Riemannian Geometry Notes

Simon Xiang

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Introduction

These are supplementary notes, following Lee's *Introduction to Riemannian Manifolds*. To get an idea of what we're studying, Chapter 1 will start from the roots and give a high level overview of the material.

Geometry as a mathematical discipline stems from Euclidian plane geometry, the stuff you learned in middle school. Its elements are points, lines, distances, angles, and areas: the notion of equivalence comes from **congruence—** two plane figures are congruent if they can be transformed into each other by a **rigid motion of the plane**, a bijective transformation from the plane to itself preserving distance. Some theorems:

Side-Side-Side Theorem. *Two Euclidian triangles are congruent iff the lengths of their corresponding sides are equal.*

Angle-Sum Theorem. *The sum of the interior angles of a Euclidian triangle is π.*

These two seemingly simple theorems illustrate two major types of results in geometry, we call them "classification theorems" and "local-to-global theorems". The SSS theorem is a *classification theorem*. Such a theorem tells us how to determine whether two objects are equivalent. Ideal classification theorems list computable invariants and says objects are equivalent iff these invariants match. The angle-sum theorem relates a local geometric property (angle measure) to a global property (being a triangle). Most of the theorems we study are *local-to-global theorems*.

After studying points and lines, we can talk about circles. Here we state two theorems, one is a classification theorem, while the other is a local-to-global theorem (it will become clear why with time).

Circle Classification Theorem. *Two circles in the Euclidian plane are congruent iff they have the same radius.*

Circumference Theorem. *The circumference of a Euclidian circle of radius R is $2\pi R$.*

1.1 Curvature

If we want to study more stuff, we'll have to talk about curves in the plane. Arbitary curves don't vibe well with things like length and radius, so we have a new basic invariant called *curvature*, defined using calculus and is a function of position on the curve.

Formally, the **curvature** of a plane curve γ is defined as $\kappa(t) = |\gamma''(t)|$, the length of the acceleration vector, when γ is given a unit-speed parametrization. This is how we think about curvature geometrically: Given a point $p = \gamma(t)$, there are several circles tangent to γ at p, namely the circles whose velocity vector at p is the same as that of γ when both are given unit-speed parametrizations. The center of these circles lie on the line passing through p orthogonal to $\gamma'(p)$. Among these circles, there is exactly one unit-speed parametrized circle whos acceleration vector at p is the same as γ, it is called the **osculating circle**. (If acceleration is zero, replace the osculating circle by a straight line, a "circle with infinite radius"). The curvature is then $\kappa(t) = 1/R$, where R is the radius of the osculating circle. The larger the curvature , the greater the acceleration, the smaller the radius, and therefore the faster the curve is turning. A circle of radius R has constant curvature $\gamma \equiv 1/R$, while a straight line has a curvature of zero. All of this makes much more sense with a figure TODO

It is often convenient to extend the definition of curvature to allow positive and negative values, we do this by choosing a continuous unit normal vector field N along the curve, and assigning the curvature a positive sign if the curve is facing the normal vector and a negative sign if it's facing away. The resulting function κ_N along the curve is then called the **signed curvature**. We state two theorems about plane curves.

Plane Curve Classification Theorem. *Suppose γ and $\widetilde{\gamma}\colon [a,b] \to \mathbb{R}^2$ are smooth, unit-speed plane curves with unit normal vector fields N and $\widetilde{N}$, and $\kappa_N(t), \kappa_{\widetilde{N}}(t)$ represent the signed curvatures at $\gamma(t)$ and $\widetilde{\gamma}(t)$, respectively. Then γ and $\widetilde{\gamma}$ are congruent by a direction-preserving congruence iff $\kappa_N(t) = \kappa_{\widetilde{N}}(t)$ for all $t \in [a,b]$.*

Total Curvature Theorem. *If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a unit-speed simple closed curve such that $\gamma'(a) = \gamma'(b)$, and N is the inward pointing normal, then*

$$\int_a^b \kappa_N(t)\,dt = 2\pi.$$

The first theorem is a classification theorem, while the second is a local-to-global theorem, relating the local property of curvature to the global (topological) property of being a simple closed curve. These generalize the circle theorems: two circles are congruent if they have the same curvature (radius), and if a circle has curvature κ and circumference C, then $\kappa C = 2\pi$.

