Differential Equations Notes

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Notes for Differential Equations (M 427J) with Dr. Tsishchanka at UT Austin. May follow lectures or textbooks depending on my mood. Source code: `https://git.simonxiang.xyz/math_notes/file/freshman_year/differential_equations/master_notes.tex.html`.

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§1 First Test Review

As you can see, I gave up taking notes for this class. It's no fun. I don't care about logistic equations or manually calculating things. If I lived in a world where all I did was proofs, life would be much better. Alas, I have a test in two days, and this is not the case. So, here we are.

We didn't cover some basic stuff that everyone should know (variation of parameters, proving uniqueness-existence, Picard iteration, series solutions), you know, basically what I signed up for this class to learn. So we'll cover those later with proper sections in my free time.

§1.1 First-order linear differential equations (homogeneous)

First order linear ODE's are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{1}$$

We solve the homogeneous case, $\frac{dy}{dt} + a(t)y = 0$ by (intuitively) dividing by y and writing $\frac{y'}{y} = \frac{dy}{dt}/y$ as $\frac{d}{dt}\ln|y(t)|$. Then it pretty much immediately follows that

$$y(t) = \exp\left(-\int a(t)\,dt\right).$$

§1.2 Initial value problem homogeneous 1st order ODE

Above gives solution sets of infinite order. Sometimes engineers care about initial value problems, that is, we want to solve equations of the form

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0. \tag{2}$$

If we just follow the same steps as earlier and integrate with bounds, we get

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)\,ds\right).$$

§1.3 Nonhomogeneous linear 1st order ODEs

They are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{3}$$

Multiply by a continuous $\mu(t)$ such that we have $\frac{dy}{dy}\mu(t) + a(t)\mu(t)y = \mu(t)b(t)$: if $\frac{d}{dt}\mu(t)y = \frac{d\mu}{dt}y + \frac{dy}{dt}\mu$, then simply replace the left half of the expression with this, and notice that they're equal if $\frac{d\mu(t)}{dt} = a(t)\mu(t)$. So $\mu(t) = c\exp\left(\int a(t)\,dt\right)$. Therefore we have

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t) \implies y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right),$$

which is the general solution.

§1.4 Initial value nonhomogeneous linear 1st order ODE

We're given something that looks like

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0. \tag{4}$$

To solve this, literally just integrate on the bounds. We get that solutions are of the form

$$y = \frac{1}{\mu(t)}\left(y_0\mu(t_0) + \int_{t_0}^{t} \mu(s)b(s)\,ds\right).$$

§1.5 Separable equations

They are of the form

$$\frac{dy}{dx} = g(x)f(y). \tag{5}$$

Because you can just do this: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$, where $h = f^{-1}$ given $f \neq 0$ on its domain. Nobody knows what a differential form actually is, but it's apparent how to solve it (nonrigorously).

§1.6 The logistic equation

I hope this doesn't show up or I'm gonna lose my mind.

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}} \tag{6}$$

§1.7 Second order linear homogenous differential equations

They are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{7}$$

By the existence-uniqueness theorem, there exists a unique solution $y(t)$ satisfying this ODE on an open interval (with given initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$). Let's define an operator by

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t).$$

This is just a natural transformation if we view maps as functors from $\mathbb{R}$ to $\mathbb{R}$ (category theory ftw). If $L[cy] = cL[y]$ and $L[y_1 + y_2] = L[y_1] + L[y_2]$ for $c \in \mathbb{R}$, $y_1, y_2 \colon \mathbb{R} \to \mathbb{R}$, we say L is a *linear operator*. You can verify that $L[y](t)$ defined above is linear. Clearly just solve for $L[y](t)$ and we get the solutions to the second-order ODE. Here's the useful thing: by this fact, we get that

$$c_1y_1(t) + c_2y_2(t)$$

is the general form of solutions to Equation (7), where $c_1, c_2 \in \mathbb{R}$ and y_1, y_2 are particular solutions to Equation (7). You can see this by evaluating $L[c_1y_1(t) + c_2y_2(t)]$ and applying linearity properties. In particular, *all* solutions to Equation (7) are of that form, by a quick application of the existence uniqueness theorem, given that the gradient vectors are linearly independent (checking this is just a quick calculation to see that the Wronskian is nonzero). We say $\{y_1, y_2\}$ is a *fundamental set* of solutions of Equation (7).

