Differential Geometry Notes

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Lecture 1

Curves and Surfaces

1.1 Curves

A curve $\mathscr{C} := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$. Curves in $\mathbb{R}^3$ are defined similarly. These are called **level curves**.

Definition 1.1. A **parametrized curve** in $\mathbb{R}^n$ is a map $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ for some α, β with $-\infty \le \alpha \le \beta \le \infty$. A parametrized curve whose image is contained in a level curve $\mathscr{C}$ is called a **parametrization** of $\mathscr{C}$.

Example 1.1. We parametrize the parabola. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the components γ_1 and γ_2 of γ must satisfy $\gamma_2(t) = \gamma_1(t)^2$ for all $t \in (\alpha, \beta)$. The parametrization $\gamma\colon (-\infty,\infty) \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$ works, as well as $\gamma(t) = (t^3, t^6)$, $\gamma(t) = (2t, 4t^2)$, and so on.

For the circle $x^2 + y^2 = 1$, we could try $x = t$, but that only hits half of S^1. What satisfies $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$? $\gamma_1(t) = \cos t$ and $\gamma_2(t) = \sin t$ do. The interval $(-\infty, \infty)$ is overkill since the map has infinite degree.

Example 1.2. Consider the *astroid* $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in \mathbb{R}$. Since $\cos^2 t + \sin^2 t = 1$ for all t, then $x = \cos^3 t,\ y = \sin^3 t$ satisfy $x^{2/3} + y^{2/3} = 1$.

A function $f : (\alpha, \beta) \to \mathbb{R}$ is **smooth** if $\frac{d^n f}{dt^n}$ exists for all $n \ge 1$ and $t \in (\alpha, \beta)$. Smoothness is preserved under addition, multiplication, composition, etc. You differentiate vector valued functions componentwise, and we denote $d\gamma/dt$ by $\dot\gamma(t)$, $d^2\gamma/dt^2$ by $\ddot\gamma(t)$, etc.

Definition 1.2. If γ is a parametrized curve, then $\dot\gamma(t)$ is the **tangent vector** of γ at the point $\gamma(t)$.

Proposition 1.1. *If the tangent vector of a parametrized curve is constant, then the image of the curve is a straight line.*

Proof. If $\dot\gamma(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is constant, then

$$\gamma(t) = \int \frac{d\gamma}{dt}\, dt = \int \mathbf{a}\, dt = t\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is another constant vector. ⊠

Example 1.3. The **limaçon** is the parametrized curve $\gamma(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t)$, $t \in \mathbb{R}$. There's a self intersection at the origin, the tangent vector is $\dot\gamma(t) = (-\sin t - 2\sin 2t, \cos t + 2\cos 2t)$. This is well defined, but takes two different values at $t = 2\pi/3$ and $t = 4\pi/3$.

1.2 Arc Length

The length of a straight line segment between two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is $||\mathbf{u} - \mathbf{v}||$, given the standard norm/inner product/metric/blah on $\mathbb{R}^n$.

Definition 1.3. The **arc-length** of a curve γ starting at $\gamma(t_0)$ is the function $s(t)$ given by

$$s(t) = \int_{t_0}^{t} ||\dot\gamma(u)||\, du.$$

Example 1.4. For a **logarithmic spiral** $\gamma(t) = (e^{kt}\cos t, e^{kt}\sin t)$, we have $\dot\gamma = (e^{kt}(k\cos t - \sin t), e^{kt}(k\sin t + \cos t))$, so $||\dot\gamma||^2 = e^{2kt}(k\cos t - \sin t)^2 + e^{2kt}(k\sin t + \cos t)^2 = (k^2+1)e^{2kt}$. Then the arc length of γ starting at $\gamma(0) = (1,0)$ is

$$s = \int_0^t \sqrt{k^2+1}\, e^{ku}\, du = \frac{\sqrt{k^2+1}}{k}(e^{kt} - 1).$$

Note that the arc-length is differentiable, that is,

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\gamma}(u)\|\,du = \|\dot{\gamma}(t)\|.$$

Definition 1.4. If $\gamma\colon(\alpha,\beta)\to\mathbb{R}^n$ is a parametrized curve, its **speed** at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be a **unit-speed** curve if $\dot{\gamma}(t)$ is a unit vector for all $t\in(\alpha,\beta)$.

Proposition 1.2. *Let $\mathbf{n}(t)$ be a unit vector that is a smooth function of parameter t. Then the dot product $\dot{\mathbf{n}}(t)\cdot\mathbf{n}(t)=0$ for all t, i.e., $\dot{\mathbf{n}}(t)$ is zero or orthogonal to $\mathbf{n}(t)$ for all t. If γ is a unit-speed curve, then $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.*

Proof. We differentiate $\mathbf{n}\cdot\mathbf{n}=1$ to get $\dot{\mathbf{n}}\cdot\mathbf{n}+\mathbf{n}\cdot\dot{\mathbf{n}}=0$, so $\dot{\mathbf{n}}\cdot\mathbf{n}=0$. ⊠

1.3 Reparametrization

A parametrized curve $\tilde{\gamma}\colon(\tilde{\alpha},\tilde{\beta})\to\mathbb{R}^n$ is a **reparametrization** of a parametrized curve $\gamma\colon(\alpha,\beta)\to\mathbb{R}^n$ if there is a smooth bijective map $\phi\colon(\tilde{\alpha},\tilde{\beta})\to(\alpha,\beta)$ (the *reparametrization map*) such that the inverse map $\phi^{-1}\colon(\alpha,\beta)\to(\tilde{\alpha},\tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$ for all $\tilde{t}\in(\tilde{\alpha},\tilde{\beta})$.

Note that since ϕ has a smooth inverse, γ is a reparametrization of $\tilde{\gamma}$, since $\tilde{\gamma}(\phi^{-1}(t))=\gamma(\phi(\phi^{-1}(t)))=\gamma(t)$ for all $t\in(\alpha,\beta)$.

Example 1.5. We can reparametrize the circle as $\tilde{\gamma}(t)=(\sin t,\cos t)$. To show this, we want to find a reparametrization map ϕ such that $(\cos\phi(t),\sin\phi(t))=(\sin t,\cos t)$. $\phi(t)=\pi/2-t$ works.

Definition 1.5. A point $\gamma(t)$ of a parametrized curve γ is called a **regular point** if $\dot{\gamma}(t)\neq\mathbf{0}$; otherwise $\gamma(t)$ is a **singular point** of γ. A curve is **regular** if all of its points are regular.

Proposition 1.3. *Any reparametrization of a regular curve is regular.*

Proof. Suppose $\tilde{\gamma}$ is a reparametrization of γ, let $t=\phi(\tilde{t})$ and $\psi=\phi^{-1}$ such that $\tilde{t}=\psi(t)$. Differentiating both sides of $\phi(\psi(t))=t$ WRT t gives $\frac{d\phi}{d\tilde{t}}\frac{d\psi}{dt}=1$. So $d\phi/d\tilde{t}$ is never zero. Since $\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$, differentiating again gives $\frac{d\tilde{\gamma}}{d\tilde{t}}=\frac{d\gamma}{dt}\frac{d\phi}{d\tilde{t}}$, so $d\tilde{\gamma}/d\tilde{t}$ is never zero, if $d\gamma/dt$ is never zero. ⊠

Proposition 1.4. *If $\gamma(t)$ is regular, then s is a smooth function of t.*

Proof. Recall that $\frac{ds}{dt}=\|\dot{\gamma}(t)\|=\sqrt{\dot{u}^2+\dot{v}^2}$. Since $f(x)=\sqrt{x}$ is *smooth* on $(0,\infty)$, along with u and v, and $\dot{u}^2+\dot{v}^2>0$ for all t (since γ is regular), s itself is also smooth. ⊠

Proposition 1.5. *A parametrized curve has a unit-speed reparametrization iff it is regular.*

Proof. Suppose a parametrized curve $\gamma\colon(\alpha,\beta)\to\mathbb{R}^n$ has a unit-speed reparametrization $\tilde{\gamma}$, with a reparametrization map ϕ. Letting $t=\phi(\tilde{t})$, we have $\tilde{\gamma}(\tilde{t})=\gamma(t)$ and so

$$\frac{d\tilde{\gamma}}{d\tilde{t}}=\frac{d\gamma}{dt}\frac{dt}{d\tilde{t}} \implies \left\|\frac{d\tilde{\gamma}}{d\tilde{t}}\right\|=\left\|\frac{d\gamma}{dt}\right\|\left|\frac{dt}{d\tilde{t}}\right|.$$

Since $\tilde{\gamma}$ is unit speed, $\|d\tilde{\gamma}/d\tilde{t}\|=1$, so $d\gamma/dt$ cannot be zero. ⊠

Lecture 2

How much does a curve curve?

Let's talk about curvature. Curvature measures the extent to which a curve isn't contained in a straight line, while torsion measures the extent to which a curve isn't contained in a plane (so planes have zero torsion). We discover that torsion and curvature determine the shape of a curve.

2.1 Curvature

Definition 2.1. If γ is a unit-speed curve with parameter t, its **curvature** $\kappa(t)$ at the point $\gamma(t)$ is defined as $\|\ddot{\gamma}(t)\|$.

Note that curvature is defined for unit speed curves in $\mathbb{R}^n$ for $n \geq 2$. The curvature of a circle is $1/R$. If γ is regular, then we can just define the curvature as $\kappa(\tilde{\gamma})$, where $\tilde{\gamma}$ is the unit speed parametrization of κ. Sometimes its hard to write down explicit parametrizations so let's find a definition that doesn't depend on that.

Proposition 2.1. *Let $\gamma(t)$ be regular in $\mathbb{R}^3$. Then*

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

Proof. Let s be a unit speed parametrization of γ. Then $\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt}$, so

$$\kappa = \left\| \frac{d^2\gamma}{ds^2} \right\| = \left\| \frac{d}{ds}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} \right) \right\| = \left\| \frac{\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} \right)}{\frac{ds}{dt}} \right\| = \left\| \frac{\frac{ds}{dt}\frac{d^2\gamma}{dt^2} - \frac{d^2s}{dt^2}\frac{d\gamma}{dt}}{\left(\frac{ds}{dt}\right)^3} \right\|.$$

Now $(ds/dt)^2 = \|\dot{\gamma}\|^2 = \dot{\gamma} \cdot \dot{\gamma}$, and differentiating WRT t gives $\frac{ds}{dt}\frac{d^2s}{dt^2} = \dot{\gamma} \cdot \ddot{\gamma}$. Combining this with our huge equation above we have

$$\kappa = \left\| \frac{\left(\frac{ds}{dt}\right)^2 \ddot{\gamma} - \frac{d^2s}{dt^2}\frac{ds}{dt}\dot{\gamma}}{\left(\frac{ds}{dt}\right)^4} \right\| = \frac{\|(\dot{\gamma} \cdot \dot{\gamma})\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}\|}{\|\dot{\gamma}\|^4}.$$

Since $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we have $\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma}) = (\dot{\gamma} \cdot \dot{\gamma})\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}$. Furthermore, $\dot{\gamma}$ and $\ddot{\gamma} \times \dot{\gamma}$ are orthogonal, so $\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\| = \|\dot{\gamma}\|\|\ddot{\gamma} \times \dot{\gamma}\|$. Therefore

$$\frac{\|(\dot{\gamma} \cdot \dot{\gamma})\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}\|}{\|\dot{\gamma}\|^4} = \frac{\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\|}{\|\dot{\gamma}\|^4} = \frac{\|\dot{\gamma}\|\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^4} = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

⊠

This formula for curvature makes sense if $\dot{\gamma} \neq 0$. So the curvature is defined for all regular points on a curve.

Example 2.1. Consider the **circular helix** defined by

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta), \quad \theta \in \mathbb{R},$$

where a and b are constants. The **radius** of the helix is a and the **pitch** is $2\pi b$. To compute the curvature, note that $\|\dot{\gamma}(\theta)\| = \sqrt{a^2 + b^2}$, so $\dot{\gamma}(\theta)$ is never zero (and hence γ is regular) for $a \neq 0$ and $b \neq 0$. The book does this with some cross product stuff, and it turns out to be constant with curvature $\frac{|a|}{a^2+b^2}$. If $b = 0$, this projects the helix onto the plane, and so the curvature is $1/|a|$. If $a = 0$, then the helix is a straight line, so the curvature is zero.

2.2 Plane curves

Say $\gamma(s)$ is unit speed in $\mathbb{R}^2$, and let $\mathbf{t} = \dot{\gamma}$ be the **tangent vector** of γ. Define $\mathbf{n}_s$, the **signed unit normal** of γ, to be the unit vector obtained by rotating $\mathbf{t}$ counterclockwise by $\pi/2$. Now $\dot{\mathbf{t}} = \ddot{\gamma}$ is orthogonal to $\mathbf{t}$, and therefore parallel to $\mathbf{n}_s$. So we have a scalar κ_s such that $\ddot{\gamma} = \kappa_s \mathbf{n}_s$. This κ_s is the **signed curvature** of γ. Since $\|\mathbf{n}_s\| = 1$, we have

$$\kappa = \|\ddot{\gamma}\| = \|\kappa_s \mathbf{n}_s\| = |\kappa_s|.$$

Positive curvature curves outward, negative curvature points inward. For γ a regular but not unit-speed curve, define $\mathbf{t}$, $\mathbf{n}_s$, and κ_s as the ones corresponding to its unit speed parametrization $\tilde{\gamma}(s)$, where s is the arc-length of γ.So $\mathbf{t} = \frac{d\gamma/dt}{ds/dt} = \frac{d\gamma/dt}{\|d\gamma/dt\|}$, $\mathbf{n}_s$ is obtained by rotating $\mathbf{t}$ counterclockwise by $\pi/2$, and

$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \kappa_s \frac{ds}{dt}\mathbf{n}_s = \kappa_s \left\| \frac{d\gamma}{dt} \right\| \mathbf{n}_s.$$

If γ is unit-speed, the direction of the tangent vector $\dot{\gamma}(s)$ is measured by $\varphi(s)$ where $\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s))$. The angle $\varphi(s)$ is not always unique, but we do always have a *smooth* choice:

Proposition 2.2. *Let $\gamma\colon(\alpha,\beta)\to\mathbb{R}^2$ be a unit-speed curve, let $s_0\in(\alpha,\beta)$, and let φ_0 satisfy*

$$\dot\gamma(s_0)=(\cos\varphi_0,\sin\varphi_0).$$

Then we have a unique smooth function $\varphi\colon(\alpha,\beta)\to\mathbb{R}$ such that $\varphi(s_0)=\varphi_0$ and that $\dot\gamma(s)=(\cos\varphi(s),\sin\varphi(s))$ for all $s\in(\alpha,\beta)$.

