Differential Geometry Notes

Simon Xiang

January 29, 2021

Notes for the Spring 2021 section of Differential Geometry (Math 365G) at UT Austin, taught by Dr. Bowen. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

January 25, 2021

1.1 Closed curves

Definition 1.1. We say $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is **T-periodic** (where $T > 0$) if $\gamma(T + t) = \gamma(t)$. We say γ is **closed** if it is T-period for some T.

A natural question to ask is whether or not we can parametrize level curves? You know what a gradient is.

Theorem 1.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x, y) \neq \vec{0}$ for all (x, y) with $f(x, y) = 0$. Then for all (x_0, y_0) with $f(x_0, y_0) = 0$, there exists a regular $\gamma\colon (\alpha, \beta) \to \mathbb{R}^2$ such that $\alpha < 0 < \beta$, $\gamma(0) = (x_0, y_0)$ and $f(\gamma(t)) = 0$ for all t.*

Note. The proof uses the inverse function theorem. Note that we can parametrize the entire curve under fairly broad conditions, that is, if $f^{-1}(0)$ is *connected* then we can choose γ to parametrize all of $f^{-1}(0)$.

Assume $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth. A **global inverse** is a map $G\colon \mathbb{R}^n \to \mathbb{R}^n$ with $F \circ G(\vec{x}) = \vec{x}$. A **local inverse** at $\vec{x}$ is a map $G\colon U_{\vec{x}} \to \mathbb{R}^n$ with $F \circ G(\vec{y}) = \vec{y}$ for all $\vec{y}$, where $U_{\vec{x}}$ is a neighborhood of $\vec{x}$. An **infinitesmal inverse** at $\vec{x}$ is a linear map A such that $(D_{\vec{x}}F) \circ A$ is the identity, where $D_{\vec{x}}F$ is the Jacobian matrix.

The Inverse Function Theorem. *If F is smooth and has an infinitesmal inverse at $\vec{x}$, then it has a smooth local inverse at $\vec{x}$.*

Theorem 1.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x, y)$ is not horizontal for all (x, y) with $f(x, y) = 0$, then there exists a regular $\gamma\colon (\alpha, \beta) \to \mathbb{R}^2$ with $\gamma(t) = (t, g(t))$ and $f(\gamma(t)) = 0$ (and $\gamma(0) = (x_0, y_0)$ like in the previous theorem).*

Proof. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x, y) = (x, f(x, y))$. Then

$$DF = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}, \quad \det(DF) = \frac{\partial f}{\partial y} \neq 0.$$

By the inverse function theorem, since DF is invertible, there exists a local smooth inverse G, where $F \circ G(x, y) = (x, y) = (G_1(x, y), f(G_1,(x, y), G_2(x, y)))$. This implies that $G_1(x, y) = x$, $f(x, G_2(x, y)) = y$. Define $\gamma(t) = (t, G_2(t, 0))$. Since F and G are smooth, γ is regular, so

$$f(\gamma(t)) = f(t, G_2(t, 0)) = 0.$$

Something happened here. ⊠

Lecture 2

January 27, 2021

2.1 Curvature

Definition 2.1. Assume γ is a unit-speed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$. Define the curvature by $\kappa(s) = \|\ddot{(}s)\| = \left\| \left(\frac{d^2}{ds^2}\gamma\right)(s) \right\|$.

Example 2.1. The circle curve $\gamma(t) = (R\cos t, R\sin t)$ is not unit speed. So $\gamma'(t) = (-R\sin t, R\cos t)$, and $\|\gamma'(t)\| = R$. The arclength $s(t) = \int_0^t R\,du = tR$, so $s^{-1}(t) = \frac{t}{R}$. A reparametrization is $\widetilde{\gamma}(t) = (R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)$.

Say $\gamma(s) = \left(R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)\right)$ for simplicity. Then $\dot{\gamma} = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right)\right)$, and $\ddot{\gamma} = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)$. So $\|\ddot{\gamma}\| = \kappa(s) = \frac{1}{R}$.

Parametrizing by arc length is painful. So we can define (if γ is regular) $\kappa = \frac{\|\dot{\kappa}\times\ddot{\kappa}\|}{\|\kappa\|^3}$. This makes life easier, since in this definition, $\kappa(t) = \kappa(s(t))$. What is a cross product?? Let $\vec{v}, \vec{w} \in \mathbb{R}^3$, then $\vec{v} \times \vec{w} \in \mathbb{R}^3$ as well. One way to find the cross product is by computing

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = (v_2 w_3 - v_3 w_2)\mathbf{i} + (-v_1 w_3 + v_3 w_1)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}.$$

The cross product is **bilinear**, that is, $(\mathbf{v}+\mathbf{u}) \times \mathbf{w} = \mathbf{v} \times \mathbf{w} + \mathbf{u} \times \mathbf{w}$, and satisfies homogeneity, and antisymmetric like the determinant. Also, $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, and finally we have the right hand rule.

We can simplify our old formula to $\frac{\|\ddot{\gamma}\| \sin\theta}{\|\dot{\gamma}\|^2}$.

Proof of the formula for curvature. Let $s(t) = \int_{t_0}^{t} \|\gamma'(u)\|\, du$ be the arc length of a curve, and $\widetilde{\gamma}(t) = \gamma(s^{-1}(t))$. So $\widetilde{\gamma}(s(t)) = \gamma(t)$. Then

$$\widetilde{\gamma}'(s(t))s'(t) = \gamma'(t) \implies \widetilde{\gamma}'(s(t)) = \frac{\gamma'(t)}{s'(t)}.$$

Then $\widetilde{\gamma}''(s(t))s'(t)^2 + \widetilde{\gamma}'(s(t))s''(t) = \gamma''(t)$ by the chain rule. So

$$\kappa(t) = \widetilde{\gamma}''(s(t)) = \frac{\gamma''(t) - \widetilde{\gamma}'(s(t))s''(t)}{s'(t)^2} = \frac{\gamma''(t) - \frac{\gamma'(t)}{s'(t)} \cdot s''(t)}{s'(t)^2}.$$

Recall that $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du$, so $s'(t) = \|\dot{\gamma}(t)\|$. We use inner products, now $s'(t)^2 = \|\dot{\gamma}(t)\|^2 = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$. So differentiating gives $2s'(t)s''(t) = 2\langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle$. Then $s''(t) = \frac{\langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle}{s'(t)}$. Plugging everything gives

$$\kappa(t) = \left\| \frac{\ddot{\gamma} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \frac{\dot{\gamma}\cdot\ddot{\gamma}}{\|\dot{\gamma}\|}}{\|\dot{\gamma}\|^2} \right\| = \left\| \left(\frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \frac{\dot{\gamma} \cdot \ddot{\gamma}}{\|\dot{\gamma}\|\|\ddot{\gamma}\|} \right) \right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \left\| \frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \cos\theta \right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \sin\theta \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2}.$$

⊠

Lecture 3

January 29, 2021

Whoops, showed up 30 min late because I was eating breakfast. We're talking about signed curvature.

3.1 Signed Curvature

The idea is to find a unit normal vector for each point on the curve. Then the signed curvature is positive if the curve bends in the direction of the unit normal vector field, and negative if it isn't.