Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1 Category Theory

Today we talk about abstract nonsense! These notes will follow Evan Chen's Napkin §60 and May's "A Concise Course in Algebraic Topology" §2.

§1.1 Motivation

Why do we talk about categories? Categories rise from objects (sets, groups, topologies) and maps between them (bijections, isomorphisms, homeomorphisms). Algebraic topology speaks of maps from topologies to groups, which makes maps between categories a suitable tool for us.

Example 1.1. Here are some examples of morphisms between objects:

- A bijective homomorphism between two groups G and H is an isomorphism. What also works is two group homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ which are mutual inverses, that is $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.
- Metric (or topological) spaces X and Y are isomorphic if there exists a continuous bijection $f\colon X \to Y$ such that f^{-1} is also continuous.
- Vector spaces V and W are isomorphic if there is a bijection $T\colon V \to W$ that's a linear map (aka, T and T^{-1} are linear maps).
- Rings R and S are isomorphic if there is a bijective ring homomorphism ϕ (or two mutually inverse ring homomorphism).

§1.2 Categories

Definition 1.1 (Category)**.** A *category* $\mathcal{A}$ consists of

- A class of *objects* , denoted $\mathrm{obj}(\mathcal{A})$.
- For any two objects $A_1, A_2 \in \mathrm{obj}(\mathcal{A})$, a class of *arrows* (also called *morphisms* or *maps* between them). Let's denote the set of arrows by $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \mathrm{obj}(\mathcal{A})$, if $f\colon A_1 \to A_2$ is an arrow and $g\colon A_2 \to A_3$ is an arrow, we can compose the two arrows to get $h = g \circ f\colon A_1 \to A_3$ an arrow, represented in the *commutative diagram* below:

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[dr, dotted, "h"] & A_2 \arrow[d, "g"] \\ & A_3 \end{tikzcd}$$

  The composition operation can be denoted as a function

$$\circ\colon \mathrm{Hom}_{\mathcal{A}}(A_2, A_3) \times \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$$

  for any three objects A_1, A_2, A_3. Composition must be associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$. In the diagram above, we say h *factors* through A_2.

- Every object $A \in \mathrm{obj}_{\mathcal{A}}$ has a special *identity arrow* $\mathrm{id}_{\mathcal{A}}$. The identity arrow has the expected properties $\mathrm{id}_{\mathcal{A}} \circ f = f$ and $f \circ \mathrm{id}_{\mathcal{A}} = f$.

Note. We can't use the word "set" to describe the class of objects because of some weird logic thing (there is no set of all sets). But you can think of a class as a set.

From now on, $A \in \mathcal{A}$ is the same as $A \in \mathrm{obj}(\mathcal{A})$. A category is *small* if it has a set of objects, and *locally small* if $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathcal{A}$.

Example 1.2 (Basic Categories)**.** Here are some basic examples of categories:

- We have the category of groups Grp.
  - The objects of Grp are groups.
  - The arrows of Grp are group homomorphisms.
  - The composition of Grp is function composition.
- Describe the category CRing (of commutative rings) in a similar way.
- Consider the category Top of topological spaces, whose arrows are continuous maps between spaces.

- Also consider the category Top_* of topological spaces with a distinguished basepoint, that is, a pair (X, x_0), $x_0 \in X$. Arrows are continuous maps $f\colon X \to Y$ with $f(x_0) = y_0$.

- Similarly, the category of (possibly infinite-dimensional) vector spaces over a field k Vect_k has linear maps for arrows. There is even a category FDVect_k of finite-dimensional vector spaces.

- Finally, we have a category Set of sets, arrows denote any map between sets.

Definition 1.2 (Isomorphism)**.** An arrow $A_1 \xrightarrow{f} A_2$ is an *isomorphism* if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \mathrm{id}_{A_2}$ and $g \circ f = \mathrm{id}_{A_1}$. We say A_1 and A_2 are *isomorphic*, denoted $A_1 \cong A_2$.

Remark 1.1. In the category Set, $X \cong Y \iff |X| = |Y|$.

In other fields, we can tell a lot about the structure of an object by looking at maps between them. In category theory, we *only* look arrows, and ignore what the objects themselves are.

Example 1.3 (Posets are Categories)**.** Let $\mathcal{P}$ be a poset. Then we can construct a category P for it as follows:

- The objects of P are elements of $\mathcal{P}$.

- We define the arrows of P as follows:
  - For every object $p \in P$, we add an identity arrow id_p, and
  - For any pair of distinct objects $p \leq q$, we add a single arrow $p \to q$.

  There are no other arrows.

- We compose arrows in the only way possible, examining the order of the first and last object.

Here's a figure depicting the category of a poset $\mathcal{P}$ on four objects $\{a, b, c, d\}$ with $a \leq b$ and $a \leq c \leq d$.

$$\begin{tikzcd}
 & {[a]} \arrow[loop above, "\mathrm{id}_a"] \arrow[dl, "a\leq b"'] \arrow[dr, "a\leq c"] \arrow[dd, "a\leq d" description] & \\
{[b]} \arrow[loop left, "\mathrm{id}_b"] & & {[c]} \arrow[loop right, "\mathrm{id}_c"] \arrow[dl, "c\leq d"] \\
 & {[d]} \arrow[loop below, "\mathrm{id}_d"] &
\end{tikzcd}$$

Note that no two distinct objects of a poset are isomorphic.

This shows that categories don't have to refer to just structure preserving maps between sets (these are called "concrete categories".

Example 1.4 (Groups as a category with one object)**.** A group G can be thought of as a category $\mathcal{G}$ with one object $*$, all of whos arrows are isomorphisms.

If the universe were structured differently and kids learned category theory before groups, symmetries transforming X into itself would be a natural extension of categories that transform X into other objects, a special case in which all the maps are invertible. Alas, this is not the right timeline.

Example 1.5 (Deriving Categories)**.** We can make categories from other categories!

(a) Given a category $\mathcal{A}$, we can construct the *opposite category* A^{op}, which is the same as $\mathcal{A}$ but with all the arrows reversed.

(b) Given categories $\mathcal{A}$ and $\mathcal{B}$, we can construct the *product category* $\mathcal{A} \times \mathcal{B}$ as follows: the objects are pairs (A, B) for $A \in \mathcal{A}$, $B \in \mathcal{B}$, and the arrows from (A_1, B_1) to (A_2, B_2) are pairs

$$\left(A_1 \xrightarrow{f} A_2, B_1 \xrightarrow{g} B_2\right).$$

The composition is just pairwise composition, and the identity is the pair of identity functions on A and B.

§1.3 Special objects in categories

Some categories have things called *initial objects*. For example the empty set $\varnothing$, the trivial group, the empty space, initial element in a poset, etc. More interestingly: the initial object of CRing is the ring $\mathbb{Z}$.

Definition 1.3 (Initial object)**.** An *initial object* of $\mathcal{A}$ is an object $A_{\text{init}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{init}}$), there is exactly one arrow from A_{init} to A.

The yang to this yin is the *terminal object*:

Definition 1.4 (Terminal object)**.** A *terminal object* of $\mathcal{A}$ is an object $A_{\text{final}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{final}}$, there is exactly one arrow from A to A_{final}.

For example, the terminal object of Set is $\{*\}$, Grp is $\{1\}$, CRing is the zero ring, Top is the single point space, and a poset its maximal element (if one exists).

§1.4 Monic and epic maps (todo)

Injectivity and surjectivity don't really make sense when talking about categories. todo

§1.5 Functors

Motivation: maps between categories, objects rising from other objects.

Example 1.6 (Basic Functors)**.** Here are some basic examples of functors:

- Given an algebraic structure (group, field, vector space) we can take its underlying set S: this is a functor from $\mathsf{Grp} \to \mathsf{Set}$ (or whatever you want to start with).
- If we have a set S, if we consider the vector space with basis S we get a functor $\mathsf{Set} \to \mathsf{Vect}$.
- Taking the power set of a set S gives a functor $\mathsf{Set} \to \mathsf{Set}$.
- Given a locally small category $\mathcal{A}$, we can take a pair of objects (A_1, A_2) and obtain a set $\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)$. This turns out to be a functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$.

Finally, the most important example (WRT this course):

- In algebraic topology, we build groups like $H_1(X)$, $\pi_1(X)$ associated to topological spaces. All these group constructions are functors $\mathsf{Top} \to \mathsf{Grp}$.

