Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

Homology

The big boy has arrived. These notes will follow Hatcher §2.1.

Remark 1.1. This is something I heard even before I enrolled in this course. The homotopy groups are easy to define, but impossible to compute and work with. The homology groups take a lot of work to define, but the resulting groups are much nicer and easier to work with.

The fundamental group is a cool tool when dealing with low-dimensional spaces (the pride and joy of UT Austin), but it doesn't do well with higher dimensional spaces, for example, it can't distinguish between the n-spheres S^n for $n \geq 2$. We can get rid of this limitation by considering the higher homotopy groups $\pi_n(X)$, which are defined in terms of maps from the n-dimensional cube I^n and homotopies $I^n \times I \to X$ of such maps. Cool things about higher homotopy groups: for X a CW complex, $\pi_n(X)$ only depends on the $(n+1)$-skeleton, and $\pi_i(S^n) = 0$ for $i < n$ and $\mathbb{Z}$ for $i = n$, as expected. However, the drawback is that they're extremely difficult to compute in general— take the "simple" task of computing $\pi_i(S^n)$ for $i > n$.

Enter the homology groups $H_n(X)$. Similar to $\pi_n(X)$, $H_n(X)$ for X a CW complex depends only on the $(n+1)$-skeleton, and for the spheres $H_i(S^n) \simeq \pi_i(S^n)$ for $1 \leq i \leq n$, but the homology groups have the advantage in that $H_i(S^n) = 0$ for $i > n$. However, everything has a price. How exactly do we define these so called homology groups? We start by motivating, then doing simplicial homology, before moving onto singular homology. Most efficient method for computing homology groups is called cellular homology. We'll also talk about Mayer-Vietoris sequences, the analogue of the van Kampens for the fundamental group.

Something interesting about homology: most of the time we only use the basic properties of homology, not the definition itself. So we could almost invoke an axiomatic approach, which will happen soon. We could also skip the algebra and talk about geometry, but then Dr. Brand would be unhappy (and so would I), so we'll approach it with a mix of the two (talk about intuition first then state the axioms later).

1.1 The big idea of homology

Issues with homotopy groups: things get really wacky because S^2 has no cells of dimension greater than 2, but some (infinitely many) of the higher homotopy groups $\pi_n(S^2)$ are nontrivial. $\langle god\, shattering\, star\, noises\rangle$ However, homology groups are (directly) related to cell structures, in that you can regard them as an algebraization of how cells of dimension n attach to cells of dimension $n-1$.

Imagine a circle with two antipodal points x and y, with four arrows a, b, c, d drawn in the direction from x to y, which we'll denote by X_1.

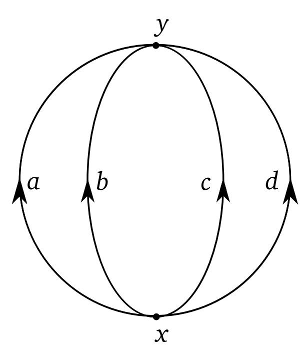

Figure 1: The graph X_1, consisting of two vertices and four edges.

Usually loops are nonabelian, so suppose we abelianize the loops. That is, the loops ab^{-1} and $b^{-1}a$ are "the same circle" (but with a different starting point), so we'll just say they're equal. Formally (not really), rechoosing the basepoint just permutes the letters cyclically, so by abelianizing we can cast off our silly worries about the basepoint. So we make the transition from loops (chosen basepoint) $\longrightarrow$ cycles (no chosen basepoint).

Now we abelian, and all the cool abelian groups use additive notation. So a cycle looks something like $a-b+c-d$ now, a linear combination of edges with integer coefficients. We'll call these linear combinations **chains** of edges. We can decompose these into cycles by several ways, eg $(a-c)+(b-d)=(a-d)+(b-c)$, so it's better just to say cycles are any LC of edges st at least one decomposition makes geometric sense. When is a chain a cycle? Cycles are distinguished by the fact that they enter and exit a vertex the same amount of times. So for an arbitrary chain $ka+lb+mc+nd$, it enters y about $k+l+m+n$ times (one for each thing) and enters x (or leaves it) $-k-l-m-n$ times. So if we want $ka+lb+mc+nd$ to be a cycle, we just need to require $k+l+m+n=0$.

