Differential Equations Notes

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Notes for Differential Equations (M 427J) with Dr. Tsishchanka at UT Austin. Only ever taken right before exams. Source code: `https://git.simonxiang.xyz/math_notes/file/freshman_year/differential_equations/master_notes.tex.html`.

Contents

Lecture 1

First Test Review

As you can see, I gave up taking notes for this class. It's no fun. I don't care about logistic equations or manually calculating things[1]. If I lived in a world where all I did was proofs, life would be much better. Alas, I have a test in two days, and this is not the case. So, here we are.

We didn't cover some basic stuff that everyone should know (variation of parameters, proving uniqueness-existence, Picard iteration, series solutions), you know, basically what I signed up for this class to learn. So we'll cover those later with proper sections in my free time.

1.1 First-order linear differential equations (homogeneous)

First order linear ODE's are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{1}$$

We solve the homogeneous case, $\frac{dy}{dt} + a(t)y = 0$ by (intuitively) dividing by y and writing $\frac{y'}{y} = \frac{dy}{dt}/y$ as $\frac{d}{dt} \ln|y(t)|$. Then it pretty much immediately follows that

$$y(t) = \exp\left(-\int a(t)\,dt\right).$$

1.2 Initial value problem homogeneous 1st order ODE

Above gives solution sets of infinite order. Sometimes engineers care about initial value problems, that is, we want to solve equations of the form

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0. \tag{2}$$

If we just follow the same steps as earlier and integrate with bounds, we get

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)\,ds\right).$$

1.3 Nonhomogeneous linear 1st order ODEs

They are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{3}$$

Multiply by a continuous $\mu(t)$ such that we have $\frac{dy}{dy}\mu(t) + a(t)\mu(t)y = \mu(t)b(t)$: if $\frac{d}{dt}\mu(t)y = \frac{d\mu}{dt}y + \frac{dy}{dt}\mu$, then simply replace the left half of the expression with this, and notice that they're equal if $\frac{d\mu(t)}{dt} = a(t)\mu(t)$. So $\mu(t) = c\exp\left(\int a(t)\,dt\right)$. Therefore we have

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t) \implies y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right),$$

which is the general solution.

[1] This was a horrible premonition...

1.4 Initial value nonhomogeneous linear 1st order ODE

We're given something that looks like

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0. \tag{4}$$

To solve this, literally just integrate on the bounds. We get that solutions are of the form

$$y = \frac{1}{\mu(t)}\left(y_0\mu(t_0) + \int_{t_0}^{t} \mu(s)b(s)\,ds\right).$$

1.5 Separable equations

They are of the form

$$\frac{dy}{dx} = g(x)f(y). \tag{5}$$

Because you can just do this: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$, where $h = f^{-1}$ given $f \neq 0$ on its domain. Nobody knows what a differential form actually is, but it's apparent how to solve it (nonrigorously).

1.6 The logistic equation

I hope this doesn't show up or I'm gonna lose my mind.

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}} \tag{6}$$

1.7 Second order linear homogenous differential equations

They are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{7}$$

By the existence-uniqueness theorem, there exists a unique solution $y(t)$ satisfying this ODE on an open interval (with given initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$). Let's define an operator by

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t).$$

This is just a natural transformation if we view maps as functors from $\mathbb{R}$ to $\mathbb{R}$ (category theory ftw). If $L[cy] = cL[y]$ and $L[y_1 + y_2] = L[y_1] + L[y_2]$ for $c \in \mathbb{R}$, $y_1, y_2 : \mathbb{R} \to \mathbb{R}$, we say L is a *linear operator*. You can verify that $L[y](t)$ defined above is linear. Clearly just solve for $L[y](t)$ and we get the solutions to the second-order ODE. Here's the useful thing: by this fact, we get that

$$c_1y_1(t) + c_2y_2(t)$$

is the general form of solutions to Equation (7), where $c_1, c_2 \in \mathbb{R}$ and y_1, y_2 are particular solutions to Equation (7). You can see this by evaluating $L[c_1y_1(t) + c_2y_2(t)]$ and applying linearity properties. In particular, *all* solutions to Equation (7) are of that form, by a quick application of the existence uniqueness theorem, given that the gradient vectors are linearly independent (checking this is just a quick calculation to see that the Wronskian is nonzero). We say $\{y_1, y_2\}$ is a *fundamental set* of solutions of Equation (7).

