380C PROBLEM SET 5

DUE WEDNESDAY, OCTOBER 6TH

Problem 1.

(a) Recall that there is a natural $G \times G$-action on G given by $(\gamma_1, \gamma_2) \bullet g := \gamma_1 g \gamma_2^{-1}$.

Show that $G \simeq (G \times G)/G$ as sets with $G \times G$-actions; on the right hand side, we are considering G as the subgroup $\{(g, g) \mid g \in G\}$ of $G \times G$.

(b) Let H and K be subgroups of G. We consider $H \times K$ acting on G via the above action of $G \times G$. It is conventional to let $H\backslash G/K$ denote the set of orbits for this action; orbits themselves are called *double cosets*.

Show that there is a canonical bijection $H\backslash G/K \simeq G\backslash(G/H \times G/K)$ (where G acts diagonally on the right hand side).

(c) Suppose G acts on a set X. Suppose that there are exactly two orbits for the action of G on $X \times X$.

Show that X is isomorphic (as a set with a G-action) to G/H for some *maximal* subgroup $H \subseteq G$, i.e., $H \subsetneq G$ and there does not exist a subgroup $H \subsetneq H_0 \subsetneq G$.

Deduce that S_n is a maximal subgroup of S_{n+1}.

Problem 2. Suppose G is a finite group and p, q are distinct primes dividing $|G|$. Suppose G has a unique p-Sylow (resp. q-Sylow) subgroup G_p (resp. G_q). Show that elements of G_p and G_q commute with each other. Deduce that if all Sylow subgroups of G are normal, then G is the product of its Sylow subgroups.

Problem 3. A group G is *nilpotent* if there is an increasing sequence of subgroups:

$$\{1\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n = G$$

such that each G_i is normal in G and each G_{i+1}/G_i is abelian.

(a) Define normal subgroups $Z_i \subseteq G$ inductively by taking $Z_0 = \{1\}$ and Z_{i+1} to be the preimage of the center $Z(G/Z_i)$ of G/Z_i under the projection $G \to G/Z_i$.

Show that G is nilpotent if and only if $Z_n = G$ for $n \gg 0$.

(b) Show that any p-group is nilpotent.

(c) Suppose G is a finite group. Show that G is nilpotent if and only if each of its Sylow subgroups is normal.

Problem 4. Suppose A is a commutative ring and $f \in A[t]$ has degree d, so $f = a_d t^d + \ldots + a_0$ with $a_d \neq 0$.

For an integer $n \geqslant 0$, let $A[t]_{<n} \subseteq A[t]$ be the set of elements of the form $\sum_{i=0}^{n-1} b_i t^i$ for $b_i \in A$.

(a) Suppose f is *monic*, i.e., $a_d = 1$. Show that the composition $A[t]_{<d} \hookrightarrow A[t] \to A[t]/f$ is a bijection.

(b) Suppose A is a domain and a_d is not a unit in A (e.g., $A = \mathbb{Z}$ and $a_d = 2$). Show that $A[t]_{<n} \to A[t]/f$ is not surjective for any n.

Problem 5.

(a) Let A be a commutative ring and let $I_1, \ldots, I_n$ be *pairwise coprime ideals* in A, meaning $I_i + I_j = A$ for every $i \neq j$. Show that the natural map:
$$A/(I_1 \cap \ldots \cap I_n) \to A/I_1 \times \ldots \times A/I_n$$
is an isomorphism.

(b) Show that for a field k and pairwise distinct elements $\lambda_1, \ldots, \lambda_n \in k$ and (arbitrary) elements $\eta_1, \ldots, \eta_n$, there exists a unique polynomial $f \in k[t]$ of degree less than n such that $f(\lambda_i) = \eta_i$ for every i. (Hint: the map $k[t] \xrightarrow{t \mapsto \lambda} k$ induces an isomorphism $k[t]/(t - \lambda) \simeq k$.)