Notes on Topological Quantum Field Theory

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Notes for my Spring 2022 DRP on Bordism and TQFTs, mentored by William Stewart. We follow Dan Freeds notes for a topics course called "*Bordism: Old and New*" (M392C) that he taught in 2012 (course url here: `https://web.ma.utexas.edu/users/dafr/M392C-2012/index.html`). Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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1 Introduction to Bordism

Review of homology: A **singular q-chain** in a space S is a formal sum of continuous maps $\Delta^q \to S$ from the standard q-simplex. There is a boundary operation ∂ on chains; a chain c is a **cycle** if $\partial c = 0$, and a **boundary** if there exists a $(q+1)$-chain b with $\partial b = c$. If S is a point then every cycle is a boundary. Bordism replaces cycles by *closed smooth manifolds* mapping continuously into S. (Here *closed* means *compact without boundary*). Chains become *compact smooth manifolds* X with a continuous map $X \to S$, and the boundary of this chain is the restriction $\partial X \to S$ to the boundary.

Example 1.1. Not every closed smooth manifold is the boundary of a compact smooth manifold. We have $Y = \mathbb{RP}^2$ *not* the boundary of a compact 3-manifold. However, it is the boundary of a noncompact 1-manifold with with boundary. To see this, first consider $\mathbb{RP}^0 \simeq \{\mathrm{pt}\}$. This is the boundary of a non-compact 1-manifold, namely the half line $[0,1)$. Here the cover $\left\{[0,\frac{1}{2}),(\frac{1}{3},\frac{2}{3}),(\frac{1}{2},\frac{3}{4}),(\frac{2}{3},\frac{4}{5}),\cdots,(\frac{n}{n+1},\frac{n+2}{n+3})\right\}$ as $n \to \infty$ has no finite subcover. This generalizes to $\mathbb{RP}^1 \simeq S^1$, which is the boundary of $S^1 \times [0,1)$ and so $\mathbb{RP}^2$ is the boundary of $\mathbb{RP}^2 \times [0,1)$. From here, we can see that every closed smooth manifold Y is the boundary of a noncompact manifold with boundary, namely $Y \times [0,1)$. todo:check What fails if Y isn't closed? If Y has boundary, then $\partial^2 = 0$, and if Y is non-compact this doesn't work. todo:check (why we need closed). also 1-manifold??

How do we prove our earlier assertion that $\mathbb{RP}^2$ is not the boundary of a compact 3-manifold? We will see this later.

1.1 Review of smooth manifolds

Definition 1.1. A **topological manifold** is a paracompact, Hausdorff topological space X such that every point of X has an open neighborhood homeomorphic to an open subset of affine space. We define *n-dimensional affine space* as $\mathbb{A}^n = \{(x^1, x^2, \cdots, x^n) \mid x^i \in \mathbb{R}\}$. The vector space $\mathbb{R}^n$ acts transitively on $\mathbb{A}^n$ by translations.

Definition 1.2. The empty set $\emptyset$ is trivially a manifold of any dimension $n \in \mathbb{Z}^{\geq 0}$. We write $\emptyset^n$ to denote the empty manifold of dimension n.

Definition 1.3. Define $\mathbb{A}^n_- = \{(x^1, x^2, \cdots, x^n) \in \mathbb{A}^n \mid x^1 \leq 0\}$. We require that coordinate charts take values in open sets of $\mathbb{A}^n_-$. Then we partition X into two disjoint subsets (both manifolds): the **interior** (points with $x^1 < 0$ in every coordinate system) and the **boundary** ∂X (points with $x^1 = 0$).

Remark 1.1. Recall the mnemonic "ONF", standing for "Outward Normal First". An outward normal in a coordinate system is represented by the first coordinate vector field $\partial/\partial x^1$, which points outward at the boundary.

Definition 1.4. At any point $p \in \partial X$ there is a canonical subspace $T_p(\partial X) \subseteq T_pX$; the quotient space $T_pX/T_p(\partial X)$ is a real line ν_p. So over the boundary there is a short exact sequence

$$0 \to T(\partial X) \to TX \xrightarrow{p} \nu \to 0$$

of vector bundles.

The vector $\partial/\partial x^1(p)$ projects to a nonzero element of ν_p, but there is no canonical basis independent of coordinate system. However, any two such vectors are in the same component of $\nu_p \setminus \{0\}$, so ν carries a canonical *orientation*. Furthermore, there is a splitting $s\colon \nu \to TX$ that assigns to a point on ν a tangent vector which lies in $T(\partial X)$, which by the quotient projection maps to 0. Therefore since $s \circ p = \mathrm{id}_{TX}$, we have $TX \simeq T(\partial X) \oplus \nu$. For example, say we have an n-manifold with boundary, then $T_pM \simeq \mathbb{R}^{n+1}$ and $T_mM \simeq \mathbb{R}^n$ for $m \in \partial X$. Since T_mM has codimension 1 we have $\nu \simeq \mathbb{R}$, which comes from $\mathbb{R}^{n+1}/\mathbb{R}^n$. We also see that $T_pM \simeq \mathbb{R}^{n+1} \simeq (T_mM \simeq \mathbb{R}^n) \oplus (\nu \simeq \mathbb{R})$.