1.8 Second order homogeneous ODE constant coefficients

General method for constant coefficients: let's say they're of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{8}$$

where a, b, c are constants and a nonzero. Then just look at the characteristic polynomial $P(r) = ar^2 + br + c$, and examine the roots r_1, r_2 such that $(r - r_1)(r - r_2) = 0$. If $r_1 \neq r_2,\ r_1, r_2 \in \mathbb{R}$, then $e^{r_1 x}, e^{r_2 x}$ are LI solutions to Equation (8) so the general solution is of the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If $r_1 = r_2 = r,\ r_1, r_2 \in \mathbb{R}$, then e^{rx}, xe^{rx} are LI solutions and the general solution is of the form

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Finally, if $r_1 \in \mathbb{C}$ (that is, $r_1 = a + bi$) for $a, b \in \mathbb{R}$), then r_2 is the complex conjugate of r_1 (that is, $r_2 = \overline{r_1} = a - bi$) and the functions $e^{ax}\cos(bx)$, $e^{ax}\sin(bx)$ are LI solutions to Equation (8) and the general solution is of the form

$$y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx).$$

1.9 Nonhomogeneous second order ODEs

Let's turn our attention to the big boy, the nonhomogeneous second order differential equation given by

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \tag{9}$$

where the functions $p(t), q(t)$ and $g(t)$ are continuous on an open interval.

Theorem 1.1. *Every solution of Equation (9) is of the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t)$$

where y_1, y_2 are LI solutions to Equation (7), $\psi(t)$ is a particular solution to Equation (9), and c_1, c_2 are constants.

Proof. We need a lemma.

Lemma 1.1. *The difference of any two solutions of Equation (9) is a solution of Equation (7).*

Proof. If y_1, y_2 are two solutions of Equation (9), then $L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0$. ⊠

Now returning to the proof of the theorem, we know $y(t)$ is a solution of Equation (9) by definition. Then by Lemma 1.1, $\phi(t) = y(t) - \psi(t)$ is a solution of Equation (7). But since every solution of Equation (7) is of the form $c_1 y_1(t) + c_2 y_2(t)$, we have

$$y(t) = \phi(t) = \psi(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t).$$

⊠

1.10 The method of judicious guessing

Is this the actual name of the method? We try to guess solutions for equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t), \tag{10}$$

where $a, b, c \in \mathbb{R}$ and $g(t)$ is of a certain form, described below.

Case 1: The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1 t + \cdots + a_n t^n.$$

It can be shown that there is a solution of the form

$$\psi(t) = \begin{cases} A_0 + A_1 t + \cdots + A_n t^n, & c \neq 0, \\ t(A_0 + A_1 t + \cdots + A_n t^n), & c = 0,\ b \neq 0, \\ t^2(A_0 + A_1 t + \cdots + A_n t^n), & c = b = 0. \end{cases}$$

Case 2: The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}.$$

Then it can be shown that there is a particular solution of the form

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is not a solution of the homogeneous equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is a solution of the homogeneous equation, but } te^{\alpha t} \text{ is not,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ and } te^{\alpha t} \text{ are both solutions of the homogeneous equation.} \end{cases}$$

Equivalently, we have

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is not a solution of the characteristic equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is one of two distinct solutions of the characteristic,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is the only solution of the characteristic equation.} \end{cases}$$

Case 3: Let $\phi(t) = u(t) + iv(t)$ be a particular solution of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{i\omega t}.$$

All you have to do is look at the real and imaginary parts to get $\operatorname{Re}(\phi(t)) = u(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\cos(\omega t)$, and similarly $\operatorname{Im}(\phi(t)) = v(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\sin(\omega t)$.

Remark 1.1. To find solutions where the function on the right is of the form $e^{2t} + e^{-3t}$ or $t\sin t + e^t$ or something like that, simply find solutions to the two componenents and add them.

Lecture 2

Examples

2.1 Homogeneous 1st order ODE

Problem. *Find the general solution of*

$$\frac{dy}{dt} + 2ty = 0.$$

Solution. $y = c\exp\left(-\int 2t\,dt\right) = c\exp\left(-t^2\right)$. ■

2.2 Homogeneous first order ODE initial value

Problem. *Find the solution of*

$$\frac{dy}{dt} + (\sin t)y = 0, \quad y(0) = \frac{3}{2}.$$

Solution. $y = \frac{3}{2}\exp\left(-\int_0^t \sin t\,dt\right) = \frac{3}{2}\exp(\cos t - 1)$. ■

Problem. *Solve*

$$\frac{dy}{dt} + e^{t^2}y = 0, \quad y(1) = 2.$$

Solution. $y = 2\exp\left(-\int_1^t e^{t^2}\,dt\right)$. This function isn't integrable (to be precise, no closed form solution exists) so we're done. ■

2.3 Nonhomogeneous first order

Problem. *Solve*
$$\frac{dy}{dt} - 2ty = t.$$

Solution. Let $\mu(t) = \exp\left(\int -2t\,dt\right) = \exp\left(-t^2\right)$. So $\frac{d}{dt}y \cdot \exp\left(-t^2\right) = \exp\left(-t^2\right)t \implies y \cdot \exp(-t^2) = -\frac{1}{2}e^{-t^2} + c \implies y = -\frac{1}{2} + ce^{t^2}$. ∎

