times I \to Y \times I$, and it also induces isomorphisms with the $(n+1)$th homology group, that is, $\widetilde{H}_{n+1}(X) \approx \widetilde{H}_n(SX)$ for all n.

Join. The cone CX consists of all line segments joining points of X to a vertex, and the suspension does the same with two vertices. In general, for spaces X, Y we can define define the space of line segments joining X to Y, called the **join** $X \star Y$. Formally, the join is the quotient space $X \times Y \times I$ with the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ for all $x \in X, y_1, y_2 \in Y$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for all $x_1, x_2 \in X, y_2 \in Y$. This collapses the subspace $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. The join can be particularly confusing to wrap your head around, so it helps to start by visualizing the join of some basic spaces and working up from there until a general understanding is achieved. This is a lot of text, so feel free to skip this and move onto the wedge sum if you're not interested in how joins work.

- Let's begin by considering the join of the singleton spaces $X = \{x_0\}, Y = \{y_0\}$. Then $X \times Y \times I$ is just an interval of the form (x_0, y_0, i) for $i \in I$, so the join identifications just identify the endpoints with themselves, which does nothing. So the join of two points is an interval.
- Now let's consider the join of a point and an interval, or $X \star Y$ where $X = \{x_0\}$ and $Y = [0, 1]$. In this case, $X \times Y \times I$ is a cube I^2 of the form (x_0, i_1, i_2) for $i_1, i_2 \in I$. The first identification joins all $i_1 \in Y$ to x_0, collapsing the left side of the square I^2 to a point. But the second identification does nothing since identifying since X is just a point (yielding $(x_0, y, 1) \sim (x_0, y, 1)$) so the join of a point and an interval is a triangle.
- In this case, let $X = \{x_0\}$ and $Y = S^1$. If $X \times Y \times I$ is the cylinder, the identification $(x_0, y_1, 0) \sim (x_0, y_2, 0)$ scrunches up the circle to x_0, while $(x_0, y, 1) \sim (x_1, y, 1)$ does nothing, resulting in a hollow top, which is precisely the cone CX of S^1. Furthermore, considering the join of S^1 with two points denoted $S^1 \star S^0$, we get the suspension top as stated earlier. In general, the join of X with a one point space is the cone CX, while the join of X with the two point space S^0 is the suspension SX.
- Consider X, Y closed intervals. Then $X \star Y$ becomes a tetrahedron: to see this, note that $X \times Y \times I$ looks like the cube I^3. If we take $(x, y_1, 0) \sim (x, y_2, 0)$, then if x varies along the interval, any y_1, y_2 will collapse into x, therefore this collapses the leftmost face of the cube into a line that looks like X. The same argument holds for Y (with the directions reversed since these are ordered pairs).
- This time, let $X = S^1$ and Y be a closed interval. $X \times Y \times I$ looks like a thick washer (precisely, a washer with thickness and length 1, with a hole of diameter 2). The identification $(x, y_1, 0) \sim (x, y_2, 0)$ will collapse the interval onto the circle, "thinning out" the left side of the washer until the radius is zero. The identification $(x_1, y, 1) \sim (x_2, y, 1)$ is a little more confusing but still understandable: it collapses every circle onto a point of the interval. So if you think of the washer as consisting of rings, start with the inner ring, smoosh it to a point, and move all the way up until all that's left is a line. The remaining space is the join $S^1 \star I$.
- Finally, consider the join of two circles $S^1 \star S^1$. This doesn't embed in $\mathbb{R}^3$, but it is still possible to understand what it looks like. $S^1 \times S^1 \times I$ is a torus times an interval: consider on one end of the interval the torus being smashed to one circle, and on the other end the other circle being smashed. Thus the result is a circle continuously deforming "sideways" into another circle. From here, you should be getting a pretty good idea of what the join of two spaces is like.

In general, the join $X \star Y$ contains a copy of X at one end and a copy of Y at the other, and every other point $(x, y, t) \in X \star Y$ is a unique line segment joining a point $x \in X$ to another point $y \in Y$, where the segment is obtained by letting t vary. We can write points of $X \star Y$ as formal linear combinations $t_1 x + t_2 y$ with $0 \le t_i \le 1$ and $t_1 + t_2 = 1$, where $0x + 1y = y$ and $1x + 0y = x$ corresponding to the identifications defining $X \star Y$. Think of t_1 as how far away x is from $X \times \{0\}$, and similarly for t_2. They must add to 1 since an interval is of length 1[6].

In the same way, we can construct an interated join $X_1 \star \cdots \star X_n$ of formal linear combinations $t_1 x_1 + \cdots + t_n x_n$ with $0 \le t_i \le 1$ and $t_1 + \cdots + t_n = 1$, omitting the terms $0x_i$. A very special case is when each X_i is just a point: as we saw earlier, the join of two points is a line, the join of three (join of line and point) is a triangle, and four a tetrahedron. In general, the join of n points is a convex polyhedron of dimension $n-1$ called a **simplex**. Formally, given that the n points form a standard basis for $\mathbb{R}^n$ we have their join the $(n-1)$-simplex

$$\Delta^{n-1} := \{(t_1, \cdots, t_n) \in \mathbb{R}^n \mid t_1 + \cdots + t_n = 1 \text{ and } t_i \ge 0\}.$$

If X and Y are CW complexes, then we have a natural CW structure on $X \star Y$ with X, Y subcomplexes, the remaining cells being the product cells of $X \times Y \times (0, 1)$.

[6] I don't have too good intuition on why they have to add up to 1 for anything greater than the 2-simplex, though...

Wedge Sum. This is one of the more important operations on spaces. Given spaces X and Y with points $x_0 \in X$ and $y_0 \in Y$, the **wedge sum** $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ with the relation identifying $x_0 \sim y_0$ to a single point. For example, $S^1 \vee S^1$ is the figure eight. Note that $S^1 \vee S^1 \vee S^1$ is *not* the chain of three cirles, but rather the fidget spinner! In general, we can form the wedge sum $\bigvee_\alpha X_\alpha$ of spaces X_α by identifying the points $x_\alpha \in X_\alpha$ to a point in $\amalg_\alpha X_\alpha$, which is why $S^1 \vee S^1 \vee S^1$ takes on a fidget spinner structure, all the points have to get identified to one. If the X_α are cell complex and the points are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex obtained from the complex $\amalg_\alpha X_\alpha$ by quotienting the subcomplex of 0-cells to a point.

For any cell complex X, the quotient X^n/X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$ with one sphere for each n-cell of X. This makes sense because turning the $(n-1)$-skeleton into a point leaves the remaining n-cells (which look like n-spheres) all connected at a single point. This will come in handy later.

Smash Product. To be honest, I haven't ever used this but apparently it comes in handy later. In a product space $X \times Y$ there are copies of X and Y given by $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X, y_0 \in Y$. The copies intersect at the point (x_0, y_0), so we can view the spaces at this intersection as the wedge sum $X \vee Y$ with the intersection point (x_0, y_0). Then the **smash product** $X \wedge Y$ is the quotient $(X \times Y)/(X \vee Y)$. A way to think about $X \wedge Y$ is a reduced version of the product $X \times Y$, obtained by collapsing the "overlapping" parts that "aren't genuinely products", the pointed spaces X and Y. The example that makes this click is the the smash product $S^1 \wedge S^1$. Since $S^1 \times S^1 = \mathbb{T}$ the torus, a wedge $S^1 \vee S^1$ is just two rings, one on the inner circle and one on the outer. So collapsing the inner circle removes the hole, and the outer brings the extra structure back to the surface, hence $S^1 \wedge S^1$ is homeomorphic to the 2-sphere. Another example: for a space X, the smash product $X \wedge S^0$ is just X, since it collapses the second copy of X to a point in the first copy.

The smash product $X \wedge Y$ is a complex if X and Y are complexes with x_0, y_0 0-cells, assuming that $X \times Y$ has the cell-complex topology as opposed to the product topology (in cases where the two are different). To generalize the previous example, $S^m \wedge S^n$ is the $(m+n)$-sphere S^{m+n}. To see why this is true, note that $S^m \times S^n$ has four cells, a 0-cell, an m-cell, an n-cell, and an $(m+n)$-cell. The wedge sum consists of identifying the m-cell and the n-cell to the 0-cell, and collapsing these three cells into the 0-cell yields a complex with one 0-cell and one $(m+n)$-cell, which is precisely the $(m+n)$-sphere S^{m+n}.

3.4 Preserving homotopy type of complexes (todo)

Here are two ways to construct homotopy equivalences. The first is by collapsing spaces.

Proposition 3.1. *If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X/A$ is a homotopy equivalence.*

The proof may or may not come. Here are some examples of how this is applied.

Example 3.6 (Graphs)**.** Suppose X is a finite graph. If two endpoints of any edge of X are distinct, we can collapse this to a point, producing a homotopy equivalent graph with one less edge. We can repeat until all edges of X become loops, then each component of X is either an isolated vertex or a wedge sum of circles.

A natural question to ask is whether two graphs, having only one vertex in each component (post contractions) can be homotopy equivalent if they aren't isomorphic graphs? By an exercise, we can reduce this down to connected graph, so this boils down to asking whether $\bigvee_m S^1$ is homotopy equivalent to $\bigvee_n S^1$ for $n \neq m$, which isn't true since their fundamental groups are different. We could have also done this by the Euler characteristic, but that only works for graphs, this better version demonstrates the power of the fundamental group.

Example 3.7. Consider X obtained from S^2 by attaching an arc A to two points, say the north and south poles. Let B be the arc on the surface of S^2 connecting the poles. Contracting A yields a crescent shape, while contracting B yields $S^2 \vee S^1$. So S^2/S^0 and $S^1 \vee S^2$ are homotopy equivalent.

TODO: torus with meridional disks necklace, attaching and mapping cones.

3.5 The real projective space $\mathbb{R}\mathrm{P}^n$

Credit to Cameron Krulewski at UChicago, who wrote up a paper on $\mathbb{R}\mathrm{P}^n$ for a Math 132 project, whose notes I am following today.

Definition 3.6 (Real projective space)**.** We define **real projective space** (or real projective n-space) $\mathbb{R}\mathrm{P}^n$ by the set of lines that pass through the origin in $\mathbb{R}^{n+1}$. Each line is determined by a nonzero vector in $\mathbb{R}^{n+1}$ unique up to scalar multiplication, and $\mathbb{R}\mathrm{P}^n$ is topologized as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$. We'll make sense of this topologization soon. For $n = 1$ this is called the real projective line, and the real projective plane for $n = 2$.

Manifolds are often talked about as subsets of $\mathbb{R}^n$, for example, we often discuss k-manifolds embedded in at most $\mathbb{R}^{2k+1}$. For $\mathbb{R}\mathrm{P}^2$ (the real projective plane), this doesn't embed in $\mathbb{R}^3$, but it does immerse. This won't make sense the higher we go up. A better way to think of abstract manifolds like $\mathbb{R}\mathrm{P}^n$ is as a **quotient space** by identifying points of another manifold.

Claim. The real projective n-space is homeomorphic to an n-sphere with antipodal points identified, that is, $\mathbb{R}\mathrm{P}^n \cong S^n/(v \sim -v)$.

Why is this true? Since $\mathbb{R}\mathrm{P}^n$ is topologized as the quotient of nonzero vectors with $v \sim \lambda v$, consider vectors of length one which gives S^n. Then $v \sim \lambda v$ becomes $v \sim -v$ due to lack of choice. To make sense of this, let's look at the cases.

(i) In the trivial case, let $n = 0$. Then $\mathbb{R}\mathrm{P}^0$, the set of lines through the origin in $\mathbb{R}$, consists of just one line $\{\mathbb{R}\}$, so it's homeomorphic to a singleton. What is S^0? It's two singletons, so if you identify them you get your expected result.

(ii) Now let's look at $n = 1$: we want to show that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to the circle S^1. Let's parametrize the lines by their slopes, that is, the angle $\tan\left(\frac{y}{x}\right)$ for any positive pair (x, y) on any given line. We choose (x, y) positive since the lines extend in both directions and looking at both would mean a redundancy. Then these lines hit every angle from 0 to π, and the x-axis given by $\mathbb{R} \times \{0\}$ has an angle of both 0 and π (identifying the two together). So we get that $\mathbb{R}\mathrm{P}^1 \cong S^1$. How is this homeomorphic to $S^1/(v \sim -v)$, as we claimed? Identifying antipodal points gets a semicircle, but the endpoints of the semicircle are also antipodal and get identified, so $S^1 \cong S^1/(v \sim -v)$.

