Miscellaneous Notes on Differentiable Manifolds

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These notes cover a variety of topics related to or required for the study of smooth manifolds, taken over Winter break 2020-2021 in preparation for my geometry overload next semester. Source files: https://git.simonxiang.xyz/math_notes/files.html

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Part I
Vector Calculus

Lecture 1

Vector Calculus Fundamentals

Here we review some basics, and cover other stuff that should be taught in a standard multivariable calculus course, but wasn't at UNT (div, grad, curl).

1.1 Tangent planes

Definition 1.1 (Tangent plane)**.** Let $z = f(x, y)$ represent some surface S in $\mathbb{R}^3$, $P = (a, b, c)$, $Q = (x, y, z)$ be points on S, and T a plane containing P. If the angle between the vector $\overrightarrow{PQ}$ and the plane T approaches zero as $Q \to P$ along S, then T is the **tangent plane** to S at P.

Since two lines determine a plane, the tangent lines from the partial derivatives will be in the tangent plane, if it exists: lines may determine the plane, but not the existence of it. However, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in nbd of (a, b) and are continuous at (a, b), then the tangent plane $z = f(x, y)$ at $(a, b, f(a, b))$ also exists. An equation for T is given by

$$A(x-a)+B(y-b)+C(z-f(a,b))=0$$

where $\mathbf{n} = (A, B, C)$ is a normal vector to T. Since T contains the tangent lines L_x and L_y, we just need vectors $\mathbf{v}_x$ and $\mathbf{v}_y$ parallel to L_x and L_y respectively, and then set $\mathbf{n} = \mathbf{v}_x \times \mathbf{v}_y$. Now the slope of L_x is $\frac{\partial f}{\partial x}(a, b)$, so $\mathbf{v}_x = (1, 0, \frac{\partial f}{\partial x}(a, b))$ is parallel to L_x. Similarly, $\mathbf{v}_y = (0, 1, \frac{\partial f}{\partial y}(a, b))$ is parallel to L_y, so the normal vector to T is given by

$$\mathbf{n} = \mathbf{v}_x \times \mathbf{v}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} = -\frac{\partial f}{\partial x}(a,b)\mathbf{i} - \frac{\partial f}{\partial y}(a,b)\mathbf{j} + \mathbf{k}.$$

So T can be represented by the equation $-\frac{\partial f}{\partial x}(a, b)(x - a) = \frac{\partial f}{\partial y}(a, b)(y - b) + z - f(a, b) = 0$, which simplifies to

$$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) - z + f(a,b) = 0.$$

In general, if the surface is defined by an equation of the form $F(x, y, z) = 0$, then the tangent plane at (a, b, c) is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

Our previous formula was just this applied to the case where $F(x, y, z) = f(x, y) - z$.

Example 1.1. To find the tangent plane to the surface $z = x^2 + y^2$ at $(1, 2, 5)$, note that for $f(x, y) = x^2 + y^2$, we have $f_x = 2x$ and $f_y = 2y$, so the equation is just $2(1)(x-1)+2(2)(y-2)-z+(1^2+2^2) = 0$, or $2x+4y-z-5 = 0$.

Example 1.2. For the surface $x^2 + y^2 + z^2 = 9$, we have $F(x, y, z) = x^2 + y^2 + z^2 - 9$, so $F_x = 2x$, $F_y = 2y$, and $F_z = 2z$. Therefore the equation for the tangent plane is $2x + 2y - z - 9 = 0$.

1.2 Directional derivatives and the gradient

Definition 1.2 (Directional derivative)**.** Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$, and $(a, b) \in D$. Let $\mathbf{v} \in \mathbb{R}^2$ be a unit vector ($\|v\| = 1$). Then the **directional derivative** of f at (a, b) in the direction of $\mathbf{v}$, denoted $D_v f(a, b)$, is defined as

$$D_v f(a,b) = \lim_{h \to 0} \frac{f((a,b)+h\mathbf{v}) - f(a,b)}{h}.$$

Note that if we write $\mathbf{v} = (v_1, v_2)$, then this expression becomes $\lim_{h\to 0}(f(a + hv_1, b + hv_2) - f(a, b))/h$.

Note that the partial derivatives f_x and f_y are just the cases where $\mathbf{v} = \mathbf{i} = (1,0)$, etc. We can express this by saying $f_x = D_i f$ and $f_y = D_j f$.

Theorem 1.1. *Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ such that f_x and f_y exists and are continuous on D. Let $(a,b) \in D$, and $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ be a unit vector. Then*

$$D_v f = v_1 f_x + v_2 f_y.$$

Proof. The proof is annoying so it has been skipped. ⊠

Definition 1.3 (Gradient)**.** For $f : \mathbb{R}^2 \to \mathbb{R}$, the **gradient** of f denoted $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

In general, if $f : \mathbb{R}^n \to \mathbb{R}$, we have $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

for $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.

Note that $D_v f = \mathbf{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$, which is the same as saying $D_v f = \mathbf{v} \cdot \nabla f$.