Problem. *Solve*
$$x\frac{dy}{dx} + y = \cos x,\ x > 0.$$

Solution. We have $\frac{dy}{dx} + \frac{y}{x} = \frac{\cos x}{x}$. So $\mu(x) = e^{|\ln(x)|} = x$ for all x strictly positive. Then $\frac{d}{dx}yx = x\frac{\cos x}{x} = \cos x$. So $xy = \sin x + c \implies y = \frac{\sin x}{x} + \frac{c}{x}$. ∎

2.4 Nonhomogeneous first order initial value

Problem. *Solve*
$$\frac{dy}{dt} + 2ty = t, \quad y(1) = 2.$$

Solution. We have $\mu(t) = e^{t^2}$. So $\frac{d}{dt}ye^{t^2} = te^{t^2} \implies ye^{t^2} = \frac{1}{2}e^{t^2} + c \implies y = \frac{1}{2} + ce^{-t^2}$. At $y(1) = 2$, we have $\frac{3}{2} = \frac{c}{e} \implies c = \frac{3}{2}e$. So the solution is $y = \frac{1}{2} + \frac{3}{2}e^{(-t^2+1)}$. ∎

Problem. *Solve*
$$\frac{dy}{dx} + xy = xe^{\frac{x^2}{2}}, \quad y(0) = 1.$$

Solution. Now $\mu(t) = e^{\frac{x^2}{2}}$. So $\frac{d}{dx}ye^{\frac{x^2}{2}} = xe^{x^2}$, and $ye^{\frac{x^2}{2}} = \frac{1}{2}e^{x^2} + c \implies y = \frac{1}{2}e^{\frac{x^2}{2}} + ce^{-\frac{x^2}{2}}$. At $y(0) = 1$, we have $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$, and the general solution is of the form $y = \frac{1}{2}e^{\frac{x^2}{2}} + \frac{1}{2}e^{-\frac{x^2}{2}} = \frac{1}{2}e^{\frac{x^2}{2}}\left(1 + e^{-x^2}\right)$. ∎

2.5 Separable equations

Problem. *Solve*
$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

Solution. $y^3 = x^3 + c \implies y = \sqrt[3]{x^3 + 3c}$. ∎

Problem. *Solve*
$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$

Solution. $\int 2y + \cos y\,dy = 2x^3 \implies y^2 + \sin y + c = 2x^3$. Now what? I think it's over. ∎

Problem. *Solve*
$$y' = x^2 y.$$

Solution. $\ln|y| = \frac{x^3}{3} + c \implies y = \pm e^{\frac{x^3}{3} + c} \implies y = Ce^{\frac{x^3}{3}}$. ∎

2.6 Second order homogeneous ODEs

Problem. *Find the solutions of*
$$\frac{d^2y}{dt^2} + y = 0.$$

Solution. Clearly two particular solutions are $y_1(t) = \cos t$, $y_2(t) = \sin t$, then by the existence uniqueness thm the general solution is of the form $y(t) = c_1\cos t + c_2\sin t$. ∎

Problem. *Calculate the Wronskian for* y_1, y_2.

Solution. Why am I doing this??? I have better things to do. ∎

2.7 Second order ODE constant coefficients

Problem. *Determine all solutions to the differential equation*

$$y'' + y' - 6y = 0$$

of the form e^{rx}.

Solution. $y' = re^{rx}, y'' = r^2e^{rx}$. So we have $e^{rx}(r^2 + r - 6) = 0$ for the differential equation. Clearly $r = 2, -3$ satisfy this equation, so the solutions are $y_1 = e^{2x}, y_2 = e^{-3x}$. These are LI, so the general solution is of the form $c_1e^{2x} + c_2e^{-3x}$. ∎

Problem. *Solve*

$$y'' + y = 0.$$

Solution. The characteristic is $r^2 + 1$, so $r_1 = i$ and $r_2 = -i$. Then solutions are of the form $c_1e^0\cos(1x) + c_2e^0\sin(1x) = c_1\cos x + c_2\sin x$. ∎

Problem. *Solve*

$$y'' + 6y' + 25y = 0.$$

Solution. The solutions to the characteristic polynomial $r^2+6r+25$ are simply $r = -3 \pm 4i$. So the general solution is of the form $c_1e^{-3x}\cos(4x) + c_2e^{-3x}\sin(4x)$. ∎

Problem. *Solve the following initial value problem:*

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, y'(0) = 4.$$

Solution. Clearly the general solution is of the form $c_1e^{-2x} + c_2xe^{-2x}$. At $y(0) = 1$, we have $1 = c_1$. Then $y' = -2e^{-2x} + c_2e^{-2x} + -2c_2xe^{-2x}$, so at $y'(0) = 4$ we have $4 = -2 + c_2$. So $c_2 = 6$, and the general solution is of the form $e^{-2x} + 6xe^{-2x}$. ∎

2.8 Nonhomogeneous second order ODEs

Problem. *Three solutions of some second-order nonhomogeneous ODE are*

$$\varphi_1(t) = t, \varphi_2(t) = t + e^t, \textit{ and } \varphi_3(t) = 1 + t + e^t.$$

Find the general solution of the equation.