(iii) For $n = 2$, this immerses in $\mathbb{R}^3$ as a weird surface called *Boy's surface*. To make sense of our claim for $n = 2$, we can use the same argument as last time, parametrizing the lines through the origin in $\mathbb{R}^3$. Then these hit the entire sphere, with each line hitting both the northern and southern hemispheres, so we can get rid of one of the hemispheres. Flattening out the remaining hemisphere, we get that this looks like $B(0, 1)$, a closed ball in the plane! Then we have to deal with lines through the origin in $\mathbb{R}^2$, which is just $\mathbb{R}\mathrm{P}^1$. This means that we can build $\mathbb{R}\mathrm{P}^2$ as $B(0, 1) \cup \mathbb{R}\mathrm{P}^1$, which hints at a CW structure we'll see soon. I've thought about how to visualize this, but there's no explanation I can fit in a simple sentence or two, so I recommend going to Wikipedia and starting from there.

(iv) In general, we can see this inductive building process holds. The sphere S^n with antipodal points identified by $v \sim -v$ is the same as a hemisphere D^n with antipodal points of $\partial D^n = S^{n-1}$ identifed, which is just $\mathbb{R}\mathrm{P}^{n-1}$. So $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ as the attaching map. It follows by induction that $\mathbb{R}\mathrm{P}^n$ has a CW structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$. For example, in the previous case with $\mathbb{R}\mathrm{P}^2 = B(0, 1) \cup \mathbb{R}\mathrm{P}^1$, this is the same as attaching a 2-cell to $\mathbb{R}\mathrm{P}^1$, which was built by attaching a 1-cell to a point, which is precisely $\mathbb{R}\mathrm{P}^0$.

Example 3.8. We can talk about the infinite union $\mathbb{R}\mathrm{P}^\infty$, since we can build $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$. So $\mathbb{R}\mathrm{P}^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}\mathrm{P}^n$ becomes a cell complex with one cell in each dimension. This can be viewed as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}^n$.

Example 3.9. We can define **complex projective n-space** $\mathbb{C}\mathrm{P}^n$ as the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Once again, $\mathbb{C}\mathrm{P}^n$ is topologized as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $v \sim \lambda v$ for $\lambda \neq 0$. Equivalently, this is quotient of $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$. Similarly, we can obtain $\mathbb{C}\mathrm{P}^n$ as a quotient of D^{2n} under $v \sim \lambda v$ for $v \in \partial D^{2n} = S^{2n-1}$, but I don't think it helps illuminate the example very much so you can read Hatcher if you want to know precisely how this identification works. Just trust me when I say that $\mathbb{C}\mathrm{P}^n$ is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a $2n$-cell via the quotient map $S^{n_2 n-1} \to \mathbb{C}\mathrm{P}^{n-1}$. So $\mathbb{C}\mathrm{P}^n$ has a cell complex structure $\mathbb{C}\mathrm{P}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with one cell for each even dimension, and similarly so does $\mathbb{C}\mathrm{P}^\infty$.

Example 3.10 (Cellular properties of projective space)**.** As stated in the last section, in $\mathbb{R}\mathrm{P}^n$ a characteristic map for e^i is the quotient map $D^i \to \mathbb{R}\mathrm{P}^i \subseteq \mathbb{R}\mathrm{P}^n$ identifying antipodal points of $\partial D^i = S^{i-1}$, and is similarly defined for $\mathbb{C}\mathrm{P}^n$. We have the subcomplexes $\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$ for $k \leq n$. These turn out to be the only subcomplexes of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

Lecture 4

The Fundamental Group

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups. These notes follow Hatcher §1.2, Lee §10, and May §1,§2.7.

4.1 Defining the fundamental group

Let X be a space, we say two paths $f, g : I \to X$ from x to y are **equivalent up to homotopy** if there exists a homotopy $h : I \times I \to X$ such that

$$h(s,0) = f(s),\ h(s,1) = g(s),\ h(0,t) = x,\ h(1,t) = y$$

for all $s, t \in I$. In other words, starting at 0 for the second interval ensures you begin at f, and evaluating at the end takes you to g. Since the starting position of f and g are the same, for the first interval starting at 0 must give x, and similarly for y. Denote the homotopy equivalence class of f as $[f]$: we say that f is a **loop** if $f(0) = f(1)$.

Definition 4.1 (Fundamental group)**.** The **fundamental group** of a space X with a basepoint x denoted $\pi_1(X,x)$ is the set of loops up to homotopy starting and ending at x.

4.2 Fundamental group of the circle(todo)

If this is a first introduction to fundamental groups, then our first fundamental group of real interest is $\pi_1(S^1) = \mathbb{Z}$. Before we do this, let's do a quick calculation to show $\pi_1(\mathbb{R}) = 0$. Take the origin as a convenient basepoint. Define $k : \mathbb{R} \times I \to \mathbb{R}$ by $k(s,t) = (1-t)s$. Then k is a homotopy from the identity to the constant map at 0. For a loop $f : I \to \mathbb{R}$ at 0, define $h(s,t) = k(f(s),t)$. Then f is equivalent to a constant c_0 by the homotopy h.

Now let's talk about circles: we can view S^1 as the circle group (let's denote it U^1), that is,

$$U^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Multiplication is continuous, so this is a topological group. Take the identity 1 as a convenient basepoint for S^1.

Theorem 4.1. *We have the fundamental group of a circle isomorphic to the integers, that is,*

$$\pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. For all $n \in \mathbb{Z}$, define a loop f_n in S^1 by $f_n(s) = e^{2\pi i n s}$. This is the same as composing the map $I \to S^1$, $s \mapsto e^{2\pi i s}$ and the nth power map on S^1. If we identiy the boundary points 0 and 1 of I, then the first map ($I \to S^1$) induces the evident identification of $I/\partial I$ with S^1. By complex exponentiation, we have $[f_m][f_n] = [f_{m+n}]$: define a homomorphism $i : \mathbb{Z} \to \pi_1(S^1, 1)$ by $i(n) = [f_n]$. We claim i is an isomorphism. The main idea is to use the fact that (locally) S^1 looks like $\mathbb{R}$, ie, S^1 is a 1-manifold.

FINISH LATER ⊠

4.3 The van Kampen Theorem (Hatcher)

Let's take a space X and say it's the union of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi\colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is often onto but in general, we can expect Φ to have a nontrivial kernel.

For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both of these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Van Kampen says under fairly broad hypotheses that this determines all of Φ.

Theorem 4.2. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi\colon *_\alpha\,(A_\alpha) \to \pi_1(X)$$

is onto. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N.$$

Example 4.1 (Wedge Sums)**.** I like the visual of the wedge sum but the terminology of the smash product. If only we could keep the `\vee` symbol ($\vee$) and say we "smash the spaces together" at a point.

We define the wedge sum $\bigvee_\alpha X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ as the disjoint union $\amalg_\alpha X_\alpha$ with all the basepoints x_α identified to a single point. If each $x_\alpha$is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Then by van Kampens theorem,

$$\Phi\colon *_\alpha\, \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$$

is an isomorphism, provided each X_α is path-connected, hence also each A_α. Therefore for a wedge sum of circles, $\pi_1(\bigvee_\alpha S^1_\alpha)$ is a free group, the free product of copies of $\mathbb{Z}$.

I know it always helps to see something done somewhere else. For me, the above definition fails to make any sense at all whatsoever. So, let's revisit van Kampens from two more lens: one from the words of Lee (*Introduction to Topological Manifolds*) and another from the categorical perspective.

4.4 The van Kampen Theorem (Lee)

Let's say we have a space X that's made up of the union of two open subsets $U, V \subseteq X$. We know how to compute the fundamental groups of U, V, and $U \cap V$ (each of which is path-connected). Every loop can be written as a product of loops in U or V (visualized as the free product $\pi_1(U) * \pi_1(V)$), but any loop in $U \cap V$ only represents a single element of $\pi_1(X)$, even though it represents two distinct elements of the free product (one in $\pi_1(U)$ and one in $\pi_1(V)$). So $\pi_1(X)$ can be though of as the quotient of this free product modulo some relations from $\pi_1(U \cap V)$ that express this redundancy.

Let's do some setup so we can state a precise version of van Kampens. Let X be a topological space and $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V \neq \emptyset$. Let $q \in U \cap V$. Then the four inclusion maps shown below,

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

induce fundamental group homomorphisms as shown below.

$$\begin{tikzcd}
 & \pi_1(U,q) \arrow[dr, "k_*"] & \\
\pi_1(U\cap V,q) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & \pi_1(X,q) \\
 & \pi 1(V,q) \arrow[ur, "l_*"'] &
\end{tikzcd}$$

Now insert the free product $\pi_1(U,q)*\pi_1(V,q)$ in the middle of the diagram and let $\iota_U\colon \pi_1(U,q)\hookrightarrow \pi_1(U,q)*\pi_1(V,q)$ and $\iota_V\colon \pi_1(V,q)\hookrightarrow \pi_1(U,q)*\pi_1(V,q)$ be the canonical injections. By the characteristic property (unique induced homomorphisms) of the free product, k_* and l_* induce a homomorphism $\Phi\colon \pi_1(U,q)*\pi_1(V,q)\to\pi_1(X,q)$ such that the right half of the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(U,q) \arrow[d, dashed, "\iota_U"] \arrow[dr, "k_*"] & \\
\pi_1(U\cap V,q) \arrow[ur, "i_*"] \arrow[r, dotted, "F"] \arrow[dr, "j_*"'] & \pi_1(U,q)*\pi_1(V,q) \arrow[r, "\Phi"] & \pi_1(X,q) \\
 & \pi_1(V,q) \arrow[u, dashed, "\iota_V"'] \arrow[ur, "l_*"'] &
\end{tikzcd}$$

Finally, define a map $F\colon \pi_1(U\cap V,q)\to\pi_1(U,q)*\pi_1(V,q)$ by setting $F(\gamma)=(i_*\gamma)^{-1}(j_*\gamma)$[7]. Let $\overline{F(\pi_1(U\cap V,q))}$ denote the *normal closure*[8] of the image of F in $\pi_1(U,q)*\pi_1(V,q)$.

Theorem 4.3 (Seifert-Van Kampen)**.** *Let X be a topological space. Suppose $U,V\subseteq X$ are open, $U\cap V=X$, and U,V, and $U\cap V$ are path-connected. Then for any $q\in U\cap V$, the homomorphism Φ is surjective, and its kernel is $\overline{F(\pi_1(U\cap V,q))}$. Therefore we have*

$$\pi_1(X,q)\cong \pi_1(U,q)*\pi_1(V,q)\big/\overline{F(\pi_1(U\cap V,q))}.$$

When the fundamental groups in question are finitely presented, the theorem has a useful reformulation in terms of generators and relations.

Corollary 4.1. *In addition to the hypothesis of van Kampen, assume that the fundamental groups of U,V, and $U\cap V$ have the following finite presentations:*

$$\begin{aligned}
\pi_1(U,q)&\cong\langle\alpha_1,\cdots,\alpha_m\mid\rho_1,\cdots,\rho_r\rangle;\\
\pi_1(V,q)&\cong\langle\beta_1,\cdots,\beta_n\mid\sigma_1,\cdots,\sigma_s\rangle;\\
\pi_1(U\cap V,q)&\cong\langle\gamma_1,\cdots,\gamma_p\mid\tau_1,\cdots,\tau_t\rangle.
\end{aligned}$$

Then $\pi_1(X,q)$ has the presentation

$$\pi_1(X,q)\cong\langle\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n\mid\rho_1,\cdots,\rho_r,\sigma_1,\cdots,\sigma_s,u_1=v_1,\cdots,u_p=v_p\rangle$$

where for each $a=1,\cdots,p$, u_a is an expression for $i_*\gamma_a\in\pi_1(U,q)$ in terms of the generators $\{\alpha_1,\cdots,\alpha_m\}$, and v_a similarly expresses $j_*\gamma_a\in\pi_1(V,q)$ in terms of $\{\beta_1,\cdots,\beta_n\}$.

4.5 The fundamental groupoid

We backtrack a little to talk about categorical nonsense. This doesn't fit too well with the section on category theory, so it's here. These will follow May §2.5.

[7] F is not a homomorphism.

[8] the *normal closure* of a set means the smallest normal subgroup that contains such set.

We often talk of pointed spaces, but it would to nice to talk about spaces without making such a choice. We define the fundamental groupoid $\prod(X)$ of a space X to be the category whose objects are the points of X and whose morphism $x \to y$ are the equivalence classes of paths from x to y[9]. Then the set of endomorphisms of the object x is the fundamental group $\pi_1(X, x)$.

