Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

§1 Homology

The big boy has arrived. These notes will follow Hatcher §2.1.

Remark 1.1. This is something I heard even before I enrolled in this course. The homotopy groups are easy to define, but impossible to compute and work with. The homology groups take a lot of work to define, but the resulting groups are much nicer and easier to work with.

The fundamental group is a cool tool when dealing with low-dimensional spaces (the pride and joy of UT Austin), but it doesn't do well with higher dimensional spaces, for example, it can't distinguish between the n-spheres S^n for $n \geq 2$.We can get rid of this limitation by considering the higher homotopy groups $\pi_n(X)$, which are defined in terms of maps from the n-dimensional cube I^n and homotopies $I^n \times I \to X$ of such maps. Cool things about higher homotopy groups: for X a CW complex, $\pi_n(X)$ only depends on the $(n+1)$-skeleton, and $\pi_i(S^n) = 0$ for $i < n$ and $\mathbb{Z}$ for $i = n$, as expected. However, the drawback is that they're extremely difficult to compute in general— take the "simple" task of computing $\pi_i(S^n)$ for $i > n$.

Enter the homology groups $H_n(X)$. Similar to $\pi_n(X)$, $H_n(X)$ for X a CW complex depends only on the $(n+1)$-skeleton, and for the spheres $H_i(S^n) \simeq \pi_i(S^n)$ for $1 \leq i \leq n$, but the homology groups have the advantage in that $H_i(S^n) = 0$ for $i > n$. However, everything has a price. How exactly do we define these so called homology groups? We start by motivating, then doing simplicial homology, before moving onto singular homology. Most efficient method for computing homology groups is called cellular homology. We'll also talk about Mayer-Vietoris sequences, the analogue of the van Kampens for the fundamental group.

Something interesting about homology: most of the time we only use the basic properties of homology, not the definition itself. So we could almost invoke an axiomatic approach, which will happen soon. We could also skip the algebra and talk about geometry, but then Dr. Brand would be unhappy (and so would I), so we'll approach it with a mix of the two (talk about intuition first then state the axioms later).

§1.1 The big idea of homology (todo)

Issues with homotopy groups: things get really wacky because S^2 has no cells of dimension greater than 2, but some (infinitely many) of the higher homotopy groups $\pi_n(S^2)$ are nontrivial. $\langle$*god shattering star noises*$\rangle$ However, homology groups are (directly) related to cell structures, in that you can regard them as an algebraization of how cells of dimension n attach to cells of dimension $n-1$.

Imagine a circle with two antipodal points x and y, with four arrows a, b, c, d drawn in the direction from x to y, which we'll denote by X_1.

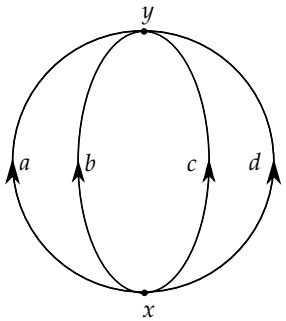

Figure 1: The graph X_1, consisting of two vertices and four edges.

Usually loops are nonabelian, so suppose we abelianize the loops. That is, the loops ab^{-1} and $b^{-1}a$ are "the same circle" (but with a different starting point), so we'll just say they're equal. Formally (not really), rechooisng the basepoint just permutes the letters cyclically, so by abelianizing we can cast off our silly worries about the basepoint. So we make the transition from loops (chosen basepoint) $\longrightarrow$ cycles (no chosen basepoint).

Now we abelian, and all the cool abelian groups use additive notation. So a cycle looks something like $a - b + c - d$ now, a linear combination of edges with integer coefficients. We'll call these linear combinations *chains* of edges. We can decompose these into cycles by several ways, eg $(a-c)+(b-d) = (a-d)+(b-c)$, so it's better just to say cycles are any LC of edges st at least one decomposition makes geometric sense. When is a chain a cycle? Cycles are distinguished by the fact that they enter and exit a vertex the same amount of times. So for an arbitrary chain $ka + lb + mc + nd$, it enters y about $k+l+m+n$ times (one for each thing) and enters x (or leaves it) $-k-l-m-n$ times. So if we want $ka + lb + mc + nd$ to be a cycle, we just need to require $k+l+m+n = 0$.

To generalize this, let C_1 be the free abelian group with a basis set $\{a, b, c, d\}$ (edges), and C_0 be the free abelian group with basis $\{x, y\}$ (vertices). Elements of C_1 are chains of edges, and elements of C_0 are linear combinations of

vertices. Define a homomorphism $\partial\colon C_1 \to C_0$ by sending each basis element to $y - x$, then $\partial(ka + lb + mc + nd) = (k + l + m + n)y - (k + l + m + n)x$, so cycles are precisely $\ker \partial$. It can be seen that $a - b$, $b - c$, and $c - d$ form a basis for $\ker \partial$, so every cycle in X_1 is a unique linear combination of these three elts. Basically, X_1 has three "holes", the three gaps in between the four edges.

Now let's attach a 2-cell to X_1 to get X_2, as seen below.