Proof. time is finite ⊠

Definition 2.2. The smooth function φ is the **turning angle** of γ determined by the condition $\varphi(s_0)=\varphi_0$.

Proposition 2.3. *Let $\gamma(s)$ be a unit-speed plane curve, and let $\varphi(s)$ be a turning angle for γ. Then $\kappa_s=d\varphi/ds$. So the signed curvature is the rate at which the tangent vector of the curve rotates.*

Proof. Now $\mathbf{t}=(\cos\varphi,\sin\varphi)$, so $\dot{\mathbf{t}}=\dot\varphi(-\sin\varphi,\cos\varphi)$. Since $\mathbf{n}_s=(-\sin\varphi,\cos\varphi)$, $\dot{\mathbf{t}}=\kappa_s\mathbf{n}_s$. ⊠

Example 2.2. Let us compute the signed curvature of the catenary with parametrization $\gamma(t)=(t,\cosh t)$. Then $\dot\gamma=(1,\sinh t)$, and

$$s=\int_0^t\sqrt{1+\sinh^2 t}\,dt=\sinh t.$$

If φ is the angle between $\dot\gamma$ and the x-axis, $\tan\varphi=\sinh t=s$, so $\sec^2\varphi\frac{d\varphi}{ds}=1$, and so

$$\kappa_s=\frac{d\varphi}{ds}=\frac{1}{\sec^2\varphi}=\frac{1}{1+\tan^2\varphi}=\frac{1}{1+s^2}.$$

Corollary 2.1. *Define the **total signed curvature** of a unit-speed* closed *curve γ of length ℓ as $\int_0^\ell\kappa_s(s)\,ds$. Then the total signed curvature of a closed plane curve is an integer multiple of 2π.*

Proof. Let γ be a unit-speed closed plane curve with length ℓ. The total signed curvature is

$$\int_0^\ell\frac{d\varphi}{ds}\,ds=\varphi(\ell)-\varphi(0),$$

where φ is a turning angle for γ. Since γ is l-periodic, $\gamma(s+\ell)=\gamma(s)$. So $\dot\gamma(s+\ell)=\dot\gamma(s)$, in particular $\dot\gamma(\ell)=\dot\gamma(0)$. So

$$(\cos\varphi(\ell),\sin\varphi(\ell))=(\cos\varphi(0),\sin\varphi(0)),$$

implying that $\varphi(\ell)-\varphi(0)$ is an integer multiple of 2π. ⊠

An **isometry** is a map $M:\mathbb{R}^2\to\mathbb{R}^2$ of the form $M=T_{\mathbf{a}}\circ\rho_\theta$, where ρ_θ is an anticlockwise rotation by an angle θ about the origin, and $T_{\mathbf{a}}$ is the translation by $\mathbf{a}$. That is,

$$\rho_\theta(x,y)=(x\cos\theta-y\sin\theta,x\sin\theta+y\cos\theta),$$
$$T_{\mathbf{a}}(\mathbf{v})=\mathbf{v}+\mathbf{a}.$$

Theorem 2.1. *Let $k\colon(\alpha,\beta)\to\mathbb{R}$ be smooth. Then we have a unit speed curve $\gamma\colon(\alpha,\beta)\to\mathbb{R}^2$ whose signed curvature is k. Furthermore, if $\widetilde\gamma\colon(\alpha,\beta)\to\mathbb{R}^2$ is another unit-speed curve with signed curvature k, there exists an isometry M of $\mathbb{R}^2$ such that $\widetilde\gamma(s)=M(\gamma(s))$ for all $s\in(\alpha,\beta)$.*

Proof. HURRY UP PLEASE IT'S TIME[1] ⊠

Example 2.3. Any regular plane curve γ whose curvature is a positive constant is part of a circle. Since $\kappa_s\pm\kappa$, the possibility of a curve having $\kappa_s=\kappa$ at some points and $\kappa_s=-\kappa$ other places is null because γ is continuous (precisely, by IVT). Given κ_s we want to find a parametrized circle with such signed curvature. By our theorem, every curve with constant curvature maps onto this circle by an isometry. This shows what we wanted to show.

A unit speed parametrization of the circle is $\gamma(s)=\left(R\cos\frac{s}{R},R\sin\frac{s}{R}\right)$, with tangent vector $\left(-\sin\frac{s}{R},\cos\frac{s}{R}\right)$ making an angle $\pi/2+s/R$ with the x-axis. So the signed curvature of γ is $\frac{d}{ds}\left(\frac{\pi}{2}+\frac{s}{R}\right)=\frac{1}{R}$. So the curve of radius $1/\kappa_s$ has signed curvature κ_s. If $\kappa_s<0$, the curve $\widetilde\gamma(s)=\left(R\cos\frac{s}{R},-R\sin\frac{s}{R}\right)$ has signed curvature $-\frac{1}{R}$.

[1] I have an exam tomorrow and am cramming material, hence all the omitted proofs.

Example 2.4. Simple curvatures can lead to complicated curves. For example, if $\kappa_s(s) = s$, taking $s_0 = 0$ we have $\varphi(s) = \int_0^s u\,du = \frac{s^2}{2}$, so

$$\gamma(s) = \left(\int_0^s \cos\left(\frac{t^2}{2}\right) dt, \int_0^s \sin\left(\frac{t^2}{2}\right) dt \right)$$

The curve is really pretty, and is called *Cornu's Spiral*. These integrals cannot be evaluated in 'elementary functions', and are called *Fresnel's integrals*. They arise when studying diffraction of light.

Remark 2.1. How do we compute κ_s in general?

(1) Compute the standard curvature and find out the sign,

(2) Use the formula $\kappa_s = d\varphi/ds$, then note that $\tan\varphi = y$-component of γ' divided by x-component of γ', and use $\sec^2\varphi \frac{d\varphi}{ds} =$ whatever,

(3) Note that $\mathbf{t} = \gamma'/||\gamma'||$ and $d\mathbf{t}/ds := \kappa_s \mathbf{n}_s$, where $\mathbf{n}_s = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\mathbf{t}$.

2.3 Space curves

Curves in $\mathbb{R}^3$ are not determined by their curvature. Let $\gamma(s)$ be unit-speed in $\mathbb{R}^3$, and $\mathbf{t} = \dot{\gamma}$ be its unit tangent. If $\kappa(s)$ is non-zero, define the **principal normal** of γ at $\gamma(s)$ to be the vector $\mathbf{n}(s) = \frac{1}{\kappa(s)}\dot{\mathbf{t}}(s)$. Since $||\dot{\mathbf{t}}|| = \kappa$, $\mathbf{n}$ is a unit vector. Since $\mathbf{t} \cdot \dot{\mathbf{t}} = 0$, $\mathbf{t}$ and $\mathbf{n}$ are actually orthogonal. Then the **binormal** vector $:= \mathbf{t} \times \mathbf{n}$ is orthogonal to both $\mathbf{t}$ and $\mathbf{n}$, so we have $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ an orthonormal basis of $\mathbb{R}^3$. Furthermore, this basis is *right-handed*, i.e.,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{t} = \mathbf{n} \times \mathbf{b}.$$

Note that $\dot{\mathbf{b}} = \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} = \mathbf{t} \times \dot{\mathbf{n}}$. Since $\dot{\mathbf{b}}$ is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$, it must be a scalar multiple of $\mathbf{n}$, that is, $\dot{\mathbf{b}} = -\tau\mathbf{n}$ for some τ. Let us call the τ the **torsion** of γ. Note that curvature and torsion are well-defined for any regular curve.

Proposition 2.4. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then*

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{||\dot{\gamma} \times \ddot{\gamma}||^2}$$

Proof. $\langle algebra \rangle$ ⊠

Example 2.5. Let us compute the torsion of the circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$. Now

$$\begin{gathered} \dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b), \\ \ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta, 0), \\ \dddot{\gamma}(\theta) = (a\sin\theta, -a\cos\theta, 0), \\ ||\dot{\gamma} \times \ddot{\gamma}||^2 = a^2(a^2 + b^2), \quad (\dot{\gamma} \times \ddot{\gamma}) \cdot (\dddot{\gamma}) = a^2 b, \end{gathered}$$

so

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{||\dot{\gamma} \times \ddot{\gamma}||^2} = \frac{a^2 b}{a^2(a^2 + b^2)} = \frac{b}{a^2 + b^2}.$$

Proposition 2.5. *Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then the image of γ is contained in a plane iff $\tau = 0$ at all points on γ.*

Proof. Suppose the image of γ is contained in a plane $\mathbf{v} \cdot \mathbf{N} = d$, where $\mathbf{N}$ is a constant vector, d is a constant scalar, and $\mathbf{v} \in \mathbb{R}^3$. Assume $\mathbf{N}$ is a unit vector, then differentiating $\gamma \cdot \mathbf{N} = d$ wrt s gives

$$\begin{aligned} \mathbf{t} \cdot \mathbf{N} = 0 &\implies \dot{\mathbf{t}} \cdot \mathbf{N} = 0. \\ &\implies \kappa\mathbf{n} \cdot \mathbf{N} = 0 \\ &\implies \mathbf{n} \cdot \mathbf{N} = 0. \end{aligned}$$

So both $\mathbf{n}$ and $\mathbf{t}$ are orthogonal to $\mathbf{N}$, which implies $\mathbf{b}$ is parallel to $\mathbf{N}$. So $\mathbf{b}(s) = \pm\mathbf{N}$ for all s. In either case, $\mathbf{b}$ is constant, so $\dot{\mathbf{b}} = 0$, and $\tau = 0$.

Now assume $\tau = 0$. Then $\dot{\mathbf{b}} = 0$, and $\mathbf{b}$ is constant. Consider

$$\frac{d}{ds}(\gamma \cdot \mathbf{b}) = \dot{\gamma} \cdot \mathbf{b} = \mathbf{t} \cdot \mathbf{b} = 0,$$

so $\gamma \cdot \mathbf{b}$ is a scalar (say d). So γ is contained in the plane $\mathbf{v} \cdot \mathbf{b} = d$. ⊠

We know that $\dot{\mathbf{t}} = \kappa\mathbf{n}$ and $\dot{\mathbf{b}} = -\tau\mathbf{n}$ by definition, but we don't know how to compute $\dot{\mathbf{n}}$. Since $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ form a right-handed orthonormal basis of $\mathbb{R}^3$, we have $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. So

$$\dot{\mathbf{n}} = \dot{\mathbf{b}} \times \mathbf{t} + \mathbf{b} \times \dot{\mathbf{t}} = -\tau\mathbf{n} \times \mathbf{t} + \kappa\mathbf{b} \times = -\kappa\mathbf{t} + \tau\mathbf{b}.$$

Theorem 2.2 (Frenet-Serret equations). *Let γ be unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then*

$$\begin{aligned} \dot{\mathbf{t}} &= \kappa\mathbf{n}, \\ \dot{\mathbf{n}} &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \dot{\mathbf{b}} &= -\tau\mathbf{n}. \end{aligned}$$

These equations are called the ***Frenet-Serret equations****. Note that the matrix in the matrix equation*

$$\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix}$$

is ***skew-symmetric****, that is, $A = -A^T$.*

Theorem 2.3. *Let $\gamma(s)$ and $\widetilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s) > 0$ and torsion $\tau(s)$ for all s. Then we have an isometry M of $\mathbb{R}^3$ such that*

$$\widetilde{\gamma}(s) = M(\gamma(s)) \quad \textit{for all } s.$$

Furthermore, if k and t are smooth function with $k > 0$ everywhere, then there is unit-speed curve in $\mathbb{R}^3$ whose curvature is k and torsion is t.

Lecture 3

Global properties of curves

3.1 Simple closed curves

Definition 3.1. A **simple closed curve** in $\mathbb{R}^2$ is a closed curve in $\mathbb{R}^2$ with no self-intersections. We say γ is **positively-oriented** if the signed unit normal $\mathbf{n}_s$ of γ points into γ° at each point of γ.

Hopf's Umlaufsatz. *The total signed curvature of a simple closed curve in $\mathbb{R}^2$ is $\pm 2\pi$.*

We already know that the total signed curvature of any closed curve is an integer multiple of 2π, the *Umlaufsatz* (German for 'rotation theorem') states that this integer must be ± 1.

3.2 The isoperimetric inequality

The **area** in a simple closed curve is given by $\mathscr{A}(\gamma) = \int_{\gamma^\circ} dx\,dy$, which can be computed by Green's theorem.