1.2 Surfaces in Space

The natural next step is to move to three dimensions, that is, the study of general curve surfaces in space (2-dimensional embedded submanifolds of $\mathbb{R}^3$). The invariant is curvature, but it gets more complicated since a surface can curve differently in different directions.

Curvature in space is described by two numbers at each point, called the **principal curvatures**. Suppose S is a surface in $\mathbb{R}^3$, p is a point on S, and N is a unit normal vector to S at p. Here's a rough outline of how to compute principal curvature.

1. Choose a plane Π passing through p and parallel to N. The intersection of Π with a neighborhood of p in S is a place curve $\gamma \subseteq \pi$ containing p.

2. Compute the signed curvature κ_N of γ at p with respect to the chosen unit normal N.

3. Repeat *for all* normal places Π. The **principal curvatures of S at p**, denoted by minimum and maximum signed curvatures obtained.

Principle curvatures give us information about geometry, but don't answer the a paramount question in Riemannian geometry: Which properties of a surface are *intrinsic*? A property of surfaces is *intrinsic* if it is preserved by *isometries*, maps between surfaces preserving lengths of curves. To see that principle curvature isn't intrinsic, consider the embedded surface S_1, S_2 in $\mathbb{R}^3$, where S_1 is the square in the xy-plane with $0 < x < \pi, 0 < y < \pi$, and S_2 is the half-cylinder $\{(x,y,z) \mid z = \sqrt{1-y^2}, 0 < x < \pi, |y| < 1\}$. The principal curvatures of S_1 are zero, while the principal curvatures of S_2 are $\kappa_1 = 0$ and $\kappa_2 = 1$. But the map sending $(x,y,0)$ to $(x, \cos y, \sin y)$ is a diffeomorphism from S_1 to S_2, and thus an isometry.

Principle curvatures may not be intrinsic, but Guass discovered that a particular combination of them is, that is, the product $K = \kappa_1\kappa_2$ (known as the *Gaussian curvature*) is intrinsic. He named it *Theorema Egregium*, meaning "remarkable theorem". To get an idea of how Gaussian curvature works, first note that the square and half-cylinder have the same Gaussian curvature of zero (which is true by *Theorema Egregium* since they are isometric). A sphere of radius R has positive Gaussian curvature $1/R^2$, since each plane intersects the sphere in a great circle of radius R, and so the principal curvatures are $\pm 1/R \implies K = \kappa_1\kappa_2 = 1/R^2$. Similarly, "dome-shaped" objects have positive Gaussian curvature, since two principal curvatures always have the same sign, while "saddle-shaped" objects have negative Gaussian curvatures.

Model spaces of surface theory have constant Gaussian curvature. We have already seen two: Euclidian space $\mathbb{R}^2$ ($K = 0$), and the sphere of radius R ($K = 1/R^2$). The most important model surface with constant negative Gaussian curvature is the **hyperbolic plane**, whicn we'll talk about later. We state two theorems, you know the drill.

Uniformization Theorem. *Every connected 2-manifold is diffeomorphic to a quotient of one of the constant-curvature model surfaces described above by a discrete group of isometries without fixed points. So every connected 2-manifold has a complete Riemannian metric with constant Gaussian curvature.*

Gauss-Bonnet Theorem. *Suppose S is a compact Riemannian 2-manifold. Then*

$$\int_S K\,dA = 2\pi\chi(S),$$

where $\chi(X)$ is the Euler characteristic of S.

The uniformization theorem replaces the problem of classifying surfaces with classifying certain discrete groups of the models. Usually the uniformization theorem is state differently and proved with complex analysis. The Gauss-Bonnet theorem is a pure theorem of differential geometry, and arguable the most fundamental and important of them all. It relates a local geometric property (curvature) with a global topological invariant (the Euler Characteristic).

Together, these theorems place strong restrictions on the types of metrics that can occur on a given surface. For example, a consequence of Gauss-Bonnet is that the only compact, connected, orientable surface that admits a metric of strictly positive Gaussian curvature is the sphere. On the other hand, if a compact, connected orientable surface has nonpositive Gaussian curvature, Gauss-Bonnet rules out the sphere, and the uniformization theorem tells us that its universal covering space is homeomorphic to the plane.

