Abstract Algebra Homework

Simon Xiang

Homework for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it (ergo the homework is semi-optional). The course follows Dummit and Foote (Third Edition). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_homework.tex.html`.

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§1.1 Problem 8 Section 1.1

Problem. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.

(a) Prove that G is a group under multiplication (roots of unity).

(b) Prove that G is not a group under addition.

§1.2 Problem 9

Problem. Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.

(a) Prove that G is a group under addition.

(b) Prove that the nonzero elements of G are a group under multiplication.

§1.3 Problem 25

Problem. Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

§1.4 Problem 5 Section 1.3

Problem. Find the order of (1 12 8 10 4)(2 13)(5 11 7)(6 9).

§1.5 Problem 15

Problem. Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition. [Use Exercise 10 and Exercise 24 of Section 1.]

§1.6 Problem 8 Section 1.4

Problem. Show that $\mathrm{GL}_n(F)$ is non-abelian for any $n \geq 2$ and any F.

§1.7 Problem 10

Problem. Let $G = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c, \in \mathbb{R},\ a \neq 0,\ c \neq 0\}$.

(a) Compute the product of $\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & c_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_2 & b_2 \\ 0 & c_2 \end{smallmatrix}\right)$ to show that G is closed under matrix multiplication.

(b) Find the matrix inverse of $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ and deduce that G is closed under inverses.

(c) Deduce that G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(d) Prove that the set of elements of G whos two diagonal entries are equal (i.e., $a = c$) is also a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

§1.8 Problem 5 Section 2.1

Problem. Prove that G cannot have a subgroup H with $|H| = n-1$, where $n = |G| > 2$.

§1.9 Problem 6

Problem. Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup* of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

§1.10 Problem 12

Problem. Let A be an abelian group and fix some $n \in \mathbb{Z}$. Prove that the following sets are subgroups of A:

(a) $\{a^n \mid a \in A\}$,

(b) $\{a \in A \mid a^n = 1\}$.

§1.11 Problem 13

Problem. Let H be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element x of H. Prove that $H = 0$ or $\mathbb{Q}$.

§1.12 Problem 2 Section 2.3

Problem. If x is an element of the finite group G and $|x| = |G|$, prove that $G = \langle x \rangle$. Give an explicit example to show that this result need not be true if G is an infinite group.

§1.13 Problem 3

Problem. Find all generators for $\mathbb{Z}/48\mathbb{Z}$.

§1.14 Problem 12

Problem. Prove that the following groups are *not* cyclic.

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$

(b) $\mathbb{Z}_2 \times \mathbb{Z}$

(c) $\mathbb{Z} \times \mathbb{Z}$.

§1.15 Problem 16

Problem. Assume $|X| = n$ and $|y| = m$. Supposed that x and y *commute*: $xy = yx$. Prove that $|xy|$ divides the least common multiple of m and n. Need this be true if x and y do *not* commute? Give an example of commuting elements x, y such that the order of xy is not equal to the least common multiple of $|x|$ and $|y|$.

§1.16 Problem 20

Problem. Let p be a prime and n be a positive integer. Show that if x is an element of the group G such that $x^{p^n} = 1$ then $|x| = p^m$ for some $m \leq n$.

§1.17 Problem 14 Section 2.4

Problem. A group H is called *finitely generated* is there is a finite set A such that $H = \langle A \rangle$.

(a) Prove that every finite group is finitely generated.

(b) Prove that $\mathbb{Z}$ is finitely generated.

(c) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. [If H is a finitely generated subgroup of $\mathbb{Q}$, show that $H \leq \langle \frac{1}{k} \rangle$, where k is the product of all the denominators which appear in a set of generators for H.]

(d) Prove that $\mathbb{Q}$ is not finitely generated.

§1.18 Problem 15

Problem. Exhibit a proper subgroup of $\mathbb{Q}$ that is not cyclic.

§1.19 Problem 3 Section 3.1

Problem. Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

§1.20 Problem 4

Problem. Prove that in the quotient group G/N, $(gN)^{\alpha} = g^{\alpha}N$ for all $\alpha \in \mathbb{Z}$.

§1.21 Problem 14

Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

(b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

(c) Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.

§1.22 Problem 24

Problem. Prove that $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

§1.23 Problem 8 Section 3.2

Problem. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.