Green's Theorem. *Let $f(x, y)$ and $g(x, y)$ be smooth and γ be a positively oriented simple closed curve. Then*

$$\int_{\gamma^\circ} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy = \int_\gamma f(x, y)\,dx + g(x, y)\,dy.$$

Proposition 3.1. *If $\gamma(t)=(x(t),y(t))$ is a positively-oriented simple closed curve in $\mathbb{R}^2$ with period T, then*

$$\mathscr{A}(\gamma)=\frac{1}{2}\int_0^T (x\dot{y}-y\dot{x})\,dt.$$

Proof. Let $f=-\frac{1}{2}y$, $g=\frac{1}{2}x$ in Green's theorem, then $\mathscr{A}(\gamma)=\frac{1}{2}\int_\gamma x\,dy-y\,dx$. ⊠

Isoperimetric Inequality. *Let γ be a simple closed curve, $\ell(\gamma)$ be its length, and $\mathscr{A}(\gamma)$ be the area contained in it. Then*

$$\mathscr{A}(\gamma)\le\frac{1}{4\pi}\ell(\gamma)^2,,$$

and equality holds iff γ is a circle.

Use something called *Wirtinger's Inequality*.

3.3 The four vertex theorem

Definition 3.2. A **vertex** of a curve $\gamma(t)$ in $\mathbb{R}^2$ is a point where its signed curvature κ_s has a stationary point, i.e., where $d\kappa_s/dt=0$.

Four Vertex Theorem. *Every simple closed curve in $\mathbb{R}^2$ has at least four vertices.*

Lecture 4

Surfaces in three dimensions

4.1 What are surfaces?

Do you know what open sets are?

Definition 4.1. A subset $U\subseteq\mathbb{R}^n$ is **open** if for $\mathbf{a}\in U$, there exists an $\varepsilon>0$ such that every $\mathbf{u}\in\mathbb{R}^n$ within a distance ε of $\mathbf{a}$ also lies in U. Using equations:

$$\mathbf{a}\in U \text{ and } \|\mathbf{u}-\mathbf{a}\|<\varepsilon \implies \mathbf{u}\in U.$$

In simpler terms, every point has a neighborhood contained in U.

$\mathbb{R}^n$ is open but the closed ball is not. Consider a map $f:X\to Y$. We say f is **continuous** at $\mathbf{a}$ if for any $\varepsilon>0$, there exists a $\delta>0$ such that for $\mathbf{u}\in X$,

$$\|\mathbf{u}-\mathbf{a}\|<\delta \implies \|f(\mathbf{u})-f(\mathbf{a})\|<\varepsilon.$$

This is equivalent to the fact that the preimage of an open set is open. See here for more details: `simonxiang.xyz/blog/topological-continuity-simplicity-in-abstraction`[2]. Homeomorphism are continuous bijections with a continuous inverse.

Definition 4.2. A set $\mathscr{S}\subseteq\mathbb{R}^3$ is a **surface** if for every $\mathbf{p}\in\mathscr{S}$, there is an open set $U\subseteq\mathbb{R}^2$ and an open set $W\subseteq\mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathscr{S}\cap W$ is homeomorphic to U. Basically, locally a surface has to look like a 2-manifold (rather, it *is* a 2-manifold). A homeomorphism $\sigma:U\to\mathscr{S}\cap W$ defined above is a **surface patch** or **parametrization** of the open subset $\mathscr{S}\cap W$ of $\mathscr{S}$. A collection of charts covering $\mathscr{S}$ formes an **atlas** of $\mathscr{S}$.

OK, we called surface patches charts in differential topology, so I will be calling surface patches charts from now on. Aren't we missing the condition that charts have to be C^∞ compatible as well to form an atlas?

[2] shameless plug

Example 4.1. A plane in $\mathbb{R}^3$ is a 2-manifold with a single chart. Let $\mathbf{a}$ lie in the plane, and $\mathbf{p}, \mathbf{q}$ be orthogonal unit vectors parallel to the plane. If $\mathbf{v}$ also lies in the plane, then $\mathbf{v}-\mathbf{a}$ is parallel to the plane, so $\mathbf{v}-\mathbf{a}=u\mathbf{p}+v\mathbf{q}$. So the surface patch is $\sigma(u,v)=\mathbf{a}+u\mathbf{p}+v\mathbf{q}$, with inverse $\sigma^{-1}(\mathbf{v})=((\mathbf{v}-\mathbf{a})\cdot\mathbf{p},(\mathbf{v}-\mathbf{a})\cdot\mathbf{q})$. This is clearly a continous homeomorphism.

Example 4.2. Why do we talk about charts? Consider a *circular cylinder*, the set of points in $\mathbb{R}^3$ a fixed distance from an axis. For example, say the circle is of radius 1 around the z-axis, which we will call the *unit cylinder.* This is defined by

$$\mathscr{S}=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1\}.$$

We can parametrize this by $\sigma(u,v)=(\cos u,\sin u,v)$. The map σ is continuous but not injective, since it's periodic. Restricting to an interval of length less than 2π gives an injective map, say $[0,2\pi]$. However, although the restriction $\sigma|_V$ where $V=\{(u,v)\in\mathbb{R}^2 \mid u\in[0,2\pi]\}$ is injective, V is not open and so $\sigma|_V$ is not a surface patch. If we restrict σ to $U=V^\circ=\{(u,v)\in\mathbb{R}^2 \mid u\in(0,2\pi)\}$, then $\sigma|_U$ is a chart. However, $\sigma|_U$ does not hit the line $x=1, y=0$ in $\mathscr{S}$, so it does not cover $\mathscr{S}$.

We need another chart to make an atlas. So consider the chart $\sigma|_{\widetilde{U}}$, where $\widetilde{U}=\{(u,v)\in\mathbb{R}^2 \mid u\in[-\pi,\pi]\}$. This covers $\mathscr{S}$ sans the line $x=-1, y=0$. Joining these two charts give an atlas, and so $\mathscr{S}$ is a surface.

Example 4.3. Say hello to your old friend S^2, defined by

$$S^2=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}.$$

A popular parametrization is given by latitude θ and longitude φ: projecting a point p on the sphere down to the xy-plane gives a point q, then θ is the angle between p and q, and φ is the angle between q and the positive x-axis. Circles corresponding to θ are called **parallels**, and those corresponding to φ are called **meridians**.

To find an explicit parametrization, we want to express p in terms of θ and φ. The z-component is $\sin\theta$ by looking at the triangle. come back to this since it's important but not essentially important, essentially parametrizing the sphere and showing it's a 2-manifold with two charts

Example 4.4. Our next (non)example is the **circular cone** with a vertex at a point $\mathbf{v}$ with an axis a straight line ℓ passing through $\mathbf{v}$, and an angle α, where $\alpha\in(0,\pi/2)$. It consists of the set of points $\mathbf{p}\in\mathbb{R}^3$ such that the straight line through $\mathbf{v}$ and $\mathbf{p}$ makes an angle α with the line ℓ. For example, for $\mathbf{v}$ the origin, ℓ the z-axis and $\alpha=\pi/4$ we have the circular cone defined by

$$\mathscr{S}=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=z^2\}.$$

If we take the image of a chart around the origin in $\mathbb{R}^2$, by path-connectedness we can find a path from c to b, where c corresponds to a point q in the upper cone and b corresponds to a point p in the lower cone. Furthermore, we can choose this path to avoid the origin, so its preimage in $\mathscr{S}$ is a path joining the hemispheres that avoids the origin. However, $\mathscr{S}\setminus\{0\}$ is not connected, and p and q lie in different connected components, so this is a contradiction.

If we delete the origin, we do get a surface $\mathscr{S}_-\cup\mathscr{S}_+$, with an atlas consisting of two charts given by the inverse of projection.

Usually a point $\mathbf{a}$ on a surface will lie in more than two charts. If we want two charts $\sigma,\widetilde{\sigma}$ to speak to each other, consider the **transition maps** $\sigma^{-1}\circ\widetilde{\sigma}$, $\widetilde{\sigma}^{-1}\circ\sigma$.

4.2 Smooth surfaces

We will use the following abbreviations:

$$\frac{\partial\mathbf{f}}{\partial u}=\partial_u\mathbf{f},\quad \frac{\partial^2\mathbf{f}}{\partial u^2}=\partial_{uu}\mathbf{f},\quad \frac{\partial^2\mathbf{f}}{\partial u\partial v}=\partial_{uv}\mathbf{f},\quad \text{etc.}$$

Answer to my question above: surface is a codeword for topological manifold, while smooth surface is a codeword for smooth manifolds. You know what smooth functions are.

Definition 4.3. A surface patch $\sigma: U\to\mathbb{R}^3$ is **regular** if it is smooth and the vectors $\partial_u\sigma,\partial_v\sigma$ are LI at all (u,v). Equivalently, the product $\partial_u\sigma\times\partial_v\sigma$ should be non-zero at each point in U.

I guess my usage of chart earlier was incorrect, it's just a local homeomorphism onto its image. Now charts are *allowable surface patches*, or a regular surface patch $\sigma\colon U\to\mathbb{R}^3$ that is a homeomorphism onto its image. An atlas is what you think it is. (We've talked about compatible charts, where is the condition that charts have to be compatible for a smooth manifold?)

Example 4.5. A plane living in $\mathbb{R}^3$ is a surface, as well as the unit cylinder and S^2.

The book states the condition I've been waiting for as a proposition, that is, the transition maps are smooth. Interesting.

Proposition 4.1. *Let U and $\widetilde{U}$ be open subsets of $\mathbb{R}^2$ and $\sigma\colon U\to\mathbb{R}^3$ be a regular surface patch. Let $\Phi\colon\widetilde{U}\to U$ be a smooth bijection with smooth inverse. Then $\widetilde{\sigma}=\sigma\circ\Phi\colon\widetilde{U}\to\mathbb{R}^3$ is a regular surface patch.*

Proof. We have $\widetilde{\sigma}$ smooth because the composition of smooth maps is smooth. For regularity, let $(u,v)=\Phi(\widetilde{u},\widetilde{v})$. By the chain rule, we have

$$\partial_{\widetilde{u}}\widetilde{\sigma}=\frac{\partial u}{\partial\widetilde{u}}\partial_u\sigma+\frac{\partial v}{\partial\widetilde{u}}\partial_v\sigma,\quad \partial_{\widetilde{v}}\sigma=\frac{\partial u}{\partial\widetilde{v}}\partial_u\sigma+\frac{\partial v}{\partial\widetilde{v}}\partial_v\sigma,$$

so

$$\partial_{\widetilde{u}}\widetilde{\sigma}\times\partial_{\widetilde{v}}\widetilde{\sigma}=\left(\frac{\partial u}{\partial\widetilde{u}}\frac{\partial v}{\partial\widetilde{v}}-\frac{\partial u}{\partial\widetilde{v}}\frac{\partial v}{\partial\widetilde{u}}\right)\partial_u\sigma\times\partial_v\sigma.$$

Note that the scalar is just the Jacobian determinant of Φ, which is nonzero ($J(\Phi^{-1})=J(\Phi)^{-1}$, so $J(\Phi)$ is invertible). We conclude that $\widetilde{\sigma}$ is regular. ⊠

We say the chart $\widetilde{\sigma}$ is a **reparametrization** of σ, and Φ is a **reparametrization map**. Note that σ is a reparametrization of $\widetilde{\sigma}$ by Φ^{-1}. Also note that any two charts are reparametrizations of each other. This is important because we don't want things to depend on our choice of chart. From now on, surface means smooth surface and charat means smooth chart (whoops I've already been doing this). We also assume surfaces are connected.

4.3 Smooth maps

Say $\mathscr{S}_1,\mathscr{S}_2$ are surfaces covered by single charts $\sigma_1\colon U_1\to\mathbb{R}^3$ and $\sigma_2\colon U_2\to\mathbb{R}^3$. Then a map $f:\mathscr{S}_1\to\mathscr{S}_2$ is **smooth** if the map $\sigma_2^{-1}\circ f\circ\sigma_1\colon U_1\to U_2$ is smooth.

$$\begin{tikzcd}
\underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[d, "\sigma_2"] \\
\underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2}
\end{tikzcd}$$

Suppose $\widetilde{\sigma}_1\colon\widetilde{U}_1\to\mathbb{R}^3$ and $\widetilde{\sigma}_2\colon\widetilde{U}_2\to\mathbb{R}^3$ are reparametrizations of σ_1 and σ_2 with reparametrization maps $\Phi_1\colon\widetilde{U}_1\to U_1$ and $\Phi_2\colon\widetilde{U}_2\to U_2$. We want to show that $\widetilde{\sigma}_2^{-1}\circ f\circ\widetilde{\sigma}_1\colon\widetilde{U}_1\to\widetilde{U}_2$ is smooth (provided the other map is smooth).

$$\begin{tikzcd}
\widetilde{U}_1 \arrow[r, "\Phi_1"] \arrow[rrr, bend left, "\widetilde{\sigma}_2^{-1}\circ f\circ\widetilde{\sigma}_1"] \arrow[dr, "\widetilde{\sigma}_1"'] & \underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[d, "\sigma_2"] & \widetilde{U}_2 \arrow[l, "\Phi_2"'] \arrow[dl, "\widetilde{\sigma}_2"] \\
& \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2} &
\end{tikzcd}$$

If we write $\widetilde{\sigma}_2^{-1}=\Phi_2^{-1}\circ\sigma_2^{-1}$ and $\widetilde{\sigma}_1=\sigma_1\circ\Phi_1$, then subsituting and applying the associative property gives the map as $\Phi_2^{-1}\circ(\sigma_2^{-1}\circ f\circ\sigma_1)\circ\Phi_1$, which is smooth if each component is. We have the middle component smooth by assumption, and the Φ_i are smooth by definition. The composition of smooth functions is also smooth:

$$\begin{tikzcd}
\underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] \arrow[rr, bend left, "(\sigma_3^{-1}\circ g\circ\sigma_2^{-1})\circ(\sigma_2\circ f\circ\sigma_1)=\sigma_3^{-1}\circ(g\circ f)\circ\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[r, dotted, "\sigma_3^{-1}\circ g\circ\sigma_2"] \arrow[d, "\sigma_2"] & \underset{\subseteq\mathbb{R}^2}{U_3} \arrow[d, "\sigma_3"] \\
\underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2} \arrow[r, "g"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_3}
\end{tikzcd}$$

We choose the same chart U_2 that gets mapped onto by U_1 and maps to U_3 since choice of chart is independent of smoothness, as we have just shown. Bijective Smooth maps $f : \mathscr{S}_1 \to \mathscr{S}_2$ with smooth inverse are called **diffeomorphisms**. A smooth map $f : \mathscr{S}_1 \to \mathscr{S}_2$ is a **local diffeomorphism** if for any $p \in \mathscr{S}_1$, we have an open $\mathscr{O} \subseteq \mathscr{S}_1$ such that $f|_{\mathscr{O}}$ is a diffeomorphism onto its image.

