# Complex Analysis Lecture Notes

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These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. I was also unhappy with the textbook, so some supplementary notes from different texts are found at the bottom of the document. Since I took these live in class, there are many mistakes and gaps: if you have questions, comments, corrections, etc, feel free to email them to me at simonxiang@utexas.edu.

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Lecture 1

## Actual notes

I want to do some real math! These notes will follow Stein and Shakarchi §1.2.

### 1.1 Continuous functions

We've already seen the standard epsilon-delta definition of continuity. An equivalent definition is the sequential definition, that is, for every sequence $\{z_1, z_2, \cdots\} \subseteq \Omega \subseteq \mathbb{C}$ such that $\lim z_n = z_0$, then $f$ is continuous at $z_0$ if $\lim f(z_n) = f(z_0)$. Since the notions for convergence of complex numbers and $\mathbb{R}^2$ is the same, $f$ of $z = x + iy$ is continuous iff it's continuously viewed as a function of two real variables $x$ and $y$. If $f$ is continuous, then the real valued function defined by $z \mapsto |f(z)|$ is clearly continuous (by the triangle inequality).

We say $f$ attains a **maximum** at the point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in \Omega.$$

The definition of a minimum is what you think it is.

**Theorem 1.1.** *A continuous function on a compact set $\Omega$ is bounded and attains a maximum and minimum on $\Omega$.*

*Proof.* Same as the any one you'd find in a Real Analysis course. ⊠

## 1.2 Holomorphic functions

Let's talk about the good stuff. Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then $f$ is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, such that the quotient is well-defined. This limit is called the **derivative of** $f$ and $z_0$, and is denoted by

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Note that $h$ approaches 0 from any direction. The function $f$ is said to be **holomorphic on** $\Omega$ if $f$ is holomorphic at every point of $\Omega$. If $C \subseteq \mathbb{C}$ is closed, then $f$ is **holomorphic on** $C$ if $f$ is holomorphic on some open set containing $C$. Finally, if $f$ is holomorphic on $\mathbb{C}$ then $f$ is said to be **entire**. Sometimes the terms **regular** or **complex differentiable** are used in place of holomorphic, but holomorphic functions are much much nicer than real values differentiable functions. Furthermore, every holomorphic function is analytic (that is, it has a power series expansion near every point), and so we also use the term **analytic** to refer to holomorphic functions. Once again, things are not as nice in Real Analysis, with infinitely differentiable real valued functions not having power series expansions.

**Example 1.1.** We have any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ entire, and $f(z) = \frac{1}{z}$ holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$. However, $f(z) = \frac{\overline{z}}{z}$ is not entire, as $\frac{f(z_0+h)-f(z_0)}{h} = \frac{\overline{h}}{h}$, which has no limit as $h \to 0$.

If we write the definition of a holomorphic function as $f$ being holomorphic at $z_0 \in \mathbb{C}$ iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0+h) - f(z_0) - ah = h\psi(h),$$

where $\psi$ is a function defined for all "small" $h$, and $\lim_{h\to 0} \psi(h) = 0$, we can see that $f$ is holomorphic implies $f$ is continuous (clearly $a = f'(z_0)$). The basic stuff, distribution over addition, product rule, quotient rule, chain rule, yada yada all apply.

## 1.3 Cauchy-Riemann equations and the Jacobian

OK, here's the difference between real and complex valued functions again. In terms of real variables, $f(z) = \overline{z}$ corresponds to $F : (x, y) \to (x, -y)$, which is differentiable in the real sense, its derivative at a point being the map corresponding to its Jacobian. In fact, $F$ is linear and is equal to its derivative at any point, and is therefore infinitely differentiable. So a complex valued function having a real derivative need not imply the complex valued function is holomorphic. We can associate complex valued functions $f = u + iv$ to the mapping $F(x, y) = (u(x, y), v(x, y))$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Recall that $F(x, y)$ is differentiable at a $P_0 = (x_0, y_0) \in \mathbb{R}^2$ if there exists a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P_0+H) - F(P_0) - J(H)|}{|H|} \to 0 \quad \text{as } |H| \to 0, H \in \mathbb{R}^2.$$

