Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1.1 Problem 1

Problem. An n-dimensional *manifold with boundary* means a Hausdorff space M, such that every $x \in M$ has a neighborhood U such that the pair (U, x) is homeomorphic to either $(\mathbb{R}^n, 0)$ or $\big(\mathbb{R}^{n-1} \times [0, \infty), 0\big)$, where in both cases 0 means $(0, \ldots, 0)$. We call x an interior or boundary point according to which of these holds. Note that this is *not* the usual use of "interior" and "boundary" from point-set topology. The set of boundary points is written ∂M.

Prove that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence.

You may use without proof the fact that no point can be both an interior and a boundary point. You may also use the following additional hypotheses, which sound like they should follow from the definition, but turn out not to.

1. ∂M has countably many components.
2. M is second countable.
3. M is metrizable.

Also, the 2-dimensional case is enough to give a complete understanding. Finally, a hint: chain together an infinite sequence of homotopies, being careful that the result makes sense and is continuous.

Remarks: informally, I think of $M \setminus \partial M$ as a sort of deformation-retract of M. But it is easy to see that if $\partial M \neq \emptyset$ then M does not actually deformation retract to $M \setminus \partial M$. Also, without the extra hypotheses, the only solution I know uses something you probably have not seen: topological dimension, which lets you build an open cover with good overlap properties.

Solution. We want to show that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence, that is, it is one of the continuous maps f or g such that $f \circ g$ is homotopic to ι_M and $g \circ f$ is homotopic to $\iota_{M \setminus \partial M}$. ■

§1.2 Problem 2

Problem (A "bad" group action)**.** Let $X = \mathbb{R}^2 \setminus 0$ where 0 is the origin. Let G be the group of homeomorphisms of X generated by the transformation $(x, y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.

(a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).

(b) Prove that G's action on X satisfies the hypothesis of the theorem from class about $\pi_1(X/G) \cong G$, namely: every $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for every $g \in G \setminus \{1\}$.

(c) Prove that Y is a manifold, except for the fact that it is *not* Hausdorff.

(When working on a theorem involving a group action, if I wonder whether some hypothesis can be omitted, checking it for this single example usually reveals the answer.)

Solution. Discrete orbit: the orbit is endowed with the discrete topology (eg $\mathbb{Z} \trianglelefteq \mathbb{R}$). What even is this group? ■

§1.3 Problem 9 Chapter 0

Problem. Show that a retract of a contractible space is contractible.

Solution. Let A be a retract of a contractible space X. Then there exists a deformation retract of X onto a point, and a retract of X onto A: denote this retract with f, and the homotopy encoding the deformation retraction as $H : X \times I \to X$, where $f|_A = \operatorname{id} A$, $H(x, 0) = \operatorname{id}_X$, $H(x, 1) = \{x_0\}$. Consider the homotopy $H' = f \circ H|_A$ from $A \times I \to A$. Then this homotopy is continuous since f and H are continuous, and is a deformation retraction onto a point since $H'(x, 0) = \operatorname{id}_A, H'(x, 1) = \{x_0\}$. Therefore A is contractible. ■

§1.4 Problem 20

Problem. Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Solution. We can contract the intersecting disk of the Klein bottle to itself, so the resulting structure resembles S^2/S^0 (the sphere with two points identified), which is homotopy equivalent to $S^1 \vee S^2$ by Example 0.8. Counting the boundary of the intersecting disk itself, this forms another S^1 identified with (now $S^1 \vee S^2$) at a point. Therefore the self-intersecting Klien bottle embedded in $\mathbb{R}^3$ has the homotopy type of $S^1 \vee S^1 \vee S^2$. ■

§1.5 Problem 17 Section 1.1

Problem. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$ (whoops, attempted this one last week).

Solution. Solution. (didn't finish) Take the family of retractions that map the first circle to it- self (identity) and wrap the second circle around the first n times, then if n in N this is an infinite family of retractions. (Not sure how to formalize this or show they're nonhomotopic). ■

§1.6 Problem 18

Problem. Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

§1.7 Problem 20

Problem. Suppose $f_t \colon X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the enter of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

§1.8 Problem 2 Section 1.2

Problem. Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.

§1.9 Problem 4

Problem. Let $X \subseteq \mathbb{R}^3$ be the finite union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.