Definition 1.5 (Functors)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* F takes every object of $\mathcal{A}$ to an object of $\mathcal{B}$. In addition, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. Refer to the commutative diagram:

$$\begin{tikzcd}
 & A_1 \arrow[dd, "f"'] & & B_1 = F(A_1) \arrow[dd, "F(f)"] & \\
\mathcal{A} \ni & \arrow[rr, dotted, "F"] & & & \in \mathcal{B} \\
 & A_2 & & B_2 = F(A_2) &
\end{tikzcd}$$

Functors also satisfy the following requirements:

- Identity arrows get sent to identity arrows, that is, for each identity arrow id_A, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
- Functors respect composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3$ are arrows in $\mathcal{A}$, then $F(g \circ f) = F(g) \circ F(f)$.

More precisely, these are covariant functors. A contravariant functor F reverses the direction of arrows, so that F sends $f \colon A_1 \to A_2$ to $F(f) \colon F(A_2) \to F(A_1)$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ instead. A category $\mathcal{A}$ has an opposite category $\mathcal{A}^{\text{op}}$ with the same objects and with $\mathcal{A}^{\text{op}}(A_1, A_2) = \mathcal{A}(A_2, A_1)$. A contravariant functor $F \colon \mathcal{A} \to \mathcal{B}$ is just a covariant functor $\mathcal{A}^{\text{op}} \to \mathcal{B}$.

Example 1.7. We have already talked about *free* and *forgetful* functors in Example 1.3: the forgetful functors are functors from spaces to sets (the underlying set of a group) and free functors are from sets to spaces (the basis set forming a vector space).

- Another example of a forgetful functor is a functor $\mathsf{CRing} \to \mathsf{Grp}$ by sending a ring R to its abelian group $(R, +)$.

- Another example of a free functor is a functor $\mathsf{Set} \to \mathsf{Grp}$ by taking the free group generated by a set S (who would have known this is free?)

Here is a cool example: functors preserve isomorphism. If two groups are isomorphic, then they must have the same cardinality. In the language of category theory, this can be expressed as such: if $G \cong H$ in Grp and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor, then $U(G) \cong U(H)$. We can generalize this to *any* functor and category!

Theorem 1.1. *If $A_1 \cong A_2$ are isomorphic objects in $\mathcal{A}$ and $F\colon \mathcal{A} \to \mathcal{B}$ is a functor then*

$$F(A_1) \cong F(A_2).$$

Proof. Let's go diagram chasing!

$$\begin{tikzcd}
\mathcal{A} \ni & A_1 \arrow[d, shift right, "f"'] \arrow[rr, dotted, "F"] & & B_1 = F(A_1) \arrow[d, shift right, "F(f)"'] & \in \mathcal{B} \\
& A_2 \arrow[u, shift right, "g"'] & & B_2 = F(A_2) \arrow[u, shift right, "F(g)"'] &
\end{tikzcd}$$

The main idea of the proof follows from the fact that functors preserve composition and the identity map. ⊠

This is very very useful for us (people who are doing algebraic topology) because functors will preserve isomorphism between spaces (we get that homotopic spaces have isomorphic fundamental groups).

Note. As a meme (or not really, but it's still funny), we can construct the category Cat whose objects are categories and arrows are functors.

§1.6 Homotopy Categories and Homotopy Equivalence

Let Top_* be the category of pointed topological spaces. Then the fundamental group gives a functor $\mathsf{Top}_* \to \mathsf{Grp}$. When we have a suitable relation of homotopy between maps in a category $\mathcal{C}$, we define the homotopy category $\mathsf{Ho}(\mathcal{C})$ to be the category sharing the same objects as $\mathcal{C}$, but morphisms the homotopy classes of maps. On Top_*, we require homotopies to map basepoint to basepoint, and we get the homotopy category hTop_* of pointed spaces.

Homotopy equivalences in $\mathcal{C}$ are isomorphisms in $\mathsf{Ho}(\mathcal{C})$. More concretely, recall that a map $f\colon X \to Y$ is a homotopy equivalence if there is a map $g\colon Y \to X$ such that both $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In the language of category theory, we can obtain the analogous notion of a pointed homotopy equivalence. Functors carry isomorphisms to isomorphisms, so then the pointed homotopy equivalence will induce an isomorphism of fundamental groups. This also holds, but less obviously, for the category of non pointed homotopy equivalences.

Theorem 1.2. *If $f\colon X \to Y$ is a homotopy equivalence, then*

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is an isomorphism for all $x \in X$.

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. By our homotopy invariance diagram, we see that the composites

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x))$$

and

$$\pi_1(Y, y) \xrightarrow{g_*} \pi_1(X, g(y)) \xrightarrow{f_*} \pi_1(Y, (f \circ g)(y))$$

are isomorphisms determined by paths between basepoints given by chosen homotopies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Then in each displayed composite, the first map is a monomorphism and the second is an epimorphism. Taking $y = f(x)$ in the second composite, we see that the second map in the first composite is an isomorphism. Therefore so is the first map, and we are done. ⊠

A space X is said to be contractible if it is homotopy equivalent to a point.

Corollary 1.1. *The fundamental group of a contractible space is zero.*

§1.7 Natural Transformations

We talked about maps between objects which led to categories, and then maps between categories which lead to functors. Now let's talk about maps between functors, the natural transformation: this is actually not too strange (recall the homotopy, a "deformation" from a map to another map).

In this case, we also want to pull a map (functor) F to another map G by composing a bunch of arrows in the target space $\mathcal{B}$.

Definition 1.6 (Natural Transformations)**.** Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be two functors. A *natural transformation* $\alpha\colon F \to G$ denoted

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B}
\end{tikzcd}$$

consists of, for each $A \in \mathcal{A}$ an arrow $\alpha_A \in \mathrm{Hom}_{\mathcal{B}}(F(A), G(A))$, which is called the component of α at A. Pictorially, it looks like this:

$$\begin{tikzcd}
 & F(A) \in \mathcal{B} \arrow[dd, "\alpha_A"] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\
 & G(A) \in \mathcal{B}
\end{tikzcd}$$

The α_A are subject to the "naturality" requirement such that for any $A_1 \xrightarrow{f} A_2$, the following diagram commutes:

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\
G(A_1) \arrow[r, "G(f)"'] & G(A_2)
\end{tikzcd}$$

The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (like in a homotopy from f to g the point $f(t)$ gets deformed to the point $g(t)$ continuously). Think of f representing the homotopy and the basepoints being $F(A_1), G(A_1)$ to $F(A_2), G(A_2)$.

Natural transformations can be composed. Take two natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & F(A) \arrow[d, "\alpha_A"] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[r, dotted, "G"] \arrow[dr, dotted, "H"'] & G(A) \arrow[d, "\beta_A"] \\
 & H(A)
\end{tikzcd}$$

We can also construct inverses: suppose α is a natural transformation such that α_A is an isomorphism for each A. Then we construct an inverse arrow β_A in the following way:

$$\begin{tikzcd}
 & F(A) \in \mathcal{B} \arrow[dd, shift right, "\alpha_A"'] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\
 & G(A) \in \mathcal{B} \arrow[uu, shift right, "\beta_A"']
\end{tikzcd}$$

We say α is a *natural isomorphism*. Then $F(A) \cong G(A)$ *naturally* in A (and β is an isomorphism too!) We write $F \cong G$ to show that the functors are naturally isomorphic.

Example 1.8. If $F\colon \mathsf{Set} \to \mathsf{Grp}$ is the free functor that sends a set to the free group on such set and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor sending a free group to its generating set, then we have a natural inclusion of $S \hookrightarrow UF(S)$. The functors F and U are left and right adjoint to each other, in the sense that we have a natural isomorphism

$$\mathsf{Grp}(F(S), A) \cong \mathsf{Set}(S, U(A))$$

for a set S and an abelian group A. This expresses the "universal property" of free objects: a map of sets $S \to U(A)$ extends uniquely to a homomorphism of groups $F(S) \to A$.

Definition 1.7. Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there are functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $FG \to \mathrm{Id}$ and $GF \to \mathrm{Id}$, where the Id are the respective identity functors.

