

Miscellaneous Notes on Differentiable Manifolds

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These notes cover a variety of topics related to or required for the study of smooth manifolds, taken over Winter break 2020-2021 in preparation for my geometry overload next semester. Source files: https://git.simonxiang.xyz/math_notes/files.html

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Part I

Euclidian Spaces

Lecture 1

Smooth Functions on a Euclidian Space

INTRODUCTION

Calculus talks about differentiation and integration on \mathbb{R} , while real analysis extends this to \mathbb{R}^n . Vector calculus talks about integrals on curves and surfaces, and now we extend these concepts to higher dimensions, the structures which we work with are called manifolds. Things become simple: gradient, curl, and divergence are cases of the exterior derivative, and the FTC for line integrals, Green's theorem, Stokes' theorem, and the divergence theorem are manifestations of the generalized Stokes' theorem.

Manifolds arise even when dealing with the space we live in, for example the set of affine motions in \mathbb{R}^3 is a 6-manifold. This is our plan: recast calculus on \mathbb{R}^n so we can generalize it to manifolds by differential forms. Working in \mathbb{R}^n first isn't necessary, but much easier, since the examples are simple. Then, we define a manifold and talk about tangent spaces, working with the idea of approximating nonlinear things with linear things, with Lie groups and Lie algebras as examples. Finally, we do calculus on manifolds, generalizing the theorems of vector calculus, with the de Rham cohomology groups as C^∞ and topological invariants.

1.1 C^∞ Versus Analytic Functions

Let's talk about C^∞ functions on \mathbb{R}^n . Write a base for \mathbb{R}^n as x^1, \dots, x^n and let $p = (p^1, \dots, p^n)$ be a point in an open set U in \mathbb{R}^n . Differential geometry uses *superscripts*, not *subscripts*, more on this later.

Definition 1.1. Let k be a nonnegative integer. A function $f : U \rightarrow \mathbb{R}$ is C^k at p if its partial derivatives $\frac{\partial^j f}{\partial x^{i_1} \dots \partial x^{i_j}}$ of all orders $j \leq k$ exist and are continuous at p . The function $f : U \rightarrow \mathbb{R}$ is C^∞ at p if it is C^k for all $k \geq 0$, that is, its partial derivatives of all orders

$$\frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}$$

exist and are continuous at p . We say f is C^k on U if it is C^k for all points in U , and the concept of C^∞ on a set U is defined similarly. When we say "smooth", we mean C^∞ .

Example 1.1.

- (i) We call C^0 functions on U continuous on U .
- (ii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^{1/3}$. Then $f'(x)$ is $\frac{1}{3}x^{-2/3}$ for $x \neq 0$ and undefined at zero, so f is C^0 but not C^1 at $x = 0$.
- (iii) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(x) = \int_0^x f(t) dt = \int_0^x t^{1/3} dt = \frac{3}{4}x^{4/3}.$$

Then $g'(x) = f(x) = \frac{1}{3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In general, we can construct functions that are C^k but not C^{k+1} at a point.

- (iv) Polynomials, the sine and cosine functions, and the exponential functions on \mathbb{R} are all C^∞ .

A function f is **real-analytic** at p if in some neighborhood of p it is equal to its Taylor series at p , that is,

$$f(x) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \dots$$

Real-analytic functions are C^∞ because you can differentiate them termwise in their region of convergence. The converse does not hold: define

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0; \\ 0 & \text{for } x \leq 0. \end{cases}$$

We can show f is C^∞ on \mathbb{R} and the derivatives $f^{(k)}(0) = 0$ for all $k \geq 0$ by induction, then the Taylor series must be zero in any neighborhood of the origin, but f is not. Then f isn't equal to its Taylor series, and we have a smooth non-analytic function.

1.2 Taylor's Theorem with Remainder

However, we have a Taylor's theorem with remainder for C^∞ functions that's good enough. Say a subset S of \mathbb{R}^n is **star-shaped** with respect to a point p in S if for every $x \in S$, the line segment from p to x lies in S .

Lemma 1.1 (Taylor's theorem with remainder). *Let f be a C^∞ function on an open subset U of \mathbb{R}^n star-shaped with respect to a point $p = (p^1, \dots, p^n)$ in U . Then there are C^∞ functions $g_1(x), \dots, g_n(x)$ on U such that*

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Proof. For any $x \in U$ the line segment $p + t(x - p)$, $0 \leq t \leq 1$ lies in U . So $f(p + t(x - p))$ is defined, and by the chain rule we have

$$\frac{d}{dt} f(p + t(x - p)) = \sum (x^i - p^i) \frac{\partial f}{\partial x^i}(p + t(x - p)).$$

Integrating both sides with respect to $t \in [0, 1]$ we have

$$f(p + t(x - p)) \Big|_0^1 = \sum (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt.$$

Let $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt$. Then $g_i(x)$ is C^∞ and the above expression simplifies to

$$f(x) - f(p) = \sum (x^i - p^i)g_i(x).$$

Furthermore, $g_i(p) = \int_0^1 \frac{\partial f}{\partial x^i}(p) dt = \frac{\partial f}{\partial x^i}(p)$. □