Solution. By our lemma, $\varphi_2 - \varphi_1 = e^t$ and $\varphi_3 - \varphi_2 = 1$ are clearly LI solutions to the nonhomogeneous equation. Then the general solution is of the form $c_1+c_2e^t+t$. Our choices of φ_i don't really matter, just trust the theorems. ∎

Problem. *Three solutions of some second order nonhomogeneous linear ODE are*

$$\phi_1(t) = t^2,\ \phi_2(t) = t^2 + e^{2t}, \textit{ and } \phi_3(t) = 1 + t^2 + 2e^{2t}.$$

Find the general solution of the equation.

Solution. Just take the difference of two, it'll work out. ∎

2.9 Judicious guessing (nonhomogeneous second order with constant coefficients)

Problem. *Find a particular solution* $\psi(t)$ *of the equation*

$$L[y] = \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = t^2.$$

Solution. Since $c \neq 0$, we have a solution $\psi(t)$ of the form $A_0 + A_1t + A_2t^2$. Plug that in and solve for the constants. ∎

Problem. *Find a particular solution*

Lecture 3

Second Test Review

I really wish I had an RREF or eigen-blah calculator for the test.

3.1 Algebraic Properties of Solutions of Linear Systems

You can convert any n-th order differential equation to a system of n first order differential equations. Just let

$$x_1 = y,\ x_2 = y',\ x_3 = y''\ ,\cdots,\ x_n = y^{(n-1)}$$

and everything will work out. Yes, you can write things as matrices. The rest of the section just goes on and on about manually plugging stuff in and solving by methods from previous sections.

3.2 Vector Spaces

If anybody wasn't annoyed enough already by the ridiculous memorization/plug and chug approach to computational lower level mathematics, here are ten axioms you should memorize about vector spaces. For $u, v, w \in V$ and $a, b, c \in \mathbb{F}$ we have the following:

(i) $u + v \in V$

(ii) $cu \in V$

(iii) $u + v = v + u$

(iv) $(u + v) + w = u + (v + w)$

(v) $\exists 0 \in V \ni 0 + u = u$

(vi) $\forall u \in V\ \exists(-u) \in V \ni u + (-u) = 0$

(vii) $1 \cdot u = u$

(viii) $a(bu) = (ab)u$

(ix) $a(u + v) = au + av$

(x) $(a + b)u = au + bu$

For example, $\mathbb{R}^n$, $\mathbb{P}^n, \mathbb{R}, \mathbb{P}, \mathbb{R}^{\mathbb{R}}, \{0\}, \mathrm{GL}_n(\mathbb{F})$ etc are all vector spaces. You know what a subspace is. And clearly subspaces form vector spaces themselves. You can show weird things are spaces by considering them as subspaces of $\mathbb{R}^n$. Also, linear combinations of vectors of a space form a subspace (and therefore a space).

3.3 Dimension of a Vector Space

Definition 3.1 (Linear independence and dependence)**.** Vectors $v_1, \cdots, v_p$ are **linearly dependent** if there exist scalars $c_1, \cdots, c_p$ not all zero such that

$$c_1 v_1 + \cdots + c_p v_p = 0. \tag{11}$$

If the only solution to Equation (11) is the trivial one, that is, $c_i = 0$ for all i, then the vectors are said to be **linearly independent**.

To show things are LI or not, just reduce them: if there's a zero column, then dep, if you can get it into REF, then LI. This is the first method you learn. Note that vectors aren't LI iff one is a linear combo of the others, if there are only two this is equivalent to one being a scalar multiple of the other.

Definition 3.2 (Dimension)**.** The **dimension** of a vector space V, denoted by $\dim V$, is the order of any basis for V. Note that we can have zero dimensional vector spaces, for example take the trivial space $\{0\}$.

