Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 380C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1 Free Groups

Not to be confused with free *abelian* groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39 and Hatcher §1.2.

§1.1 Words and Reduced Words

Let A_i be a set of elements (not necessarily finite). We say A is an *alphabet* and think of the $a_i \in A$ as *letters*. Symbols of the form a_i^n are *syllables* and *words* are a finite string of syllables. We denote the *empty word* 1 as the word with no syllables.

Example 1.1. Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to a_i^{m+1} (*elementary contractions*) or replacing a_i^0 by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a *reduced word*.

Example 1.2. The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

§1.2 Free Groups

Denote the set of all reduced words from our alphabet A as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words w_1 and w_2 in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of w_2 onto w_1.

Example 1.3. If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet A. Man, the weather outside today is nice.

Definition 1.1 (Free Group)**.** The group $F[A]$ described above is the *free group generated by A*.

Sometimes we have a group G and a generating set $A = \{a_i \mid \in I\}$, and we want to know whether or not G is *free* on $\{a_i\}$, that is, G is the free group generated by $\{a_i\}$.

Definition 1.2 (Free Generators)**.** If G is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi\colon G \to F[A]$ such that $\phi(a_i) = a_i$, then G is *free on A*, and the a_i are *free generators of* G. A group is *free* if it is free on some nonempty set A.

Oh you'll be free... free indeed...

Example 1.4. $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on n letters" as opposed to the "free group on n generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

Example 1.5. $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

Theorem 1.1. *If G is free on A and B, then A and B have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

Proof. Refer "to the literature". ☒

Definition 1.3 (Rank)**.** If G is free on A, then the number of letters in A is the *rank of the free group G.*

Theorem 1.2. *Two free groups are isomorphic if and only if they have the same rank.*

Proof. Immediate. ☒

Theorem 1.3. *A nontrivial proper subgroup of a free group is free.*

Proof. Back "to the literature". ☒

Example 1.6. Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

§1.3 Homomorphisms of Free Groups

Theorem 1.4. *Let G be generated by $A = \{a_i \mid \in I\}$ and let G' be any group. If $a_i{}'$ for $i \in I$ are any elements in G' not necessarily distinct, then there is at most one homomorphism $\phi\colon G \to G'$ such that $\phi(a_i) = a_i{}'$. If G is free on A, then there is exactly one such homomorphism.*

Proof. Let ϕ be a homomorphism from G into G'such that $\phi(a_i) = a_i{}'$. Then any $x \in G$ can be written as a finite product of the generators a_i, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the a_i not necessarily distinct. Since ϕ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i{}'$.

Now suppose that G is free on A, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi\colon G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in G, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ⊠

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

Corollary 1.1. *Every group G' is a homomorphic image of a free group G.*

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 1.4 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i{}'$. Clearly the image of G under ψ is all of G'. ⊠

Only the free group on one letter is abelian.

§1.4 Free Products of Groups

Definition 1.4 (Free Products). As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of w_1 and the beginning of w_2 belong to the same G_α? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of a and b are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, ...$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha \colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi \colon *_\alpha G_\alpha \to H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under φ.

Example 1.7. For a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

§2 Van Kampen's Theorem

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups.