Algebraic Topology II Lecture Notes

Simon Xiang

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1 Homology groups

Some review. A semi-simplicial set is a functor $X_\bullet\colon (\mathsf{Ord}_<)^{\mathrm{op}} \to \mathsf{Set}$, so that the co-face map $\varepsilon^i\colon [n-1] \to [n]$ induces a **face map** $\partial^i\colon X_n \to X_{n-1}$. These satisfy the **face relations**: $\partial^i \circ \partial^j = \partial^{j-1} \circ \partial^i$ when $i < j$.

Definition 1.1. In any category $\mathcal{C}$, a **semi-simplicial object** of $\mathcal{C}$ is a functor $X_\bullet\colon (\mathsf{Ord}_<)^{\mathrm{op}} \to \mathcal{C}$. Similarly, a **simplicial object** is a functor $X_\bullet\colon (\mathsf{Ord}_\le)^{\mathrm{op}} \to \mathcal{C}$.

For example, we have (semi-)simplicial spaces or (semi-)simplicial groups, etc. We'll focus on the csae where $\mathcal{C} = \mathsf{Mod}_R$, the category of modules over a commutative ring R, especially the case $R = \mathbb{Z}$, or $\mathsf{Mod}_\mathbb{Z} = \mathsf{Ab}$. A semi-simplicial R-module consists of:

- Some R-modules A_n $(n \ge 0)$,
- Face homomorphisms $\partial^i\colon A_n \to A_{n-1}$ among face relations.

How do these relate to simplicial sets? We can turn any (semi-)simplicial set $X_\bullet$ into a (semi-)simplicial R-module $[n] \mapsto R^{X_n}$, applying the free functor $\mathsf{Set} \to \mathsf{Mod}_R$.

$$(\mathsf{Ord}_<)^{\mathrm{op}} \to \mathsf{Set} \xrightarrow{\text{free}} \mathsf{Mod}_R$$

We make a momentous definition.

Definition 1.2 (Boundary operator)**.** For $n \ge 1$, define the **boundary operator** $\partial = \sum_{i=0}^n (-1)^i \partial^i$, $A_n \to A_{n-1}$.

Lemma 1.1. *$\partial \circ \partial = 0$, as maps $A_n \to A_{n-2}$.*

One of the problems on the problem set was to use the face relations to prove Lemma 1.1. Hence we get a **chain complex** of R-modules $\cdots \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \to 0$. Reminder: a chain complex of R-modules is a family of R-modules $\{C_n\}_{n\in\mathbb{Z}}$ with R-linear maps $\partial : C_n \to C_{n-1}$ such that $\partial \circ \partial = 0$. In our case, take $A_n = 0$ when $n < 0$. This is also called a "non-negative chain complex". Set $C_* = \bigoplus_{n\in\mathbb{Z}} C_n$, which leads to an operator $\partial : C_* \to C_*$, which by definition is a **graded** R-module. ∂ has **degree** -1, i.e. carries C_n into C_{n-1}.

There is a category of chain complexes Ch_R. Morphisms $(C_*, \partial)g \to (D_X, \partial')$ are maps $f : C_* \to D_*$ of degree zero ($f(C_n) < D_n$) such that $\partial' \circ f = f \circ \partial$. This leads to a functor $\mathsf{ssMod}_r \to \mathsf{Ch}_R$, where ssMod_R is the category of semi-simplicial R-modules. We usually query chain complexes vias their **homology**. Namely, set

$$H_n(C_*, \partial) := \frac{\ker(\partial : C_n \to C_{n-1})}{\operatorname{im}(\partial : C_{n+1} \to C_n)}.$$

So H_n is a subquotient of C_n, where $\ker \partial$ presents "n-cycles" and $\operatorname{im} \partial$ represents "n-boundaries".

Remark 1.1. If $A_\bullet$ is a **simplicial** R-module, its chain complex (A_*, ∂) has a **chain subcomplex** DA_*, the *degenerate* chains. Namely, $DA_n = \sum$ images of the degeneracy maps $A_{n-1} \to A_n$. Some facts:

- $\partial(DA_n) \subseteq DA_{n-1}$.
- (DA_*, ∂) is **acyclic**, i.e. $H_*(DA_*, \partial) = 0$.

It follows that $A_* \to A_*/DA_*$ (called the **normalized** chain complex) induces an isomorphism on homology. To see this, we use the long exact sequence on homology induced by the short exact sequence $0 \to DA_* \to A_* \to A_*/DA_* \to 0$. So one has the choice to work with normalized chains if one wishes, without affecting homology. This idea comes up all over the place; we'll apply it next to singular homology, but also includes simplicial homology, Čech cohomology of spaces equipped with open coveres, group homology (bar complex), the homology of Lie algebras, the Hochschild homology of associative algebras, it's all over the place.

A question that comes up regularly is "why do chain complexes show up everywhere in mathematics (or algebra)"? A number of principles apply, but this construction of a chain complex from a simplicial module encompasses a whole lot of them. In many cases we have the option of working with standard or normalized chains.

1.1 Singular homology

A space X leads to a simplicial set $C(|\Delta^\bullet|, X), [n] \mapsto \{\text{continuous maps } |\Delta^n| \to X\}$. Here $\partial^i : C(|\Delta^n|, X) \to C(|\Delta^{n-1}|, X)$, $\partial^i \sigma = \sigma|_{\mathrm{conv}(v_0, \cdots, \hat{v}_i, \cdots, v_n)}$. Let us draw this out.

- For $n = 1$, $\partial^0 \sigma = \partial|_{\{v_1\}}$, $\partial^1 \sigma = \sigma|_{\{v_0\}}$.
- For $n = 2$, $\partial^0(\sigma)$ todo:figure.

We turn this simplicial set into a simplicial R-module $S_\bullet(X;R) = R^{C(|\Delta^\bullet|, X)}$. So $S_n(X;R) = \{\sum n_\sigma \sigma\}$ for $n_\sigma \in R, \sigma : |\Delta^n| \to X$. Then $\partial = \sum_{i=0}^n (-1)^i \partial^i : S_n(X;R) \to S_{n-1}(X;R)$. For instance, $\partial(\sigma)$ that maps $(v_0, v_1) \to X$ is equal to $\sigma(v_1) - \partial(v_0)$. Similarly, $\partial(\sigma)$ mapping the 2-simplex into X is todo:see figure. From this we get **singular homology groups** $H_n(X;R) := H_n(S_*(X;R), \sigma)$. These end up being the key invariants of spaces. If $f : X \to Y$ is a map of spaces, set $Sf : S_*(X) \to S_*(Y)$, $\sigma \mapsto f \circ \sigma$ so we get $f_* : S_*X \to S_*Y$ a map of chain complexes.

The problem with this construction is that its utterly opaque. It's extremely natural and we can write down many properties about it, but what the heck does this mean. The singular chain complex is absolutely vast, but it turns out the homology is computable.