We could also write

$$F(P_0+H) - F(P_0) = J(H) + |H|\Psi(H),$$

where $|\Psi(H)| \to 0$ as $|H| \to 0$. Very similar to what just happened above. Then the transformation $J$ is unique and called the **derivative** of $F$ at $P_0$. Given that $F$ is differentiable and the partial derivatives exist, we have

$$J = J_F(x,y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

**Note.** In the case of complex differentiation, the derivative is a complex number $f'(z_0)$, but for real derivatives, it's a matrix.

However, there is a way to link these two notions. Let's consider the limit of $h \in \mathbb{R}$, that is, $h = h_1 + ih_2$ with $h_2 = 0$. Then if $z_0 = x_0 + iy_0$, we have

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0). \end{aligned}$$

Similarly, for $h \in \mathbb{C} \setminus \mathbb{R}$, say $h = ih_2$ ($h$ is purely imaginary), we have

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0+h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i}\frac{\partial f}{\partial y}(z_0). \end{aligned}$$

So if $f$ is holomorphic, then we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Now $f = u + iv$, so $\partial f/\partial x = \partial u/\partial x + i\partial v/\partial x$, and similarly $\partial f/\partial y = \partial u/\partial y + i\partial v/\partial y$. Separate the real and imaginary parts and note that $1/i = -i$, then we get

$$\begin{aligned} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{1}{i}\left(\frac{\partial u}{\partial y}\right) + \frac{1}{i}\cdot i\left(\frac{\partial v}{\partial y}\right) \implies \\ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \end{aligned}$$

which implies the following nontrivial relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

The relations described in Equation (1) are known as the **Cauchy-Riemann** equations, which link real and complex analysis. Here we state the converse of the Cauchy-Riemann theorems in an important theorem.

**Theorem 1.2.** *Suppose $f = u + iv$ is a complex-valued function on an open set $\Omega$. If $u$ and $v$ are continuously differentiable and satisfy the Cauchy-Riemann equations for all $\omega \in \Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \partial f/\partial z$.*

*Proof.* Recall that $u(x+h_1, y+h_2) - u(x,y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h)$ and $v(x+h_1, y+h_2) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h)$, where $\psi_j(h) \to 0$ (for $j \in \{1,2\}$) as $|h|$ tends to 0, and $h = h_1 + ih_2$. Then by the Cauchy-Riemann equations we have

$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi(h),$$

where $\psi(h) = \psi_1(h) + \psi_2(h) \to 0$, as $|h| \to 0$. Therefore $f$ is holomorphic and $f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$. ⊠

## 1.4 Power series

I was wondering when we were going to cover these? The prime example of a power series is the complex **exponential** function, defined for $z \in \mathbb{C}$ by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

When $z$ is real, this is the same as the usual exponential function, and the series converges absolutely for all $z \in \mathbb{C}$. To see this, note that $\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}$, so compare $|e^z|$ to $\sum |z|^n/n! = e^z < \infty$. This shows that the series defining $e^z$ is uniformly convergent in every disk in $\mathbb{C}$. We'll prove that $e^z$ is entire, and that its derivative can be found by termwise differentiation of the power series expansion. Hence

$$(e^z)' = \sum_{n=0}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z$$

for $m = n-1$, which shows that $e^z$ is its own derivative. In contrast, the geometric series $\sum_{n=0}^{\infty} z^n$ converges absolutely only in the disk $|z| < 1$, and the sum is equal to $\frac{1}{1-z}$, which is holomorphic in the punctured plane $\mathbb{C} \setminus \{1\}$. The proof is the same as the one for real variables: observe that

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z},$$

then note that if $|z| < 1$ we have $\lim_{N\to\infty} z^{N+1} = 0$.

**Definition 1.1** (Power series). In general, a **power series** is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$.