If $n = 1$ and $p = 0$, this lemma says that $f(x) = f(0) + xf_1(x)$ for a C^∞ function $f_1(x)$. Applying repeatedly gives $f_i(x) = f_i(0) + xf_{i+1}(x)$, where f_i, f_{i+1} are C^∞ functions. So

$$\begin{aligned} f(x) &= f(0) + x(f_1(0) + xf_2(x)) \\ &= f(0) + xf_1(0) + x^2(f_2(0) + xf_3(x)) \\ &\vdots \\ &= f(0) + f_1(0)x + f_2(0)x^2 + \dots + f_i(0)x^i + f_{i+1}(x)x^{i+1}. \end{aligned}$$

If we differentiate this expression k times, we get $f^{(k)}(0) = k!f_k(0)$, which simplifies to $f_k(0) = \frac{1}{k!}f^{(k)}(0)$ for $k = 1, 2, \dots, i$. Note that balls are star-shaped, and since U is open there exists an $\varepsilon > 0$ such that $p \in B(p, \varepsilon) \subseteq U$. So when a function's domain is restricted to $B(p, \varepsilon)$, f is defined on a star-shaped neighborhood of p and Taylor's theorem with remainder applies.

Tangent Vectors in \mathbb{R}^n as Derivations

Vectors at a point p are usually represented by columns of points or arrows stemming from p . A vector at p is tangent to a surface if it lies in the tangent plane at p , the limiting position of the secant planes through p . This kind of definition assumes we live in \mathbb{R}^n and would not work for a large class of manifolds. We will find a generalization that works for manifolds.

2.1 The Directional Derivative

We usually visualize the tangent space $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ as the vector space of all arrows emanating from p . This can be identified with the vector space \mathbb{R}^n . We write points as $p = (p^1, \dots, p^n)$ and vectors v in $T_p(\mathbb{R}^n)$ as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{or} \quad \langle v^1, \dots, v^n \rangle.$$

We denote the standard basis for \mathbb{R}^n or $T_p(\mathbb{R}^n)$ by $\{e_1, \dots, e_n\}$. Then $v = \sum v^i e_i$. Sometimes we denote $T_p(\mathbb{R}^n)$ by $T_p \mathbb{R}^n$. Elements of $T_p(\mathbb{R}^n)$ are called **tangent vectors** (or simply **vectors**) at p in \mathbb{R}^n .

The line through a point $p = (p^1, \dots, p^n)$ with direction $\langle v_1, \dots, v_n \rangle$ in \mathbb{R}^n has parametrization $c(t) = (p^1 + tv^1, \dots, p^n + tv^n)$, with i th component $c^i(t) = p^i + tv^i$. If f is C^∞ in a neighborhood of p in \mathbb{R}^n and v is a tangent vector at p , the **directional derivative** of f in the direction v at p is defined to be

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t)).$$

By the chain rule,

$$D_v f = \sum_{i=1}^n \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Note that $D_v f$ is a number, since we evaluate partial derivatives at a point p . We write $D_v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p$ for the operator that sends a function f to the number $D_v f$. Often times we omit the subscript p .

2.2 Germs of Functions

If two functions agree on some neighborhood of a point p , they will have the same directional derivatives at p . This suggests introducing an equivalence relation on the C^∞ functions defined in some neighborhood of p . Consider the set of pairs (f, U) , where U is a neighborhood of p and $f : U \rightarrow \mathbb{R}$ is a C^∞ function. We say (f, U) is equivalent to (g, V) if there is an open set $W \subseteq U \cap V$ containing p such that $f = g$ when restricted to W . The equivalence class of (f, U) is called the **germ** of f at p . We write $C_p^\infty(\mathbb{R}^n)$ or simply C_p^∞ if there is no possibility of confusion, for the set of all germs of C^∞ functions on \mathbb{R}^n at p .

Example 2.1. The functions $f(x) = \frac{1}{1-x}$ with domain $\mathbb{R} \setminus \{1\}$ and $g(x) = \sum_{n=0}^{\infty} x^n$ with domain $(-1, 1)$ have the same germ at any point p in the open interval $(-1, 1)$.

An **algebra** over a field K is a vector space A over K with a multiplication map $\mu : A \times A \rightarrow A$, usually written $\mu(a, b) = a \times b$, such that we all $a, b, c \in A$ and $r \in K$,

- (i) $(a \times b) \times c = a \times (b \times c)$ (associativity),
- (ii) $(a + b) \times c = a \times c + b \times c$ and $a \times (b + c) = a \times b + a \times c$ (distributivity),
- (iii) $r(a \times b) = (ra) \times b = a \times (rb)$ (homogeneity).

Equivalently, an algebra over a field K is a ring A also a K -vector space such that ring multiplication satisfies the homogeneity condition. So an algebra has three operations: the addition and multiplication of a ring, and the scalar multiplication of a vector space. Usually we write ab in place of $a \times b$.

Addition and multiplication of functions induce operations on C_p^∞ , making it into an algebra over \mathbb{R} .

2.3 Derivations at a Point

A map $L : V \rightarrow W$ between vector spaces over a field K is called a **linear map** or a **linear operator** if for any $r \in K$ and $u, v \in V$,

(i) $L(u + v) = L(u) + L(v)$;

(ii) $L(rv) = rL(v)$.

Such a map can also be called *K-linear*.