We say "groupoid" because a group is simply a groupoid with only one object (the class of morphisms or symmetries on an object). However, the category of groupoids has several objects. We also defined groupoids as categories whose morphisms are all isomorphisms. If morphisms are functors, then we have the category Grpd of groupoids. So we can see $\prod$ as a functor $\mathsf{Top}_* \to \mathsf{Grpd}$.

Let's talk about skeletons. We have the skeleton of a category $\mathscr{C}$ denoted by $\mathrm{sk}(\mathscr{C})$. This is a "full" subcategory with one object from each isomorphism class of objects of $\mathscr{C}$, "full" meaning that the morphisms between two objects of $\mathrm{sk}(\mathscr{C})$ are all of the morphisms between these objects in $\mathscr{C}$. The inclusion functor $J\colon \mathrm{sk}(\mathscr{C}) \to \mathscr{C}$ is an equivalence of categories. We can find an inverse functor $F\colon \mathscr{C} \to \mathrm{sk}(\mathscr{C})$ by letting $F(A)$ be the unique object in $\mathrm{sk}(\mathscr{C})$ that is isomorphic to A, choosing an isomorphism $\alpha_A\colon A \to F(A)$, and defining $F(f) = \alpha_B \circ f \circ \alpha_A^{-1}$ for a morphism $f : A \to B$ in $\mathscr{C}$. Choose α to be the identity morphisms if $A \in \mathrm{sk}(\mathscr{C})$, then $FJ = \mathrm{id}_{\mathrm{sk}(\mathscr{C})}$; the α_A specify a natural isomorphism $\alpha\colon \mathrm{id} \to JF$.

A category is connected if any two objects can be connected by a sequence of morphisms. Then a groupoid is connected iff any two of its objects are isomorphic. The group of endomorphisms of any object C is then a skeleton of $\mathscr{C}$, so we can generalize our results about skeletons to give the relationship between a fundamental group and a fundamental groupoid of a path connected space X.

Proposition 4.1. *Let X be a path connected space. Then for each $x \in X$, the inclusion $\pi_1(X, x) \to \prod(X)$ is an equivalence of categories.*

Proof. View $\pi_1(X, x)$ as a groupoid with on object x: then $\pi_1(X, x)$ is a skeleton of $\prod(X)$ and we are done. ⊠

May's presentation and proofs are very concise and elegant. I like this.

Lecture 5

Covering Spaces

Today we talk about covering spaces, another central topic in algebraic topology. The notes will follow various texts, including Hatcher, Lee, and May.

5.1 Some preliminary definitions

Sometimes we need to know what words mean so we can talk about big concepts. These notes will follow May §3. We can talk about the theory of covering spaces on *locally contractible* spaces that are path-connected, that is, spaces with a base of contractible spaces, that is, open sets that are contractible when viewed as a space under the subspace topology. However, to get the full picture, we must talk about *locally path-connected* spaces.

Definition 5.1 (Locally path-connected). A space X is *locally path-connected* if for any $x \in X$ and any neighborhood U of x, there exists a smaller neighborhood V of x, with each of whose points can be connected to x by a path in U. We could also say X has a base consisting of open sets that are path-connected (under the subspace topology).

Note that if X is connected and locally path-connected, then it is path-connected. From now on[10], we assume that spaces are connected and locally path-connected. Let's look at how May defines covering spaces.

[9] Recall Example 1.4 of a group being realized as a category with all its arrows isomorphisms.
[10] By this, we mean any sections following May.

Definition 5.2 (Covering space)**.** A map $p\colon E \to B$ is a covering (or cover, covering space) if it is onto and if each point $b \in B$ has an open neighborhood V such that each component of $p^{-1}(V)$ is open in E and is mapped homeomorphically onto V by p. We say that a path connected open subset V with this property is a fundamental neighborhood of B. We call E the total space, B the base space, and $F_b = p^{-1}(b)$ a fiber of the covering p.

Some notes: in other texts, we have

- covering → covering map,
- U is a fundamental neighborhood → U is evenly covered,
- total space → covering space,
- base space → ??,
- $F_b = p^{-1}(b)$ is a fiber of $p \longrightarrow F_b$ is the preimage of b (points) in the union of sheets of $\widetilde{X}$ over U_b.

Another definition that will come in handy when classifying covering spaces is the notion of something being semilocally simply-connected, that is, given a "hole" (of genus one), we can always find a neighborhood contained in that hole such that the fundamental group induced by the inclusion map is trivial in π_1 of the entire space.

Definition 5.3 (Semilocally simply-connected)**.** A space X is *semilocally simply-connected* if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.

We'll define this again when we need it, and talk a little more about what it means for a space to be semilocally simply-connected.

This is kind of out of place, but now we'll state Lebesgue's number lemma. It's useful when dealing with compact metric spaces.

Lemma 5.1 (Lebesgue's number lemma)**.** *If a metric space (X,d) is compact and we have an open cover of X, then there exists a $\delta > 0$ such that every subset of X having a diameter less than δ is contained in some member of the cover. We say δ is the* Lebesgue number *of such cover.*

Proof. If the subcover is trivial then any $\delta > 0$ will suffice. Otherwise, if $\bigcup_{i\in I} A_i$ is a finite subcover, then for $i \in I$, define $C_i := X \setminus A_i$ (note that C_i is nonempty since the subcover is nontrivial). Define a function

$$f : X \to R, \quad x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

Since f is continuous on a compact set, it obtains a minimum δ. The key thing to note is that every x is in some A_i, so by the extreme value theorem $\delta > 0$. To show that this δ is indeed the Lebesgue number of the cover, let $x_0 \in Y$, where $\operatorname{diam}(Y) < \delta$, such that $Y \subseteq B(x_0,\delta)$. Since $f(x_0) \geq \delta$, there exists at least one i such that $d(x_0, C_i) \geq \delta$. But then $B(x_0,\delta) \subseteq A_i$, and so $Y \subseteq A_i$. ⊠

5.2 Covering spaces

These notes will follow Hatcher §1.3.

We've already seen these briefly when we calculated $\pi_1(S^1)$, using the projection $\mathbb{R} \to S^1$ of a helix onto a circle. Covering spaces can be used to calculated fundamental groups of other spaces as well, but the connection runs much deeper than this. We can talk about algebraic aspects of the fundamental group through the geometric language of covering spaces, exemplified in one of the main results in this section: a one to one correspondence between connected covering spaces of a space X and subgroups of $\pi_1(X)$ (spoilers, smh). This is really really similar to Galois theory, where we looked at the towers of field extensions and related them to the subgroup lattice of the Galois group of automorphisms[11].

[11] I actually know this! Thank goodness for an entire semester of algebra to understand an example.

Definition 5.4 (Covering space)**.** A *covering space* of a space X is a space $\widetilde{X}$ together with a map $p: \widetilde{X} \to X$ (we say p is a *covering map*) satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p. Then we say U is *evenly covered* and the disjoint open sets in $\widetilde{X}$ that project homeomorphically to U by p are called *sheets* of $\widetilde{X}$ over U.

If U is connected these sheets are the connected components of $p^{-1}(U)$ so they're uniquely determined by U. If U is not connected, however, the decomposition of U into sheets may not be unique. $p^{-1}(U)$ is allowed to be empty, so p doesn't have to be onto. The number of sheets over U can be given by the cardinality of $p^{-1}(x)$, given $x \in U$. This number is a constant if X is connected.

Example 5.1. A prototypical example (or way to wrap your head around) this section is the helix embedded in $\mathbb{R}^3$: if you think of it projecting on a circle, then $p^{-1}(U)$ is just $\amalg_\alpha U_\alpha$, where each U_α corresponds to the U of a coil or wind of the helix.

Example 5.2. Another example is the helicoid surface $S \subseteq \mathbb{R}^3$ given by $(s\cos 2\pi t, s\sin 2\pi t, t)$ for $(s,t) \in (0,\infty)\times\mathbb{R}$. This projects onto $\mathbb{R}^2 \setminus \{0\}$ via the map $(x,y,z) \mapsto (x,y)$, and defines a covering space $p: S \to \mathbb{R}^2 \setminus \{0\}$ since each point of $\mathbb{R}^2 \setminus \{0\}$ is contained in an open disk U in $\mathbb{R}^2 \setminus \{0\}$ with $p^{-1}(U)$ consisting of countably many disjoint open disks in S projecting homeomorphically onto U. (I can't really see this example...)

Example 5.3. We also have the map $p: S^1 \to S^1$, $p(z) = z^n$ where we view z as a complex number with $|z| = 1$ and n any positive integer[12]. This projection is as described in the footnote, but intersects itself in $n-1$ points (that one can't really imagine as intersections). To see this without the defect, embed S^1 in the boundary torus of a solid torus $S^1 \times D^2$ such that it winds n times monotonically around the S^1 factor without self-intersections, then restrict the projection $S^1 \times D^2 \to S^1 \times \{0\}$ to this embedded circle. What?

We usually restrict our attention to connected covering spaces, as these contain all the interesting examples.

5.3 The covering spaces of $S^1 \vee S^1$ (todo figures)

Covering spaces of $S^1 \vee S^1$ form a rich family that demonstrate the general theory very concretely. For convenience, let $X = S^1 \vee S^1$. View it as a graph with one vertex and two edges, with the edges labeled a and b.

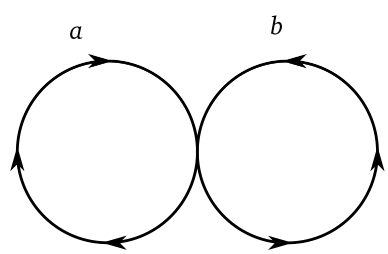


Figure 2: The graph of $S^1 \times S^1$.

Let $\widetilde{X}$ be any other graph with four edges connected to each point, like X at its singular vertex, and that each edge has been assigned an orientation like the ones assigned to each edge of X. That is, for each vertex there are two a-edges and b-edges oriented toward and away from the vertex. Help I can't include figures that are the proper size! Let's call $\widetilde{X}$ a 2-oriented graph.

Given a 2-oriented graph $\widetilde{X}$ we can construct a map $p: \widetilde{X} \to X$ that sends all vertices of $\widetilde{X}$ to the vertex of X, and all edges of $\widetilde{X}$ to the edge of X with the same label. Say p is a homeomorphism on the regions bounded by the edges, and preserves the orientation of the edges. Then p is a covering map. Conversely, every covering space of

[12]Something about the order of z I realized when thinking about this example: z^n means z coils around in S^1 n times. So if z was a fifth root of unity, the covering space would be a circle with five coils projecting onto S^1. Now what if z has infinite order. Can z even have infinite order? I'm not entirely sure...
Edit: elements that are irrational multiples of 2π have infinite order. So does that mean it never winds back to itself? How is this isomorphic to S^1?

$\widetilde{X}$ is a graph that inherits a 2-orientation from X. It can be shown that every graph with four edges at each vertex can be 2-oriented: the proof follows from graph theory. We could also generalize this to n-oriented graphs, which are covering spaces of the wedge sum of n circles.

How would we generate a simply-connected covering space of X? Start with the open intervals $(-1, 1)$ in $\mathbb{R}^2$ (one per coordinate axis). Then for a fixed λ, $0 < \lambda < 1/2$, say $\lambda = 1/3$, adjoin four open segments of length 2λ to the ends of the previous segments, and shift each back by a distance of λ. These new adjoined segments are perpendicular and bisected by the old ones: continue with four more new segments of distance $2\lambda^2$ at a distance λ^2 to the (now 12) end segments, and so on. Then at the n-th iteration, we would be adding open segments of length λ^{2n-1} at a distance λ^{n-1} from the previous endpoints. Then the union of the segments is a graph (the Cayley graph of the fundamental group of S^1!), with vertices the intersections, labeling horizontal edges a and orienting them to the right, and vertical edges b, orienting them upward.

This covering space is called the *universal cover* of X, because it covers every connected covering space of X.

5.4 More on covering spaces

These notes will follow Lee §11.

The definition of a covering space is the same as Hatcher except: the covering space $\widetilde{X}$ must be connected. Once again, the only interesting covering spaces are connected ones, and so we eliminate the need to frit fret around about details when introducing new theorems and just make sure covering spaces are connected in the definition.

Example 5.4. The exponential quotient map $\varepsilon\colon \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a covering map. Another example: define $E\colon \mathbb{R}^n \to \mathbb{T}^n$ by

$$E(x_1, \cdots, x_n) = (\varepsilon(x_1), \cdots, \varepsilon(x_n)).$$

We will show in an exercise that a product of covering maps is a covering map. So E is a covering map.

