Differential Geometry Notes

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Lecture 1

Surfaces in three dimensions

1.1 What are surfaces?

Do you know what open sets are?

Definition 1.1. A subset $U \subseteq \mathbb{R}^n$ is **open** if for $\mathbf{a} \in U$, there exists an $\varepsilon > 0$ such that every $\mathbf{u} \in \mathbb{R}^n$ within a distance ε of $\mathbf{a}$ also lies in U. Using equations:

$$\mathbf{a} \in U \text{ and } \|\mathbf{u} - \mathbf{a}\| < \varepsilon \implies \mathbf{u} \in U.$$

In simpler terms, every point has a neighborhood contained in U.

$\mathbb{R}^n$ is open but the closed ball is not. Consider a map $f : X \to Y$. We say f is **continuous** at $\mathbf{a}$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for $\mathbf{u} \in X$,

$$\|\mathbf{u} - \mathbf{a}\| < \delta \implies \|f(\mathbf{u}) - f(\mathbf{a})\| < \varepsilon.$$

This is equivalent to the fact that the preimage of an open set is open. See here for more details: `simonxiang.xyz/blog/topological-continuity-simplicity-in-abstraction`[1]. Homeomorphism are continuous bijections with a continuous inverse.

Definition 1.2. A set $\mathscr{S} \subseteq \mathbb{R}^3$ is a **surface** if for every $\mathbf{p} \in \mathscr{S}$, there is an open set $U \subseteq \mathbb{R}^2$ and an open set $W \subseteq \mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathscr{S} \cap W$ is homeomorphic to U. Basically, locally a surface has to look like a 2-manifold (rather, it *is* a 2-manifold). A homeomorphism $\sigma : U \to \mathscr{S} \cap W$ defined above is a **surface patch** or **parametrization** of the open subset $\mathscr{S} \cap W$ of $\mathscr{S}$. A collection of charts covering $\mathscr{S}$ formes an **atlas** of $\mathscr{S}$.

OK, we called surface patches charts in differential topology, so I will be calling surface patches charts from now on. Aren't we missing the condition that charts have to be C^∞ compatible as well to form an atlas?

Example 1.1. A plane in $\mathbb{R}^3$ is a 2-manifold with a single chart. Let $\mathbf{a}$ lie in the plane, and $\mathbf{p}, \mathbf{q}$ be orthogonal unit vectors parallel to the plane. If $\mathbf{v}$ also lies in the plane, then $\mathbf{v} - \mathbf{a}$ is parallel to the plane, so $\mathbf{v} - \mathbf{a} = u\mathbf{p} + v\mathbf{q}$. So the surface patch is $\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$, with inverse $\sigma^{-1}(\mathbf{v}) = ((\mathbf{v} - \mathbf{a}) \cdot \mathbf{p}, (\mathbf{v} - \mathbf{a}) \cdot \mathbf{q})$. This is clearly a continous homeomorphism.

Example 1.2. Why do we talk about charts? Consider a *circular cylinder*, the set of points in $\mathbb{R}^3$ a fixed distance from an axis. For example, say the circle is of radius 1 around the z-axis, which we will call the *unit cylinder.* This is defined by

$$\mathscr{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

We can parametrize this by $\sigma(u, v) = (\cos u, \sin u, v)$. The map σ is continuous but not injective, since it's periodic. Restricting to an interval of length less than 2π gives an injective map, say $[0, 2\pi]$. However, although the restriction $\sigma|_V$ where $V = \{(u, v) \in \mathbb{R}^2 \mid u \in [0, 2\pi]\}$ is injective, V is not open and so $\sigma|_V$ is not a surface patch. If we restrict σ to $U = V^\circ = \{(u, v) \in \mathbb{R}^2 \mid u \in (0, 2\pi)\}$, then $\sigma|_U$ is a chart. However, $\sigma|_U$ does not hit the line $x = 1, y = 0$ in $\mathscr{S}$, so it does not cover $\mathscr{S}$.

We need another chart to make an atlas. So consider the chart $\sigma|_{\widetilde{U}}$, where $\widetilde{U} = \{(u, v) \in \mathbb{R}^2 \mid u \in [-\pi, \pi]\}$. This covers $\mathscr{S}$ sans the line $x = -1, y = 0$. Joining these two charts give an atlas, and so $\mathscr{S}$ is a surface.

Example 1.3. Say hello to your old friend S^2, defined by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

A popular parametrization is given by latitude θ and longitude φ: projecting a point p on the sphere down to the xy-plane gives a point q, then θ is the angle between p and q, and φ is the angle between q and the positive x-axis. Circles corresponding to θ are called **parallels**, and those corresponding to φ are called **meridians**.

To find an explicit parametrization, we want to express p in terms of θ and φ. The z-component is $\sin\theta$ by looking at the triangle. come back to this since it's important but not essentially important, essentially parametrizing the sphere and showing it's a 2-manifold with two charts

[1] shameless plug

Example 1.4. Our next (non)example is the **circular cone** with a vertex at a point $\mathbf{v}$ with an axis a straight line ℓ passing through $\mathbf{v}$, and an angle α, where $\alpha \in (0, \pi/2)$. It consists of the set of points $\mathbf{p} \in \mathbb{R}^3$ such that the straight line through $\mathbf{v}$ and $\mathbf{p}$ makes an angle α with the line ℓ. For example, for $\mathbf{v}$ the origin, ℓ the z-axis and $\alpha = \pi/4$ we have the circular cone defined by

$$\mathscr{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}.$$

If we take the image of a chart around the origin in $\mathbb{R}^2$, by path-connectedness we can find a path from c to b, where c corresponds to a point q in the upper cone and b corresponds to a point p in the lower cone. Furthermore, we can choose this path to avoid the origin, so its preimage in $\mathscr{S}$ is a path joining the hemispheres that avoids the origin. However, $\mathscr{S} \setminus \{0\}$ is not connected, and p and q lie in different connected components, so this is a contradiction.

If we delete the origin, we do get a surface $\mathscr{S}_- \cup \mathscr{S}_+$, with an atlas consisting of two charts given by the inverse of projection.

Usually a point $\mathbf{a}$ on a surface will lie in more than two charts. If we want two charts $\sigma, \widetilde{\sigma}$ to speak to each other, consider the **transition maps** $\sigma^{-1} \circ \widetilde{\sigma}$, $\widetilde{\sigma}^{-1} \circ \sigma$.

1.2 Smooth surfaces

We will use the following abbreviations:

$$\frac{\partial \mathbf{f}}{\partial u} = \partial_u \mathbf{f}, \quad \frac{\partial^2 \mathbf{f}}{\partial u^2} = \partial_{uu} \mathbf{f}, \quad \frac{\partial^2 \mathbf{f}}{\partial u \partial v} = \partial_{uv} \mathbf{f}, \quad \text{etc.}$$

Answer to my question above: surface is a codeword for topological manifold, while smooth surface is a codeword for smooth manifolds. You know what smooth functions are.

Definition 1.3. A surface patch $\sigma \colon U \to \mathbb{R}^3$ is **regular** if it is smooth and the vectors $\partial_u \sigma, \partial_v \sigma$ are LI at all (u, v). Equivalently, the product $\partial_u \sigma \times \partial_v \sigma$ should be non-zero at each point in U.

