Differential Topology Notes

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Contents

Lecture 1

March 30, 2021

todo:brouwer stuff (not hard)

1.1 Mod 2 degree: first attempt

Fix a positive integer n. Let X be a compact n-manifold and Y a connected n-manifold. Suppose $f: X \to Y$ is smooth. If $q \in Y$ is a regular value, then $f^{-1}(q)$ is a 0-dimensional submanifold (by the preimage theorem). The degree counts the number of points in $f^{-1}(q)$.

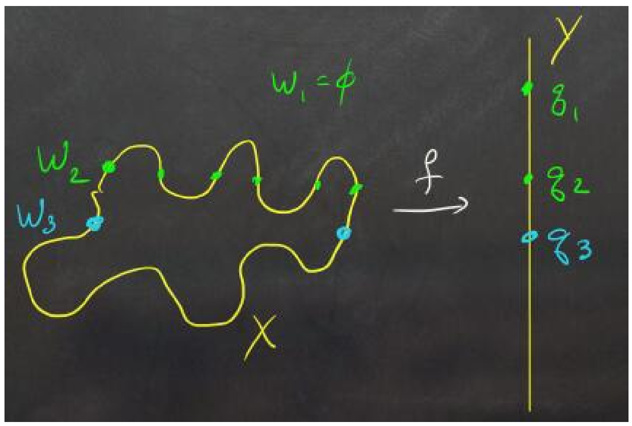

Figure 1: The mod 2 degree is independent of the regular value q_i.

The standard degree depends on the regular value q; in Figure 1, you can see the degree go from $0 \to 6 \to 2$ as $q_1 \to q_2 \to q_3$. However, mod 2 the degree is constant, so it's independent of q. Examine the inverse images W_1 and W_2 of the closed intervals $[q_1, q_2]$ and $[q_2, q_3]$; note that W_1 and W_2 are 1-dimensional compact manifolds with boundary todo:see lec 13 for proof. In fact, W_1 is a bordism from $f^{-1}(q_1)$ to $f^{-1}(q_2)$, while W_2 is a bordism from $f^{-1}(q_2)$ to $f^{-1}(q_3)$.

The fact that this degree mod 2 is invariant follows from todo:see classificaiton of 1-manifold lec: number of boundary points of a compact 1-manifold is even. As t varies through $[q_1, q_3]$, we can see the birth and death of preimage pairs as we pass through critical values.

Definition 1.1. A **smooth homotopy** of maps $X \to Y$ between manifolds (without boundary) is a smooth map $F: [0,1] \times X \to Y$. We write $F_t : X \to Y$ for the restriction of F to $\{t\} \times X$.

Theorem 1.1. *Fix $n \in \mathbb{Z}^{>0}$ and let X be a compact n-manifold, Y a connected n-manifold, and $f: X \to Y$ a smooth map. Then*

(1) *The mod 2 cardinality $\# f^{-1}(q) \pmod 2$ of the inverse image of a regular value $q \in Y$ is independent of q.*
(2) *If $F: [0,1] \times X \to Y$ is a smooth homotopy of maps, and $q \in Y$ a simultaneous regular value of F, F_0, and F_1, then $\# F_0^{-1}(q) \equiv \# F_1^{-1}(q) \pmod 2$.*

Proof. For (2), note that the simultaneous regular values of F, F_0, F_1 exist by Sard's theorem. Observe that $\partial([0,1]\times X) = \{0\} \times X \amalg \{1\} \times X$, so $\partial F = F_0 \amalg F_1$. todo:by some theorem, we have $W := F^{-1}(q)$ a 1-dimensional submanifold of $[0,1] \times X$, and

$$\partial W = W \cap (\{0\} \times X) \amalg W \cap (\{1\} \times X) = \{0\} \times F_0^{-1}(q) \amalg \{1\} \times F_1^{-1}(q).$$

Since $\# \partial W$ is even, it follows that $\# F_0^{-1}(q) \equiv F_1^{-1}(q) \pmod 2$. ⊠

todo:finish

1.2 The mod 2 winding number

Let A be $(n+1)$-dimensional affine space with V acting on A by translations, X be a compact n-manifold. Let $f : X \to A, q \in A \setminus f(x)$. Define $w_q : X \to S = S(v) \subseteq V, p \mapsto \frac{f(p)-q}{\|f(p)-q\|}$.

