Notes on Topological Quantum Field Theory

Simon Xiang

April 29, 2022

Notes for my Spring 2022 DRP on Bordism and TQFTs, mentored by William Stewart. We follow Dan Freeds notes for a topics course called "*Bordism: Old and New*" (M392C) that he taught in 2012 (course url here: `https://web.ma.utexas.edu/users/dafr/M392C-2012/index.html`). Source files: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

1 Introduction to Bordism

Review of homology: A **singular q-chain** in a space S is a formal sum of continuous maps $\Delta^q \to S$ from the standard q-simplex. There is a boundary operation ∂ on chains; a chain c is a **cycle** if $\partial c = 0$, and a **boundary** if there exists a $(q+1)$-chain b with $\partial b = c$. If S is a point then every cycle is a boundary. Bordism replaces cycles by *closed smooth manifolds* mapping continuously into S. (Here *closed* means *compact without boundary*). Chains become *compact smooth manifolds* X with a continuous map $X \to S$, and the boundary of this chain is the restriction $\partial X \to S$ to the boundary.

Example 1.1. Not every closed smooth manifold is the boundary of a compact smooth manifold. We have $Y = \mathbb{RP}^2$ *not* the boundary of a compact 3-manifold. However, it is the boundary of a noncompact 3-manifold with with boundary. To see this, first consider $\mathbb{RP}^0 \simeq \{\mathrm{pt}\}$. This is the boundary of a non-compact 1-manifold, namely the half line $[0,1)$. Here the cover $\big\{[0,\frac{1}{2}),(\frac{1}{3},\frac{2}{3}),(\frac{1}{2},\frac{3}{4}),(\frac{2}{3},\frac{4}{5}),\cdots,(\frac{n}{n+1},\frac{n+2}{n+3})\big\}$ as $n \to \infty$ has no finite subcover. This generalizes to $\mathbb{RP}^1 \simeq S^1$, which is the boundary of $S^1 \times [0,1)$ and so $\mathbb{RP}^2$ is the boundary of $\mathbb{RP}^2 \times [0,1)$. From here, we can see that every closed smooth manifold Y is the boundary of a noncompact $(n+1)$-manifold with boundary, namely $Y \times [0,1)$. What fails if Y isn't closed? If Y has boundary, then $\partial^2 = 0$, and if Y is non-compact this doesn't work.

How do we prove our earlier assertion that $\mathbb{RP}^2$ is not the boundary of a compact 3-manifold? We will see this later.

1.1 Review of smooth manifolds

Definition 1.1. A **topological manifold** is a paracompact, Hausdorff topological space X such that every point of X has an open neighborhood homeomorphic to an open subset of affine space. We define *n-dimensional affine space* as $\mathbb{A}^n = \{(x^1, x^2, \cdots, x^n) \mid x^i \in \mathbb{R}\}$. The vector space $\mathbb{R}^n$ acts transitively on $\mathbb{A}^n$ by translations.

Definition 1.2. The empty set $\emptyset$ is trivially a manifold of any dimension $n \in \mathbb{Z}^{\geq 0}$. We write $\emptyset^n$ to denote the empty manifold of dimension n.

Definition 1.3. Define $\mathbb{A}^n_- = \{(x^1, x^2, \cdots, x^n) \in \mathbb{A}^n \mid x^1 \leq 0\}$. We require that coordinate charts take values in open sets of $\mathbb{A}^n_-$. Then we partition X into two disjoint subsets (both manifolds): the **interior** (points with $x^1 < 0$ in every coordinate system) and the **boundary** ∂X (points with $x^1 = 0$).

Remark 1.1. Recall the mnemonic "ONF", standing for "Outward Normal First". An outward normal in a coordinate system is represented by the first coordinate vector field $\partial/\partial x^1$, which points outward at the boundary.

Definition 1.4. At any point $p \in \partial X$ there is a canonical subspace $T_p(\partial X) \subseteq T_pX$; the quotient space $T_pX/T_p(\partial X)$ is a real line ν_p. So over the boundary there is a short exact sequence

$$0 \to T(\partial X) \to TX \xrightarrow{p} \nu \to 0$$

of vector bundles.

The vector $\partial/\partial x^1(p)$ projects to a nonzero element of ν_p, but there is no canonical basis independent of coordinate system. However, any two such vectors are in the same component of $\nu_p \setminus \{0\}$, so ν carries a canonical *orientation*. Furthermore, there is a splitting $s\colon \nu \to TX$ that assigns to a point on ν a tangent vector which lies in $T(\partial X)$, which by the quotient projection maps to 0. Therefore since $s \circ p = \mathrm{id}_{TX}$, we have $TX \simeq T(\partial X) \oplus \nu$. For example, say we have an n-manifold with boundary, then $T_pM \simeq \mathbb{R}^{n+1}$ and $T_mM \simeq \mathbb{R}^n$ for $m \in \partial X$. Since T_mM has codimension 1 we have $\nu \simeq \mathbb{R}$, which comes from $\mathbb{R}^{n+1}/\mathbb{R}^n$. We also see that $T_pM \simeq \mathbb{R}^{n+1} \simeq (T_mM \simeq \mathbb{R}^n) \oplus (\nu \simeq \mathbb{R})$.

Definition 1.5. Let X be a manifold with boundary. A **collar** of the boundary is an open set $U \subset X$ which contains ∂X and a diffeomorphism $(-\varepsilon, 0] \times \partial X \to U$ for some $\varepsilon > 0$.

Theorem 1.1. *The boundary ∂X of a manifold X with boundary has a collar.*

Proof. todo:this? ⊠

Let $\{X_1, X_2, \cdots\}$ be a countable collection of manifolds. We form a new manifold $X_1 \amalg X_2 \amalg \cdots$, the **disjoint union** of $X_1, X_2, \cdots$. As a set it is the disjoint union of the underlying sets for the manifolds. A question is how to precisely define this; what is $X \amalg X$, for example? A solution is to fix some $\mathbb{A}^\infty$ and regard all manifolds embedded in it. Replace X_i by $\{i\} \times X_i$, then define the disjoint union to be the ordinary union of subsets of $\mathbb{A}^\infty$. We could also use a universal property; a disjoint union of $X_1, X_2, \cdots$ is a manifold Z and collection of smooth maps $\iota_i : X_i \to Z$ such that for any manifold Y and collection $f_i : X_i \to Y$ of smooth maps, there exists a unique map $f : Z \to Y$ such that for each i the diagram

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & Z \arrow[d, dotted, "f"] \\ & Y \end{tikzcd}$$

commutes.

1.2 Bordism

Definition 1.6. Let Y_0, Y_1 be closed n-manifolds. A **bordism** $(X, (\partial X)_0 \amalg (\partial X)_1, \theta_0, \theta_1)$ from Y_0 to Y_1 is a compact $(n+1)$-manifold X with boundary, a decomposition $\partial X = (\partial X)_0 \amalg (\partial X)_1$ of its boundary, and embeddings $\theta_0 : [0, +1) \times Y_0 \to X, \theta_1 : (-1, 0] \times Y_1 \to X$ such that $\theta_i((0, Y_i)) = (\partial X)_i, i = 0, 1$.

The map θ_i is a diffeomorphism onto its image, which is a collar neighborhood of $(\partial X)_i$. The reason why we add the collar neighborhoods is to make it easier to glue bordisms; without them we could say a bordism X from Y_0 to Y_1 is a compact $(n+1)$-manifold with boundary $Y_0 \amalg Y_1$.

Definition 1.7. Let $(X, (\partial X)_0 \amalg (\partial X)_1, \theta_0, \theta_1)$ be a bordism from Y_0 to Y_1. The **dual bordism** from Y_1 to Y_0 is $(X^\vee, (\partial X^\vee)_0 \amalg (\partial X^\vee)_1, \theta_0^\vee, \theta_1^\vee)$ where $X^\vee = X$, the decomposition of the boundary is swapped so $(\partial X^\vee)_0 = (\partial X)_1$ and $(\partial X^\vee)_1 = (\partial X)_0$, and

$$\begin{aligned} \theta_0^\vee(t, y) &= \theta_1(-t, y), && t \in [0, +1),\ y \in Y_1, \\ \theta_1^\vee(t, y) &= \theta_0(-t, y), && t \in (-1, 0],\ y \in Y_0. \end{aligned}$$

Think of the dual bordism $X^\vee$ as the original bordism X "turned around", and view it as a bordism from $Y_1^\vee$ to $Y_0^\vee$, where for naked manifolds we set $Y_i^\vee = Y_i$. When manifolds have tangential structure, this will not necessarily be the case.