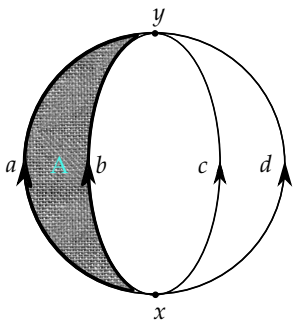

Figure 2: X_1 with a 2-cell attached, denoted X_2. Have you ever seen a 2-cell that looks like cloth?

The 2-cell is attached along the cycle $a - b$, forming the 2-skeleton X_2. Now the cycle is trivial (homotopically), which suggest we form a quotient by factoring out the subgroup generated by $a - b$. For example, $a - c$ and $b - c$ are now equivalent, since they're homotopic in X_2. Algebraically, we define a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the infinite cyclic group generated by A, and $\partial_2(A) = a - b$. ∂_1 is the boundary homomorphism, defined earlier. We are interested in $\ker \partial_1 / \operatorname{im} \partial_2$, that is, the 1-dimensional cycles modulo the boundaries (multiples of $a - b$). Remember, factor groups collapse everything we don't like to the identity. This quotient group is the *homology group* $H_1(X_2)$. If we were to talk about X_1, since it has no 2-cells C_2 is simply zero, so $H_1(X_1) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1$, which is free abelian on three generators. $H_1(X_2)$ is free abelian on two generators ($b - c$ and $c - d$), which expresses the geometric observation that there are two holes remaining after filling one of them in with the 2-cell A.

Let's go farther. Add another 2-cell to the pre-existing 2-cell A, to get the 3-complex X_3.

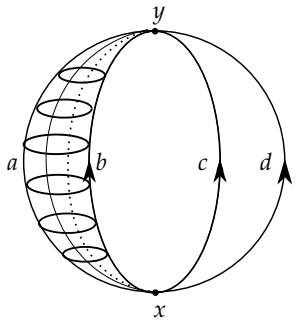

Figure 3: The 3-complex X_3, formed by attaching a 2-cell to X_2.

This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B, and the boundary homomorphism $\partial_2\colon C_2 \to C_1$ sends A, B to $a - b$. $H_1(X_3) = \ker \partial_1 / \operatorname{im} \partial_2 = H_1(X_2)$, but now ∂_2 has a nontrivial kernel (the infinite cyclic group generated by $A - B$). We view $A - B$ as a 2d cycle generating $H_2(X_3) = \ker \partial_2 \simeq \mathbb{Z}$. The second homology detects the 2d "hole" in X_3.

Unfortunately the diagrams will have to stop now, but let's go even farther and make the complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere by A and B, creating a chain group C_3 generated by C. The boundary homomorphism $\partial_3\colon C_3 \to C_2$ that sends C to $A - B$ should be seen as the boundary of C, similar to how $a - b$ is the boundary of A. Now we have a sequence of boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and $H_2(X_4) = \ker \partial_2 / \operatorname{im} \partial_3$ is now trivial. $H_3(X_4) = \ker \partial_3 = 0$, note that $H_1(X_4) = H_1(X_3) \simeq \mathbb{Z} \times \mathbb{Z}$, so this is the only homology group of X_4 that isn't trivial.

You can pretty much see where this is going. For a cell complex X, we have chain groups $C_n(X)$ free abelian with basis the n-cells of X, with boundary homomorphisms $\partial_n\colon C_n(X) \to C_{n-1}(X)$, by which we define the homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. So what's the problem? It's how to define ∂_n in general— for $n = 1$ this is easy, it's the vertex head minus the one at the tail. For $n = 2$, it still isn't hard per say, if the cell is attached on a loop of edges,

just take the cycle of edges, keeping in mind orientation. This is much tricker for higher dimension cells, even with restrictions to polyhedral cells and nice attaching maps we still have to worry about orientation and stuff.

So what do we do? Use triangles, of course. We can subdivide arbitary polyhedra into certain special types of polyhedra called simplices (what we talked about in class day 1), so there isn't any loss of generality (but there is a loss of efficiency). This gives rise to our more basic *simplicial homology*, which deals with cell complexes from simplices. However, we are still quite limited in what we can do.

So, what do we really do this time? Make things less simple, and make your life difficult by considering the collection of all possible continuous maps of simplices into a space X (wow). The chain groups $C_n(X)$ are tremendously large, but the quotients $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, the *singular homology groups*, are much smaller and easier to work with[1]. For example, in the examples above the singular homology groups coincide with the ones computed from cellular chains. Furthermore (as we will see later), singular homology lets us define these nice cellular homology groups for *all* cell complexes, which solves the issue of how to define boundary maps for cellular chains.

§1.2 Simplicial homology and Δ-complexes

I have a feeling we're gonna be typing a lot of `\Delta`'s. So basically, the only thing cool kids talk about is singular homology, but it's kinda complicated so we gotta talk about the inferior version for those who have the brain capacity of a literal ape[2], simplicial homology, first. We talk about simplicial homology in the domain of Δ-complexes. Take the standard fundamental polygons with orientation for $\mathbb{T}^2$, $\mathbb{R}P^2$, and the Klein bottle K.

[1] For reasonably "nice" spaces X, of course.

[2] The book simply says "primitive" version, so I used my imagination a little bit.