To generalize this, let C_1 be the free abelian group with a basis set $\{a,b,c,d\}$ (edges), and C_0 be the free abelian group with basis $\{x,y\}$ (vertices). Elements of C_1 are chains of edges, and elements of C_0 are linear combinations of vertices. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element to $y-x$, then $\partial(ka+lb+mc+nd)=(k+l+m+n)y-(k+l+m+n)x$, so cycles are precisely $\ker\partial$. It can be seen that $a-b$, $b-c$, and $c-d$ form a basis for $\ker\partial$, so every cycle in X_1 is a unique linear combination of these three elts. Basically, X_1 has three "holes", the three gaps in between the four edges.

Now let's attach a 2-cell to X_1 to get X_2, as seen below.

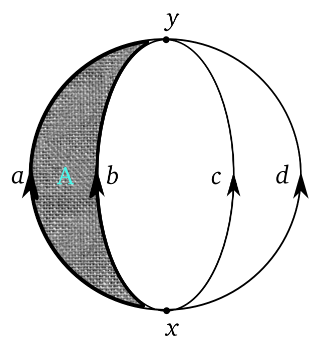

Figure 2: X_1 with a 2-cell attached, denoted X_2. Have you ever seen a 2-cell that looks like cloth?

The 2-cell is attached along the cycle $a-b$, forming the 2-skeleton X_2. Now the cycle is trivial (homotopically), which suggest we form a quotient by factoring out the subgroup generated by $a-b$. For example, $a-c$ and $b-c$ are now equivalent, since they're homotopic in X_2. Algebraically, we define a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the infinite cyclic group generated by A, and $\partial_2(A)=a-b$. ∂_1 is the boundary homomorphism, defined earlier. We are interested in $\ker\partial_1/\operatorname{im}\partial_2$, that is, the 1-dimensional cycles modulo the boundaries (multiples of $a-b$). Remember, factor groups collapse everything we don't like to the identity. This quotient group is the **homology group** $H_1(X_2)$. If we were to talk about X_1, since it has no 2-cells C_2 is simply zero, so $H_1(X_1)=\ker\partial_1/\operatorname{im}\partial_2=\ker\partial_1$, which is free abelian on three generators. $H_1(X_2)$ is free abelian on two generators ($b-c$ and $c-d$), which expresses the geometric observation that there are two holes remaining after filling one of them in with the 2-cell A.

Let's go farther. Add another 2-cell to the pre-existing 2-cell A, to get the 3-complex X_3.

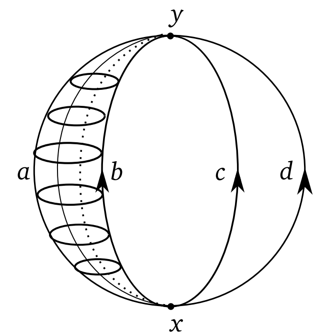

Figure 3: The 3-complex X_3, formed by attaching a 2-cell to X_2.

This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B, and the boundary homomorphism $\partial_2 \colon C_2 \to C_1$ sends A, B to $a - b$. $H_1(X_3) = \ker \partial_1 / \operatorname{im} \partial_2 = H_1(X_2)$, but now ∂_2 has a nontrivial kernel (the infinite cyclic group generated by $A - B$). We view $A - B$ as a 2d cycle generating $H_2(X_3) = \ker \partial_2 \simeq \mathbb{Z}$. The second homology detects the 2d "hole" in X_3.

Unfortunately the diagrams will have to stop now, but let's go even farther and make the complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere by A and B, creating a chain group C_3 generated by C. The boundary homomorphism $\partial_3 \colon C_3 \to C_2$ that sends C to $A - B$ should be seen as the boundary of C, similar to how $a - b$ is the boundary of A. Now we have a sequence of boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and $H_2(X_4) = \ker \partial_2 / \operatorname{im} \partial_3$ is now trivial. $H_3(X_4) = \ker \partial_3 = 0$, note that $H_1(X_4) = H_1(X_3) \simeq \mathbb{Z} \times \mathbb{Z}$, so this is the only homology group of X_4 that isn't trivial.