Lemma 1.1. *Bordism defines an equivalence relation.*

Proof. For any closed manifold Y, the manifold $X = [0, 1] \times Y$ is a bordism from Y to Y. Formally, set $(\partial X) + 0 = \{0\} \times Y, (\partial X)_1 = \{1\} \times Y$, and simple diffeomorphisms $[0, 1) \to [0, \frac{1}{3}), (-1, 0] \to (\frac{2}{3}, 1]$ to construct our θ_i. Symmetry is given by the dual bordism; if X is a bordism from Y_0 to Y_1, then $X^\vee$ is a bordism from Y_1 to Y_0.

For transitivity let X be a bordism $Y_0 \to Y_1$, and X' a bordism from Y_1 to Y_2. Define a new manifold $X'' = X \amalg X / \sim$, where for $(a, b), (c, d) \in X \amalg X'$, if either $a, d \in Y_1$, then $(a, b) \sim (c, d)$. todo:how exactly is this a manifold? bourbaki: https://math.stackexchange.com/questions/496571/under-what-conditions-the-quotient-space-of-a-ma basically E is a closed submanifold of $M \times M$ (true since $E = (\partial M)_1 = (\partial M')_0$ which are manifolds by def. the projection is also a submersion. diffeomorphic. okay how do we show the smooth structure? we did it in office, will maybe write down later.

⊠

Example 1.2. If $f : M \to N$ is a diffeomorphism between manifolds, then consider the mapping cylinder $Mf = ([0,1] \times M) \amalg_f N$, a smooth manifold with boundary $M \times \{0\} \cup N \times \{1\}$. So diffeomorphic manifolds are bordant.

Let Ω_n denote the set of equivalence classes of n-manifolds under the equivalence relation of bordism. We use the term **bordism class** for an element of Ω_n. Note that $\emptyset^0$ (empty manifold) is a distinct element of Ω_n, so Ω_n is a **pointed set**.

1.3 Disjoint union and the abelian group structure

The disjoint union and cartesian product give Ω_n more structure.

Definition 1.8. A **commutative monoid** is a set with a commutative, associative composition law and identity element. An **abelian group** is a commutative monoid in which every element has an inverse.

Disjoint unions of manifolds pass to bordism classes: if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \amalg Y_1$ is bordant to $Y_0' \amalg Y_1'$ (take the disjoint union of the bordisms as manifolds). So $(\Omega_n, \amalg)$ is a commutative monoid.

Lemma 1.2. *$(\Omega_n, \amalg)$ is an abelian group with identity $\emptyset^n$. Furthermore, for $Y \in \Omega_n$, $Y \amalg Y$ is null-bordant.*

Proof. Let $Y \in \Omega_n$. Consider the manifold $X = [0,1] \times Y$; this gives a bordism between $Y \amalg Y$ and $\emptyset^n$, with $(\partial X)_0 = Y \amalg Y$ and $(\partial X)_1 = \emptyset^n$. So $Y = Y^{-1}$ and we are done. ⊠

Proposition 1.1. *$\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ with generator* pt.

Proof. Compact 0-manifolds are finite disjoint unions of points. Lemma 1.2 implies that the disjoint union of two points is a boundary, so this is zero in Ω_0. To show that pt is not the boundary of a compact 1-manifold without boundary, this follows from the classification of 1-manifolds with boundary; they are a finite disjoint union of circles and closed intervals, so its boundary has an even number of points. ⊠

Proposition 1.2. *$\Omega_1 = 0$ and $\Omega_2 = \mathbb{Z}/2\mathbb{Z}$ with generator $\mathbb{R}\mathrm{P}^2$.*

Proof. By the classification of compact 1-manifolds, closed 1-manifolds are finite disjoint unions of circles, which bound disks (and so they are null-bordant). Therefore $\Omega_1 = 0$. Now recall the classification theorem for 2-manifolds, which states that there are two connected families; oriented and unoriented surfaces. For oriented surfaces, they are either 2-spheres or connected sum of tori (genus g surfaces). Spheres bound the 3-ball, and genus g surfaces go to genus g handlebodies.

Any unoriented surface is a **connected sum** of $\mathbb{R}\mathrm{P}^2$'s. It suffices to prove that $\mathbb{R}\mathrm{P}^2$ does not bound and $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ bounds. Why? We claim that $M_1 \amalg M_2$ is bordant to $M_1 \# M_2$. To see this, consider $M_1 \times I \cup M_2 \times I$. Then for points v, w in $M_1 \times \{0\}, M_2 \times \{0\}$, cut out a half-neighborhood (half ball) D_+^n out of both, and glue along the boundary half-sphere S_+^{n-1}. Then the top copy is $M_1 \# M_2$, the bottom copy is $M_1 \amalg M_2$, the entire construction is a manifold, and so the two are bordant. Therefore $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is bordant to $\mathbb{R}\mathrm{P}^2 \amalg \mathbb{R}\mathrm{P}^2 \amalg \mathbb{R}\mathrm{P}^2$ which is bordant to $\mathbb{R}\mathrm{P}^2$, and so on.

For the former, suppose that X is a compact manifold with $\partial X = \mathbb{R}\mathrm{P}^2$. Then consider the **double** $D = X \cup_{\mathbb{R}\mathrm{P}^2} X$, constructed by gluing two copies of X along $\mathbb{R}\mathrm{P}^2$. We have $\chi(D) = 2\chi(X) - \chi(\mathbb{R}\mathrm{P}^2) = 2\chi(X) - 1$ by Hatcher 2.2.21, which is odd. However, closed odd-dimensional manifolds have zero euler characteristic. It remains to show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$; this is true by inclusion-exclusion (on counting cells).

Now $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to the Klein bottle K, which has a map $K \to S^1$, a fiber bundle with fiber S^1. Then there is an associated fiber bundle with fiber D^2, a compact 3-manifold with boundary K. ⊠

1.4 Cartesian product and the ring structure

Definition 1.9.

(i) A **commutative ring** R is an abelian group $(+,0)$ with a second commutative, associative composition law $(\cdot)$ with identity (1) which distributes over the first: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ for all $r_1, r_2, r_3 \in R$.

(ii) A $\mathbb{Z}$-**graded commutative ring** is a commutative ring S which as an abelian group is a direct sum $S = \bigoplus_{n\in\mathbb{Z}} S_n$ of abelian group such that $S_{n_1} \cdot S_{n_2} \subseteq S_{n_1+n_2}$. In other words, you can multiply two elements in S_{n_1}, S_{n_2} to get an element in $S_{n_1+n_2}$.

Elements in $S_n \subset S$ are called **homogeneous of degree** n; an element of S is a finite sum of homogeneous elements.

Example 1.3. The integers $\mathbb{Z}$ form a commutative ring, and for any commutative ring R there is a polynomial ring $S = R[x]$ in a single variable which is $\mathbb{Z}$-graded. To define this grading, we need to assign a degree to the indeterminate x, usually 1; in this case S_n is the abelian group of homogeneous polynomials of degree n in x. More generally, there is a $\mathbb{Z}$-graded polynomial ring $R[x_1, \cdots, x_k]$ in any number of indeterminates with any assigned integer degrees $\deg x_k \in \mathbb{Z}$.

Define

$$\Omega = \bigoplus_{n\in\mathbb{Z}^{\geq 0}} \Omega_n.$$

Formally, define $\Omega_{-m} = 0$ for $m > 0$. The Cartesian product of manifolds is compatible with bordism; if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \times Y_1$ is bordant to $Y_0' \times Y_1'$. To see this, let M_0, M_1 be the bordisms with $\partial M_0 = Y_0 \amalg Y_0'$, $\partial M_1 = Y_1 \amalg Y_1'$. Then the bordism between $Y_0 \times Y_1$ and $Y_0' \times Y_1'$ is given by $M_0 ?? M_1$ todo:not $M_0 \times M_1$, dimension. not $\amalg$, counterex. help- check pictures. So this passes to a commutative, associative binary composition on Ω.