Definition 1.5. Let X be a manifold with boundary. A **collar** of the boundary is an open set $U \subset X$ which contains ∂X and a diffeomorphism $(-\varepsilon, 0] \times \partial X \to U$ for some $\varepsilon > 0$.

Theorem 1.1. *The boundary ∂X of a manifold X with boundary has a collar.*

Proof. todo:this? ⊠

Let $\{X_1, X_2, \cdots\}$ be a countable collection of manifolds. We form a new manifold $X_1 \amalg X_2 \amalg \cdots$, the **disjoint union** of $X_1, X_2, \cdots$. As a set it is the disjoint union of the underlying sets for the manifolds. A question is how to precisely define this; what is $X \amalg X$, for example? A solution is to fix some $\mathbb{A}^\infty$ and regard all manifolds embedded in it. Replace X_i by $\{i\} \times X_i$, then define the disjoint union to be the ordinary union of subsets of $\mathbb{A}^\infty$. We could also use a universal property; a disjoint union of $X_1, X_2, \cdots$ is a manifold Z and collection of smooth maps $\iota_i : X_i \to Z$ such that for any manifold Y and collection $f_i : X_i \to Y$ of smooth maps, there exists a unique map $f : Z \to Y$ such that for each i the diagram

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & Z \arrow[d, dotted, "f"] \\ & Y \end{tikzcd}$$

commutes.

1.2 Bordism

Definition 1.6. Let Y_0, Y_1 be closed n-manifolds. A **bordism** $(X, (\partial X)_0 \amalg (\partial X)_1, \theta_0, \theta_1)$ from Y_0 to Y_1 is a compact $(n+1)$-manifold X with boundary, a decomposition $\partial X = (\partial X)_0 \amalg (\partial X)_1$ of its boundary, and embeddings $\theta_0 : [0, +1) \times Y_0 \to X$, $\theta_1 : (-1, 0] \times Y_1 \to X$ such that $\theta_i((0, Y_i)) = (\partial X)_i$, $i = 0, 1$.

The map θ_i is a diffeomorphism onto its image, which is a collar neighborhood of $(\partial X)_i$. The reason why we add the collar neighborhoods is to make it easier to glue bordisms; without them we could say a bordism X from Y_0 to Y_1 is a compact $(n+1)$-manifold with boundary $Y_0 \amalg Y_1$.

Definition 1.7. Let $(X, (\partial X)_0 \amalg (\partial X)_1, \theta_0, \theta_1)$ be a bordism from Y_0 to Y_1. The **dual bordism** from Y_1 to Y_0 is $(X^\vee, (\partial X^\vee)_0 \amalg (\partial X^\vee)_1, \theta_0^\vee, \theta_1^\vee)$ where $X^\vee = X$, the decomposition of the boundary is swapped so $(\partial X^\vee)_0 = (\partial X)_1$ and $(\partial X^\vee)_1 = (\partial X)_0$, and

$$\begin{aligned} \theta_0^\vee(t, y) &= \theta_1(-t, y), && t \in [0, +1),\ y \in Y_1, \\ \theta_1^\vee(t, y) &= \theta_0(-t, y), && t \in (-1, 0],\ y \in Y_0. \end{aligned}$$

Think of the dual bordism $X^\vee$ as the original bordism X "turned around", and view it as a bordism from $Y_1^\vee$ to $Y_0^\vee$, where for naked manifolds we set $Y_i^\vee = Y_i$. When manifolds have tangential structure, this will not necessarily be the case.

Lemma 1.1. *Bordism defines an equivalence relation.*

Proof. For any closed manifold Y, the manifold $X = [0, 1] \times Y$ is a bordism from Y to Y. Formally, set $(\partial X) + 0 = \{0\} \times Y, (\partial X)_1 = \{1\} \times Y$, and simple diffeomorphisms $[0, 1) \to [0, \frac{1}{3}), (-1, 0] \to (\frac{2}{3}, 1]$ to construct our θ_i. Symmetry is given by the dual bordism; if X is a bordism from Y_0 to Y_1, then $X^\vee$ is a bordism from Y_1 to Y_0.

