# Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html.

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## §1 October 5, 2020

Last time: I missed a section on group extensions. I hope they're similar to field extensions, and splitting fields.

### §1.1 Composition series of groups

Preview: Jordan Holder theorem.

**Definition 1.1** (Composition series)**.** Let $G$ be a group. Then the *composition series* of $G$ is a sequence of subgroups such that

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = 1,$$

where $G_i/G_{i+1}$ is simple for all $i$. The $G_i/G_{i+1}$ are called *composition factors* of $G$, and $r$ is the length of the composition series.

Question: do we know the composition series exist? Are they unique? Some information about the composition series are their length ($r$) and their composition factors. The existence is obvious once you think about it: take $G_1$ a maximal normal subgroup, that is, $G_1$ is not contained in a normal subgroup $H \trianglelefteq G$ such that $H \neq G$. Then $G_2$ is a maximal normal subgroup of $G_1$, etc. The reason why we want $G_1$ maximal is so that the factor group $G_0/G_1$ is simple. So we know that the composition series exists. What about uniqueness? That turns out to fail.

**Example 1.1.** Take $S_4 \triangleright A_4 \triangleright \{1, (12)(34), (13)(24), (14)(23)\}$ (note that two transpositions have order 2, so the last group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ since there are four elements of order 2). From here, you can find two distince composition series $\{1, (12)(34)\} \triangleright 1$ and $\{1, (13)(24)\} \triangleright 1$, showing that uniqueness of composition series does not hold.

### §1.2 The Jordan-Hölder theorem

**Theorem 1.1** (Jordan-Hölder theorem)**.** *Any two composition series of $G$ are* equivalent*: that is, they have the same length and the same composition factors up to reordering.*

*Proof.* We do this proof by induction. Recall the second isomorphism theorem, which says that if we have $A$ and $B$ are subgroups of a group $G$ such that one of them (say $B$) is normal in $G$, then $AB \leq G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \simeq A/A \cap B$. Say we have two composition series of $G$ such that

$$\begin{tikzcd}
A_0 = G \arrow[d, dash] & B_0 = G \arrow[d, dash] \\
A_1 \arrow[d, dash] & B_1 \arrow[d, dash] \\
A_2 & B_2 \\
\vdots & \vdots \\
A_r = 1 & B_s = 1
\end{tikzcd}$$

Assume $r \leq s$. Use induction on $\min(r, s)$. If $r = 1$, then $G$ is simple implies $s = 1$ (this is the base case). Now assume $r > 1$: if $A_1 = B_1$, then we can use the induction hypothesis on $A_1 = B_1 = G$. Now if $A_1 \neq B_1$, then $A_1B_1 = A_0$ or $B_0 = G$ by the maximality of $A_1$, since $A_1B_1 \trianglelefteq G$. Define $C_2 = A_1 \cap B_1$, then we construct an intermediate series as follows:

$$\begin{tikzcd}
A_0 = G \arrow[d, dash] & & B_0 = G \arrow[d, dash] \\
A_1 \arrow[dr, dash] & & B_1 \arrow[dl, dash] \\
\vdots & & \vdots \\
A_r = 1 & C_2 \arrow[d, dash] & B_s = 1 \\
& C_3 & \\
& \vdots & \\
& C_t = 1 &
\end{tikzcd}$$

Note that $A_1/C_2 \simeq A_1B_1/B_1 \simeq B_0/B_1$, $B_1/C_2 \simeq A_0/A_1$. Use the induction hypothesis to compare the branch of $A_1 = C_1$ into $A_2$ and $C_2$, which implies $r = t$ and $A_i/A_{i+1}$ corresponds to $C_j/C_{j+1}$ up to reordering, for $i, j \geq 2$. $\langle visible\ confusion \rangle$: similarly, we can do the same thing with the branch $B_1$ into $C_2$ and $B_2$, all the way down to $C_r = 1$ and $B_s = 1$. Then by the induction hypothesis, $r = s$ and $C_j/C_{j+1} \simeq B_i/B_{i+1}$, where $\{j_i \mid i = 1 \cdots r\} = \{1 \cdots r\}$. $\boxtimes$

For finite simple groups, this tells us nothing. Are we talking about the classification of finite simple groups now?? They have been classified up to isomorphism as $\mathbb{Z}/p\mathbb{Z}$ for $p$ a prime, $A_n$ for $n \geq 5$, group of Lie type $\mathrm{PSL}(n, q)$ (the quotient by diagonal matrices), and 26 sporadic groups.

## §1.3 Solvable groups

From Dr. Ciperiani's point of view, these are the most beautiful groups. For $G$ finite, we have $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = 1$, $G_i/G_{i+1}$ simple. $G$ is *solvable* if and only if $G_i/G_{i+1}$ is cyclic, which implies they're isomorphic to cyclic groups of prime order, since they are simple. An equivalent definition is that $G$ has a subnormal series with abelian quotients, ie for $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = 1$, $G_i/G_{i+1}$ abelian. The other direction is really easy if we have the classification of finitely generated abelian groups.