Abstract Algebra Lecture Notes

Simon Xiang

Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html`.

Contents

§1 September 2, 2020

Last time: Homomorphisms, Isormorphisms, Automorphisms, trivial maps.

§1.1 The Symmetric Group Rises from the Automorphism Group

Example 1.1 (Group of Automorphisms)**.** Let X be a finite set. Let

$$S_x := \{f \colon X \to X \mid f \text{ is bijective}\}$$

Bijections on X preserve X: think of this set as the *group of automorphisms* on X, defined as $\mathrm{Aut}(X)$. The group operation is simply function composition. Then the identity element is the identity map, and the inverse of any $f \in S_x$ is $f^{-1} \in S_x$.

Assume that $g \colon X \to Y$ is a bijection. Then g gives rise to a homomorphism $\phi_g \colon S_y \to S_x$, $f \mapsto g^{-1}fg$. Verify that this map is well defined and a group homomorphism. Is ϕ_g an isomorphism? If $\phi_g^{-1} \colon S_x \to S_y$ were well-defined, then ϕ_g is a bijection. Consider $S_y(\phi_g) \to S_x(\phi_g^{-1}) \to S_y$, $f \mapsto g^{-1}fg \mapsto g(g^{-1}fg)g^{-1} = (gg^{-1})f(gg^{-1}) = f$. So $\phi_{g^{-1}} \colon S_x \to S_y$, $h \mapsto (g^{-1})^{-1}fg^{-1} = gfg^{-1}$.

Conclusion. Two finite sets X, Y have the same cardinality if there exists a bijection $g \colon X \to Y$. This bijection gives rise to the map $\phi_g \colon S_y \to S_x$ an isomorphism, so the group of automorphisms S_x depends only on the size of the group (when X is a finite set). Let $|X| = n$, then $S_x \simeq S_n$.

§1.2 On the Symmetric Group S_n

A cycle in S_n: $(\alpha_x, ..., \alpha_k)$ is a k-cycle. $\alpha_1, ..., \alpha_k \in \{1, ..., n\}$, $a_i \neq a_j$ $\forall i \neq j$. We have

$$(\alpha_1, ..., \alpha_k)(m) = \begin{cases} m & \text{if } m \neq \alpha_i \ \forall i = 1, ..., k \\ \alpha_{i+1} & \text{if } m = \alpha_i,\ i \in \{1, ..., k-1\} \\ \alpha_1 & \text{if } m = \alpha_k. \end{cases}$$

Definition 1.1 (Transpositions)**.** A *transposition* is a 2-cycle in S_n, denoted

$$(\alpha_1 \alpha_2),$$

where $\alpha_1 \neq \alpha_2$.

Definition 1.2. Two cycles $(\alpha_1, ..., \alpha_k)(\beta_1, ..., \beta_m)$ are *disjoint* if $\alpha_i \neq \beta_j$ for all $i \in \{1, ..., k\}$, $j \in \{1, ..., m\}$. Disjoint cycles commute, that is,

$$(\alpha_1, ..., \alpha_k)(\beta_1, ..., \beta_m) = (\beta_1, ..., \beta_m)(\alpha_1, ..., \alpha_k)$$

Lemma 1.1. *Every element $s \in S_n$ can be written uniquely (up to reordering) as a product of disjoint cycles.*

Proof. **Step 1**: Let $s \in S_n$. If $s = \mathrm{id}_{\{1,\dots,n\}}$, then $s = 1_{S_n}$. We have $s \neq 1_{S_n} \implies I_0(\neq \emptyset) := \{1 \leq k \leq n,\, s(k) \neq k\}$. Define $k_1 := \min I_0$. Then

$$\iota_1 := (k_1\, s(k_1)\, s^2(k_1)\, ...)$$

is an e_1-cycle where

$$\begin{cases} s^{e_1}(k_1) = k_1 \\ e_1 = \min\{d \in \mathbb{N} \mid s^d(k_1) = k_1\}. \end{cases}$$

Step 2: Now

$$I_1 = I_0 \setminus \{k_1, ..., s^{e_1}(k_1)\}.$$

If $I_1 = \emptyset$, we are done: $s = c$. If $I_1 \neq \emptyset$: $k_2 = \min I_1$. Set $\iota_2 = (k_2\, s(k_2)\, ...)$ an e_2-cycle where $s^{e_2}(k_2) = k_2)$, $e_2 = \min\{d \in \mathbb{N} \mid s^d(k_2) = k_2\}$.

Note. c_1, c_2 are disjoint cycles.

Step 3: $I_2 = i_1 \setminus \{k_2, s(k_2), ..., s^{e_2 - 1}(k_2)\}$. If $I_2 = \emptyset$ then we are done, verify $s = c_1 c_2$. If $I_2 \neq \emptyset$ then $k_3 = \min I_2$. Repeat the steps until $I_j = \emptyset \implies s = c_1 ... c_j$ disjoint cycles by construction. Verify the uniqueness in your free time. ⊠

Note. We will show next time that every finite group is a subgroup of S_n for some $n \in \mathbb{N}$ (Cayley's Theorem). This shows the importants of permutation groups: they contain all the information you need to know about groups.