Definition 1.2. The **mod 2 winding number** is given by

$$W_2(f_1 q) = \deg_2 w_q \in \mathbb{Z}/2\mathbb{Z}.$$

Remark 1.1.

- $w_2(f,q)$ depends only on $[q] \in \pi_0(A \setminus f(x))$,
- $w_2(f,q)$ is unchanged under smooth homotopies of f which do not contain q in the image.

There are two methods to compute $w_2(f,q)$.

Theorem 1.2. *If W is a compact $(n+1)$-manifold with $\partial W = X$, $F : W \to A$ such that $\partial F = f$, suppose $q \in A \setminus f(X)$ is a regular of F. Then $w_2(f,q) = \#F^{-1}(q) \pmod 2$.*

Proof. todo:free time ⊠

Theorem 1.3. *Let $z = z_q(\xi)$. If $f \,\overline{\pitchfork}\, z$, then $w_2(f,q) = \#_2(f,z)$ in $Y = A \setminus \{q\}$.*

1.3 The Jordan Brouwer theorem

This is the famous topological fact that's notoriously hard to prove. Say we embed S^1 in $\mathbb{R}^2$. Then the embedding has two components, a bounded interior and an unbounded exterior.

The Jordan curve theorem. *Suppose $X \subseteq A$ (where A is affine space) is a compact connected hypersurface (submanifold of codimension 1). Then $A \setminus X$ has two path components D_0, D_1, exactly one of which, say D_1, is bounded. The closure $\overline{D_1}$ is a compact manifold with boundary with $\partial \overline{D_1} = X$. Finally, if $q \in D_j$, then $w_2(i_X, q) = j \pmod 2$, where $i_x : X \to A$ denotes the inclusion.*

Proof. todo: ⊠

Seems like Borsuk Ulam is going in the notes.

Corollary 1.1. *There does not exists an embedding $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{A}^3$.*

Lecture 2

March 25, 2021

todo:a lot of unclean notes commented out, also read everything about pertrubing to get transverse intersection

2.1 Mod 2 degree (again)

todo:complete last time proof

Proposition 2.1. *Let X be a compact connected manifold. Then id_X is not smoothly homotopic to a constant map.*

Proof. The mod 2 degree is defined for maps $X \to X$, and $\deg_2 \mathrm{id}_X = 1$, since every point of X is a regular value with a single inverse image point. On the other hand, the constant map $X \to X$ with value $p \in X$ has any $q \neq p$ as a regular value with empty inverse image, so the mod two degree of a constant map is zero. ⊠

Proposition 2.2. *Let n be a positive integer, W a compact $(n+1)$-dimensional manifold with boundary, Y a connected n-dimensional manifold, and $F : W \to Y$ a smooth map. Then the mod two degree of the restriction of F to the boundary vanishes, or $\deg_2 \partial F = 0$.*

Proof. Let $q \in Y$ be a simultaneous regular value of $F, \partial F$. Then $F^{-1}(q) \subseteq W$ is a compact 1-dimensional with $\partial F^{-1}(q) = F^{-1}(q) \cap \partial W$. Now apply todo:fact that boundary of 1 maniofld is even ⊠

Proposition 2.3. *Let X be a compact n-manifold. Then there exists $f : X \to S^n$ such that $\deg_2 f = 1$.*

Proof. todo: ⊠

2.2 Mod 2 intersection theory

Let Y be a smooth manifold and $X, Z \subseteq Y$ submanifolds of complementary dimension: $\dim X + \dim Z = \dim Y$. We want to define the *intersection number* of X and Z in Y by counting the elements of $X \cap Z \subseteq Y$. An issue is that this intersection may be infinite; let $Y = \mathbb{A}^r$ and $X = Z = \{(x, 0) \mid x \in \mathbb{R}\} \subseteq \mathbb{A}^2$. So we need to perturb one of the submanifolds to achieve a transverse intersection. Our techniques allow us to perturb maps, so pertrub the inclusion $i_X : X \hookrightarrow Y$. So we can generalize the setup to an arbitrary smooth map $f : X \to Y$. todo:corollary from last lec implies that we can homotope f to a map $g : X \to Y$ such that $g \pitchfork Z$, and so $g^{-1}(Z) \subseteq X$ is a 0-dimensional submanifold. We want this set to be finite, so we add that X must be *compact* in the conditions. We also want the number of points mod 2 in $g^{-1}(Z)$ to be independent of perturbation, which requires $Z \subseteq Y$ be *closed*.