§1.8 Second order ODE constant coefficients

General method for constant coefficients: let's say they're of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{8}$$

where a, b, c are constants and a nonzero. Then just look at the characteristic polynomial $P(r) = ar^2 + br + c$, and examine the roots r_1, r_2 such that $(r - r_1)(r - r_2) = 0$. If $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, then e^{r_1x}, e^{r_2x} are LI solutions to Equation (8) so the general solution is of the form

$$y = c_1e^{r_1x} + c_2e^{r_2x}.$$

If $r_1 = r_2 = r$, $r_1, r_2 \in \mathbb{R}$, then e^{rx}, xe^{rx} are LI solutions and the general solution is of the form

$$y = c_1e^{rx} + c_2xe^{rx}.$$

Finally, if $r_1 \in \mathbb{C}$ (that is, $r_1 = a + bi$) for $a, b \in \mathbb{R}$), then r_2 is the complex conjugate of r_1 (that is, $r_2 = \overline{r_1} = a - bi$) and the functions $e^{ax}\cos(bx)$, $e^{ax}\sin(bx)$ are LI solutions to Equation (8) and the general solution is of the form

$$y = c_1e^{ax}\cos(bx) + c_2e^{ax}\sin(bx).$$

§1.9 Nonhomogeneous second order ODEs

Let's turn our attention to the big boy, the nonhomogeneous second order differential equation given by

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \tag{9}$$

where the functions $p(t), q(t)$ and $g(t)$ are continuous on an open interval.

Theorem 1.1. *Every solution of Equation (9) is of the form*

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t)$$

where y_1, y_2 are LI solutions to Equation (7), $\psi(t)$ is a particular solution to Equation (9), and c_1, c_2 are constants.

Proof. We need a lemma.

Lemma 1.1. *The difference of any two solutions of Equation (9) is a solution of Equation (7).*

Proof. If y_1, y_2 are two solutions of Equation (9), then $L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0.$ ⊠

Now returning to the proof of the theorem, we know $y(t)$ is a solution of Equation (9) by definition. Then by Lemma 1.1, $\phi(t) = y(t) - \psi(t)$ is a solution of Equation (7). But since every solution of Equation (7) is of the form $c_1 y_1(t) + c_2 y_2(t)$, we have

$$y(t) = \phi(t) = \psi(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t).$$

$\boxtimes$

§1.10 The method of judicious guessing

Is this the actual name of the method? We try to guess solutions for equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t), \tag{10}$$

where $a, b, c \in \mathbb{R}$ and $g(t)$ is of a certain form, described below.

Case 1: The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1 t + \cdots + a_n t^n.$$

It can be shown that there is a solution of the form

$$\psi(t) = \begin{cases} A_0 + A_1 t + \cdots + A_n t^n, & c \neq 0, \\ t(A_0 + A_1 t + \cdots + A_n t^n), & c = 0,\ b \neq 0, \\ t^2(A_0 + A_1 t + \cdots + A_n t^n), & c = b = 0. \end{cases}$$

Case 2: The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}.$$

Then it can be shown that there is a particular solution of the form

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is not a solution of the homogeneous equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is a solution of the homogeneous equation, but } te^{\alpha t} \text{ is not,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ and } te^{\alpha t} \text{ are both solutions of the homogeneous equation.} \end{cases}$$

Equivalently, we have

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is not a solution of the characteristic equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is one of two distinct solutions of the characteristic,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is the only solution of the characteristic equation.} \end{cases}$$

Case 3: Let $\phi(t) = u(t) + iv(t)$ be a particular solution of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{i\omega t}.$$

All you have to do is look at the real and imaginary parts to get $\mathrm{Re}(\phi(t)) = u(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\cos(\omega t)$, and similarly $\mathrm{Im}(\phi(t)) = v(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\sin(\omega t)$.

