

Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html.

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§1 September 30, 2020

Last time: if we proved that if $n \geq 5$, then A_n is simple in S_n . Today we'll talk about products of groups. We've pretty much seen this before, let's breeze through it.

§1.1 Direct products

Recall direct products: we start with two group G_1 and G_2 . The cartesian product $G_1 \times G_2$ is the direct product of G_1 and G_2 , we associate this with a binary operation componentwise by letting

$$(g_1, g_2)(g'_1, g'_2) = (g_1g'_1, g_2g'_2).$$

You can verify that this is well defined and the group satisfies the group axioms. We often use these to understand bigger groups by letting them be the direct product of smaller groups. If H, K are two normal subgroups such that

$$\begin{cases} H \cap K = \{1_G\} \\ HK = G, HK := \{hk \mid h \in H, k \in K\} \end{cases}$$

as sets, then $G \simeq H \times K$. How do we show this? We need a map: simply take $(h, k) \mapsto hk$. To verify that it's a group homomorphism, we need normality: surjectivity is by the second condition, and injectivity follows from the fact that the intersection is trivial.

§1.2 Semidirect products

Now let's talk about semidirect products. These are a little more interesting, a generalization of direct products. Let G be a group and H, K two subgroups of G such that only one of them (say H) is normal in G , their intersection is the identity, and $HK = G$. Then G is the *semidirect product* of H and K in G , ie, $G \simeq H \rtimes K$. We could also write the conditions as

$$\begin{cases} H \trianglelefteq G \\ H \cap K = \{1_G\} \\ HK = G. \end{cases}$$

Remark 1.1. We have $H \rtimes K = H \times K$ as sets. How will define a multiplication on this set? We have

$$(h, k)(h', k') = hkh'k' = h(kh'k^{-1})kk' = (h(kh'k^{-1}), kk')$$

since $(kh'k^{-1}) \in H$ by the normality of H . Notice that the conjugation action of K on H determines the product operation on $H \rtimes K$. Here $\varphi: K \rightarrow \text{Aut } H, k \mapsto (h \mapsto khk^{-1})$ an automorphism. More generally, if we have two groups G_1, G_2 and a homomorphism $\varphi: G_2 \rightarrow \text{Aut } G_1$, then we can define the corresponding semidirect product $G_1 \rtimes G_2 = G_1 \times G_2$ (as sets) by

$$(g_1, g_2)(g'_1, g'_2) = (g_1\varphi(g_2)(g'_1), g_2g'_2).$$

From here, it's easy to see that if φ is the trivial map, then this is simply the direct product of the two sets, that is, $\varphi(G_2) = \{\text{id}_{G_1}\} \iff G_1 \rtimes G_2 \simeq G_1 \times G_2$.

We use this concept to understand bigger groups in terms of their subgroups. Consider groups of order pq where p, q are distinct primes. G has a p -Sylow P and a q -Sylow Q . If Q is normal, then $n_q \equiv 1 \pmod{q}, n_q \mid p \implies n_q = 1$. $|Q| = q \implies Q \simeq \mathbb{Z}/q\mathbb{Z}$ and $|P| = p \implies P \simeq \mathbb{Z}/p\mathbb{Z}$. Together, we have $Q \cap P = \{1_G\}, (|Q|, |P|) = 1, |QP| = \frac{|Q| \cdot |P|}{|Q \cap P|} = \frac{pq}{1}$. So we have $Q \trianglelefteq G, Q \cap P = \{1_G\}, QP = G$, and we conclude that $G = Q \rtimes P$.

To understand this fully we need to look at the homomorphisms $\varphi: P \rightarrow \text{Aut } Q$. We have

$$\varphi: \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/q\mathbb{Z}) \simeq (\mathbb{Z}/q\mathbb{Z})^*, (n \mapsto kn) \mapsto k,$$

where $q \nmid k$. This map is uniquely determined up to isomorphisms of $\mathbb{Z}/p\mathbb{Z}$ itself. We have two possibilities:

1. $q \not\equiv 1 \pmod{p}$. Then $\varphi(P) = \{1_G\}$. Hence $G \simeq Q \times P$.
2. $q \equiv 1 \pmod{p}$. Then φ ?? Something happened, but I gotta run to my next class.

§2 October 2, 2020

Today: we're finishing what we should have finished last week. I don't really know what's going on... should have paid more attention during the Sylow unit. We have seen that $n_p = 1$ or $n_q = 1$ for P or Q normal in G . Furthermore, $P \cap Q = \{1_G\}$ since $|P|$ and $|Q|$ are coprime. This implies that PQ has order $|P| \cdot |Q| = |G|$, and these imply that $G = PQ$. Together, we have that G is a semidirect product of P and Q (depending on which one is normal).

Example 2.1. If $|G| = 2^2 \cdot 3$ for G a group, let $p = 2$ and $q = 3$. Then P is a 2-Sylow implies P is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, Q a 3-Sylow implies $Q \cong \mathbb{Z}/3\mathbb{Z}$. We know that $n_2 = 1$ or 3 , $n_3 = 1$ or 4 , and $n_2 = 1$ or $n_3 = 1$. Then $n_2 = n_3 = 1 \implies P, Q \trianglelefteq G, P \cap Q = 1, PQ = G$. Together these imply that $G \cong P \times Q$. I'm not entirely sure what happened here either...