Example 1.1. Let $x \in X$. We can define a constant map $c_x^n : |\Delta^n| \to X$, the constant n-simplex about x. We have $\partial^i c_x^n = c_x^{n-1}$. So

$$\partial c_x^n = \left(\sum (-1)^i\right) c_x^{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ c_x^{n-1} & \text{if } n \text{ even.} \end{cases}$$

For example, if $X = \{*\}$, these constant simplices are the only maps to X that we have. We get that $S_*(*) = \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$ (here $R = \mathbb{Z}$ for simplicity). So $H_n(*) = \mathbb{Z}$ if $n = 0$ and 0 if $n > 0$.

todo:poincare lemma

2 Classifying spaces for categories and groups

This topic will probably take a couple of lectures to do totally.

Aim. Construct this **classifying space** BG for a group G.

Significance. The classifying space classifies principal G-bundles (equivalently, regular covering spaces with deck group G). The other key reason is that $H_*(BG)$ (or cohomology) is a very interesting invariant of the group G, called **group homology**.

This fits nicely into the simplicial approach we have been doing. We follow the approach laid out in todo:rerefence: Segal (1967), classifying spaces and spectral sequences. We won't touch the part of the paper that concerns spectral sequences.

We attach to any small category $\mathcal{C}$ a space $B\mathcal{C}$. How to do this? Note that any totally ordered set $(S, <)$ can be made into a category, where objects are elements of S, with unique morphism $s \to s'$ in this category if $s \leq s'$ (and no morphisms if $s > s'$). We attach to $\mathcal{C}$ its **nerve** $N\mathcal{C}$, a simplicial set

$$N\mathcal{C}(S, <) = \{\text{functors } S \to \mathcal{C}\}.$$

So $(N\mathcal{C})_n = \{\text{functors } [n \ \to \mathcal{C}\}$. For $f : [n] \to [n]$, we get a map $(N\mathcal{C})_n \to (N\mathcal{C})_m, F \mapsto F \circ f$. Concretely, what is $(N\mathcal{C})_0$? This is defined as $(N\mathcal{C})_0 = \{\text{functors } [0] \to \mathcal{C}\} = \text{ob}(\mathcal{C})$. For $(N\mathcal{C}_1) = \{\text{functors } [1] \to \mathcal{C}\}$, this amounts to $\{X_0 \xrightarrow{f} X_1\}$ morphisms in $\mathcal{C}$. Going on, $(N\mathcal{C})_2 = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2\}$. In general, $(N\mathcal{C})_n = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\}$.

The face maps $\partial_i \colon (N\mathcal{C})_n \to (N\mathcal{C})_{n-1}$ omit X_i. For example, $\partial^1(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3) = (X_0 \xrightarrow{f_2 \circ f_1} X_2 \xrightarrow{f_3} X_3$. Degeneracy maps $(N\mathcal{C})_n \xrightarrow{\eta^i} (N\mathcal{C})_{n+1}$ put in identity maps. For example, for $\mathcal{C} = [n]$, $N\mathcal{C} = \mathsf{Ord}_{\leq}(\cdot, [n]) = \Delta^n_{SS}$.

Definition 2.1. The **classifying space** $B\mathcal{C} = |N\mathcal{C}|$, or the geometric realization of the nerve.

For a cellular string of $B\mathcal{C}$, k-cells correspond to non-degenerate k-simplices. For example, $B[n] = |N[n]| = |\Delta^n_{SS}| = \Delta^n|$.

2.1 Groups

Let G be a group. View G as a category with $\text{ob}(G) = \{*\}$, morphisms $\{*\} \to \{*\}$ parametrizing elements of the group ($\text{mor}(*, *) = G$). Then composition in the category is composition in G. The nerve $(NG)_n = G^n$, since $(NG)_n$ corresponding to strings of composable morphisms, but all morphisms are composable since they send one object to one object. The face maps satisfy $\partial^i(g_1, \cdots, g_n) = (g_1, .., g_{i+1}g_i, \cdots, g_n)$, and the degeneracy maps insert copies of e (the identity of G). From here we get $BG = |NG|$, the geometric realization of this nerve.

2.2 Products

Another "clean" story. For $\mathcal{C}, \mathcal{D}$ small categories, we get another small category $\mathcal{C} \times \mathcal{D}$, where $\text{ob}(\mathcal{C} \times \mathcal{D})$, $\text{mor}(\mathcal{C} \times \mathcal{D}) = \text{mor}\mathcal{C} \times \text{mor}\mathcal{D}$. It is easy to check that $N(\mathcal{C} \times \mathcal{D}) \cong N\mathcal{C} \times N\mathcal{D}$ as a product of simplicial sets. So $B(\mathcal{C} \times \mathcal{D}) = |N\mathcal{C} \times N\mathcal{D}|$ which leads to a natural map $|N\mathcal{C}| \times |N\mathcal{D}| = B\mathcal{C} \times B\mathcal{D}$. This map is a homeomorphism if (e.g.)[1] $\mathcal{C}$ is finite as a CW complex.

The observation from todo:referencec segal paper is the following; take $\mathcal{C} = [1] = \{\bullet \to \bullet\}$. We have seen that $M[1] = |\Delta^1| = I$. So $B([1] \times \mathcal{D}) \xrightarrow{\cong} I \times B\mathcal{D}$. What is this category $[1] \times \mathcal{D}$? Objects are pairs $(0, X), (1, X)$ for $X \in \text{ob}\mathcal{D}$. Morphisms $([1] \times \mathcal{D})((0, X), (0, Y)) = \mathcal{D}(X, Y), ((1, X), (1, Y))$, and $([1] \times \mathcal{D})((0, X), (1, Y)) = \mathcal{D}(X, Y)$. In other words, two copies of $\mathcal{D}$ with an arrow in between them.

Lemma 2.1. *If $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$ are functors, then a natural transformation $F_0 \implies F_1$ determines a homotopy $I \times B\mathcal{C} \to B\mathcal{D}$ from $BF_0 \colon B\mathcal{C} \to B\mathcal{D}$ to BF_1.*

So the classfying space turns natural transformations into homotopies.

Proof. A functor $H \colon [1] \times \mathcal{D} \to \mathcal{D}$ determines functors $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$, since $[1] \times \mathcal{C} = \mathcal{C}_0 \to \mathcal{C}_1$. The arrow gives us a natural transformation $\theta_H \colon F_0 \to F_1$; how so? We have a distinguished morphism $(0, X) \xrightarrow{e_x^{0 \to 1}} (1, X)$ for $(0, X) \in \text{ob}\mathcal{C}$. Set $\theta_H(X) = H(e_X^{0 \to 1}) \in \mathcal{D}(F_0 X, F_1 X)$. It an exercise to verify that conversely, this data determines a functor $H_\theta \colon [1] \times \mathcal{C} \to \mathcal{D}$. So $H_\theta \colon [1] \times \mathcal{C} \to \mathcal{D}$, and $BH_\theta \colon B([1] \times \mathcal{C}) \simeq I \times B\mathcal{C} \to B\mathcal{D}$, which is the desired homotopy. ⊠

Corollary 2.1. *An equivalence of categories $F \colon \mathcal{C} \to \mathcal{D}$ induces a homotopy equivalence $B\mathcal{C} \xrightarrow{BF} B\mathcal{D}$.*

We will see next time that the classifying space of an abelian group is a topological group. We will also apply Corollary 2.1 to see that BG has contractible universal cover.