You can pretty much see where this is going. For a cell complex X, we have chain groups $C_n(X)$ free abelian with basis the n-cells of X, with boundary homomorphisms $\partial_n \colon C_n(X) \to C_{n-1}(X)$, by which we define the homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. So what's the problem? It's how to define ∂_n in general— for $n = 1$ this is easy, it's the vertex head minus the one at the tail. For $n = 2$, it still isn't hard per say, if the cell is attached on a loop of edges, just take the cycle of edges, keeping in mind orientation. This is much tricker for higher dimension cells, even with restrictions to polyhedral cells and nice attaching maps we still have to worry about orientation and stuff.

So what do we do? Use triangles, of course. We can subdivide arbitary polyhedra into certain special types of polyhedra called simplices (what we talked about in class day 1), so there isn't any loss of generality (but there is a loss of efficiency). This gives rise to our more basic **simplicial homology**, which deals with cell complexes from simplices. However, we are still quite limited in what we can do.

So, what do we really do this time? Make things less simple, and make your life difficult by considering the collection of all possible continuous maps of simplices into a space X (wow). The chain groups $C_n(X)$ are tremendously large, but the quotients $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, the **singular homology groups**, are much smaller and easier to work with[1]. For example, in the examples above the singular homology groups coincide with the ones computed from cellular chains. Furthermore (as we will see later), singular homology lets us define these nice cellular homology groups for *all* cell complexes, which solves the issue of how to define boundary maps for cellular chains.

1.2 The structure of Δ-complexes

I have a feeling we're gonna be typing a lot of `\Delta`'s. So basically, the only thing cool kids talk about is singular homology, but it's kinda complicated so we gotta talk about the inferior version for those who have the brain capacity of a literal ape[2], simplicial homology, first. We talk about simplicial homology in the domain of Δ-complexes. Take the standard fundamental polygons with orientation for $\mathbb{T}^2$, $\mathbb{R}P^2$, and the Klein bottle K. Cut

[1] For reasonably "nice" spaces X, of course.

[2] The book simply says "primitive" version, so I used my imagination a little bit.

the squares in half with a diagonal to get two triangles, from here we can get the original shape by identifying in pairs. We can do this with any n-gon, decomposing it into $n-2$ base triangles. So we can make any closed surface from triangles, furthermore, we could also make a larger class of spaces that aren't surfaces by allowing more than two edges to be glued together at the same time.

The idea of a Δ-complex is to generalize these constructions to n-dimensions. The n-dimensional triangle is the n-simplex, the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that don't lie in a hyperplane of dimension less than n, where by "hyperplane" we mean the set of solutions to a system of linear equations. We could also just say that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are LI. The v_i are **vertices** of the simplex, and the simplex itself is $[v_0, \cdots, v_n]$.

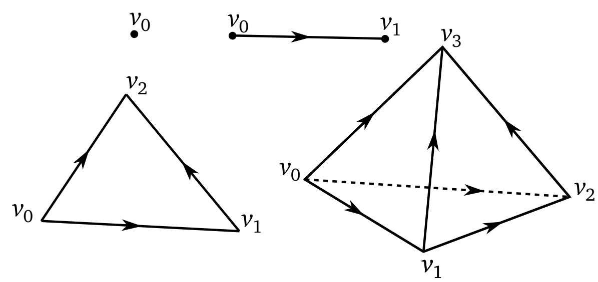

Figure 4: The 0-simplex to the 3-simplex, respectively (with ordered vertices and oriented edges).

For example, we have the standard n-simplex given by

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\},$$

whose vertices are the unit vectors along the coordinate axes. Think of this as taking the unit vectors, and drawing a triangle from each of their endpoints. This works because the difference vectors are LI. For homology, orientation of vertices is really important, so n-simplex really means n-simplex with an ordering on its vertices. Ordering the vertices will determine an orientation on its subscripts, as can be seen in Figure 4. This also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other simplex $[v_0, \cdots, v_n]$ that preserves the order of the vertices, given by

$$(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i.$$

We say the coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i \in [v_0, \cdots, v_n]$. Deleting a vertex of a n-simplex yields something that spans an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$. We'll adopt the following convention: *The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.* That sounds reasonable enough. We say the union of all faces of Δ^n is the **boundary** of Δ^n, written $\partial \Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is equal to $\Delta^n \setminus \partial \Delta^n$, the interior of Δ^n.

