Differential Topology Notes

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Contents

Lecture 1

April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

A lot of definitions, here's the new ones:

Definition 1.1. A **subalgebra** of an algebra A is a linear subspace $A' \subseteq A$ containing 1 such that $a_1' a_2' \in A'$ for all $a_1', a_2' \in A'$. A **2-sided ideal** $I \subseteq A$ is a linear subspace such that $AI = I$ and $IA = I$. A $\mathbb{Z}$**-grading** of an algebra A is a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that $A^{k_1} A^{k_2} \subseteq A^{k_1+k_2}$ for all $k_1, k_2 \in \mathbb{Z}$. If A is a $\mathbb{Z}$-graded algebra and $a \in A^k, k \in \mathbb{Z}^{>0}$, then a is **decomposable** if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \cdots, a_k \in A^1$. If not, a is **indecomposable**.

1.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra $\bigotimes V$, thought of as the "free algebra generated by V".

Definition 1.2. Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map $i: V \to A$ such that for all (B, T) of an algebra B and a linear map $T: V \to B$ such that φ_T is a homomorphism of algebras.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "T"'] & & A \arrow[dl, dotted, "\exists!"', "\varphi_T"] \\ & B & \end{tikzcd}$$

(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If $(\xi \neq 0) \in V$ and $i(\xi) = 0$, set $B = \mathbb{R} \oplus \mathbb{R}\xi$ and define $\xi^2 = 0$.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "\pi"'] & & A \arrow[dl, dotted, "\varphi"] \\ & \mathbb{R} \otimes \mathbb{R}\xi & \end{tikzcd}$$

Note that $\pi|_{\mathbb{R}\xi} = \mathrm{id}$. But $\xi = \pi(\xi) = \varphi_1(\xi) = 0$, a contradiction. Furthermore, A has a canonical $\mathbb{Z}$-grading. $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$, $T_\lambda: V \to V$ is scalar multiplication, $\varphi_\lambda: A \to A$ is a homomoprhism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

Definition 1.3. Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map $b: V \times V'' \to X$ such that for all (W, B),

$$\begin{tikzcd} V' \times V'' \arrow[rr, "b"] \arrow[dr, "B"'] & & X \arrow[dl, "\exists!"', "T_B"] \\ & W & \end{tikzcd}$$

We denote $X = V' \otimes V''$, and $b(\xi', \xi'') = \xi' \otimes \xi''$, $\xi' \in V', \xi'' \in V''$.

If S' is a basis of V', S'' a basis of V'', then $S' \times S''$ is a basis of $V' \otimes V''$, where

$$S' \times S'' \cong \{\xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S''\}.$$

Note that $\bigotimes$ is "commutative" and "associative" with unit $\mathbb{R}$, so

$$\begin{gathered} \mathbb{R} \otimes V \to V \\ V_1 \otimes V_2 \to V_2 \otimes V_1 \\ (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), \end{gathered}$$

forming what we call a **symmetric monoidal category**. We write $\otimes^1 V = V$, $\otimes^2 V = V \otimes V$, $\otimes^3 V = V \otimes V \otimes V$ and so on. We also write $\otimes^0 V = \mathbb{R}$, and sometimes replace $\otimes^n V$ with $V^{\otimes n}$.

1.2 Existence of tensor algebras

Let V be a vector space, and $A = \bigoplus_{k=0}^{\infty} \otimes^k V$. Let $i \colon V \hookrightarrow A$ be the inclusion into $\otimes' V = V$.

Claim. (A, i) is a tensor algebra over V.

To see this, note that

$$\underbrace{\xi_1 \otimes \cdots \otimes \xi_k)}_{\in \otimes^k V} \cdot_A \underbrace{\eta_1 \otimes \cdots \otimes \eta_\ell}_{\in \otimes^\ell V} = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that $A = \otimes' V$ is *not* commutative.

1.3 The Exterior Algebra

We want to impose todo:come back

Lecture 2

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todo:see notes on chapter 21, multivariate analysis

2.1 Exterior algebra of a direct sum

Definition 2.1. Let V be a vector space. An **exterior algebra** (E, j) over V is an algebra E and a linear map $j \colon V \to E$ satisfying $j(\xi)^2 = 0$ for all $\xi \in V$ such that for all pairs (B, T) consisting of an algebra B and a linear map $T \colon V \to B$ satisfying $T(\xi)^2 = 0$ for all $\xi \in V$, there exists a unique algebra homomorphism $\varphi \colon E \to B$ such that $T = \varphi \circ j$.

Let L_1, L_2 be linear, and $\bigwedge^*(L_1 \oplus L_2 = V)$.