

Differential Equations Notes

Math 427J (53550)

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1 First Order Linear Differential Equations (8/27/20)

Definition 1.1 (Order). We have the *order* of a differential equation the highest derivative of a function y that appears in the equation. For example, the order of the differential equation

$$\frac{dy}{dt} = 3y^2 \sin(t + y)$$

is 1, while the order of

$$\frac{d^3 y}{dt^3} = e^{-y} + t + \frac{d^2 y}{dt^2}$$

is 3. We would call the first example a *first-order differential equation* and the second a *third order differential equation*.

Definition 1.2 (Solution). The *solution* of a differential equation is a continuous function $y(t)$ that together with its derivatives satisfies the given relationship.

Example 1.1. The function

$$y(t) = 2 \sin t - \frac{1}{3} \cos 2t$$

is the solution of the second-order differential equation

$$\frac{d^2 y}{dt^2} + y = \cos 2t$$

since

$$\begin{aligned} & \frac{d^2}{dt^2} \left(2 \sin t - \frac{1}{3} \cos 2t \right) + \left(2 \sin t - \frac{1}{3} \cos 2t \right) \\ &= \left(-2 \sin t + \frac{4}{3} \cos 2t \right) + 2 \sin t - \frac{1}{3} \cos 2t = \cos 2t. \end{aligned}$$

Goal: Given a differential equation of the form

$$\frac{dy}{dt} = f(t, y)$$

and the function $f(t, y)$, find all functions $y(t)$ that satisfy the equation above.

What we have: As of now, all we can solve is a differential equation of the form

$$\frac{dy}{dt} = g(t)$$

given $g(t)$ is integrable. Very sad!

Definition 1.3 (Linear ODE). The general first-order linear differential equation is of the form

$$\frac{dy}{dt} + a(t)y = b(t),$$

where $a(t)$ and $b(t)$ are continuous (assumed to be functions of time).

Definition 1.4 (Homogeneous Linear ODE). The equation

$$\frac{dy}{dt} + a(t)y = 0$$

is called the *homogeneous* first-order linear differential equation, and the previous definition is called the *nonhomogeneous* first-order linear differential equation for $b(t)$ not necessarily zero.

Example 1.2. Let us solve the homogeneous first-order linear differential equation. Rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t).$$

Second, note that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt} \ln |y(t)|.$$

Then we can write the differential equation in the form

$$\frac{d}{dt} \ln |y(t)| = -a(t),$$

so we have

$$\ln |y(t)| = - \int a(t) dt + c_1.$$

Continuing on,

$$|y(t)| = \exp \left(- \int a(t) dt + c_1 \right) = c \exp \left(- \int a(t) dt \right)$$

or

$$\left| y(t) \exp \left(\int a(t) dt \right) \right| = c.$$

Now $y(t) \exp \left(\int a(t) dt \right)$ is continuous and we know its absolute value is constant which implies that the function itself is constant (which follows from the IVT, assuming $g(t_1) = c$ and $g(t_2) = -c$ for g a function, c a constant. So we have $y(t) \exp \left(\int a(t) dt \right) = c$, or

$$y(t) = c \exp \left(- \int a(t) dt \right). \quad (1)$$

Equation (1) is the *general solution* of the homogeneous equation. Note that there exist infinitely many solutions since for all c we have a distinct $y(t)$.

Example 1.3. To solve the Linear ODE

$$\frac{dy}{dt} + 2ty = 0,$$

simply apply Equation (1) to yield

$$y(t) = c \exp \left(- \int 2t dt \right) = c \exp (-t^2).$$

(This is taking too long! I'll type notes with less rigor next time).

Usually scientists are not interested in the general solution given by Equation (1), rather we look for solutions to a specific $y(t)$ which at some time t_0 has the value y_0 , or we want to determine a $y(t)$ such that

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0.$$

Please accept the derivation that a general solution to this type of problem is

$$y(t) = y_0 \exp \left(- \int_{t_0}^t a(s) ds \right) \quad (2)$$

without proof (there is nothing of interest about the derivation process).

Example 1.4. To solve

$$\frac{dy}{dt} + (\sin t)y = 0, \quad y(0) = \frac{3}{2},$$

let $a(t) = \sin t$, $t_0 = 0$, $y_0 = \frac{3}{2}$. Then

$$y(t) = \frac{3}{2} \exp \left(- \int_0^t \sin s ds \right) = \frac{3}{2} \exp (\cos t - 1).$$

Example 1.5. To solve the initial value problem

$$\frac{dy}{dt} + \exp(t^2)y = 0, \quad y(1) = 2,$$

simply PLUG IT IN (reese) to get

$$y(t) = 2 \exp \left(- \int_1^t e^{s^2} ds \right).$$

Recall that this is the *Gaussian Integral* and can be solved by a change to double integration by polar coordinates (yielding $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$), but in general has no closed form solution.