Topology Notes

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These notes were transcribed from my physical lecture notes for the Spring 2020 undergraduate/graduate section of Topology (Math 4500) at UNT, taught by Dr. Fishman, which I took while I was at TAMS. Source files: https://git.simonxiang.xyz/math_notes/files.html

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Wow, these are pre-corona notes. This really brings me back. I miss Dr. Fishman already.

Metric spaces. Let $x, y \in \mathbb{R}$, and $d(x, y)$ denote the distance $|x - y|$, where $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Properties of d include:

(1) $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and $d(x, y) = 0$ iff $x = y$ (non-negativity),
(2) $d(x, y) = d(y, x)$ (symmetry),
(3) $d(x, z) \le d(x, y) + d(y, z)$ (triangle inequality).

Definition 1.1. Let X be a nonempty set. A function $d\colon X \times X \to \mathbb{R}$ is a **metric** if the following (previous) axioms are satisfied for every $x, y, z \in X$.

In $\mathbb{R}^n$, $\vec{x} = (x_1, \cdots, x_n), \vec{y} = (y_1, \cdots, y_n)$ (we use $(-)$ to denote vectors and $\langle - \rangle$ to denote sequences). Then $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Theorem 1.1. *$d(\vec{x}, \vec{y})$ is a metric on $\mathbb{R}^n$.*

Proof. It is clear that $d(\vec{x}, \vec{y})$ satisfies conditions (1) and (2). To show that $d(\vec{x}, \vec{y})$ satisfies condition (3), recall that $\|\vec{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$, where $\|\vec{x}\|$ is the **norm** of $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Note that $\|\vec{x} - \vec{y}\| = \|(x_1 - y_1, \cdots, x_n - y_n)\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d(\vec{x}, \vec{y})$. Let $\vec{p} = (a_1, \cdots, a_n)$ and $\vec{q} = (b_1, \cdots, b_n)$. Then by the Cauchy-Schwarz inequality, $\sum_{i=1}^n |a_i b_i| \le \|\vec{p}\| \cdot \|\vec{q}\|$. To prove this, assume $\vec{p} \neq 0$ and $\vec{q} \neq 0$. Notice that $0 \le (x - y)^2 = x^2 + y^2 - 2xy$, or $x^2 + y^2 \ge 2xy$ for all $x, y \in \mathbb{R}^n$. Then $2 \cdot \frac{|a_i|}{\|\vec{p}\|} \cdot \frac{|b_i|}{\|\vec{q}\|} \le \frac{|a_i|^2}{\|\vec{p}\|^2} + \frac{|b_i|^2}{\|\vec{q}\|^2}$. So the statement

$$2 \cdot \sum_{i=1}^n \frac{|a_i b_i|}{\|\vec{p}\| \cdot \|\vec{q}\|} \le \sum_{i=1}^n \frac{|a_i|^2}{\|\vec{p}\|^2} + \sum_{i=1}^n \frac{|b_i|^2}{\|\vec{q}\|^2}$$

must also hold for every $i, 1 \le i \le n$, which is equivalent to

$$2 \cdot \sum_{i=1}^n \frac{|a_i \cdot b_i|}{\|\vec{p}\| \cdot \|\vec{q}\|} \le \frac{1}{\|\vec{p}\|^2} \sum_{i=1}^n |a_i|^2 + \frac{1}{\|\vec{q}\|} \sum_{i=1}^n \sum_{i=1}^n |b_i|^2 = \frac{\|\vec{p}\|^2}{\|\vec{p}\|^2} + \frac{\|\vec{q}\|^2}{\|\vec{q}\|^2} = 2.$$

This implies that $\sum_{i-1}^n |a_i b_i| \le \|\vec{p}\| \cdot \|\vec{q}\|$, and we are done with showing the Cauchy-Schwarz inequality. Now we need to prove Minkowski's inequality. Let $\vec{p} = (a_1, \cdots, a_n)$ and $\vec{q} = (b_1, \cdots, b_n)$. Then Minkowski's inequality states that

$$\|\vec{p} + \vec{q}\| \le \|\vec{p}\| + \|\vec{q}\|.$$

Let $\|\vec{p} + \vec{q}\| \neq 0$ (otherwise we are done). Recall the triangle inequality, $|a_i + b_i| \le |a_i| + |b_i|$ for every $a_i, b_i \in \mathbb{R}$. Then let $a_i = x - y, b_i = y - z$, yielding $|x - z| \le |x - y| + |y - z|$. Observe that

$$\|\vec{q} + \vec{q}\|^2 = \sum_{i=1}^n |a_i + b_i|^2 = \sum_{i=1}^n |a_i + b_i||a_i + b_i| \le \sum_{i=1}^n |a_i + b_i|(|a_i| + |b_i|)$$

by the triangle inequality. Then

$$\sum_{i=1}^n |a_i + b_i|(|a_i| + |b_i|) = \sum_{i=1}^n |a_i + b_i||a_i| + \sum_{i=1}^n |a_i + b_i||b_i|,$$

and by the Cauchy-Schwarz inequality, this is less than or equal to $\|\vec{p} + \vec{q}\| \cdot \|\vec{p}\| + \|\vec{p} + \vec{q}\| \cdot \|\vec{q}\|$. Then $\|\vec{p} + \vec{q}\|^2 \le \|\vec{p} + \vec{q}\|(\|\vec{p}\| + \|\vec{q}\|)$ which implies $\|\vec{p} + \vec{q}\| \le \|\vec{p}\| + \|\vec{q}\|$. Now $d(\vec{x}, \vec{z}) = \|\vec{x} - \vec{z}\| = \|\vec{x} - \vec{y} + \vec{y} - \vec{z}\| \le \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$ by definition, and we are done. ⊠

Example 1.1. Let $X \neq \emptyset$. We define the **discrete metric** on X as following:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Example 1.2 (ℓ_2 space)**.** By definition,

$$\ell_2 = \left\{ \langle x_1, \cdots, x_n \rangle, x_i \in \mathbb{R} \,\middle|\, \sum_{i=1}^{\infty} x_i^2 < \infty \right\}.$$

For example, $\langle 1, 0, 1, \cdots \rangle \notin \ell_2$, $\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \rangle \in \ell_2$, and $\langle 0, 0, 0, \cdots \rangle \in \ell_2$. Let $x, y \in \ell_2$. Define a metric $d(x,y)$ as follows: $d(x,y) := \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$. We know that $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \|\vec{x}_n - \vec{y}_n\| \leq \|\vec{x}_n\| + \|\vec{y}_n\| \leq \sum_{i=1}^{\infty} |x_i|^2 + \sum_{i=1}^{\infty} |y_i|^2 < \infty$ in ℓ_2. Then $\sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$ converges by the monotone convergence theorem.