Miscellaneous Notes on Linear Algebra

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Who ever suffered from learning too much linear algebra? These notes will seek to fill in my linear algebra gaps. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

Inner-Product Spaces

What is an inner product?? Let's find out.

1.1 Inner Products

The length of a vector x is the **norm** of x, denoted $\|x\|$. If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we have $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Note that the norm is not linear. For $x, y \in \mathbb{R}^n$, the **dot product** of x and y, denoted $x \cdot y$, is defined by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. Note that this is a number, not a vector. Clearly $x \cdot x = \|x\|^2$ for all $x \in \mathbb{R}^n$, which implies $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$ ($x \cdot x = 0$ only if x is the zero vector). The map that sends $x \in \mathbb{R}^n$ to $x \cdot y$ in $\mathbb{R}$ for fixed y is linear since $\mathbb{R}$ is a field. The dot product is also commutative, since $\mathbb{R}$ is.

Inner products generalize dot products. Recall that $|\lambda|^2 = \lambda\overline{\lambda}$ for $\lambda \in \mathbb{C}$. For $z \in \mathbb{C}^n$, we define the norm of z by $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. We take the modulus of z_i since we want the result to be nonnegative. Note that $\|z\|^2 = z_1\overline{z_1} + \cdots z_n\overline{z_n}$. We want to think of $\|z\|^2$ as the inner product of z with itself, like in $\mathbb{R}^n$. This suggests we define the inner product of $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$ with z as $w_1\overline{z_1} + \cdots + w_n\overline{z_n}$. We expect the inner product of w with z equal the complex conjugate of the inner product of z with w. With this motivation in mind, let us define inner products.

Definition 1.1 (Inner product)**.** An **inner product** on an F-vector space V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in F$ such that

(i) $\langle v, v\rangle \geq 0$ for all $v \in V$; (**positivity**)

(ii) $\langle v, v\rangle = 0$ iff $v = 0$; (**definiteness**)

(iii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$; (**additivity in first slot**)

(iv) $\langle av, w\rangle = a\langle v, w\rangle$ for all $a \in F$ and all $v, w, \in V$; (**homogeneity in first slot**)

(v) $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$. (**conjugate symmetry**).

For real numbers, condition (v) simply becomes $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$. An **inner product space** is a vector space V along with an inner product on V.

Example 1.1. The most important example is the **Euclidian inner product** on $\mathbf{F}^n$ (Axler uses $\mathbf{F}$ to denote either $\mathbb{C}$ or $\mathbb{R}$). We define an inner product on $\mathbf{F}^n$ by

$$\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n)\rangle = w_1\overline{z_1} + \cdots w_n\overline{z_n}.$$

An example of another inner product on $\mathbf{F}^n$ is defined by $\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n)\rangle = c_1 w_1\overline{z_1} + \cdots + c_n w_n\overline{z_n}$ for c_i positive constants. The case where $c_i = 1$ for all i is simply the standard Euclidian inner product.

Example 1.2. Consider the vector space $\mathscr{P}_m(\mathbf{F})$, the polynomial ring over $\mathbf{F}$ of polynomials with degree at most m. We can define an inner product on $\mathscr{P}_m(\mathbf{F})$ by

$$\langle p, q\rangle = \int_0^1 p(x)\overline{q(x)}\, dx.$$

For fixed $w \in V$, the function that takes v to $\langle v, w\rangle$ is a linear map $V \to \mathbf{F}$. So $\langle 0, w\rangle = 0$, and by condition (v) $\langle w, 0\rangle = 0$ as well. Furthermore, $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ and $\langle u, av\rangle = \overline{a}\langle u, v\rangle$ hold as well: This second condition is known as conjugate homogeneity in the second slot.

1.2 Norms

For $v \in V$, we define the **norm** of v, denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v\rangle}$. For example, if $p \in \mathscr{P}_m(\mathbf{F})$, then $\|p\| = \sqrt{\int_0^1 |p(x)|^2\,dx}$. Some properties: $\|v\| = 0$ iff $v = 0$, and $\|av\| = |a|\|v\|$. To see this, note that $\|av\|^2 = \langle av, av\rangle = a\langle v, av\rangle = a\overline{a}\langle v, v\rangle = |a|^2\|v\|^2$, taking square roots gives us our result. This illustrates a general idea: working with norms squared is easier than working directly with norms.