Example 2.1. Consider $Y = \mathbb{A}^2, Z = \{(x, 0) \mid x \in \mathbb{R}^{\neq 0}\} \subseteq \mathbb{A}^2$, and $X = \{(x, y) \mid (x-1)^2 + y^2 = 1\}$. Then $\#(X \cap Z) = 1$, but any small nonzero translation of X intersects Z in 2 points.

Setup. Here, X is a compact manifold, Y is a manifold, $Z \subseteq Y$ is a *closed* submanifold, $f : X \to Y$ is smooth, and $\dim X + \dim Z = \dim Y$.

Lemma 2.1. *Let $g_0, g_1 : X \to Y$ be smoothly homotopic maps satisfying $g_0, g_1 \pitchfork Z$. Then $\#g_0^{-1}(Z) = \#g_1^{-1}(Z)$.*

Definition 2.1. Define the **mod 2 intersection number** $\#_2(f, Z) = \#g^{-1}(Z)$, where $g \simeq f$ is any smoothly homotopic map such that $g \pitchfork Z$. Such map exists by todo:corollary in lec 16, and the intersection number is independent of choice of g by Lemma 2.1.

Remark 2.1. If $X \subseteq Y$ is a compact submanifold and $f = i_X$ is the inclusion, then we write $\#_2(X, Z) = \#_2(Z, X)$. This is not symmetric for X compact and Z closed, but if Z is compact, then $\#_2(X, Z) = \#_2(Z, X)$. We can prove this by letting $\Delta \subseteq Y \times Y$ be the diagonal submanifold, then

$$\#_2^Y(X, Z) = \#_2^Y(Z, X) = \#_2^{Y \times Y}(i_X \times i_Z, \Delta).$$

Proposition 2.4. *Given our setup,*

(1) If $f_0 \simeq f_1$ are smoothly homotopic, then $\#_2(f_0, Z) = \#(f_1, Z)$.

(2) If W is a compact $(n+1)$-dimensional manifold with boundary $\partial W = X$, and $F : W \to Y$ a smooth map such that $\partial F = f$, then $\#_2(f, Z) = 0$.

2.3 Examples

Example 2.2. Let $Y = S^1 \times S^1$, and consider the submanifolds $X = S^1 \times \{0\}$ and $\mathbb{Z} = \{0\} \times S^1$. Then $\#_2(X, Z) = 1$. On the other hand, $\#_2(X, X) = \#_2(Z, Z) = 0$. You can organize these mod 2 intersection numbers into a 2×2 *intersection matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Example 2.3. Let $Y = \mathbb{RP}^2$ be the real projective plane, and $X = \mathbb{RP}^1 \subseteq \mathbb{RP}^2$ a projective line. Then $\#(X, X) = 1$. To compute this, perturb the inclusion $i : \mathbb{RP}^1 \to \mathbb{RP}^2$ to achieve transversality with the given line X, something we can achieve by choosing a transverse line. In terms of $\mathbb{RP}^2 = \mathbb{P}(\mathbb{R}^3)$, a projective line is a 2-dimensional subspace of $\mathbb{R}^3$, and two transverse 2-dimensional subspaces intersect in a 1-dimensional subspaces. That is, two projective lines intersect.

Theorem 2.1. *The 2-torus $S^1 \times S^1$ is not diffeomorphic to the 2-sphere S^2.*

Proof. If there is a diffeomorphism, we can find two 1-dimensional submani ⊠

Lecture 3

March 30, 2021

todo:the days are completely off

Setup. Let n denote a positive integer, A a real affine space of dimension $n+1$, V be a tangent space to A equipped wih an inner product, X be a compact n-manifold, and $f: X \to A$ a smooth map.