I guess my usage of chart earlier was incorrect, it's just a local homeomorphism onto its image. Now charts are *allowable surface patches*, or a regular surface patch $\sigma \colon U \to \mathbb{R}^3$ that is a homeomorphism onto its image. An atlas is what you think it is. (We've talked about compatible charts, where is the condition that charts have to be compatible for a smooth manifold?)

Example 1.5. A plane living in $\mathbb{R}^3$ is a surface, as well as the unit cylinder and S^2.

The book states the condition I've been waiting for as a proposition, that is, the transition maps are smooth. Interesting.

Proposition 1.1. *Let U and $\widetilde{U}$ be open subsets of $\mathbb{R}^2$ and $\sigma \colon U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi \colon \widetilde{U} \to U$ be a smooth bijection with smooth inverse. Then $\widetilde{\sigma} = \sigma \circ \Phi \colon \widetilde{U} \to \mathbb{R}^3$ is a regular surface patch.*

Proof. We have $\widetilde{\sigma}$ smooth because the composition of smooth maps is smooth. For regularity, let $(u, v) = \Phi(\widetilde{u}, \widetilde{v})$. By the chain rule, we have

$$\partial_{\widetilde{u}} \widetilde{\sigma} = \frac{\partial u}{\partial \widetilde{u}} \partial_u \sigma + \frac{\partial v}{\partial \widetilde{u}} \partial_v \sigma, \quad \partial_{\widetilde{v}} \widetilde{\sigma} = \frac{\partial u}{\partial \widetilde{v}} \partial_u \sigma + \frac{\partial v}{\partial \widetilde{v}} \partial_v \sigma,$$

so

$$\partial_{\widetilde{u}} \widetilde{\sigma} \times \partial_{\widetilde{v}} \widetilde{\sigma} = \left(\frac{\partial u}{\partial \widetilde{u}} \frac{\partial v}{\partial \widetilde{v}} - \frac{\partial u}{\partial \widetilde{v}} \frac{\partial v}{\partial \widetilde{u}} \right) \partial_u \sigma \times \partial_v \sigma.$$

Note that the scalar is just the Jacobian determinant of Φ, which is nonzero ($J(\Phi^{-1}) = J(\Phi)^{-1}$, so $J(\Phi)$ is invertible). We conclude that $\widetilde{\sigma}$ is regular. ⊠

We say the chart $\widetilde{\sigma}$ is a **reparametrization** of σ, and Φ is a **reparametrization map**. Note that σ is a reparametrization of $\widetilde{\sigma}$ by Φ^{-1}. Also note that any two charts are reparametrizations of each other. This is important because we don't want things to depend on our choice of chart. From now on, surface means smooth surface and charat means smooth chart (whoops I've already been doing this). We also assume surfaces are connected.

1.3 Smooth maps

Say $\mathscr{S}_1, \mathscr{S}_2$ are surfaces covered by single charts $\sigma_1 \colon U_1 \to \mathbb{R}^3$ and $\sigma_2 \colon U_2 \to \mathbb{R}^3$. Then a map $f : \mathscr{S}_1 \to \mathscr{S}_2$ is **smooth** if the map $\sigma_2^{-1} \circ f \circ \sigma_1 \colon U_1 \to U_2$ is smooth.

$$\begin{tikzcd}
\underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[d, "\sigma_2"] \\
\underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2}
\end{tikzcd}$$

Suppose $\widetilde{\sigma}_1 \colon \widetilde{U}_1 \to \mathbb{R}^3$ and $\widetilde{\sigma}_2 \colon \widetilde{U}_2 \to \mathbb{R}^3$ are reparametrizations of σ_1 and σ_2 with reparametrization maps $\Phi_1 \colon \widetilde{U}_1 \to U_1$ and $\Phi_2 \colon \widetilde{U}_2 \to U_2$. We want to show that $\widetilde{\sigma}_2^{-1} \circ f \circ \widetilde{\sigma}_1 \colon \widetilde{U}_1 \to \widetilde{U}_2$ is smooth (provided the other map is smooth).

$$\begin{tikzcd}
\widetilde{U}_1 \arrow[r, "\Phi_1"] \arrow[dr, "\widetilde{\sigma}_1"'] \arrow[rrr, bend left, "\widetilde{\sigma}_2^{-1}\circ f\circ\widetilde{\sigma}_1"] & \underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[d, "\sigma_2"] & \widetilde{U}_2 \arrow[l, "\Phi_2"'] \arrow[dl, "\widetilde{\sigma}_2"] \\
 & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2} &
\end{tikzcd}$$

If we write $\widetilde{\sigma}_2^{-1} = \Phi_2^{-1} \circ \sigma_2^{-1}$ and $\widetilde{\sigma}_1 = \sigma_1 \circ \Phi_1$, then subsituting and applying the associative property gives the map as $\Phi_2^{-1} \circ (\sigma_2^{-1} \circ f \circ \sigma_1) \circ \Phi_1$, which is smooth if each component is. We have the middle component smooth by assumption, and the Φ_i are smooth by definition. The composition of smooth functions is also smooth:

$$\begin{tikzcd}
\underset{\subseteq\mathbb{R}^2}{U_1} \arrow[r, dotted, "\sigma_2^{-1}\circ f\circ\sigma_1"] \arrow[d, "\sigma_1"] \arrow[rr, bend left, "(\sigma_3^{-1}\circ g\circ\sigma_2^{-1})\circ(\sigma_2\circ f\circ\sigma_1)=\sigma_3^{-1}\circ(g\circ f)\circ\sigma_1"] & \underset{\subseteq\mathbb{R}^2}{U_2} \arrow[r, dotted, "\sigma_3^{-1}\circ g\circ\sigma_2"] \arrow[d, "\sigma_2"] & \underset{\subseteq\mathbb{R}^2}{U_3} \arrow[d, "\sigma_3"] \\
\underset{\subseteq\mathbb{R}^3}{\mathscr{S}_1} \arrow[r, "f"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_2} \arrow[r, "g"] & \underset{\subseteq\mathbb{R}^3}{\mathscr{S}_3}
\end{tikzcd}$$

We choose the same chart U_2 that gets mapped onto by U_1 and maps to U_3 since choice of chart is independent of smoothness, as we have just shown. Bijective Smooth maps $f : \mathscr{S}_1 \to \mathscr{S}_2$ with smooth inverse are called **diffeomorphisms**. A smooth map $f : \mathscr{S}_1 \to \mathscr{S}_2$ is a **local diffeomorphism** if for any $p \in \mathscr{S}_1$, we have an open $\mathscr{O} \subseteq \mathscr{S}_1$ such that $f|_{\mathscr{O}}$ is a diffeomorphism onto its image.