Proposition 4.2. *Let $f : \mathscr{S}_1 \to \mathscr{S}_2$ be a local diffeomorphism. If f is injective and σ_1 a smooth chart on $\mathscr{S}_1$, then $f \circ \sigma_1$ is a smooth chart on $\mathscr{S}_2$.*

Example 4.6. Consider the map from the yz-plane to the unit cylinder $\mathscr{S}$, wrapping each line parallel to the axis around the cylinder at height z. This map is defined by $f(0, y, z) = (\cos y, \sin y, z)$. This is not injective, so not a diffeomorphism, but is a local diffeomorphism. Parametrizing by the chart $\pi(u, v) = (0, u, v)$ and using the atlas $\{\sigma|_U, \sigma|_{\tilde{U}}\}$ of $\mathscr{S}$, let $p = (0, a, b)$ be a point in the yz-plane. If a is not an even multiple of 2π, then we have an $n \in \mathbb{Z}$ such that $2\pi n < a < 2(n+1)\pi$ and

$$f(\pi(u, v)) = \sigma(u - 2\pi n, v) \quad \text{if} \quad 2\pi n < u < 2(n+1)\pi.$$

So f is a diffeomorphism from the open set $\mathscr{O} = \{(0, y, z) \mid 2\pi n < y < 2(n+1)\pi\}$ of the plane to the open set $f(\mathscr{O}) = \{(x, y, z) \in \mathscr{S} \mid x \neq 1\}$. We use the other chart if a is not an odd multiple of π.

4.4 Tangents and derivatives

A **tangent vector** to a surface S[3] is the tangent vector at p of some curve in S. The **tangent space** T_pS at p is the set of all tangent vectors to S at p.

Proposition 4.3. *Let $\sigma : U \to \mathbb{R}^3$ be a chart containing some $p \in S$, and (u, v) be coordinates in U. The tangent space T_pS is the subspaces of $\mathbb{R}^3$ spanned by $\partial_u\sigma, \partial_v\sigma$.*

Proof. Let $\gamma(t) = \sigma(u(t), v(t))$ be smooth. Then by the chain rule we have $\dot{\gamma} = \partial_u\sigma\dot{u} + \partial_v\sigma\dot{v}$. So $\dot{\gamma}$ is a linear combination of $\partial_u\sigma$ and $\partial_v\sigma$. Conversely, any vector in a subspace spanned by $\partial_u\sigma$ and $\partial_v\sigma$ is of the form $\lambda\partial_u\sigma + \mu\partial_u\sigma$. Define $\gamma(t) = \sigma(u_0 + \lambda t, v_0 + \mu t)$. Then γ is smooth in S, and at $t = 0$ we have $\dot{\gamma} = \lambda\partial_u\sigma + \mu\partial_v\sigma$. So every vector in the space is the tangent vector of some curve. ⊠

Denote the vectors $\partial_u\sigma, \partial_v\sigma$ that span T_pS as the **parameter curves** on the surface. Suppose $f : S \to \tilde{S}$ is smooth: the derivative should measure show a point $f(p) \in \tilde{S}$ changes when p moves to a nearby point, say q, of S. If p and q are close, the line near them should be tangent to S at p. So we expect that the derivative of f at p associates to any tangent vector to S at p a tangent vector to $\tilde{S}$ at $f(p)$. In other words, the derivative of f should be a map $D_pf : T_pS \to T_{f(p)}\tilde{S}$.

Definition 4.4. Let $w \in T_pS$ be a tangent vector to S at p. Then w is the tangent vector at p of a curve γ in S passing through p, say $w = \dot{\gamma}(t_0)$. Then $\tilde{\gamma} = f \circ \gamma$ is a curve in $\tilde{S}$ passing through $f(p)$ when $t = t_0$, so $\tilde{w} = \dot{\tilde{\gamma}}(t_0) \in T_{f(p)}\tilde{S}$. We say the **derivative** D_pf of f at $p \in S$ is the map $D_pf : T_pS \to T_{f(p)}\tilde{S}$ such that $D_pf(w) = \tilde{w}$ for any tangent vector $w \in T_pS$.

Proposition 4.4. *The derivative is linear.*

Proposition 4.5.

[3] Surfaces are now denoted S, I'm tired of typing `\mathcal S`.

(i) If S is a surface and $p \in S$, then the derivative of the identity at p is $\mathrm{id}\colon T_pS \to T_pS$.

(ii) Chain rule, $D_p(f_2 \circ f_1) = D_{f_1(p)} f_2 \circ D_p f_1$.

(iii) If $f : S_1 \to S_2$ *is a diffeomorphism then* $D_p f : T_p S_1 \to T_{f(p)} S_2$ *is invertible.*

Proposition 4.6. *Let* $f : S \to \widetilde{S}$ *be smooth. Then* f *is a local diffeomorphism iff for all* $p \in S$, $D_p f : T_p S \to T_{f(p)} \widetilde{S}$ *is invertible.*

todo:section on orientability

Lecture 5

Examples of surfaces

5.1 Level surfaces

Theorem 5.1. *Let* $S \subseteq \mathbb{R}^3$ *such that for each* $p \in S$, *there is a* $W_p \subseteq \mathbb{R}^3$ *open and a smooth* $f : W \to \mathbb{R}$ *such that*

(i) $S \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$,

(ii) ∇f *is nonvanishing at* p.

Then S is a smooth surface.

Example 5.1. We can construct S^2 in this manner by letting $W = \mathbb{R}^3$ and considering the single function $f(x, y, z) = x^2 + y^2 + z^2 - 1$, since the gradient $\nabla f = (2x, 2y, 2z)$ is nonvanishing.

Example 5.2. Consider the cone cut out by $f(x, y, z) = x^2 + y^2 - z^2$, which vanishes at the origin. Then the cone minus the origin is a surface.

5.2 Quadric surfaces

Definition 5.1. A **quadric** is the a subset of $\mathbb{R}^3$ defined by an equation of the form

$$v^t A v + b^t v + c = 0,$$

where $v = (x, y, z)$, A is a constant symmetric 3×3, $b \in \mathbb{R}^3$ is constant, $c \in \mathbb{R}$ is a scalar. Explicitly, for $A = \left(\begin{smallmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{smallmatrix}\right)$, $b = (b_1, b_2, b_3)$ we have

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 xy + 2a_5 yz + 2a_6 xz + b_1 x + b_2 y + b_3 z + c = 0.$$

Some quadrics which are not surfaces include $x^2 + y^2 + z^2 = 0,\ x^2 + y^2 = 0,\ xy = 0$.

Theorem 5.2. *Up to isometry, every non-empty quadric with not all zero coefficients can be transformed into one of the following:*

(i) Ellipsoid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$.

(ii) One sheeted hyperboloid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$.

(iii) Two sheeted hyperboloid: $\frac{z^2}{r^2} - \frac{z^2}{p^2} - \frac{y^2}{q^2} = 1$.

(iv) Elliptic paraboloid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$.

(v) Hyperbolic paraboloid: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$.

(vi) *Quadric cone:* $\frac{x^2}{p^2}+\frac{y^2}{q^2}-\frac{z^2}{r^2}=0$.

(vii) *Elliptic cylinder:* $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$.

(viii) *Hyperbolic cylinder:* $\frac{x^2}{p^2}-\frac{y^2}{q^2}=1$.

(ix) *Parabolic cylinder:* $\frac{x^2}{p^2}=y$.

(x) *Plane:* $x=0$.

(xi) *Two parallel planes:* $x^2=p^2$.

(xii) *Two intersection planes:* $\frac{x^2}{p^2}-\frac{y^2}{q^2}=0$.

(xiii) *Straight line:* $\frac{x^2}{p^2}+\frac{y^2}{q^2}=0$.

(xiv) *Point:* $\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}=0$.

Proof. oh save me there's a proof todo:this ⊠

Corollary 5.1. *Every nonempty quadric of types (i)-(x) is a surface (for (vi) remove the vertex).*

5.3 Ruled surfaces and surfaces of revolution

Definition 5.2. A **ruled surface** is a surface that is a union of straight lines, called the **rulings** (or **generators**) of the surface.

If the rulings are all parallel, then S is a **generalized cylinder**. We don't want a curve that passes through all the lines to be tangent to the rulings (intersect transversely??). Some noninteresting stuff happened.

5.4 Compact surfaces

Example 5.3. You know what a compact set is. Some examples:

- Spheres are compact subspaces of $\mathbb{R}^{n+1}$. They are clearly bounded, and closed since the complement $\mathbb{R}^{n+1}\setminus S^n = B^{n+1}\cup(\mathbb{R}^{n+1}\setminus D^{n+1})$, a union of two open sets (where B^{n+1} is the open n-ball and D^{n+1} is the closed n-disk).
- Planes are not compact since they're unbounded. Neither are open disks since they're open.
- In our $\mathbb{R}^3$, the torus and the other genus n surfaces are also compact. These turn out to classify compact 2-manifolds up to diffeomorphism:

  Theorem 5.3. *Up to diffeomorphism, the only compact* 2*-manifolds are the genus n surfaces for $n\geq 0$.*

A corollary of this:

Corollary 5.2. *Every compact surface is orientable.*

Proof. By the Jordan separation theorem we can separate a genus n surface into a bounded interior and unbounded exterior. Define the unit normal at each point on the surface to point toward the exterior. ⊠

5.5 Triply orthogonal systems

We skipped this section.

5.6 Applications of the IFT

This section too.

Lecture 6

The first fundamental form

Now we do some geometry.

6.1 Lengths of curves on surfaces

Definition 6.1. Let $p \in S$. The **first fundamental form** of S at p associates to tangent vectors $v, w \in T_pS$ the scalar $\langle v, w\rangle_{p,S} = v \cdot w$.

This is an inner product for S at p. If you know how tensors work, at each point we can write $g = g_{ij}^{(x)} dx^i \otimes dx^j$, and we can calculate the coefficients of the metric tensor by $g_{ij} = g(\partial_i, \partial_j)_x = \langle \partial_i, \partial_j\rangle_x$. Traditionally, for $\sigma(u,v)$ a patch, if $p \in \operatorname{im}\sigma$ we have $\partial_u\sigma, \partial_v\sigma$ spanning T_pS. Define $du\colon T_pS \to \mathbb{R},\ dv\colon T_pS \to \mathbb{R}$, by

$$du(v) = \lambda, \quad dv(v) = \mu \quad \text{if } v = \lambda\partial_u\sigma + \mu\partial_v\sigma.$$

(These are 1-forms, the basis vectors of the cotangent space.) Then $\langle v, v\rangle = \lambda^2\langle\partial_u\sigma, \partial_u\sigma\rangle + 2\lambda\mu\langle\partial_u\sigma, \partial_v\sigma\rangle + \mu^2\langle\partial_v\sigma, \partial_v\sigma\rangle$. If we write $E = \|\partial_u\sigma\|^2,\ F = \partial_u\sigma\cdot\partial_v\sigma,\ G = \|\partial_v\sigma\|^2$, then

$$\langle v, v\rangle = E\lambda^2 + 2F\lambda\mu + G\mu^2 = Edu^2 + 2Fdudv + Gdv^2.$$

If $\gamma \subseteq \operatorname{im}\sigma$, $\gamma(t) = \sigma(u(t), v(t))$ for u, v smooth. Then $\dot\gamma = \dot u\partial_u\sigma + \dot v\partial_v\sigma$, so $\langle\dot\gamma, \dot\gamma\rangle = E\dot u^2 + 2F\dot u\dot v + G\dot v^2$, and the length of γ is given by $\int (E\dot u^2 + 2F\dot u\dot v + G\dot v^2)^{1/2}\,dt$.

Example 6.1. For a plane $\sigma(u,v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$ for unit vectors $\mathbf{p} \perp \mathbf{q}$, we have $\partial_u\sigma = \mathbf{p}, \partial_v\sigma = \mathbf{q}$, then $E = \|\partial_u\sigma\|^2 = \|\mathbf{p}\|^2 = 1, F = \partial_u\sigma\cdot\partial_v\sigma = 0, G = \|\mathbf{q}\|^2 = 1$. So $g_{ij} = \delta^i_j$, and the first fundamental form is $du^2 + dv^2$.

Example 6.2. From now on we write $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$. For a surface of revolution of the form $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, since $\partial_u\sigma = (\dot f\cos v, \dot f\sin v, \dot g)$, $\partial_v\sigma = (-f\sin v, f\cos v, 0)$, we have $g_{11} = \dot f^2 + \dot g^2 = 1$ (since the curve is unit speed), $g_{12} = g_{21} = 0, g_{22} = f^2$. So the fff is $du^2 + f(u)^2dv^2$. If we take $u = \theta, v = \varphi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$, this gives us the fff of S^2 as $d\theta^2 + \cos^2\theta\, d\phi^2$.

Example 6.3. A generalized cylinder $\sigma(u,v) = \gamma(u) + v\mathbf{a}$ as $g_{ij} = \delta^i_j$, and so the fff is $du^2 + dv^2$ as well.