Testing for absolute convergence involves looking at the sum $\sum_{n=0}^{\infty} |a_n||z|^n$, and if $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for some $z_0$, then it converges for all $z$ in the disk $|z| \le |z_0|$. Now we prove a theorem that there always exists an open disk (possibly empty)[1] on which the power series converges absolutely.

**Theorem 1.3.** *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that*

*(i) If $|z| < R$ the series converges absolutely.*

*(ii) If $|z| > R$ the series diverges.*

*Moreover, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$, then $R$ is given by Hadamard's formula*

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}.$$

We say that $R$ is the **radius of convergence** of the power series, and the region $|z| < R$ is the **disc of convergence** . In particular, $R = \infty$ for exp, and $R = 1$ for the geometric series.

*Proof.* Let $L = 1/R$, and suppose that $L \neq 0, \infty$[2]. If $|z| < R$, choose a small $\varepsilon > 0$ such that

$$(L+\varepsilon)|z| = \frac{1+\varepsilon R}{R} \cdot \text{“}R\text{”} = r < 1.$$

By the definition of $L$, we have that $|a_n|^{\frac{1}{n}} \le L + \varepsilon$ for all large $n$, therefore

$$|a_n||z|^n \le \{(L+\varepsilon)|z|\}^n = r^n.$$

Comparing this with the geometric series $\sum r^n$ shows that $\sum a_n z^n$ converges. If $|z| > R$, then a similar argument shows that there exists a sequence in terms of the series whose absolute value goes to infinity, so the series diverges. ⊠

[1] Wait, this theorem is pretty lame then.

[2] These two cases are exercises for the reader.

**Remark 1.1.** Things are much more delicate when we deal with the boundary of the disk of convergence, where $|z| = R$. Here, we can have either convergence or divergence.

More examples include the **trigonometric functions**, defined by

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

which agree with the usual cosine and sine of a real argument when $z \in \mathbb{R}$. A simple calculation yields

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

These are the **Euler formulas** for cosine and sine.

**Theorem 1.4.** *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. The derivative of $f$ is also a power series obtained by differentiating term by term the series for $f$, that is,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

*Moreover, $f'$ has the same radius of convergence as $f$.*

*Proof.* To show that $f'$ has the same radius of convergence as $f$, refer to Hadamard's formula. Indeed, $\lim_{n\to\infty} n^{1/n} = 1$, and therefore

$$\limsup |a_n|^{\frac{1}{n}} = \limsup |n a_n|^{\frac{1}{n}},$$

so $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence, and therefore so do $\sum a_n x^n$ and $\sum n a_n z^{n-1}$. Now for the first assertion, we need to show that

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

gives a description of $f'$. Let $R$ be the radius of convergence of $f$, and suppose that $|z_0| < r < R$. Write $f(z) = S_N(z) + E_N(z)$, where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Some crazy stuff happened, and we'll omit the rest of the proof. ⊠

**Corollary 1.1.** *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

**Definition 1.2** (Power series not centered at the origin)**.** We can define a power series centered at $z_0 \in \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The disk of convergence is now centered at $z_0$ and the radius is still given by Hadamard's formula. In fact, if $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f$ is simply obtained by translating $g$, where $f(z) = g(w)$, $w = z - z_0$. So everything we know about $g$ works for $f$, in particular, by the chain rule we have

$$f'(z) = g'(w) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

A function $f$ defined on an open set $\Omega$ is **analytic** (or has a **power series expansion**) if at a point $z_0 \in \Omega$ there exists a power series $\sum a_n (z - z_0)^n$ centered at $z_0$, with positive radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z$ in a nbd of $z_0$. If $f$ has a power series expansion for all $z_0 \in \Omega$, then $f$ is **analytic on $\Omega$**. By Theorem 1.4, an analytic function on $\Omega$ is also holomorphic there. Next chapter we prove a deep theorem that every holomorphic function is analytic, which is why we use the terms holomorphic and analytic interchangeably.