Example 1.3. To find $D_v f(1,2)$ where $f : (x,y) \mapsto xy^2 + x^3 y$ and $\mathbf{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$, note that $\nabla f = (y^2 + 3x^2 y, 2xy + x^3)$, so

$$D_v f(1,2) = \mathbf{v} \cdot \nabla f(1,2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot (10,5) = \frac{15}{\sqrt{2}}.$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives f_x and f_y, then f is **continuously differentiable.** Say f is continuously differentiable with $\nabla f \neq 0$, $c \in \operatorname{im} f$, and $\mathbf{v} \in \mathbb{R}^2$ be a unit vector tangent to the contour $f(x,y) = c$. Since $\mathbf{v}$ is tangent to the constant contour, the rate of change in the direction of $\mathbf{v}$ is zero, or $D_v f = 0$. We also know $D_v f = \mathbf{v} \cdot \nabla f = \|\mathbf{v}\| \|\nabla f\| \cos\theta$, where θ is the angle between $\mathbf{v}$ and ∇f. Since $\|\mathbf{v}\| = 1$, $D_v = \|\nabla f\| \cos\theta = 0$, and since ∇f is nonzero, $\cos\theta = 0$, and therefore $\theta = 90°$. We conclude that $\nabla f \perp \mathbf{v}$, which says that ∇f is *normal* to the contour.

In general, for a unit vector $\mathbf{v} \in \mathbb{R}^2$ we have $D_v f = \|\nabla f\| \cos\theta$. At a point (x,y) the length $\|\nabla f\|$ is fixed, and $D_v f$ varies with θ. The maximum value of $D_v f$ is when $\theta = 0$ such that $\cos\theta = 1$, and the smallest value is when $\theta = \pi$ such that $\cos\theta = -1$. So f increases the fastest in the direction of ∇f (this is the case $\theta = 0$) and slowest in the direction of $-\nabla f$. We can formulate our findings as a theorem.

Theorem 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, and $\nabla f \neq 0$. Then*

(a) The gradient ∇f is normal to any level curve $f(x,y) = c$.

(b) The value of f increases the fastest in the direction of ∇f.

(c) The value of f decreases the fastest in the direction of $-\nabla f$.

Example 1.4. In which direction does $f : (x,y) \mapsto xy^2 + x^3 y$ increase the fastest from the point $(1,2)$? What about the fastest rate of decrease?

Solution. We have $\nabla f = (y^2 + 3x^2 y, 2xy + x^3)$, so $\nabla f(1,2) = (10,5) \neq 0$. So a unit vector in that direction is $\mathbf{v} = \frac{\nabla f}{\|\nabla f\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$, similarly $\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right)$ in the direction of $-\nabla f$. You can fill in the rest. ■

1.3 Line integrals

Here we review the concept of a line integral.

Definition 1.4 (Line integral)**.** For a function $f:\mathbb{R}^2\to\mathbb{R}$ and a curve $C\subseteq\mathbb{R}^2$ parametrized by $x=x(t), y=y(t), a\le t\le b$, the **line integral** of $f(x,y)$ along C is

$$\int_C f(x,y)\,ds=\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt,$$

where $s=s(t)=\int_a^t\sqrt{x'(u)^2+y'(u)^2}\,du$ denotes the arc length of the curve. So $ds=s'(t)dt=\sqrt{x'(t)^2+y'(t)^2}dt$ by the FTC.

Some basic things: traversing the curve in the opposite direction doesn't change anything. We can also define a line integral with respect to x as opposed to s, where $\int_C f(x,y)\,ds=\int_a^b f(x(t),y(t))x'(t)\,dt$. For the physically inclined, you can think of the line integral as work done by a force moving along a curve.

Some of these constructions seem similar: we can work toward generalizing this. Define a function $f:\mathbb{R}^2\to\mathbb{R}^2$ by $f:(x,y)\mapsto P\mathbf{i}+Q\mathbf{j}$ where $P,Q:\mathbb{R}^2\to\mathbb{R}$, then f is a **vector field** on $\mathbb{R}^2$. This function takes in points and outputs vectors. For a curve C with parametrization $x=x(t), y=y(t), a\le t\le b$, let $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$ be the position vector for a point $(x(t),y(t))$ on C. Then $\mathbf{r}'(t)=x'(t)\mathbf{i}+y'(t)\mathbf{j}$ and so

$$\begin{aligned}\int_C P(x,y)\,dx+\int_C Q(x,y)\,dy&=\int_a^b P(x(t),y(t))x'(t)\,dt+\int_a^b Q(x(t),y(t))y'(t)\,dt\\&=\int_a^b\big(P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)\big)\,dt\\&=\int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)\,dt.\end{aligned}$$

This leads us to the following definition.

Definition 1.5. For a vector field $\mathbf{f}=P\mathbf{i}+Q\mathbf{j}$ and a curve C with parametrization $y=y(t), a\le t\le b$, the **line integral** of f along C is

$$\int_C \mathbf{f}\cdot d\mathbf{r}=\int_C P(x,y)\,dx+\int_C Q(x,y)\,dy=\int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)\,dt,$$

where $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$ is the position vector for points on C.