6.2 Isometries of surfaces

The plane and generalized cylinder have the same fff, which is because they're isometric (intrinsic curvature?). However, this doesn't hold for the sphere, because you can't "wrap" a piece of paper around it.

Definition 6.2. If S_1, S_2 are surfaces, a smooth map $f : S_1 \to S_2$ is a **local isometry** if it takes any curve in S_1 to a curve of the same length in S_2. If $f : S_1 \to S_2$ is a local isometry, then S_1 and S_2 are **locally isometric**.

We will see that every local isometry is a local diffeomorphism, and a global diffeomorphism that is also a local isometry is just an **isometry**. Let $f : S_1 \to S_2$ be smooth and $p \in S_1$. For $v, w \in T_pS_1$, define

$$f^*\langle v, w\rangle_p = \langle D_pf(v), D_pf(w)\rangle_{f(p)}.$$

Then $f^*\langle\ ,\ \rangle_p$ is a symmetric bilinear form because the inner product is symmetric and the derivative is bilinear.

Theorem 6.1. *A smooth map $f : S_1 \to S_2$ is a local isometry iff the symmetric bilinear forms $\langle\ ,\ \rangle_p$ and $f^*\langle\ ,\ \rangle_p$ on T_pS_1 are equal for all $p \in S_1$.*

Proof. For γ_1 a curve in S_1, a curve has length $\int_{t_0}^{t_1}\langle\dot\gamma_1, \dot\gamma_1\rangle^{1/2}\,dt$. Then the length of $\gamma_2 = f\circ\gamma_1$ is

$$\int_{t_0}^{t_1}\langle\dot\gamma_2, \dot\gamma_2\rangle^{1/2}\,dt = \int_{t_0}^{t_1}\langle Df(\dot\gamma_1), Df(\dot\gamma_1)\rangle^{1/2}\,dt = \int_{t_0}^{t_1} f^*\langle\dot\gamma_1, \dot\gamma_1\rangle^{1/2}\,dt.$$

So if $\langle\ ,\ \rangle_p$ and $f^*\langle\ ,\ \rangle_p$ are the same, curves have the same length. OTOH, suppose the integrals are euqal, then the integrands are the same, that is, $\langle\dot\gamma, \dot\gamma\rangle = f^*\langle\dot\gamma, \dot\gamma\rangle$. So $\langle v, v\rangle = f^*\langle v, v\rangle$ since tangent vectors are tangent to a curve. ⊠

Since f is a local isometry iff $\langle D_p f(v), D_p f(w)\rangle_{f(p)} = \langle v, w\rangle_p$, $D_p f$ is then an isometry. Then every local isometry is a local diffeomorphism, since if it wasn't we would have a nontrivial $v \in T_p S_1$ such that $v \in \ker(D_p f)$, or $D_p f(v) = 0$. But

$$0 \neq \langle v, v\rangle_p \langle D_p f(v), D_p f(v)\rangle_{f(p)} = \langle 0, 0\rangle_p = 0,$$

a contradiction.

Corollary 6.1. *A local diffeomorphism is a local isometry iff for any surface patch σ_1 of S_1, the patches σ_1 and $f \circ \sigma_1$ of S_1 and S_2 respectively have the same fff.*

Proof. Same process as Theorem 6.1. ⊠

This actually shows that if $p \in S_1$ is in $\operatorname{im}\sigma_1$, then σ_1 and $f \circ \sigma_1$ have the same fff at p iff $D_p f$ is an isometry. It follows that if $p \in \operatorname{im}\sigma_2$, then σ_1 and $f \circ \sigma_1$ have the same fff at p iff the same is true of σ_2 and $f \circ \sigma_2$.

Example 6.4. The unit cylinder, the generalized cone, and the plane are locally isometric.

It turns out there is another class of circles locally isometric to a plne, called **tangent developables**, the tangent bundle to a curve in $\mathbb{R}^3$.

6.3 Conformal mappings of surfaces

Conformal mappings don't preserve length, but they do preserve angle. Say $\gamma, \tilde{\gamma} \subseteq S$ and $p \in \gamma \cap \tilde{\gamma}$. We define the **angle** θ of intersection of γ and $\tilde{\gamma}$ at p as the angle between the tangent vectors $\dot{\gamma}$ and $\dot{\tilde{\gamma}}$ (at $t = t_0$, $t = \tilde{t}_0$ respectively). Then

$$\cos\theta = \frac{\dot{\gamma} \cdot \dot{\tilde{\gamma}}}{\|\dot{\gamma}\|\|\dot{\tilde{\gamma}}\|} = \frac{\langle \dot{\gamma}, \dot{\tilde{\gamma}}\rangle}{\langle \dot{\gamma}, \dot{\gamma}\rangle^{1/2}\langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}\rangle^{1/2}}.$$

If γ and $\tilde{\gamma}$ lie in a surface patch σ such that $\gamma(t) = \sigma(u(t), v(t))$ and $\tilde{\gamma}(t) = \sigma(\tilde{u}(t), \tilde{v}(t))$ for some smooth $u, v, \tilde{u}, \tilde{v}$, then

$$\cos\theta = \frac{g_{11}\dot{u}\dot{\tilde{u}} + g_{12}(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + g_{22}\dot{v}\dot{\tilde{v}}}{(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2)^{1/2}(g_{11}\dot{\tilde{u}}^2 + 2g_{12}\dot{\tilde{u}}\dot{\tilde{v}} + g_{22}\dot{\tilde{v}}^2)^{\frac{1}{2}}}.$$

Example 6.5. The **parameter curves** on a surface patch $\sigma(u, v)$ can be parametrized by $\gamma(t) = \sigma(u_0, t)$, $\tilde{\gamma}(t) = \sigma(t, v_0)$, where u_0 is the constant value of u and t_0 is the constant value of v. Then $u(t) = t_0$, $v(t) = t$, and $\dot{u} = 0$, $\dot{v} = 1$. Similarly, $\tilde{u}(t) = t$, $\tilde{v}(t) = v_0$, and $\dot{\tilde{u}} = 1$, $\dot{\tilde{v}} = 0$. These parameter curves intersect at $\sigma(u_0, v_0)$, so $\cos\theta = g_{11}/\sqrt{g_{12}g_{22}}$. In particular, the parameter curves are orthogonal iff $g_{11} = 0$.

Definition 6.3. For S_1, S_2 surfaces, a **conformal map** $f : S_1 \to S_2$ is a local diffeomorphism such that for γ_1, γ_2 two curves on S^1 intersecting at a point $p \in S_1$, the angle of intersection of γ_1 and γ_2 at p is equal to the angle of intersection of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ at $f(p)$, if $\tilde{\gamma}_1, \tilde{\gamma}_2$ are the images of the curves under f.

So as we said earlier, conformal mappings preserve angle. Note that the angle is only well defined when both curves are regular.

Theorem 6.2. *A local diffeomorphism $f : S_1 \to S_2$ is conformal iff there is a function $\lambda\colon S_1 \to \mathbb{R}$ such that*

$$f^*\langle v, w\rangle_p = \lambda(p)\langle v, w\rangle_p \quad \textit{for all } p \in S_1 \textit{ and } v, w \in T_p S_1.$$

Proof. todo:come back ⊠

Corollary 6.2. *A local diffeomorphism $f : S_1 \to S_2$ is conformal iff for any surface patch σ of S_1, the fff of the patches of S_1 and $f \circ \sigma$ of S_2 are proportional (i.e., differ by a scalar multiple).*

Example 6.6. If $g_{ij}dx^i \otimes dx^j$ denotes the usual metric on $\mathbb{R}^2$ and $\mathring{g}_{ij}dx^i \otimes dx^j$ denotes the round metric on S^2, we have

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathring{g}_{ij} = \begin{pmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{pmatrix} = \frac{4}{(1+u^2+v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{4}{(1+u^2+v^2)^2} g_{ij}.$$

So stereographic projection is a conformal mapping.

A natural question to ask is when we have a conformal map between two surfaces. It turns out this is always true locally.

Theorem 6.3. *Every surface has an atlas consisting of conformal surfaces patches.*

No proof, it's too hard (even do Carmo doesn't prove it!).

6.4 Equiareal maps and a theorem of Archimedes

Recall that parameter curves $u \mapsto \sigma(u, v_0)$ and $v \mapsto \sigma(u_0, v)$ are the ones that if you take their derivative, you get the basis for T_pS denoted $\{\partial_u, \partial_v\}$. Fixing $(u_0, v_0) \in U$, we get a parallelogram with edges $\partial_u\sigma\Delta u, \partial_v\sigma\Delta v$ (corresponding to a small change in u, v on the surface) and $u = u_0+\Delta u$, $v = v_0+\Delta v$. The area of this parallelogram is $\|\partial_u\sigma\Delta u \times \partial_v\sigma\Delta v\| = \|\partial_u\sigma \times \partial_v\sigma\|\Delta u\Delta v$.

Definition 6.4. The **area** $\mathscr{A}_\sigma(R)$ of the part $\sigma(R)$ of a surface patch $\sigma\colon U \to \mathbb{R}^3$ corresponding to a region $R \subseteq U$ is defined by

$$\mathscr{A}_\sigma(R) = \int_R \|\partial_u\sigma \times \partial_v\sigma\|\, du\, dv.$$

If R is contained in a rectangle entirely contained in U, then the area will be finite. It turns out this cross product is easily computable.

Proposition 6.1. $\|\partial_u\sigma \times \partial_v\sigma\| = (EG - F^2)^{1/2} = \sqrt{g_{11}g_{22} - (g_{12})^2}$.

Proof. Recall that $(a \times b)\cdot(c \times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$. Then

$$\|\partial_u\sigma \times \partial_v\sigma\| = (\partial_u\sigma \times \partial_v\sigma)\cdot(\partial_u\sigma \times \partial_v\sigma) = (\partial_u\sigma\cdot\partial_v\sigma)(\partial_v\sigma\cdot\partial_v\sigma) - (\partial_u\sigma\cdot\partial_v\sigma)^2 = EG - F^2.$$ ⊠

So our definition of area is $\mathscr{A}_\sigma(R) = \int_R (EG - F^2)^{1/2}\, du\, dv$. Sometimes we denote $(EG - F^2)^{1/2}\, du\, dv$ by $d\mathscr{A}_\sigma$. Is this definition invariant under our choice of chart?

Proposition 6.2. *The area of a surface patch is invariant under reparametrization.*

Proof. Let $\sigma\colon U \to \mathbb{R}^3$ be a surface patch and $\widetilde{\sigma}\colon \widetilde{U} \to \mathbb{R}^3$ be a reparametrization of σ, with reparametrization map $\Phi\colon \widetilde{U} \to U$. If $\Phi(\widetilde{u}, \widetilde{v}) = (u, v)$, we have $\widetilde{\sigma}(\widetilde{u}, \widetilde{v}) = \sigma(u, v)$. Let $\widetilde{R} \subseteq \widetilde{U}$ be a region, and let $R = \Phi(\widetilde{R}) \subseteq U$. We want to show that

$$\int_R \|\partial_u\sigma \times \partial_v\sigma\|\, du\, dv = \int_{\widetilde{R}} \|\partial_{\widetilde{u}}\widetilde{\sigma} \times \partial_{\widetilde{v}}\widetilde{\sigma}\|\, d\widetilde{u}\, d\widetilde{v}.$$

We have seen that $\partial_{\widetilde{u}}\widetilde{\sigma} \times \partial_{\widetilde{v}}\widetilde{\sigma} = \det(J(\Phi))\partial_u\sigma \times \partial_v\sigma$, so the RHS becomes $\int_{\widetilde{R}} |\det(J(\Phi))|\, \|\partial_u\sigma \times \partial_v\sigma\|\, d\widetilde{u}\, d\widetilde{v}$. Apply the change of variables formula for double integrals and this becomes $\int_R \|\partial_u\sigma \times \partial_v\sigma\|\, du\, dv$, and we are done. ⊠

Definition 6.5. A local diffeomorphism $f : S_1 \to S_2$ is **equiareal** if it takes any region in S_1 to a region of the same area in S_2. We assume that the area of the regions are small enough to be contained in a single chart.

Now we state an analogue of Theorem 6.1.

Theorem 6.4. *A local diffeomorphism $f : S_1 \to S_2$ is equiareal iff for any surface patch $\sigma(u, v)$ on S_1, the fff's E_i, F_i, G_i for $i \in \{1, 2\}$ satisfy*

$$E_1G_1 - F_1^2 = E_2G_2 - F_2^2.$$

Proof. Exercise. ⊠

Think of S^2 sitting inside a unit cylinder. You can "flatten" S^2 by considering the unique line that goes through each point $p \in S^2$ and the z-axis (excluding the poles). Then this line intersects the cylinder at a point q; let f be the map that sends $p \mapsto q$. Explicitly, say $p = (x, y, z)$ and $q = (X, Y, Z)$. Since the line connecting p and q is parallel to the xz-plane, we have $z = Z$ and $(X, Y) = \lambda(x, y)$ for λ a scalar. Since (X, Y, Z) is on the cylinder we know $1 = X^2 + Y^2 = \lambda^2(x^2 + y^2)$, so $\lambda = \pm(x^2 + y^2)^{-1/2}$. Taking the plus sign, we have

$$f(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z\right).$$

Archimedes' Theorem. *The map f is an equiareal diffeomorphism.*

Proof. Consider the standard atlas for S^2 (minus the poles) given by $\sigma(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, defined on $\{\theta,\varphi \mid \theta \in [-\pi/2,\pi/2], \varphi \in [0,2\pi]\}$ and $\{\theta,\varphi \mid \theta \in [-\pi/2,\pi/2], \varphi \in [-\pi,\pi]\}$. The image of σ under f is $\widetilde{\sigma}(\theta,\varphi) = (\cos\varphi, \sin\varphi, \sin\theta)$. This gives an atlas for the cylinder in between the planes $z=1, z=-1$ (denote this surface C), defined on the same open sets as S^2. We want to show that $E_1G_1 - F_1^2 = E_2G_2 - F_2^2$, so we can apply Theorem 6.4.