§1.8 The Yoneda lemma (todo)

Definition 1.8 (The functor category)**.** The *functor category* of two categories $\mathcal{A}$ and $\mathcal{B}$, denoted $[\mathcal{A}, \mathcal{B}]$ is defined as follows:

- The objects of $[\mathcal{A}, \mathcal{B}]$ are (covariant) functors $F\colon \mathcal{A} \to \mathcal{B}$, and
- The morphisms are natural transformations $\alpha\colon F \to G$.

todo

§1.9 Limits and colimits (todo)

Let $\mathcal{D}$ be a small category and $\mathcal{C}$ be any category. A $\mathcal{D}$-shaped diagram in $\mathcal{C}$ is a functor $F\colon \mathcal{D} \to \mathcal{C}$. A morphism $F \to F'$ of $\mathcal{D}$-shaped diagrams is a natural transformation, and we have the category $\mathcal{D}[\mathcal{C}]$ of $\mathcal{D}$ shaped diagrams in $\mathcal{C}$. Any object C of $\mathcal{C}$ determines the constant diagram

§2 Free Groups and Group Theory

Not to be confused with free *abelian* groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39,40 and Hatcher §1.2.

I've decided to expand this section to include any miscellaneous group theory that I may not have covered/forgot. What texts they follow will probably be at the beginning of each subsection.

§2.1 Words and Reduced Words

Let A_i be a set of elements (not necessarily finite). We say A is an *alphabet* and think of the $a_i \in A$ as *letters*. Symbols of the form a_i^n are *syllables* and *words* are a finite string of syllables. We denote the *empty word* 1 as the word with no syllables.

Example 2.1. Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to a_i^{m+1} (*elementary contractions*) or replacing a_i^0 by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a *reduced word*.

Example 2.2. The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

§2.2 Free Groups

Denote the set of all reduced words from our alphabet A as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words w_1 and w_2 in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of w_2 onto w_1.

Example 2.3. If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet A. Man, the weather outside today is nice.

Definition 2.1 (Free Group)**.** The group $F[A]$ described above is the *free group generated by A*.

Sometimes we have a group G and a generating set $A = \{a_i \mid \in I\}$, and we want to know whether or not G is *free* on $\{a_i\}$, that is, G is the free group generated by $\{a_i\}$.

Definition 2.2 (Free Generators)**.** If G is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi\colon G \to F[A]$ such that $\phi(a_i) = a_i$, then G is *free on A*, and the a_i are *free generators of G*. A group is *free* if it is free on some nonempty set A.

Oh you'll be free... free indeed...

Example 2.4. $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on n letters" as opposed to the "free group on n generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

Example 2.5. $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

Theorem 2.1. *If G is free on A and B, then A and B have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

Proof. Refer "to the literature". ☒

Definition 2.3 (Rank)**.** If G is free on A, then the number of letters in A is the *rank of the free group G*.

Theorem 2.2. *Two free groups are isomorphic if and only if they have the same rank.*

Proof. Immediate. ☒

Theorem 2.3. *A nontrivial proper subgroup of a free group is free.*

Proof. Back "to the literature". ☒

Example 2.6. Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

§2.3 Homomorphisms of Free Groups

Theorem 2.4. *Let G be generated by $A = \{a_i \mid \in I\}$ and let G' be any group. If $a_i{}'$ for $i \in I$ are any elements in G' not necessarily distinct, then there is at most one homomorphism $\phi\colon G \to G'$ such that $\phi(a_i) = a_i{}'$. If G is free on A, then there is exactly one such homomorphism.*

Proof. Let ϕ be a homomorphism from G into G'such that $\phi(a_i) = a_i{}'$. Then any $x \in G$ can be written as a finite product of the generators a_i, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the a_i not necessarily distinct. Since ϕ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i{}'$.

Now suppose that G is free on A, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi\colon G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in G, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ⊠

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

Corollary 2.1. *Every group G' is a homomorphic image of a free group G.*

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 2.4 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i{}'$. Clearly the image of G under ψ is all of G'. ⊠

Only the free group on one letter is abelian.

§2.4 Free Products of Groups

Definition 2.4 (Free Products)**.** As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of w_1 and the beginning of w_2 belong to the same G_α? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of a and b are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, \ldots$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha \colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi \colon *_\alpha G_\alpha \to H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under φ.

Example 2.7. For a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

§2.5 Group Presentations

Apparently, I never took group theory. Let's talk about group presentations!

Motivation: form a group by giving generators and having them follow certain relations. We want the group as free (free indeed) as possible on these generators.

Example 2.8. Suppose G has generators x and y and is *free except for the relation* $xy = yx$, or $xyx^{-1}y^{-1} = 1$. This makes sure G is abelian, and so G is isomorphic to $F[\{x, y\}]$ modulo its commutator subgroup, the smallest normal subgroup containing $xyx^{-1}y^{-1}$. This is because any normal subgroup containing $xyx^{-1}y^{-1}$ gives rise to an abelian factor group and thus contians the commutator subgroup (by a previous theorem).

This example illustrates what we want: let $F[A]$ be a free group and we want a new group as free as possible, with certain equations satisfied. We can always write these equations with the RHS equal to 1, so we consider the equations to be $r_i = 1$ for $i \in I$, where $r_i \in F[A]$. If $r_i = 1$, then

$$x(r_i^n)x^{-1} = 1$$

for any $x \in F[A]$, $n \in \mathbb{Z}$. Any product of elements equal to 1 again equals 1, so any finite product of the form

$$\prod_j x_j \left(r_{i_j}^{n_j}\right) x_j^{-1}$$

where r_{i_j} need not be distinct equals 1 in the new group. It can be seen that the set of all these finite products is a normal subgroup R of $F[A]$. Then any group that looks like $F[A]$ given $r_i = 1$ also has $r = 1$ for all $r \in R$. But $F[A]/R$ looks like $F[A]$, except that R has been collapsed to form the identity 1. Hence the group we are after is (at least isomorphic to) $F[A]/R$. We can view this group as described by the generating set A and the set $\{r_i \mid i \in I\}$, abbreviated $\{r_i\}$.

Definition 2.5 (Group Presentations)**.** Let A be a set and $\{r_i\} \subseteq F[A]$. Let R be the least normal subgroup of $F[A]$ containing the r_i. An isomorphism ϕ of $F[A]/R$ onto a group G is a *presentation of* G. The sets A and $\{r_i\}$ give a *group presentation*. The set A is the set of *generators for the presentation* and each r_i is a *relator*. Each $r \in R$ is a *consequence of* $\{r_i\}$. An equation $r_i = 1$ is a *relation*. A *finite presentation* is one in which both A and $\{r_i\}$ are finite sets.

Refer back to Example 2.1: $\{x, y\}$ is our set of generators and $xyx^{-1}y^{-1}$ is the only relator. The equation $xyx^{-1}y^{-1} = 1$ or $xy = yx$ is a relation— this was an example of a finite presentation.

§2.6 Free Abelian Groups (todo)

todo

§2.7 Semidirect products and Commutators(todo)

I had an existential crisis when Dr. Allcock said to simply observe that one group is a semidirect of another by such and such group. These notes will follow Dummit and Foote §5.

The direct product is what you think it is: the set of n-tuples with the group operation done componentwise.

Definition 2.6 (Commutators)**.** Let G be a group and $x, y \in G$. Let A, B be nonempty subsets of G. Then

1. Define $[x, y] = x^{-1}y^{-1}xy$ as the *commutator* of x and y.
2. Define $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$, the group generated[1] by commutators of elements from A and B.
3. Define $G' = [G : G] = \langle [x, y] \mid x, y \in G \rangle$, the subgroup of G generated by commutators of elements from G, called the *commutator subgroup* of G.

[1]Generators and presentations are starting to blur the line for me...

The commutator of x and y is 1 iff x and y commute, hence the name.