Proposition 1.3. *$(\Omega, \amalg, \times)$ is a $\mathbb{Z}$-graded ring. A homogeneous element of degree $n \in \mathbb{Z}$ is represented by a closed manifold of dimension n.*

Proof. todo:?? dk what the bordism of product is. show it's compatible?? ⊠

In his Ph.D. thesis Thom todo:references proved the following theorem.

Theorem 1.2 (Thom)**.** *There is an isomorphism $\Omega \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, x_6, x_8, \cdots$ where there is a polynomial generator of degree k for each positive integer k not of the form $2^i - 1$. Furthermore, if k is even, then x_k is represented by $\mathbb{R}\mathrm{P}^k$.*

Dold later constructed manifolds representing the odd degree generators, which are fiber bundles over $\mathbb{R}\mathrm{P}^m$ will fiber $\mathbb{C}\mathrm{P}^\ell$. Working out Ω_{10}, or 10-manifolds up to bordism, we have generator $\mathbb{R}\mathrm{P}^{10}$. todo:?? don't know much about 10-manifolds- just check generators and combine

Thom proved that the **Stiefel-Whitney numbers** determine the bordism class of a closed manifold. The **Steifel-Whitney classes** $w_i(Y) \in H^i(Y; \mathbb{Z}/2\mathbb{Z})$ are examples of **characteristic classes** of the tangent bundle; we will discuss this stuff later. Any closed n-manifold Y has a **fundamental class** $[Y] \in H_n(Y; \mathbb{Z}/2\mathbb{Z})$. If $x \in H^\bullet(Y; \mathbb{Z}/2\mathbb{Z})$, the pairing $\langle x, [Y]\rangle$ produces a number in $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.3. *The Stiefel-Whitney numbers*

$$\langle w_{i_1}(Y) \smile w_{i_2}(Y) \smile \cdots \smile w_{i_k}(Y), [Y]\rangle \in \mathbb{Z}/2\mathbb{Z}$$

determine the bordism class of a closed n-manifold Y.

That is to say, if two closed n-manifolds Y_0, Y_1 have the same Stiefel-Whitney numbers, then they are bordant. Notice that not all naively possible nonzero Stiefel-Whitney numbers can be nonzero. For example, $\langle w_1(Y), [Y]\rangle$ vanishes for any closed 1-manifold Y. Also, the theorem implies that a closed n-manifold is the boundary of a compact $(n+1)$-manifold iff all the Stiefel-Whitney numbers of Y vanish. If it is a boundary, it is immediate that the Stiefel-Whitney numbers vanish; the converse is not obvious.

2 Orientations, framings, and the Pontrjagin-Thom construction

Thom made a profound link between geometric topology (bordism rings) to homotopy theory. The geometric side is the set of *framed* bordism classes of submanifolds of a fixed manifold M; the homotopical side is the set of homotopy classes of maps from M into a sphere.

First we review orientations.

2.1 Orientations

Let V be a real vector space of dimension $n > 0$. A *basis* of V is a linear isomorphism $b\colon \mathbb{R}^n \to V$. Let $\mathcal{B}(V)$ denote the set of all bases of V. The group $\mathrm{GL}_n(\mathbb{R})$ of linear isomorphisms of $\mathbb{R}^n$ acts **simply transitively** (or regular, transitively and freely) on the right of $\mathcal{B}(V)$ by composition; if $b\colon \mathbb{R}^n \to V$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ are isomorphisms, then so is $b \circ g\colon \mathbb{R}^n \to V$. We say $\mathcal{B}(V)$ is a **right $\mathrm{GL}_n(\mathbb{R})$-torsor**. For $b \in \mathcal{B}(V)$ the map $g \mapsto b \circ g$ is a bijection from $\mathrm{GL}_n(\mathbb{R})$ to $\mathcal{B}(V)$, we use it to topologize $\mathcal{B}(V)$. Since $\mathrm{GL}_n(\mathbb{R})$ has two components, so does $\mathcal{B}(V)$.

Definition 2.1. An **orientation** of V is a choice of component of $\mathcal{B}(V)$.

Recall that components of $\mathrm{GL}_n(R)$ are distinguished by the determinant homomorphism $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^{\neq 0}$, where the identity component consists of $g \in \mathrm{GL}_n(\mathbb{R})$ with $\det(g) > 0$, and the other component consists of g with $\det(g) < 0$. On the other hand, an isomorphism $b\colon \mathbb{R}^n \to V$ does not have a numerical determinant. Rather, its determinant lives in the **determinant line** Det. Define

$$\operatorname{Det} V = \{\varepsilon\colon \mathcal{B}(V) \to \mathbb{R} \mid \varepsilon(b \circ g) = \det(g)^{-1}\varepsilon(b) \text{ for all } b \in \mathcal{B}(V), g \in \mathrm{GL}_n(\mathbb{R})\}.$$

todo:do not understand this def

Remark 2.1. Prove these as exercises:

(i) Construct a canonical isomorphism $\operatorname{Det} V \xrightarrow{\cong} \bigwedge^n V$ of the determinant line with the highest exterior power. The latter is often taken as the definition.

(ii) Prove that an orientation is a choice of component of component of $\operatorname{Det} V \setminus \{0\}$. More precisely, construct a map $\mathcal{B}(V) \to \operatorname{Det} V \setminus \{0\}$ which induces a bijection on components.

(iii) Construct the "determinant" of an arbitrary linear map $b\colon \mathbb{R}^n \to V$ as an element of $\operatorname{Det} V$. Show it is nonzero iff b is invertible.

(iv) More generally, construct the determinant of a linear map $T\colon V \to W$ as a linear map $\det T\colon \operatorname{Det} V \to \operatorname{Det} W$, assuming $\dim V = \dim W$.

(v) Show a canonical $\{\pm 1\}$-torsor associated to a vector space can also be defined as

$$\mathfrak{o}(V) = \{\varepsilon\colon \mathcal{B}(V) \to \{\pm 1\} \mid \varepsilon(b \circ g) = \operatorname{sign}\det(g)^{-1}\varepsilon(b) \text{ for all } b \in \mathcal{B}(V), g \in \mathrm{GL}_n(\mathbb{R})\}.$$

In short, an orientation of V is a point of $\mathfrak{o}(V)$.

Proof. Attempts: todo:check

(i) The isomorphism is given by sending $\{e_1, \cdots, e_n\}$ to $e_1 \wedge \cdots \wedge e_n$? What is this def of Det?

(ii) For a basis $\{e_i\} \in \mathcal{B}(V)$, consider the bijection $e_1, \cdots, e_n \mapsto e_1 \wedge \cdots \wedge e_n$. This gives a partition of $\mathcal{B}(V)$ by the two components of $\operatorname{Det} V \setminus \{0\}$ (since $\mathrm{GL}_n(\mathbb{R})$ is partitioned the same way).

(iii) same process?

(iv) ?

$\boxtimes$

There is a unique 0-dimensional vector space consisting of a single element, the zero vector. The unique basis is the empty set, so $\operatorname{Det} 0$ is canonically isomorphic to $\mathbb{R}$ by definition todo:isn't there only one map? and $\mathfrak{o}(V)$ is canonically isomorphic to $\{\pm 1\}$. Note that $\bigwedge^0(0) = \mathbb{R}$ as $\bigwedge^0 V = \mathbb{R}$ for *any* real vector space V. The real line $\mathbb{R}$ has a canonical orientation: the component $\mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0}$. We denote this orientation as "+". The opposite orientation is denoted "-".

Exercise 2.1. Suppose $0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$ is a short exact sequence of finite dimensional vector spaces. Construct a canonical isomorphism $\operatorname{Det} V'' \otimes \operatorname{Det} V' \to \operatorname{Det} V$. Note the order: quotient before sub.

Proof. todo:? forgot how to work with tensor $\boxtimes$

Let X be a smooth manifold and $V \to X$ a finite rank real vector bundle. For each $x \in X$ there is associated to the fiber V_x over x a canonical $\{\pm 1\}$-torsor $\mathfrak{o}(V)_x$— a two element set—which has the two descriptions in (ii) of the exercise.

Exercise 2.2. Use local trivializations of $V \to X$ to construct local trivializations of $\mathfrak{o}(V) \to X$, where $\mathfrak{o}(V) = \coprod_{x \in X} \mathfrak{o}(V)_x$.

