Algebraic Topology Homework

Simon Xiang

This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source code: `https://git.simonxiang.xyz/math_notes/file/freshman_year/algebraic_topology/master_homework.tex.html`

Contents

§1 August 29, 2020: Homework 1

Hatcher Chapter 0 (p. 18): 1, 3ab, 17,
Hatcher Section 1.1 (p. 38): 3, 6, 7, 16.

§1.1 Question 1

Problem 1. Suppose X, Y are compact Hausdorff spaces and $f\colon X \to Y$ is continuous and onto. Define $\sim$ as the equivalence relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

(a) Prove the quotient space $X/\sim$ is Hausdorff.

(b) Use this to show that the induced map $X/\sim \to Y$ is a homeomorphism.

(c) Show that identifying the ends of the interval gives S^1.

(d) Give a cooler example.

Solution. We do this by

(a) Let $[a], [b]$ be elements (equivalence classes) of the quotient space $X/\sim$. We want to separate $[a]$ and $[b]$ by open sets: since (note X is T_4). Canonical projection map is a quotient map: therefore it maps closed sents onto closed (and preimage of closed is also closed)

■

§1.2 Problem 1 Chapter 0

Problem. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

§1.3 Problem 3a

Problem. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

§1.4 Problem 3b

Problem. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

§1.5 Problem 17a

Problem. Show that the mapping cylinder of every map $f\colon S^1 \to S^1$ is a CW complex.

§1.6 Problem 17b

Problem. Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

§1.7 Problem 3 Section 1.1

Problem. For a path-connected space X, show that $\pi(X)$ is abelian if and only if all basepoint-change homeomorphisms β_h depend only on the endpoints of the path h.

§1.8 Problem 6

Problem. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $\left(S^1, s_0\right) \to (X, x_0)$. Let $\left[S^1, X\right]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to \left[S^1, X\right]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $\left[S^1, X\right]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

§1.9 Problem 7

Problem. Define $f\colon S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on both boundary circles. [Consider what f does to the map $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.

§1.10 Problem 17

Problem. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.