Proposition 2.1. *The factor group G/G' is abelian. Furthermore, G/G' is the largest abelian quotient of G in the sense that if $H \trianglelefteq G$ and G/H is abelian, then $G' \leq H$. Conversely, if $G' \leq H$, then $H \trianglelefteq G$ and G/H is abelian.*

Proof. Let $xG', yG' \in G/G'$. Since the commutator $[x, y] \in G'$ collapses to zero, we have

$$\begin{aligned}(xG')(yG') &= (xy)G' \\ &= (yx[x,y])G' \\ &= (yx)G' \\ &= (yG')(xG').\end{aligned}$$

So G/G' is abelian. Now supposed $H \trianglelefteq G$ and G/H is abelian. Then for all $x, y \in G$ we have $(xH)(yH) = (yH)(xH)$, so

$$\begin{aligned}1H &= (xH)(xH)^{-1}(yH)(yH)^{-1} \\ &= (xH)^{-1}(yH)^{-1}(xH)(yH) \\ &= (x^{-1}y^{-1}xy)H \\ &= [x,y] \in H,\end{aligned}$$

which implies $[x, y] \in H$ for all $x, y \in G$. So $G' \leq H$. Conversely, if $G' \leq H$, then every subgroup of G/G' is normal, in particular, $H/G' \trianglelefteq G/G'$. We have $H \trianglelefteq G$ by the Lattice Isomorphism Theorem, and by the Third Isomorphism Theorem, we have

$$G/H \cong (G/G')/(H/G').$$

Since G/H is isomorphic to a quotient of the abelian group G/G', G/H must be abelian. ⊠

Why does this work? We mod out by the stuff we don't like: in this case, all the commutators collapse to the identity, so elements in the quotient group commute.

§3 The Fundamental Group

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups. These notes follow Hatcher §1.2, Lee §10, and May §2.7.

§3.1 Fundamental group of the circle(todo)

If this is a first introduction to fundamental groups, then our first fundamental group of real interest is $\pi_1(S^1) = \mathbb{Z}$. Before we do this, let's do a quick calculation to show $\pi_1(\mathbb{R}) = 0$. Take the origin as a convenient basepoint. Define $k\colon \mathbb{R} \times I \to \mathbb{R}$ by $k(s,t) = (1-t)s$. Then k is a homotopy from the identity to the constant map at 0. For a loop $f\colon I \to \mathbb{R}$ at 0, define $h(s,t) = k(f(s),t)$. Then f is equivalent to a constant c_0 by the homotopy h.

Now let's talk about circles: we can view S^1 as the circle group (let's denote it U^1), that is,

$$U^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Multiplication is continuous, so this is a topological group. Take the identity 1 as a convenient basepoint for S^1.

Theorem 3.1. *We have the fundamental group of a circle isomorphic to the integers, that is,*

$$\pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. For all $n \in \mathbb{Z}$, define a loop f_n in S^1 by $f_n(s) = e^{2\pi i n s}$. This is the same as composing the map $I \to S^1$, $s \mapsto e^{2\pi i s}$ and the nth power map on S^1. If we identiy the boundary points 0 and 1 of I, then the first map $(I \to S^1)$ induces the evident identification of $I/\partial I$ with S^1. By complex exponentiation, we have $[f_m][f_n] = [f_{m+n}]$: define a homomorphism $i\colon \mathbb{Z} \to \pi_1(S^1, 1)$ by $i(n) = [f_n]$. We claim i is an isomorphism. The main idea is to use the fact that (locally) S^1 looks like $\mathbb{R}$, ie, S^1 is a 1-manifold.

FINISH LATER ⊠

§3.2 The van Kampen Theorem (Hatcher)

Let's take a space X and say it's the union of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi\colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is often onto but in general, we can expect Φ to have a nontrivial kernel.

For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both of these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Van Kampen says under fairly broad hypotheses that this determines all of Φ.

Theorem 3.2. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi\colon *_\alpha\, (A_\alpha) \to \pi_1(X)$$

is onto. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N.$$

Example 3.1 (Wedge Sums)**.** I like the visual of the wedge sum but the terminology of the smash product. If only we could keep the `\vee` symbol ($\vee$) and say we "smash the spaces together" at a point.

We define the wedge sum $\bigvee_\alpha X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ as the disjoint union $\amalg_\alpha X_\alpha$ with all the basepoints x_α identified to a single point. If each $x_\alpha$is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Then by van Kampens theorem,

$$\Phi\colon *_\alpha\, \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$$

is an isomorphism, provided each X_α is path-connected, hence also each A_α. Therefore for a wedge sum of circles, $\pi_1(\bigvee_\alpha S^1_\alpha)$ is a free group, the free product of copies of $\mathbb{Z}$.

I know it always helps to see something done somewhere else. For me, the above definition fails to make any sense at all whatsoever. So, let's revisit van Kampens from two more lens: one from the words of Lee (*Introduction to Topological Manifolds*) and another from the categorical perspective.

§3.3 The van Kampen Theorem (Lee)

Let's say we have a space X that's made up of the union of two open subsets $U, V \subseteq X$. We know how to compute the fundamental groups of U, V, and $U \cap V$ (each of which is path-connected). Every loop can be written as a product of loops in U or V (visualized as the free product $\pi_1(U) * \pi_1(V)$), but any loop in $U \cap V$ only represents a single element of $\pi_1(X)$, even though it represents two distinct elements of the free product (one in $\pi_1(U)$ and one in $\pi_1(V)$). So $\pi_1(X)$ can be though of as the quotient of this free product modulo some relations from $\pi_1(U \cap V)$ that express this redundancy.

Let's do some setup so we can state a precise version of van Kampens. Let X be a topological space and $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V \neq \emptyset$. Let $q \in U \cap V$. Then the four inclusion maps shown below,

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

induce fundamental group homomorphisms as shown below.

$$\begin{tikzcd} & \pi_1(U,q) \arrow[dr, "k_*"] & \\ \pi_1(U \cap V, q) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & \pi_1(X,q) \\ & \pi1(V,q) \arrow[ur, "l_*"'] & \end{tikzcd}$$

Now insert the free product $\pi_1(U,q) * \pi_1(V,q)$ in the middle of the diagram and let $\iota_U \colon \pi_1(U,q) \hookrightarrow \pi_1(U,q) * \pi_1(V,q)$ and $\iota_V \colon \pi_1(V,q) \hookrightarrow \pi_1(U,q) * \pi_1(V,q)$ be the canonical injections. By the characteristic property (unique induced homomorphisms) of the free product, k_* and l_* induce a homomorphism $\Phi \colon \pi_1(U,q) * \pi_1(V,q) \to \pi_1(X,q)$ such that the right half of the following diagram commutes:

$$\begin{tikzcd} & \pi_1(U,q) \arrow[d, "\iota_U"] \arrow[dr, "k_*"] & \\ \pi_1(U \cap V, q) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] \arrow[r, dotted, "F"] & \pi_1(U,q) * \pi_1(V,q) \arrow[r, "\Phi"] & \pi_1(X,q) \\ & \pi_1(V,q) \arrow[u, "\iota_V"'] \arrow[ur, "l_*"'] & \end{tikzcd}$$

Finally, define a map $F \colon \pi_1(U \cap V, q) \to \pi_1(U,q) * \pi_1(V,q)$ by setting $F(\gamma) = (i_*\gamma)^{-1}(j_*\gamma)$[2]. Let $\overline{F(\pi_1(U \cap V, q))}$ denote the *normal closure*[3] of the image of F in $\pi_1(U,q) * \pi_1(V,q)$.

Theorem 3.3 (Seifert-Van Kampen). *Let X be a topological space. Suppose $U, V \subseteq X$ are open, $U \cap V = X$, and U, V, and $U \cap V$ are path-connected. Then for any $q \in U \cap V$, the homomorphism Φ is surjective, and its kernel is $\overline{F(\pi_1(U \cap V, q))}$. Therefore we have*

$$\pi_1(X,q) \cong \pi_1(U,q) * \pi_1(V,q) / \overline{F(\pi_1(U \cap V, q))}.$$

When the fundamental groups in question are finitely presented, the theorem has a useful reformulation in terms of generators and relations.

Corollary 3.1. *In addition to the hypothesis of van Kampen, assume that the fundamental groups of U, V, and $U \cap V$ have the following finite presentations:*

$$\begin{aligned} \pi_1(U,q) &\cong \langle \alpha_1, \cdots, \alpha_m \mid \rho_1, \cdots, \rho_r \rangle; \\ \pi_1(V,q) &\cong \langle \beta_1, \cdots, \beta_n \mid \sigma_1, \cdots, \sigma_s \rangle; \\ \pi_1(U \cap V, q) &\cong \langle \gamma_1, \cdots, \gamma_p \mid \tau_1, \cdots, \tau_t \rangle. \end{aligned}$$

[2] F is not a homomorphism.

[3] the *normal closure* of a set means the smallest normal subgroup that contains such set.