From before, we know that $E_1 = 1, F_1 = 0$, and $G_1 = \cos^2\theta$ for σ. For $\widetilde{\sigma}$, we have $\partial_\theta\widetilde{\sigma} = (0,0,\cos\theta)$, $\partial_\varphi\widetilde{\sigma} = (-\sin\varphi,\cos\varphi,0)$. So $E_2 = \cos^2\theta, F_2 = 0$, and $G_2 = 1$. Therefore

$$E_1G_1 - E_2G_2 + F_2^2 - F_1^2 = (\cos^2\theta - \cos^2\theta) + 0 - 0 = 0. \qquad \boxtimes$$

Definition 6.6. A **spherical triangle** is a triangle on a sphere whose sides are arcs of great circles.

Theorem 6.5. *The area of a spherical triangle on the S^2 with internal angles α, β, and γ is $\alpha+\beta+\gamma-\pi$.*

Proof. By Archimedes' theorem, if we cut out a lune by two planes with angle θ, it has area 2θ. If A, B, C are the points on the spherical triangle that correspond to α,β,γ, consider the lune connecting A and A' with area 2α, where A' is the antipodal point of A. Note that each lune can be decomposed into $ABC+$ an extra region. So adding gives

$$3\mathscr{A}(ABC) + \mathscr{A}(\text{three extra regions}) = 2\alpha + 2\beta + 2\gamma.$$

If you draw a picture, it can be seen that the three extra regions plus $\mathscr{A}(ABC)$ actually make up the hemisphere containing A in the interior and $\overset{\frown}{BC}$ in the boundary. So

$$\begin{aligned} 2\mathscr{A}(ABC) + \overbrace{\mathscr{A}(ABC) + \mathscr{A}(\text{three extra regions})}^{\text{has area } 2\pi} &= 2\alpha + 2\beta + 2\gamma \implies \\ 2\mathscr{A}(ABC) + 2\pi &= 2\alpha + 2\beta + 2\gamma \implies \\ \mathscr{A}(ABC) &= \alpha + \beta + \gamma - \pi. \end{aligned} \qquad \boxtimes$$

We will eventually see a generalization of this result where S^2 becomes an arbitrary surface and great circles become arbitrary curves.

Lecture 7

Curvature of surfaces

How "curved" is a surface? We will see something new called the second fundamental form. It turns out a surface is determined up to isometry by its first and second fundamental forms, like how a unit-speed plane curve is determined up to isometry by its signed curvature.

7.1 The second fundamental form

We work with oriented surfaces. Suppose σ is a surface patch in $\mathbb{R}^3$ with standard unit normal $\mathbf{N}$. As (u,v) change by $\Delta u, \Delta v$ and becomes $(u+\Delta u, v+\Delta v)$, the surface moves away from $T_{\sigma(u,v)}S$ by a distance $(\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)) \cdot \mathbf{N}$. By Taylor's theorem,

$$\sigma(u+\Delta, v+\Delta v) - \sigma(u,v) = \partial_u\sigma\Delta u + \partial_v\sigma\Delta v + \frac{1}{2}(\partial_{uu}\sigma(\Delta u)^2 + 2\partial_{uv}\sigma\Delta u\Delta v + \partial_{vv}\sigma(\Delta v)^2) + \text{remainder},$$

where (remainder)$/((\Delta u)^2 + (\Delta v)^2)$ tends to zero as $(\Delta u)^2 + (\Delta v)^2$ tends to zero. Since $\partial_u\sigma, \partial_v\sigma$ are tangent to the surface (hence orthogonal to $\mathbf{N}$), the deviation from the tangent plane becomes

$$\frac{1}{2}\left(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right) + \text{remainder},$$

where

$$L = \partial_{uu}\sigma \cdot \mathbf{N}, \quad M = \partial_{uv}\sigma \cdot \mathbf{N}, \quad N = \partial_{vv}\sigma \cdot \mathbf{N}.$$

We see that $L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2$ is the analogue for the surface of the curvature $\kappa(\Delta t)^2$ in the case of a curve.

Definition 7.1. The expression $L\,du^2 + 2M\,dudv + N\,dv^2$ is the **second fundamental form** of the surface patch. From now on, sff is used in place of "second fundamental form". The corresponding symmetric bilinear form on the tangent plane is defined by

$$\langle\langle v, w\rangle\rangle = L du(v)du(w) + M(du(v)dv(w) + du(w)dv(v)) + N dv(v)dv(w).$$

Example 7.1. Consider the plane, since $\partial_u\sigma, \partial_v\sigma$ are both constant, we have $\partial_{uu}\sigma = \partial_{uv}\sigma = \partial_{vv}\sigma = 0$. So the sff of a plane is zero.

Example 7.2. Given a surface of revolution $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, assume $f(u) > 0$ for all u and $u \mapsto (f(u), 0, g(u))$ is unit-speed (so $\dot{f}^2 + \dot{g}^2 = 1$). Since $\partial_u\sigma = (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \partial_v\sigma = (-f\sin v, f\cos v, 0)$, we have

$$\begin{aligned}
\partial_u\sigma \times \partial_v\sigma &= (-f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}), \\
\|\partial_u\sigma \times \partial_v\sigma\| &= f, \\
\mathbf{N} &= \frac{\partial_u\sigma \times \partial_v\sigma}{\|\partial_u\sigma \times \partial_v\sigma\|} = (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}), \\
\partial_{uu}\sigma &= (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}), \\
\partial_{uv}\sigma &= (-\dot{f}\sin v, \dot{f}\cos v, 0), \\
\partial_{vv}\sigma &= (-f\cos v, -f\sin v, 0), \\
L &= \partial_{uu}\cdot\mathbf{N} = \dot{f}\ddot{g} - \ddot{f}\dot{g}, \\
M &= \partial_{uv}\sigma\cdot\mathbf{N} = 0, \\
N &= \partial_{vv}\sigma\cdot N = f\dot{g} \implies \\
\text{sff } &= (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2.
\end{aligned}$$

If the surface is S^2 where $u = \theta, v = \varphi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$, we have the sff of S^2 as $d\theta^2 + \cos^2\theta\, d\varphi^2$. This turns out to be the same as the fff of S^2. If the surface is the unit cylinder, then $f(u) = 1, g(u) = u$, so $L = M = 0$, $N = 1$, so the sff is dv^2.

7.2 The Gauss and Weingarten maps

An approach to defining curvature of surfaces is by considering the rate at which $\mathbf{N}$ varies. The values of $\mathbf{N}$ at S are recorded by its **Gauss map** $\mathscr{G}$, defined by

$$\mathscr{G}_S : S \to S^2, \quad p \mapsto \mathbf{N}_p.$$

The rate at which $\mathbf{N}$ varies is measured by $D_p\mathscr{G} : T_pS \to T_{\mathscr{G}(p)}S^2$. Since $T_{\mathscr{G}(p)} \perp \mathbf{N}_p$, this is equivalent to T_pS and so the Gauss map is an endomorphism $T_pS \to T_pS$.

Definition 7.2. Let $p \in S$. The **Weingarten map** $\mathscr{W}_{p,S}$ of S at p is defined by $\mathscr{W} = -D_p\mathscr{G}$. The **second fundamental form** of S at $p \in S$ is the bilinear form T_pS given by

$$\langle\langle v, w\rangle\rangle_{p,S} = \langle \mathscr{W}(v), w\rangle_{p,S}, \quad v, w \in T_pS.$$

Lemma 7.1. *Let $\sigma(u,v)$ be a surface patch with standard unit normal $\mathbf{N}(u,v)$. Then*

$$\mathbf{N}_u \cdot \partial_u\sigma = -L, \quad \mathbf{N}_u \cdot \partial_v\sigma = \mathbf{N}_v \cdot \partial_u\sigma = -M, \quad \mathbf{N}_v \cdot \partial_v\sigma = -N.$$

Proof. We have $\mathbf{N}\cdot\partial_u\sigma = \mathbf{N}\cdot\partial_v\sigma = 0$. Differentiate wrt u to get $\mathbf{N}_u \cdot \partial_u\sigma = -\mathbf{N}\cdot\partial_{uu} = -L$, and the other equalities follow in a similar manner. ⊠

Proposition 7.1. *Let $p \in S$, $\sigma(u,v)$ be a surface patch of S such that $p \in \operatorname{im}\sigma$, and let $L\,du^2 + 2M\,du\,dv + N\,dv^2$ be the sff of σ. Then for any $v, w \in T_pS$,*

$$\langle\langle v, w\rangle\rangle = L\,du(v)du(w) + M(du(v)dv(w) + du(w)dv(v)) + N\,dv(v)dv(w).$$

Proof. Since both sides of the equation are bilinear forms, we just have to check that they agree when v and w are $\partial_u\sigma$ or $\partial_v\sigma$. This boils down to showing

$$\langle\langle \partial_u\sigma, \partial_u\sigma\rangle\rangle = L, \quad \langle\langle \partial_u\sigma, \partial_v\sigma\rangle\rangle = \langle\langle \partial_v\sigma, \partial_u\sigma\rangle\rangle = M, \quad \langle\langle \partial_v\sigma, \partial_v\sigma\rangle\rangle = N.$$

Let $\sigma(u_0, v_0) = p$. Then

$$\mathscr{W}(\partial_u\sigma) = -\frac{d}{du}\bigg|_{u=u_0} \mathscr{G}(\sigma(u, v_0)) = \frac{d}{du}\bigg|_{u=u_0} \mathbf{N}(u, v_0) = -\mathbf{N}_u,$$

where $\mathbf{N}$ is the standard unit normal of σ. Similarly, $\mathscr{W}(\partial_v\sigma) = -\mathbf{N}_v$. So

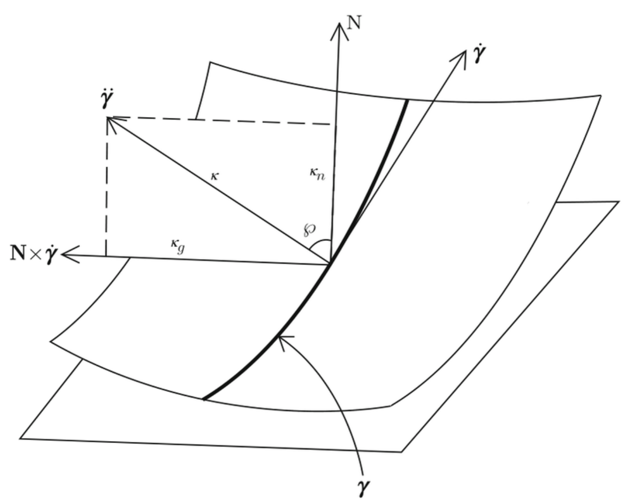
$$\langle\langle \partial_u\sigma, \partial_u\sigma\rangle\rangle = \langle \mathscr{W}(\partial_u\sigma), \partial_u\sigma\rangle = -\mathbf{N}_u \cdot \partial_u\sigma = L.$$

The other equations follow in a similar manner. ⊠

Corollary 7.1. *The sff is a symmetric bilinear form. Equivalently, the Weingarten endomorphism is self-adjoint.*

7.3 Normal and geodesic curvatures

If γ is a unit speed curve on an oriented surface S, then $\dot\gamma$ is a unit vector, and therefore a tangent vector to S. So $\dot\gamma \perp \mathbf{N}$, and $\dot\gamma, \mathbf{N}$, and $\mathbf{N}\times\dot\gamma$ are mutually orthogonal. Since $\ddot\gamma \perp \dot\gamma$, it must be a linear combination of $\mathbf{N}$ and $\mathbf{N}\times\dot\gamma$, where $\ddot\gamma = \kappa_n\mathbf{N} + \kappa_g(\mathbf{N}\times\dot\gamma)$.



Definition 7.3. The scalars κ_n and κ_g are the **normal curvature** and **geodesic curvature** of γ, respectively.

Note that these change sign when $\mathbf{N}$ does, so in general only the magnitudes of κ_n and κ_s are well defined.

Proposition 7.2. *We have*

$$\begin{aligned} &\kappa_n = \ddot\gamma\cdot\mathbf{N}, \quad \kappa_g = \ddot\gamma\cdot(\mathbf{N}\times\dot\gamma), \\ &\kappa^2 = \kappa_n^2 + \kappa_g^2, \\ &\kappa_n = \kappa\cos\psi, \quad \kappa_g = \pm\kappa\sin\psi, \end{aligned}$$

where κ is the curvature of γ and ψ is the angle between $\mathbf{N}$ and the principal normal $\mathbf{n}$ of γ.

Proof. To show $\kappa_n = \ddot{\gamma} \cdot \mathbf{N}$, note that $\ddot{\gamma} = \kappa_n \mathbf{N} + \kappa_g (\mathbf{N} \times \dot{\gamma})$, so $\ddot{\gamma} \cdot \mathbf{N} = \kappa_n \mathbf{N} \cdot \mathbf{N} + \kappa_g (\mathbf{N} \times \dot{\gamma}) \cdot \mathbf{N}$, which implies that $\ddot{\gamma} \cdot \mathbf{N} = \kappa_n$ since the vectors $\mathbf{N}$ and $\mathbf{N} \times \dot{\gamma}$ are orthogonal. A similar process shows that $\kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})$. todo:not sure how to show $\kappa^2 = \kappa_n^2 + \kappa_g^2$? Since $\ddot{\gamma} = \kappa n$, todo:? ⊠

As a unit speed parameter t is changed to another parameter $\pm t + c$, then $\kappa_n \mapsto \kappa_n$ and $\kappa_g \mapsto \pm\kappa_g$, so κ_n is well defined for any regular curve while κ_g is only well defined up to sign.