Example 5.5. Define a map $\pi\colon S^n \to \mathbb{R}P^n$ (where $n \geq 1$) by sending each point x in the sphere to the line through the origin and x, thought of as a point in $\mathbb{R}P^n$. Then π is a covering map, and the fiber of each point in $\mathbb{R}P^n$ is a pair of antipodal points $\{x, -x\}$.

Lemma 5.2 (Elementary properties of covering maps)**.** *Every covering map is a local homeomorphism, an open map, and a quotient map. An injective covering map is a homeomorphism.*

Proof. Left as an exercise to the reader. ⊠

Proposition 5.1. *For any covering map $p\colon \widetilde{X} \to X$, the cardinality of each fiber $p^{-1}(q)$ is the same for any fiber.*

Proof. If U is any evenly covered open set in X, each sheet in $p^{-1}(U)$ contains exactly one point of each fiber. Then for any $q, q' \in U$, there are one-to-one correspondences

$$p^{-1}(q) \longleftrightarrow \{\text{sheets of}\, p^{-1}(U)\} \longleftrightarrow p^{-1}(q'),$$

which shows that the number of sheets is constant on U. It follows that the set of points $q' \in X$ such that $p^{-1}(q')$ has the same cardinality as $p^{-1}(q)$ is open. Now let $q \in X$, and let A be the set of points in X whose fibers have the same cardinality as $p^{-1}(q)$. Then A is open, and $X \setminus A$ is open since it's a union of open sets (one open set for each cardinality not equal to $p^{-1}(q)$). Since X is connected and nonempty, we have $A = X$. ⊠

If $p\colon \widetilde{X} \to X$ is a covering map, then the cardinality of any fiber is the *number of sheets* of the covering. For example, the n-th power map $(S^1 \to S^1)$ is an n-sheeted covering, $\pi\colon S^n \to \mathbb{R}P^n$ is a two sheeted covering, and $\varepsilon\colon \mathbb{R} \to S^1$ is a countably sheeted covering.

5.5 Lifting properties

Here we'll talk about some important lifting properties, that we discussed when we proved that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. Recall: if $p\colon \widetilde{X} \to X$ is a covering map and $\varphi\colon B \to X$ is any continuous map, a *lift* of φ is a continuous map $\widetilde{\varphi}\colon B \to \widetilde{X}$ such that $p \circ \widetilde{\varphi} = \varphi$. See the commutative diagram below for reference.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ B \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Proposition 5.2 (Unique lifting property). *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose B is connected, $\varphi\colon B \to X$ is continuous, and $\widetilde{\varphi}_1 \widetilde{\varphi}_2\colon B \to \widetilde{X}$ are lifts of φ that agree at some point of B. Then $\widetilde{\varphi}_1 \equiv \widetilde{\varphi}_2$, that is, lifts are unique.*

Proof. We show that the set
$$\mathscr{S} = \{b \in B \mid \widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)\}$$
is both open and closed in B, contradicting the connectedness of B if $\mathscr{S}$ is a proper nontrivial subset of B. We conclude that $\mathscr{S}$ must be all of B since $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ agree at a point (so $\mathscr{S}$ is nontrivial) and therefore $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are unique.

Let $b \in \mathscr{S}$ and $U \subset X$ be an evenly covered neighborhood of $\varphi(b)$, and let U_α be the component of p^{-1} containing $\widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)$. On the neighborhood $V = \widetilde{\varphi}_1^{-1}(U_\alpha) \cap \widetilde{\varphi}_2^{-1}(U_\alpha)$ of b, we have $\varphi = p \circ \widetilde{\varphi}_1 = p \circ \widetilde{\varphi}_2$. Since p is 1-1 on U_α, this implies $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$ on V, so $\mathscr{S}$ is open.

OTOH, for $b \notin \mathscr{S}$, if U is an evenly covered neighborhood of $\varphi(b)$, there are disjoint components U_1, U_2 of $p^{-1}(U)$ containing $\widetilde{\varphi}_1(b)$, $\widetilde{\varphi}_2(b)$ such that p is a homeomorphism from each U_i to U. Letting $V = \widetilde{\varphi}_1^{-1}(U_1) \cap \widetilde{\varphi}_2^{-1}(U_2)$, we conclude that $\widetilde{\varphi}_1 \neq \widetilde{\varphi}_2$ on V, so $\mathscr{S}$ is closed. This proof is much easier to follow if you trace everything out with all the inverse relations on the commutative diagram above. ⊠

Proposition 5.3 (Path lifting property). *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose $f\colon I \to X$ is any path, and $\widetilde{q}_0 \in \widetilde{X}$ is any point in the fiber of p over $f(0)$. Then there exists a unique lift $\widetilde{f}\colon I \to \widetilde{X}$ of f such that $\widetilde{f}(0) = \widetilde{q}_0$.*

Proof. By the Lebesgue number lemma, n can be chosed large enough that p maps each subinterval $[k/n, (k+1)/n]$ into an evenly covered open subset of X. Starting with $\widetilde{f}(0) = \widetilde{q}_0, \widetilde{f}$ is defined inductively by choosing an evenly covered neighborhood U_k containing $f[k/n, (k+1)/n]$, a local section[13] $\sigma_k\colon U_k \to \widetilde{X}$ such that $\sigma_k(f(k/n)) = \widetilde{f}(k/n)$, and setting $f = \sigma_k \circ f$ on $[k/n, (k+1)/n]$. Because $p \circ \widetilde{f} = (p \circ \sigma_k) \circ f = f$, this is indeed a lift, and it is unique by the unique lifting property. ⊠

Proposition 5.4 (Homotopy lifting property). *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose $f_0, f_1\colon I \to X$ are path homotopic, and $\widetilde{f}_0, \widetilde{f}_1\colon I \to \widetilde{X}$ are lifts of f_0 and f_1 such that $\widetilde{f}_0(0) = \widetilde{f}_1(0)$. Then $\widetilde{f}_0 \sim \widetilde{f}_1$.*

Proof. If $H\colon f_0 \sim f_1$ is a path homotopy, by the Lebesgue number lemma we can choose n large enough that H maps each square of side $\frac{1}{n}$ into an evenly covered open set. Labeling the squares $S_{ij} = [i/n, (i+1)/n] \times [j/n, (j+1)/n]$, we define a lift $\widetilde{H}$ of H square by square along the bottom row, then the next row, and so on by induction. On each square S_{ij}, set $\widetilde{H} = \sigma \circ H$, for an appropiate local section σ chosen such that the new definition of $\widetilde{H}$ matches the previous one at the corner point $(i/n, j/n)$. Then since two such definitions agree on a line segment (by restricting H to it), they are equal by the unique lifting property.

On the left-hand and right-hand edges of $I \times I$, where $s = 0$ or $s = 1$, $\widetilde{H}$ is a lift of the constant loop and therefore constant. The restriction $\widetilde{H}_0$ to the bottom edge where $t = 0$ is a lift of f_0 starting at $\widetilde{f}_0(0)$, and therefore is equal to $\widetilde{f}_0$, similarly $\widetilde{H}_1 = \widetilde{f}_1$. Therefore $\widetilde{H}$ is the required path homotopy between $\widetilde{f}_0$ and $\widetilde{f}_1$. ⊠

[13] A *local section* of a continuous map is a continuous right inverse defined on some open subset. This exists here by Lee's Lemma 11.7, which shows the existence of local sections of covering maps.

5.6 Connections to the fundamental group

Back to Hatcher §1.3.

Here are some applications of the lifting properties with respect to the fundamental group.

Proposition 5.5. *The map $p_*\colon \pi_1(\widetilde{X},\widetilde{x}_0) \to \pi_1(X,x_0)$ induced by a covering space $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ in $\pi_1(X,x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

Proof. An element of the kernel of p_* is represented by a loop $\widetilde{f}_0\colon I \to \widetilde{X}$ with a homotopy $f_t\colon I \to X$ of $f_0 = p\widetilde{f}_0$ to the trivial loop f_1. By the homotopy lifting property, there is a lifted homotopy of loops $\widetilde{f}_t$ starting with $\widetilde{f}_0$ and ending with a constant loop. Basically, since elements of the kernel start with the same point, and there exist unique lifts to them that are nullhomotopic, we conclude the kernel is trivial and p_* is 1-1.

For the second part of the proposition, loops at x_0 lifting to loops at $\widetilde{x}_0$ represent elements of the image of $p_*\colon \pi_1(\widetilde{X},\widetilde{x}_0) \to \pi_1(X,x_0)$. Conversely, a loop representing an element of the image p_* is homotopic to a loop having such a lift, and by the homotopy lifting property, this loop must also have such a lift. ⊠

Proposition 5.6. *The number of sheets (cardinality of a fiber) of a covering space $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ with X and $\widetilde{X}$ path-connected equals the index of $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ in $\pi_1(X,x_0)$.*

Proof. For a loop g in X based at x_0, let $\widetilde{g}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h\cdot g$ with $[h] \in H = p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ has the lift $\widetilde{h}\cdot\widetilde{g}$ ending at the same point as $\widetilde{g}$ since $\widetilde{h}$ is a loop ($\widetilde{h}$ denotes the same lift as $\widetilde{g}$, just of h instead). All this is saying is that you can lift a product of loops by a product of loops, and we're choosing one loop to the in the image subgroup of p_*. Then we can define a function Φ from cosets $H[g]$ to $p^{-1}(x_0)$ by sending $H[g]$ to $\widetilde{g}(1)$. $H[g]$ denotes $h\cdot g$, where $h \in H$, the coset of g. If you think about it, these are cosets since we just vary g: and so the number of cosets is the index of the subgroup $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ in $\pi_1(X,x_0)$. Now we just have to show Φ is a bijection to complete the proof.

Φ is onto by the path-connectedness of $\widetilde{X}$, since $\widetilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ by a path $\widetilde{g}$ projecting to a loop g at x_0. To show Φ is 1-1, note that $\Phi(H[g_1]) = \Phi(H[g_2])$ implies that $g_1\cdot\overline{g_2}$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ and hence $H[g_1] = H[g_2]$. ⊠

Question: for a continuous map $\varphi\colon Y \to X$, does φ admit a lift $\widetilde{\varphi}$ to a covering space $\widetilde{X}$ of X? The lifting criterion can help us out.

Theorem 5.1 (Lifting criterion)**.** *Suppose we are given a covering space $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ and a map $f\colon (Y,y_0) \to (X,x_0)$ with Y path-connected and locally path-connected. Then a lift $\widetilde{f}\colon (Y,y_0) \to (\widetilde{X},\widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y,y_0)) \subseteq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.*

Proof. If a lift $\widetilde{f}$ exists, then $p\widetilde{f} = f$, so $f_* = p_*\widetilde{f}_*$ ⊠

5.7 Classification of covering spaces (todo split it up)

How can we catch all the covering spaces? This whole topic deals closely with its analogue in algebra, Galois theory, with a 1-1 correspondence between connected covering spaces of X (towers of field extensions) and subgroups of $\pi_1(X)$ (subgroups of $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$). This comes from the function that assigns each covering space $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ to the subgroup $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ of $\pi_1(X,x_0)$.

Definition 5.5 (Semilocally simply-connected)**.** A space X is semilocally simply-connected if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.

Basically, the fundamental group of U is trivial *inside* the fundamental group of X, that is, loops in $\pi_1(U,x)$ are nullhomotopic in X (not necessarily U, if that were the case, U would be locally simply connected). Intuitively, there are lower bounds on the size of holes (genus-wise): if theres a hole, you can find a neighborhood smaller than it so that loops are still trivial. For example, take the Hawaiian earring: loops here are very very small, and at the base every neighborhood will contain a hole, so it's not semilocally simply-connected (a "bad" space)[14].

Proposition 5.7. *If X is a path-connected, locally path-connected, and semilocally simply-connected space, then for every subgroup H of $\pi_1(X, x_0)$, there is a covering space $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$.*

We'll prove Proposition 5.7 much later, let's talk about it first. If $H = 1$, then $\widetilde{X}$ is simply-connected, so $\widetilde{X}$ is the universal cover of X. Now given an evenly covered open set U, any loop in U will lift to a sheet in $\widetilde{X}$, which implies it's nullhomotopic in $\widetilde{X}$, and therefore nullhomotopic in X (we don't know if it's nullhomotopic in U), we can see this just by projecting the loop with p. This implies that if $U \overset{\iota}{\hookrightarrow} X$ denotes the inclusion of U in X, then $\iota_*(\pi_1(U)) = 1$ in $\pi_1(X)$. So X must be semilocally simply-connected. The following claim shows why we need these claims for X to be a "nice" space.