1.3 Curvature in Higher Dimensions

Curvature becomes a lot more complicated in higher dimensions since manifolds can curve in all sorts of crazy ways. Our first issue is that in general, Riemannian manifolds don't present themselves as embedded submanifolds of Euclidian space. So we can't cut out curves by intersecting planes. However, **geodesics**—curves that are the shortest path between two points, help with our case. Examples are straight lines in Euclidian space and great circles on a sphere.

Suppose M is an n-dimensional Riemannian manifold. The most fundamental fact about geodesics is that given any $p \in M$ and any vector v tangent to M at p, there is a unique geodesic starting at p with inital velocity v. Here's a brief recipe for computing curvatures at some $p \in M$:

1. Choose a 2-dimensional subspace Π of the tangent space to M at p.
2. Look at all the geodesics through p whose initial velocities lie in Π. It turns out that near p these sweep out a certain 2-dimensional submanifold S_Π of M, which inherits a Riemannian metric from M.
3. Compute the Gaussian curvature of S_Π at p, which *Theorema Egregium* tells us can be computed from the inherited Riemannian metric. This associates a number, denoted sec(Π), called the **sectional curvature** of M at p, with the plane Π.

So the "curvature of M at p has to be interpreted as a map $\sec\colon \{\text{2-planes in } T_pM\} \to \mathbb{R}$. We again have three classes of constant (sectional) curvature model spaces: $\mathbb{R}^n$ with its Euclidian metric (for which $\sec \equiv 0$); the n-sphere of radius R, with the Riemannian metric metric inherited from $\mathbb{R}^{n+1}$ ($\sec \equiv 1/R^2$); and hyperbolic space of radius R (with $\sec \equiv -1/R^2$). Unfortunately, we have no satisfactory uniformization theorem for Riemannian manifolds in higher dimensions. In general, it is *not* true that every manifold has a metric of constant sectional curvature.

Characterization of Constant-Curvature Metrics. *The complete, connected, n-dimensional Riemannian manifolds of constant sectional curvature are, up to isometry, exactly the Riemannian quotients of the form $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is a Euclidian space, sphere, or hyperbolic space with constant sectional curvature, and Γ is a discrete group of isometrics of $\widetilde{M}$ acting freely on $\widetilde{M}$.*

On the other hand, we have a number of power local-to-global theorems, which can be thought of generalizations of Gauss-Bonnet in various directions. They are consequences of the fact that positive curvature makes geodesics converge, while negative curvature makes them spread out.

Cartan-Hadamard Theorem. *Suppose M is a complete, connected Riemannian n-manifold with all sectional curvatures less than or equal to zero. Then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*

Myer's Theorem. *Suppose M is a complete, connected Riemannian manifold with all sectional curvatures bounded below by a positive constant. Then M is compact and has a finite fundamental group.*

You can see that these theorems generalize the uniformization and Gauss-Bonnet, although not their full strength. Our goal with this course is to prove the three aforementioned theorems, among others; it is a primary goal of Riemannian geometry to improve upon and generalize the results of surface theory to higher dimensions.

Part II
Class Notes

Lecture 2

January 19, 2021

What is Riemannian geometry?? Consider $\{(x,y) \mid x^2+y^2<1\}$, this is a coordinate chart. This doesn't tell us anything about the geometry of the surface, since $z=\sqrt{1-x^2-y^2}$ and $z=x^2-y^2$ have the same coordinates. Geometry is extra data on a surface, besides the topological data. Our basic extra ingredient is the inner product, which allows us to talk about length, which tells us all sorts of things about the manifold.

2.1 Introduction to curvature

"In fact, with the inherited inner product on $\mathbb{R}^2$, this curved string is *not* curved!"

The map sending a curved string to a straight line is an isometry. So there are two different notions of curvature: extrinsic curvature talks about to what extent is something bent relative to something else, and intrinsic talks about what happens if you look locally. Locally, all 1-manifolds are isometric to $\mathbb{R}$.