Proposition 7.3. *If γ is unit-speed on an oriented surface S, its normal curvature is given by $\kappa_n = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle$. If σ is a surface patch of S and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ, then we also have $\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$.*

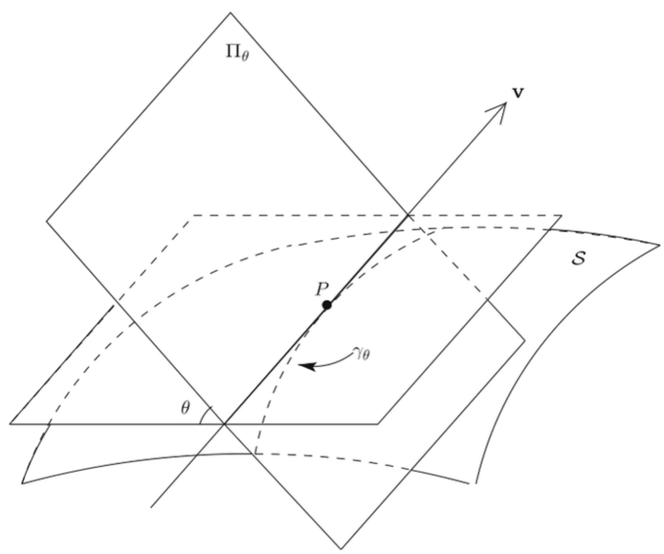
So two curves that interset at p and have the parallel tangent vectors there have the same normal curvature at p. todo:how?

Proof. Since $\dot{\gamma}$ is tangent to S, $\mathbf{N} \cdot \dot{\gamma} = 0$. So $\mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma}$. Note that $\dot{\mathbf{N}} = \frac{d}{dt}\mathscr{G}(\gamma(t)) = -\mathscr{W}(\dot{\gamma})$, therefore

$$\kappa_n = \mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma} = \langle \mathscr{W}(\dot{\gamma}), \dot{\gamma} \rangle = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle.$$ ⊠

While the normal curvature depends on the sff, the geodesic curvature only depends on the fff.

Meusnier's Theorem. *Let $p \in S$ and $v \in T_pS$. Let Π_θ be the plane spanned by v and making an angle θ with T_pS. Suppose Π_θ intersects S at a curve with curvature κ_θ. Then $\kappa_\theta \sin\theta$ is independent of θ.*



Proof. Suppose γ_θ is a unit speed parametrization of the aforementioned curve. Then at $p, \dot{\gamma}_\theta = \pm v$, so $\ddot{\gamma}_\theta$ is perpendicular to v and parallel to Π_θ. Then $\psi = \pi/2 - \theta$, so $\kappa_n = \kappa_\theta \cos(\pi/2 - \theta) = \kappa_\theta \sin\theta$, but κ_n does not depend on θ. ⊠

An important case is where γ is a **normal section** of the surface, i.e., γ is the intersection of the surface with a plane Π that is orthogonal to the tangent plane of the surface at each point of γ.

Corollary 7.2. *The curvature, normal curvature κ_n, and geodesic curvature κ_g of a normal section of a surface are related by $\kappa_n = \pm\kappa$, $\kappa_g = 0$.*

Proof. We have $\kappa_n = \kappa \sin\theta$ where $\theta = \pm\pi/2$. For the second part, $\kappa^2 = \kappa_n^2 + \kappa_g^2$, and since $\kappa = \pm\kappa_n$, we must have $\kappa_g = 0$. ⊠

7.4 Parallel transport and the covariant derivative

Lecture 8

Gaussian, mean, and principal curvatures

8.1 Gaussian and mean curvatures

Definition 8.1. If $\mathscr{W}$ is the Weingarten map of an oriented surface at a point $p \in S$, then the Gaussian curvature K and the mean curvature H of S at p are defined by

$$K = \det \mathscr{W}, \qquad H = \frac{1}{2}\operatorname{tr}(\mathscr{W}).$$

Note that K is defined for any surface, while H is only defined up to sign on a non-orientable surface. Define

$$\mathscr{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathscr{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Proposition 8.1. *For σ a surface patch of an oriented surface S, the matrix $\mathscr{W}_{p,S}$ wrt the basis $\{\sigma_u, \sigma_v\}$ of T_pS is $\mathscr{F}_I^{-1}\mathscr{F}_{II}$.*

Corollary 8.1. *We have*

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Example 8.1. Recall for the surface of revolution $\sigma(u,v) = (f(u), \cos v, f(u)\sin v, g(u))$, assuming $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everyhwere,

$$E = 1,\ F = 0,\ G = f^2, L = \dot{f}\ddot{g} - \ddot{f}\dot{g},\ M = 0, N = f\dot{g}.$$

Then

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2} = \frac{-\ddot{f}f}{f^2} = -\frac{\ddot{f}}{f}.$$

Theorem 8.1. *Let $\sigma : U \to \mathbb{R}^3$ be a surface patch, $(u_0, v_0) \in U$, and $\delta > 0$ such that R_δ the closed disk of length δ about (u_0, v_0) os contained in U. Then*

$$\lim_{\delta \to 0} \frac{\mathscr{A}_N(R_\delta)}{A_\sigma(R_\delta)} = |K|,$$

where K is the Gaussian curvature of σ at $\sigma(u_0, v_0)$.

Example 8.2. A plane has constant unit normal, so $\mathscr{G}(R)$ is a point and has no area, and the plane has zero Gaussian curvature everywhere. This shows every point on the sphere is an umbilic.

8.2 Principal curvatures of a surface

Proposition 8.2. *Let $p \in S$. Then we have scalars κ_1, κ_2 and a basis $\{t_1, t_2\}$ of T_pS such that*

$$\mathscr{W}(t_1) = \kappa_1 t_1, \quad \mathscr{W}(t_2) = \kappa_2 t_2.$$

If $\kappa_1 \neq \kappa_2$, then $\langle t_1, t_2 \rangle = 0$.

We say κ_1, κ_2 are the **principal curvatures** of S, and t_1, t_2 are the **principal vectors** corresponding to the κ_i. Points where $\kappa_1 = \kappa_2$ are called **umbilics**, and p is an umbilic iff $W_{p,S}$ is a scalar multiple of the identity, or every tangent vector is principal. If p is not an umbilic, then the principal vectors are orthogonal.

Corollary 8.2. *If $p \in S$, then there is an orthonormal basis of T_pS consisting of principal vectors.*

Proposition 8.3. *If κ_1, κ_2 are the principal curvatures of a surface, the mean and Gaussian curvatures are given by*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1 \kappa_2.$$

Proof. Using the basis of principal vectors, $\mathscr{W} = \left(\begin{smallmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{smallmatrix}\right)$. Our result immediately follows. ⊠

Euler's Theorem. *Let γ be a curve on an oriented surface S, and κ_1, κ_2 be the principal curvatures of σ, with non-zero principal vectors t_1 and t_2. Then, the normal curvature of γ is*

$$\kappa_n = Kk_1 \cos^2\theta + Kk_2 \sin^2\theta,$$

where θ is the oriented angle $\widehat{t_1\dot{\gamma}}$.

Corollary 8.3. *The principal curvatures at a point are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Furthermore, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.*

Proof. If κ_1, κ_2 are disticnct, WLOG suppose $\kappa_1 > \kappa_2$. Then

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = \kappa_1 - (\kappa_1 - \kappa_2)\sin^2\theta,$$

then $\kappa_n \le \kappa_1$ with equality iff $\theta = 0$ or π, i.e., iff $\dot{\gamma}$ is parallel to t_1. If $\kappa_1 = \kappa_2$, then the normal curvature of every curve is equal to κ_1 by Euler's theorem and every tangent vector to the surface is a principal vector. ⊠

How do we compute principal curvature? Since $\mathscr{W} = \mathscr{F}_I^{-1} F_{II}$, we want to solve $\det(\mathscr{F}_I^{-1} F_{II} - \kappa I) = 0$, and a tangent vector $t = \xi\sigma_u = \eta\sigma_v$ is a principal vector if

$$(\mathscr{F}_I^{-1}\mathscr{F}_{II} - \kappa I)\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

Write $\mathscr{F}_I^{-1}\mathscr{F}_{II} - \kappa I$ as $\mathscr{F}_I^{-1}(\mathscr{F}_{II} - \kappa\mathscr{F}_I)$.

Proposition 8.4. *The principal curvatures of the root of the equation*

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0,$$

and the principal vectors corresponding to principal curvature κ are the tangent vectors $t = \xi\sigma_u + \eta\sigma_v$ such that

$$\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

Example 8.3. For S^2, we know $E = 1, F = 1, G = \cos^2\theta$, and $L = 1, M = 0, N = \cos^2\theta$. Then solving $(1 - \kappa)(\cos^2\theta - \kappa\cos^2\theta) = 0$ gives $\kappa = 1$.

Example 8.4. Consider the unit cylinder $\sigma(u, v) = (\cos v, \sin v, u)$. Then $E = 1, F = 0, G = 1$ and $L = 0, M = 0, N = 1$. Then solving

$$\begin{vmatrix} 0 - \kappa & 0 \\ 0 & 1 - \kappa \end{vmatrix} = 0$$

gives $\kappa = 0, 1$. Any principal vector t_1 corresponding to $\kappa_1 = 1$ satisfies $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi_1 \\ \eta_1 \end{smallmatrix}\right) = 0$, so $\xi_1 = 0$ and t_1 is a multiple of $(-\sin v, \cos v, 0)$. Similarly, any principal vector corresponding to $\kappa_2 = 0$ is a multiple of $\sigma_u = (0, 0, 1)$.

Proposition 8.5. *Let S be a connected surface of which every point is an umbilic. Then $\mathscr{S}$ is an open subset of a plane or a sphere.*

What do the values of the principal curvatures tell us about a surface? Assume p is the origin and T_pS is the xy-plane, $t_1 = (1, 0, 0)$ and $t_2 = (0, 1, 0)$ are principal corresponding to principal curvatures κ_1 and κ_2, and $\mathbf{N} = (0, 0, 1)$. Something, then near p, S is approximated by the quadric surface $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$.

(i) Both the $\kappa_i > 0$ or $\kappa_i < 0$. Then the equation ($z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$) is of an elliptic paraboloid and p is an **elliptic point** of the surface.
(ii) The κ_i have opposite sign. Then the equation is of a hyperbolic paraboloid and p is a **hyperbolic point** of the surface.
(iii) One of the κ_i is zero and the other nonzero. Then the equation is of a parabolic cylinder and p is a **parabolic point** of the surface.
(iv) Both the $\kappa_i = 0$. Then the equation is of a plane and p is a **planar point** of the surface. In this case, we need to analyze higher order terms.

Example 8.5. For the torus

$$\sigma(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta),$$

then the fff is $b^2 d\theta^2 + (a + b\cos\theta)^2 d\varphi^2$ and the sff is $b\, d\theta^2 + (a + b\cos\theta)\cos\theta\, d\varphi^2$. So $\kappa_1 = \frac{1}{b}$, $\kappa_2 = \frac{\cos\theta}{a+b\cos\theta}$. Since $\kappa_1 > 0$ everywhere, the point $\sigma(\theta, \varphi)$ is elliptic, parabolic, or hyperbolic according to whether $\kappa_2 > 0, = 0$, or < 0 resp.

8.3 Surfaces of constant Gaussian curvature

Lecture 9

Geodesics

9.1 Basics

Definition 9.1. A curve γ on S is a **geodesic** if $\ddot{\gamma}(t)$ is zero or perpendicular to the $T_{\gamma(t)}S$, i.e., parallel to its unit normal for all t. Equivalently, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Proposition 9.1. *Geodesics have constant speed.*

Proposition 9.2. *A unit-speed curve on a surface is a geodesic iff its geodesic curvature is zero everywhere.*

Proposition 9.3. *Straight lines are geodesics, as well as normal sections of surfaces. Recall normal sections C of S are the intersection of a plane Π with S, such that $\Pi \perp S$ for all $c \in C$.*

Example 9.1. Great circles are geodesics.

9.2 The geodesic equations

Theorem 9.1. *A curve γ on S is a geodesic iff for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a chart σ of S, we have*

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_U\dot{v}^2), \tag{1}$$

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2). \tag{2}$$

These differential equations are called the ***geodesic equations****.*

Proposition 9.4. *Given the setup above, γ is a geodesic iff*

$$\ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 = 0, \tag{3}$$

$$\ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 = 0. \tag{4}$$

Proposition 9.5. *Let $p \in S$, t be tangent to S at p. Then there exists a unqiue unit-speed geodesic γ on S passing through p with tangent vector t. In other words, geodesics are locally unique.*

Example 9.2. Straight lines are the only geodesics in the plane, and great circles are the only geodesics on the sphere.

Corollary 9.1. *Any local isometry sends geodesics to geodesics.*

9.3 Geodesics on surfaces of revolution

Proposition 9.6. *On a surface of revolution $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$,*

(i) Every meridian is a geodesic.

(ii) A parallel $u = u_0$ is a geodesic iff $\frac{df}{du} = 0$ when $u = u_0$, ie u_0 is a stationary point of f.

Clairaut's Theorem. *Let γ be a unit-speed curve on a surface of revolution S, let $\rho : S \to \mathbb{R}$ be the distance of a point of S from the axis of rotation, and let ψ be the angle between $\dot{\gamma}$ and the meridians of S. If γ is a geodesic, then $\rho \sin\psi$ is constant along γ. Conversely, if $\rho \sin\psi$ is constant along γ, and if not part of γ is part of some parallel of S, then γ is a geodesic.*

Example 9.3. Look in book for geodesics on the pseudosphere and one-sheeted hyperboloid.