Definition 2.2. (i) An **orientation** of a real vector bundle $V \to X$ is a section of $\mathfrak{o}(V) \to X$.

(ii) If $o : X \to \mathfrak{o}(V)$ is an orientation, then the **opposite orientation** is the section $-o : X \to \mathfrak{o}(V)$.

(iii) An **orientation** of a manifold is an orientation of its tangent bundle $TX \to X$.

Note that orientations may or may not exist, that is, a vector bundle $V \to X$ may or may not be orientable. The notation $-o$ uses the fact that $\mathfrak{o}(V) \to X$ is a principal $\{\pm 1\}$-bundle: $-o$ is the result of acting -1 on o.

Exercise 2.3. Construct the **determinant line bundle** $\operatorname{Det} V \to X$ by carrying out the determinant construction pointwise and proving local trivializations exist. Show that a nonzero section of $\operatorname{Det} V \to X$ determines an orientation.

todo:bordism invariant?

2.2 Oriented bordism

We discuss bordism on manifolds with orientation. Now the manifolds Y_0, Y_1 both carry orientations, as well as the bordism X, and the embeddings θ_0, θ_1 must be orientation preserving todo:check this. In the dual case, $Y^\vee \neq Y$, but rather $-Y$, the manifold with the opposite orientation. The reversal ensures $\theta_0^\vee$ and $\theta_1^\vee$ are orientation preserving.

Denote the set of oriented bordism classes of n-manifolds as Ω_n^{SO}. There is an oriented bordism ring Ω^{SO}. Some facts:

Theorem 2.1. *(i) There is an isomorphism*

$$\mathbb{Q}[y_4, y_8, y_{12}, \cdots] \xrightarrow{\cong} \Omega^{SO} \otimes \mathbb{Q}$$

under which y_{4k} maps to the oriented bordism class of the complex project space $\mathbb{CP}^{2k}$.

(ii) All torsion in Ω^{SO} is of order 2.

(iii) There is an isomorphism

$$\mathbb{Z}[z_4, z_8, z_{12}, \cdots] \xrightarrow{\cong} \Omega^{SO}/\textit{torsion}.$$

Let's discuss Ω^{SO} in lower dimensions.

- $\Omega_0^{SO} \cong \mathbb{Z}$, the generator being an oriented point pt_+.
- $\Omega_1^{SO} = 0$. Every closed oriented 1-manifold is a finite disjoint union of circles S^1, and $S^1 = \partial D^2$.

todo:rest

2.3 Framed bordism and the Pontrjagin-Thom construction

Fix a closed m-dimensional manifold M. Let $Y \subseteq M$ be a submanifold. Recall that on Y there is a short exact sequence of vector bundles

$$0 \to TY \to TM|_Y \to \nu \to 0$$

where ν is the quotient bundle and is called the **normal bundle** of Y in M.

Definition 2.3. A **framing** of the submanifold $Y \subset M$ is a trivialization of the normal bundle ν.

Recall a trivialization of ν is an isomorphism of vector bundles $\mathbb{R}^q \to \nu$, where q is the codimension of Y in M. Equivalently, it is a global basis of sections of ν.

Let N be a manifold of dimension q and $f : M \to N$ smooth. Suppose $p \in N$ is a *regular value* of f and fix a basis $e_1, \cdots, e_q$ of T_pN. Then $Y := f^{-1}(p) \subset M$ is a submanifold and the basis $e_1, \cdots, e_q$ pulls back to a basis of the normal bundle at each point $y \in Y$.

todo:lecture 13 back to front, lec 2 exercrses, tensor stuff

3 Categories

Definition 3.1. A **category** C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0, y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y, y)$, and for each triple of objects a composition law

$$\circ : C(y_1, y_2) \times C(y_0, y_1) \to C(y_0, y_2)$$

such that $\circ$ is associative and id_y is an identity for $\circ$. This means that for all $f \in C(y_0, y_1)$ we have

$$\mathrm{id}_{y_1} \circ f = f \circ \mathrm{id}_{y_0} = f$$

and for all $f_1 \in C(y_0, y_1), f_2 \in C(y_1, y_2)$ and $f_3 \in C(y_2, y_3)$ we have

$$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1).$$

We use the notation $y \in C$ for an object of C and $f : y_0 \to y_1$ for a morphism $f \in C(y_0, y_1)$.

Definition 3.2. Let C be a category.

(i) A morphism $f \in C(y_0, y_1)$ is **invertible** (or an **isomorphism**) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$.

(ii) If every morphism in C is invertible, then we call C a **groupoid**.

Reformulation. We can reformulate this definition in terms of sets and functions. A category C then consists of a set C_0 of objects, a set C_1 of functions, and structure maps $i\colon C_0 \to C_1$, $s, t\colon C_1 \to C_0$, $c\colon C_1 \times_{C_0} C_1 \to C_1$ satisfying certain conditions. The map i attaches to each object y the identity morphism, the structure maps s, t assign to $f: y_0 \to y_1$ the source $s(f) = y_0$ and the target $t(f) = y_1$, and c is composition. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$ (so composition is valid).

Example 3.1. Some examples of categories:

- Let C be a category with one object, i.e., $C_0 = \{*\}$. Then C_1 is a set with an identity element and an associative composition law. This is called a **monoid**. A groupoid with a single object is a **group**.
- Suppose C is a category with only identity maps. Then C is given canonically by the set C_0 of objects, and we identify the category C as this specific set.
- Let S be a set and G a group acting on S. There is an associated groupoid $C = S//G$ with objects $C_0 = S$ and morphisms $C_1 = G \times S$. The source map is projection to the first factor todo:second? and the target map is the action $G \times S \to S$. Since $g_1(g_2 s) = (g_1 g_2)(s)$ this gives associativity, and the fact that $\mathrm{id}_G\, s = s$ gives our identity for $s \in S$ as (id_G, s).
- Assuming we overcome the set theoretic stuff, there is a category Set whose objects are sets and whose morphisms are functions.
- There are lots of subcategories of Set, including the category Ab of abelian groups. Objects $A \in \mathsf{Ab}$ are abelian groups and morphisms $f: A_0 \to A_1$ are homomorphisms of abelian groups. Similarly, there is a category Vect_k of vector spaces over a field, a category of rings, and a category of R-modules for a fixed ring R (note that Ab is the case where $R = \mathbb{Z}$). Each of these categories is special since the hom-sets are abelian groups. There is also a category Top where objects are topological spaces Y and morphisms $f: Y_0 \to Y_1$ are continuous maps.
- Let Y be a topological space. The simplest invariant is the *set* $\pi_0 Y$, which imposes the equivalence relation on Y that points $y_0, y_1 \in Y$ are equivalent if there exists a continuous path connecting them, i.e., a continuous map $\gamma\colon [0,1] \to Y$ satisfying $\gamma(0) = y_0, \gamma(1) = y_1$. The **fundamental groupoid** $C = \pi_{\le 1} Y$ is defined as follows. The objects $C_0 = Y$ are the points of Y. The hom-set $C(y_0, y_1)$ is the set of homotopy clsases of maps $\gamma\colon [0,1] \to Y$ satisfying $\gamma(0) = y_0, \gamma(1) = y_1$. The homotopies are taken "rel boundary", which means endpoints are fixed in a homotopy. Explicitly, a homotopy is a map

  $$\Gamma\colon [0,1] \times [0,1] \to Y$$

  such that $\Gamma(s, 0) = y_0$ and $\Gamma(s, 1) = y_1$ for all $s \in [0,1]$. Note that the *automorphism group* $C(y, y)$ todo:need to be auto? since in groupoid morphisms are invertible by def is the fundamental group $\pi_1(Y, y)$. So $\pi_{\le 1} Y$ encodes both $\pi_0 Y$ and all of the fundamental groups.

Exercise 3.1. Given a groupoid C use the morphisms to define an equivalence relation on the objects and so a set $\pi_0 C$ of equivalence classes. Can you do the same for a category that is not a groupoid?