We distinguish Definition 1.4 and Definition 1.5 by calling one the line integral of a *scalar field* and the other the line integral of a *vector field*. Which one corresponds to which should be clear from context. We use the notation $d\mathbf{f}=\mathbf{r}'(t)dt=dx\mathbf{i}+dy\mathbf{j}$ to denote the **differential** of $\mathbf{r}$. Often we denote $\int_C P(x,y)\,dx+\int_C Q(x,y)\,dy$ by $\int_C P(x,y)\,dx+Q(x,y)\,dy$ for convenience. $P(x,y)dx+Q(x,y)dy$ is known as a **differential form**. For $F:\mathbb{R}^2\to\mathbb{R}$, the **differential** of F is $dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy$. A form is **exact** if it's the form of some function F.

Recall that $\mathbf{r}'(t)$ is a tangent vector to points on C in the direction of C. C is smooth, therefore $\mathbf{r}'(t)\neq 0$ on $[a,b]$ and so the unit tangent vector to C at a point is given by $\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. This naturally leads us to the following theorem:

Theorem 1.3. *For a vector field* $\mathbf{f}=P\mathbf{i}+Q\mathbf{j}$ *and a smooth curve* C *parametrized on* $[a,b]$ *with position vector* $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$*, we have*

$$\int_C \mathbf{f}\cdot d\mathbf{r}=\int_C \mathbf{f}\cdot\mathbf{T}ds,$$

where $\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

This also works for piecewise smooth curves. If $C = C_1 \cup C_2 \cup \cdots \cup C_n$, then

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \int_{C_1} \mathbf{f} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{f} \cdot d\mathbf{r}_2 + \cdots + \int_{C_n} \mathbf{f} \cdot d\mathbf{r}_n.$$

Example 1.5. Evaluate $\int_C (x^2+y^2)\,dx + 2xy\,dy$ on the curves $x=t,\ y=2t$ and $x=t,\ y=2t^2$ for $t \in [0,1]$.

Solution. For the first curve, note that $x'(t)=1$ and $y'(t)=2$, so

$$\int_C (x^2+y^2)\,dx + 2xy\,dy = \int_0^1 (t^2+4t^2)x'(t) + 2t(2t)y'(t)\,dt = \int_0^1 5t^2 + 8t^2\,dt = \frac{13}{3}.$$

Similarly, for the second curve $x'(t)=1$ and $y'(t)=4t$, and so

$$\int_C (x^2+y^2)\,dx + 2xy\,dy = \int_0^1 (t^2+4t^4) + (2t\cdot 2t^2\cdot 4t)\,dt = \int_0^1 t^2 + 20t^4\,dt = \frac{13}{3}.$$

■

1.4 Closed curves and conservative vector fields

Recall that $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$ for line integrals of scalar fields, but for vector fields this does not hold, namely, $\int_{-C} \mathbf{f} \cdot dr = -\int_C \mathbf{f} \cdot dr$. Recall that our definition of line integrals depends on the parametrization of the curve: what if we parametrize C by some alternative parametrization? Then our definition would not be well defined, and this would be very bad. Thankfully, this is not the case, as long as the orientation of C is invariant under parametrization.

Theorem 1.4. *Let $\mathbf{f} = P\mathbf{i} + Q\mathbf{j}$ be a vector field and C be a smooth curve parametrized by $x = x(t), y = y(t)$ for $t \in [a,b]$. Say $t = \alpha(u)$ for $u \in [c,d]$ such that $a = \alpha(c), b = \alpha(d)$, and $\alpha'(u) > 0$ on the interval (c,d). Then $\int_C \mathbf{f} \cdot d\mathbf{r}$ has the same value for the alternate parametrization $x = \tilde{x}(u) = x(\alpha(u)), y = \tilde{y}(u) = y(\alpha(u)), u \in [c,d]$.*

Proof. Not too interesting, chain rule and u-sub. ⊠

A **closed curve** is a loop, and a **simple closed curve** has no self-intersections, and we denote line integrals along closed curves with $\oint$.

Theorem 1.5. *In a region R, the line integral $\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of path between two points of R iff $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$ for $C \subseteq R$ a closed curve.*

Proof. Split $C = C_1 \cup -C_2$ and go from there. ⊠

This doesn't completely determine path independence, but it does relate some things. We work toward a more practical condition for independence of path.

Theorem 1.6 (Chain Rule)**.** *If $z = f(x,y)$ is a continuously differentiable function of x and y, and both $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then Z is a differentiable function of t, and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Theorem 1.7. *Let $\mathbf{f} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on some region R, where $P, Q\colon \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Let $C \subseteq R$ be a smooth curve with parametrization $x = x(t), y = y(t), t \in [a,b]$. Suppose we have a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla F = \mathbf{f}$ on R. Then*

$$\int_C \mathbf{f} \cdot d\mathbf{r} = F(B) - F(A),$$

where $A = (x(a), y(a))$ and $B = (x(b), y(b))$ are the endpoints of C. So the line integral depends only on the endpoints.

Proof. We have

$$\begin{aligned}
\int_C \mathbf{f}\cdot d\mathbf{r} &= \int_a^b \big(P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)\big)\,dt \\
&= \int_a^b \left(\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}\right)dt \text{ (since } \nabla F=\mathbf{f} \implies \frac{\partial F}{\partial x}=P \text{ and } \frac{\partial F}{\partial y}=Q) \\
&= \int_a^b F'(x(t),y(t))\,dt \\
&= F(x(t),y(t))\Big|_a^b = F(B)-F(A).
\end{aligned}$$

⊠

We can think of Theorem 1.7 as the line integral version of the FTC. A function $F:\mathbb{R}^2\to\mathbb{R}$ such that $\nabla F=\mathbf{f}$ is a **potential** for $\mathbf{f}$. A **conservative** vector field is one that has a potential.