[1] This is overkill, locally finite is enough, but this case will do for us.

3 The universal cover of BG is contractible

Some reminders.

Definition 3.1. An **aspherical space** is a path-connected CW complex X whose universal cover $\widetilde{X}$ is contractible.

The reason for the name is that the higher homotopy groups are the same as the universal cover, so there are no non-trivial maps between spheres.

Example 3.1. Some examples:

- point
- S^1 with universal cover $\mathbb{R} \to S^1$, $\pi_1(S^1) \cong \mathbb{Z}$
- $\underbrace{S^1 \times \cdots \times S^1}_{n}$ with universal cover $\mathbb{R}^n$, $\pi_1((S^1)^n) = \mathbb{Z}^n$
- The genus g surface Σ_g with universal cover $\mathbb{D} \cong \mathbb{R}^2$, $\pi_1 \cong \langle A_1, \cdots, A_g, B_1, \cdots, B_g \mid [A_1, B_1] \cdots [A_g, B_g] = 1 \rangle$.
- $\bigvee^n S^1$, whose universal cover is an ∞ tree, $\pi_1 = F_n$.
- $\mathbb{RP}^\infty$, whose universal cover is $S^\infty = \bigcup S^n$. It is a fact that S^∞ is contractible! We have $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$. There is a generalization with infinite lens spaces, with fundamental group $\mathbb{Z}/n$.

Last time we saw that a small category $\mathcal{C}$ has

- a nerve $N\mathcal{C}$ (simplicial set)
- a classifying space $B\mathcal{C} = |N\mathcal{C}|$.

Example 3.2. A group G defines a category G with one object and morphisms parametrized by group elements, hence NG and BG. Here, $(NG)_n = G^n$. We didn't actually need inverses to do this, it works for monoids as well.

Remark 3.1 (From Riccardo, who isn't here)**.** I didn't catch it rip.

Variant. Let G be a monoid. It has a **translation category** tG, whose objects are elements of G ($\mathrm{ob}(tG) = G$) and has morphisms $\mathrm{mor}_{tG}(g_1, g_2) = \{h \in G \mid hg_1 = g_2\}$. If G is a group, all morphisms in tG are isomorphisms.

Proposition 3.1. *The classifying space $B(tG)$ is contractible when G is a group.*

Proof. We saw last time that if a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, then $BF\colon B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence. This was the punchline from last time. Take the trivial category $[1]$ with one object and one morphism, this category has a trivial classifying space (a point). There is a functor $F\colon tG \to [1]$ which sends $F(g) = *, F(g_1 \xrightarrow{h} g_2) = \mathrm{id}_*$. This category is the terminal object among categories. More interestingly, there is a functor in the opposite direction $G\colon [1] \to tG$, sending $g(*) = e, g(\mathrm{id}_*) = (e \xrightarrow{e} e)$.

We claim that G is an equivalence (with F as an inverse equivalence). To show a functor is an equivalence, we show it is fully faithful and essentially surjective. Essential surjectivity means that every object in the target category tG is isomorphic to one in the image of the functor. $g \underset{t_G}{\cong} e$ since $e \xrightarrow{g} g$, and it is an isomorphism since we go in the other direction by g^{-1}. Fully faithful means that G is bijective on morphism sets; $[1](*,*) = \{\mathrm{id}_*\}$, and again $tG(e,e) = \{e \xrightarrow{e} e\}$, and G bijectively maps between these two sets. Finally we get that $G\colon tG \to [1]$ induces a homotopy equivalence $B(tG) \to \{\mathrm{pt}\}$. Therefore $B(tG)$ is contractible. ⊠

Set $EG = B(tG)$. The next thing is that there is a functor $P\colon tG \to G$. On objects, $P(g) = *$, $P(g_1 \xrightarrow{h} g_2) = h$. This induces a map $p = BP\colon EG \to BG$ (the p stands for projection).

Note. Note that $N(tG)_0 = \mathrm{ob}(tG) = G$, and

$$N(tG)_n = \left\{ \left| g_0 \xrightarrow{g_1} g_1 g_0 \xrightarrow{g_2} g_2 g_1 g_0 \to \cdots \xrightarrow{g_n} g_n \cdots g_0 \right| \right\} \cong G^{n+1}$$

in a sequence $(g_1, g_1, \cdots, g_n)$. Compare this with $(NG)_n = G^n$. Face and degree maps are like those in NG. We have $NP\colon N(tG) \to NG$, $G^{n+1} \to G^n$, where $(g_0, \cdots, g_n) \mapsto (g_1, \cdots, g_n)$.

Theorem 3.1. *EG comes with a free and proper right G-action, covering id_{BG} such that $P\colon EG/G \to BG$ is a homeomorphism.*

$$\begin{tikzcd} EG \arrow[rr, "\gamma"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & EG \arrow[dl, "p"] \\ & BG & \end{tikzcd}$$

Furthermore, $p\colon EG \to BG$ is a (universal) covering map.

Corollary 3.1. *BG is aspherical.*

Proof of Theorem 3.1. G acts (strictly) on the right on the category tG; to $g \in G$, attach $F_\gamma\colon tG \to tG$ such that $F_{\gamma'\gamma} = F_\gamma \circ F_{\gamma'}$, $F_e = \mathrm{id}_{tG}$. Then $F_\gamma(g) = g\gamma, F_\gamma(g_1 \xrightarrow{h} g_2) = (g_1\gamma \xrightarrow{h} g_2\gamma)$. F_γ then induces a G-action on $N(tG)$, hence on EG. Here $G^{n+1} = N(tG)_n \xrightarrow{\gamma} N(tG)_n = G^{n+1}$, $(g_0, \cdots, g_n) \mapsto (g_0\gamma, g_1, \cdots, g_n$, hence amounts to a map of simplicial sets. This action on $N(tG)_n$ is free, and also is free on the non-degenerate simplices in $N(tG)_n$, so the G-action on EG is free. The categorical G-action on tG commutes with P, and it follows that $p\colon EG/G \to BG$ is bijective.

The final thing is that the G-action on EG is proper. This means that the map $EG \times G \to EG \times EG, (x, \gamma) \mapsto (x, x \cdot \gamma)$ is proper (pre-image of compact is compact). It follows that EG/G is Hausdorff, and the projection $EG \to EG/G$ is a covering map. It further follows that $EG/G \xrightarrow{P} BG$ is also a covering map (and bijective), hence a homeomorphism. todo:notes ⊠

4 Classifying spaces for topological categories and groups

Last time we got classifying spaces for small categories, then groups. Today we do it for topological categories, and get a special class of spaces called "Eilenberg-Maclane" spaces, which arise from cohomology functors. We explain them in terms of principle bundles, which may be slightly sketchy.