Define $x \sim y$ for $x, y \in C_0$ if there exists some $f \in C_1$ with $s(f) = x, t(f) = y$. This defines an equivalence relation because $x \sim x$ by the identity morphism, $x \sim y$ implies $y \sim x$ by the inverse morphism f^{-1}, and $x \sim y, y \sim z$ implies $x \sim z$ by composition $g \circ f$ for $s(g) = y, t(g) = z$. Reflexivity fails if C is not a groupoid. todo:simple check

3.1 Functors and natural transformations

Definition 3.3. Let C, D be categories.

(i) A **functor** or **homomorphism** $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0, F_1\colon C_1 \to D_1$ which commute with the structure maps (preserving composition and taking identities to identities).

(ii) Suppose $F, G\colon C \to D$ are functors. A **natural transformation** η from F to G is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f : y_0 \to y_1) \in C_1$ the diagram

$$\begin{tikzcd} Fy_0 \arrow[r, "Ff"] \arrow[d, "\eta(y_0)"'] & Fy_1 \arrow[d, "\eta(y_1)"] \\ Gy_0 \arrow[r, "Gf"] & Gy_1 \end{tikzcd}$$

commutes. We write $\eta\colon F \to G$, and depict natural transformations in diagrams as following

$$\begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow\eta"] & D \end{tikzcd}$$

often with a double arrow.

(iii) A natural transformation $\eta\colon F \to G$ is an **isomorphism** if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.

Example 3.2. Show that for fixed categories C, D there is a category $\mathrm{Hom}(C, D)$ whose objects are functors and whose morphisms are natural transformations.

To do this, we need to show that natural transformations are associative and unital. The identity is given by assigning to each $y \in C_0$ the identity map $\mathrm{id}_y \in C_1$, resulting in an identity transformation $\mathrm{id}_F\colon C_0 \to C_1$. For associativity, todo:diagram

Example 3.3. todo:double dual

Example 3.4. Let Y be a topological space and $\pi\colon Z \to Y$ a covering space. Then there is a functor $F_\pi\colon \pi_{\leq 1}Y \to \mathrm{Set}, y \to \pi^{-1}(y)$ which maps each point of y to the fiber over y. Any path $\gamma\colon [0,1] \to Y$ "lifts" to an isomorphism $\widetilde{\gamma}\colon \pi^{-1}(y_0) \to \pi^{-1}(y_1)$, and the isomorphism is unchanged under homotopy. A map

$$\begin{tikzcd} Z_0 \arrow[rr, "\varphi"] \arrow[dr, "\pi_0"'] & & Z_1 \arrow[dl, "\pi_1"] \\ & Y & \end{tikzcd}$$

of covering spaces induces a natural transformation $\eta_\varphi\colon F_{\pi_0} \to F_{\pi_1}$. todo:associates to each hom class a map of fibers $\pi_0^{-1}(y) \to \pi_1^{-1}(y)$?

Definition 3.4. Let C, D be categories. A functor $F\colon C \to D$ is an **equivalence** if there exists a functor $G\colon D \to C$, and natural isomorphisms $G \circ F \to \mathrm{id}_C$ and $F \circ G \to \mathrm{id}_D$.

Proposition 3.1. *A functor $F\colon C \to D$ is an equivalence iff:*

(i) For each $d \in D$ there exists a $c \in C$ and an isomorphism $(f(c) \to d) \in D$; and

(ii) For each $c_1, c_2 \in C$ the map of hom-sets $F\colon C(c_1, c_2) \to D(F(c_1), F(c_2))$ is a bijection.

If F satisfies (i) it is said to be **essentially surjective** and if it satisfies (ii) it is **fully faithful**.

Proof. Say a functor F is essentially surjective and fully faithful. Then consider a functor $G\colon D \to C$ defined as follows; $G\colon d \mapsto f(c)$ by the isomorphism from essential surjectivity, and $G\colon D(d_1 = F(c_1), d_2 = F(c_2)) \to C(c_1, c_2)$ by the bijection from F being fully faithful. The converse is also easy; since the isomorphism $G \circ F \to \mathrm{id}_C$ is natural we get a bijection of hom sets, and the natural isomorphism $F \circ G \to \mathrm{id}_D$ gives a bijection $f(c) \to d$ for each $d \in D$. todo:fix this, choice of c, detail natural transformations ⊠

3.2 Symmetric monoidal categories

A category is an enhanced version of a set, and a **symmetric monoidal category** is an enhanced version of a commutative monoid.

Definition 3.5. If C', C'' are categories, then there is a **Cartesian product** category $C = C' \times C''$. The set of objects is the Cartesian product $C_0 = C_0' \times C_0''$ and the set of morphisms is likewise the Cartesian product $C_1 = C_1' \times C_1''$. The structure maps consist of i mapping componentwise to the respective identities, composition mapping componentwise, and respective source and target maps.

Definition 3.6. Let C be a category. A **symmetric monoidal structure** on C consists of an object $1_C \in C$, a functor $\otimes\colon C \otimes C \to C$, and natural isomorphisms

$$\begin{tikzcd} C \times C \times C \arrow[r, bend left, "-\otimes(-\otimes-)"{name=U}] \arrow[r, bend right, "(-\otimes-)\otimes-"'{name=D}] & C \arrow[from=D, to=U, phantom, "\Uparrow\alpha"] \end{tikzcd} ,$$

$$\begin{tikzcd} C \times C \arrow[r, bend left, "(-\otimes-)\circ\tau"{name=U}] \arrow[r, bend right, "-\otimes-"'{name=D}] & C \arrow[from=D, to=U, phantom, "\Uparrow\sigma"] \end{tikzcd} ,$$

and

$$\begin{tikzcd} C \arrow[r, bend left, "\mathrm{id}_C"{name=U}] \arrow[r, bend right, "1_C\otimes-"'{name=D}] & C \arrow[from=D, to=U, phantom, "\Uparrow\iota"] \end{tikzcd} .$$

The quintuple $(1_C, \otimes, \alpha, \sigma, \iota)$ is required to satisfy the axioms below. The functor τ is transposition $\tau\colon C \times C \to C \times C, y_1, y_2 \mapsto y_2, y_1$. A crucial axiom is that $\sigma^2 = \mathrm{id}$. So for any $y_1, t_2 \in C$, the composition

$$y_1 \otimes y_2 \xrightarrow{\sigma} y_2 \otimes y_1 \xrightarrow{\sigma} y_1 \otimes y_2$$

is $\mathrm{id}_{y_1 \otimes y_2}$. The other axioms express compatibility conditions among the extra data. For example, we require that for all $y_1, y_2 \in C$ the diagram

$$\begin{tikzcd} & (1_C \otimes y_1) \otimes y_2 \arrow[dl] \arrow[dr, "\iota"] & \\ 1_C \otimes (y_1 \otimes y_2) \arrow[rr] & & y_1 \otimes y_2 \end{tikzcd}$$

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring α, σ, ι with identity maps. These compositions have domain a tensor product of objects $y_1, \cdots, y_n$ and any number of identity objects 1_C—ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesizeed arbitrarily. Coherence theorems show there is a small set of conditions which need to be verified; then arbitrary diagrams of the sort envisioned commute.

Symmetric monoidal *functors* are homomorphisms between symmetric monoidal categeories, but as is typical for categories we express the fact that the identity maps to the identity and tensor product to tensor products through *data*, not as a condition. This leads to higher order conditions.