Corollary 1.1. *If a vector field* $\mathbf{f}$ *has a potential in a region* R*, then* $\oint_C \mathbf{f}\cdot d\mathbf{r}=0$ *for any closed curve* $C\subseteq R$*, in other words,* $\oint_C \nabla F\cdot d\mathbf{r}=0$ *for any* $F:\mathbb{R}^2\to\mathbb{R}$.

Example 1.6. Show that the line integral $\int_C (x^2+y^2)\,dx+2xy\,dy$ is path independent.

Solution. We want to show that $\mathbf{f}$ is conservative, that is, we want to find a potential F such that

$$\frac{\partial F}{\partial x}=x^2+y^2 \quad\text{and}\quad \frac{\partial F}{\partial y}=2xy.$$

If $\frac{\partial F}{\partial x}=x^2+y^2$, then $F=\frac{1}{3}x^3+xy^2+g(y)$ for some function $g(y)$. This satisfies $\frac{\partial F}{\partial y}=2xy$ if $g'(y)=0$, so g is a constant (say $g=0$). Then a potential F exists, where $F(x,y)=\frac{1}{3}x^3+xy^2$. So $\int_C (x^2+y^2)\,dx+2xy\,dy$ is path independent. By Theorem 1.7, we can also see that any value of $\int_C \mathbf{f}\cdot dr=\frac{13}{3}$ for C from $(0,0)$ to $(1,2)$ since

$$\int_C \mathbf{f}\cdot d\mathbf{f}=F(1,2)-F(0,0)=\frac{1}{3}+4=\frac{13}{3}.$$

■

1.5 Green's Theorem

We now examine a way of evaluating line integrals on smooth vector fields (a vector field whose components P and Q are smooth) on simple closed curves.

Green's Theorem. *Let* R *be a region in* $\mathbb{R}^2$ *whose boundary is a simply closed curve* C *which is piecewise smooth. Let* $\mathbf{f}=P\mathbf{i}+Q\mathbf{j}$ *be a smooth vector field defined on both* R *and* C*. Then*

$$\oint_C \mathbf{f}\cdot d\mathbf{r}=\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA.$$

Proof. We prove Green's theorem for a simple region R, where $C=C_1\cup C$. We can write C in two distinct ways, one where C_1 is the curve $y=y_1(x)$ from the farthest horizontal points X_1 and X_2 (C_2 is similarly defined) and the other where C_1 is the curve $x=x_1(y)$ from the farthest vertical points Y_2 to Y_1. Integrate P around C where

C_1 is $y = y_1(x)$, then

$$\begin{aligned}
\oint_C P(x,y)\,dx &= \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx \\
&= \int_a^b P(x,y_1(x))\,dx + \int_b^a P(x,y_2(x))\,dx \\
&= \int_a^b (P(x,y_1(x)) - P(x,y_2(x)))\,dx \\
&= -\int_a^b \left(P(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)} \right) dx \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy\,dx \\
&= -\iint\limits_R \frac{\partial P}{\partial y}\,dA
\end{aligned}$$

A similar calculation shows that $\int_C Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$ by integrating along C where $C_1 = x_1(y)$. This finishes the proof. Of course, we can generalize this if we wish. ⊠

Example 1.7. To evaluate $\int_C (x^2+y^2)\,dx + 2xy\,dy$ where C is the boundary enclosed by $y = 2x$ and $y = 2x^2$, by Green's Theorem we have

$$\oint_C (x^2+y^2)\,dx + 2xy\,dy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint\limits_R (2y-2y)dA = 0.$$

Of course we already knew this, since **f** has a potential function.

Example 1.8. To see where Green's Theorem does not hold, let **f** be defined by $P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$ on a punctured disk homeomorphic to the annulus given by $R = \{(x,y) \mid 0 < x^2+y^2 \le 1\}$. An exercise shows that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 2\pi$, but since both $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are equal to $\frac{y^2-x^2}{(x^2+y^2)^2}$ we have $\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0$. The key thing is that the we integrated over a region not contained in R. If we outright define R as an annulus with boundary, then this works.

As seen in the example, we can extend Green's Theorem for multiply connected closed regions, just subdivide while preserving orientation and use the fact that it still works for curves $C_1 = C_2$. Say we have a smooth potential F in R of a vector field **f**, then $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$, so $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$ implies that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in R. Conversely, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in R, then $\oint_C \mathbf{f}\cdot d\mathbf{r} = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0$. Then for a simply connected region R, the following are equivalent:

(a) $\mathbf{f} = P\mathbf{i} + Q\mathbf{j}$ has a smooth potential $F : \mathbb{R}^2 \to \mathbb{R}$,

(b) $\int_C \mathbf{f}\cdot d\mathbf{r}$ is independent of path for curves $C \subseteq R$,

(c) $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for every simple closed curve $C \subseteq R$,

(d) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in R (in this case, the differential form $P\,dx + Q\,dy$ is exact).