Now to 2-manifolds. We have the circle, half-sphere, saddle, and half-cylinder discussed earlier. We can't really tell the difference between a cylinder and the plane, they're isometric. Extrinsically they're different, but intrinsically they're the same. The sphere is different— a circle has circumference $2\pi R$, that is, the circumference is the set of points of distance R away. What is the circumference of a sphere? Use spherical coordinates: a circle of radius R is a latitude line, and the length is 2π times the distance of the great circle cut out at R, or $2\pi\sin(R)$ if you draw out the angles. Approximately, $2\pi\sin(R)\approx 2\pi(R-\frac{R^3}{6})$. So circles are too small!

This means that spheres are in some sense, sphere shaped. Another thing is the area: the area of the cap is $A=\int 2\pi\sin(R)dR = 2\pi(1-\cos(R)) = 2\pi(\frac{R^2}{2}-\frac{R^4}{4!}+\cdots) = \pi R^2 - \frac{\pi R^4}{12}+\cdots$. So circumferences and areas are a little bit too small. If we worked in $z=x^2+y^2$, we would find that circumferences and areas would be a little bit too big. This is why you can't flatten and orange peel, or an accurate scaled map of the world preserving angles and area.

2.2 Dual Space

Suppose V is an n-dimensional vector space, with basis $\mathscr{E}=\{e_1,\cdots,e_n\}$. Then we can write any $v\in V$ as the sum $\sum v^i e_i$. So we have a natural correspondence between vectors v and coordinates $\{v^i\}$ where $v^i\in\mathbb{R}^n$. From now on, we use $v^i e_i$ to denote $\sum v^i e_i$, this is called Einstein summation notation.

The dual space $V^*=\mathrm{Hom}(V,\mathbb{R})$ is the space of linear functionals from V into the base field, the reals. One element of V^* is the function that assigns each vector to its ith coordinate $\phi^i(v)=v^i$. So we have a nice set of transformations $\{\phi^1,\phi^2,\cdots,\phi^n\}$.

Claim. The set $\{\phi^1,\phi^2,\cdots,\phi^n\}$ forms an n-dimensional basis for V^*, called the *dual basis* to $\mathscr{E}$.

Proof. Let $\alpha=\alpha_i\phi^i$. Suppose $\alpha=0$, then for all $v, \alpha(v)=0$. So $\alpha(e_j)=0$, and $\alpha_i\phi^i(e_j)=0$. Now $\phi^i(e_j)=\delta^i_j$, so $\alpha_i\delta^i_j=0$ and therefore $\alpha_j=0$. Now define $\alpha_j=\alpha(e_j)$. Applying this to a vector v gives $(\alpha_j\phi^j)(v)=\alpha_j(\phi^j(v))=\alpha_j v^j$. Then $\alpha(v)=\alpha(v^je_j)=v^j\alpha(e_j)=v^j\alpha_j$, and these are equal. We conclude that $\alpha=\alpha_j v^j$. ⊠

Summary: write arbitrary elements of V as v^ie_i, and the dual space as $\alpha=\alpha_j\phi^j$. We have $\phi^j(v)=v^j$, $\alpha(e_i)=\alpha_i$, and $\alpha(v)=\alpha_i v^i$. This is why we call V^* the dual space: pairing elements together in either order gives a number.

On this vein, $V^{**} = V$, where the basis $\{e_1, \cdots, e_n\}$ is dual to $\{\phi^1, \cdots, \phi^n\}$. V is the space of contravariant vectors, while V^* is the space of covectors, which are covariant in a categorical sense.

Suppose we have a new basis $\widetilde{\mathscr{E}} = \{\widetilde{e}_1, \cdots, \widetilde{e}_n\}$. There must be some change of basis matrix, that is, $\widetilde{e}_i = A_i^j e_j$, similarly we have a new dual basis with $\widetilde{\phi}^i = B^i_j \phi^j$. If we have a vector $v = v^i e_i = \widetilde{v}^i \widetilde{e}_i$, we have $\widetilde{v}^i = \widetilde{\phi}^i(v) = B^i_j \phi^j(v) = B^i_j v^j$. When your basis vectors get big, your coordinates get small, so they transform oppositely. This is the origin of the term contravariant. We claim that $A_i^j B_j^k = \delta_i^k = B_i^j A_j^k$. To see this, note that $\delta_i^k = \widetilde{\phi}^k(\widetilde{e}_i) = \widetilde{\phi}^k(A_i^j e_j) = \phi^\ell B_\ell^k (A_i^j e_j) = B_\ell^k A_i^j \delta_j^\ell = B_j^k A_i^j = A_i^j B_j^k$.