Definition 3.7. Let C, D be symmetric monoidal categories. A **symmetric monoidal functor** $F : C \to D$ is a functor with two additional pieces of data, namely an isomorphism

$$1_D \longrightarrow F(1_C)$$

and a natural isomorphism

$$\begin{tikzcd} C \times C \arrow[r, bend left, "F(-\otimes-)"] \arrow[r, bend right, "F(-)\otimes F(-)"'] \arrow[r, phantom, "\Uparrow\psi"] & C \end{tikzcd} .$$

There are many conditions on this data. The first condition expresses compatibility with the associativity morphisms: for all $y_1, y_2, y_3 \in C$ the diagram

$$\begin{tikzcd} (F(y_1)\otimes F(y_2))\otimes F(y_3) \arrow[r, "\psi"] \arrow[d, "\alpha_D"'] & F(y_1\otimes y_2)\otimes F(y_3) \arrow[d, "\psi"] \\ F(y_1)\otimes(F(y_2)\otimes F(y_3)) \arrow[d, "\psi"'] & F((y_1\otimes y_2)\otimes y_3) \arrow[d, "F(\alpha_C)"] \\ F(y_1)\otimes F(y_2\otimes y_3) \arrow[r, "\psi"] & F(y_1\otimes(y_2\otimes y_3)) \end{tikzcd}$$

Note the use of ψ vs $F(\alpha_C)$ or α_D (working in the domain vs codomain). Next there is compatibility with the identity data ι: for all $y \in C$ we require that

$$\begin{tikzcd} F(1_C)\otimes F(y) \arrow[r, "\psi"] & F(1_C\otimes Y) \arrow[d, "F(\iota)"] \\ 1_D\otimes F(y) \arrow[u, "1_D\to F(1_C)"] \arrow[r, "\iota_D"] & F(y) \end{tikzcd}$$

todo:so ψ combines and $F(\psi)$ splits? check usage of domain and target categories (ι_D). distinction: since these are natural transformations, is the notation abuse. it should be mapping to an iso. so "composition" is really sending the source/target (same), not the map itself.. The final condition expresses compatibility with the symmetry σ: for all $y_1, y_2 \in C$ the diagram

$$\begin{tikzcd} F(y_1)\otimes F(y_2) \arrow[r, "\psi"] \arrow[d, "\sigma_D"'] & F(y_1\otimes y_2) \arrow[d, "F(\sigma_C)"] \\ F(y_2)\otimes F(y_1) \arrow[r, "\psi"] & F(y_2\otimes y_1) \end{tikzcd}$$

commutes.

Example 3.5. The prototypical example of a symmetric monoidal category is $(\mathsf{Vect}_k, \otimes, k)$ where $\otimes$ denotes the standard tensor product. To show this, we need to check the conditions; we claim our object $1_{\mathsf{Vect}_k} \in \mathsf{Vect}_k$ is given by k, the functor $\otimes\colon \mathsf{Vect}_k \otimes \mathsf{Vect}_k \to \mathsf{Vect}_k$ is the standard tensor product, and the natural isomorphisms exist.

First we need to show $\otimes\colon \mathsf{Vect}_k \otimes \mathsf{Vect}_k \to \mathsf{Vect}_k$ is a functor. It maps objects componentwise by sending $V, W \xrightarrow{\otimes} V \otimes W$, and similarly it sends maps $f, g \xrightarrow{\otimes} f \otimes g$. Recall that for $f : U \to V$ linear and W a vector space, $f \otimes W : U \otimes W \to V \otimes W$ is the unique linear map satisfying $(f \otimes W)(u \otimes w) = f(u) \otimes w$ ($W \otimes f$ is similarly defined), and for $g : W \to Z$ we have $(f \otimes g)\colon U \otimes W \to V \otimes Z$ the unique linear map satisfying $(f \otimes g)(u \otimes w) = f(u) \otimes g(w)$. To show functorality, let $f : V \to W, g : W \to X, \ell : M \to N, h : N \to L$ be linear maps. We want to show that

$$\overset{\otimes((g,h)\circ(f,\ell))=}{\underset{V\otimes M\to X\otimes L}{(g\circ f)\otimes(h\circ\ell)}} = \underset{V\otimes M\to W\otimes N\to X\otimes L}{(g\otimes h)\circ(f\otimes\ell)}$$

Writing it out is the hard part. It follows from the definitions that for $v \in V, m \in M$, we have

$$(g \circ f) \otimes (h \circ \ell)(v, m) = g(f(v)) \otimes h(\ell(m))$$

and

$$(g \otimes h) \circ (f \otimes \ell)(v, m) = (g \otimes h)(f(v) \otimes \ell(m)) = g(f(v)) \otimes h(\ell(m)).$$

So the two are equal. The tensor product also plays well with identity maps since for $V, W \in \mathsf{Vect}_\mathsf{k}$, $\mathrm{id}_{V,W} : (V, W) \to (V, W)$, we have $\otimes(\mathrm{id}_{V,W}) = \otimes(V, W) = V \otimes W = \mathrm{im}_{\mathrm{id}_{V\otimes W}}(V \otimes W)$.

Now to show the natural associativity isomorphism α, associate to (U, V, W) the canonical associativity isomorphism $\alpha \colon (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. This is given by

To show that this is really a natural transformation, let $U, V, W, U', V', W' \in \mathsf{Vect}_\mathsf{k}$ with a map $f : (U, V, W) \to (U', V', W')$. Then the naturality square

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "((-\otimes-)\otimes-)f"] \arrow[d, "\alpha_{U,V,W}"'] & (U' \otimes V') \otimes W' \arrow[d, "\alpha_{U',V',W'}"] \\
U \otimes (V \otimes W) \arrow[r, "(-\otimes(-\otimes-))f"] & U' \otimes (V' \otimes W')
\end{tikzcd}$$

certainly commutes. Following an object $(u \otimes v) \otimes w$ across the top map first gives $(f(u) \otimes f(v)) \otimes f(w) = (u' \otimes v') \otimes w'$, then composing with the right map gives $u' \otimes (v' \otimes w')$. Following this element through the left map gives $u \otimes (v \otimes w)$, then composing with the bottom map gives $f(u) \otimes (f(v) \otimes f(w)) = u' \otimes (v' \otimes w')$. Therefore associativity is a natural isomorphism.

Our natural isomorphism σ is given by the isomorphism $\sigma_{V,W} : V \otimes W \cong W \otimes V$ which sends $v \otimes w \mapsto w \otimes v$. Drawing out the naturality square, for $V, W, V', W' \in \mathsf{Vect}_\mathsf{k}, f : (V, W) \to (V', W')$, we have

$$\begin{tikzcd}
V \otimes W \arrow[r, "(-\otimes-)f"] \arrow[d, "\sigma_{V,W}"'] & V' \otimes W' \arrow[d, "\sigma_{V',W'}"] \\
W \otimes V \arrow[r, "((-\otimes-)\tau)f"] & W' \otimes V'
\end{tikzcd}$$

which commutes by a quick check. It is also easy to check that $\sigma^2 = \mathrm{id}$ since $v \otimes w \xrightarrow{\sigma} w \otimes v \xrightarrow{\sigma} v \otimes w$. Finally, the identity transformation is given by $\iota_V : k \otimes V \to V$. To show this is an isomorphism, consider the bilinear projection $\pi \colon k \times V \to V, (k, v) \mapsto v$. Then by the universal property, the diagram

$$\begin{tikzcd}
k \times V \arrow[r, "(k,v)\mapsto k\otimes v"] \arrow[dr, "\pi"'] & k \otimes V \arrow[d, dotted, "\exists!\iota"] \\
 & V
\end{tikzcd}$$

commutes, therefore ι is an isomorphism. Once again, for $f : V \to W$ the naturality square is given by

$$\begin{tikzcd}
k \otimes V \arrow[r, "k\otimes f"] \arrow[d, "\iota_V"'] & k \otimes W \arrow[d, "\iota_W"] \\
V \arrow[r, "f"] & W
\end{tikzcd}$$

which clearly commutes. Therefore $(\mathsf{Vect}_\mathsf{k}, \otimes, k)$ is a symmetric monoidal category.

Definition 3.8. Let C, D be symmetric monoidal categories and $F, G \colon C \to D$ symmetric monoidal functors. Then a **symmetric monoidal natural transformation** $\eta\colon F \to G$ is a natural transformation such that the diagrams

$$\begin{tikzcd} & F(1_C) \arrow[dd, "\eta(1_C)"] \\ 1_D \arrow[ur] \arrow[dr] & \\ & G(1_C) \end{tikzcd}$$

and

$$\begin{tikzcd} F(y_1)\otimes F(y_2) \arrow[r, "\psi"] \arrow[d, "\eta\otimes\eta"'] & F(y_1\otimes y_2) \arrow[d, "\eta"] \\ G(y_1)\otimes G(y_2) \arrow[r, "\psi"] & G(y_1\otimes y_2) \end{tikzcd}$$

commute for all $y_1, y_2 \in C$.

3.3 Bordism Categories

Fix a nonnegative integer n.

Definition 3.9. Suppose $X, X'\colon Y_0 \to Y_1$ are bordisms between closed $(n-1)$-manifolds Y_0, Y_1. A **diffeomorphism** $F\colon X \to X'$ is a diffeomorphism of manifolds with boundary which commutes with p, θ_0, θ_1. So we have a diagram

$$\begin{tikzcd} X \arrow[dd, "F"'] \arrow[dr, "p"] & \\ & \{0,1\} \\ X' \arrow[ur, "p'"'] & \end{tikzcd}$$

and similar commutative diagrams involving the θ's.