1.6 Surface Integrals and the Divergence Theorem

Similar to how curves are parametrized with a variable t, we can parametrize surfaces in $\mathbb{R}^3$ with two variables u, v, in essence a continuous map $f : I^2 \to \mathbb{R}^3$. In this case, a position vector is given by $\mathbf{r}(u,v) = x(u,v) + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

for $(u,v)\in R$ (where R is a region in $\mathbb{R}^2$). Then define the partial derivatives as $\frac{\partial \mathbf{r}}{\partial u}(u,v)=\frac{\partial x}{\partial u}(u,v)\mathbf{i}+\frac{\partial y}{\partial u}(u,v)\mathbf{j}+\frac{\partial z}{\partial u}(u,v)\mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v}(u,v)$ analogously. Tangent vectors to points on vertical gridlines are given by $\frac{\partial \mathbf{r}}{\partial v}$, and similarly for horizontal gridlines.

Take a rectangle at a point (u,v) with width Δu and height Δv, so it has area $\Delta u \Delta v$. It gets mapped onto some surface Σ by a parametrization, with small enough Δ (say $\Delta\sigma$) is approximately the area of the rectangle. Recall that $\frac{\partial \mathbf{r}}{\partial u}\approx\frac{\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)}{\Delta u}$ and $\frac{\partial \mathbf{r}}{\partial v}\approx\frac{\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v)}{\Delta v}$, then the surface area is approximately

$$\|(\mathbf{r}(u+\Delta u)-\mathbf{r}(u,v))\times(\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v))\|\approx\left\|\left(\Delta u\frac{\partial \mathbf{r}}{\partial u}\right)\times\left(\Delta v\frac{\partial \mathbf{r}}{\partial v}\right)\right\|=\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\Delta u\Delta v.$$

This is just taking the norm of the cross product of the two vectors that make up the rectangle, nothing special. So the surface area of a surface Σ is the sum of the $\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\Delta u\Delta v$ over the rectangles in R, therefore

$$S=\iint\limits_R\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|du\,dv.$$

We'll notate this $\iint\limits_\Sigma d\sigma=\iint\limits_R\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|du\,dv$, which is a special case of a *surface integral* over a surface Σ.

Definition 1.6. Let Σ be a surface in $\mathbb{R}^3$ parametrized by $x=x(u,v),\ y=y(u,v),\ z=z(u,v)$ for (u,v) in some region $R\subseteq\mathbb{R}^2$. Let $\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$ be the position vector for Σ, and let f be a function on some subset of $\mathbb{R}^3$ containing Σ. Then the **surface integral** of f over Σ is

$$\iint\limits_\Sigma f\,d\sigma=\iint\limits_R f(x(u,v),y(u,v),z(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|du\,dv.$$

In particular, the surface area S of Σ is equal to $\iint\limits_\Sigma 1\,d\sigma$.

Since $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangent to the surface Σ, $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ is perpendicular to the tangent plane at each point of Σ, so the surface integral of a function f can be expressed as $\iint_R f\,\|\mathbf{n}\|\,d\sigma$, where $\mathbf{n}=\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ is a **normal vector** to Σ. An **outward unit normal vector** points away from the "top" part of the surface. Let's define surface integrals of three dimensional vector fields.

Definition 1.7. Let Σ be a surface in $\mathbb{R}^3$ and let $\mathbf{f}=f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$ be a vector field defined on some subset of $\mathbb{R}^3$ containing Σ. The **surface integral** of $\mathbf{f}$ over Σ is

$$\iint\limits_\Sigma \mathbf{f}\cdot d\sigma=\iint\limits_\Sigma \mathbf{f}\cdot\mathbf{n}\,d\sigma,$$

where $\mathbf{n}$ is the outward unit normal vector to Σ.

Example 1.9. Let's find the surface area T of a torus, created by revolving a circle of radius a in the yz-plane around the z-axis, at a distance b from the z-axis. Say points on longitudinal circles make an angle of u and meridional circles an angle of v with their midpoints. Then we can parametrize the torus as

$$x=(b+a\cos u)\cos v,\quad y=(b+a\cos u)\sin v,\quad z=a\sin u,\quad 0\le u\le 2\pi,\quad 0\le v\le 2\pi.$$

So $\frac{\partial \mathbf{r}}{\partial u}=-a\sin u\cos v\,\mathbf{i}-a\sin u\sin v\,\mathbf{j}+a\cos u\,\mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v}=-(b+a\cos u)\sin v\,\mathbf{i}+(b+a\cos u)\cos v\,\mathbf{j}+0\mathbf{k}$, therefore the cross product $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial r}{\partial v}$ is $-a(b+a\cos u)\cos v\cos u\,\mathbf{i}-a(b+a\cos u)\sin v\cos u\,\mathbf{j}-a(b+a\cos u)\sin u\,\mathbf{k}$, which has magnitude $a(b+a\cos u)$. So the surface area S is equal to $\iint_\Sigma 1\,d\sigma$, which is equal to

$$\int_0^{2\pi}\int_0^{2\pi}\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|du\,dv=\int_0^{2\pi}\int_0^{2\pi}a(b+a\cos u)\,du\,dv=\int_0^{2\pi}\left(abu+a^2\sin u\Big|_{u=0}^{u=2\pi}\right)dv,$$

which simplifies to $4\pi^2ab$.