Consider a small category $\mathcal{C}$. We have a pair of sets $\mathrm{ob}(\mathcal{C}), \mathrm{mor}(\mathcal{C})$, which come with structural maps $s, t\colon \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ (source and target maps), where $s\mathcal{C}(X, Y) = X$ and $t\mathcal{C}(X, Y) = Y$. There is an identity morphism $e\colon \mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C}), X \mapsto e_X \in \mathcal{C}(X, X)$. Finally, there is composition $\mathrm{mor}\mathcal{C} \times_{\mathrm{ob}\mathcal{C}} \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$, which is the set $\{(f, g) \in \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) \mid tg = sf\}$ (which is associative and unital). This is a way of structuring the small category, and a **topological** category $\mathcal{C}$ is one where $\mathrm{ob}(\mathcal{C}), \mathrm{mor}(\mathcal{C})$ are *spaces*, and the four structural morphism are *continuous*.

Example 4.1. A topological group (or monoid) G defines a 1-object topological category.

To a topological category $\mathcal{C}$, we can attach to it a "classifying space" $B\mathcal{C}$. Consider the nerve $N_\bullet \mathcal{C}$, which is the simplicial set with $N\mathcal{C}_n = \{\text{functors}[n] \to \mathcal{C}\} = \text{chains of } n \text{ composable morphisms} = \mathrm{mor}(\mathcal{C}) \times_{\mathrm{ob}(\mathcal{C})} \cdots \times_{\mathrm{ob}(\mathcal{C})} \mathrm{mor}(\mathcal{C})$ n times, $N_0(\mathcal{C}) = \mathrm{ob}(\mathcal{C})$. This is more than a simplicial set, it's a **simplicial space**, ie. a functor $\left(\mathsf{Ord}_\leq\right)^{\mathrm{op}} \to \mathsf{Top}$. A simplicial space $X_\bullet$ still has a geometric realization $|X_\bullet|$, where $|X_\bullet| = X_{\mathrm{pre}}/\sim$. Last time we had $X_{\mathrm{pre}} = \coprod_{n \geq 0} X_n \times |\Delta^n|$; this time the X_n are spaces parametrizing n-simplices. For $\theta\colon [m] \to [n]$ increasing, we identify $(\theta^*\sigma, x) \sim (\sigma, \theta_* x)$.

From here, define the classifying space of our category to be the geometric realization of the nerve, or $B\mathcal{C} := |N_\bullet\mathcal{C}|$. For $X_\bullet$ a simplicial space, $|X_\bullet|$ is Hausdorff, but not a CW complex in general. The natural map $|(X \times Y)_\bullet| \to |X_\bullet| \times |Y_\bullet|$ is still a continuous bijection, and a homeomorphism when $|X_\bullet|$ is *compact*. In general, $|X \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet| \xrightarrow{\text{id}} K(|X_\bullet| \times |Y_\bullet|)$. Here K is the compactly generated topology, where a subset of closed if its intersection with a compact subspace is is closed in the original topology. We tacitly apply "K" to products as needed.

What do natural transformations induce? Suppose we have *continuous* functors $F_0, F_0 \colon \mathcal{C} \to \mathcal{D}\mathcal{D}$ between topological categories, $\theta \colon \mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, and a *continuous* natural transformation $F_0 \implies F_1$. This still induces a homotopy between $BF_0 \simeq BF_1 \colon B\mathcal{C} \to B\mathcal{D}$. The same argument applies: F_0, F_1, θ lead to a continuous function $[1] \times \mathcal{C} \to \mathcal{D}$, which implies $B([1] \times \mathcal{C}) \to B\mathcal{D}, B[1] = I, [1] = I \times B\mathcal{C}$.

4.1 Classifying spaces of topological groups

We apply this to topological groups. For now, let G be a topological monoid, which leads to the spaces $N_\bullet G, BG$. Like in the discrete discussion, we can also form the translation category tG, where $\mathrm{ob}(tG) = G, tG(g_0, g_1) = \{h \in G \mid hg_1 = g_2\}$, again a topological category under the subspace topology. We get a functor $tG \xrightarrow{P} G$ which induces $p = BP$, $B(tG) \to BG$. It is tradition to set $EG = B(tG)$, with projection $EG \xrightarrow{p} BG$. Arguing as before, $EG = B(tG) \simeq B[1] = \text{point}$. So we find that EG is contractible when G is a topological *group*.

4.2 Eilenberg-Maclane spaces

For G, H topological monoids, we have $B(G \times H) \xrightarrow{\cong} BG \times BH, t(G \times H) = tG \times tH$. So $E(G \times H) \xrightarrow{\cong} EG \times EG$. Now say that G is s commutative. Then the multiplication $m \colon G \times G \to G$ is actually a homomorphism. So we get a map $Bm \colon BG \times BG \simeq B(G \times G) \to BG$. BG now has this binary operation Bm which is associative, because m is associative and B is functorial. The unit $\{1\} \xrightarrow{e} G$ induces $Be \colon B\{1\} = \{\mathrm{pt}\} \to BG$, i.e. BG comes with a basepoint (0-simplex) Be. This is a unit for Bm. So BG is a commutative topological monoid. We can iterate and define $B^n G = B(B^{n-1}G)$. This is essentially the definition of an Eilenberg-Maclane space.

Definition 4.1 (E-M spaces)**.** For π a group made discrete, define a space $K(\pi, 1) = B\pi$. For π an abelian group, define $K(\pi, n) = B^n\pi$ for $n \geq 1$.

For a commutative topological *group* G, inversion $i \colon G \to G, g \mapsto g^{-1}$ is a continuous homomorphism. This induces $B_i \colon BG \to BG$ showing that BG is a topological abelian group. This applies to E-M spaces $K(\pi, n)$ where π is abelian. We save the principal bundle story for next time. Next time, we will outline (but not give a full proof) that if π is a discrete group, $K(\pi, 1) = B\pi$ can be characterized up to homotopy equivalence as any *aspherical space* X with $\pi_1(X) \simeq \pi$. For instance, $K(\mathbb{Z}, 1) \simeq S^1$ since $\pi_1(S^1) \simeq \mathbb{Z}$. Key points about $K(\pi, n)$;

- The homotopy groups $\pi_i K(\pi, n)$ turn out to be trivial if $i \neq n$, and π if $i = n$. From the perspective of homotopy groups these are a basic gadget.
- We will discuss singular cohomology a lot going forward. The point is that $H^n(X; \mathbb{Z})$ is identified with homotopy classes of maps $[X, K(\pi, n)]$. So Eilenberg-Maclane spaces are called representing spaces for cohomology functors.

4.3 I missed a lecture

about principal bundles

5 Chapter II: Homology and cohomology

We spend a while talking about simplicial sets and things you can do with them, but now we talk about homology and cohomology. This is a consolidation of foundations, with further techniques that may or may not have been covered (homology of products, *co*homology, product structures (cross, cup, cap), etc). Primarily we will talk about singular (co) homology, but also simplicial and cellular (and possibly Čech) as well.