We can kinda visualize vectors as columns and covectors as rows. Then e_i is a column with 1 in the ith while ϕ^j is a row in the jth slot. Applying α to e_i gives α_i, while applying ϕ^j to v gives v^i.

2.3 Tensors

A (k, ℓ) tensor eats k-vectors and ℓ-covectors, and pops out a number. These should be multilinear in each slot. In terms of coordinates, $T(v, w) = T(v^i e_i, w^j e_j) = v^i w^i T(e_i, e_j)$. We define $T_{ij} = T(e_i, e_j)$, so $T(v, w)$ becomes $T_{ij} v^i w^j$. This is a $(2, 0)$-tensor. What is a $(0, 2)$-tensor? This is essentially a matrix, where $S(v, \alpha) = S_j^j v^i \alpha_j$. If you don't give a tensor enough information, it turns into a tensor of lower rank, which works the same in matrix multiplication. If you have a doubly covariant tensor, it turns vectors into covectors. If you have a doubly contravariant tensor, it turns covectors into vectors. Next time, we'll talk about the most important doubly covariant tensor, the inner product.

Lecture 3

January 21, 2021

3.1 A Basis for Tensors

Let's continue the algebra from yesterday. Recall a tensor takes two two vectors as input, denoted $T(v, w) = T(v^i e_i, w^j e_j) = v^i w^j T(e_i, e_j) = \left(\sum_{i,j}\right) T_{ij} v^i w^{ij}$. If $V = \mathbb{R}^2$, what's an example of a covariant 2-tensor? The standard inner product $\langle v \mid w \rangle = v^1 w^1 + v^2 w^2$ works. Using the notation $g(v, w)$, we have $g_{11} = g(e_1, e_1) = 1, g_{12} = 0, g_{21} = 0, g_{22} = 1$. In general, for an inner product $g_{ij} = g_{ij}$, and if you think of as a matrix, this will be a positive definite matrix, or $g_{ij} v^i v^j \geq 0$ if $v \neq 0$ (all eigenvectors are positive). For example, the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with positive eigenvectors} \quad \begin{bmatrix} \frac{1 \pm \sqrt{5}}{2} \\ 1 \end{bmatrix}$$

works. Another interesting tensor is the determinant $A(v, w) = v^1 w^2 - v^2 w^1 = -A(w, v)$. In this case, $A_{12} = -A_{21} = 1$ and $A_{11} = A_{22} = 0$.

The natural next question is: what is the space of covariant 2-tensors on $\mathbb{R}^2$? This is a vector space, what would the basis be? It's denoted $\phi^i \otimes \phi^j(e_k, e_\ell)$. Given two tensors $S(v_1, \cdots, v_n)$ and $T(w_1, \cdots, w_m)$, we define $S \otimes T(v_1, \cdots, v_n, w_1, \cdots, w_m) := S(v_1, \cdots, v_n) T(w_1, \cdots, w_m)$. So $\phi^i \otimes \phi^j(e_k, e_\ell) =$?? missed this portion. So $T = T_{ij} \phi^i \otimes \phi^j$, and $T(e_k, e_\ell)$.

Consider the inner product g, we can think of it as taking a vector and turning it into a covector. We have $g(v, \cdot)(v) \neq 0$, so g induces a map onto the dual space $g : V \to V^*$, which is an isomorphism. So $\alpha = g(v, \cdot), \alpha_j = g_{ij} v^i$. You can think of this as lowering the indices. The matrix that *raises* the indices is the inverse matrix, denoted g^{ij}, where $g^{ij} g_{jk} = \delta_k^i$.