Example 1.10. Let's calculate a surface integral. Hopefully this shouldn't take too much time. Let $\mathbf{f} = yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$ and Σ be the plane $x+y+z=1$ bounded by $x \geq 0,\ y \geq 0,\ z \geq 0$. The outward unit normal vector is $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and Σ is parametrized by $z = 1-(u+v)$ for $u \in [0,1]$, $v \in [0,1-u]$ (call this region R) by projecting Σ onto the xy-plane. Then $\mathbf{f}\cdot\mathbf{n} = (yz, xz, xy)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}(yz+xz+xy) = \frac{1}{\sqrt{3}}((x+y)z+xy) = \frac{1}{\sqrt{3}}((u+v)(1-(u+v))+uv) = \frac{1}{\sqrt{3}}((u+v)-(u+v)^2+uv)$ for $(u,v)\in R$. For $\mathbf{r} = u\mathbf{j}+v\mathbf{j}+(1-(u+v))\mathbf{k}$ we have $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = (1,0,-1)\times(0,1,-1) = (1,1,1)$, so $\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| = \sqrt{3}$. Then

$$\iint\limits_{\Sigma} \mathbf{f}\cdot\mathbf{n}\,d\sigma = \int_0^1\int_0^{1-u} \frac{1}{\sqrt{3}}((u+v)-(u+v)^2+uv)\sqrt{3}\,dv\,du = \frac{1}{8}.$$

Computing surface integrals can be tedious. If Σ is a **closed surface** (ie bounds a solid in $\mathbb{R}^3$, or is a 2-manifold), the Divergence Theorem makes this easier for us.

Divergence Theorem. *Let Σ be a closed surface in $\mathbb{R}^3$ that bounds a solid S, and let $\mathbf{f} = f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$ be a vector field defined on some subset of $\mathbb{R}^3$ containing Σ. Then*

$$\iint\limits_{\Sigma} \mathbf{f}\cdot d\sigma = \iiint\limits_{S} \operatorname{div}\mathbf{f}\,dV,$$

*where $\operatorname{div}\mathbf{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$ is the **divergence** of $\mathbf{f}$.*

Proof. The proof is similar to Green's Theorem, first being proved for when S is bounded above and below by one surface, and laterally by a number of surfaces. Then extend the proof to a general solid. ⊠

Example 1.11. To evaluate the surface integral $\iint_\Sigma \mathbf{f}\cdot d\sigma$ where $\mathbf{f} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\Sigma = S^2$ using the Divergence Theorem, note that $\operatorname{div}\mathbf{f} = 1+1+1 = 3$, so

$$\iint\limits_{\Sigma} \mathbf{f}\cdot d\sigma = \iiint\limits_{S} \operatorname{div}\mathbf{f}\,dV = 3\iiint\limits_{S} 1\,dV + 3\,\mathrm{vol}(S) = 4\pi.$$

Note. Warning, physics will follow. The surface integral $\iint_\Sigma \mathbf{f}\cdot d\Sigma$ is often called the **flux** of $\mathbf{f}$ through Σ. If $\mathbf{f}$ represents the velocity of a field of a fluid, a positive flux means a net flow out of the fluid (in the direction of n), and similarly negative flux means net inward flow in the direction of $-\mathbf{n}$.

Divergence can be interpreted as how much a vector field diverges from a point, which makes more sense with the following definition equivalent to the Divergence Theorem:

$$\operatorname{div}\mathbf{f}(x,y,z) = \lim_{V\to 0}\frac{1}{V}\iint\limits_{\Sigma} \mathbf{f}\cdot d\sigma,$$

where V is the volume enclosed by Σ around (x,y,z). Taking the limit as $V\to 0$ means taking smaller and smaller neighborhoods around (x,y,z). The limit is the ratio of flux through a surface to the volume enclosed by the surface, giving a rough measure of a flow "leaving" a point. Vector fields with zero divergence are called *solenoidal* fields.

Corollary 1.2. *If the flux of a vector field $\mathbf{f}$ is zero through every closed surface containing a given point, then $\operatorname{div}\mathbf{f} = 0$ at such point.*

Proof. At a point (x,y,z) we have $\operatorname{div}\mathbf{f}(x,y,z) = \lim_{V\to 0}\frac{1}{V}\iint_\Sigma \mathbf{f}\cdot d\sigma$, but the surface integral is zero by assumption, so the limit is also zero, and we are done. ⊠

Note. Sometimes

$$\oiint\limits_{\Sigma} f(x,y,z)\,d\sigma \quad\text{and}\quad \oiint\limits_{\Sigma} \mathbf{f}\cdot d\sigma$$

are used to denote surface integrals of scalar and vector fields, respectively, over closed surfaces. In physics, you often see $\oint\limits_{\Sigma}$ instead of $\oiint\limits_{\Sigma}$ (it's just `\oiint` as opposed to `\oint`??).

1.7 Stokes' Theorem

This is the good stuff. Let's generalize things.

