# Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html.

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## §1 September 21, 2020

Last time: Representation Theory. Recall that if $X$ is finite and we have a group $G$ acting on $X$, then we have a representation $\varphi\colon G \to \mathrm{GL}(V)$, where $V = \bigoplus_{x \in X} \mathbb{F}e_x$ for $\mathbb{F}$ a field. Recall again that the matrix corresponding to $\varphi(g)$ consists of 0's and 1's. When does the following hold?

$$\varphi(g) = \begin{pmatrix} 1 & \cdots \\ \vdots & \ddots \end{pmatrix}$$

Note that $\varphi(g)_{ii} = 1 \iff gx_i = x_i$, $\varphi(g)_{ii} = 0 \iff gx_i \neq x_i$. Let $\chi := \operatorname{char} \varphi$. Then $\chi(g) = \operatorname{tr} \varphi(g) = |\{x \in X \mid gx = x\} = x^g$. Note that $\chi(g)$ is an integer.

### §1.1 Not entirely sure what happened today...

**Theorem 1.1.** *Let $G$ be a group, $X$ a finite set such that $G$ acts on $X$. Let $\chi$ be the character of the representation induced from the action of $G$ on $X$. Then the number of orbits is equal to*

$$\frac{1}{|G|} \sum_{g \in G} \chi(g).$$

*Proof.* Consider
$$S = \{(x, g) \mid x \in X, g \in G \,\text{such that}\, gx = x\}.$$
Computer the number of $\#S$ in two different ways:

1. Fix $g \in G$. Then $\#\{(x, g) \in S \mid g \,\text{is fixed}\} = \#x^g = \chi(g)$. Define the set above as $S_g$: then $S = \amalg_{g \in G} S_g$ which implies $\#S = \sum_{g \in G} \chi(g)$.

2. Let $S = \{(x, g) \mid x \in X,\ g \in G, gx = x\}$. Fix $x$ such that $S_x = \{(x, g) \in S \mid x \,\text{is fixed}\}$. Then the number of $S_x$'s is equal to $|G_x|$ where $G_x$ denotes the stabilizer of $x$. Recall $x' < g_0 \cdot x \implies G_{x'} = g_0 G_x g_0^{-1}$. Then

$$\begin{aligned} S = \amalg_{x \in X} S_x \implies \#S &= \sum_{x \in X} \#S_x \\ &= \sum_{x \in X} |G_x| \\ &= \sum_{\text{distinct orbits}}^{\text{missed some stuff}} \end{aligned}$$

Then (1) and (2) together imply the number of orbits is equal to $\frac{1}{|G|} \sum_{g \in G} \chi(g)$. ⊠

**Corollary 1.1.** *Let $G$ act on $X$ transitively. Assume that $|X| > 1$. Then there exists a $g \in G$ such that fixes no element of $x$ (ie, $\#x^g = 0$).*

*Proof.* We have by the theorem that the number of orbits is equal to $\frac{1}{|G|} \sum_{g \in G} |x^g|$. Since we only have one orbit (since the action is transitive), $|G| = \sum_{g \in G} \#X^g$ and the number of $x^g \in \mathbb{N}$, together these imply that the number of $X^g$ is equal to 1. This is false since the number of $X^{1_G} = |X| > 1$, therefore the number of $x^g = 0$ for some $g \in G$. ⊠

**Corollary 1.2.** *If $H$ is a proper subgroup of $G$ and $G$ is finite, then*

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

*Proof.* Let $G$ act on $G/H$ by left multiplication. Let $k \in G$. Then $g \in G_{kH} \iff gkH = kH \iff gk \in kH \iff g \in kHk^{-1}$. Let $g \in G$: then $X^g = \{kH \mid g \in kHk^{-1}\}$. If $G = \bigcup_{k \in G} kHk^{-1}$, then for every $g \in G$, there exists some $k_0$ such that $g \in k_0 H k_0^{-1}$. This subsequently implies that for all $g \in G$, $x^g \ni k_0 H$ for some $k_0$, contradicting Corollary one and two (insert ref later, just the previous two). ⊠