9.4 Geodesics as shortest paths

Embed γ in a smooth family of curves on σ passing through p and q. By "family", we mean a curve γ^τ on σ, such that for each $\tau \in (-\delta, \delta)$,

(i) there is an $\varepsilon > 0$ such that $\gamma^\tau(t)$is defined for all $t \in (-\varepsilon, \varepsilon)$ and all $\tau \in (-\delta, \delta)$;

(ii) for some a, b with $-\varepsilon < a < b < \varepsilon$, we have
$$\gamma^\tau(a) = p \quad \text{and} \quad \gamma^\tau(b) = q \quad \text{for all} \quad \tau \in (-\delta, \delta);$$

(iii) the map from the rectangle $(-\delta, \delta) \times (-\varepsilon, \varepsilon)$ into $\mathbb{R}^3$ given by $(\tau, t) \mapsto \gamma^\tau(t)$ is smooth;

(iv) $\gamma^0 = \gamma$.

The length of the part of γ^τ between p and q is defined by
$$\mathscr{L}(\tau) = \int_a^b \|\dot{\gamma}^\tau\|\, dt,$$
where the dot denotes $\frac{d}{dt}$.

Theorem 9.2. *The unit-speed curve γ is a geodesic iff $\frac{d}{dt}\mathscr{L}(\tau) = 0$ when $\tau = 0$ for all families of curves γ^τ with $\gamma^0 = \gamma$.*

Lecture 10

Gauss' Theorema Egregium

10.1 The Gauss and Codazzi-Maidarni equations

Codazzi-Maidarni Equations.

$$L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}, \tag{5}$$
$$M_v - N_u = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}. \tag{6}$$

Gauss Equations.

$$EK = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 \tag{7}$$
$$FK = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \tag{8}$$
$$FK = (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \tag{9}$$
$$GK = (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^1_{22}\Gamma^1_{11} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22} \tag{10}$$

Theorem 10.1. *Let $\sigma : U \to \mathbb{R}^3$ and $\tilde{\sigma} : U \to \mathbb{R}^3$ be surface paths with the same fff and sff, then there is a direct isometry M of $\mathbb{R}^3$ such that $\tilde{\sigma} = M(\sigma)$.*

10.2 Gauss' remarkable theorem

Gauss' Theorema Egregium. *The Gaussian curvature of a surface is preserved by local isometries. Explicitly, if S_1, S_2 are surfaces and $f : S_1 \to S_2$ is a local isometry, then for any $p \in S_1$, the Gaussian curvature of S_1 at p is equal to the Gaussian curvature of S_2 at $f(p)$.*

Corollary 10.1. *Any map of any region of the surface of the earth must distort distances.*

We have a nasty expression for K in terms of determinants, look in the book. This can be simplified:

Corollary 10.2. *(i) If $F = 0$,*

$$K = -\frac{1}{2\sqrt{EG}} \left\{ \frac{\partial}{\partial u} \left(\frac{G_U}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right\}$$

(ii) If $E = 1$ and $F = 0$,

$$K = \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

Lecture 11

Hyperbolic geometry

11.1 Upper half-plane model

A parametrization of the pseudosphere is

$$\widetilde{\sigma}(v, w) = \left(\frac{1}{w} \cos v, \frac{1}{w} \sin v, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1} w \right),$$

where $w > 1$. Geodesics are arcs of circles and straight lines in the vw-plane that orthogonally intersect the v-axis. The fff is

$$\frac{dv^2 + dw^2}{w^2}.$$

A natural question is "is there a surface corresponding to the whole half-plane $w > 0$ with this fff?" A theorem of Hilbert says that there is no surface with constant negative Gaussian curvature that is "geodesically complete", ie where geodesics can be extended ifninitely in both directions. Identify $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way such that

$$\mathscr{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Proposition 11.1. *Hyperbolic angles are the same as Euclidian angles.*

This is true because the metric is conformal. Similarly, hyperbolic lines are just the geodesics in $\mathscr{H}$.

Proposition 11.2. *The geodesics in $\mathscr{H}$ are the half-lines parallel to the imaginary axis and the semicircles with centers on the real axis.*

Proposition 11.3. *There is a unique hyperbolic line passing through any two distinct points of $\mathscr{H}$. Furthermore, the parallel axiom does not hold in $\mathscr{H}$.*

Proposition 11.4. *The hyperbolic distance between two points $a, b \in \mathscr{H}$ is given by*

$$d_{\mathscr{H}}(a, b) = 2 \tanh^{-1} \frac{|b - a|}{|b - \overline{a}|}.$$

Theorem 11.1. *Let P be an n-polygon with internal angles α_i. Then the hyperbolic area of the polygon is*

$$\mathscr{A}(P) = (n-2)\pi - \left(\sum_{i=1}^{n} \alpha_i \right).$$

In particular, the triangle has area $\pi - \alpha - \beta - \gamma$, contrasting with the the Euclidian formula $\alpha + \beta + \gamma = \pi$, and the spherical formula $\alpha + \beta + \gamma - \pi =$ area.

11.2 Isometries of $\mathscr{H}$

Proposition 11.5. *Let l_1, l_2 be hyperbolic lines in H, and let z_1, z_2 be points on l_1, l_2 resp. Then we have an isometry of H taking l_1 to l_2 and z_1 to z_2.*

11.3 Poincaré disk model

Definition 11.1. The Poincaré disk model D_P of hyperbolic geometry is the disk D equipped with the metric

$$\frac{4(dv^2 + dw^2)}{(1 - v^2 - w^2)^2}.$$

Proposition 11.6. *Let Γ be a circle that intersects C orthogonally, then inversion in Γ is an isometry of D_P. Reflections along lines passing through the origin (and therefore orthogonal to C) are also isometries.*

We know that the distance between two points is $d_{D_p}(a, b) = d_H(P^{-1}(a), P^{-1}(b))$, $a, b \in D_P$.

Proposition 11.7. *For $a, b \in D_P$, we have*

$$d_{D_p} = 2\tanh^{-1} \frac{|b - a|}{|1 - \overline{a}b|}.$$

Proposition 11.8. *Hyperbolic lines in D_P are the lines and circles that intersect C orthogonally.*

Lecture 12

Minimal surfaces

The goal is to find a surface of minimal area with a fixed curve as its boundary.

12.1 Plateau's problem

We study a family of surface patches $\sigma^\tau : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is open and independent of τ, and τ lies in some open interval $(-\delta, \delta)$ for some $\delta > 0$. Let $\sigma = \sigma^0$. The **surface variation** of the family is the function $\varphi : U \to \mathbb{R}^3$ given by

$$\varphi = \dot{\sigma}^\tau|_{t=0}\,.$$

Here, a dot denotes $\frac{d}{d\tau}$. Define the area function $A(\tau)$ to be

$$A(\tau) = \int_{\text{int}}$$

todo:stuff

Definition 12.1. A **minimal surface** is a surface whose mean curvature is zero everywhere.

Corollary 12.1. *If a surface S has least area among all surfaces with the same boundary curve, then S is a minimal surface.*

12.2 Examples of minimal surfaces

Proposition 12.1. *Any minimal surface of revolution is either an open subset of a plane or a catenoid.*

Proposition 12.2. *Any ruled minimal surface is an open subset of a plane or a helicoid.*

Example 12.1. Ennerper's surface given by

$$\sigma(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\right)$$

it looks cool

Example 12.2. Scherk's surface is given by $z = \ln\left(\frac{\cos y}{\cos x}\right)$. This surface only exists when $\cos x$ and $\cos y$ are both greater or less than zero.

Lecture 13

The Gauss-Bonnet theorem

GB relates a topological invariant, the Euler characteristic χ, to Gaussian curvature, which changes wildly under diffeomorphisms. This interplay between geometry and topology shows up a lot in life.

Gauss-Bonnet Theorem.

$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi\chi(M).$$

Please keep this in mind as you see the special cases.

13.1 Gauss-Bonnet for simple closed curves

This is the simplest version of GB.

Theorem 13.1. *Let $\gamma(s)$ be a unit-speed simple closed curve on a patch σ of length $\ell(\gamma)$, and assume that γ is positively-oriented. Then*

$$\int_0^{\ell(\gamma)} \kappa_g ds = 2\pi - \int_{int(\gamma)} K\,dA_\sigma.$$

13.2 Gauss-Bonnet for curvilinear polygons

Definition 13.1. A **curvilinear polygon** in $\mathbb{R}^2$ is a continuous map $\pi\colon \mathbb{R} \to \mathbb{R}^2$ such that for some real number T and some points $0 = t_0 < t_1 < \cdots < t_n = T$,

(i) $\pi(t) = \pi(t')$ iff $t' - t$ is an integer multiple of T,
(ii) π is smooth on each subinterval $(t_0, t_1), (t_1, t_2,) \cdots, (t_{n-1}, t_n)$,
(iii) The one-sided derivatives

$$\dot{\pi}^-(t_i) = \lim_{t\uparrow t_i} \frac{\pi(t) - \pi(t_i)}{t - t_i}, \quad \dot{\pi}^+(t_i) = \lim_{t\downarrow t_i} \frac{\pi(t) - \pi(t_i)}{t - t_i}$$

exist for $1 \le i \le n$ and are non-zero and not parallel.

The points $\gamma(t_i)$ are called **vertices** of the curvilinear polygon π, and the segments corresponding to (t_{i_1}, t_i) are called its **edges**.

Theorem 13.2. *Let γ be a positively-oriented unit-speed curvilinear polygon with n edges on a surface σ, and let α_i be the interior angles at its vertices. Then*

$$\int_0^{\ell(\gamma)} \kappa_G\,ds = \sum_{i=1}^n \alpha_i - (n-2)\pi - \int_{int(\gamma)} K\,dA.$$

Corollary 13.1. *If γ is a curvilinear polygon with n edges each of which is the arc of a geodesic, then the internal angles α_i of the polygon satsify*

$$\sum_{i=1}^n \alpha_i = (n-2)\pi + \int_{int(\gamma)} K\,dA.$$

13.3 Gauss-Bonnet for compact surfaces

Definition 13.2. Let S be a surface with atlas $\{(\sigma_i \mid U_i \to \mathbb{R}^3)\}_{i\in I}$. A **triangulation** of S is a collection of curvilinear polygons, each of which is contained (along with its interior) in one of the $\gamma_i(U_i)$ such that

(i) Every point of S is in at least one of the curvilinear polygons,
(ii) Two curvilinear polygons are either disjoint, or either intersection is a common edge or common vertex,
(iii) Each edge is an edge of exactly two polygons.

Example 13.1. You can triangulate S^2 into eight polygons by dividing it by three orthonormal coordinate planes.

Theorem 13.3. *Every compact surface has a finite triangulation.*

Definition 13.3. The **Euler number** (or Euler characteristic) of a compact surface S with finitely many polygons is

$$\chi = V - E + F.$$

For example, the triangulation of S^2 above has Euler characteristic $\chi = 6 - 12 + 8 = 2$. χ is actually a topological invariant and doesn't depend on the triangulation (see homology), for example “inflating” a tetrahedron to triangulate S^2 gives $\chi = 4 - 6 + 4 = 2$.

Theorem 13.4. *Let S be a compact surface. Then for any triangulation of S,*

$$\int_S K\, dA = 2\pi\chi.$$

So the Euler number χ of a triangulation of a compact surface S depends only on S and not on the choice of triangulation.

Example 13.2. For S^2, $\int_S K\, dA = 4\pi$. Since $K = 1$, the LHS is just the area of the sphere. However, deform S^2; K goes crazy but deforming doesn't change the triangulation, so $\int_S K\, dA$ is still 4π. This shows any two diffeomorphic compact surfaces have the same Euler number.

Theorem 13.5. *The Euler characteristic of the compact surface T_g of genus g is $2 - 2g$.*

Proof. We know it holds for S^2. For $\mathbb{T}^1$, triangulate the square appropiately and identify. Then proceed by induction, stitching subsequent tori on by surgery. This identification is done by removing a curvilinear n-gon from T_g and gluing edges. ⊠

Corollary 13.2.

$$\int_{T_g} K\, dA = 4\pi(1 - g).$$

13.4 Holonomy and Gaussian curvature

Proposition 13.1. *Let γ be unit-speed along a patch σ and v be nonzero parallel vector field along γ. Let φ be the oriented angle $\widehat{\dot{\boldsymbol{\gamma}}}$[4] from $\dot{\gamma}$ to v. Then the geodesic curvature of γ is*

$$\kappa_g = -\frac{d\varphi}{ds}.$$

Proposition 13.2. *Let γ be a positively-oriented unit-speed simple closed curve on a surface σ, let κ_g be the geodesic curvature of γ, and let v be a non-zero parallel vector field along γ. Then, going once around γ, v rotates through an angle*

$$2\pi - \int_0^{\ell(\gamma)} \kappa_g\, ds,$$

where $\ell(\gamma)$ is the length of γ. This angle is called the **holonomy** around γ, and is denoted by h_γ.

[4] What is this formatting?? It's suppose to be a hat over $\dot{\gamma}v$.

Theorem 13.6. *Let γ be a positively oriented simple closed curve on a surface path σ, let h_γ be the holonomy around γ, and let K be the Gaussian curvature of σ. Then*

$$h_\gamma = \int_{int(\gamma)} K\, dA.$$

We can use this to help find the Gaussian curvature at a point p of a surface S: if γ is a small positively oriented simple closed curve on the surface containing p in its interior, the Gaussian curvature of S at p will be approximately

$$\frac{h_\gamma}{\text{Area}(\text{int}(\gamma))}.$$

Proposition 13.3. *Suppose a surface S has the property that for any two points $p, q \in S$, the parallel transport Π_γ^{pq} is independent of the curve γ joining p and q. Then S is flat.*

The converse is generally not true, but if we assume S is simply-connected, then it is.