5.1 Singular homology

Let X be a space. This leads to a singular simplicial set $S_\bullet(X)$, with $S_n(X) = \mathbb{Z}^{\{\text{maps } |\Delta^n| \xrightarrow{\sigma} X\}}$. Recall that $|\Delta^n| = \operatorname{conv}(v_0, \cdots, v_n) \subseteq \mathbb{R}^n$. The face maps $\partial_i : S_n(X) \to S_{n-1}(X)$ are defined by $\partial_i \sigma|_{\operatorname{conv}(v_0,\cdots,\hat{v}_i,\cdots,v_n)}$. There are also the degeneracy maps $\eta^i(\sigma) = \{|\Delta^{n+1}| \xrightarrow{\Sigma^i} |\Delta^n| \xrightarrow{\sigma} X\}$, where $|\Delta^{n+1}| \to |\Delta^n|$ is the ith codegeneracy map. Then we have the singular chain complex, where $S_*(X) = \bigoplus_{n\in\mathbb{Z}} S_n(X)$ (where $S_q X = 0$ for $q < 0$), with $\partial = \sum (-1)^i \partial_i$ taking the degree down by one, or $\partial : S_n X \to S_{n-1} X$. The crucial feature is that $\partial \circ \partial = 0$, which means we get a chain complex. Here $H_n(X) = H_n(S_*(X))$, and homology is defined as $Z_n(X)/B_n(X)$; the $Z_n(X)$ are n-cycles equal to $\ker(\partial : S_n \to S_{n-1})$, and the $B_n(X)$ are n-boundaries equal to $\operatorname{im}(\partial : S_{n+1} \to S_n)$. The indexing satisfies $B_n X \subseteq Z_n X \subseteq S_n X$, which reflect the fact that $\partial^2 = 0$.

Maps $f : X \to Y$ functorially induce $S_*(f) = f_* : S_* X \to S_* Y$, $f_*\sigma = f \circ \sigma$, where $g_* f_* = (g \circ f)_*$, and we get maps on homology, and so on. There is a variant with singular chains with coefficicents in a commutative ring R. This is the same idea but now with a simplical R-module that sends $n \mapsto R^{\{\sigma : |\Delta^n| \to X\}}$. This leads to $S_*(X;R)$ which is a chain complex of R-modules, where $S_*(X;R) = S_*(X) \otimes_{\mathbb{Z}} R, \partial = \partial \otimes \mathrm{id}_R$. So we get homology with coefficients $H_*(X;R)$. An n-cycle over $\mathbb{Z}$ and n-boundary over $\mathbb{Z}$ is also an n-boundary or n-cycle over R. Hence we get a natural map $H_*(X) \otimes R \to H_*(X;R)$ of R-modules. We will study the issue of "is this map an isomorphism"? (It isn't in general).

Very few calculations of $H_*(X)$ can be done "by hand". We can easily compute that if $X = \{\text{point}\}$, $S_*(X) = (\cdots \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$, $H_* X = H_0 X = \mathbb{Z}$. More generally, if $X \subseteq \mathbb{R}^n$ is star-shaped, $H_* X = H_0 X = \mathbb{Z}$. The generator is a constant $|\Delta^0| \to X$. To show this, assume we have some simplex mapping into a star-shaped set. We can homotope it to a constant simplex at the origin (star-shaped), and this process defines a chain homotopy equivalence on the singular chains of X and the basepoint. Beyond this, thee is next to nothing one can do besides rolling up your sleeves.

5.2 Simplicial homology

We probably already know some more techniques for computing homology, like Mayer-Vietoris (which is not obvious, since it requires proving a locality for singular chains). For this course, we discuss simplicial homology, of the geometric realization of a simplicial set.

Let $X_\bullet$ be a simplicial set, $X = |X_\bullet|$. We then have two simplicial abelian groups; $C_\bullet(X_\bullet)$, where $C_n(X_\bullet) = \mathbb{Z}^{X_n}$. OTOH we have $S_\bullet(X)$, where $S_n(X) = \mathbb{Z}^{C(|\Delta^n|,X)}$. Therefore we have two chain complexes; $C_*(X_\bullet)$ the *simplicial chains*, and $S_*(X)$ the *singular chains*. The assertion is that $C_\bullet(X_\bullet)$ is contained in $S_\bullet(X)$. Each $\delta \in X_n$ defines a "characteristic map" $\chi_\delta : |\Delta^n| \to X = X_{\text{pre}}/\sim$, where $X_{\text{pre}} = \coprod_n X_n \times |\Delta^n|$. So χ_δ takes a point t in the simplex and maps it to the equivalence class $[(\delta, t)] \in X_{\text{pre}}$, or $\chi_\delta(t) = [(\delta, t)]$. This leads to a map of simplicial abelian groups $C_\bullet(X_\bullet) \xrightarrow{\chi} S_\bullet(X), \delta \mapsto \chi_\delta$, which subsequently leads to a map of chain complexes

$$\chi_* : C_*(X) \to S_*(X)$$

a chain map.

Theorem 5.1. *χ_* is a **quasi-isomorphism**, that is to say, it is a chain map inducing an isomorphism on homology.*

We have a subcomplex $D_*(X_\bullet) \subseteq C_*(X_\bullet)$, where the $D_*(X_\bullet)$ are degenerate simplicial chains (images of degeneracy maps) and the $C_*(X_\bullet)$ are simplicial chains. We have *normalized chains* $N_*(X_\bullet) = C_*(X_\bullet)/D_*(X_\bullet)$ (or non-degenerate) simplicial chains with induced boundary operation. An algebraic fact (valid for all simplicial abelian groups) is that the quotient map $C_*(X_\bullet) \xrightarrow{\text{quotient}} N_*(X_\bullet)$ is a quasi-isomorphism. So

$$\begin{tikzcd} H_n(C_*(X_\bullet)) \arrow[r, "\chi_*", "\cong"'] \arrow[d, "\cong"'] & H_n(X) \arrow[dl, bend left, "\cong"] \\ H_n(N_*(X_\bullet)) & \end{tikzcd}$$

Variant. Suppose $X_\bullet^{\text{semi}}$ is a semi-simplicial set, then $\chi_*\colon N_*(X_\bullet) = \mathbb{Z}^{X_\bullet} \to S_*(|X|)$ is still a quasi-isomorphism.

Example 5.1. Consider the Klein bottle $K = |K_\bullet|$, the geometric realization of a semi-simplical set $K_\bullet$. todo:find the picture somehow We have $K_0 = \{*\}, K_1 = \{a, b, c\}, K_2 = \{A, B\}$. So $\partial A = b - c + a, \partial B = a - b + c$. Furthermore $\partial a = * - * = 0$, and so $\partial b, \partial c = 0$ as well. Then

$$N_*\left(0 \to \mathbb{Z}^{\{A,B\}} \xrightarrow{\partial} \mathbb{Z}^{\{a,b,c\}} \xrightarrow{\partial} \mathbb{Z}^* \to 0\right)$$

Reading off the cycles, note that $\partial A, \partial B$ are LI so there is no kernel and $Z_2 = 0$. We have $Z_1 = \mathbb{Z}^{\{a,b,c\}}$, and $Z_0 = \mathbb{Z}^{\{*\}}$. So

$$H_n(K) = \begin{cases} 0 & n \geq 2, \\ \mathbb{Z}_{[*]} & n = 0 \\ \frac{\mathbb{Z}^{\{a,b,c\}}}{a+b-c, a-b+c} = \mathbb{Z}/2_{[a]} \oplus \mathbb{Z}_{[b]} & n = 1 \end{cases}$$

What does it mean to calculate a homology group in full? Not only do we calculate the homology as some abelian groups, we specify a generator for the cycles.