§2 Examples

§2.1 Homogeneous 1st order ODE

Problem. *Find the general solution of*

$$\frac{dy}{dt} + 2ty = 0.$$

Solution. $y = c\exp\left(-\int 2t\,dt\right) = c\exp\left(-t^2\right)$. ■

§2.2 Homogeneous first order ODE initial value

Problem. *Find the solution of*

$$\frac{dy}{dt} + (\sin t)y = 0, \quad y(0) = \frac{3}{2}.$$

Solution. $y = \frac{3}{2}\exp\left(-\int_0^t \sin t\, dt\right) = \frac{3}{2}\exp(\cos t - 1)$. ■

Problem. *Solve*

$$\frac{dy}{dt} + e^{t^2} y = 0, \quad y(1) = 2.$$

Solution. $y = 2\exp\left(-\int_1^t e^{t^2}\, dt\right)$. This function isn't integrable (to be precise, no closed form solution exists) so we're done. ■

§2.3 Nonhomogeneous first order

Problem. *Solve*

$$\frac{dy}{dt} - 2ty = t.$$

Solution. Let $\mu(t) = \exp\left(\int -2t\, dt\right) = \exp\left(-t^2\right)$. So $\frac{d}{dt}y \cdot \exp\left(-t^2\right) = \exp\left(-t^2\right) t \implies y \cdot \exp(-t^2) = -\frac{1}{2}e^{-t^2} + c \implies y = -\frac{1}{2} + ce^{t^2}$. ■

Problem. *Solve*

$$x\frac{dy}{dx} + y = \cos x,\ x > 0.$$

Solution. We have $\frac{dy}{dx} + \frac{y}{x} = \frac{\cos x}{x}$. So $\mu(x) = e^{|\ln(x)|} = x$ for all x strictly positive. Then $\frac{d}{dx}yx = x\frac{\cos x}{x} = \cos x$. So $xy = \sin x + c \implies y = \frac{\sin x}{x} + \frac{c}{x}$. ■

§2.4 Nonhomogeneous first order initial value

Problem. *Solve*

$$\frac{dy}{dt} + 2ty = t, \quad y(1) = 2.$$

Solution. We have $\mu(t) = e^{t^2}$. So $\frac{d}{dt}ye^{t^2} = te^{t^2} \implies ye^{t^2} = \frac{1}{2}e^{t^2} + c \implies y = \frac{1}{2} + ce^{-t^2}$. At $y(1) = 2$, we have $\frac{3}{2} = \frac{c}{e} \implies c = \frac{3}{2}e$. So the solution is $y = \frac{1}{2} + \frac{3}{2}e^{(-t^2+1)}$. ■

Problem. *Solve*

$$\frac{dy}{dx} + xy = xe^{\frac{x^2}{2}}, \quad y(0) = 1.$$

Solution. Now $\mu(t) = e^{\frac{x^2}{2}}$. So $\frac{d}{dx}ye^{\frac{x^2}{2}} = xe^{x^2}$, and $ye^{\frac{x^2}{2}} = \frac{1}{2}e^{x^2} + c \implies y = \frac{1}{2}e^{\frac{x^2}{2}} + ce^{-\frac{x^2}{2}}$. At $y(0) = 1$, we have $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$, and the general solution is of the form $y = \frac{1}{2}e^{\frac{x^2}{2}} + \frac{1}{2}e^{-\frac{x^2}{2}} = \frac{1}{2}e^{\frac{x^2}{2}}\left(1 + e^{-x^2}\right)$. ■