Claim. If X is path-connected, locally path-connected, and semilocally simply-connected, then there exists a universal cover of X.

Proof. We prove this by directly constructing a universal cover of X through the fundamental groupoid. First assume that X has a universal cover $\widetilde{X} \xrightarrow{p} X$. Let $\widetilde{x}_0 \in \widetilde{X}$. Then for some other $\widetilde{x} \in \widetilde{X}$, there is a unique path homotopy class of paths from $\widetilde{x}_0$ to $\widetilde{x}$. So points in $\widetilde{X}$ are in a 1-1 correspondence of path homotopy classes of paths starting at $\widetilde{x}_0$. But by the path lifting property, these are all homotopic.

Let's turn this around and define the universal cover of X by its path homotopy classes, that is, let

$$\widetilde{X} := \{[f] \in \Pi_1(X) \mid f(0) = x_0\},$$

where $\Pi_1(X)$ denotes the fundamental groupoid of X. The covering is given by $p : \widetilde{X} \to X$, $[f] \mapsto f(1)$. We want to define a topology on $\widetilde{X}$ that makes p continuous and a covering map. To do this, we define a basis $\mathscr{B}$ and check to see if the inverse image of open sets in the *basis* are continuous. Albin, 24 min lecutre 8 unfinished ⊠

Now that we've proved that for every subgroup we have a covering space, the next question is how many covering spaces per subgroup? We have two covering spaces $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are *equivalent* if there is a homeomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$, or such that the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

If so, it's easy to see that this is an equivalence relation.

Theorem 5.2. *The covering spaces $(\widetilde{X}_i, \widetilde{x}_i) \xrightarrow{p_i} (X, x_0)$, where $i \in \{1, 2\}$ and X is path-connected, locally path-connected are equivalent if and only if $p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

So it turns out the answer to the question above is just one.

Proof. One direction is easy: look at the diagram of induced fundamental groups, and notice that the homeomorphism induces an isomorphism on the subgroups of $\pi_1(X)$. The other direction is more interesting. Let $H_1 = p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$ and $H_2 = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. Since $H_1 \subseteq H_2$ and p_2 is a covering map and X is path-connected and locally path-connected, there exists a lift of $\widetilde{p}_1$ to a map $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ by the lifting criterion, making

[14]Does anyone reading this know of a space that's path-connected but not locally path-connected? I know of many counterexamples for the converse, but without a counterexample to the implication I don't see why local path-connectedness is a necessary condition on top of path-connectedness.

the diagram commute. Similarly, $H_2 \subseteq H_1$, so there's a lift of p_2 to a map $g\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ making the appropiate diagram commute. In particular, we have

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "g\circ f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_1, \widetilde{x}_1) \arrow[dl, "p_1"] \\
& (X, x_0) &
\end{tikzcd}$$

Since the identity is also a lift of p_1 to $(\widetilde{X}_1, \widetilde{x}_1)$, by uniqueness of lifts we have $g \circ f$ equal to the identity, that is, $g \circ f = \mathrm{id}_{\widetilde{X}_1}$. Similarly, we have $f \circ g = \mathrm{id}_{\widetilde{X}_2}$. So f is a homeomorphism. [15] ⊠

Now for the theorem we all came here for.

Theorem 5.3. *Let X be a path-connected, locally path-connected, and semilocally simply-connected space. Then there is a bijection between the coverings $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ up to equivalence and the subgroups of $\pi_1(X, x_0)$. This bijection is given by $p \mapsto p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, we also have a 1-1 correspondence between the non pointed covering spaces $\widetilde{X} \xrightarrow{p} X$ and the conjugacy classes of subgroups, given by the same map $p \mapsto [p_*(\pi_1(\widetilde{X}))]$.*

It's important that we have a covering space and choice of basepoint: if we change the basepoint, we might not necessarily give the same group. Changing the basepoint gives a conjugacy isomorphisms between fundamental groups. This conjugacy isomorphism might give rise to different subgroups, conjugating by some element of the group possibly gives a different subgroup. Hence the second part of the theorem.

It turns out there's an equivalence of posets between covers of a space (X, x_0) (for X a "nice" space) and subgroups of $\pi_1(X, x_0)$, known as the Galois correspondence. The partial order is given by defining two elements to be comparable if one is a cover of another.

5.8 Actions on the fibers

If $p\colon \widetilde{X} \to X$ a cover, $\alpha \in \pi_1(X, x_0)$, define $L_\alpha \in \mathrm{Sym}(p^{-1}(x_0))$ by $L_\alpha \widetilde{x} = \widetilde{\alpha}(0)$, where $\widetilde{\alpha}$ is the lift of x to a path ending at $\widetilde{x}$. We have $L_{\alpha\beta} = L_\alpha \circ L_\beta$, since $L_{\alpha\beta}(\widetilde{x}) = \widetilde{\alpha\beta}(0) = L_\alpha(\widetilde{\beta}(0)) = L_\alpha(L_\beta(\widetilde{x}))$. This is why we defined $L_\alpha(\widetilde{x})$ starting at the left endpoint 0. Albin lecture 9, 36 minutes

Lecture 6

Homology

The big boy has arrived. These notes will follow Hatcher §2.1.

Remark 6.1. This is something I heard even before I enrolled in this course. The homotopy groups are easy to define, but impossible to compute and work with. The homology groups take a lot of work to define, but the resulting groups are much nicer and easier to work with.

The fundamental group is a cool tool when dealing with low-dimensional spaces (the pride and joy of UT Austin), but it doesn't do well with higher dimensional spaces, for example, it can't distinguish between the n-spheres S^n for $n \geq 2$. We can get rid of this limitation by considering the higher homotopy groups $\pi_n(X)$, which are defined in terms of maps from the n-dimensional cube I^n and homotopies $I^n \times I \to X$ of such maps. Cool things about higher homotopy groups: for X a CW complex, $\pi_n(X)$ only depends on the $(n+1)$-skeleton, and $\pi_i(S^n) = 0$ for $i < n$ and

[15] I don't understand where f came from: how can we guarantee its existence?

$\mathbb{Z}$ for $i = n$, as expected. However, the drawback is that they're extremely difficult to compute in general— take the "simple" task of computing $\pi_i(S^n)$ for $i > n$.

Enter the homology groups $H_n(X)$. Similar to $\pi_n(X)$, $H_n(X)$ for X a CW complex depends only on the $(n+1)$-skeleton, and for the spheres $H_i(S^n) \simeq \pi_i(S^n)$ for $1 \leq i \leq n$, but the homology groups have the advantage in that $H_i(S^n) = 0$ for $i > n$. However, everything has a price. How exactly do we define these so called homology groups? We start by motivating, then doing simplicial homology, before moving onto singular homology. Most efficient method for computing homology groups is called cellular homology. We'll also talk about Mayer-Vietoris sequences, the analogue of the van Kampens for the fundamental group.

Something interesting about homology: most of the time we only use the basic properties of homology, not the definition itself. So we could almost invoke an axiomatic approach, which will happen soon. We could also skip the algebra and talk about geometry, but then Dr. Brand would be unhappy (and so would I), so we'll approach it with a mix of the two (talk about intuition first then state the axioms later).

6.1 The big idea of homology

Issues with homotopy groups: things get really wacky because S^2 has no cells of dimension greater than 2, but some (infinitely many) of the higher homotopy groups $\pi_n(S^2)$ are nontrivial. $\langle god\, shattering\, star\, noises \rangle$ However, homology groups are (directly) related to cell structures, in that you can regard them as an algebraization of how cells of dimension n attach to cells of dimension $n-1$.

Imagine a circle with two antipodal points x and y, with four arrows a, b, c, d drawn in the direction from x to y, which we'll denote by X_1.

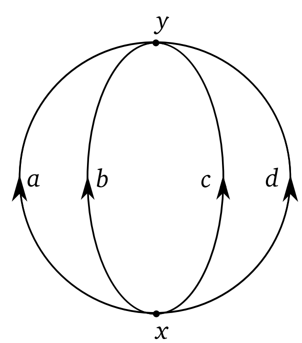


Figure 3: The graph X_1, consisting of two vertices and four edges.

Usually loops are nonabelian, so suppose we abelianize the loops. That is, the loops ab^{-1} and $b^{-1}a$ are "the same circle" (but with a different starting point), so we'll just say they're equal. Formally (not really), rechoosing the basepoint just permutes the letters cyclically, so by abelianizing we can cast off our silly worries about the basepoint. So we make the transition from loops (chosen basepoint) $\longrightarrow$ cycles (no chosen basepoint).

Now we abelian, and all the cool abelian groups use additive notation. So a cycle looks something like $a-b+c-d$ now, a linear combination of edges with integer coefficients. We'll call these linear combinations **chains** of edges. We can decompose these into cycles by several ways, eg $(a-c)+(b-d) = (a-d)+(b-c)$, so it's better just to say cycles are any LC of edges st at least one decomposition make geometric sense. When is a chain a cycle? Cycles are distinguished by the fact that they enter and exit a vertex the same amount of times. So for an arbitrary chain $ka + lb + mc + nd$, it enters y about $k + l + m + n$ times (one for each thing) and enters x (or leaves it) $-k-l-m-n$ times. So if we want $ka + lb + mc + nd$ to be a cycle, we just need to require $k + l + m + n = 0$.

To generalize this, let C_1 be the free abelian group with a basis set $\{a, b, c, d\}$ (edges), and C_0 be the free abelian group with basis $\{x, y\}$ (vertices). Elements of C_1 are chains of edges, and elements of C_0 are linear combinations of vertices. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element to $y - x$, then $\partial(ka + lb + mc + nd) = (k + l + m + n)y - (k + l + m + n)x$, so cycles are precisely $\ker \partial$. It can be seen that $a-b$, $b-c$, and $c-d$ form a basis for $\ker \partial$, so every cycle in X_1 is a unique linear combination of these three elts. Basically, X_1 has three "holes", the three gaps in between the four edges.

Now let's attach a 2-cell to X_1 to get X_2, as seen below.

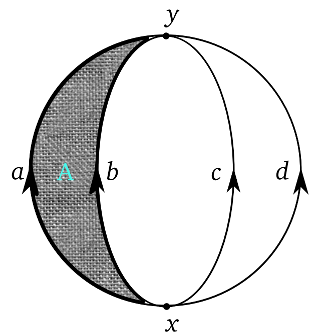


Figure 4: X_1 with a 2-cell attached, denoted X_2. Have you ever seen a 2-cell that looks like cloth?

The 2-cell is attached along the cycle $a-b$, forming the 2-skeleton X_2. Now the cycle is trivial (homotopically), which suggest we form a quotient by factoring out the subgroup generated by $a-b$. For example, $a-c$ and $b-c$ are now equivalent, since they're homotopic in X_2. Algebraically, we define a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the infinite cyclic group generated by A, and $\partial_2(A) = a-b$. ∂_1 is the boundary homomorphism, defined earlier. We are interested in $\ker \partial_1 / \operatorname{im} \partial_2$, that is, the 1-dimensional cycles modulo the boundaries (multiples of $a-b$). Remember, factor groups collapse everything we don't like to the identity. This quotient group is the **homology group** $H_1(X_2)$. If we were to talk about X_1, since it has no 2-cells C_2 is simply zero, so $H_1(X_1) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1$, which is free abelian on three generators. $H_1(X_2)$ is free abelian on two generators ($b-c$ and $c-d$), which expresses the geometric observation that there are two holes remaining after filling one of them in with the 2-cell A.

Let's go farther. Add another 2-cell to the pre-existing 2-cell A, to get the 3-complex X_3.

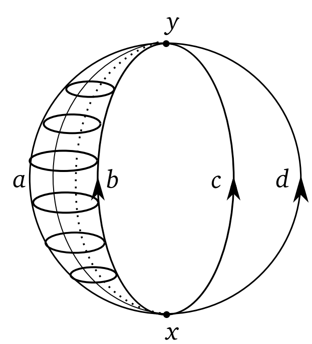


Figure 5: The 3-complex X_3, formed by attaching a 2-cell to X_2.

This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B, and the boundary homomorphism $\partial_2 \colon C_2 \to C_1$ sends A, B to $a-b$. $H_1(X_3) = \ker \partial_1 / \operatorname{im} \partial_2 = H_1(X_2)$, but now ∂_2 has a nontrivial kernel (the infinite cyclic group generated by $A-B$). We view $A-B$ as a 2d cycle generating $H_2(X_3) = \ker \partial_2 \simeq \mathbb{Z}$. The second homology detects the 2d "hole" in X_3.