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is onto, and each point of X is in the image of exactly one restriction $\sigma_\alpha | \mathring{\Delta}^n$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear order-preserving homeomorphism.
3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

A consequence of (3) is that X can be built as a quotient space of a collection of disjoint simplices Δ_α^n, one for each $\sigma_\alpha \colon \Delta^n \to X$, the quotient space obtained by idenfying each face of a Δ_α^n with the Δ_β^{n-1} corresponding to the restriction σ_β of σ_α to the face in question. You can think of this as basically cell complexes, attaching 0-simplices (cells) to 1-simplices and 2-simplices, and so on.

In general, we can make Δ-complexes from collections of disjoint simplices by identifying various subspaces spanned by subsets of the vertices, with identifications performed by the canonical order-preserving linear homeomorphisms. Note that if we think of a Δ-complex X as a quotient space of disjoint simplices, then X must be Hausdorff. Each restriction $\sigma_\alpha | \mathring{\Delta}^n$ is a homeomorphism onto its image by condition (3), which is an open simplex in X. Then these open simplices are the cells e^n_α of a CW complex structure on X with the σ_α's as characteristic maps (we won't use this fact yet).

1.3 Simplicial homology

Goal: define simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Formally, we can write elements of $\Delta_n(X)$ as finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$, called **n-chains.** We could also write $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha \colon \Delta^n \to X$ is the characteristic map of e^n_α, with image the closure of e^n_α. Such a sum can be thought of as a finite collection, or 'chain', of n-simplices in X.

Take a look at $\partial[v_0, v_1] = [v_1] - [v_0]$, $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$, and $\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$. Naïvely, one might assume the boundary of an n-simplex to be the sum of the faces delete a point, denoted by $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ where v_i is the vertex to be deleted. However, note the signs to take orientations into account, it just happens that they work out based on the position of v_i. So we have

$$\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots v_n].$$

Keeping this in mind, let's define a **boundary homomorphism** $\partial_n \colon \Delta_n(X) \to \Delta_{n-1}(X)$ for X a general Δ-complex by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \; \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Lemma 1.1. *The composition* $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ *is zero.*

Proof. Note that

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \; \sigma | [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^i (-1)^{j-1} \; \sigma | [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n].$$

Then after switching i and j in the second term, it becomes the negative of the first. Alternate proof from Dr. Allcock: note that $\partial\sigma := \sum_{i=0}^n (-1)^i \sigma \circ [v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Then

$$\partial\partial\sigma = \sum_{i=0}^n (-1)^i \partial\left(\sigma \circ [v_0, \cdots, \hat{v}_i, \cdots, v_n]\right),$$

which distributes because C_{n-1} is free on {singular $(n-1)$-simplex}. So defining any function {singular $(n-1)$-simplex} $\to C_{n-2}$ extends to a $\mathbb{Z}$-linear map $C_{n-1} \to C_{n-2}$. Then

$$\partial\partial\sigma = \sum_{i=0}^n (-1)^i \left(\sum_{j=0}^{i-1} \sigma \circ [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n](-1)^j + \sum_{j=i+1}^n \sigma \circ [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n](-1)^{j-1} \right),$$

which is equal to zero by cancellation[3]. ⊠

What we have here is a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n\partial_{n+1} = 0$ for all n. This is called a **chain complex**. Note that we've extended the sequence to 0, with $\partial_0 = 0$. The equation $\partial_n\partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{im}\partial_{n+1} \subseteq \ker\partial_n$, so we can define the $\mathbf{n^{th}}$ **homology group** of the chain complex as $H_n = \ker\partial_n / \operatorname{im}\partial_{n+1}$. Elements of $\ker\partial_n$ are called **cycles** and elements of $\operatorname{im}\partial_{n+1}$ are called boundaries. Cosets of $\operatorname{im}\partial_{n+1}$ in H_n are called **homology classes**. Two cycles representing the same homology class are said to be **homologous**, that is, their difference is a boundary. When $C_n = \Delta_n(X)$, the homology group $\ker\partial_n / \operatorname{im}\partial_{n+1}$ will be denoted by $H^\Delta_n(X)$ and called the $\mathbf{n^{th}}$ **simplicial homology group** of X.