For transitivity let X be a bordism $Y_0 \to Y_1$, and X' a bordism from Y_1 to Y_2. Define a new manifold $X'' = X \amalg X / \sim$, where for $(a, b), (c, d) \in X \amalg X'$, if either $a, d \in Y_1$, then $(a, b) \sim (c, d)$. todo:how exactly is this a manifold? bourbaki: https://math.stackexchange.com/questions/496571/under-what-conditions-the-quotient-space-of-a-ma basically E is a closed submanifold of $M \times M$ (true since $E = (\partial M)_1 = (\partial M')_0$ which are manifolds by def. the projection is also a submersion. diffeomorphic. okay how do we show the smooth structure?

⊠

Example 1.2. If $f: M \to N$ is a diffeomorphism between manifolds, then consider the mapping cylinder $Mf = ([0,1] \times M) \amalg_f N$, a smooth manifold with boundary $M \times \{0\} \cup N \times \{1\}$. So diffeomorphic manifolds are bordant.

Let Ω_n denote the set of equivalence classes of n-manifolds under the equivalence relation of bordism. We use the term **bordism class** for an element of Ω_n. Note that $\emptyset^0$ (empty manifold) is a distinct element of Ω_n, so Ω_n is a **pointed set**.

1.3 Disjoint union and the abelian group structure

The disjoint union and cartesian product give Ω_n more structure.

Definition 1.8. A **commutative monoid** is a set with a commutative, associative composition law and identity element. An **abelian group** is a commutative monoid in which every element has an inverse.

Disjoint unions of manifolds pass to bordism classes: if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \amalg Y_1$ is bordant to $Y_0' \amalg Y_1'$ (take the disjoint union of the bordisms as manifolds). So $(\Omega_n, \amalg)$ is a commutative monoid.

Lemma 1.2. *$(\Omega_n, \amalg)$ is an abelian group with identity $\emptyset^n$. Furthermore, for $Y \in \Omega_n$, $Y \amalg Y$ is null-bordant.*

Proof. Let $Y \in \Omega_n$. Consider the manifold $X = [0,1] \times Y$; this gives a bordism between $Y \amalg Y$ and $\emptyset^n$, with $(\partial X)_0 = Y \amalg Y$ and $(\partial X)_1 = \emptyset^n$. Inverses are unique because if we took another manifold M not bordant to Y, we can't have a manifold with boundary $M \amalg Y$ by definition, so we cannot do the empty manifold decomposition. todo:check So $Y = Y^{-1}$ and we are done. ⊠

Proposition 1.1. *$\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ with generator* pt.

Proof. Compact 0-manifolds are finite disjoint unions of points. Lemma 1.2 implies that the disjoint union of two points is a boundary, so this is zero in Ω_0. To show that pt is not the boundary of a compact 1-manifold without boundary, this follows from the classification of 1-manifolds with boundary; they are a finite disjoint union of circles and closed intervals, so its boundary has an even number of points. ⊠

Proposition 1.2. *$\Omega_1 = 0$ and $\Omega_2 = \mathbb{Z}/2\mathbb{Z}$ with generator $\mathbb{R}\mathrm{P}^2$.*

Proof. By the classification of compact 1-manifolds, closed 1-manifolds are finite disjoint unions of circles, which bound disks (and so they are null-bordant). Therefore $\Omega_1 = 0$. Now recall the classification theorem for 2-manifolds, which states that there are two connected families; oriented and unoriented surfaces. For oriented surfaces, they are either 2-spheres or connected sum of tori (genus g surfaces). Spheres bound the 3-ball, and genus g surfaces go to genus g handlebodies. todo:check

Any unoriented surface is a **connected sum** of $\mathbb{R}\mathrm{P}^2$'s. It suffices to prove that $\mathbb{R}\mathrm{P}^2$ does not bound and $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ does not bound. For the former, suppose that X is a compact manifold with $\partial X = \mathbb{R}\mathrm{P}^2$. Then consider the **double** $D = X \cup_{\mathbb{R}\mathrm{P}^2} X$, constructed by gluing two copies of X along $\mathbb{R}\mathrm{P}^2$. We have $\chi(D) = 2\chi(X) - \chi(\mathbb{R}\mathrm{P}^2) = 2\chi(X) - 1$ by Hatcher 2.2.21, which is odd. However, closed odd-dimensional manifolds have zero euler characteristic. It remains to show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$; this is true by inclusion-exclusion (on counting cells). todo:check

Similarly $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^2) = 1$. The same argument applies; suppose there exists some compact X with $\partial X = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$, examine χ of the double, and we are done. todo:hatcher 3.3.6(b). where does this argument fail? right, existence of X. but it does bound... ??

Correct argument; $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to the Klein bottle K, which has a map $K \to S^1$, a fiber bundle with fiber S^1. Then there is an associated fiber bundle with fiber D^2, a compact 3-manifold with boundary K. ⊠

1.4 Cartesian product and the ring structure

Definition 1.9.