Unfortunately the diagrams will have to stop now, but let's go even farther and make the complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere by A and B, creating a chain group C_3 generated by C. The boundary homomorphism $\partial_3 \colon C_3 \to C_2$ that sends C to $A-B$ should be seen as the boundary of C, similar to how $a-b$ is the boundary of A. Now we have a sequence of boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and $H_2(X_4) = \ker \partial_2 / \operatorname{im} \partial_3$ is now trivial. $H_3(X_4) = \ker \partial_3 = 0$, note that $H_1(X_4) = H_1(X_3) \simeq \mathbb{Z} \times \mathbb{Z}$, so this is the only homology group of X_4 that isn't trivial.

You can pretty much see where this is going. For a cell complex X, we have chain groups $C_n(X)$ free abelian with basis the n-cells of X, with boundary homomorphisms $\partial_n \colon C_n(X) \to C_{n-1}(X)$, by which we define the homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. So what's the problem? It's how to define ∂_n in general— for $n = 1$ this is easy, it's the vertex head minus the one at the tail. For $n = 2$, it still isn't hard per say, if the cell is attached on a loop of edges, just take the cycle of edges, keeping in mind orientation. This is much tricker for higher dimension cells, even with restrictions to polyhedral cells and nice attaching maps we still have to worry about orientation and stuff.

So what do we do? Use triangles, of course. We can subdivide arbitary polyhedra into certain special types of polyhedra called simplices (what we talked about in class day 1), so there isn't any loss of generality (but there is a loss of efficiency). This gives rise to our more basic **simplicial homology**, which deals with cell complexes from simplices. However, we are still quite limited in what we can do.

So, what do we really do this time? Make things less simple, and make your life difficult by considering the collection of all possible continuous maps of simplices into a space X (wow). The chain groups $C_n(X)$ are tremendously large, but the quotients $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, the **singular homology groups**, are much smaller and easier to work with[16]. For example, in the examples above the singular homology groups coincide with the ones computed from cellular chains. Furthermore (as we will see later), singular homology lets us define these nice cellular homology groups for *all* cell complexes, which solves the issue of how to define boundary maps for cellular chains.

6.2 The structure of Δ-complexes

I have a feeling we're gonna be typing a lot of `\Delta`'s. So basically, the only thing cool kids talk about is singular homology, but it's kinda complicated so we gotta talk about the inferior version for those who have the brain capacity of a literal ape[17], simplicial homology, first. We talk about simplicial homology in the domain of Δ-complexes. Take the standard fundamental polygons with orientation for $\mathbb{T}^2$, $\mathbb{R}P^2$, and the Klein bottle K. Cut the squares in half with a diagonal to get two triangles, from here we can get the original shape by identifying in pairs. We can do this with any n-gon, decomposing it into $n-2$ base triangles. So we can make any closed surface from triangles, furthermore, we could also make a larger class of spaces that aren't surfaces by allowing more than two edges to be glued together at the same time.

The idea of a Δ-complex is to generalize these constructions to n-dimensions. The n-dimensional triangle is the n-simplex, the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that don't lie in a hyperplane of dimension less than n, where by "hyperplane" we mean the set of solutions to a system of linear equations. We could also just say that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are LI. The v_i are **vertices** of the simplex, and the simplex itself is $[v_0, \cdots, v_n]$.

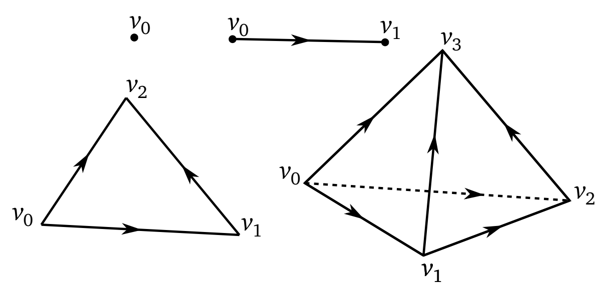


Figure 6: The 0-simplex to the 3-simplex, respectively (with ordered vertices and oriented edges).

For example, we have the standard n-simplex given by

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\},$$

whose vertices are the unit vectors along the coordinate axes. Think of this as taking the unit vectors, and drawing a triangle from each of their endpoints. This works because the difference vectors are LI. For homology, orientation of vertices is really important, so n-simplex really means n-simplex with an ordering on its vertices. Ordering the

[16] For reasonably "nice" spaces X, of course.
[17] The book simply says "primitive" version, so I used my imagination a little bit.

vertices will determine an orientation on its subscripts, as can be seen in Figure 6. This also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other simplex $[v_0, \cdots, v_n]$ that preserves the order of the vertices, given by

$$(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i.$$

We say the coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i \in [v_0, \cdots, v_n]$. Deleting a vertex of a n-simplex yields something that spans an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$. We'll adopt the following convention: *The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.* That sounds reasonable enough. We say the union of all faces of Δ^n is the **boundary** of Δ^n, written $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is equal to $\Delta^n \setminus \partial\Delta^n$, the interior of Δ^n.

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha|\mathring{\Delta}^n$ is onto, and each point of X is in the image of exactly one restriction $\sigma_\alpha|\mathring{\Delta}^n$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear order-preserving homeomorphism.
3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

A consequence of (3) is that X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha \colon \Delta^n \to X$, the quotient space obtained by idenfying each face of a Δ^n_α with the Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face in question. You can think of this as basically cell complexes, attaching 0-simplices (cells) to 1-simplices and 2-simplices, and so on.

In general, we can make Δ-complexes from collections of disjoint simplices by identifying various subspaces spanned by subsets of the vertices, with identifications performed by the canonical order-preserving linear homeomorphisms. Note that if we think of a Δ-complex X as a quotient space of disjoint simplices, then X must be Hausdorff. Each restriction $\sigma_\alpha | \mathring{\Delta}^n$ is a homeomorphism onto its image by condition (3), which is an open simplex in X. Then these open simplices are the cells e^n_α of a CW complex structure on X with the σ_α's as characteristic maps (we won't use this fact yet).

6.3 Simplicial homology

Goal: define simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Formally, we can write elements of $\Delta_n(X)$ as finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$, called **n-chains.** We could also write $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha \colon \Delta^n \to X$ is the characteristic map of e^n_α, with image the closure of e^n_α. Such a sum can be thought of as a finite collection, or 'chain', of n-simplices in X.

Take a look at $\partial[v_0, v_1] = [v_1] - [v_0]$, $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$, and $\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$. Naïvely, one might assume the boundary of an n-simplex to be the sum of the faces delete a point, denoted by $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ where v_i is the vertex to be deleted. However, note the signs to take orientations into account, it just happens that they work out based on the position of v_i. So we have

$$\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots v_n].$$

Keeping this in mind, let's define a **boundary homomorphism** $\partial_n \colon \Delta_n(X) \to \Delta_{n-1}(X)$ for X a general Δ-complex by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \, \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Lemma 6.1. *The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.*

Proof. Note that

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \, \sigma | [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^i (-1)^{j-1} \, \sigma | [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n].$$

Then after switching i and j in the second term, it becomes the negative of the first. Alternate proof from Dr. Allcock: note that $\partial\sigma := \sum_{i=0}^{n}(-1)^i\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,v_n]$. Then

$$\partial\partial\sigma = \sum_{i=0}^{n}(-1)^i\partial\left(\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,v_n]\right),$$

which distributes because C_{n-1} is free on {singular $(n-1)$-simplex}. So defining any function {singular $(n-1)$-simplex} $\to C_{n-2}$ extends to a $\mathbb{Z}$-linear map $C_{n-1}\to C_{n-2}$. Then

$$\partial\partial\sigma = \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{i-1}\sigma\circ[v_0,\cdots,\hat{v}_j,\cdots,\hat{v}_i,\cdots,v_n](-1)^j + \sum_{j=i+1}^{n}\sigma\circ[v_0,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_n](-1)^{j-1}\right),$$

which is equal to zero by cancellation[18]. ⊠

What we have here is a sequence of homomorphisms of abelian groups

$$\cdots\to C_{n+1}\xrightarrow{\partial_{n+1}}C_n\xrightarrow{\partial_n}C_{n-1}\to\cdots\to C_1\xrightarrow{\partial_1}C_0\xrightarrow{\partial_0}0$$

with $\partial_n\partial_{n+1}=0$ for all n. This is called a **chain complex**. Note that we've extended the sequence to 0, with $\partial_0=0$. The equation $\partial_n\partial_{n+1}=0$ is equivalent to the inclusion $\operatorname{im}\partial_{n+1}\subseteq\ker\partial_n$, so we can define the **nth homology group** of the chain complex as $H_n=\ker\partial_n/\operatorname{im}\partial_{n+1}$. Elements of $\ker\partial_n$ are called **cycles** and elements of $\operatorname{im}\partial_{n+1}$ are called boundaries. Cosets of $\operatorname{im}\partial_{n+1}$ in H_n are called **homology classes**. Two cycles representing the same homology class are said to be **homologous**, that is, their difference is a boundary. When $C_n=\Delta_n(X)$, the homology group $\ker\partial_n/\operatorname{im}\partial_{n+1}$ will be denoted by $H_n^{\Delta}(X)$ and called the **nth simplicial homology group** of X.

6.4 Homological algebra

We'll take this section to digress a bit and talk about some homological algebra. These notes will follow May §12.

Let R be a commutative ring: the main example is $R=\mathbb{Z}$. A **chain complex** over R is a sequence of R-modules

$$\cdots\to X_{i+1}\xrightarrow{d_{i+1}}X_i\xrightarrow{d_i}X_{i-1}\to\cdots$$

such that $d_i\circ d_{i+1}=0$ for all i (abbreviated $d=d_i$). A **cochain complex** over R is an analogous sequence

$$\cdots\to Y^{i-1}\xrightarrow{d^{i-1}}Y^i\xrightarrow{d^i}Y^{i+1}\to\cdots$$

with $d^i\circ d^{i+1}=0$. Usually $X_i=0$ for $i<0$ and $Y^i=0$ for $i<0$ (or else $\{X_i,d_i\}\to\{X^{-i},d^{-i}\}$, making chain and cochain complexes equivalent). An element of the kernel of d_i is a **cycle** and an element of the image of d_{i+1} is a **boundary**. This makes a lot more sense if you picture the boundary map d_i as removing a vertex to get an $n-1$ simplex each time. We say two cycles are **homologous** if their difference is a boundary, and write $B_i(X)\subseteq Z_i(X)\subseteq X_i$ for the submodules of boundaries and cycles, respectively. Then we can define the **ith homology group** $H_i(X)$ as the quotient module $Z_i(X)/B_i(X)$, and write $H_*(X)$ for the sequence of R-modules $H_i(X)$. To get things straight, we've defined things the following way:

$$Z_i(X) = \text{cycles} := \ker d_i\subseteq X_i$$
$$B_i(X) = \text{boundaries} := \operatorname{im} d_{i+1}\subseteq X_i.$$

[18] The proof from Dr. Allcock was for singular homology, but the idea is the same.

A **chain map** $f : X \to X'$ of chain complexes is a sequence of maps of R-modules $f_i : X_i \to X'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$ for all i. That is, the following diagram commutes for all i[19]:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r, "d_i"] \arrow[d, "f_i"] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & X'_{i+1} \arrow[r, "d'_{i+1}"] & X'_i \arrow[r, "d'_i"] & X'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

It follows that $f_i(B_i(X)) \subseteq B_i(X')$ and $f_i(Z_i(X)) \subseteq Z_i(X')$. Therefore we have that f induces a map of R-modules $f_* = H_i(f) \colon H_i(X) \to H_i(X')$. A **chain homotopy** $s \colon f \simeq g$ between chain maps $f, g \colon X \to X'$ is a sequence of homomorphisms $s_i : X_i \to X'_{i+1}$ such that

$$d'_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

for all i. Chain homotopy is an equivalence relation (this was an exercise) since if $t \colon g \simeq h$, then $s + t = \{s_i + t_i\}$ is a chain homotopy $f \simeq h$.

Lemma 6.2. *Chain homotopic maps induce the same homomorphism of homology groups.*

Proof. Let $s \colon f \simeq g$, $f, g \colon X \to X'$. If $x \in Z_i(X)$, then $f_i(x) - g_i(x) = d'_{i+1} s_i(x)$ such that $f_i(x)$ and $g_i(x)$ are homologous. ⊠

A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of modules is **exact** if $\operatorname{im} f = \ker g$. If $M' = 0$, then g is a monomorphism; if $M'' = 0$, then f is an epimorphism. We proved this as an exercise! A longer sequence is exact if it is exact at each position. A **short exact sequence** of chain complexes is a sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

that is exact in each degree. Here 0 denotes that chain complex that is the 0 module in each degree.

