Math Club Lectures

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The UT Math Club meets weekly and invites speakers to give talks every Tuesday at 5:00 PM! Here are some notes I've TEX'd up from some of them (not all).

Contents

§1 The Borsuk-Ulam Theorem (9/15/20)

Today's speaker is Hannah Turner, a 6th year Ph.D student. We'll be talking about the Borsuk Ulam Theorem!

§1.1 Continuous Maps

We talk about maps from n-dimensional spheres to $\mathbb{R}^n$. Usually we talk about maps $f\colon \mathbb{R} \to \mathbb{R}$ that are continuous, "don't lift your pencil". In topology, preimage of open sets are open, AKA for $f\colon X \to Y$, points are close in Y imply sets are close in X. For the scope of this talk, assume topological spaces are metrizable.

Definition 1.1 (Sphere)**.** We have $\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$ for $x_i \in \mathbb{R}$. We define the *sphere* notated S^{n-1} as the set

$$\{x_i \mid |x_i| = 1\},$$

or the set of points that are a distance 1 from the origin. For example, $S^1 \subseteq \mathbb{R}^2$, $S^2 \subseteq \mathbb{R}^3$.

Let talk about maps $S^1 \to \mathbb{R}$. Deform the circle into squiggly things then smash it. Or you can turn it into a square then squish it. Yay for deformation retractions! Also: S^1 is compact, so it maps onto a closed and bounded interval. Note this map isn't onto.

§1.2 The Borsuk-Ulam Theorem

Theorem 1.1 (Borsuk-Ulam)**.** *Any map $f\colon S^n \to \mathbb{R}^n$ sends two antipodal points $(v \sim -v)$ in S^n to the same point in $\mathbb{R}^n$.*

Example 1.1. Any map $S^1 \xrightarrow{f} \mathbb{R}$ sends two antipodal points in S^1 to the same point in $\mathbb{R}$. Look at $g(x) = f(x) - f(-x)$, where $g\colon S^1 \to \mathbb{R}$. Our new goal: show that $g(x)$ has a zero (this shows BU for $n = 1$). Pick our favorite point $x_0 \in S^1$, and assume $g(x_0) \neq 0$. So $g(x_0)$ is either positive or negative, that is $g(x_0) > 0$ or $g(x_0) < 0$.

Assume $g(x_0) > 0$: what happends to $-x_0$, the antipodal point?

$$g(-x_0) = f(-x_0) - f(-(-x_0)) = f(-x_0) - f(x_0) = -(f(x_0) - f(-x_0)) = -g(x_0).$$

The $g(-x_0) < 0$. Now we apply the IVT, but we have to be a little careful. For the usual $\mathbb{R} \xrightarrow{f} \mathbb{R}$, say $f(x) = 5$, $f(y) = 7$, we hit every value in between 5 and 7. What's important: S^1 is *path-connected* (so the IVT still applies, since f is a function from a path-connected space into $\mathbb{R}$). Then there exists some $x \in S^1$ such that $g(x) = 0$, finishing the example.

The proof in higher dimensions is more difficult. There are three flavors:

1. Algebraic Topology: Assign an algebraic invariant. Weird equation: $H_*(\mathbb{R}P_i^n \mathbb{F}_2)$
2. Combinatorics: Tucker's Lemma,
3. Set covering (Lusternik-Schnirelmann): For S^n, any $n+1$ open sets covering one of the sets must contain antipodal points (in at least one of the covering sets).

§1.3 Corollaries of BU

Definition 1.2 (Homeomorphisms)**.** A *homeomorphism* is a continuous function $f\colon X \to Y$ which has a continuous inverse $f^{-1}\colon Y \to X$, $f \circ f^{-1} = \mathrm{id}_X$.

Example 1.2. A map which is not injective cannot have an inverse! Because then one point would map to two, breaking the rules and causing society to fall into a complete collapse.

Example 1.3. Take the map from the half open interval to the circle, that is, $f\colon [0,1) \to S^1$. f is continuous, has an inverse, but the inverse isn't continuous. Intuition: points at the place where the "endpoints" are identified are now very far away in the preimage of the inverse. So f is a bijection but its inverse is not continuous, so f is NOT a homeomomorphism.

Corollary 1.1. *There is no homeomorphism from $S^n \to \mathbb{R}^n$. Any continuous function $f\colon S^n \to \mathbb{R}^n$ has $f(x) = f(-x)$, not even one to one!*

§1.4 Pancakes!

Corollary 1.2 (Pancake Theorem)**.** *Any two disks in the place can be cut exactly in half by one slice. This includes weirdly shaped disks! In general, if we have n amount of n-dimensional blobs, we would have an n-dimensional hyperplane (locally homeo to $\mathbb{R}^{n-1}$) in $\mathbb{R}^n$ that slices each n-dimensional blob exactly in half.*

Proof. Sketch of a proof: take our 3 objects A_1, A_2, A_3. Something about normal vectors and perpendicular planes. Measure the volume? (Measures??) Pick the plane that gives half of the sandwich. Repeat for every plane in the sphere, call each plane P_x (where half of the sandwich is on each side of any P_x). Define a map $f\colon S^2 \to \mathbb{R}^2$ by $x \mapsto (\mathrm{vol}(A_2)$ on the positive side of P_x, $\mathrm{vol}(A_3)$ on the positive side of $P_x)$. We know there are x_0 and $-x_0$ with $f(x_0) = f(-x_0)$ by BU. Man, I wish I could TEX figures in real time. So

$$\begin{aligned} x_0 &\mapsto (\mathrm{vol}(A_2)P^+_{x_0}, \mathrm{vol}(A_3)P^+_{x_0}), \\ -x_0 &\mapsto (\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}), \end{aligned}$$

which are equal. The point is, we get the same plane but we're looking at it from two different directions, because $(\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}) = \big(\mathrm{vol}(A_2)P^-_{x_0}, \mathrm{vol}(A_3)P^-_{x_0}\big)$. $\mathrm{vol}(A_2)$ is cut in half by P_{x_0}, $\mathrm{vol}(A_3)$ is cut in half by P_{x_0}. ⊠