Definition 3.10. Fix $n \in \mathbb{Z}^{\geq 0}$. The **bordism category** $\mathsf{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category defined as follows.

(i) Objects are closed $(n-1)$-manifolds.

(ii) The hom-set $\mathsf{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $X\colon Y_0 \to Y_1$.

(iii) Composition of morphisms is by gluing.

(iv) For each Y the bordism $[0,1]\times Y$ is $\mathrm{id}_Y\colon Y \to Y$.

(v) The monoidal product is disjoint union.

(vi) The empty manifold $\emptyset^{n-1}$ is the tensor unit (for the symmetric monoidal structure).

Definition 3.11. An **isotopy** is a smooth map $F\colon [0,1]\times Y \to Y$ such that $F(t,-)\colon Y \to Y$ is a diffeomorphism for all $t \in [0,1]$. A **pseudoisotopy** is a diffeomorphism $\widetilde{F}\colon [0,1]\times Y \to [0,1]\times Y$ which preserves the submanifolds $\{0\}\times Y$ and $\{1\}\times Y$. todo:intuition

Equivalently, an isotopy is a path in $\operatorname{Diff} Y$. Diffeomorphisms f_0, f_1 are said to be *isotopic* if there exists an isotopy $F\colon f_0 \to f_1$. Isotopies form an equivalence relation, and the set of isotopy classes is $\pi_0 \operatorname{Diff} Y$, often called the **mapping class group** of Y. An isotopy induces a pseudoisotopy

$$\begin{aligned} \widetilde{F}\colon [0,1]\times Y &\to [0,1]\times Y \\ (t,y) &\mapsto (t, F(t,y)) \end{aligned}$$

We say $\widetilde{F}\colon f_0 \to f_1$ if the induced diffeomorphisms of Y on the boundary of $[0,1]\times Y$ are f_0 and f_1.

Example 3.6. Let Y be a closed $(n-1)$-manifold and $f: Y \to Y$ a diffeomorphism. There is an associated bordism $X_f = [0,1]\times Y$. For $F\colon f_0 \to f_1$ an isotopy, then the bordisms X_{f_0} and X_{f_1} are equal in the hom-set $\mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y)$. In summary, there is a homomorphism $\pi_0(\mathrm{Diff}\, Y) \to \mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y)$, not necessarily injective. todo:inutition. pesuoisitopy = diffeo = equiv in bord

3.4 Examples of bordism categories

Example 3.7. There is a unique (-1)-dimensional manifold—the empty manifold $\emptyset^{-1}$—so $\mathsf{Bord}_{\langle -1,0\rangle}$ is a category with a single object, hence a monoid. The monoid is the set of morphisms $\mathsf{Bord}_{\langle -1,0\rangle}(\emptyset^{-1},\emptyset^{-1})$ under composition. The symmetric monoidal structure gives a second composition law, but it is the same as the first (and is necessarily commutative).

Namely, the monoid consists of diffeomorphism classes of closed 0-manifolds, so finite unions of points. The set of diffeomorphism classes is $\mathbb{Z}^{\geq 0}$, and composition/disjoint union both induce addition in $\mathbb{Z}^{\geq 0}$.

todo:bord SO- diffeo clsases of pt+ and pt-. shouldn't it be a two object category then?

Example 3.8. Now consider $\mathrm{Bord}_{\langle 1,2\rangle}$. Objects are closed 1-manifolds, or finite unions of circles. The cylinder can be interpreted as a bordism $X: (S^1)^{\amalg 2} \to \emptyset^1$ (right macaroni); the dual bordism $X^\vee$ is a map $X^\vee\colon \emptyset^1 \to (S^1)^{\amalg 2}$ (left macaroni). Let $\rho: S^1 \to S^1$ be reflection, $f = 1 \amalg \rho$ the indicated diffeomorphism of $(S^1)^{\amalg 2}$, and X_f the associated bordism (istopy). Then

$$X \circ X_{\mathrm{id}} \circ X^\vee \simeq \text{torus}$$
$$X \circ X_f \circ X^\vee \simeq \text{Klein bottle}$$

These diffeomorphisms become equations in the monoid $\mathrm{Bord}_{\langle 1,2\rangle}(\emptyset^1,\emptyset^1)$ of diffeomorphism classes of closed 2-manifolds.

4 Topological quantum field theory

In the same way we study abstract groups via their representations, in the same way we study bordism categories via representations.

Definition 4.1. Fix $n \in \mathbb{Z}^{\geq 0}$. Let C be a symmetric monoidal category. An **n-dimensional topological quantum field theory with values in C** is a symmetric monoidal functor

$$F: \mathrm{Bord}_{\langle n-1,n\rangle} \to C.$$

Recall that homology theory gives us symmetric monoidal functors $H_q\colon (\mathsf{Top},\amalg) \to (\mathsf{Ab},\oplus)$ for $q \in \mathbb{Z}^{\geq 0}$. One should think of the direct sum as *classical*; for *quantum* field theories we will use instead the tensor product. In vague terms, "quantization", the passage from classical to quantum, is a sort of exponentiation turning sums to products.

Example 4.1. Typical "linear" choices for C are:

(i) the symmetric monoidal category $(\mathsf{Vect}_k,\otimes)$ of vector spaces over a field k,
(ii) the symmetric category $({}_R\mathsf{Mod},\otimes)$ of left modules over a commutative ring R,
(iii) the symmetric monoidal category $(\mathsf{Ab},\otimes)$ of abelian groups under the tensor product.

We could also take the codomain to be a bordism category, which is decidedly nonlinear. For example, if M is a closed k-manifold, then there is a symmetric monoidal functor $-\times M\colon \mathsf{Bord}_{\langle n-1,n\rangle} \to \mathsf{Bord}_{\langle n+k-1,n+k\rangle}$. Let $F\colon \mathsf{Bord}_{\langle n+k-1,n+k\rangle} \to$ be any $(n+k)$-dimensional TQFT, then composition with $-\times M$ gives an n-dimensional TQFT, the **dimensional reduction of F along M**.

$$\mathsf{Bord}_{\langle n-1,n\rangle} \xrightarrow{-\times M} \mathsf{Bord}_{\langle n+k-1,n+k\rangle} \xrightarrow{F} C$$

Remark 4.1. If M is a closed oriented n-manifold, we can view M as a bordism $\emptyset^{n-1} \to \emptyset^{n-1}$, determining a morphism in $\mathsf{Bord}_{\langle n-1,n\rangle}$. If Z is an n-dimensional TQFT, then M determines a map $Z(M)\colon Z(\emptyset) \to Z(\emptyset)$. Now Z preserves unit objects as a tensor functor, that is, $Z(\emptyset)$ is *canonically* isomorphic to the base field k. Consequently, we can think of $Z(M)$ as an element of the endomorphism ring $\mathrm{Hom}_{\mathsf{Vect}_k}(k,k)$, that is, an element of k. So the functor Z assigns a *number* to every closed oriented manifold of dimension n.

4.1 TQFTs as a symmetric monoidal category

Fix $B = \mathsf{Bord}_{\langle n-1,n\rangle}$ and a symmetric monoidal category C. We explain that TQFTs $F\colon B \to C$ are objects in a symmetric monoidal category, with morphisms being symmetric monoidal natural transformations. The monoidal product of theories is defined by

$$(F_1 \otimes F_2)(Y) = F_1(Y) \otimes F_2(Y)$$
$$(F_1 \otimes F_2)(X) = F_1(X) \otimes F_2(X)$$

for all objects $Y \in B$ and morphisms $(X\colon Y_0 \to Y_1) \in B$. The tensor unit $\mathbf{1}$ is the trivial theory

$$1(Y) = 1_C, \quad 1(X) = \mathrm{id}_{1_C}$$

for all $Y \in B$ and $(X\colon Y_0 \to Y_1) \in B$. The symmetric monoidal categories of TQFTs is then denoted as

$$\mathrm{TQFT}_n = \mathrm{TQFT}_n[C] = \mathrm{Hom}^{\otimes}\left(\mathsf{Bord}_{\langle n-1,n\rangle}, C\right).$$

Example 4.2. Suppose $\eta\colon \mathbf{1} \to \mathbf{1}$ in TQFT_n. Then for all $Y \in \mathsf{Bord}_{\langle n-1,n\rangle}$, we have $\eta(Y) \in C(1_C, 1_C) = \mathrm{End}(1_C)$. If $C = \mathsf{Ab}$, then $\mathrm{End}(1_C) = \mathbb{Z}$, so η is a numerical invariant of closed $(n-1)$-manifolds. Furthermore if $X\colon Y_0 \to Y_1$ then by naturality we find that $\eta(Y_0) = \eta(Y_1)$. This shows that η factors down to a homomorphism of monoids

$$\eta\colon \Omega_{n-1} \to \mathrm{End}(1_C).$$

Now every element of Ω_{n-1} is invertible ($Y \amalg Y = \emptyset$), so the image of η consists of invertible elements. We say, simply, that η is invertible. todo:related to assigning each manifold a number? idea: η is an $(n-1)$ diemnsional theory, assigning numbers to $(n-1)$ manifolds, preserved by equivalence class, bordism invariant.

Theorem 4.1. *A morphism $(\eta\colon F \to G) \in \mathrm{TQFT}_n$ is invertible, and TQFT_n is a groupoid.*

The two statements are equivalent.

Central problem. Given a dimension n and a codomain category C we can ask to “compute” the groupoid $\mathrm{TQFT}_n[C]$. This is a vague problem whose solution is an equivalent groupoid which is “simpler” than the groupoid of topological quantum field theories. It has a nice answer when $n = 1$. In the oriented case it is a generalization of the theorem that Ω_0^{SO} is the free abelian group with a single generator pt_+. There is also a nice answer in the oriented case for $n = 2$.

4.2 Finiteness

Proposition 4.1. *Let $F: \mathsf{Bord}_{\langle n-1,n\rangle} \to \mathsf{Vect}_{\mathbb{C}}$ be a TQFT. Then for all $Y \in \mathsf{Bord}_{\langle n-1,n\rangle}$ the vector space $F(Y)$ is finite dimensional.*

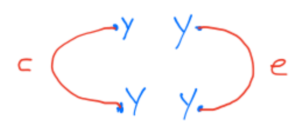

Figure 1: Evaluation and coevaluation

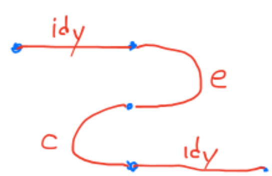

Figure 2: The S-diagram

Proof. Fix $Y \in \mathsf{Bord}_{\langle n-1,n\rangle}$ and let $V = F(Y)$. Let $c: \emptyset^{n-1} \to Y \amalg Y$ and $e: Y \amalg Y \to \emptyset^{n-1}$ be the bordisms in Figure 1. The manifold Y is depicted as a point, and each bordism has underlying manifold $[0,1] \times Y$. The composition depicted in Figure 2 is diffeomorphic to the identity bordism $\mathrm{id}_Y: Y \to Y$, which maps to $\mathrm{id}_V: V \to V$ under F. OTOH, the compositon maps to

$$V \xrightarrow{\mathrm{id}_V \otimes F(c)} V \otimes V \otimes V \xrightarrow{F(e) \otimes \mathrm{id}_V} V.$$

Let the value of $F(c): \mathbb{C} \to V \otimes V$ on $1 \in \mathbb{C}$ be $\sum_i v_i' \otimes v_i''$ for some finite set of vectors $v_i', v_i'' \in V$. Then equating with the identity map, we get that for all $\xi \in V$, $\xi = \sum_i e(\xi, v_i')v_i''$, and so v_i'' spans V. Then V is finite dimensional, and we are done. ⊠

It is clear to see that $F(c): \mathbb{C} \to V \otimes V$ and $F(e): V \otimes V \to \mathbb{C}$ are inverse bilinear forms, since both their compositions are diffeomorphic to the identity. Explicitly, $F(c): 1 \mapsto \sum_i v_i' \otimes v_i''$, and sending this through $F(e)$ results in 1.

4.3 Duality

Definition 4.2. Let C be a symmetric monoidal cateogry and $y \in C$.

(i) **Duality data** for y is a triple of data $(y^\vee, c, e)$ in which $y^\vee$ is an object of C and c, e are morphisms $c: 1_C \to y \otimes y^\vee, e: y^\vee \otimes y \to 1_C$. We require that the compositions

$$y \xrightarrow{c \otimes \mathrm{id}_y} y \otimes y^\vee \otimes y \xrightarrow{\mathrm{id}_y \otimes e}$$

and

$$y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes c} y^\vee \otimes y \otimes y^\vee \xrightarrow{e \otimes \mathrm{id}_{y^\vee}} y^\vee$$

be identity maps. If duality data exists for y, then we say that y is **dualizable**.

(ii) A **morphism of duality data** $(y^\vee, c, e) \to (\widetilde{y}^\vee, \widetilde{c}, \widetilde{e})$ is a morphism $y^\vee \xrightarrow{f} \widetilde{y^\vee}$ such that the diagrams

$$\begin{tikzcd} & y \otimes y^\vee \arrow[dd, "\mathrm{id}_Y \otimes f"] \\ 1_C \arrow[ur, "c"] \arrow[dr, "\widetilde{c}"'] & \\ & y \otimes \widetilde{y^\vee} \end{tikzcd}$$

and

$$\begin{tikzcd} y^\vee \otimes y \arrow[dd, "f \otimes \mathrm{id}_Y"'] \arrow[dr, "e"] & \\ & 1_C \\ \widetilde{y^\vee} \otimes y \arrow[ur, "\widetilde{e}"'] & \end{tikzcd}$$

commute. c is called **coevaluation** and e is called **evaluation**.

We now express the uniqueness of duality data. We cannot say there is a unique object as duality data is an object in a category, however, we do have the strongest form of uniqueness possible in a category: duality data is unique up to unique isomorphism.

Definition 4.3. Let C be a category.

(i) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ is either empty or contains a unique element, we say that C is a **discrete groupoid** . todo:what about $y_0 \neq y_1$ aren't there two choices? not equivalent to objects (only identity map)

(ii) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ has a unique element, we say that C is **contractible**. todo:check: contractible $\subseteq$ discrete groupoids

A discrete groupoid is equivalent to a set. A contractible groupoid is equivalent to a category with one object and one morphism, the categorical analog of a point. We prove this second assertion.

Proof. Let G be a contractible groupoid. Consider a functor $F\colon G \to \mathrm{pt}$, where $g_i \mapsto \mathrm{pt}$ for every $g_i \in G$ and $(g_i \xrightarrow{ij} g_j) \mapsto \mathrm{id}_{\mathrm{pt}}$ for every morphism $ij \in \mathrm{Mor}(G)$. There is only one $d \in D$ so essential surjectivity is clear. In the same way, every pair of elements only has one morphism between them, so this functor is well-defined and a bijection (the target category only has one morphism as well, the identity $\mathrm{id}_{\mathrm{pt}}$). So F is essentially surjective and fully faithful, and we are done. todo:check

The reason why this argument fails for discrete groupoids is that *for every* $c_1, c_2 \in C$, there has to be a bijection of hom sets onto its image. Sometimes $\mathrm{Hom}(g_1, g_2)$ will be empty in a discrete groupoid, hence there is no bijection $\emptyset \to \{*\}$. ⊠

Proposition 4.2. *Let C be a symmetric monoidal category and $y \in C$. Then the category of duality data for y is either empty or contractible.*

Proof. todo: ⊠

Definition 4.4. Let $y_0, y_1 \in C$ be dualizable objects in a symmetric monoidal category and $f : y_0 \to y_1$ a morphism. The **dual morphism** $f^\vee : y_1^\vee \to y_0^\vee$ is the composition

$$y_1^\vee \xrightarrow{\mathrm{id}_{y_1^\vee} \otimes c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{\mathrm{id}_{y_1^\vee} \otimes f \otimes \mathrm{id}_{y_0^\vee}} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_1 \otimes \mathrm{id}_{y_0^\vee}} y_0^\vee$$

Exercise 4.1. For $C = \mathsf{Vect}$, let $T : M \to N$ be a linear map.