6 Homotopy-invariance and locality for simplicial chains

Before we discuss that stuff, we will discuss...

6.1 Reduced homology

The *augmented singular chain complex* of X is the following:

$$\widetilde{S}_*(X) = \left(\cdots \xrightarrow{\partial} \underset{2}{S_2(X)} \xrightarrow{\partial} \underset{1}{S_1(X)} \xrightarrow{\partial} \underset{0}{S_0(X)} \xrightarrow{\varepsilon} \underset{-1}{\mathbb{Z}} \to 0 \to \cdots\right)$$

The map ε is called the *augmentation*, which augments the chain complex. For $\sigma\colon |\Delta^0| \to X \to$, $\varepsilon\left(\sum n_i \sigma_i\right) = \sum n_i$. Why is this a chain complex? For $\sigma\colon |\Delta^1| \to X$, $\varepsilon \circ \partial\sigma = \varepsilon\left(\sigma|_{\{v_1\}} - \sigma|_{\{v_0\}}\right) = 1 - 1 = 0$. So $\widetilde{S}_*(S)$ really is a chain complex. We write $\widetilde{H}_q(X) = H_q(\widetilde{S}_*(X))$, which is called **reduced homology** (or *augmented homology*). There is an obvious chain map $\widetilde{S}_*X \to S_*X$:

$$\begin{tikzcd} \cdots \arrow[r] & S_*X \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & S_*X \arrow[r] & 0 & \end{tikzcd}$$

Note that the other way around doesn't result in a chain map. So there is a map $\widetilde{H}_*(X) \to H_*X$ resulting from this chain map. Note that $\widetilde{S}_*$ and $\widetilde{H}_*$ are *functorial in X* and don't require a basepoint.

It is clear that $\widetilde{H}_q(X) \xrightarrow{\cong} H_q(X)$ for every $q > 0$ (almost by definition). The interesting this is what happens at the bottom of the complex? Another obvious thing is that $H_*(\emptyset) = 0$, $\widetilde{H}_*(\emptyset) = \mathbb{Z}_{-1}$. In general, $H_0(X) \cong \mathbb{Z}^{\pi_0(X)}$ (where π_0 is the set of path-components of X). For $x \in X$, the constant 0-simplex $c_x^0 \colon |\Delta^0| \to X$ at x, we have $[c_x^0] \mapsto [x] \in \mathbb{Z}^{\pi_0(X)}$. As for $\widetilde{H}^0(X)$ (with $X \neq \emptyset$), fix a basepoint $b \in X$. We get that $\widetilde{H}_0(X) \cong \mathbb{Z}^{\pi_0(X)\setminus\{b\}}$, that is, $[c_x^0 - c_b^0] \mapsto [x]$. In particular, for X path-connected, $\widetilde{H}_0(X) = 0$, and for $X \subseteq \mathbb{R}^n$ star-shaped, the reduced homology $\widetilde{H}_*(X) = 0$.

6.2 Two fundamental principles

There are two fundamental principles;

(1) The homotopy principle

(2) The locality principle

What are they? First we discuss the homotopy principle. A map $f : X \to Y$ induces a map $f_* = S_*f : S_*X \to S_*Y$ and so a map $f_* = H_*f : H_*X \to H_*Y$. What does a homotopy $F : I \times X \to Y$ induce? It induces $F_* : S_*(I \times X) \to S_*Y$, but we want an answer in terms of S_*X. The answer is the **homotopy principle**, which says that F induces a *chain homotopy* $h_F \colon S_*(X) \to S_{*+1}(Y)$ from f_* to g_* (or $F|_{\{0\}\times X} = f$, $F|_{\{1\}\times X} = g$). In other words, $\partial \circ h_F + h_F \circ \partial = g_* - f_*, S_*X \to S_*Y$. It follows that $f_* = g_* \colon H_*X \to H_*Y$, or the maps induced on homology are *equal*. Hence if $h \colon X \to Y$ and $k \colon Y \to X$ are homotopy inverses, the induced maps on homology $h_* \colon H_*X \to H_*Y, k_* \colon H_*Y \to H_*X$ are inverse isomorphisms. This tells is that homology is an invariant of homotopy type.

Example 6.1. If X is contractible, then $H_*X = H_0\mathbb{Z} = \mathbb{Z}$, since this is true of a point.

Let us move on to locality. Let X be a space, and $\mathcal{U} = \{U_i \mid i \in I\}$ be a "cover" (keep in mind the practical case of a covering by two subsets). In a sense, we want $X = \bigcup_{i\in I} \operatorname{int}(U_i)$. There are inclusions $u_i \colon U_i \to X$, so we get a map on singular chains $\bigoplus_i S_*(U_i) \to S_*(X)$. We call this map $u = \bigoplus (u_i)_*$. Define $S_*^{\mathcal{U}}(X) = \operatorname{im}(u)$, which is a subcomplex of $S_*(X)$.

Theorem 6.1 (Locality)**.** *The inclusion $S_*^{\mathcal{U}}(X) \to S_*X$ is a quasi-isomorphism. That is to say, it induces isomorphisms $H_q(S_*^{\mathcal{U}}(X)) \xrightarrow{\cong} H_q(X)$.*

The brief idea is that we can take our chains and subdivide them by barycentric subdivision, which have smaller domains. Iterate this process to get maps on sufficiently tiny simplices, which will land in one of the sets of our open cover. On the other hand, the overall simplex is homologous to the signed sum of the subdivided simplices (difference is a boundary), which proves that cycles and boundaries in S_*X is homologous to a cycle/boundary in $S_*^{\mathcal{U}}(X)$. One of the consequences of this is the exactness of the Mayer-Vietoris sequence.

Corollary 6.1 (Mayer-Vietoris)**.** *Let $\mathcal{U} = \{U, V\}$. Here*

$$\begin{tikzcd} U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & U \arrow[d, hook, "k"] \\ V \arrow[r, hook, "\ell"'] & X \end{tikzcd}$$

Then we get a short exact sequence of chain complexes

$$0 \longrightarrow S_*(U \cap V) \xrightarrow{i_* \oplus j_*} S_*U \oplus S_*V \xrightarrow{k_* - \ell_*} S_*^{\mathcal{U}}(X) \longrightarrow 0.$$

This leads to a long exact sequence on H_*, given by

$$\to \cdots H_q(U \cap V) \to H_qU \oplus H_qV \to H_q(S_*^{\mathcal{U}}X) = H_q(X) \xrightarrow{\beta} H_{q-1}(U \cap V) \to \cdots$$

where the equality comes from locality. There is a reduced variant where we replace S_* by $\widetilde{S}_*$, which is still exact, leading to a Mayer-Vietoris sequence for $\widetilde{H}_*$.