Now that we have a basis for 2-tensors, what about a basis for $\tau^{k,\ell}$ (space of tensors)? It's going to be

$$\{\phi^{i_1} \otimes \phi^{i_2} \otimes \cdots \otimes \phi^{i_k} \otimes e_{j_1} \otimes \cdots \otimes_{j_\ell}\}.$$

So $T = T_{i_1 \cdots i_k}^{j_1 \cdots j_\ell} \phi^{i_1} \otimes \cdots \otimes e_{j_\ell} = T(e_{i_1}, \cdots, e_k, \phi^{j_1}, \cdots, \phi^{j_\ell})$ this may be wrong

3.2 Trace of a Matrix

Now let's talk about the trace of a matrix, defined as $\operatorname{Tr}(M) = M^i_i$. If we change our basis by $\widetilde{e}_i = A^j_i e_j$, recall that this gives rise to a dual basis $\widetilde{\phi}^i = B^i_j \phi^j$. In our new basis, $\widetilde{M}^i_j = B^i_k A^\ell_j M^k_\ell = B^i_k M^k_\ell A^\ell_j$, this is kind of like PMP^{-1} that we did in linear algebra. But this doesn't really work with tensors, what about $2, 3, 4, 5$-tensors? That's why we're doing it this way.

We have the trace $\operatorname{Tr} = \widetilde{M}^i_i = B^i_k M^k_\ell A^\ell_i$. But $B^i_k A^\ell_i = \delta^\ell_k$, since these two are inverse matrices. So $\operatorname{Tr} = \delta^k_\ell M^k_\ell = M^k_k$, which is the old definition of trace. We can apply this to higher rank tensors: suppose we have a tensor that takes in a covector and two vectors, denoted $T(\alpha, v, w)$. Given a tensor $S(w) = T(\phi^i, e_i, w)$, so $S_i = T^j_{ji}$. What happens if you change basis, that is, is $T(\phi^i, e_i, w) = T(\widetilde{\phi}^i, \widetilde{e}_i, w)$?

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For fixed w, let's define a new tensor, $U(\alpha, v) = T(\alpha, v, w)$. Let's take the trace of this, we've already shown that the trace of a $(1,1)$-tensor doesn't depend on basis. So the w comes along for the ride. This is a slick solution, since $\operatorname{Tr} U$ is well defined.

3.3 Tangent Vectors

Enough about about tensors. Now let's talk about manifolds. Usually the vector space at a point p we care about is the tangent space $T_p(M)$ at that point. What's a tangent vector? In $\mathbb{R}^n$, consider a point p: what is a tangent vector there?

1. An arrow, add them head to tail.

2. An element of $\mathbb{R}^n$, a list of n numbers. This is pretty much an arrow, just take the coefficients and impose them on the standard basis.

3. The equivalence class of curves $\gamma(t)$ with $\gamma(0) = p$.

4. Velocity. Consider all possible parametrized curves through a point, and identify all curves with the same velocity at time zero. So we mean the equivalence class of curves $\gamma(t)$ with $\gamma(0) = p$. The beauty of this third definition is it makes sense on any manifold. So we can sonsider the curves going through this point, and take them up to equivalence.

5. Directional Derivative. For $f : \mathbb{R}^n \to \mathbb{R}$, we can take $\left.\frac{df \cdot \gamma(t)}{dt}\right|_{t=0} = \left.\frac{d\gamma^i}{dt}\right|_{t=0} \left.\frac{\partial f}{\partial x^i}\right|_p$. The partial derivaties $\{\partial_1, \cdots, \partial_n\}$ gives a basis for this vector space.

6. Derivations. A derivation at p is a map $D\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ with the following properties:

   (1) $D(af + bg) = aD(f) + bD(g)$,

   (2) $D(fg) = f(p)D(g) + D(f)g(p)$.

   The partial derivatives $\partial_i|_p$ are derivations.

Claim. We have $\{\partial_i\}$ a basis for the set of derivations at p, denoted $\mathscr{D}_p(\mathbb{R}^n)$.

This idea of thinking about tangent vectors as derivations carries over very nicely to abstract manifolds, which is why we care. We want to show a couple of things:

1. To show $D(\text{constant}) = 0$, note that $D(cf) = cD(f) = cD(f) + f(p)D(c)$, so we need D of a constant to be zero.