Two vectors $u, v \in V$ are **orthogonal** if $\langle u, v\rangle = 0$. The zero vector is orthogonal to every vector, and the only vector orthogonal to itself. Assume $V = \mathbb{R}^2$, now let us state a 2500 year old theorem.

Pythagorean Theorem. *If u, v are orthogonal vectors in V, then*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. Exercise. ⊠

Suppose $u, v \in V$. We want to write u as a scalar multiple of v plus a vector w orthogonal to v. Let $a \in \mathbf{F}$ be a scalar, then $u = av + (u - av)$. We need to choose a such that v is orthogonal to $u - av$, in other words, we want $0 = \langle u - av, v\rangle = \langle u, v\rangle - a\|v\|^2$. So we should choose $a = \langle u, v\rangle / \|v\|^2$ (where $v \neq 0$). Then

$$u = \frac{\langle u, v\rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v\rangle}{\|v\|^2} v\right).$$

Cauchy-Schwarz Inequality. *If $u, v \in V$, then*

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality iff one of u, v is a scalar multiple of the other.

Proof. Let $u, v \in V$, and assume $v \neq 0$. Consider $u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$, where w is orthogonal to v. By the Pythagorean theorem, we have

$$\|u\|^2 = \left\|\frac{\langle u, v\rangle}{\|v\|^2} v\right\|^2 + \|w\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Multiply both sides, take a square root, and we are done. This is an equality iff $w = 0$, but this is true iff u is a multiple of v. ⊠

Triangle Inequality. *If $u, v \in V$, then*

$$\|u+v\| \leq \|u\| + \|v\|.$$

This is an equality iff one of u, v is a nonnegative multiple of the other.

Proof. Let $u, v \in V$. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} = \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \leq \|u\|^2 + \|v^2\| + 2\|u\|\|v\| = (\|u\| + \|v\|)^2.$$

The inequality step frollows from Cauchy-Schwartz, where $2\operatorname{Re}\langle u, v\rangle \leq 2|\langle u, v\rangle|$. Taking square roots gives the triangle inequality. This is an equality iff the two inequalities above are equalities, which is true iff $\langle u, v\rangle = \|u\|\|v\|$. ⊠

Parallelogram Equality. *If $u, v \in V$, then*

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

Proof. Exercise. ⊠

1.3 Orthonormal Bases

A list $(e_1, \cdots, e_m)$ of vectors in V is orthonormal if $\langle e_j, e_k \rangle = 0$ when $j \neq k$ and equals 1 when $j = k$, for $j, k \in \{1, \cdots, m\}$. Orthonormal lists are nice.

Proposition 1.1. *If $(e_1, \cdots, e_m)$ is an orthonormal list of vectors in V, then*

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \cdots, a_m \in \mathbf{F}$.

Proof. Since each e_j has norm 1, this follows from repeated applications of the Pythagorean theorem. ⊠

Corollary 1.1. *Every orthonormal list of vectors is linearly independent.*

An **orthonormal basis** of V is an orthonormal list of vectors in V that forms a basis for V. The standard basis is a good example. If we find an orthonormal list of length $\dim V$, then this is automatically an orthonormal basis of V (since they must be LI). In general, given a basis $(e_1, \cdots, e_n)$ of V and a vector $v \in V$, we know there is some choice of scalars $a_1, \cdots, a_m$ such that $v = a_1 e_1 + \cdots a_n e_n$, but finding the a_j's can be difficult. This is not the case for an orthonormal basis.

Theorem 1.1. *Suppose $(e_1, \cdots, e_n)$ is an orthonormal basis of V. Then*

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$$

for every $v \in V$.

Proof. Let $v \in V$. Since $(e_1, \cdots, e_n)$ is a basis of V, there exist scalars $a_1, \cdots, a_n$ such that $v = a_1 e_1 + \cdots + a_n e_n$. Taking the inner product of both sides with e_j, we get $\langle v, e_j \rangle = a_j$. The second part follows from the first proposition and our previous result. ⊠