[3] The proof from Dr. Allcock was for singular homology, but the idea is the same.

1.4 Homological Algebra

We'll take this section to digress a bit and talk about some homological algebra. These notes will follow May §12.

Let R be a commutative ring: the main example is $R = \mathbb{Z}$. A **chain complex** over R is a sequence of R-modules

$$\cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i (abbreviated $d = d_i$). A **cochain complex** over R is an analogous sequence

$$\cdots \to Y^{i-1} \xrightarrow{d^{i-1}} Y^i \xrightarrow{d^i} Y^{i+1} \to \cdots$$

with $d^i \circ d^{i+1} = 0$. Usually $X_i = 0$ for $i < 0$ and $Y^i = 0$ for $i < 0$ (or else $\{X_i, d_i\} \to \{X^{-i}, d^{-i}\}$, making chain and cochain complexes equivalent). An element of the kernel of d_i is a **cycle** and an element of the image of d_{i+1} is a **boundary**. This makes a lot more sense if you picture the boundary map d_i as removing a vertex to get an $n-1$ simplex each time. We say two cycles are **homologous** if their difference is a boundary, and write $B_i(X) \subseteq Z_i(X) \subseteq X_i$ for the submodules of boundaries and cycles, respectively. Then we can define the **ith homology group** $H_i(X)$ as the quotient module $Z_i(X)/B_i(X)$, and write $H_*(X)$ for the sequence of R-modules $H_i(X)$. To get things straight, we've defined things the following way:

$$\begin{aligned} Z_i(X) &= \text{cycles} := \ker d_i \subseteq X_i \\ B_i(X) &= \text{boundaries} := \operatorname{im} d_{i+1} \subseteq X_i. \end{aligned}$$

A **chain map** $f: X \to X'$ of chain complexes is a sequence of maps of R-modules $f_i: X_i \to X'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$ for all i. That is, the following diagram commutes for all i[4]:

$$\begin{tikzcd} \cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r, "d_i"] \arrow[d, "f_i"] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\ \cdots \arrow[r] & X'_{i+1} \arrow[r, "d'_{i+1}"] & X'_i \arrow[r, "d'_i"] & X'_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

It follows that $f_i(B_i(X)) \subseteq B_i(X')$ and $f_i(Z_i(X)) \subseteq Z_i(X')$. Therefore we have that f induces a map of R-modules $f_* = H_i(f): H_i(X) \to H_i(X')$. A **chain homotopy** $s: f \simeq g$ between chain maps $f, g: X \to X'$ is a sequence of homomorphisms $s_i: X_i \to X'_{i+1}$ such that

$$d'_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

for all i. Chain homotopy is an equivalence relation (this was an exercise) since if $t: g \simeq h$, then $s + t = \{s_i + t_i\}$ is a chain homotopy $f \simeq h$.

Lemma 1.2. *Chain homotopic maps induce the same homomorphism of homology groups.*

Proof. Let $s: f \simeq g$, $f, g: X \to X'$. If $x \in Z_i(X)$, then $f_i(x) - g_i(x) = d'_{i+1}s_i(x)$ such that $f_i(x)$ and $g_i(x)$ are homologous. ⊠

[4] May's diagram showed much less, but I feel this illustrates the idea much better: it also makes following around the chain homotopy homomorphisms easier.

A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of modules is **exact** if $\operatorname{im} f = \ker g$. If $M' = 0$, then g is a monomorphism; if $M'' = 0$, then f is an epimorphism. We proved this as an exercise! A longer sequence is exact if it is exact at each position. A **short exact sequence** of chain complexes is a sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

that is exact in each degree. Here 0 denotes that chain complex that is the 0 module in each degree.