## 1.5 Integration along curves

**Definition 1.3** (Parametrized curve)**.** A **parametrized curve** is a function $z(t)\colon [a,b] \to \mathbb{C}$, where $a,b \in \mathbb{R}$. We say a parametrized curve is **smooth** if $z'(t)$ exists and is continuous on $[a,b]$ and $z'(t) \neq 0$ for $t \in [a,b]$.

At the endpoints $t = a$ and $t = b$, we interpret $z'(a)$ and $z'(b)$ as the one-sided limits

$$z'(a) = \lim_{h \to 0, h>0} \frac{z(a+h) - z(a)}{h} \quad \text{and} \quad \lim_{h \to 0, h<0} \frac{z(b+h) - z(b)}{h}.$$

We call these the right-hand derivative of $z(t)$ at $a$ and the left-hand derivative of $z(t)$ at $b$, respectively[3]. Similarly, we say the parametrized curve is **piecewise-smooth** if $z$ is continuous on $[a,b]$, and there exist points $\{a_0, a_1, \cdots, a_n\}$ such that

$$a = a_0 < a_1 < \cdots < a_n = b,$$

where $z(t)$ is smooth on $[a_k, a_{k+1}]$ for $1 \leq k \leq n$. This differs from the standard definition of a smooth curve in that the right and left hand derivatives at $a_k$ may differ for some $1 \leq k \leq n-1$. Why don't we just say parametrizations are just paths (like in the definition of path-connectedness)? That would save me a lot of typing, because the word "path" is much shorter than the word "parametrization"[4].

We say two parametrizations $z\colon [a,b] \to \mathbb{C}$ and $\widetilde{z}\colon [c,d] \to \mathbb{C}$ are **equivalent** if there exists a continouously differentiable bijection $s \mapsto t(s)$ from $[c,d]$ to $[a,b]$ such that $t'(s) > 0$ and $\widetilde{z}(x) = z(t(s))$. The fact that we require the derivative to be positive says that the orientation is preserved: as $s$ walks on the path from $c$ to $d$, $t(s)$ walks on the path from $a$ to $b$. The family of all parametrizations that are equivalent to $z(t)$ determine a **smooth curve** $\gamma \subseteq \mathbb{C}$, which is the image of $[a,b]$ under $z$ with the given orientation. We can define a curve $\gamma^-$ obtained from $\gamma$ by reversing the orientation, for example consider the parametrization $z^-\colon [a,b] \to \mathbb{R}^2$ defined by

$$z^-(t) = z(b + a - t).$$

We can also define a **piecewise-smooth curve** in the same way: let $z(a)$ and $z(b)$ be the end-points of the curve (independent of parametrization). Then $\gamma$ begins at $z(a)$ and ends at $z(b)$. A curve is **closed** if $z(a) = z(b)$ for any parameterization (it forms a loop), and is **simple** if it isn't self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$ (of course, we make an exception if it's closed). From now on, we'll call a piecewise-smooth curve a **curve** for brevity, since we don't really care if it's not piecewise-smooth.

**Example 1.2.** The standard example of a curve is a circle. Consider the circle

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

The **positive orientation** (counterclockwise) is given by $z(t) = z_0 + re^{it}$ while the **negative orientation** is given by $z(t) = z_0 + re^{-it}$, for $t \in [0, 2\pi]$. When we talk about circles $C$, we'll usually be talking about the positively oriented circle.

Now let's talk about integration along curves! A key theorem says that if a complex valued function is holomorphic in the interior of a closed circle $\gamma$, then

$$\int_\gamma f(z)\,dz = 0.$$

A version of this theorem is called *Cauchy's Theorem*, which we'll talk about later. Given a smooth curve $\gamma \subseteq \mathbb{C}$ parametrized by $z\colon [a,b] \to \mathbb{C}$ and $f$ a continuous function on $\gamma$, we define the **integral of $f$ along $\gamma$** by

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

[3] Intuitively, shouldn't it be the other way around? Like left-hand derivative corresponds to $a$, since it's on the "left-hand side" of the interval...
[4] Thank you for the observation, very cool.