Then $\pi_1(X,q)$ has the presentation

$$\pi_1(X,q) \cong \langle \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \mid \rho_1, \cdots, \rho_r, \sigma_1, \cdots, \sigma_s, u_1 = v_1, \cdots, u_p = v_p \rangle$$

where for each $a = 1, \cdots, p$, u_a is an expression for $i_*\gamma_a \in \pi_1(U,q)$ in terms of the generators $\{\alpha_1, \cdots, \alpha_m\}$, and v_a similarly expresses $j_*\gamma_a \in \pi_1(V,q)$ in terms of $\{\beta_1, \cdots, \beta_n\}$.

§3.4 The fundamental groupoid

We backtrack a little to talk about categorical nonsense. This doesn't fit too well with the section on category theory, so it's here. These will follow May §2.5.

We often talk of pointed spaces, but it would to nice to talk about spaces without making such a choice. We define the fundamental groupoid $\prod(X)$ of a space X to be the category whose objects are the points of X and whose morphism $x \to y$ are the equivalence classes of paths from x to y[4]. Then the set of endomorphisms of the object x is the fundamental group $\pi_1(X,x)$.

We say "groupoid" because a group is simply a groupoid with only one object (the class of morphisms or symmetries on an object). However, the category of groupoids has several objects. We also defined groupoids as categories whose morphisms are all isomorphisms. If morphisms are functors, then we have the category Grpd of groupoids. So we can see $\prod$ as a functor $\mathsf{Top}_* \to \mathsf{Grpd}$.

Let's talk about skeletons. We have the skeleton of a category $\mathcal{C}$ denoted by $\mathrm{sk}(\mathcal{C})$. This is a "full" subcategory with one object from each isomorphism class of objects of $\mathcal{C}$, "full" meaning that the morphisms between two objects of $\mathrm{sk}(\mathcal{C})$ are all of the morphisms between these objects in $\mathcal{C}$. The inclusion functor $J\colon \mathrm{sk}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of categories. We can find an inverse functor $F\colon \mathcal{C} \to \mathrm{sk}(\mathcal{C})$ by letting $F(A)$ be the unique object in $\mathrm{sk}(\mathcal{C})$ that is isomorphic to A, choosing an isomorphism $\alpha_A\colon A \to F(A)$, and defining $F(f) = \alpha_B \circ f \circ \alpha_A^{-1}$ for a morphism $f\colon A \to B$ in $\mathcal{C}$. Choose α to be the identity morphisms if $A \in \mathrm{sk}(\mathcal{C})$, then $FJ = \mathrm{id}_{\mathrm{sk}(\mathcal{C})}$; the α_A specify a natural isomorphism $\alpha\colon \mathrm{id} \to JF$.

A category is connected if any two objects can be connected by a sequence of morphisms. Then a groupoid is connected iff any two of its objects are isomorphic. The group of endomorphisms of any object C is then a skeleton of $\mathcal{C}$, so we can generalize our results about skeletons to give the relationship between a fundamental group and a fundamental groupoid of a path connected space X.

Proposition 3.1. *Let X be a path connected space. Then for each $x \in X$, the inclusion $\pi_1(X,x) \to \prod(X)$ is an equivalence of categories.*

Proof. View $\pi_1(X,x)$ as a groupoid with on object x: then $\pi_1(X,x)$ is a skeleton of $\prod(X)$ and we are done. ⊠

May's presentation and proofs are very concise and elegant. I like this.

[4]Recall Example 1.4 of a group being realized as a category with all its arrows isomorphisms.

§4 Covering Spaces

Today we talk about covering spaces, another central topic in algebraic topology. The notes will follow various texts, including Hatcher, Lee, and May.

§4.1 Some preliminary definitions

Sometimes we need to know what words mean so we can talk about big concepts. These notes will follow May §3. We can talk about the theory of covering spaces on *locally contractible* spaces that are path-connected, that is, spaces with a base of contractible spaces, that is, open sets that are contractible when viewed as a space under the subspace topology. However, to get the full picture, we must talk about *locally path-connected* spaces.

Definition 4.1 (Locally path-connected)**.** A space X is *locally path-connected* if for any $x \in X$ and any neighborhood U of x, there exists a smaller neighborhood V of x, with each of whose points can be connected to x by a path in U. We could also say X has a base consisting of open sets that are path-connected (under the subspace topology).

Note that if X is connected and locally path-connected, then it is path-connected. From now on[5], we assume that spaces are connected and locally path-connected. Let's look at how May defines covering spaces.

Definition 4.2 (Covering space)**.** A map $p\colon E \to B$ is a covering (or cover, covering space) if it is onto and if each point $b \in B$ has an open neighborhood V such that each component of $p^{-1}(V)$ is open in E and is mapped homeomorphically onto V by p. We say that a path connected open subset V with this property is a fundamental neighborhood of B. We call E the total space, B the base space, and $F_b = p^{-1}(b)$ a fiber of the covering p.

Some notes: in other texts, we have

- covering $\to$ covering map,
- U is a fundamental neighborhood $\to$ U is evenly covered,
- total space $\to$ covering space,
- base space $\to$??,
- $F_b = p^{-1}(b)$ is a fiber of p $\longrightarrow$ F_b is the preimage of b (points) in the union of sheets of $\widetilde{X}$ over U_b.

Another definition that will come in handy when classifying covering spaces is the notion of something being semilocally simply-connected, that is, given a "hole" (of genus one), we can always find a neighborhood contained in that hole such that the fundamental group induced by the inclusion map is trivial in π_1 of the entire space.

Definition 4.3 (Semilocally simply-connected)**.** A space X is *semilocally simply-connected* if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

We'll define this again when we need it, and talk a little more about what it means for a space to be semilocally simply-connected.

This is kind of out of place, but now we'll state Lebesgue's number lemma. It's useful when dealing with compact metric spaces.

Lemma 4.1 (Lebesgue's number lemma)**.** *If a metric space (X, d) is compact and we have an open cover of X, then there exists a $\delta > 0$ such that every subset of X having a diameter less than δ is contained in some member of the cover. We say δ is the* Lebesgue number *of such cover.*

Proof. If the subcover is trivial then any $\delta > 0$ will suffice. Otherwise, if $\bigcup_{i \in I} A_i$ is a finite subcover, then for $i \in I$, define $C_i := X \setminus A_i$ (note that C_i is nonempty since the subcover is nontrivial). Define a function

$$f\colon X \to R, \quad x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

Since f is continuous on a compact set, it obtains a minimum δ. The key thing to note is that every x is in some A_i, so by the extreme value theorem $\delta > 0$. To show that this δ is indeed the Lebesgue number of the cover, let $x_0 \in Y$, where $\operatorname{diam}(Y) < \delta$, such that $Y \subseteq B(x_0, \delta)$. Since $f(x_0) \geq \delta$, there exists at least one i such that $d(x_0, C_i) \geq \delta$. But then $B(x_0, \delta) \subseteq A_i$, and so $Y \subseteq A_i$. ⊠

[5] By this, we mean any sections following May.

§4.2 Covering spaces

These notes will follow Hatcher §1.3.

We've already seen these briefly when we calculated $\pi_1(S^1)$, using the projection $\mathbb{R} \to S^1$ of a helix onto a circle. Covering spaces can be used to calculated fundamental groups of other spaces as well, but the connection runs much deeper than this. We can talk about algebraic aspects of the fundamental group through the geometric language of covering spaces, exemplified in one of the main results in this section: a one to one correspondence between connected covering spaces of a space X and subgroups of $\pi_1(X)$ (spoilers, smh). This is really really similar to Galois theory, where we looked at the towers of field extensions and related them to the subgroup lattice of the Galois group of automorphisms[6].

Definition 4.4 (Covering space)**.** A *covering space* of a space X is a space $\widetilde{X}$ together with a map $p\colon \widetilde{X} \to X$ (we say p is a *covering map*) satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p. Then we say U is *evenly covered* and the disjoint open sets in $\widetilde{X}$ that project homeomorphically to U by p are called *sheets* of $\widetilde{X}$ over U.

If U is connected these sheets are the connected components of $p^{-1}(U)$ so they're uniquely determined by U. If U is not connected, however, the decomposition of U into sheets may not be unique. $p^{-1}(U)$ is allowed to be empty, so p doesn't have to be onto. The number of sheets over U can be given by the cardinality of $p^{-1}(x)$, given $x \in U$. This number is a constant if X is connected.