Example 6.2. We can now prove by induction that $\widetilde{H}_*(S^n) \simeq \mathbb{Z}^n_{[\omega_n]}$. The generator or fundamental class is given by this; suppose we take $|\Delta^{n+1}| \cong D^{n+1}$. Then the geometric boundary $\Sigma_n = \partial|\Delta^{n+1}| \xrightarrow{\cong} S^n$. What we are really looking at is $\widetilde{H}_*(\Sigma_n)$, which will do us just fine. Let $\iota_{n+1} = \mathrm{id}_{|\Delta^{n+1}|} \in S_{n+1}(|\Delta^{n+1}|)$. Then $\omega_n = \partial\iota_{n+1}$ viewed as living in $S_n(\Sigma_n)$. So $\partial\omega_n = 0$ when viewed as living in the ambient simplex.

The claim is that $\mathbb{Z}[\omega_n] = \widetilde{H}_*(S^n)$. By reduced Mayer-Vietoris (splitting the sphere in half then thickening), we get $\widetilde{H}_*(S^n) \underset{\beta}{\xrightarrow{\cong}} \widetilde{H}_{*-1}(S^{n-1})$. This is familiar, but it also leads to a proof of our claim; todo:ok

7 Filtrations and homology

Discussion section on Monday, 3-4 PM in room 12.166. The topology seminar has booked the room until half past three but it is unlikely they'll stay in there that long.

We previously asserted that the following chain maps are quasi-isomorphisms:

- For $A_\bullet$ a simplicial abelian group, we have the quotient map $A_* \to NA_* = A_*/DA_*$ of the simplicial chains mod the degenerate chains a quasi-isomorphism.
- For $X_\bullet$ a simplicial set, the map from simplicial chains $C_*(X_\bullet) = \mathbb{Z}X_* \to S_*(|X_\bullet|)$.

We lay out a homological framework for proving such assertions.

7.1 Mapping cones

There is a homological and topological version of this concept, and it is really the homological version we want to use. But we start with the topological version for context. Let $f : X \to Y$ be a map up spaces, then define the **mapping cone** by

$$Cf := (I \times X) \amalg Y/\sim, \quad (0,x) \sim f(x), (1,x) \sim (1,x')$$

for all $x, x' \in X$. In other words, this is the mapping cylinder with the top collapsed to a point, hence the cone. Then there is a pushout diagram

$$\begin{tikzcd}
X \arrow[r, hook, "x\mapsto(0,x)"] \arrow[d, "f"'] & I \times X/\{1\} \times X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & Cf \arrow[dr, dotted, "\exists!"] & \\
 & & Z
\end{tikzcd}$$

A map $Cf \to Z$ corresponds to $g : Y \to Z, h : I \times X \to Z$, where h is a nullhomotopy of $g \circ f$. Now we get to the homological version, where there is a close analogy to the topological situation. We have $f : C_* \to D_*$ a map of chain complexes. This leads to a new chain complex cone(f) over some ring R, where

$$(\mathrm{cone}f)_n = C_{n-1} \oplus D_n, \quad \delta : (\mathrm{cone}f)_n \to (\mathrm{cone}f)_{n-1}, \quad \delta(x,y) = (-\partial_C x, fx + \partial_D y).$$

From here we can see $\delta^2 = 0$. There is a notion of a shift of a chain complex; for $m \in \mathbb{Z}$, $C_*[x] = C_{n+m}$. The boundary operator is $(-1)^m = (-1)^m\partial_C$. By construction, there is a short exact sequence of complexes

$$0 \longrightarrow D_* \longrightarrow \mathrm{cone}f \longrightarrow C_*[-1] \longrightarrow 0$$

which leads to a long exact sequence in homology, given by

$$\cdots \longrightarrow H_q D \longrightarrow H_q(\mathrm{cone} f) \longrightarrow H_{q-1} C \xrightarrow[\text{map } \beta]{\text{connecting}} H_{q-1} D \longrightarrow \cdots$$

The connecting map β actually turns out to be the induced map on homology Hf. What is $\beta[x]$? We have $\partial_C x = 9$, then we left x to $(x, 0) \in \mathrm{cone} f$. Apply δ, then we get $(-\partial_C x, fx)$. Project to D_x to get $[fx]$. This is all obtained by following the general formula to find the connecting map of a long exact sequence.

Lemma 7.1. *f is a quasi-isomorphism iff $H_*(\mathrm{cone} f) = 0$.*

Proof. Use the LES. Recall that it's given by

$$H_* D \to H_* \mathrm{conef} \to H_{*-1} C \xrightarrow{Hf} H_{*-1} D \to \cdots$$

Subsitute zero in the appropiate places and apply exactness. ⊠

So mapping cones convert quasi-isomorphisms into acyclic complexes. We haven't actually made the connection to the topological version; a chain map has the form $\mathrm{cone} f \xrightarrow{h+g} E_*$, where $g : D_* \to E_*$, $h : C_{*-1} \to E_*$. This is a chain map iff g is a chain map and h is a chain homotopy from 0 to $g \circ f$.

Remark 7.1. Suppose we take a map of spaces $f : X \to Y$. Then there is an induced map $f_* : S_* X \to S_* Y$, so we can take $\mathrm{cone}(f_*)$. On the other hand, we could also take the singular chains of the topological cone $S_*(Cf)$. We claim there is a chain map from $\mathrm{cone}(f_* : S_* X \to S_* Y) \to S_*(Cf)$. We have $Y \xhookrightarrow{i} Cf$, which implies there ia a map $i_* : S_* Y \to S_*(Cf)$. So there is a chain nullhomotopy of $i_* f_*$, which leads to $h : S_* X \to S_{*+1}(Cf)$, a nullhomotopy of $i_* f_*$. Moreover, this map is actually a quasi-isomorphism (we will need excision to prove this part).

7.2 Filtered complexes

Let C_* be a chain complex, and take an increasing filtration on it. This means we have a sequence of subcomplexes

$$\cdots \subset F_p \subset C_* \subset F_{p+1} C_* \subset \cdots \subset C_*$$

Why are filtrations worth talking about? They come up all over the place.

Example 7.1. Say X is a space with nested subspaces $X_p \subset X_{p+1} \subset \cdots \subset X$. Then the singular chains $S_* X$ are filtered by $F_p(S_* X) = \mathrm{im}(S_* X_p \to S_* X)$. For example, CW complexes are filtered by their skeleta, and preimages of the skeleta of the base of fiber bundles give an interesting filtration (leading to the Serre spectral sequence).

