Algebraic Topology Homework

Math 382C

Simon Xiang

Dr. Allock, M 382C
The University of Texas at Austin
August 24, 2020

Homework 0

Problem 1. Prove that the finite product of manifolds is a manifold.

Proof. We prove $M \times N$ is a manifold, where M is an m-manifold and N is an n-manifold, which is a sufficient condition for the finite product

$$\prod_{i=1}^{n} M_i$$

to be a manifold for M_i a m_i-manifold, $m_i \in \mathbb{N}$. First, note that the product of two T_2 spaces is T_2. Take τ_1, τ_2 to be topological spaces, and let X be their product. We have two distinct points $(a, b), (c, d)$ in X, which we can separate by open sets $X_1 \times U_2$, $X_1 \times V_2 \in X$ for $X_1 \in \tau_1$, $U_2, V_2 \in \tau_2$ if $a = c$ (which implies $b \neq d$), and $U_1 \times X_2$, $V_1 \times X_2$ for $U_1, V_1 \in \tau_1$, $X_2 \in \tau_2$ if $a \neq c$.

Now let (a, b) be in X, where a is in τ_1 and b is in τ_2. Then there exist $U_1 \in \tau_1$, $U_2 \in \tau_2$ such that $a \in U_1$, $b \in U_2$, and U_1 homeomorphic to $\mathbb{R}^m$, U_2 homeomorphic to $\mathbb{R}^n$. Simply take the open set $U_1 \times U_2 \in X$ containing the point (a, b), and define the homeomorphism $f : M \times N \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) = (g(x), h(y))$, where g and h are the homeomorphisms of U_1 onto $\mathbb{R}^m$ and U_2 onto $\mathbb{R}^n$ respectively. Clearly f is continuous since g and h are continuous, and by the same logic f^{-1} exists and is continuous, and is given by $f^{-1}(x, y) = (g^{-1}(x), h^{-1}(y))$ (whose components are continuous since g and h are homeomorphisms).

Finally, we have $\mathbb{R}^m \times \mathbb{R}^n$ homeomorphic to $\mathbb{R}^{m+n} = \mathbb{R}^{n+m}$, so we conclude the product manifold $M \times N$ is indeed an $n + m$-manifold. ⊠

Problem 2. Prove that a manifold is connected if and only if it is path-connected.

Proof. First, note that every path-connected space is connected. By way of contradiction, assume that a path-connected topological space (X, τ) is not connected, that is, there exist $U, V \in \tau$ such that $U \cap V = \emptyset$ and $U \cup V = X$.

Recall that an *interval* is a set $I \subset \mathbb{R}$ such that for all $a, b \in I$, $a < x < b$ implies $x \in I$. Furthermore, all intervals are connected (we omit the proof for brevity). Since τ is path-connected, for all $x, y \in X$ there exists a path $f : [a, b] \to X$ such that f is continuous and $f(a) = x$ and $f(b) = y$. Now the image of the path denoted $f([a, b])$ is connected, since the image of a connected set under a continuous function is connected. Choose $x \in U$ and $y \in V$: then the path f cannot connect x and y since $U \cap V = \emptyset$, and $f([a, b])$ must either be fully contained in U or V. Therefore path-connected spaces (and manifolds) are connected, proving the reverse implication.

For the forward implication, let $a \in M$. Consider X, the set of points that are path-connected to a. Note that $a \in X$ so $X \neq \emptyset$ (this is important). We claim

X and X^c are open: to see this, let $x \in X$. Then we have an open neighborhood of x homeomorphic to $\mathbb{R}^n$, let us denote its image under the homeomorphism f as $U \subset \mathbb{R}^n$. We can find a convex neighborhood of $f(x)$ denoted $B(f(x), \epsilon) \subset U$ that is path-connected by definition. Since path-connectedness is preserved under a continuous map, the inverse image of the convex neighborhood containing $f(x)$ under the homeomorphism f denoted $f^{-1}(B(f(x), \epsilon))$ is path-connected. Note that $x \in f^{-1}(B(f(x), \epsilon))$, so there exists a path between every point in $f^{-1}(B(f(x), \epsilon))$ and x, therefore $x \in f^{-1}(B(f(x), \epsilon)) \subset X$ and is open. Since for all $x \in X$ we have $x \in X^\circ$, we conclude X is open. A similar argument follows for the fact that X^c is open: simply examine $y \in X^c$ and $B(f(y), \delta)$ instead.

We reach the final stage of this proof. By assumption, our manifold M is connected. This is equivalent to the fact that the only subsets of M that are both open and closed are M and $\emptyset$: if there existed an $A \subset M$ that were both open and closed, then $A \cap A^c = \emptyset$ and $A \cup A^c = M$, contradicting the fact that M is connected. Now we have constructed a path-connected set X that is both open and closed— both X and X^c are open, and $X \neq \emptyset$ as stated earlier in the proof. We conclude that $X = M$, and so the manifold M is path-connected. $\boxtimes$

Problem 3. Suppose a finite group G acts on a manifold M. Suppose the action is *free*, meaning that only the identity element has any fixed points. Then the orbit space M/G is also a manifold. ("Lying in the same G-orbit" is an equivalence relation on M. M/G means the set of equivalence classes. The topology on M induces one on M/G, which is the one you must work with.)

Proof. Why helppppppp

G acts on M: a map $* : G \times M \to M \ni ex = x \forall x \in M$, $*(g1, g2)x = *(g1 * (g2, x)) \forall g_1, g_2 \in G$, $x \in M$ or alternatively $(g_1 g_2)x = g_1(g_2 x) \forall x \in M$, $g_1, g_2 \in G$. M is a G-set.

Free action: $g \in G \wedge \exists x \in X \ni gx = x \implies g = e$.

Orbit of $x \in X$: $Gx = \{gx \mid g \in G\}$ for some $x \in X$. x $\sim$ y iff $\exists g \in G \ni gx = y \implies$ orbits are equivalence classes under this relation.

Orbit space: set of all equivalence classes (under the same orbit relation), denoted X/G (also called quotient, space of coinvariants).

G-orbit $\boxtimes$