Proposition 1.1. *A short exact sequence of chain complexes naturally gives rise to a LES of R-modules*

$$\cdots \to H_q(X') \xrightarrow{f} H_q(X) \xrightarrow{g_*} H_q(X'') \xrightarrow{\partial} H_{q-1}(X') \to \cdots .$$

Proof. Let $[x]$ denote the homology class of a cycle x. Define the "connecting homomorphism" $\partial : H_q(X'') \to H_{q-1}(X')$ by $\partial[x''] = [x']$, where $f(x') = d(x)$ for some x such that $g(x) = x''$. There exists such an x because g is an epimorphism, and x' exists because $gd(x) = dg(x) = 0$. Use a "diagram chase" to verify that ∂ is well defined and the sequence is exact. Naturality means that a commutative diagram of short exact sequences of chain complexes gives rise to a commutative diagram of long exact sequences of R-modules. The big idea is the naturality of the connecting homomorphism, which is left as an exercise to the reader. ⊠

1.5 Singular homology

These notes will follow Massey §2 and the rest of Hatcher §2.1.

Let's define $H_0(X)$ as such: let $Z_0(X) = C_0(X)$ and $H_0(X) = Z_0(X)/B_0(X) = C_0(X)/B_0(X)$. Anoter way we could do this is is defining $C_n(X) = \{0\}$ for $n < 0$, then defining $\partial_n : C_n(X) \to C_{n-1}(X)$ in the only possible way for $n \le 0$ (i.e., $\partial_n = 0$ for $n \le 0$), and finally defining $Z_n(X) = \ker \partial_0$. In general, we could define $Z_n(X) = \ker \partial_n$for all integers n, $B_n(X) = \partial_{n-1}(C_{n+1}(X)) \subseteq Z_n(X)$, and $H_n(X) = Z_n(X)/B_n(X)$ for all n, with $H_n(X) = \{0\}$ for $n < 0$.

Now let's define (not really, we'll ignore the definition) the reduced 0-dimensional homology group $\widetilde{H}_0(X)$. Let's define a homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$, often called the *augmentation*, made by the typical barycentric coordinate sum $\varepsilon : \sum_i n_i \sigma_i \mapsto \sum_i n_i$. Then $\varepsilon \circ \partial_1 = 0$: to do this, show that $\varepsilon(\partial_1(T)) = 0$ for some 1-cube (not hard)[5]. Then we can define $\widetilde{Z}_0(X) = \ker \varepsilon$, and

$$\widetilde{H}_0(X) = \widetilde{Z}_0(X)/B_0(X).$$

We say that $\widetilde{H}_0(X)$ is the **reduced 0-dimensional homology group** of X. To avoid weird stuff happening, assume $X \neq \emptyset$. It's often convenient to set $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$.

JK, back to Hatcher. Some examples of simplicial homology:

Example 1.1. Let $X = S^1$, with one vertex v and an edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map ∂_1 is zero since $\partial e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \ge 2$ since there are no simplices in these dimensions. Therefore

$$H_n^{\Delta}(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{for } n \ge 2. \end{cases}$$

Example 1.2. Let $X = \mathbb{T}$, the torus with a Δ-complex structure having one vertex, three edges a, b, and c,, and two 2-simplices U and L. Since $\partial_1 = 0$, $H_0^{\Delta}(\mathbb{T}) \simeq \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(\mathbb{T})$, it follows that $H_1^{\Delta}(\mathbb{T}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^{\Delta}(\mathbb{T})$ is equal to $\ker \partial_2$, which is infinite cyclic generated by $U - L$. So

$$H_n^{\Delta}(\mathbb{T}) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z} & \text{for } n = 0, 2, \\ 0 & \text{for } n \ge 3. \end{cases}$$

[5] I'm glossing over formal stuff because everywhere else uses triangles instead of cubes. I just want results!