Definition 1.8. For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and curve $C \subseteq \mathbb{R}^3$ parametrized by $x = x(t),\ y = y(t), z = z(t),\ t \in [a, b]$, the **line integral** of f along C with respect to arc length s is

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

The line integral of f along C with respect to x is $\int_C f\,dx = \int_a^b f(x(t), y(t), z(t))x'(t)\,dt$, and the line integrals of f with respect to y and z are similarly defined.

Definition 1.9. For a vector field $\mathbf{f} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ for $P, Q, K : \mathbb{R}^3 \to \mathbb{R}$ and a curve $C \subseteq \mathbb{R}^3$ with smooth parametrization $x = x(t),\ y = y(t),\ z = z(t),\ t \in [a, b]$, the **line integral** of $\mathbf{f}$ along C is

$$\int_C f \cdot d\mathbf{r} = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz = \int_a^b \mathbf{f}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)dt,$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position vector for C.

Theorem 1.8. *For a vector field* $\mathbf{f}$*, we have*

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \mathbf{f} \cdot \mathbf{T}\,ds$$

where $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ *is the unit tangent vector for* C*.*

Theorem 1.9. *For a vector field* $\mathbf{f}$ *with* P, Q, R *continuously differentiable functions on a solid* S*, if there exists an* $F : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\nabla F = \mathbf{f}$ *on* S*, then for* A, B *endpoints of a curve* C *we have*

$$\int_C \mathbf{f} \cdot d\mathbf{r} = F(B) - F(A).$$

Corollary 1.3. *For any* $F : \mathbb{R}^3 \to \mathbb{R}$*, we have* $\oint_C \nabla F \cdot d\mathbf{r} = 0$*.*

Now that we've finished the theorem spam, let's talk about generalizing Green's Theorem to **orientable** surfaces, which require the existence of a continuous nonzero vector field $\mathbf{N}$ in $\mathbb{R}^3$ normal to the surface (i.e. perpendicular to the the tangent plane for all points of the surface). We call $\mathbf{N}$ a *normal vector field*. For an orientable surface Σ with boundary C, choose a unit normal vector $\mathbf{n}$ with the surface on the left, we say $\mathbf{n}$ is a *positive unit normal vector* and that C is traversed $\mathbf{n}$-positively.

Stoke's Theorem. *Let* Σ *be an orientable surface in* $\mathbb{R}^3$ *whose boundary is a simple closed curve* C*, and for* $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ *let* $\mathbf{f} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *be a smooth vector field defined on a subset of* $\mathbb{R}^3$ *containing* Σ*. Then*

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_\Sigma (\operatorname{curl}\mathbf{f}) \cdot \mathbf{n}\,d\sigma,$$

where

$$\operatorname{curl}\mathbf{f} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k},$$

$\mathbf{n}$ *is a positive unit normal vector over* Σ*, and* C *is traversed* $\mathbf{n}$*-positively.*

Proof. Homology! Omitted because we'll come back to this during diff top. ⊠

You can think of $\oint_C \mathbf{f} \cdot d\mathbf{r}$ as the **circulation** of $\mathbf{f}$ around C, for example, if $\mathbf{E}$ represents the electrostatic field due to a point charge, then $\operatorname{curl}\mathbf{E} = 0$, so the circulation $\oint_C \mathbf{E} \cdot d\mathbf{r} = 0$ by Stokes' Theorem. Such vector fields are called irrotational fields. The term "curl" was made to talk about electromagnetism, which measures something called *circulation density*. We can see this by considering the following definition for curl:

$$\mathbf{n} \cdot (\operatorname{curl}\mathbf{f})(x, y, z) = \lim_{S \to 0} \frac{1}{S} \oint_C \mathbf{f} \cdot d\mathbf{r},$$

where S is the surface area of a surface Σ containing a point (x, y, z) with boundary curve C and positive unit normal vector $\mathbf{n}$. Imagine C shrinking to encapsulate (x, y, z), causing S to approach zero. The ratio of circulation to surface area is what makes the curl a rough measure of circulation density.

Theorem 1.10. *Let S be a simply connected solid region in $\mathbb{R}^3$. Then the following are equivalent:*

(a) $\mathbf{f} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *has a smooth potential F in S,*

(b) $\int_C \mathbf{f} \cdot d\mathbf{r}$ *is path independent of curves C in S,*

(c) $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$ *for every simple closed curve C in S,*

(d) $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, *and* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ *in S (i.e.* $\operatorname{curl}\mathbf{f} = 0$ *in S, or the differential form $P\,dx + Q\,dy + R\,dz$ is exact).*

1.8 Div, grad, curl

A wrap up section. For $f : \mathbb{R}^n \to \mathbb{R}$, ∇f is a vector-falued function $\mathbb{R}^n \to \mathbb{R}^n$, so we can "apply" the del operator to f to get a new function. Think of $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ as a "vector" in $\mathbb{R}^3$, this doesn't make sense on its own but we apply these to functions (to get $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ for example). We denote the divergence $\operatorname{div}\mathbf{f}$ as $\nabla \cdot \mathbf{f}$, the dot product of $\mathbf{f}$ with ∇, the reasoning can be seen below:

$$\nabla \cdot \mathbf{f} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \operatorname{div}\mathbf{f}.$$

Similarly, we write $\operatorname{curl}\mathbf{f}$ as the cross product $\nabla \times \mathbf{f}$, this is a fairly straightforward calculation. The divergence of the gradient $\nabla \cdot \nabla f$ has a special name, the **Laplacian** of f, where $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Theorem 1.11. *The curl of the gradient is zero, or* $\nabla \times (\nabla f) = 0$ *for any smooth* $f : \mathbb{R}^3 \to \mathbb{R}$.