Proposition 1.2. *Let $f : \mathscr{S}_1 \to \mathscr{S}_2$ be a local diffeomorphism. If f is injective and σ_1 a smooth chart on $\mathscr{S}_1$, then $f \circ \sigma_1$ is a smooth chart on $\mathscr{S}_2$.*

Example 1.6. Consider the map from the yz-plane to the unit cylinder $\mathscr{S}$, wrapping each line parallel to the axis around the cylinder at height z. This map is defined by $f(0, y, z) = (\cos y, \sin y, z)$. This is not injective, so not a diffeomorphism, but is a local diffeomorphism. Parametrizing by the chart $\pi(u, v) = (0, u, v)$ and using the atlas $\{\sigma|_U, \sigma|_{\widetilde{U}}\}$ of $\mathscr{S}$, let $p = (0, a, b)$ be a point in the yz-plane. If a is not an even multiple of 2π, then we have an $n \in \mathbb{Z}$ such that $2\pi n < a < 2(n+1)\pi$ and

$$f(\pi(u, v)) = \sigma(u - 2\pi n, v) \quad \text{if} \quad 2\pi n < u < 2(n+1)\pi.$$

So f is a diffeomorphism from the open set $\mathscr{O} = \{(0, y, z) \mid 2\pi n < y < 2(n+1)\pi\}$ of the plane to the open set $f(\mathscr{O}) = \{(x, y, z) \in \mathscr{S} \mid x \neq 1\}$. We use the other chart if a is not an odd multiple of π.

1.4 Tangents and derivatives

A **tangent vector** to a surface S[2] is the tangent vector at p of some curve in S. The **tangent space** T_pS at p is the set of all tangent vectors to S at p.

Proposition 1.3. *Let $\sigma : U \to \mathbb{R}^3$ be a chart containing some $p \in S$, and (u, v) be coordinates in U. The tangent space T_pS is the subspaces of $\mathbb{R}^3$ spanned by $\partial_u\sigma, \partial_v\sigma$.*

Proof. Let $\gamma(t) = \sigma(u(t), v(t))$ be smooth. Then by the chain rule we have $\dot\gamma = \partial_u\sigma\dot u + \partial_v\sigma\dot v$. So $\dot\gamma$ is a linear combination of $\partial_u\sigma$ and $\partial_v\sigma$. Conversely, any vector in a subspace spanned by $\partial_u\sigma$ and $\partial_v\sigma$ is of the form $\lambda\partial_u\sigma + \mu\partial_u\sigma$. Define $\gamma(t) = \sigma(u_0 + \lambda t, v_0 + \mu t)$. Then γ is smooth in S, and at $t = 0$ we have $\dot\gamma = \lambda\partial_u\sigma + \mu\partial_v\sigma$. So every vector in the space is the tangent vector of some curve. ⊠

Denote the vectors $\partial_u\sigma, \partial_v\sigma$ that span T_pS as the **parameter curves** on the surface. Suppose $f : S \to \widetilde S$ is smooth: the derivative should measure show a point $f(p) \in \widetilde S$ changes when p moves to a nearby point, say q, of S. If p and q are close, the line near them should be tangent to S at p. So we expect that the derivative of f at p associates to any tangent vector to S at p a tangent vector to $\widetilde S$ at $f(p)$. In other words, the derivative of f should be a map $D_pf : T_pS \to T_{f(p)}\widetilde S$.

Definition 1.4. Let $w \in T_pS$ be a tangent vector to S at p. Then w is the tangent vector at p of a curve γ in S passing through p, say $w = \dot\gamma(t_0)$. Then $\widetilde\gamma = f \circ \gamma$ is a curve in $\widetilde S$ passing through $f(p)$ when $t = t_0$, so $\widetilde w = \dot{\widetilde\gamma}(t_0) \in T_{f(p)}\widetilde S$. We say the **derivative** D_pf of f at $p \in S$ is the map $D_pf : T_pS \to T_{f(p)}\widetilde S$ such that $D_pf(w) = \widetilde w$ for any tangent vector $w \in T_pS$.

Proposition 1.4. *The derivative is linear.*

Proposition 1.5.

(i) If S is a surface and $p \in S$, then the derivative of the identity at p is $\mathrm{id} : T_pS \to T_pS$.

(ii) Chain rule, $D_p(f_2 \circ f_1) = D_{f_1(p)}f_2 \circ D_pf_1$.

(iii) If $f : S_1 \to S_2$ is a diffeomorphism then $D_pf : T_pS_1 \to T_{f(p)}S_2$ is invertible.

Proposition 1.6. *Let $f : S \to \widetilde S$ be smooth. Then f is a local diffeomorphism iff for all $p \in S$, $D_pf : T_pS \to T_{f(p)}\widetilde S$ is invertible.*

todo:section on orientability

Lecture 2

Examples of surfaces

2.1 Level surfaces

Theorem 2.1. *Let $S \subseteq \mathbb{R}^3$ such that for each $p \in S$, there is a $W_p \subseteq \mathbb{R}^3$ open and a smooth $f : W \to \mathbb{R}$ such that*

(i) $S \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$,

(ii) ∇f is nonvanishing at p.

Then S is a smooth surface.

Example 2.1. We can construct S^2 in this manner by letting $W = \mathbb{R}^3$ and considering the single function $f(x, y, z) = x^2 + y^2 + z^2 - 1$, since the gradient $\nabla f = (2x, 2y, 2z)$ is nonvanishing.

Example 2.2. Consider the cone cut out by $f(x, y, z) = x^2 + y^2 - z^2$, which vanishes at the origin. Then the cone minus the origin is a surface.

[2] Surfaces are now denoted S, I'm tired of typing `\mathcal S`.

2.2 Quadric surfaces

Definition 2.1. A **quadric** is the a subset of $\mathbb{R}^3$ defined by an equation of the form

$$v^t A v + b^t v + c = 0,$$

where $v = (x, y, z)$, A is a constant symmetric 3×3, $b \in \mathbb{R}^3$ is constant, $c \in \mathbb{R}$ is a scalar. Explicitly, for $A = \begin{pmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{pmatrix}$, $b = (b_1, b_2, b_3)$ we have

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 xy + 2a_5 yz + 2a_6 xz + b_1 x + b_2 y + b_3 z + c = 0.$$

Some quadrics which are not surfaces include $x^2 + y^2 + z^2 = 0,\ x^2 + y^2 = 0,\ xy = 0$.

Theorem 2.2. *Up to isometry, every non-empty quadric with not all zero coefficients can be transformed into one of the following:*

(i) *Ellipsoid:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$.

(ii) *One sheeted hyperboloid:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$.

(iii) *Two sheeted hyperboloid:* $\frac{z^2}{r^2} - \frac{z^2}{p^2} - \frac{y^2}{q^2} = 1$.

(iv) *Elliptic paraboloid:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$.

(v) *Hyperbolic paraboloid:* $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$.

(vi) *Quadric cone:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$.

(vii) *Elliptic cylinder:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.

(viii) *Hyperbolic cylinder:* $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.

(ix) *Parabolic cylinder:* $\frac{x^2}{p^2} = y$.

(x) *Plane:* $x = 0$.

(xi) *Two parallel planes:* $x^2 = p^2$.

(xii) *Two intersection planes:* $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 0$.

(xiii) *Straight line:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 0$.

(xiv) *Point:* $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 0$.

Proof. oh save me there's a proof todo:this ⊠

Corollary 2.1. *Every nonempty quadric of types (i)-(x) is a surface (for (vi) remove the vertex).*

2.3 Ruled surfaces and surfaces of revolution

Definition 2.2. A **ruled surface** is a surface that is a union of straight lines, called the **rulings** (or **generators**) of the surface.

If the rulings are all parallel, then S is a **generalized cylinder**. We don't want a curve that passes through all the lines to be tangent to the rulings (intersect transversely??). Some noninteresting stuff happened.