Example 4.1. A prototypical example (or way to wrap your head around) this section is the helix embedded in $\mathbb{R}^3$: if you think of it projecting on a circle, then $p^{-1}(U)$ is just $\amalg_\alpha U_\alpha$, where each U_α corresponds to the U of a coil or wind of the helix.

Example 4.2. Another example is the helicoid surface $S \subseteq \mathbb{R}^3$ given by $(s\cos 2\pi t, s\sin 2\pi t, t)$ for $(s,t) \in (0,\infty) \times \mathbb{R}$. This projects onto $\mathbb{R}^2 \setminus \{0\}$ via the map $(x,y,z) \mapsto (x,y)$, and defines a covering space $p\colon S \to \mathbb{R}^2 \setminus \{0\}$ since each point of $\mathbb{R}^2 \setminus \{0\}$ is contained in an open disk U in $\mathbb{R}^2 \setminus \{0\}$ with $p^{-1}(U)$ consisting of countably many disjoint open disks in S projecting homeomorphically onto U. (I can't really see this example...)

Example 4.3. We also have the map $p\colon S^1 \to S^1$, $p(z) = z^n$ where we view z as a complex number with $|z| = 1$ and n any positive integer[7]. This projection is as described in the footnote, but intersects itself in $n-1$ points (that one can't really imagine as intersections). To see this without the defect, embed S^1 in the boundary torus of a solid torus $S^1 \times D^2$ such that it winds n times monotonically around the S^1 factor without self-intersections, then restrict the projection $S^1 \times D^2 \to S^1 \times \{0\}$ to this embedded circle. What?

We usually restrict our attention to connected covering spaces, as these contain all the interesting examples.

§4.3 The covering spaces of $S^1 \vee S^1$ (todo figures)

Covering spaces of $S^1 \vee S^1$ form a rich family that demonstrate the general theory very concretely. For convenience, let $X = S^1 \vee S^1$. View it as a graph with one vertex and two edges, with the edges labeled a and b.

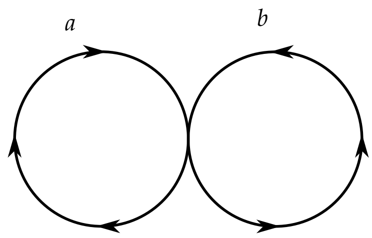

Figure 1: The graph of $S^1 \times S^1$.

Let $\widetilde{X}$ be any other graph with four edges connected to each point, like X at its singular vertex, and that each edge has been assigned an orientation like the ones assigned to each edge of X. That is, for each vertex there are two a-edges and b-edges oriented toward and away from the vertex. Help I can't include figures that are the proper size! Let's call $\widetilde{X}$ a 2-oriented graph.

[6] I actually know this! Thank goodness for an entire semester of algebra to understand an example.
[7] Something about the order of z I realized when thinking about this example: z^n means z coils around in S^1 n times. So if z was a fifth root of unity, the covering space would be a circle with five coils projecting onto S^1. Now what if z has infinite order. Can z even have infinite order? I'm not entirely sure...
Edit: elements that are irrational multiples of 2π have infinite order. So does that mean it never winds back to itself? How is this isomorphic to S^1?

Given a 2-oriented graph $\widetilde{X}$ we can construct a map $p\colon \widetilde{X} \to X$ that sends all vertices of $\widetilde{X}$ to the vertex of X, and all edges of $\widetilde{X}$ to the edge of X with the same label. Say p is a homeomorphism on the regions bounded by the edges, and preserves the orientation of the edges. Then p is a covering map. Conversely, every covering space of X is a graph that inherits a 2-orientation from X. It can be shown that every graph with four edges at each vertex can be 2-oriented: the proof follows from graph theory. We could also generalize this to n-oriented graphs, which are covering spaces of the wedge sum of n circles.

How would we generate a simply-connected covering space of X? Start with the open intervals $(-1, 1)$ in $\mathbb{R}^2$ (one per coordinate axis). Then for a fixed λ, $0 < \lambda < 1/2$, say $\lambda = 1/3$, adjoin four open segments of length 2λ to the ends of the previous segments, and shift each back by a distance of λ. These new adjoined segments are perpendicular and bisected by the old ones: continue with four more new segments of distance $2\lambda^2$ at a distance λ^2 to the (now 12) end segments, and so on. Then at the n-th iteration, we would be adding open segments of length λ^{2n-1} at a distance λ^{n-1} from the previous endpoints. Then the union of the segments is a graph (the Cayley graph of the fundamental group of S^1!), with vertices the intersections, labeling horizontal edges a and orienting them to the right, and vertical edges b, orienting them upward.

This covering space is called the *universal cover* of X, because it covers every connected covering space of X.

§4.4 More on covering spaces

These notes will follow Lee §11.

The definition of a covering space is the same as Hatcher except: the covering space $\widetilde{X}$ must be connected. Once again, the only interesting covering spaces are connected ones, and so we eliminate the need to frit fret around about details when introducing new theorems and just make sure covering spaces are connected in the definition.

Example 4.4. The exponential quotient map $\varepsilon\colon \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a covering map. Another example: define $E\colon \mathbb{R}^n \to \mathbb{T}^n$ by

$$E(x_1, \cdots, x_n) = (\varepsilon(x_1), \cdots, \varepsilon(x_n)).$$

We will show in an exercise that a product of covering maps is a covering map. So E is a covering map.

Example 4.5. Define a map $\pi\colon S^n \to \mathbb{R}P^n$ (where $n \geq 1$) by sending each point x in the sphere to the line through the origin and x, thought of as a point in $\mathbb{R}P^n$. Then π is a covering map, and the fiber of each point in $\mathbb{R}P^n$ is a pair of antipodal points $\{x, -x\}$.

Lemma 4.2 (Elementary properties of covering maps). *Every covering map is a local homeomorphism, an open map, and a quotient map. An injective covering map is a homeomorphism.*

Proof. Left as an exercise to the reader. ⊠

Proposition 4.1. *For any covering map $p\colon \widetilde{X} \to X$, the cardinality of each fiber $p^{-1}(q)$ is the same for any fiber.*

Proof. If U is any evenly covered open set in X, each sheet in $p^{-1}(U)$ contains exactly one point of each fiber. Then for any $q, q' \in U$, there are one-to-one correspondences

$$p^{-1}(q) \longleftrightarrow \{\text{sheets of } p^{-1}(U)\} \longleftrightarrow p^{-1}(q'),$$

which shows that the number of sheets is constant on U. It follows that the set of points $q' \in X$ such that $p^{-1}(q')$ has the same cardinality as $p^{-1}(q)$ is open. Now let $q \in X$, and let A be the set of points in X whose fibers have the same cardinality as $p^{-1}(q)$. Then A is open, and $X \setminus A$ is open since it's a union of open sets (one open set for each cardinality not equal to $p^{-1}(q)$). Since X is connected and nonempty, we have $A = X$. ⊠

If $p\colon \widetilde{X} \to X$ is a covering map, then the cardinality of any fiber is the *number of sheets* of the covering. For example, the n-th power map $(S^1 \to S^1)$ is an n-sheeted covering, $\pi\colon S^n \to \mathbb{R}P^n$ is a two sheeted covering, and $\varepsilon\colon \mathbb{R} \to S^1$ is a countably sheeted covering.

§4.5 Lifting properties

Here we'll talk about some important lifting properties, that we discussed when we proved that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. Recall: if $p\colon \widetilde{X} \to X$ is a covering map and $\varphi\colon B \to X$ is any continuous map, a *lift* of φ is a continuous map $\widetilde{\varphi}\colon B \to \widetilde{X}$ such that $p \circ \widetilde{\varphi} = \varphi$. See the commutative diagram below for reference.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ B \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Proposition 4.2 (Unique lifting property)**.** *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose B is connected, $\varphi\colon B \to X$ is continuous, and $\widetilde{\varphi}_1 \widetilde{\varphi}_2\colon B \to \widetilde{X}$ are lifts of φ that agree at some point of B. Then $\widetilde{\varphi}_1 \equiv \widetilde{\varphi}_2$, that is, lifts are unique.*

Proof. We show that the set

$$\mathcal{S} = \{b \in B \mid \widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)\}$$

is both open and closed in B, contradicting the connectedness of B if $\mathcal{S}$ is a proper nontrivial subset of B. We conclude that $\mathcal{S}$ must be all of B since $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ agree at a point (so $\mathcal{S}$ is nontrivial) and therefore $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are unique.

Let $b \in \mathcal{S}$ and $U \subset X$ be an evenly covered neighborhood of $\varphi(b)$, and let U_α be the component of p^{-1} containing $\widetilde{\varphi}_1(b) = \widetilde{\varphi}_2(b)$. On the neighborhood $V = \widetilde{\varphi}_1^{-1}(U_\alpha) \cap \widetilde{\varphi}_2^{-1}(U_\alpha)$ of b, we have $\varphi = p \circ \widetilde{\varphi}_1 = p \circ \widetilde{\varphi}_2$. Since p is 1-1 on U_α, this implies $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$ on V, so $\mathcal{S}$ is open.

OTOH, for $b \notin \mathcal{S}$, if U is an evenly covered neighborhood of $\varphi(b)$, there are disjoint components U_1, U_2 of $p^{-1}(U)$ containing $\widetilde{\varphi}_1(b)$, $\widetilde{\varphi}_2(b)$ such that p is a homeomorphism from each U_i to U. Letting $V = \widetilde{\varphi}_1^{-1}(U_1) \cap \widetilde{\varphi}_2^{-1}(U_2)$, we conclude that $\widetilde{\varphi}_1 \neq \widetilde{\varphi}_2$ on V, so $\mathcal{S}$ is closed. This proof is much easier to follow if you trace everything out with all the inverse relations on the commutative diagram above. ⊠

Proposition 4.3 (Path lifting property)**.** *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose $f\colon I \to X$ is any path, and $\widetilde{q}_0 \in \widetilde{X}$ is any point in the fiber of p over $f(0)$. Then there exists a unique lift $\widetilde{f}\colon I \to \widetilde{X}$ of f such that $\widetilde{f}(0) = \widetilde{q}_0$.*

Proof. By the Lebesgue number lemma, n can be chosed large enough that p maps each subinterval $[k/n, (k+1)/n]$ into an evenly covered open subset of X. Starting with $\widetilde{f}(0) = \widetilde{q}_0, \widetilde{f}$ is defined inductively by choosing an evenly covered neighborhood U_k containing $f[k/n, (k+1)/n]$, a local section[8] $\sigma_k\colon U_k \to \widetilde{X}$ such that $\sigma_k(f(k/n)) = \widetilde{f}(k/n)$, and setting $f = \sigma_k \circ f$ on $[k/n, (k+1)/n]$. Because $p \circ \widetilde{f} = (p \circ \sigma_k) \circ f = f$, this is indeed a lift, and it is unique by the unique lifting property. ⊠

Proposition 4.4 (Homotopy lifting property)**.** *Let $p\colon \widetilde{X} \to X$ be a covering map. Suppose $f_0, f_1\colon I \to X$ are path homotopic, and $\widetilde{f}_0, \widetilde{f}_1\colon I \to \widetilde{X}$ are lifts of f_0 and f_1 such that $\widetilde{f}_0(0) = \widetilde{f}_1(0)$. Then $\widetilde{f}_0 \sim \widetilde{f}_1$.*

Proof. If $H\colon f_0 \sim f_1$ is a path homotopy, by the Lebesgue number lemma we can choose n large enough that H maps each square of side $\frac{1}{n}$ into an evenly covered open set. Labeling the squares $S_{ij} = [i/n, (i+1)/n] \times [j/n, (j+1)/n]$, we define a lift $\widetilde{H}$ of H square by square along the bottom row, then the next row, and so on by induction. On each square S_{ij}, set $\widetilde{H} = \sigma \circ H$, for an appropiate local section σ chosen such that the new definition of $\widetilde{H}$ matches the previous one at the corner point $(i/n, j/n)$. Then since two such definitions agree on a line segment (by restricting H to it), they are equal by the unique lifting property.

On the left-hand and right-hand edges of $I \times I$, where $s = 0$ or $s = 1$, $\widetilde{H}$ is a lift of the constant loop and therefore constant. The restriction $\widetilde{H}_0$ to the bottom edge where $t = 0$ is a lift of f_0 starting at $\widetilde{f}_0(0)$, and therefore is equal to $\widetilde{f}_0$, similarly $\widetilde{H}_1 = \widetilde{f}_1$. Therefore $\widetilde{H}$ is the required path homotopy between $\widetilde{f}_0$ and $\widetilde{f}_1$. ⊠

§4.6 Connections to the fundamental group

Back to Hatcher §1.3.

Here are some applications of the lifting properties with respect to the fundamental group.

Proposition 4.5. *The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

Proof. An element of the kernel of p_* is represented by a loop $\widetilde{f}_0\colon I \to \widetilde{X}$ with a homotopy $f_t\colon I \to X$ of $f_0 = p\widetilde{f}_0$ to the trivial loop f_1. By the homotopy lifting property, there is a lifted homotopy of loops $\widetilde{f}_t$ starting with $\widetilde{f}_0$ and ending with a constant loop. Basically, since elements of the kernel start with the same point, and there exist unique lifts to them that are nullhomotopic, we conclude the kernel is trivial and p_* is 1-1.

For the second part of the proposition, loops at x_0 lifting to loops at $\widetilde{x}_0$ represent elements of the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Conversely, a loop representing an element of the image p_* is homotopic to a loop having such a lift, and by the homotopy lifting property, this loop must also have such a lift. ⊠

Proposition 4.6. *The number of sheets (cardinality of a fiber) of a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with X and $\widetilde{X}$ path-connected equals the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$.*

[8] A *local section* of a continuous map is a continuous right inverse defined on some open subset. This exists here by Lee's Lemma 11.7, which shows the existence of local sections of covering maps.

Proof. For a loop g in X based at x_0, let $\widetilde{g}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h \cdot g$ with $[h] \in H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ has the lift $\widetilde{h} \cdot \widetilde{g}$ ending at the same point as $\widetilde{g}$ since $\widetilde{h}$ is a loop ($\widetilde{h}$ denotes the same lift as $\widetilde{g}$, just of h instead). All this is saying is that you can lift a product of loops by a product of loops, and we're choosing one loop to the in the image subgroup of p_*. Then we can define a function Φ from cosets $H[g]$ to $p^{-1}(x_0)$ by sending $H[g]$ to $\widetilde{g}(1)$. $H[g]$ denotes $h \cdot g$, where $h \in H$, the coset of g. If you think about it, these are cosets since we just vary g: and so the number of cosets is the index of the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$. Now we just have to show Φ is a bijection to complete the proof.

Φ is onto by the path-connectedness of $\widetilde{X}$, since $\widetilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ by a path $\widetilde{g}$ projecting to a loop g at x_0. To show Φ is 1-1, note that $\Phi(H[g_1]) = \Phi(H[g_2])$ implies that $g_1 \cdot \overline{g_2}$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ and hence $H[g_1] = H[g_2]$. ⊠

Question: for a continuous map $\varphi\colon Y \to X$, does φ admit a lift $\widetilde{\varphi}$ to a covering space $\widetilde{X}$ of X? The lifting criterion can help us out.

Theorem 4.1 (Lifting criterion). *Suppose we are given a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. If a lift $\widetilde{f}$ exists, then $p\widetilde{f} = f$, so $f_* = p_*\widetilde{f}_*$ ⊠

§4.7 Classification of covering spaces (todo split it up)

How can we catch all the covering spaces? This whole topic deals closely with its analogue in algebra, Galois theory, with a 1-1 correspondence between connected covering spaces of X (towers of field extensions) and subgroups of $\pi_1(X)$ (subgroups of $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$). This comes from the function that assigns each covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$.

Definition 4.5 (Semilocally simply-connected). A space X is semilocally simply-connected if for all $x \in X$, there exists a neighborhood U_x containing x such that the inclusion map $U \hookrightarrow X$ induces the trivial map, that is, $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Basically, the fundamental group of U is trivial *inside* the fundamental group of X, that is, loops in $\pi_1(U, x)$ are nullhomotopic in X (not necessarily U, if that were the case, U would be locally simply connected). Intuitively, there are lower bounds on the size of holes (genus-wise): if theres a hole, you can find a neighborhood smaller than it so that loops are still trivial. For example, take the Hawaiian earring: loops here are very very small, and at the base every neighborhood will contain a hole, so it's not semilocally simply-connected (a "bad" space)[9].