§2.5 Separable equations

Problem. *Solve*

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

Solution. $y^3 = x^3 + c \implies y = \sqrt[3]{x^3 + 3c}$. ■

Problem. *Solve*

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$

Solution. $\int 2y + \cos y\, dy = 2x^3 \implies y^2 + \sin y + c = 2x^3$. Now what? I think it's over. ■

Problem. *Solve*

$$y' = x^2 y.$$

Solution. $\ln|y| = \frac{x^3}{3} + c \implies y = \pm e^{\frac{x^3}{3}+c} \implies y = Ce^{\frac{x^3}{3}}$. ■

§2.6 Second order homogeneous ODEs

Problem. *Find the solutions of*

$$\frac{d^2y}{dt^2} + y = 0.$$

Solution. Clearly two particular solutions are $y_1(t) = \cos t$, $y_2(t) = \sin t$, then by the existence uniqueness thm the general solution is of the form $y(t) = c_1 \cos t + c_2 \sin t$. ■

Problem. *Calculate the Wronskian for y_1, y_2.*

Solution. Why am I doing this??? I have better things to do. ■

§2.7 Second order ODE constant coefficients

Problem. *Determine all solutions to the differential equation*

$$y'' + y' - 6y = 0$$

of the form e^{rx}.

Solution. $y' = re^{rx}, y'' = r^2e^{rx}$. So we have $e^{rx}(r^2 + r - 6) = 0$ for the differential equation. Clearly $r = 2, -3$ satisfy this equation, so the solutions are $y_1 = e^{2x}, y_2 = e^{-3x}$. These are LI, so the general solution is of the form $c_1e^{2x} + c_2e^{-3x}$. ■

Problem. *Solve*

$$y'' + y = 0.$$

Solution. The characteristic is $r^2 + 1$, so $r_1 = i$ and $r_2 = -i$. Then solutions are of the form $c_1e^0 \cos(1x) + c_2e^0 \sin(1x) = c_1 \cos x + c_2 \sin x$. ■

Problem. *Solve*

$$y'' + 6y' + 25y = 0.$$

Solution. The solutions to the characteristic polynomial $r^2 + 6r + 25$ are simply $r = -3 \pm 4i$. So the general solution is of the form $c_1e^{-3x} \cos(4x) + c_2e^{-3x} \sin(4x)$. ■

Problem. *Solve the following initial value problem:*

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, y'(0) = 4.$$

Solution. Clearly the general solution is of the form $c_1e^{-2x} + c_2xe^{-2x}$. At $y(0) = 1$, we have $1 = c_1$. Then $y' = -2e^{-2x} + c_2e^{-2x} + -2c_2xe^{-2x}$, so at $y'(0) = 4$ we have $4 = -2 + c_2$. So $c_2 = 6$, and the general solution is of the form $e^{-2x} + 6xe^{-2x}$. ■

§2.8 Nonhomogeneous second order ODEs

Problem. *Three solutions of some second-order nonhomogeneous ODE are*

$$\varphi_1(t) = t, \varphi_2(t) = t + e^t, \text{ and } \varphi_3(t) = 1 + t + e^t.$$

Find the general solution of the equation.

Solution. By our lemma, $\varphi_2 - \varphi_1 = e^t$ and $\varphi_3 - \varphi_2 = 1$ are clearly LI solutions to the nonhomogeneous equation. Then the general solution is of the form $c_1 + c_2e^t + t$. Our choices of φ_i don't really matter, just trust the theorems. ■

Problem. *gotta find more problems for these*

§2.9 Judicious guessing (nonhomogeneous second order with constant coefficients)

Problem. *Find a particular solution $\psi(t)$ of the equation*

$$L[y] = \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = t^2.$$

Solution. Since $c \neq 0$, we have a solution $\psi(t)$ of the form $A_0 + A_1t + A_2t^2$. Plug that in and solve for the constants. ■