2.4 Compact surfaces

Example 2.3. You know what a compact set is. Some examples:

- Spheres are compact subspaces of $\mathbb{R}^{n+1}$. They are clearly bounded, and closed since the complement $\mathbb{R}^{n+1} \setminus S^n = B^{n+1} \cup (\mathbb{R}^{n+1} \setminus D^{n+1})$, a union of two open sets (where B^{n+1} is the open n-ball and D^{n+1} is the closed n-disk).
- Planes are not compact since they're unbounded. Neither are open disks since they're open.
- In our $\mathbb{R}^3$, the torus and the other genus n surfaces are also compact. These turn out to classify compact 2-manifolds up to diffeomorphism:

  Theorem 2.3. *Up to diffeomorphism, the only compact 2-manifolds are the genus n surfaces for $n \geq 0$.*

A corollary of this:

Corollary 2.2. *Every compact surface is orientable.*

Proof. By the Jordan separation theorem we can separate a genus n surface into a bounded interior and unbounded exterior. Define the unit normal at each point on the surface to point toward the exterior. ⊠

2.5 Triply orthogonal systems

We skipped this section.

2.6 Applications of the IFT

This section too.

Lecture 3

The first fundamental form

Now we do some geometry.

3.1 Lengths of curves on surfaces

Definition 3.1. Let $p \in S$. The **first fundamental form** of S at p associates to tangent vectors $v, w \in T_pS$ the scalar $\langle v, w\rangle_{p,S} = v \cdot w$.

This is an inner product for S at p. If you know how tensors work, at each point we can write $g = g_{ij}^{(x)} dx^i \otimes dx^j$, and we can calculate the coefficients of the metric tensor by $g_{ij} = g(\partial_i, \partial_j)_x = \langle \partial_i, \partial_j\rangle_x$. Traditionally, for $\sigma(u, v)$ a patch, if $p \in \operatorname{im}\sigma$ we have $\partial_u\sigma, \partial_v\sigma$ spanning T_pS. Define $du\colon T_pS \to \mathbb{R}$, $dv\colon T_pS \to \mathbb{R}$, by

$$du(v) = \lambda, \quad dv(v) = \mu \quad \text{if } v = \lambda\partial_u\sigma + \mu\partial_v\sigma.$$

(These are 1-forms, the basis vectors of the cotangent space.) Then $\langle v, v\rangle = \lambda^2\langle\partial_u\sigma, \partial_u\sigma\rangle + 2\lambda\mu\langle\partial_u\sigma, \partial_v\sigma\rangle + \mu^2\langle\partial_v\sigma, \partial_v\sigma\rangle$. If we write $E = \|\partial_u\sigma\|^2,\ F = \partial_u\sigma \cdot \partial_v\sigma,\ G = \|\partial_v\sigma\|^2$, then

$$\langle v, v\rangle = E\lambda^2 + 2F\lambda\mu + G\mu^2 = Edu^2 + 2Fdudv + Gdv^2.$$

If $\gamma \subseteq \operatorname{im}\sigma$, $\gamma(t) = \sigma(u(t), v(t))$ for u, v smooth. Then $\dot\gamma = \dot u\partial_u\sigma + \dot v\partial_v\sigma$, so $\langle\dot\gamma, \dot\gamma\rangle = E\dot u^2 + 2F\dot u\dot v + G\dot v^2$, and the length of γ is given by $\int (E\dot u^2 + 2F\dot u\dot v + G\dot v^2)^{1/2}\, dt$.

Example 3.1. For a plane $\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$ for unit vectors $\mathbf{p} \perp \mathbf{q}$, we have $\partial_u\sigma = \mathbf{p}, \partial_v\sigma = \mathbf{q}$, then $E = \|\partial_u\sigma\|^2 = \|\mathbf{p}\|^2 = 1, F = \partial_u\sigma \cdot \partial_v\sigma = 0, G = \|\mathbf{q}\|^2 = 1$. So $g_{ij} = \delta^i_j$, and the first fundamental form is $du^2 + dv^2$.

Example 3.2. From now on we write $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$. For a surface of revolution of the form $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, since $\partial_u \sigma = (\dot{f}\cos v, \dot{f}\sin v, \dot{g})$, $\partial_v \sigma = (-f\sin v, f\cos v, 0)$, we have $g_{11} = \dot{f}^2 + \dot{g}^2 = 1$ (since the curve is unit speed), $g_{12} = g_{21} = 0, g_{22} = f^2$. So the fff is $du^2 + f(u)^2 dv^2$. If we take $u = \theta, v = \varphi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$, this gives us the fff of S^2 as $d\theta^2 + \cos^2\theta d\phi^2$.

Example 3.3. A generalized cylinder $\sigma(u,v) = \gamma(u) + v\mathbf{a}$ as $g_{ij} = \delta^i_j$, and so the fff is $du^2 + dv^2$ as well.

3.2 Isometries of surfaces

The plane and generalized cylinder have the same fff, which is because they're isometric (intrinsic curvature?). However, this doesn't hold for the sphere, because you can't "wrap" a piece of paper around it.

Definition 3.2. If S_1, S_2 are surfaces, a smooth map $f : S_1 \to S_2$ is a **local isometry** if it takes any curve in S_1 to a curve of the same length in S_2. If $f : S_1 \to S_2$ is a local isometry, then S_1 and S_2 are **locally isometric**.

We will see that every local isometry is a local diffeomorphism, and a global diffeomorphism that is also a local isometry is just an **isometry**. Let $f : S_1 \to S_2$ be smooth and $p \in S_1$. For $v, w \in T_pS_1$, define

$$f^*\langle v, w\rangle_p = \langle D_pf(v), D_pf(w)\rangle_{f(p)}.$$

Then $f^*\langle\ ,\ \rangle_p$ is a symmetric bilinear form because the inner product is symmetric and the derivative is bilinear.

Theorem 3.1. *A smooth map $f : S_1 \to S_2$ is a local isometry iff the symmetric bilinear forms $\langle\ ,\ \rangle_p$ and $f^*\langle\ ,\ \rangle_p$ on T_pS_1 are equal for all $p \in S_1$.*

Proof. For γ_1 a curve in S_1, a curve has length $\int_{t_0}^{t_1} \langle\dot{\gamma}_1, \dot{\gamma}_1\rangle^{1/2}\, dt$. Then the length of $\gamma_2 = f \circ \gamma_1$ is

$$\int_{t_0}^{t_1} \langle\dot{\gamma}_2, \dot{\gamma}_2\rangle^{1/2}\, dt = \int_{t_0}^{t_1} \langle Df(\dot{\gamma}_1), Df(\dot{\gamma}_1)\rangle^{1/2}\, dt = \int_{t_0}^{t_1} f^*\langle\dot{\gamma}_1, \dot{\gamma}_1\rangle^{1/2}\, dt.$$

So if $\langle\ ,\ \rangle_p$ and $f^*\langle\ ,\ \rangle_p$ are the same, curves have the same length. OTOH, suppose the integrals are euqal, then the integrands are the same, that is, $\langle\dot{\gamma}, \dot{\gamma}\rangle = f^*\langle\dot{\gamma}, \dot{\gamma}\rangle$. So $\langle v, v\rangle = f^*\langle v, v\rangle$ since tangent vectors are tangent to a curve. ⊠

Since f is a local isometry iff $\langle D_pf(v), D_pf(w)\rangle_{f(p)} = \langle v, w\rangle_p$, D_pf is then an isometry. Then every local isometry is a local diffeomorphism, since if it wasn't we would have a nontrivial $v \in T_pS_1$ such that $v \in \ker(D_pf)$, or $D_pf(v) = 0$. But

$$0 \neq \langle v, v\rangle_p \langle D_pf(v), D_pf(v)\rangle_{f(p)} = \langle 0, 0\rangle_p = 0,$$

a contradiction.