Proposition 6.1. *A short exact sequence of chain complexes naturally gives rise to a LES of R-modules*

$$\cdots \to H_q(X') \xrightarrow{f} H_q(X) \xrightarrow{g_*} H_q(X'') \xrightarrow{\partial} H_{q-1}(X') \to \cdots .$$

Proof. Let $[x]$ denote the homology class of a cycle x. Define the "connecting homomorphism" $\partial : H_q(X'') \to H_{q-1}(X')$ by $\partial[x''] = [x']$, where $f(x') = d(x)$ for some x such that $g(x) = x''$. There exists such an x because g is an epimorphism, and x' exists because $gd(x) = dg(x) = 0$. Use a "diagram chase" to verify that ∂ is well defined and the sequence is exact. Naturality means that a commutative diagram of short exact sequences of chain complexes gives rise to a commutative diagram of long exact sequences of R-modules. The big idea is the naturality of the connecting homomorphism, which is left as an exercise to the reader. ⊠

6.5 Singular homology

These notes will follow Massey §2 and the rest of Hatcher §2.1.

Let's define $H_0(X)$ as such: let $Z_0(X) = C_0(X)$ and $H_0(X) = Z_0(X)/B_0(X) = C_0(X)/B_0(X)$. Anoter way we could do this is is defining $C_n(X) = \{0\}$ for $n < 0$, then defining $\partial_n \colon C_n(X) \to C_{n-1}(X)$ in the only possible way for $n \le 0$ (i.e., $\partial_n = 0$ for $n \le 0$), and finally defining $Z_n(X) = \ker \partial_0$. In general, we could define $Z_n(X) = \ker \partial_n$for all integers n, $B_n(X) = \partial_{n-1}(C_{n+1}(X)) \subseteq Z_n(X)$, and $H_n(X) = Z_n(X)/B_n(X)$ for all n, with $H_n(X) = \{0\}$ for $n < 0$.

[19] May's diagram showed much less, but I feel this illustrates the idea much better: it also makes following around the chain homotopy homomorphisms easier.

Now let's define (not really, we'll ignore the definition) the reduced 0-dimensional homology group $\widetilde{H}_0(X)$. Let's define a homomorphism $\varepsilon\colon C_0(X)\to\mathbb{Z}$, often called the *augmentation*, made by the typical barycentric coordinate sum $\varepsilon\colon \sum_i n_i\sigma_i \mapsto \sum_i n_i$. Then $\varepsilon\circ\partial_1 = 0$: to do this, show that $\varepsilon(\partial_1(T)) = 0$ for some 1-cube (not hard)[20]. Then we can define $\widetilde{Z}_0(X) = \ker\varepsilon$, and

$$\widetilde{H}_0(X) = \widetilde{Z}_0(X)/B_0(X).$$

We say that $\widetilde{H}_0(X)$ is the **reduced 0-dimensional homology group** of X. To avoid weird stuff happening, assume $X \neq \emptyset$. It's often convenient to set $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$.

JK, back to Hatcher. Some examples of simplicial homology:

Example 6.1. Let $X = S^1$, with one vertex v and an edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map ∂_1 is zero since $\partial e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \geq 2$ since there are no simplices in these dimensions. Therefore

$$H_n^{\Delta}(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0,1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

Example 6.2. Let $X = \mathbb{T}$, the torus with a Δ-complex structure having one vertex, three edges a, b, and c,, and two 2-simplices U and L. Since $\partial_1 = 0$, $H_0^{\Delta}(\mathbb{T}) \simeq \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a+b-c\}$ is a basis for $\Delta_1(\mathbb{T})$, it follows that $H_1^{\Delta}(\mathbb{T}) \simeq \mathbb{Z}\oplus\mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^{\Delta}(\mathbb{T})$ is equal to $\ker\partial_2$, which is infinite cyclic generated by $U - L$. So

$$H_n^{\Delta}(\mathbb{T}) \simeq \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z} & \text{for } n = 0,2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

Let's talk about **singular homology**. A **singular n-simplex** in a space X is just a map $\sigma\colon \Delta^n \to X$. The word 'singular' is used to imply that the map doesn't have to be nice (look like a simplex) but can have weird 'singularities'. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$, called **n-chains** (more precisely, singular n-chains) are finite formal sums $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i\colon \Delta^n \to X$. A boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ is defined by the same formula as before:

$$\partial_n(\sigma) = \sum_i (-1)^i \big| [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Then $\sigma|[v_0, \cdots, \hat{v}_i, \cdots v_n]$ is a map $\Delta^{n-1} \to X$, that is , a singular $(n-1)$-simplex. We also have $\partial_n\partial_{n+1} = 0$ (more concisely $\partial^2 = 0$), so we define the singular homology group $H_n(x) = \ker\partial_n/\operatorname{im}\partial_{n+1}$. Singular chain groups tend to be really large (often uncountable), but modding out makes the homology groups easier to work with.

Proposition 6.2. *For a space X, there is an isomorphism $H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$, where X_α denotes the path-components of X.*

Proof. Since a singular simplex always has a path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. This is preserved by the boundary maps ∂_n and similarly $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$. ⊠

Proposition 6.3. *If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. We have $H_0(X)/\operatorname{im}\partial_1$ since $\partial_0 = 0$. Define a homomorphism $\varepsilon\colon C_0(X) \to \mathbb{Z}$ by $\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$. This is onto if $X \neq \emptyset$: we claim that $\ker\varepsilon = \operatorname{im}\partial_1$ if X is path-connected, and hence ε induces an isomoprhism $H_0(X) \approx \mathbb{Z}$. To see that this is true, observe that $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$ since for a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have $\varepsilon\partial_1(\sigma) = \varepsilon\,(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$. To show that $\ker\varepsilon \subseteq \operatorname{im}\partial_1$, suppose $\varepsilon\left(\sum_i n_i\sigma_i\right) = 0$, so $\sum_i n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i\colon I \to X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i\colon [v_0, v_1] \to X$, then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence $\partial\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$ since $\sum_i n_i = 0$. So $\sum_i n_i\sigma_i$ is a boundary, which shows that $\ker\varepsilon \subseteq \partial_1$. ⊠

[20] I'm glossing over formal stuff because everywhere else uses triangles instead of cubes. I just want results!

Proposition 6.4. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof. In this case there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_i (-1)^i \sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. So we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternately isomorphisms and trivial maps, except for the last $\mathbb{Z}$. So the homology groups of this complex are trivial for every group besides $H_0 \simeq \mathbb{Z}$. ⊠

Sometimes weird stuff happens with $H_0(X)$, as can be seen in Proposition 6.4. To avoid this, we can talk about the **reduced homology groups** $\widetilde{H}_n(X)$, defined to be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the same one as in our earlier proposition[21]. Since $\varepsilon\partial_1 = 0$, ε vanishes on $\operatorname{im}\partial_1$ adn hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. Obviously $H_n(X) \simeq \widetilde{H}_n(X)$ for $n > 0$.

6.6 Exact sequences

Definition 6.1 (Exact sequences)**.** A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be **exact** if $\ker \alpha_n = \operatorname{im}_{n+1}$ for each n.

The inclusions $\operatorname{im}\alpha_{n+1} \subseteq \ker \alpha_n$ are equivalent to $\alpha_n \alpha_{n+1} = 0$, so the sequence is a chain complex, and the opposite inclusions $\ker\alpha_n \subseteq \operatorname{im}\alpha_{n+1}$ say that the homology groups of this chain complex are trivial. We can express a number of basic algebraic concepts in terms of exact sequences, for example:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker\alpha = 0$, i.e., α is injective.

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{im}\alpha = B$, i.e., α is surjective.

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism, by (i) and (ii).

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective, and $\ker\beta = \operatorname{im}\alpha$, so β induces an isomorphism $C \simeq B/\operatorname{im}\alpha$. This can be written as $C \simeq B/A$ if we think of α as an inclusion of A as a subgroup of B.

An exact sequence $0 \to A \to B \to C \to 0$ as in (iv) is called a **short exact sequence**. These turn out to be the perfect tool for stuff, in particular, relating the homolofy groups of a space, a subspace, and the associated quotient space.

Theorem 6.1. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0,$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

Proof. Basically, construct ∂. The idea is that an element $x \in \widetilde{H}_n(X/A)$ can be represented by a chain α in X with $\partial\alpha$ a cycle in A whose homology class is $\partial x \in \widetilde{H}_{n-1}(A)$. The full proof will come later. Pairs of spaces (X,A) that satisfy the hypothesis of the theorem will be called **good pairs**[22]. ⊠

Corollary 6.1. $\widetilde{H}_n(S^n) \simeq \mathbb{Z}$ *and* $\widetilde{H}_i(S^n) = 0$ *for* $i \neq n$.

[21] My clever references aren't working??

[22] *We're a good pair, you and I...*

Proof. For $n > 0$ take the good pair $(X,A) = (D^n, S^{n-1})$ so $X/A = S^n$. Since D^n is contractible the terms $\widetilde{H}_i(D^n)$ in the LES for this pair are zero. Then by the exactness of the sequence the maps $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\widetilde{H}_0(S^n) = 0$. Then our result follows by induction on n, in which the base case of S^0 holds by Proposition 6.2 and Proposition 6.4. ⊠

Lemma 6.3. *Every continuous map $h\colon D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.*

Proof. This was actually an earlier theorem in Hatcher. As you can see, this will lead into Brouwer's fixed point theorem. Suppose that $h(x) \neq x$ for all $x \in D^2$. Then we can define a map $r : D^2 \to S^1$ by letting $r(x)$ be the point of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2. Now r is continuous, furthermore, $r(x) = x$ if $x \in S^1$. So r is a retraction of D^2 onto S^1, but no such retraction exists: let f_0 be a loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop, for example $f_t(s) = (1-t)f_0(s) + tx_0$ for x_0 the basepoint of f_0. Since the retraction r is the identity on S^1, the composition rf_t is a homotopy in S^1 from $rf_0 = f_0$ to the constant loop at x_0: but this contradicts the fact that $\pi_1(S^1)$ is nonzero. ⊠

Corollary 6.2 (Brouwer's fixed point theorem)**.** *∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.*

Proof. If $r : D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for $i\colon \partial D^n \to D^n$ the inclusion map. The composition $\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\widetilde{H}_{n_1}(\partial D^n) \simeq \mathbb{Z}$. But i_* and r_* are both 0 since $\widetilde{H}_{n-1}(D^n) = 0$, and we have a contradiction. For the fixed point portion, just replace π_1 with H_n in Lemma 6.3 and we're good. ⊠

6.7 Relative homology (todo)

Sometimes ignoring things makes things easier, for example arithmetic modulo n (ignoring multiples of n). Relative homology is another example: in this case, we ignore all singular chains in a subspace of a given space.

Given a space X and a subspace $A \subseteq X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$, thus chains in A are trivial in $C_n(X,A)$. Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$. Then we have a sequence of boundary maps

$$\cdots \to C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \to \cdots$$

The relation $\partial^2 = 0$ holds since it held before (then holds for quotients).

Definition 6.2 (Relative homology groups)**.** Given the chain complex above, the homology groups $\ker\partial / \operatorname{im}\partial$ of the chain complex are the **relative homology groups** $H_n(X,A)$. We can see the following:

- Elements of $H_n(X,A)$ are represented by **relative cycles**: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle is trivial in $H_n(X,A)$ iff it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

These properties make precise the intuitive idea that $H_n(X,A)$ is 'homology of X modulo A'.

Goal: show that the relative homology groups $H_n(X,A)$ for any pair (X,A) fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

To do this, we'll go on our first diagram chase. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

where i is the inclusion map and j is the quotient map. If we let n vary and draw the short exact sequences vertically instead of horizontally, we have a large commutative diagram like the one below, where the columns are exact and the rows are chain complexes denoted by A, B, and C.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A diagram like this is called a **short exact sequence of chain complexes**. We'll show that this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where $H_n(A)$ denotes the homology group $\ker\partial/\operatorname{im}\partial$ at A_n in the chain complex, $H_n(B)$ and $H_n(C)$ similarly defined. To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, let $c \in C_n$ be a cycle. Then since j is onto, $c = j(b)$ for some $b \in B_n$. Then $\partial b \in B_{n-1}$ is also in $\ker j$ since $j(\partial b) = \partial j(b) = \partial c = 0$.