2. If $f(p) = g(p) = 0$, then $D(fg) = 0$, this follows from the product rule.

3. Taylor series, we have $f(x) = f(p) + \partial_i f(p)(x^i - p^i) +$ higher order terms. Then $D(f) = 0 + \partial_1 f(p) D(x^i) + 0 = D^i \partial_1 f(p)$, where $D = D^i \partial_i$.

This works for analytic functions. There is some cheating going on, but we don't need the entire Taylor series for the most part. So every derivation is a linear combination of partial derivatives. From now on, think of vectors as a combination of partial derivatives, or inducing curves along a vector field.

A vector field $V^i(x)\frac{\partial}{\partial x^i}$ is a bunch of coefficients in combination with the basis vectors of partial derivatives. How do we change coordinates here? We do this by the chain rule, that is,

$$\text{for } \{x\} \longleftrightarrow \{y\} \quad \text{we have} \quad \frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}.$$

Let's try this in $\mathbb{R}^2$ with $(r,\theta) \leftrightarrow (x,y)$. For $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$ and $\widetilde{e}_1 = \frac{\partial}{\partial r}, \widetilde{e}_2 \frac{\partial}{\partial \theta}$, recall that $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, and $x = r\cos\theta$, $y = r\sin\theta$. So

$$\frac{\partial y}{\partial r} = \cos\theta = \frac{x}{\sqrt{x^2+y^2}},\ \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{\sqrt{x^2+y^2}},\ \frac{\partial x}{\partial \theta} = -r\sin\theta = -y,\ \frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

So now we can convert between the two bases, that is, we have the change of basis matrices

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix}.$$

Lecture 4

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What is the tangent vector of a point on an arbitrary manifold, like $\mathbb{CP}^2$? Arrows? No. Equivalence class of curves? This works for manifolds. Consider a curve through p, and another curve that looks the same locally (we have to apply the fact that manifolds look locally like $\mathbb{R}^n$). Derivations also work perfectly.

4.1 Basis for a Tangent Space

We know what a basis is for tensors, which comes from a basis for the dual space. What's the basis for the dual space of tangent vectors? Suppose we have a function $f : M \to \mathbb{R}$. Then $df(V) := v(f)$. Looking at $\mathbb{R}^n$, we have $(df)\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial f}{\partial x^1}$, and similarly $(df)\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i} = \partial_j f$. So $(df) = \frac{\partial f}{\partial x^j}\phi^j$. Note that we have $(dx^i)\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$. So on a manifold with coordinates $x^1, \cdots, x^n$, the basis for $T_p(M) = \{\frac{\partial}{\partial x^i}\}$. The basis for the dual space $T_p^*(M) = \{dx^i\}$.

We deal with a particular 2-covariant tensor all the time, which is the metric. If we have the notion of an inner product at every point, with a metric $g = g_{ij}dx^i \otimes dx^j$, where $g_{ij}^{(x)} = g(\partial_i, \partial_j)_x = \langle \partial_i, \partial_j \rangle_x$. We will spend a ridiculous amount of time talking about this tensor at a point x. Let's play around with the manifold of the upper hemisphere, given by $\{(x,y,z) \mid x^2+y^2+z^2 = 1, z > 0\}$. Some possibilities for coordinates:

1. x, y, where $z = \sqrt{1-x^2-y^2}$,
2. θ, ϕ, where θ measures the angle from the north pole, and ϕ measures the longitude. So $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$.

How do we find a metric? We know $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = \frac{-x(dy/dt)-y(dy/dt)}{\sqrt{1-x^2-y^2}}$. So we get the vectors $(1,0,-\frac{x}{z}),(0,1,-\frac{y}{z})$. So $g_{11} = 1+\frac{x^2}{z^2} = 1+\frac{x^2}{1-x^2=y^2} = \frac{1-y^2}{1-x^2-y^2}$, $g_{12} = \frac{xy}{z^2} = \frac{xy}{1-x^2-y^2}$, $g_{21} = g_{12}, g_{22} = \frac{1-x^2}{1-x^2-y^2}$.

Now lets move onto spherical coordinates. This is a bunch of trig derivatives, look at the recorded lecture in your free time.