Corollary 3.1. *A local diffeomorphism is a local isometry iff for any surface patch σ_1 of S_1, the patches σ_1 and $f \circ \sigma_1$ of S_1 and S_2 respectively have the same fff.*

Proof. Same process as Theorem 3.1. ⊠

This actually shows that if $p \in S_1$ is in $\operatorname{im}\sigma_1$, then σ_1 and $f \circ \sigma_1$ have the same fff at p iff D_pf is an isometry. It follows that if $p \in \operatorname{im}\sigma_2$, then σ_1 and $f \circ \sigma_1$ have the same fff at p iff the same is true of σ_2 and $f \circ \sigma_2$.

Example 3.4. The unit cylinder, the generalized cone, and the plane are locally isometric.

It turns out there is another class of circles locally isometric to a plne, called **tangent developables**, the tangent bundle to a curve in $\mathbb{R}^3$.

3.3 Conformal mappings of surfaces

Conformal mappings don't preserve length, but they do preserve angle. Say $\gamma, \tilde{\gamma} \subseteq S$ and $p \in \gamma \cap \tilde{\gamma}$. We define the **angle** θ of intersection of γ and $\tilde{\gamma}$ at p as the angle between the tangent vectors $\dot{\gamma}$ and $\dot{\tilde{\gamma}}$ (at $t = t_0$, $t = \tilde{t}_0$ respectively). Then

$$\cos\theta = \frac{\dot{\gamma}\cdot\dot{\tilde{\gamma}}}{\|\dot{\gamma}\|\|\dot{\tilde{\gamma}}\|} = \frac{\langle\dot{\gamma},\dot{\tilde{\gamma}}\rangle}{\langle\dot{\gamma},\dot{\gamma}\rangle^{1/2}\langle\dot{\tilde{\gamma}},\dot{\tilde{\gamma}}\rangle^{1/2}}.$$

If γ and $\tilde{\gamma}$ lie in a surface patch σ such that $\gamma(t) = \sigma(u(t), v(t))$ and $\tilde{\gamma}(t) = \sigma(\tilde{u}(t), \tilde{v}(t))$ for some smooth $u, v, \tilde{u}, \tilde{v}$, then

$$\cos\theta = \frac{g_{11}\dot{u}\dot{\tilde{u}} + g_{12}(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + g_{22}\dot{v}\dot{\tilde{v}}}{(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2)^{1/2}(g_{11}\dot{\tilde{u}}^2 + 2g_{12}\dot{\tilde{u}}\dot{\tilde{v}} + g_{22}\dot{\tilde{v}}^2)^{\frac{1}{2}}}.$$

Example 3.5. The **parameter curves** on a surface patch $\sigma(u, v)$ can be parametrized by $\gamma(t) = \sigma(u_0, t)$, $\tilde{\gamma}(t) = \sigma(t, v_0)$, where u_0 is the constant value of u and t_0 is the constant value of v. Then $u(t) = t_0$, $v(t) = t$, and $\dot{u} = 0$, $\dot{v} = 1$. Similarly, $\tilde{u}(t) = t$, $\tilde{v}(t) = v_0$, and $\dot{\tilde{u}} = 1$, $\dot{\tilde{v}} = 0$. These parameter curves intersect at $\sigma(u_0, v_0)$, so $\cos\theta = g_{11}/\sqrt{g_{12}g_{22}}$. In particular, the parameter curves are orthogonal iff $g_{11} = 0$.

Definition 3.3. For S_1, S_2 surfaces, a **conformal map** $f : S_1 \to S_2$ is a local diffeomorphism such that for γ_1, γ_2 two curves on S^1 intersecting at a point $p \in S_1$, the angle of intersection of γ_1 and γ_2 at p is equal to the angle of intersection of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ at $f(p)$, if $\tilde{\gamma}_1, \tilde{\gamma}_2$ are the images of the curves under f.

So as we said earlier, conformal mappings preserve angle. Note that the angle is only well defined when both curves are regular.

Theorem 3.2. *A local diffeomorphism $f : S_1 \to S_2$ is conformal iff there is a function $\lambda\colon S_1 \to \mathbb{R}$ such that*

$$f^*\langle v, w\rangle_p = \lambda(p)\langle v, w\rangle_p \quad \textit{for all } p \in S_1 \textit{ and } v, w \in T_pS_1.$$

Proof. todo:come back ⊠

Corollary 3.2. *A local diffeomorphism $f : S_1 \to S_2$ is conformal iff for any surface patch σ of S_1, the fff of the patches of S_1 and $f \circ \sigma$ of S_2 are proportional (i.e., differ by a scalar multiple).*

Example 3.6. If $g_{ij}dx^i \otimes dx^j$ denotes the usual metric on $\mathbb{R}^2$ and $\mathring{g}_{ij}dx^i \otimes dx^j$ denotes the round metric on S^2, we have

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathring{g}_{ij} = \begin{pmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{pmatrix} = \frac{4}{(1+u^2+v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{4}{(1+u^2+v^2)^2}g_{ij}.$$

So stereographic projection is a conformal mapping.

A natural question to ask is when we have a conformal map between two surfaces. It turns out this is always true locally.

Theorem 3.3. *Every surface has an atlas consisting of conformal surfaces patches.*

No proof, it's too hard (even do Carmo doesn't prove it!).

3.4 Equiareal maps and a theorem of Archimedes

Recall that parameter curves $u \mapsto \sigma(u, v_0)$ and $v \mapsto \sigma(u_0, v)$ are the ones that if you take their derivative, you get the basis for T_pS denoted $\{\partial_u, \partial_v\}$. Fixing $(u_0, v_0) \in U$, we get a parallelogram with edges $\partial_u\sigma\Delta u, \partial_v\sigma\Delta v$ (corresponding to a small change in u, v on the surface) and $u = u_0 + \Delta u$, $v = v_0 + \Delta v$. The area of this parallelogram is $\|\partial_u\sigma\Delta u \times \partial_v\sigma\Delta v\| = \|\partial_u\sigma \times \partial_v\sigma\|\Delta u\Delta v$.

Definition 3.4. The **area** $\mathscr{A}_\sigma(R)$ of the part $\sigma(R)$ of a surface patch $\sigma : U \to \mathbb{R}^3$ corresponding to a region $R \subseteq U$ is defined by

$$\mathscr{A}_\sigma(R) = \int_R \|\partial_u\sigma \times \partial_v\sigma\|\, du\, dv.$$

If R is contained in a rectangle entirely contained in U, then the area will be finite. It turns out this cross product is easily computable.

Proposition 3.1. $\|\partial_u\sigma \times \partial_v\sigma\| = (EG-F^2)^{1/2} = \sqrt{g_{11}g_{22}-(g_{12})^2}$.