6.8 Homology with coefficients (todo)

6.9 Degrees of maps $S^n \to S^n$ (todo)

6.10 Cellular homology

Following Pierre Albin lecture 19 and Hatcher for more technical things. Recall that if X is a Δ-complex then $H_*^\Delta(X) \simeq H_*(X)$, and that $H_*^\Delta(X)$ is easy to compute and $H_*(X)$ is easy to prove theorems about. In an ideal world, we would like a similar equivalence for when X is a CW complex since they're much more applicable, but we ran into an issue when figuring out how to define the boundary maps. What we're going to do is defined a chain complex $C_n^{\mathrm{CW}}(X)$, and we want it to be free abelian on the n-cells of X.

Lemma 6.4. *If X is a CW complex, then:*

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.

(c) The map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof. Statement (a) follows immediately from the fact that (X^n, X^{n-1}) is a good pair and X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X (it does!). Next consider the following part of the LES of the pair (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

If $k \neq n$ the last term is zero by (a) so the middle map is surjective, while if $k \neq n-1$ then the first term is zero so the middle map is injective. Now look at the inclusion-induced homomorphisms:

$$H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1})$$

It follows that all of these maps are isomorphisms, except that the map to $H_k(X^k)$ may not be surjective and the map from $H_k(X^k)$ may not be injective. Then the first part the sequence gives (b) since $H_k(X^0) = 0$ when $k > 0$. The last part gives (c) when X is finite-dimensional. The proof when X is infinite-dimensional requires a little more work. ⊠

Let X be a CW complex. What we want is a boundary map $C^{\text{CW}}_{n+1}(X) \xrightarrow{\partial^{\text{CW}}_{n+1}} C^{\text{CW}}_n(X)$. By Lemma 6.4, we have

$$\begin{tikzcd}
& C^{\text{CW}}_{n+1}(X) \arrow[rr, "\partial^{\text{CW}}_{n+1}"] \arrow[d, equal] & & C^{\text{CW}}_n(X) \arrow[d, equal] \\
& H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] & H_n(X^n) \arrow[r, "\partial_n"] & H_n(X^n, X^{n-1})
\end{tikzcd}$$

The equalities are from Lemma 6.4, and the boundary maps between homology groups are from the LES of the good pair (X^n, X^{n-1}). Then this naturally extends to the diagram shown in Figure 7.

$$\begin{tikzcd}
& & & & 0 & & \\
& 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
& & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "\partial^{\text{CW}}_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "\partial^{\text{CW}}_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
& & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
& & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Figure 7: The diagram for cellular homology.

In this diagram, $\partial^{\text{CW}}_{n+1}$ and ∂^{CW}_n are defined as the compositions $j_n\partial_{n+1}$ and $j_{n-1}\partial_n$, which are just 'relativizations' of the boundary maps ∂_{n+1} and ∂_n. The composition $\partial^{\text{CW}}_n \partial^{\text{CW}}_{n+1}$ contains two successive maps in one of the exact sequences, hence is zero (since image maps onto kernel maps onto zero by exactness). The horizonal row in the diagram is a chain complex, called the **cellular chain complex** of X, since $H_n(X^n, X^{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n-cells of X. The resulting homology groups are called the **cellular homology groups** of X. We temporarily denote them $H^{\text{CW}}_n(X)$.

Theorem 6.2. $H^{CW}_n(X) \simeq H_n(X)$.

Proof. We can identify $H_n(X)$ with $H_n(X^n)/\operatorname{im}\partial_{n+1}$ by a simple application of the FHT and exactness. Since j_n is injective, it maps $\operatorname{im}\partial_{n+1}$ isomorphically onto $\operatorname{im}(j_n\partial_{n+1}) = \operatorname{im}\partial^{\text{CW}}_{n+1}$ and $H_n(X^n)$ isomorphically onto $\operatorname{im} j_n = \ker\partial_n$. Since j_{n-1} is injective, $\ker\partial_n = \ker\partial^{\text{CW}}_n$. So j_n induces an isomorphism of the quotient $H_n(X^n)/\operatorname{im}\partial_{n+1} \simeq H_n(X)$ onto $\ker\partial^{\text{CW}}_n/\operatorname{im}\partial^{\text{CW}}_{n+1} = H^{\text{CW}}_n(X)$. ⊠

Some immediate applications:

(i) $H_n(X) = 0$ if X is a CW complex with no 0-cells.

(ii) More generally, if X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements. For since $H_n(X^n, X^{n-1})$ is free abelian on k generators, the subgroup $\ker\partial^{\text{CW}}_n$ must be generated by at most k elements, hence also the quotient $\ker\partial^{\text{CW}}_n/\operatorname{im}\partial^{\text{CW}}_{n+1}$.

(iii) If X is a CW complex having no two of its cells in adjacet dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X. This is because the cellular boundary maps ∂_n^{CW} are automatically zero in this case.

Example 6.3. For $\mathbb{C}\mathrm{P}^n$ having a CW structure with one cell of each even dimension $2k \leq 2n$, we have

$$H_i(\mathbb{C}\mathrm{P}^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \cdots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Another example is $S^n \times S^n$ with $n > 1$, using the product CW structure consisting of a 0-cell, two n-cells, and a $2n$-cell.

Proposition 6.5 (Cellular boundary formula). *We have*

$$\partial_n^{CW}(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.

Here we identify the cells e_α^n and e_β^{n-1} with generators of the corresponding summands of the cellular chain groups. The summation in the formula contains only finitely many terms since the attaching map of e_α^n has compact image, so this image meets only finitely many cells e_β^{n-1}. From now on, we'll denote ∂_n^{CW} by d_n.

TODO commutative diagram and justification for cellular boundary formula

Example 6.4. Let M_g be the closed orientable surface of genus g with its usual CW structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

As observed above, d_1 must be 0 since there is only one 0-cell. Also, d_2 is 0 because each a_i or b_i appears with its inverse in $[a_1, b_1] \cdots [a_g, b_g]$, so the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps. Since d_1 and d_2 are both zero, the homology groups of M_g are the same as the cellular chain groups, namely, $\mathbb{Z}$ in dimensions 0 and 2, and $\mathbb{Z}^{2g}$ in dimension 1.

6.11 Axioms for homology

Let's take a formal viewpoint at some properties that all homology theories share.

Definition 6.3. A (reduced) **homology theory** assigns a sequence of abelian groups $\widetilde{h}_n(X)$ and a sequence of homomorphisms $f_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ to each nonempty CW complex X and each map $f : X \to Y$ between chain complexes. These groups and homomorphisms satisfy $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$, and the following axioms:

(1) If f is homotopic to g, that is $f \simeq g : X \to Y$, then $f_* = g_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$.

(2) There are boundary homomorphisms $\partial : \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ defined for each CW pair (X, A),[23] fitting into an exact sequence

$$\cdots \xrightarrow{\partial} \widetilde{h}_n(A) \xrightarrow{i_*} \widetilde{h}_n(X) \xrightarrow{q_*} \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

where i is the inclusion and q is the quotient map. Furthermore the boundary maps are natural (as in natural transformation): For $f : (X, A) \to (Y, B)$ inducing a quotient map $\overline{f} : X/A \to Y/B$, there are commutative diagrams

$$\begin{array}{ccc} \widetilde{h}_n(X/A) & \xrightarrow{\partial} & \widetilde{h}_{n-1}(A) \\ \downarrow{\scriptstyle \bar{f}_*} & & \downarrow{\scriptstyle f_*} \\ \widetilde{h}_n(Y/B) & \xrightarrow{\partial} & \widetilde{h}_{n-1}(B) \end{array}$$

[23] A **CW pair** (X, A) is just a CW complex X equipped with a subcomplex inclusion $A \hookrightarrow X$.

(3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the direct sum map $\bigoplus_\alpha i_{\alpha*} : \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$ is an isomorphism for all n.

Some notes on this new definition for homology. Negative values of n are allowed, in our standard singular homology theory they became zero by definition but there exist some interesting homology theories with nontrivial groups in negative dimensions. Also, the third axioms follows from the first two in the case of a finite wedge sum, but not an infinite one.

You can also give axioms for unreduced homology theories, suppose we have relative groups $h_n(X, A)$, define the absolute groups by $h_n(X) = h_n(X, \emptyset)$. Then axiom (2) splits into two, one about long exact sequences with natural boundary maps, and one about excision (eg $h_n(X,A) \approx h_n(X/A, A/A)$ for CW pairs). In axiom (3) replace wedge sum with disjoint union. The axioms are essentially the same as the ones proposed seventy years ago in [Eilenberg & Steenrod 1952], besides the fact that (3) was omitted (to focus on finite CW complexes). There was also an additional axiom called the *dimension axiom* specifying that the groups $h_n(\text{point})$ are zero for $n \neq 0$. At the time there were no interesting homology theories for which the dimension axiom doesn't hold, but now we have *bordism* in which bordism groups of a point are nonzero in infinitely many dimensions.

Reduced and unreduced homology theories are essentially equivalent. We can get a reduced theory $\widetilde{h}$ from an unreduced theory h by setting $\widetilde{h}_n(X)$ equal to the kernel of the canonical map $h_n(X) \to h_n(\text{point})$. For the other direction, set $h_n(X) = \widetilde{h}_n(X \amalg \text{point})$. You can show that these two transformation are each others inverses. We have $h_n(X) \approx \widetilde{h}_n(X) \oplus h_n(x_0)$ for any point $x_0 \in X$, since the LES of the pair (X, x_0) splits via the retraction of X onto x_0. Also note that $\widetilde{h}_n(x_0) = 0$ for all n, just look at the LES of the pair (x_0, x_0). TODO more stuff on coefficients

We can also get Mayer-Vietoris sequences from the axioms. For a CW complex $X = A \cup B$ with A, B subcomplexes, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces a commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & h_{n+1}(B, A\cap B) & \longrightarrow & h_n(A\cap B) & \longrightarrow & h_n(B) & \longrightarrow & h_n(B, A\cap B) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\approx} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\approx} & & \\
\cdots & \longrightarrow & h_{n+1}(X, A) & \longrightarrow & h_n(A) & \longrightarrow & h_n(X) & \longrightarrow & h_n(X, A) & \longrightarrow & \cdots
\end{array}$$

The vertical maps are isomorphisms since $B/(A\cap B) = X/A$. Then a diagram like this with every third vertical map an isomorphism gives rise to a LES with the remaining nonisomoprhic terms, which looks like

$$\cdots \to h_n(A\cap B) \xrightarrow{\varphi} h_n(A) \oplus h_n(B) \xrightarrow{\psi} h_n(X) \xrightarrow{\partial} h_{n-1}(A\cap B) \to \cdots$$

(This is left as an exercise to the reader.)

Lecture 7

Homotopy theory

Here comes a long block of Hatcher exposition, read if interested, skip if not.

We have met the first homotopy group already, the fundamental group $\pi_1(X)$. The higher dimensional analogues $\pi_n(X)$ are the *homotopy groups*, which have some similarities to the homology groups: $\pi_n(X)$ is abelian for $n \geq 2$, and there are relative homotopy groups fitting into a LES similar to homology. However, neither Seifert-van Kampen's nor excision holds, making the homotopy groups much harder to compute.

However, these groups are still important: one reason is *Whitehead's theorem*, which states that a map between CW complexes inducing isomorphisms on the homotopy groups is a homotopy equivalence. However the stronger statement that if two complexes have isomorphic homotopy groups then they're homotopy equivalent is false usually, aside from the case where we only have one nontrivial homotopy group— these spaces are called *Eilenberg-MacLane spaces*.

Another more direct connection between homology and homotopy is the *Hurewicz theorem*, which says that the first nonzero homotopy group $\pi_n(X)$ of a simply-connected space X is isomorphic to the first nonzero homology group $\widetilde{H}_n(X)$. Though excision doesn't always hold, in some important special cases it does for a range of dimensions. This leads to the idea of *stable homotopy groups*, the beginning of stable homotopy theory. If you figure out how to compute the stable homotopy groups of spheres, you can pick up your Fields medal at the door.

We'll also talk a little about fiber bundles which somewhat generalize the idea of covering spaces for higher homotopy groups, purely to lead toward fibrations. These allow us to describe how the homotopy type of a CW complex is inductively built up from its homotopy groups by forming 'twisted products' of Eilenberg-MacLane spaces, which is the notion of a *Postnikov tower*.

Let I^n be the n-cube, and the boundary ∂I^n be the subspace of points with at least one coordinate equal to 0 or 1.

Definition 7.1 (Higher homotopy groups)**.** For a pointed space X, x_0, define the **n-th homotopy group** $\pi_n(X, x_0)$ as the set of homotopy classes $f : (I^n, \partial I^n) \to (X, x_0)$ where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t.

This extends to π_0 by letting I^0 be a point and ∂I^0 be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.