380C PROBLEM SET 3

DUE WEDNESDAY, SEPTEMBER 15TH

Problem 1. Let p be a prime and let i be an integer. Let S_i denote the set of subsets of $\mathbb{Z}/p$ of order i. Note that $\mathbb{Z}/p$ naturally acts on S_i (via the action of $\mathbb{Z}/p$ on itself).

Show that the fixed point set $S_i^{\mathbb{Z}/p}$ is empty for $0 < i < p$. Deduce that p divides $\binom{p}{i}$.

Problem 2. Let $\varphi : G \to H$ be a homomorphism. Define the *kernel* $\mathrm{Ker}(\varphi)$ *of* φ as the subgroup $\{g \in G \mid \varphi(g) = 1\}$ of G.

(a) Show that $\mathrm{Ker}(\varphi)$ is a normal subgroup of G.
(b) Show that there exists a unique homomorphism $\widetilde{\varphi} : G/\mathrm{Ker}(\varphi) \to H$ fitting into a commutative diagram:

$$\begin{tikzcd} G \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ G/\mathrm{Ker}(\varphi) \arrow[r, "\widetilde{\varphi}"] & H. \end{tikzcd}$$

(c) Suppose that φ is surjective. Show that $\widetilde{\varphi}$ is an isomorphism.
(d) Show that the image $\mathrm{Image}(\varphi)$ is a subgroup of H. Show that $\widetilde{\varphi}$ induces an isomorphism $G/\mathrm{Ker}(\varphi) \simeq \mathrm{Image}(\varphi)$.

Problem 3. Let H be a subgroup of G. Let $N_G(H) \subseteq G$ denote the *normalizer* of H in G, which is the set $\{g \in G \mid gHg^{-1} = H\}$. Note that $N_G(H)$ is a subgroup of G that contains H as a normal subgroup, and is in fact the maximal subgroup of G in which H is normal.

Let $\pi : G \to G/H$ denote the projection map. Show that $g \in G$ lies in $N_G(H)$ if and only if $\pi(g)$ lies in the fixed point set $(G/H)^H$. Then show that the natural map $N_G(H)/H \to (G/H)^H$ is a bijection.

Problem 4. Let G be a finite group and let $g \in G$.

(a) Show that $g^{|G|} = 1$.
(b) Let $\mathrm{ord}(g)$ be the *order* of g, i.e., the minimal positive integer r such that $g^r = 1$. Show that r divides $|G|$.

Problem 5. Let G be a finite group of odd order. Show that the map:

$$G \to G$$
$$g \mapsto g^2$$

is a bijection.