Proof. Recall that $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)$. Then

$$\|\partial_u\sigma \times \partial_v\sigma\| = (\partial_u\sigma \times \partial_v\sigma)\cdot(\partial_u\sigma \times \partial_v\sigma) = (\partial_u\sigma\cdot\partial_v\sigma)(\partial_v\sigma\cdot\partial_v\sigma)-(\partial_u\sigma\cdot\partial_v\sigma)^2 = EG-F^2. \qquad \boxtimes$$

So our definition of area is $\mathscr{A}_\sigma(R) = \int_R (EG-F^2)^{1/2}\,du\,dv$. Sometimes we denote $(EG-F^2)^{1/2}\,du\,dv$ by $d\mathscr{A}_\sigma$. Is this definition invariant under our choice of chart?

Proposition 3.2. *The area of a surface patch is invariant under reparametrization.*

Proof. Let $\sigma\colon U\to\mathbb{R}^3$ be a surface patch and $\widetilde{\sigma}\colon\widetilde{U}\to\mathbb{R}^3$ be a reparametrization of σ, with reparametrization map $\Phi\colon\widetilde{U}\to U$. If $\Phi(\widetilde{u},\widetilde{v}) = (u,v)$, we have $\widetilde{\sigma}(\widetilde{u},\widetilde{v}) = \sigma(u,v)$. Let $\widetilde{R}\subseteq\widetilde{U}$ be a region, and let $R = \Phi(\widetilde{R})\subseteq U$. We want to show that

$$\int_R \|\partial_u\sigma\times\partial_v\sigma\|\,du\,dv = \int_{\widetilde{R}} \|\partial_{\widetilde{u}}\widetilde{\sigma}\times\partial_{\widetilde{v}}\widetilde{\sigma}\|\,d\widetilde{u}\,d\widetilde{v}.$$

We have seen that $\partial_{\widetilde{u}}\widetilde{\sigma}\times\partial_{\widetilde{v}}\widetilde{\sigma} = \det(J(\Phi))\partial_u\sigma\times\partial_v\sigma$, so the RHS becomes $\int_{\widetilde{R}} |\det(J(\Phi))|\,\|\partial_u\sigma\times\partial_v\sigma\|\,d\widetilde{u}\,d\widetilde{v}$. Apply the change of variables formula for double integrals and this becomes $\int_R \|\partial_u\sigma\times\partial_v\sigma\|\,du\,dv$, and we are done. $\boxtimes$

Definition 3.5. A local diffeomorphism $f: S_1\to S_2$ is **equiareal** if it takes any region in S_1 to a region of the same area in S_2. We assume that the area of the regions are small enough to be contained in a single chart.

Now we state an analogue of Theorem 3.1.

Theorem 3.4. *A local diffeomorphism $f: S_1\to S_2$ is equiareal iff for any surface patch $\sigma(u,v)$ on S_1, the fff's E_i, F_i, G_i for $i\in\{1,2\}$ satisfy*

$$E_1G_1-F_1^2 = E_2G_2-F_2^2.$$

Proof. Exercise. $\boxtimes$

Think of S^2 sitting inside a unit cylinder. You can "flatten" S^2 by considering the unique line that goes through each point $p\in S^2$ and the z-axis (excluding the poles). Then this line intersects the cylinder at a point q; let f be the map that sends $p\mapsto q$. Explicitly, say $p = (x,y,z)$ and $q = (X,Y,Z)$. Since the line connecting p and q is parallel to the xz-plane, we have $z = Z$ and $(X,Y) = \lambda(x,y)$ for λ a scalar. Since (X,Y,Z) is on the cylinder we know $1 = X^2+Y^2 = \lambda^2(x^2+y^2)$, so $\lambda = \pm(x^2+y^2)^{-1/2}$. Taking the plus sign, we have

$$f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right).$$

Archimedes' Theorem. *The map f is an equiareal diffeomorphism.*

Proof. Consider the standard atlas for S^2 (minus the poles) given by $\sigma(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, defined on $\{\theta,\varphi \mid \theta\in[-\pi/2,\pi/2], \varphi\in[0,2\pi]\}$ and $\{\theta,\varphi \mid \theta\in[-\pi/2,\pi/2], \varphi\in[-\pi,\pi]\}$. The image of σ under f is $\widetilde{\sigma}(\theta,\varphi) = (\cos\varphi,\sin\varphi,\sin\theta)$. This gives an atlas for the cylinder in between the planes $z=1, z=-1$ (denote this surface C), defined on the same open sets as S^2. We want to show that $E_1G_1-F_1^2 = E_2G_2-F_2^2$, so we can apply Theorem 3.4.

From before, we know that $E_1 = 1, F_1 = 0$, and $G_1 = \cos^2\theta$ for σ. For $\widetilde{\sigma}$, we have $\partial_\theta\widetilde{\sigma} = (0,0,\cos\theta)$, $\partial_\varphi\widetilde{\sigma} = (-\sin\varphi,\cos\varphi,0)$. So $E_2 = \cos^2\theta, F_2 = 0$, and $G_2 = 1$. Therefore

$$E_1G_1-E_2G_2+F_2^2-F_1^2 = (\cos^2\theta-\cos^2\theta)+0-0 = 0. \qquad \boxtimes$$

Definition 3.6. A **spherical triangle** is a triangle on a sphere whose sides are arcs of great circles.

Theorem 3.5. *The area of a spherical triangle on the S^2 with internal angles α,β, and γ is $\alpha+\beta+\gamma-\pi$.*

Proof. By Archimedes' theorem, if we cut out a lune by two planes with angle θ, it has area 2θ. If A, B, C are the points on the spherical triangle that correspond to α, β, γ, consider the lune connecting A and A' with area 2α, where A' is the antipodal point of A. Note that each lune can be decomposed into $ABC+$ an extra region. So adding gives

$$3\mathscr{A}(ABC) + \mathscr{A}(\text{three extra regions}) = 2\alpha + 2\beta + 2\gamma.$$

If you draw a picture, it can be seen that the three extra regions plus $\mathscr{A}(ABC)$ actually make up the hemisphere containing A in the interior and $\overset{\frown}{BC}$ in the boundary. So

$$\begin{aligned} 2\mathscr{A}(ABC) + \overbrace{\mathscr{A}(ABC) + \mathscr{A}(\text{three extra regions})}^{\text{has area } 2\pi} &= 2\alpha + 2\beta + 2\gamma \implies \\ 2\mathscr{A}(ABC) + 2\pi &= 2\alpha + 2\beta + 2\gamma \implies \\ \mathscr{A}(ABC) &= \alpha + \beta + \gamma - \pi. \end{aligned}$$

⊠

We will eventually see a generalization of this result where S^2 becomes an arbitrary surface and great circles become arbitrary curves.

Lecture 4

Curvature of surfaces

How "curved" is a surface? We will see something new called the second fundamental form. It turns out a surface is determined up to isometry by its first and second fundamental forms, like how a unit-speed plane curve is determined up to isometry by its signed curvature.