How do we know this doesn't depend on the parametrization of $\gamma$? Say $\widetilde{z}$ is an equivalent parametrization of $z$, then

$$\int_a^b f(z(t))z'(t)\,dt = \int_c^d f(z(t(s)))z'(t(s))t'(s)\,ds = \int_c^d f(\widetilde{z}(s))\widetilde{z}'(s)\,ds,$$

proving that $\int_\gamma f(z)\,dz$ is well-defined. Now if $\gamma$ is piecewise-smooth, given a piecewise-smooth parametrization $z(t)$ we have

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,dt.$$

We can also define the **length** of the smooth curve $\gamma$ as $\text{length}(\gamma) = \int_a^b |z'(t)|\,dt$. Apply the same arguments as before to get that $\text{length}(\gamma)$ is parametrization independent and that if $\gamma$ is piecewise smooth, $\text{length}(\gamma)$ is the sum of the lengths of the smoooth components.

**Proposition 1.1.** *Let $\gamma$ be a curve, and $f, g$ be functions. Then*

1. *Integration is a linear operation, that is, for $\alpha, \beta \in \mathbb{C}$ we have*

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$

2. *For $\gamma^-$ the curve representing the reverse orientation of $\gamma$, we have*

$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$

3. *The following inequality holds:*

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma)$$

*Proof.* Basically, do it yourself.

1. Follows from the definition and linearity of the Riemann integral.
2. Exercise for the reader.
3. Note that

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\,dt \le \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma).$$

⊠

Suppose $f$ is define on some open set $\Omega \in \mathbb{C}$. A **primitive** for $f$ on $\Omega$ is a function $F$ that's holomorphic on $\Omega$ such that $F'(z) = f(z)$ for all $z \in \Omega$. Now let's look at the Fundamental Theorem of Calculus again.

**Theorem 1.5.** *If a continuous function $f$ has a primitive $F$ in $\Omega$, and $\gamma$ is a curve in $\Omega$ starting at $\omega_1$ and ending at $\omega_2$, then*

$$\int_\gamma f(z)\,dz = F(\omega_2) - F(\omega_1).$$

*Proof.* Good thing we have the FTC from Real Analysis. If $z(t)\colon [a,b] \to \mathbb{C}$ is a parametrization of $\gamma$ with $z(a) = \omega_1$ and $z(b) = \omega_2$, then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b F'(z(t))z'(t)\,dt \\ &= \frac{d}{dt}F(z(t))dt \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

Clearly we're done if $\gamma$ is smooth. If $\gamma$ is only piecewise-smooth, then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\ &= F(z(a_n)) - F(z(a_0)) \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

⊠

**Corollary 1.2.** *If $\gamma$ is a closed curve in some open $\gamma \in \mathbb{C}$ and $f$ is continuous and has a primitive in $\Omega$, then*

$$\int_\gamma f(z)\,dz = 0.$$

*Proof.* Note that if $\gamma$ is a closed curve, then $\omega_1 = \omega_2$. ⊠

**Example 1.3.** We can use this corollary to show that functions don't have primitives. For example, $f(z) = 1/z$ doesn't have a primitive in $\mathbb{C} \setminus \{0\}$, since if $C$ is the unit circle parametrized by $z(t) = e^{it}$ when $0 \le t \le 2\pi$, we have

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i \ne 0.$$

**Corollary 1.3.** *If $f$ is holomorphic on a region $\Omega$ and $f' = 0$, then $f$ is a constant function.*

*Proof.* Let $\omega_0 \in \Omega$. We WTS that $f(\omega) = f(\omega_0)$ for all $\omega \in \Omega$: since $\Omega$ is connected we can find a curve (path) joining $\omega_0$ and $\omega$. Clearly $f$ is a primitive for $f'$. Then

$$\int_\gamma f'(z)\,dz = f(\omega) - f(\omega_0) = 0$$

by assumption, which implies that $f(\omega) = f(\omega_0)$, finishing the proof. ⊠