Let's talk about **singular homology**. A **singular n-simplex** in a space X is just a map $\sigma\colon \Delta^n \to X$. The word 'singular' is used to imply that the map doesn't have to be nice (look like a simplex) but can have weird 'singularities'. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$, called **n-chains** (more precisely, singular n-chains) are finite formal sums $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i\colon \Delta^n \to X$. A boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ is defined by the same formula as before:

$$\partial_n(\sigma) = \sum_i (-1)^i \Big| [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Then $\sigma|[v_0, \cdots, \hat{v}_i, \cdots v_n]$ is a map $\Delta^{n-1} \to X$, that is , a singular $(n-1)$-simplex. We also have $\partial_n\partial_{n+1} = 0$ (more concisely $\partial^2 = 0$), so we define the singular homology group $H_n(x) = \ker\partial_n / \operatorname{im}\partial_{n+1}$. Singular chain groups tend to be really large (often uncountable), but modding out makes the homology groups easier to work with.

Proposition 1.2. *For a space X, there is an isomorphism $H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$, where X_α denotes the path-components of X.*

Proof. Since a singular simplex always has a path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. This is preserved by the boundary maps ∂_n and similarly $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$. ⊠

Proposition 1.3. *If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. We have $H_0(X)/\operatorname{im}\partial_1$ since $\partial_0 = 0$. Define a homomorphism $\varepsilon\colon C_0(X) \to \mathbb{Z}$ by $\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$. This is onto if $X \neq \emptyset$: we claim that $\ker\varepsilon = \operatorname{im}\partial_1$ if X is path-connected, and hence ε induces an isomoprhism $H_0(X) \approx \mathbb{Z}$. To see that this is true, observe that $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$ since for a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have $\varepsilon\partial_1(\sigma) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$. To show that $\ker\varepsilon \subseteq \operatorname{im}\partial_1$, suppose $\varepsilon\left(\sum_i n_i\sigma_i\right) = 0$, so $\sum_i n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i\colon I \to X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i\colon [v_0, v_1] \to X$, then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence $\partial\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$ since $\sum_i n_i = 0$. So $\sum_i n_i\sigma_i$ is a boundary, which shows that $\ker\varepsilon \subseteq \partial_1$. ⊠

Proposition 1.4. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof. In this case there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_i(-1)^i\sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. So we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternately isomorphisms and trivial maps, except for the last $\mathbb{Z}$. So the homology groups of this complex are trivial for every group besides $H_0 \simeq \mathbb{Z}$. ⊠

Sometimes weird stuff happens with $H_0(X)$, as can be seen in Proposition 1.4. To avoid this, we can talk about the **reduced homology groups** $\widetilde{H}_n(X)$, defined to be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the same one as in our earlier proposition[6]. Since $\varepsilon\partial_1 = 0$, ε vanishes on $\operatorname{im}\partial_1$ adn hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. Obviously $H_n(X) \simeq \widetilde{H}_n(X)$ for $n > 0$.

1.6 Exact sequences

Definition 1.1 (Exact sequences)**.** A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be **exact** if $\ker\alpha_n = \operatorname{im}_{n+1}$ for each n.

[6] My clever references aren't working??

The inclusions $\operatorname{im}\alpha_{n+1} \subseteq \ker\alpha_n$ are equivalent to $\alpha_n\alpha_{n+1} = 0$, so the sequence is a chain complex, and the opposite inclusions $\ker\alpha_n \subseteq \operatorname{im}\alpha_{n+1}$ say that the homology groups of this chain complex are trivial. We can express a number of basic algebraic concepts in terms of exact sequences, for example:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker\alpha = 0$, i.e., α is injective.

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{im}\alpha = B$, i.e., α is surjective.

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism, by (i) and (ii).

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective, and $\ker\beta = \operatorname{im}\alpha$, so β induces an isomorphism $C \simeq B/\operatorname{im}\alpha$. This can be written as $C \simeq B/A$ if we think of α as an inclusion of A as a subgroup of B.

An exact sequence $0 \to A \to B \to C \to 0$ as in (iv) is called a **short exact sequence**. These turn out to be the perfect tool for stuff, in particular, relating the homolofy groups of a space, a subspace, and the associated quotient space.