4.1 The second fundamental form

We work with oriented surfaces. Suppose σ is a surface patch in $\mathbb{R}^3$ with standard unit normal $\mathbf{N}$. As (u, v) change by $\Delta u, \Delta v$ and becomes $(u+\Delta u, v+\Delta v)$, the surface moves away from $T_{\sigma(u,v)}S$ by a distance $(\sigma(u+\Delta u, v+\Delta v) - \sigma(u, v)) \cdot \mathbf{N}$. By Taylor's theorem,

$$\sigma(u+\Delta, v+\Delta v) - \sigma(u, v) = \partial_u\sigma\Delta u + \partial_v\sigma\Delta v + \frac{1}{2}(\partial_{uu}\sigma(\Delta u)^2 + 2\partial_{uv}\sigma\Delta u\Delta v + \partial_{vv}\sigma(\Delta v)^2) + \text{ remainder},$$

where (remainder)$/((\Delta u)^2 + (\Delta v)^2)$ tends to zero as $(\Delta u)^2 + (\Delta v)^2$ tends to zero. Since $\partial_u\sigma, \partial_v\sigma$ are tangent to the surface (hence orthogonal to $\mathbf{N}$), the deviation from the tangent plane becomes

$$\frac{1}{2}\left(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right) + \text{ remainder},$$

where

$$L = \partial_{uu}\sigma \cdot \mathbf{N}, \quad M = \partial_{uv}\sigma \cdot \mathbf{N}, \quad N = \partial_{vv}\sigma \cdot \mathbf{N}.$$

We see that $L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2$ is the analogue for the surface of the curvature $\kappa(\Delta t)^2$ in the case of a curve.

Definition 4.1. The expression $L\,du^2 + 2M\,dudv + N\,dv^2$ is the **second fundamental form** of the surface patch. From now on, sff is used in place of "second fundamental form". The corresponding symmetric bilinear form on the tangent plane is defined by

$$\langle\langle v, w\rangle\rangle = Ldu(v)du(w) + M(du(v)dv(w) + du(w)dv(v)) + Ndv(v)dv(w).$$

Example 4.1. Consider the plane, since $\partial_u\sigma, \partial_v\sigma$ are both constant, we have $\partial_{uu}\sigma = \partial_{uv}\sigma = \partial_{vv}\sigma = 0$. So the sff of a plane is zero.

Example 4.2. Given a surface of revolution $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, assume $f(u) > 0$ for all u and $u \mapsto (f(u), 0, g(u))$ is unit-speed (so $\dot{f}^2 + \dot{g}^2 = 1$). Since $\partial_u\sigma = (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \partial_v\sigma = (-f\sin v, f\cos v, 0)$, we have

$$\begin{aligned}
\partial_u\sigma \times \partial_v\sigma &= (-f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}),\\
\|\partial_u\sigma \times \partial_v\sigma\| &= f,\\
\mathbf{N} &= \frac{\partial_u\sigma \times \partial_v\sigma}{\|\partial_u\sigma \times \partial_v\sigma\|} = (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}),\\
\partial_{uu}\sigma &= (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}),\\
\partial_{uv}\sigma &= (-\dot{f}\sin v, \dot{f}\cos v, 0),\\
\partial_{vv}\sigma &= (-f\cos v, -f\sin v, 0),\\
L &= \partial_{uu}\cdot\mathbf{N} = \dot{f}\ddot{g} - \ddot{f}\dot{g},\\
M &= \partial_{uv}\sigma\cdot\mathbf{N} = 0,\\
N &= \partial_{vv}\sigma\cdot N = f\dot{g} \implies\\
\text{sff } &= (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2.
\end{aligned}$$

If the surface is S^2 where $u = \theta, v = \varphi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$, we have the sff of S^2 as $d\theta^2 + \cos^2\theta d\varphi^2$. This turns out to be the same as the fff of S^2. If the surface is the unit cylinder, then $f(u) = 1, g(u) = u$, so $L = M = 0$, $N = 1$, so the sff is dv^2.

4.2 The Gauss and Weingarten maps

An approach to defining curvature of surfaces is by considering the rate at which **N** varies. The values of **N** at S are recorded by its **Gauss map** $\mathscr{G}$, defined by

$$\mathscr{G}_S : S \to S^2, \quad p \mapsto \mathbf{N}_p.$$

The rate at which **N** varies is measured by $D_p\mathscr{G} : T_pS \to T_{\mathscr{G}(p)}S^2$. Since $T_{\mathscr{G}(p)} \perp \mathbf{N}_p$, this is equivalent to T_pS and so the Gauss map is an endomorphism $T_pS \to T_pS$.

Definition 4.2. Let $p \in S$. The **Weingarten map** $\mathscr{W}_{p,S}$ of S at p is defined by $\mathscr{W} = -D_p\mathscr{G}$. The **second fundamental form** of S at $p \in S$ is the bilinear form T_pS given by

$$\langle\langle v, w\rangle\rangle_{p,S} = \langle \mathscr{W}(v), w\rangle_{p,S}, \quad v, w \in T_pS.$$

Lemma 4.1. *Let $\sigma(u,v)$ be a surface patch with standard unit normal $\mathbf{N}(u,v)$. Then*

$$\mathbf{N}_u \cdot \partial_u\sigma = -L, \quad \mathbf{N}_u \cdot \partial_v\sigma = \mathbf{N}_v \cdot \partial_u\sigma = -M, \quad \mathbf{N}_v \cdot \partial_v\sigma = -N.$$

Proof. We have $\mathbf{N}\cdot\partial_u\sigma = \mathbf{N}\cdot\partial_v\sigma = 0$. Differentiate wrt u to get $\mathbf{N}_u \cdot \partial_u\sigma = -\mathbf{N}\cdot\partial_{uu} = -L$, and the other equalities follow in a similar manner. ⊠

Proposition 4.1. *Let $p \in S$, $\sigma(u,v)$ be a surface patch of S such that $p \in \operatorname{im}\sigma$, and let $L\,du^2 + 2M\,du\,dv + N\,dv^2$ be the sff of σ. Then for any $v, w \in T_pS$,*

$$\langle\langle v, w\rangle\rangle = L\,du(v)du(w) + M(du(v)dv(w) + du(w)dv(v)) + N dv(v)dv(w).$$

Proof. Since both sides of the equation are bilinear forms, we just have to check that they agree when v and w are $\partial_u\sigma$ or $\partial_v\sigma$. This boils down to showing

$$\langle\langle\partial_u\sigma, \partial_u\sigma\rangle\rangle = L, \quad \langle\langle\partial_u\sigma, \partial_v\sigma\rangle\rangle = \langle\langle\partial_v\sigma, \partial_u\sigma\rangle\rangle = M, \quad \langle\langle\partial_v\sigma, \partial_v\sigma\rangle\rangle = N.$$

Let $\sigma(u_0, v_0) = p$. Then

$$\mathscr{W}(\partial_u\sigma) = -\frac{d}{du}\bigg|_{u=u_0}\mathscr{G}(\sigma(u, v_0)) = \frac{d}{du}\bigg|_{u=u_0}\mathbf{N}(u, v_0) = -\mathbf{N}_u,$$

where $\mathbf{N}$ is the standard unit normal of σ. Similarly, $\mathscr{W}(\partial_v\sigma) = -\mathbf{N}_v$. So