3.4 Applications of Linear Algebra to Differential Equations

Theorem 3.1 (Existence-uniqueness). *There exists exactly one solution to the IVP*

$$\dot{x} = Ax, \quad x(t_0) = x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}.$$

Theorem 3.2. *Let $x^{(1)}, x^{(2)}, \cdots, x^{(k)}$ be k solutions for $\dot{x} = Ax$. Then for some t_0 we have $x^{(1)}, \cdots, x^{(k)}$ LI solutions iff $x^{(1)}(t_0), \cdots, x^{(k)}(t_0)$ are LI vectors in $\mathbb{R}^n$.*

Remark 3.1. There is no square matrix A with constant entries such that $x^{(1)}(t)$, $x^{(2)}(t)$ are solutions of $\dot{x} = Ax$.

3.5 The Theory of Determinants

Do I even need to take notes? Recall that if A is triangular then we can just multiply along the diagonal (be lazy! –Dr. Tran). Swapping two rows (WLOG, since for columns note that $\det A = \det A^T$) gives a negative determinant, multiplying rows by a scalar k gives $k \det A$, adding two rows does nothing.

3.6 Solutions of Simultaneous Linear Equations

Do you know how to multiply matrices? Do you know about noncommutative rings? (OHO big scary) Do you know that cancellation only holds in integral domains (which $M_{n\times n}(\mathbb{R})$ isn't due to the existence of zero divisors)? OK good. Also, can you invert matrices?

Theorem 3.3 (Cramer's rule). *Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by*

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i \in \mathbb{N}.$$

To make sense of this, imagine replacing the column vector representing x_i in the matrix with the vector b and taking its determinant, then dividing by $\det A$.

3.7 Linear Transformations

If you've heard of 3Blue1Brown, then you probably know that matrices just represent linear transformations (maps) $\mathbb{R}^n \to \mathbb{R}^m$. Hence the name, transformations are linear, so $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all $u, v \in V$ and c scalars.

Theorem 3.4. *The equation $Ax = b$ has a unique solution if the columns of A are LI, and has either no solution or infinitely many solutions if the columns of A are linearly dependent.*

Elementary row operations are adding a multiple of a row to another, multiplying a row by a nonzero constant, and interchanging two rows. REF is where zero rows are at the bottom, leading entries are to the right of the leading entry of the row above it, and all entries in a column below a leading entry are zero. RREF is when the leading entries are all pivots (1), and each leading 1 is the only nonzero entry in its column.

Definition 3.3. A **pivot position** in a matrix is a location in A that corresponds to a leading 1 in the RREF of A. A pivot column is a column that contains a pivot, variables that correspond to pivot columns are **basic variables**, while the other variables are **free variables**. A system having free variables means infinitely many solutions.

Lemma 3.1. *The columns of an $n \times n$ matrix are LI iff $\det A \neq 0$.*

Proof. Here's a neat proof.

(1) The cols of A are LI iff $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$ by Theorem 3.4.

(2) The equation $Ax = b$ has a unique solution x for all b iff the linear transformation $T(x) = Ax$ has an inverse.

(3) This this is true iff A^{-1} exists.

(4) Which is true iff $\det A \neq 0$.

$\boxtimes$

Theorem 3.5. *The equation $Ax = b$ has a unique solution $x = A^{-1}b$ if $\det A \neq 0$. $Ax = b$ has either no solutions or infinitely many solutions if $\det A = 0$.*

Corollary 3.1. *The equation $Ax = 0$ has a nontrivial solution iff $\det A = 0$.*

3.8 The Eigenvalue-Eigenvector Method of Finding Solutions

Definition 3.4 (Eigenblah)**.** An **eigenvector** of an $n \times n$ matrix is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ, we say λ is an **eigenvalue** of A. The set of all solutions of the equation $(A - \lambda I) = 0$ is the **eigenspace** of A corresponding to λ. $\det(A - \lambda I)$ is the **characteristic polynomial** of A and $\det(A - \lambda I) = 0$ is the **characteristic equation** of A.

For $\dot{x} = Ax$, we want to find n LI solutions $x^1(t), \cdots, x^n(t)$.

Theorem 3.6. *We have $x(t) = e^{\lambda t}v$ a solution of $\dot{x} = Ax$ iff λ is an eigenvalue and v is an eigenvector of A.*

Proof. Exponential functions are invariant under reduction, so let's make an educated guess that $x(t) = e^{\lambda t}v$ is a solution. Since $\frac{d}{dt}e^{\lambda t}v = \lambda e^{\lambda t}v$ and $A(e^{\lambda t}v) = e^{\lambda t}Av$, we have $\dot{x} = \lambda e^{\lambda t}v = Ax = e^{\lambda t}Av$, so $Av = \lambda v$, hence $x(t) = e^{\lambda t}v$ is a solution of $\dot{x} = Ax$ iff λ is an eigenvalue and v is an eigenvector of A. $\boxtimes$

3.9 Complex Roots

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $v = v^1 + iv^2$, then $x(t) + e^{\lambda t}v$ is a complex valued solution of the differential equation $\dot{x} = Ax$.