Proposition 4.7. *If X is a path-connected, locally path-connected, and semilocally simply-connected space, then for every subgroup H of $\pi_1(X, x_0)$, there is a covering space $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$.*

We'll prove Proposition 4.7 much later, let's talk about it first. If $H = 1$, then $\widetilde{X}$ is simply-connected, so $\widetilde{X}$ is the universal cover of X. Now given an evenly covered open set U, any loop in U will lift to a sheet in $\widetilde{X}$, which implies it's nullhomotopic in $\widetilde{X}$, and therefore nullhomotopic in X (we don't know if it's nullhomotopic in U), we can see this just by projecting the loop with p. This implies that if $U \overset{\iota}{\hookrightarrow} X$ denotes the inclusion of U in X, then $\iota_*(\pi_1(U)) = 1$ in $\pi_1(X)$. So X must be semilocally simply-connected. The following claim shows why we need these claims for X to be a "nice" space.

Claim. If X is path-connected, locally path-connected, and semilocally simply-connected, then there exists a universal cover of X.

Proof. We prove this by directly constructing a universal cover of X through the fundamental groupoid. First assume that X has a universal cover $\widetilde{X} \xrightarrow{p} X$. Let $\widetilde{x}_0 \in \widetilde{X}$. Then for some other $\widetilde{x} \in \widetilde{X}$, there is a unique path homotopy class of paths from $\widetilde{x}_0$ to $\widetilde{x}$. So points in $\widetilde{X}$ are in a 1-1 correspondence of path homotopy classes of paths starting at $\widetilde{x}_0$. But by the path lifting property, these are all homotopic.

Let's turn this around and define the universal cover of X by its path homotopy classes, that is, let

$$\widetilde{X} := \{[f] \in \Pi_1(X) \mid f(0) = x_0\},$$

where $\Pi_1(X)$ denotes the fundamental groupoid of X. The covering is given by $p\colon \widetilde{X} \to X$, $[f] \mapsto f(1)$. We want to define a topology on $\widetilde{X}$ that makes p continuous and a covering map. To do this, we define a basis $\mathscr{B}$ and check to see if the inverse image of open sets in the *basis* are continuous. Albin, 24 min lecutre 8 unfinished ⊠

[9] Does anyone reading this know of a space that's path-connected but not locally path-connected? I know of many counterexamples for the converse, but without a counterexample to the implication I don't see why local path-connectedness is a necessary condition on top of path-connectedness.

Now that we've proved that for every subgroup we have a covering space, the next question is how many covering spaces per subgroup? We have two covering spaces $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are *equivalent* if there is a homeomorphism $f\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$, or such that the following diagram commutes:

$$\begin{tikzcd} (\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

If so, it's easy to see that this is an equivalence relation.

Theorem 4.2. *The covering spaces $(\widetilde{X}_i, \widetilde{x}_i) \xrightarrow{p_i} (X, x_0)$, where $i \in \{1, 2\}$ and X is path-connected, locally path-connected are equivalent if and only if $p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

So it turns out the answer to the question above is just one.

Proof. One direction is easy: look at the diagram of induced fundamental groups, and notice that the homeomorphism induces an isomorphism on the subgroups of $\pi_1(X)$. The other direction is more interesting. Let $H_1 = p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$ and $H_2 = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. Since $H_1 \subseteq H_2$ and p_2 is a covering map and X is path-connected and locally path-connected, there exists a lift of $\widetilde{p}_1$ to a map $f\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ by the lifting criterion, making the diagram commute. Similarly, $H_2 \subseteq H_1$, so there's a lift of p_2 to a map $g\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ making the appropiate diagram commute. In particular, we have

$$\begin{tikzcd} (\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "g \circ f"] \arrow[dr, "p_1"'] & & (\widetilde{X}_1, \widetilde{x}_1) \arrow[dl, "p_1"] \\ & (X, x_0) & \end{tikzcd}$$

Since the identity is also a lift of p_1 to $(\widetilde{X}_1, \widetilde{x}_1)$, by uniqueness of lifts we have $g \circ f$ equal to the identity, that is, $g \circ f = \mathrm{id}_{\widetilde{X}_1}$. Similarly, we have $f \circ g = \mathrm{id}_{\widetilde{X}_2}$. So f is a homeomorphism. [10] ⊠

Now for the theorem we all came here for.

Theorem 4.3. *Let X be a path-connected, locally path-connected, and semilocally simply-connected space. Then there is a bijection between the coverings $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{p} (X, x_0)$ up to equivalence and the subgroups of $\pi_1(X, x_0)$. This bijection is given by $p \mapsto p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, we also have a 1-1 correspondence between the non pointed covering spaces $\widetilde{X} \xrightarrow{p} X$ and the conjugacy classes of subgroups, given by the same map $p \mapsto [p_*(\pi_1(\widetilde{X}))]$.*

It's important that we have a covering space and choice of basepoint: if we change the basepoint, we might not necessarily give the same group. Changing the basepoint gives a conjugacy isomorphisms between fundamental groups. This conjugacy isomorphism might give rise to different subgroups, conjugating by some element of the group possibly gives a different subgroup. Hence the second part of the theorem.

It turns out there's an equivalence of posets between covers of a space (X, x_0) (for X a "nice" space) and subgroups of $\pi_1(X, x_0)$, known as the Galois correspondence. The partial order is given by defining two elements to be comparable if one is a cover of another.

§4.8 Actions on the fibers

If $p\colon \widetilde{X} \to X$ a cover, $\alpha \in \pi_1(X, x_0)$, define $L_\alpha \in \mathrm{Sym}(p^{-1}(x_0))$ by $L_\alpha \widetilde{x} = \widetilde{\alpha}(0)$, where $\widetilde{\alpha}$ is the lift of x to a path ending at $\widetilde{x}$. We have $L_{\alpha\beta} = L_\alpha \circ L_\beta$, since $L_{\alpha\beta}(\widetilde{x}) = \widetilde{\alpha\beta}(0) = L_\alpha(\widetilde{\beta}(0)) = L_\alpha(L_\beta(\widetilde{x}))$. This is why we defined $L_\alpha(\widetilde{x})$ starting at the left endpoint 0. Albin lecture 9, 36 minutes

[10] I don't understand where f came from: how can we guarantee its existence?

§5 Common Topological Structures

We'll take this section to digress a little bit and explore some examples of our favorite spaces that we work with a lot in topology.

§5.1 Manifolds (todo)

§5.2 Cell complexes (todo)

§5.3 The real projective plane $\mathbb{R}P^n$ (todo)

Credit to Cameron Krulewski at UChicago, who wrote up a paper on $\mathbb{R}P^n$ for a Math 132 project, whose notes I am following today.

Manifolds are often talked about as subsets of $\mathbb{R}^n$, for example, we often discuss k-manifolds embedded in at most $\mathbb{R}^{2k+1}$. What is the real projective n-space $\mathbb{R}P^n$ exactly? It's the space of lines through the origin in $\mathbb{R}^{n+1}$. For $\mathbb{R}P^2$ (the real projective plane), this doesn't embed in $\mathbb{R}^3$, but it does immerse. This won't make sense the higher we go up. A better way to think of abstract manifolds like $\mathbb{R}P^n$ is as a **quotient space** by identifying points of another manifold.

Claim. The real projective n-space is homeomorphic to an n-sphere with antipodal points identified, that is, $\mathbb{R}P^n \cong S^n/(v \sim -v)$.

Why is this true? Let's look at the cases. In the trivial case, let $n = 0$. Then $\mathbb{R}P^0$ consists of just one line $\{\mathbb{R}\}$, so it's hoemomorphic to a singleton. What is S^0? It's two singletons, so if you identify them you get your expected result.

Now let's look at $n = 1$: we want to show that $\mathbb{R}P^1$ is homeomorphic to the circle S^1. Let's parametrize the lines by their slopes, that is, the angle $\tan\left(\frac{y}{x}\right)$ for any positive pair (x, y) on any given line. We choose (x, y) positive since the lines extend in both directions and looking at both would mean a redundancy. Then these lines hit every angle from 0 to π, and the x-axis given by $\mathbb{R} \times \{0\}$ has an angle of both 0 and π (identifying the two together). So we get that $\mathbb{R}P^1 \cong S^1$. How is this homeomorphic to $S^1/(v \sim -v)$, as we claimed? Identifying antipodal points gets a semicircle, but the endpoints of the semicircle are also antipodal and get identified, so suprisingly $S^1 \cong S^1/(v \sim -v)$.