3.1 Mod 2 winding number

Choose $q \in A \setminus f(X)$ ($f(X) \neq A$ be Sard's theorem). Let $S = S(V) \subseteq V$ be the n-sphere of unit norm vectors. Define $w_q : X \to S$, $p \mapsto \frac{f(p)-q}{\|f(p)-q\|}$.

Definition 3.1. The **mod 2 winding number** of f about q is

$$W_2(f,q) = \deg_2 w_q.$$

Lecture 4

April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

A lot of definitions, here's the new ones:

Definition 4.1. A **subalgebra** of an algebra A is a linear subspace $A' \subseteq A$ containing 1 such that $a'_1 a'_2 \in A'$ for all $a'_1, a'_2 \in A'$. A **2-sided ideal** $I \subseteq A$ is a linear subspace such that $AI = I$ and $IA = I$. A $\mathbb{Z}$**-grading** of an algebra A is a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that $A^{k_1} A^{k_2} \subseteq A^{k_1+k_2}$ for all $k_1, k_2 \in \mathbb{Z}$. If A is a $\mathbb{Z}$-graded algebra and $a \in A^k, k \in \mathbb{Z}^{>0}$, then a is **decomposable** if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \cdots, a_k \in A^1$. If not, a is **indecomposable**.

4.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra $\bigotimes V$, thought of as the "free algebra generated by V".

Definition 4.2. Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map $i: V \to A$ such that for all (B, T) of an algebra B and a linear map $T: V \to B$ such that φ_T is a homomorphism of algebras.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "T"'] & & A \arrow[dl, dotted, "\exists!" ', "\varphi_T"] \\ & B & \end{tikzcd}$$

(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If $(\xi \neq 0) \in V$ and $i(\xi) = 0$, set $B = \mathbb{R} \oplus \mathbb{R}\xi$ and define $\xi^2 = 0$.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "\pi"'] & & A \arrow[dl, dotted, "\varphi"] \\ & \mathbb{R} \otimes \mathbb{R}\xi & \end{tikzcd}$$

Note that $\pi|_{\mathbb{R}\xi} = \mathrm{id}$. But $\xi = \pi(\xi) = \varphi_1(\xi) = 0$, a contradiction. Furthermore, A has a canonical $\mathbb{Z}$-grading. $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$, $T_\lambda \colon V \to V$ is scalar multiplication, $\varphi_\lambda \colon A \to A$ is a homomoprhism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

Definition 4.3. Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map $b \colon V \times V'' \to X$ such that for all (W, B),

$$\begin{tikzcd} V' \times V'' \arrow[rr, "b"] \arrow[dr, "B"'] & & X \arrow[dl, "\exists!", "T_B"'] \\ & W & \end{tikzcd}$$

We denote $X = V' \otimes V''$, and $b(\xi', \xi'') = \xi' \otimes \xi''$, $\xi' \in V', \xi'' \in V''$.

If S' is a basis of V', S'' a basis of V'', then $S' \times S''$ is a basis of $V' \otimes V''$, where

$$S' \times S'' \cong \{\xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S''\}.$$

Note that $\bigotimes$ is "commutative" and "associative" with unit $\mathbb{R}$, so

$$\begin{gathered} \mathbb{R} \otimes V \to V \\ V_1 \otimes V_2 \to V_2 \otimes V_1 \\ (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), \end{gathered}$$

forming what we call a **symmetric monoidal category**. We write $\otimes^1 V = V$, $\otimes^2 V = V \otimes V$, $\otimes^3 V = V \otimes V \otimes V$ and so on. We also write $\otimes^0 V = \mathbb{R}$, and sometimes replace $\otimes^n V$ with $V^{\otimes n}$.

4.2 Existence of tensor algebras

Let V be a vector space, and $A = \bigoplus_{k=0}^{\infty} \otimes^k V$. Let $i \colon V \hookrightarrow A$ be the inclusion into $\otimes' V = V$.

Claim. (A, i) is a tensor algebra over V.

To see this, note that

$$\underbrace{\xi_1 \otimes \cdots \otimes \xi_k)}_{\in \otimes^k V} \cdot_A \underbrace{\eta_1 \otimes \cdots \otimes \eta_\ell}_{\in \otimes^\ell V} = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that $A = \otimes' V$ is *not* commutative.

4.3 The Exterior Algebra

We want to impose todo:come back