Lemma 3.2. *Let $z(t) = x(t) + iy(t)$ be a complex valued solution of $\dot{x} = Ax$. Then $\operatorname{Re} z$ and $\operatorname{Im} z$ (AKA $x(t)$ and $y(t)$) are both real valued solutions of $\dot{x} = Ax$, and are LI.*

3.10 Equal Roots

What if the characteristic has a root with multiplicity greater than one? Suppose $A_{n\times n}$ has $k < n$ LI eigenvectors, then the differential equation $\dot{x} = Ax$ only has k LI solutions of the form $e^{\lambda t}$, and our goal is to find $n - k$ more LI solutions. This is how we'll do it: recall that $x(t) = e^{at}c$ is a solution of the scalar differential equation $\dot{x} = ax$ for every constant c. What we want: $x(t) = e^{At}v$ a solution for $\dot{x} = Ax$ for all constant vectors v. But e^{At} isn't defined if A is a matrix, turns out this isn't too bad of an issue. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, define $e^{At} := I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!} + \cdots$. This converges, and we can differentiate termwise to get

$$\frac{d}{dt}e^{At} = A = A^2 + \frac{A^3t^2}{2!} + \cdots + \frac{A^{n+1}t^n}{n!} + \cdots = Ae^{At},$$

which implies that $e^{At}v$ is a solution of $\dot{x} = Ax$ for every constant vector v, since $\frac{d}{dt}e^{At}v = Ae^{At}v = A(e^{At}v)$. Warning! $e^{At+Bt} = e^{At}e^{Bt} \iff AB = BA$. This leads us to our general algorithm for finding more solutions:

(1) Find all eigenvalues and eigenvectors of A: if there are n of them, we are done.

(2) If we only found $k < n$ eigenvalues and eigenvectors, then to find additional solutions, we pick an eigenvalue λ of A and find all vectors v such that $(A - \lambda I)^2 v = 0, \quad$ but $(A - \lambda I)v \neq 0$. For each such vector v,

$$e^{At}v = e^{\lambda t}e^{(A-\lambda I)t} = e^{\lambda t}\left[v + t(A - \lambda I)v\right]$$

is an additional solution of $\dot{x} = Ax$. The equation holds since $e^{At}v = e^{(A-\lambda I+\lambda I)t}v = e^{\lambda t}e^{(A-\lambda I)t}v$, which is true because the matrices commute. We do this for all eigenvalues of A.

(3) If we still don't have enough solutions, then we find all vectors v such that $(A-\lambda I)v^3 = 0$, but $(A-\lambda I)v^2 \neq 0$. For each such v,

$$e^{At}v = e^{\lambda t}\left[v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v\right]$$

is an additional solution of $\dot{x} = Ax$. The reason why this algorithm works is because if $(A-\lambda I)^m = 0$, then the series $e^{(A-\lambda I)t}v$ terminates after m terms. Indeed, if $(A-\lambda I)^m = 0$, then $(A-\lambda I)^{m+\ell}v = 0$ as well, since $(A-\lambda I)^{m+\ell}v = (A-\lambda I)^{\ell}\left[(A-\lambda I)^m v\right] = 0$. As a consequence, $e^{(A-\lambda I)t}v = v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^m v$, which implies that

$$e^{At}v = e^{\lambda} = e^{\lambda t}e^{(A-\lambda I)t}v = e^{\lambda t}\left[v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}v\right].$$

(4) We continue in this manner until, hopefully, we obtain n LI solutions.

3.11 Fundamental Matrix Solutions; e^{At}

If $x^1(t)\langle\cdots x^n(t)\rangle$ are n LI solutions of the differential equation $\dot{x} = Ax$, then every solution $x(t)$ can be written in the form $x(t) = c_1x^1(t) + c_2x^2(t) + \cdots + c_nx^n(t)$. Let $X(t)$ be the matrix whose columns are $x^1(t), \cdots, x^n(t)$. Then we can write the previous equation in the form $x(t) = X(t)c$, where $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Definition 3.5 (Fundamental matrix). A matrix $X(t)$ is called a **fundamental matrix solution** of $\dot{x} = Ax$ if its columns form a set of n LI solutions.