Theorem 1.1. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0,$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

Proof. Basically, construct ∂. The idea is that an element $x \in \widetilde{H}_n(X/A)$ can be represented by a chain α in X with $\partial\alpha$ a cycle in A whose homology class is $\partial x \in \widetilde{H}_{n-1}(A)$. The full proof will come later. Pairs of spaces (X, A) that satisfy the hypothesis of the theorem will be called **good pairs**[7]. ⊠

Corollary 1.1. *$\widetilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0$ for $i \neq n$.*

Proof. For $n > 0$ take the good pair $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. Since D^n is contractible the terms $\widetilde{H}_i(D^n)$ in the LES for this pair are zero. Then by the exactness of the sequence the maps $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\widetilde{H}_0(S^n) = 0$. Then our result follows by induction on n, in which the base case of S^0 holds by Proposition 1.2 and Proposition 1.4. ⊠

Lemma 1.3. *Every continuous map $h\colon D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.*

Proof. This was actually an earlier theorem in Hatcher. As you can see, this will lead into Brouwer's fixed point theorem. Suppose that $h(x) \neq x$ for all $x \in D^2$. Then we can define a map $r : D^2 \to S^1$ by letting $r(x)$ be the point of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2. Now r is continuous, furthermore, $r(x) = x$ if $x \in S^1$. So r is a retraction of D^2 onto S^1, but no such retraction exists: let f_0 be a loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop, for example $f_t(s) = (1-t)f_0(s) + tx_0$ for x_0 the basepoint of f_0. Since the retraction r is the identity on S^1, the composition rf_t is a homotopy in S^1 from $rf_0 = f_0$ to the constant loop at x_0: but this contradicts the fact that $\pi_1(S^1)$ is nonzero. ⊠

Corollary 1.2 (Brouwer's fixed point theorem)**.** *∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.*

Proof. If $r\colon D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for $i\colon \partial D^n \to D^n$ the inclusion map. The composition $\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\widetilde{H}_{n_1}(\partial D^n) \simeq \mathbb{Z}$. But i_* and r_* are both 0 since $\widetilde{H}_{n-1}(D^n) = 0$, and we have a contradiction. For the fixed point portion, just replace π_1 with H_n in Lemma 1.3 and we're good. ⊠

[7] *We're a good pair, you and I...*

1.7 Relative homology

Sometimes ignoring things makes things easier, for example arithmetic modulo n (ignoring multiples of n). Relative homology is another example: in this case, we ignore all singular chains in a subspace of a given space.

Given a space X and a subspace $A \subseteq X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$, thus chains in A are trivial in $C_n(X,A)$. Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$. Then we have a sequence of boundary maps

$$\cdots \to C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \to \cdots$$

The relation $\partial^2 = 0$ holds since it held before (then holds for quotients).

Definition 1.2 (Relative homology groups)**.** Given the chain complex above, the homology groups $\ker\partial/\operatorname{im}\partial$ of the chain complex are the **relative homology groups** $H_n(X,A)$. We can see the following:

- Elements of $H_n(X,A)$ are represented by **relative cycles**: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle is trivial in $H_n(X,A)$ iff it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

These properties make precise the intuitive idea that $H_n(X,A)$ is 'homology of X modulo A'.

Goal: show that the relative homology groups $H_n(X,A)$ for any pair (X,A) fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

To do this, we'll go on our first diagram chase. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

where i is the inclusion map and j is the quotient map. If we let n vary and draw the short exact sequences vertically instead of horizontally, we have a large commutative diagram like the one below, where the columns are exact and the rows are chain complexes denoted by A, B, and C.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A diagram like this is called a **short exact sequence of chain complexes**. We'll show that this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where $H_n(A)$ denotes the homology group $\ker\partial/\operatorname{im}\partial$ at A_n in the chain complex, $H_n(B)$ and $H_n(C)$ similarly defined. To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, let $c \in C_n$ be a cycle. Then since j is onto, $c = j(b)$ for some $b \in B_n$. Then $\partial b \in B_{n-1}$ is also in $\ker j$ since $j(\partial b) = \partial j(b) = \partial c = 0$.