Proof. We have

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}.$$

Since f is smooth, the mixed partial derivatives cancel, and this expression is equal to zero. ⊠

Corollary 1.4. *If a vector field* $\mathbf{f}(x, y, z)$ *has a potential, then* $\operatorname{curl}\mathbf{f} = 0$.

Theorem 1.12. *The divergence of the curl is zero, or* $\nabla \cdot (\nabla \times \mathbf{f}) = 0$ *for smooth vector fields* $\mathbf{f}(x, y, z)$.

Proof. We have

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{f}) &= \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \\ &= \frac{\partial^2 Q}{\partial z \partial x} + \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 P}{\partial z \partial y}\right) + \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x}\right) - \frac{\partial^2 Q}{\partial x \partial z} \\ &= 0 \quad \text{since } \mathbf{f} \text{ is smooth.} \end{aligned}$$

⊠

Corollary 1.5. *The flux of the curl of a smooth vector field* $\mathbf{f}(x, y, z)$ *through any closed surface is zero.*

Part II
Euclidian Spaces

Lecture 2

Smooth Functions on a Euclidian Space

INTRODUCTION

Calculus talks about differentiation and integration on $\mathbb{R}$, while real analysis extends this to $\mathbb{R}^n$. Vector calculus talks about integrals on curves and surfaces, and now we extend these concepts to higher dimensions, the structures which with we work with are called manifolds. Things become simple: gradient, curl, and divergence are cases of the exterior derivative, and the FTC for line integrals, Green's theorem, Stokes' theorem, and the divergence theorem are manifestations of the generalized Stokes' theorem.

Manifolds arise even when dealing with the space we live in, for example the set of affine motions in $\mathbb{R}^3$ is a 6-manifold. This is our plan: recast calculus on $\mathbb{R}^n$ so we can generalize it to manifolds by differential forms. Working in $\mathbb{R}^n$ first isn't necessary, but much easier, since the examples are simple. Then, we define a manifold and talk about tangent spaces, working with the idea of approximating nonlinear things with linear things, with Lie groups and Lie algebras as examples. Finally, we do calculus on manifolds, generalizing the theorems of vector calculus, with the de Rham cohomology groups as C^∞ and topological invariants.

2.1 C^∞ Versus Analytic Functions

Let's talk about C^∞ functions on $\mathbb{R}^n$. Write a base for $\mathbb{R}^n$ as $x^1, \cdots, x^n$ and let $p = (p^1, \cdots, p^n)$ be a point in an open set U in $\mathbb{R}^n$. Differential geometry uses *superscripts*, not subscripts, more on this later.

Definition 2.1. Let k be a nonnegative integer. A function $f : U \to \mathbb{R}$ is C^k at p if its partial derivatives $\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$ of all orders $j \le k$ exist and are continuous at p. The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k for all $k \ge 0$, that is, its partial derivatives of all orders

$$\frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

exist and are continuous at p. We say f is C^k on U if it is C^k for all points in U, and the concept of C^∞ on a set U is defined similarly. When we say "smooth", we mean C^∞.

Example 2.1.

(i) We call C^0 functions on U continuous on U.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then $f'(x)$ is $\frac{1}{3}x^{-2/3}$ for $x \neq 0$ and undefined at zero, so f is C^0 but not C^1 at $x = 0$.

(iii) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^{1/3}\,dt = \frac{3}{4}x^{4/3}.$$

Then $g'(x) = f(x) = \frac{1}{3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In general, we can construct functions that are C^k but not C^{k+1} at a point.

(iv) Polynomials, the sine and cosine functions, and the exponential functions on $\mathbb{R}$ are all C^∞.

A function f is **real-analytic** at p if in some neighborhood of p it is equal to its Taylor series at p, that is,

$$f(x) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \cdots$$

Real-analytic functions are C^∞ because you can differentiate them termwise in their region of convergence. The converse does not hold: define

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0; \\ 0 & \text{for } x \le 0. \end{cases}$$

We can show f is C^∞ on $\mathbb{R}$ and the derivatives $f^{(k)}(0) = 0$ for all $k \ge 0$ by induction, then the Taylor series must be zero in any neighborhood of the origin, but f is not. Then f isn't equal to its Taylor series, and we have a smooth non-analytic function.

2.2 Taylor's Theorem with Remainder

However, we have a Taylor's theorem with remainder for C^∞ functions that's good enough. Say a subset S of $\mathbb{R}^n$ is **star-shaped** with respect to a point p in S if for every $x \in S$, the line segment from p to x lies in S.

Lemma 2.1 (Taylor's theorem with remainder)**.** *Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p^1, \cdots, p^n)$ in U. Then there are C^∞ functions $g_1(x), \cdots, g_n(x)$ on U such that*

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Proof. Since U is star-shaped with respect to p, for any $x \in U$ ⊠