Theorem 3.7. *Let $X(t)$ be a fundamental matrix solution of the differential equation $\dot{x} = Ax$. Then $e^{At} = X(t)X^{-1}(0)$. In other words, the product of any fundamental solution with its inverse at $t = 0$ gives e^{At}.*

Lemma 3.3. *A matrix $X(t)$ is a fundamental matrix solution of $\dot{x} = Ax$ iff $\dot{X}(t) = AX(t)$ and* $\det X(0) \neq 0$. *We can rewrite the first condition to say that $A = \dot{X}(t)X^{-1}(t)$.*

Lecture 4

Examples

4.1 Algebraic Properties of Solutions of Linear Systems

Problem. *Convert the following differential equation into a system of two first order differential equations:*

$$4\frac{d^2y}{dt^2}+\frac{dy}{dt}+3y=0$$

Solution. Let $x_1 = y$, $x_2 = y'$. Then $x_1' = x_2$ and $x_2' = \frac{-x_2-3x_1}{4}$. ∎

Problem. *Convert the following IVP.*

$$y'''+(y')^2+3y=e^t; \quad y(0)=1,\ y'(0)=0,\ y''(0)=0.$$

Solution. Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Then $x_1' = x_2$, $x_2' = x_3$, $x_3' = e^t - x_2^2 - 3x_1$, given $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 0$. ∎

4.2 Vector Spaces

Problem. *For a, b scalars, show that the set of all matrices H of the form below is a vector space.*

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix},$$

so for all $h \in H$, h is a linear combination of vectors in $\mathbb{R}^4$ and is therefore a vector space. ∎

4.3 Dimension of a Vector Space

Example 4.1. Let V be the set of all solutions of the differential equation

$$\frac{d^2x}{dt^2}-x=0.$$

Since solutions are of the form $x(t) = c_1e^t + c_2e^{-t}$, we have that $x_1(t) = e^t$ and $x_2(t) = e^{-t}$ span V. Note that $\dim(V) = 2$.

4.4 Applications of Linear Algebra to Differential Equations

Example 4.2. The vectors $x^{(1)}(t) = \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix}$ and $x^{(2)}(t) = \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$ are LI since $x^{(1)}(0) = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$ and $x^{(2)}(0) = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$, which are not scalar multiples of each other.

4.5 The Theory of Determinants

zzz

4.6 Solutions of Simultaneous Linear Equations

Example 4.3. To solve $Ax = b$ where $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$, by Cramer's rule we have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-10}{10} = -1, \quad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = -\frac{10}{10} = -1 \implies x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

4.7 Linear Transformations

Example 4.4. You can show transformations are linear by encoding them with a matrix, for example, considering the mapping $T : x \mapsto \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 \end{bmatrix}$, we show this is linear by noting that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ encodes this transformation. Also, the standard matrix for clockwise rotation about the origin for an angle ϕ is given by $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

Problem. *When does the equation $Ax = 0$ have a nontrivial solution?*

$$A = \begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution. We want $\det A = 0$, since if not A would be invertible and so would uniquely have the trivial solution. So $\lambda = 1$. ■

4.8 The Eigenvalue-Eigenvector Method of Finding Solutions

Example 4.5. To find the general solution of $\dot{x} = Ax$ where $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$, note that $p(\lambda) = \lambda^2 - 6\lambda + 5$, so the eigenvalues are $\lambda = 1$ and $\lambda = 5$. The corresponding eigenvectors are then $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, therefore by Theorem 3.6 they're solutions, and since any LC of solutions are also solutions, we have that the solutions are of the form

$$x(t) = c_1 e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 e^t - 2c_2 e^{5t} \\ 3c_1 e^t + c_2 e^{5t} \end{bmatrix}.$$

4.9 Complex Roots

Example 4.6. To solve the IVP $\dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, note that the characteristic is $(1-\lambda)(\lambda^2 - \lambda + 2)$, and so the eigenvalues are $\lambda = 1, 1 \pm i$. For $\lambda = 1$, the corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and so $x^1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the differential equation. For $\lambda = 1 + i$, an eigenvector is $\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$, so $x(t) = e^{(1+i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

is a complex-valued solution of $\dot{x} = Ax$. Now $e^{(1+i)t} = e^t(\cos\theta + i\sin\theta)$, so we can rewrite this solution as

$$\begin{bmatrix} 0 \\ ie^t(\cos\theta + i\sin\theta) \\ e^t(\cos\theta + i\sin\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ -e^t\sin\theta + ie^t\cos\theta \\ e^t\cos\theta + ie^t\sin\theta \end{bmatrix} = e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + ie^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}.$$

By Lemma 3.2, $x^2(t) = e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$ and $x^3(t) = e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$. The solutions $x^i(t)$ for $i \in \{1,2,3\}$ are LI since $x^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x^2(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $x^3(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ which are clearly LI (they form a standard basis for $\mathbb{R}^3$). So a solution must be of the form

$$x(t) = c_1e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + c_3e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}.$$

At $t = 0$, note that $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so we conclude that

$$x(t) = e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} = e^t\begin{bmatrix} 1 \\ \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{bmatrix}.$$

Just pretend that I didn't switch around t and θ for the entire problem.