(i) A **commutative ring** R is an abelian group $(+,0)$ with a second commutative, associative composition law $(\cdot)$ with identity (1) which distributes over the first: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ for all $r_1, r_2, r_3 \in R$.

(ii) A $\mathbb{Z}$**-graded commutative ring** is a commutative ring S which as an abelian group is a direct sum $S = \bigoplus_{n\in\mathbb{Z}} S_n$ of abelian group such that $S_{n_1} \cdot S_{n_2} \subseteq S_{n_1+n_2}$. In other words, you can multiply two elements in S_{n_1}, S_{n_2} to get an element in $S_{n_1+n_2}$.

Elements in $S_n \subset S$ are called **homogeneous of degree** n; an element of S is a finite sum of homogeneous elements.

Example 1.3. The integers $\mathbb{Z}$ form a commutative ring, and for any commutative ring R there is a polynomial ring $S = R[x]$ in a single variable which is $\mathbb{Z}$-graded. To define this grading, we need to assign a degree to the indeterminate x, usually 1; in this case S_n is the abelian group of homogeneous polynomials of degree n in x. More generally, there is a $\mathbb{Z}$-graded polynomial ring $R[x_1, \cdots, x_k]$ in any number of indeterminates with any assigned integer degrees $\deg x_k \in \mathbb{Z}$.

Define

$$\Omega = \bigoplus_{n\in\mathbb{Z}^{\geq 0}} \Omega_n.$$

Formally, define $\Omega_{-m} = 0$ for $m > 0$. The Cartesian product of manifolds is compatible with bordism; if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \times Y_1$ is bordant to $Y_0' \times Y_1'$. To see this, let M_0, M_1 be the bordisms with $\partial M_0 = Y_0 \amalg Y_0'$, $\partial M_1 = Y_1 \amalg Y_1'$. Then the bordism between $Y_0 \times Y_1$ and $Y_0' \times Y_1'$ is given by $M_0??M_1$ todo:not $M_0 \times M_1$, dimension. not $\amalg$, counterex. help. So this passes to a commutative, associative binary composition on Ω.

Proposition 1.3. *$(\Omega, \amalg, \times)$ is a $\mathbb{Z}$-graded ring. A homogeneous element of degree $n \in \mathbb{Z}$ is represented by a closed manifold of dimension n.*

Proof. todo:?? dk what the bordism of product is. show it's compatible?? ⊠

In his Ph.D. thesis Thom todo:references proved the following theorem.

Theorem 1.2 (Thom)**.** *There is an isomorphism $\Omega \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, x_6, x_8, \cdots$ where there is a polynomial generator of degree k for each positive integer k not of the form $2^i - 1$. Furthermore, if k is even, then x_k is represented by* $\mathbb{RP}^k$. *todo:does this vibe with $\Omega_2 = \mathbb{Z}/2\mathbb{Z}$? shouldn't it be $\mathbb{Z}/2\mathbb{Z}[x_2]$? or is that just $\mathbb{Z}/2\mathbb{Z}$*

Dold later constructed manifolds representing the odd degree generators, which are fiber bundles over $\mathbb{RP}^m$ will fiber $\mathbb{CP}^\ell$. Working out Ω_{10}, or 10-manifolds up to bordism, we have generator $\mathbb{RP}^{10}$. todo:?? don't know much about 10-manifolds

Thom proved that the **Stiefel-Whitney numbers** determine the bordism clsas of a closed manifold. The **Steifel-Whitney classes** $w_i(Y) \in H^i(Y; \mathbb{Z}/2\mathbb{Z})$ are examples of **characteristic classes** of the tangent bundle; we will discuss this stuff later. Any closed n-manifold Y has a **fundamental class** $[Y] \in H_n(Y; \mathbb{Z}/2\mathbb{Z})$. If $x \in H^\bullet(Y; \mathbb{Z}/2\mathbb{Z})$, the pairing $\langle x, [Y]\rangle$ produces a number in $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.3. *The Stiefel-Whitney numbers*

$$\langle w_{i_1}(Y) \smile w_{i_2}(Y) \smile \cdots \smile w_{i_k}(Y), [Y]\rangle \in \mathbb{Z}/2\mathbb{Z}$$

determine the bordism class of a closed n-manifold Y.

That is to say, if two closed n-manifolds Y_0, Y_1 have the same Stiefel-Whitney numbers, then they are bordant. Notice that not all naively possible nonzero Stiefel-Whitney numbers can be nonzero. For example, $\langle w_1(Y), [Y] \rangle$ vanishes for any closed 1-manifold Y. Also, the theorem implies that a closed n-manifold is the boundary of a compact $(n+1)$-manifold iff all the Stiefel-Whitney numbers of Y vanish. If it is a boundary, it is immediate that the Stiefel-Whitney numbers vanish; the converse is not obvious. todo:I'm not supposed to get this right yet?

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$