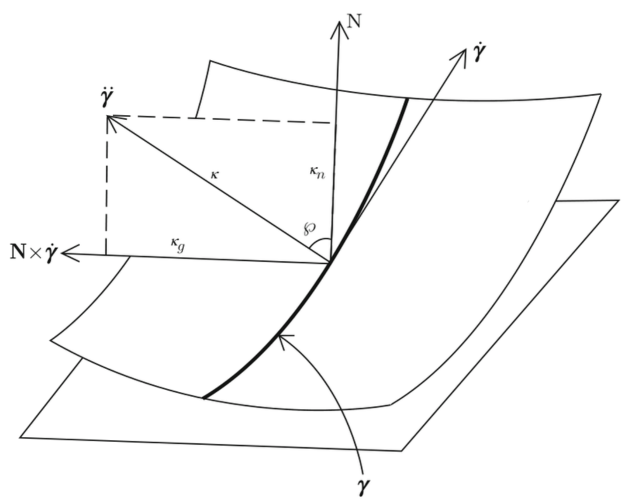
$$\langle\langle \partial_u\sigma, \partial_u\sigma\rangle\rangle = \langle \mathscr{W}(\partial_u\sigma), \partial_u\sigma\rangle = -\mathbf{N}_u \cdot \partial_u\sigma = L.$$

The other equations follow in a similar manner. ⊠

Corollary 4.1. *The sff is a symmetric bilinear form. Equivalently, the Weingarten endomorphism is self-adjoint.*

4.3 Normal and geodesic curvatures

If γ is a unit speed curve on an oriented surface S, then $\dot\gamma$ is a unit vector, and therefore a tangent vector to S. So $\dot\gamma \perp \mathbf{N}$, and $\dot\gamma, \mathbf{N}$, and $\mathbf{N}\times\dot\gamma$ are mutually orthogonal. Since $\ddot\gamma \perp \dot\gamma$, it must be a linear combination of $\mathbf{N}$ and $\mathbf{N}\times\dot\gamma$, where $\ddot\gamma = \kappa_n\mathbf{N} + \kappa_g(\mathbf{N}\times\dot\gamma)$.



Definition 4.3. The scalars κ_n and κ_g are the **normal curvature** and **geodesic curvature** of γ, respectively.

Note that these change sign when $\mathbf{N}$ does, so in general only the magnitudes of κ_n and κ_s are well defined.

Proposition 4.2. *We have*

$$\begin{aligned}
\kappa_n &= \ddot\gamma\cdot\mathbf{N}, \quad \kappa_g = \ddot\gamma\cdot(\mathbf{N}\times\dot\gamma),\\
\kappa^2 &= \kappa_n^2 + \kappa_g^2,\\
\kappa_n &= \kappa\cos\psi, \quad \kappa_g = \pm\kappa\sin\psi,
\end{aligned}$$

where κ is the curvature of γ and ψ is the angle between $\mathbf{N}$ and the principal normal $\mathbf{n}$ of γ.

Proof. To show $\kappa_n = \ddot\gamma\cdot\mathbf{N}$, note that $\ddot\gamma = \kappa_n\mathbf{N} + \kappa_g(\mathbf{N}\times\dot\gamma)$, so $\ddot\gamma\cdot\mathbf{N} = \kappa_n\mathbf{N}\cdot\mathbf{N} + \kappa_g(\mathbf{N}\times\dot\gamma)\cdot\mathbf{N}$, which implies that $\ddot\gamma\cdot\mathbf{N} = \kappa_n$ since the vectors $\mathbf{N}$ and $\mathbf{N}\times\dot\gamma$ are orthogonal. A similar process shows that $\kappa_g = \ddot\gamma\cdot(\mathbf{N}\times\dot\gamma)$. todo:not sure how to show $\kappa^2 = \kappa_n^2 + \kappa_g^2$? Since $\ddot\gamma = \kappa n$, todo:? ⊠

As a unit speed parameter t is changed to another parameter $\pm t + c$, then $\kappa_n \mapsto \kappa_n$ and $\kappa_g \mapsto \pm\kappa_g$, so κ_n is well defined for any regular curve while κ_g is only well defined up to sign.

Proposition 4.3. *If γ is unit-speed on an oriented surface S, its normal curvature is given by $\kappa_n = \langle\langle\dot\gamma,\dot\gamma\rangle\rangle$. If σ is a surface patch of S and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ, then we also have $\kappa_n = L\dot u^2 + 2M\dot u\dot v + N\dot v^2$.*

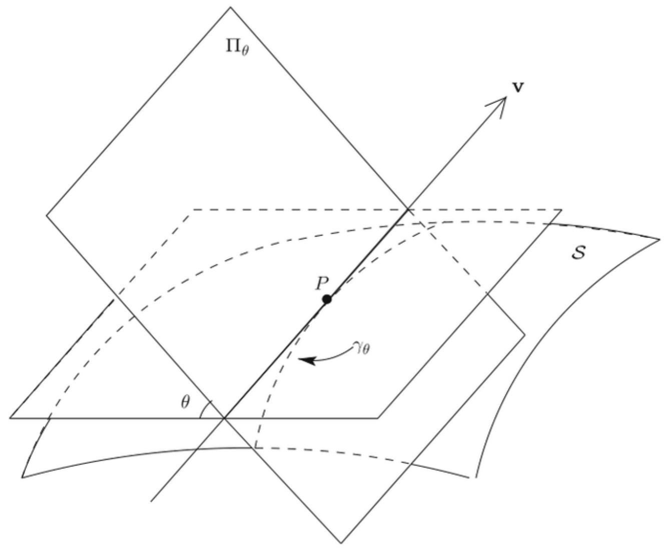
So two curves that interset at p and have the parallel tangent vectors there have the same normal curvature at p. todo:how?

Proof. Since $\dot{\gamma}$ is tangent to S, $\mathbf{N} \cdot \dot{\gamma} = 0$. So $\mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma}$. Note that $\dot{\mathbf{N}} = \frac{d}{dt}\mathscr{G}(\gamma(t)) = -\mathscr{W}(\dot{\gamma})$, therefore

$$\kappa_n = \mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma} = \langle \mathscr{W}(\dot{\gamma}), \dot{\gamma} \rangle = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle. \quad \boxtimes$$

While the normal curvature depends on the sff, the geodesic curvature only depends on the fff.

Meusnier's Theorem. *Let $p \in S$ and $v \in T_pS$. Let Π_θ be the plane spanned by v and making an angle θ with T_pS. Suppose Π_θ intersects S at a curve with curvature κ_θ. Then $\kappa_\theta \sin\theta$ is independent of θ.*



Proof. Suppose γ_θ is a unit speed parametrization of the aforementioned curve. Then at $p, \dot{\gamma}_\theta = \pm v$, so $\ddot{\gamma}_\theta$ is perpendicular to v and parallel to Π_θ. Then $\psi = \pi/2 - \theta$, so $\kappa_n = \kappa_\theta \cos(\pi/2 - \theta) = \kappa_\theta \sin\theta$, but κ_n does not depend on θ. ⊠

An important case is where γ is a **normal section** of the surface, i.e., γ is the intersection of the surface with a plane Π that is orthogonal to the tangent plane of the surface at each point of γ.

Corollary 4.2. *The curvature, normal curvature κ_n, and geodesic curvature κ_g of a normal section of a surface are related by $\kappa_n = \pm\kappa,\ \kappa_g = 0$.*

Proof. We have $\kappa_n = \kappa \sin\theta$ where $\theta = \pm\pi/2$. For the second part, $\kappa^2 = \kappa_n^2 + \kappa_g^2$, and since $\kappa = \pm\kappa_n$, we must have $\kappa_g = 0$. ⊠