4.10 Equal Roots

To solve $\dot{x} = Ax$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, note that in the characteristic $(1-\lambda)^2$, the eigenvalue 1 has a multiplicity of two, and the eigenvector corresponding to it is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $x^1(t) = e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is one solution. Applying step (2) of the algorithm, we want to find all solutions for $(A-\lambda I)^2v = 0$ with $\lambda = 1$: note that $A-\lambda I$ is a zero divisor when squared, so we conveniently choose $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, since it isn't a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and hence doesn't satisfy the equation $(A-\lambda I)v = 0$. Therefore,

$$x^2(t) = e^{At}v = e^{\lambda t}[v + t(A-\lambda I)v] = e^t\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

is a second solution of $\dot{x} = Ax$. So $x(t) = c_1x^1(t) + c_2x^2(t) = c_1e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2e^t\begin{bmatrix} 1 \\ 2t \end{bmatrix} = \begin{bmatrix} c_2e^t \\ c_1e^t + 2c_2te^t \end{bmatrix}$.

Example 4.7. We want to find three LI solutions for $\dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. The characteristic is $(1-\lambda)^2(2-\lambda)$, so the solutions are $\lambda = 1$ with a multiplicity of two, and $\lambda = 2$. So $x^1(t) = e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and

$x^2(t) = e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix}$ are two LI solutions. Now we want to find all solutions for $(A-\lambda I)^2 v = 0$, let $\lambda = 1$. Then

$$(A-\lambda I)^2 = \begin{bmatrix}(1-\lambda)^2 & 2(1-\lambda) & 0\\ 0 & (1-\lambda)^2 & 0\\ 0 & 0 & (2-\lambda)^2\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

when evaluated. Then for $v = \begin{bmatrix}0\\1\\0\end{bmatrix}$ this is zero but not zero at $(A-\lambda I)v$, since v is LI with the other eigenvectors. Therefore

$$x^3(t) = e^{At}v = e^{\lambda t}\left[v + t(A-\lambda I)v\right] = e^t\begin{bmatrix}t\\1\\0\end{bmatrix}$$

is a third solution for the equation $\dot{x} = Ax$, and so a general solution is of the form

$$c_1 e^t\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_3 e^t\begin{bmatrix}t\\1\\0\end{bmatrix} = \begin{bmatrix}c_1 e^t + c_3 t e^t\\ c_3 e^t\\ c_2 e^{2t}\end{bmatrix}.$$

4.11 Fundamental Matrix Solutions; e^{At}

Example 4.8. To find a fundamental matrix solution for the system

$$\dot{x} = \begin{bmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{bmatrix}x,$$

since we have

$$e^t\begin{bmatrix}-1\\4\\1\end{bmatrix}, \quad e^{3t}\begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{and} \quad e^{-2t}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

three LI solutions, it can be seen that the fundamental matrix is

$$X(t) = \begin{bmatrix}-e^t & e^{3t} & -e^{-2t}\\ 4e^t & 2e^{3t} & e^{-2t}\\ e^t & e^{3t} & e^{-2t}\end{bmatrix}.$$

Example 4.9. We'll show that $X(t) = e^{2t}\begin{bmatrix}4 & 1+4t\\ 2 & 2t\end{bmatrix}$ is a fundamental matrix solution for some A, and we'll also determine A. Now $X(0) = \begin{bmatrix}4 & 1\\ 2 & 0\end{bmatrix}$ and thus has nonzero determinant. If $X(t)$ is a fundamental matrix solution of $\dot{x} = Ax$, then $A = \dot{X}(t)X^{-1}(t)$. We have

$$\dot{X}(t) = \left(e^{2t}\right)'\begin{bmatrix}4 & 1+4t\\ 2 & 2t\end{bmatrix} + e^{2t}\begin{bmatrix}4' & (1+4t)'\\ 2' & (2t)'\end{bmatrix} = 2e^{2t}\begin{bmatrix}4 & 3+4t\\ 2 & 1+2t\end{bmatrix}, \quad X^{-1}(t) = -\frac{1}{2e^{2t}}\begin{bmatrix}2t & -1-4t\\ -2 & 4\end{bmatrix}.$$

So

$$
\begin{aligned}
A &= \dot{X}(t)X^{-1}(t) \\
&= 2e^{2t}\begin{bmatrix} 4 & 3+4t \\ 2 & 1+2t \end{bmatrix} \cdot \left(-\frac{1}{2e^{2t}}\right)\begin{bmatrix} 2t & -1-4t \\ -2 & -4 \end{bmatrix} \\
&= -\begin{bmatrix} 4 & 3+4t \\ 2 & 1+2t \end{bmatrix}\begin{bmatrix} 2t & -1-4t \\ -2 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}.
\end{aligned}
$$