From $\{F_p C_*\}$ a filtered chain complex, we get an associated graded chain complex

$$\mathrm{gr}(C_*) = \bigoplus_{p \in \mathbb{Z}} \mathrm{gr}_p(C_*) = F_p C_* / F_{p-1} C_*.$$

Even though $\mathrm{gr} C$ forgets a lot of information, it still knows a lot. Say we have a finite filtration $\cdots \subset 0 \subset \cdots \subset 0 \subset F_p C_* \subset \cdots \subset F_{p'} C_* = C_* \subset C_* \subset \cdots$

Lemma 7.2. *If $\{F_p C_*\}$ is a finite filtration and $H_*(\mathrm{gr} C) = 0$, then $H_*(C) = 0$.*

Proof. The number of steps $p' - p$ is finite. The idea is to induct on the number of steps. Use the LES to get our result. ⊠

Theorem 7.1. *If $f : C_* \to D_*$ is a chain map that preserving filtrations $\{F_p C_*\}$ and $\{F_p D_*\}$, and if the induced map $\mathrm{gr} C_* \to \mathrm{gr} D_*$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof. $\mathrm{cone} f_*$ inherits a filtration from whose associated graded is acyclic by assumption. By the lemma, $\mathrm{cone} f$ is acyclic, which exactly means that f is a quasi-isomorphism. If you wanted you could also prove this directly with the five-lemma. ⊠

8 The degenerate simplicial chain complex is acyclic

Let's get to it.

8.1 Degenerate simplicial chains

Consider a simplicial abelian group A, with corresponding chain complex $(A_*, \partial = \sum(-1)^i \partial^i)$. There are degeneracy maps $\sigma^i : A_n \to A_{n+1}$. The images of σ^i span a subcomplex $DA_* \subset A_*$, e.g. in $S_*(X)$. So the degenerate 2-simplices $|\Delta^2| \xrightarrow{\sigma^0} |\Delta^1| \xrightarrow{\sigma} X$, where σ^0 is just projection. We can do this for any σ^i as long as the map remains increasing.

Theorem 8.1. $H_q(DA_*) = 0$ *for every* q*, hence the projection* $A \to NA_* = A_*/DA_*$ *is a quasi-isomorphism.*

Proof. Find a filtration $\{F_p(DA_*)\}$ that is *bounded*; for every n, the filtration $F_p(DA_n)$ (an abelian group) has finitely many steps. By the same argument for finite filtrations, if $H_*(\operatorname{gr} DA_*) = 0$ then $H_*(DA_*) = 0$. We need relations between face and degeneracy operators. There are face maps $\partial^i : A_n \to A_{n-1}$ and boundary maps $\sigma^i : A_n \to A_{n+1}$ for $i \in [n]$. Then

$$\partial^i \sigma^j = \begin{cases} \sigma^{j-1}\partial^i & \text{if } i < j, \\ \mathrm{id}, & \text{if } i \in \{j, j+1\}, \\ \sigma^j \partial^{i-1}, & \text{if } i > j+1. \end{cases}$$

Using these relations, we see that degenerate chains really do form a subcomplex. We have

$$\begin{aligned} \partial \circ \sigma^j &= \sum_i (-1)^i \partial^i \sigma^j \\ &= \sum_{i<j} (-1)^i \sigma^{j-1} \partial^i + \sum_{i>j-1} (-1)^i \sigma^j \partial^{i-1} \in DA_*. \end{aligned}$$

So DA_* is a subcomplex. Filter DA_* by saying $F_p(DA_n) = \operatorname{im}\sigma^0 + \cdots + \operatorname{im}\sigma^p \subset DA_n$ (if $0 \le p \le n$). So $F_p DA_n = 0$ for $p < 0$ and DA_n for $p > n$. This is a bounded filtration, since $\partial(F_p) \subset F_p$.

To show that DA_* is acyclic, it suffices to show that $\operatorname{gr}(F_\bullet DA_*) = \bigoplus F_p/F_{p-1}$ is acyclic. An element of F_p/F_{p_1} takes the form $\sigma^p x \pmod{F_{p-1}}$. We construct a chain contraction of $\operatorname{gr}_* = F_p/F_{p-1}$, i.e. a map $(\operatorname{gr}_p) \xrightarrow{h} (\operatorname{gr}_\varphi)_{*+1}$ such that $\partial h + h\partial = \mathrm{id}$. This is a chain homotopy between our complex and the identity map, which certainly shows that our complex is acyclic. The natural guess for h is a degeneracy map, namely σ_p. Let $y = \sigma^p x$. Then $\partial y = \partial(\sigma^p x) = \sum_{i>p+1} (-1)^i \sigma^p \sigma^{i-1} x \pmod{F_{p-1}}$. We also compute $\partial(\sigma^p) = \partial(\sigma^p \sigma^p x) = (-1)^{p+1} y - \sigma^p(\partial y)$ $\pmod{F_{p-1}}$ (the steps for this calculation were omitted). This tells us that $h = (-1)^{p+1}\sigma^p$ is a chain contraction. ⊠

8.2 Relative homology

Let X be a space and A be a subspace with inclusion $i : A \to X$. Then we get an injection $i_* : S_*A \to S_*X$. Define the **relative chains** to be $S_*(X,A) = S_*X/S_*A = \operatorname{coker} i_*$. Then we have a short exact sequence of complexes

$$0 \to S_*A \xrightarrow{i_*} S_*X \to S_*(X,A) \to 0$$

by construction, which gives us a long exact sequence

$$\cdots \to H_qA \xrightarrow{i_*} H_qX \to H_q(X,A) \xrightarrow{\partial} H_{q-1}A \to \cdots$$

on homology. What is the connecting map ∂? For $[c] \in H_q(X,A)$, we take a relative cycle (boundary is zero relative to the singular chains of A 0, and let $\partial[c] = [\partial c]$ for $c \in S_{q-1}(A)$. So we get a boundary of one degree lower by identifying boundaries.

Example 8.1. If A is a deformation retract of X, then $i+*: S_*A \to S_*X$ is a quasi-isomorphism (because i is a homotopy equivalence and so i_* is a chain homotopy equivalence). So the long exact sequence tells us that $H_*(X,A)=0$. In general, the relative homology tells us the failure of the inclusion to be a homotopy equivalence.

Remark 8.1. There exists a quasi-isomorphism $\mathrm{cone}(i_*) = (S_{*-1}A \oplus S_*X) \xrightarrow{(0,\text{quotient})} S_*(X,A)$.

There is a reduced version $i_* : \widetilde{S}_*(A) \to \widetilde{S}_*(X)$, with

$$\begin{tikzcd}
\cdots \arrow[r] & S_1(A) \arrow[r] \arrow[d, "i_*"'] & S_0(A) \arrow[r] \arrow[d, "i_*"'] & \mathbb{Z} \arrow[r] \arrow[d, "="'] & 0 \\
\cdots \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Then $\mathrm{coker}(i_*) = S_*(X,A)$. In other words, there is a short exact sequence of complexes

$$0 \to \widetilde{S}_*A \xrightarrow{i_*} \widetilde{S}_*X \to S_*(X,A) \to 0$$

which gives us a long exact sequence

$$\cdots \to \widetilde{H}_qA \xrightarrow{i_*} \widetilde{H}_qX \to H_q(X,A) \xrightarrow{\partial} \widetilde{H}_{q-1}A \to \cdots$$

on reduced homology.

Example 8.2. Let $X = |\Delta^n|$, and $A = \Sigma_{n-1}$ (the geometric boundary of $|\Delta^q|$), corresponding to (D^n, S^{n-1}). Consider $H_*(|\Delta^n|, \Sigma_{n-1})$. todo:finish