380C PROBLEM SET 4

DUE WEDNESDAY, SEPTEMBER 29TH

Problem 1. For p prime and $G = S_p$ the symmetric group on p letters, find n_p, the number of p-Sylow subgroups. Using the Sylow theorems, deduce *Wilson's theorem*: $(p-1)! = -1 \bmod p$.

Problem 2. For G a group and H a subgroup, the *index* of H in G is $|G/H|$. (Often, the index is denoted $[G : H]$.)

Let G be (possibly infinite) group and let H be a subgroup of finite index. Show that there is a subgroup $K \subseteq H \subseteq G$ with K finite index and normal in G.

Problem 3. Show that any subgroup of index 2 is normal.

Problem 4. Let p be a prime and let G be a finite group of order $n = p^r m$ for $p \nmid m$. In this problem, we give an alternative proof of the first Sylow theorem.

Throughout, we let X_i denote the set of all subsets $S \subseteq G$ of order $i > 0$. We consider G acting on X_i via left translation, i.e., by using the left action of G on itself and hence on subsets of itself.

(a) Show that every orbit of G on X_i has size $\geqslant \frac{n}{i}$ with equality if and only if the orbit $\mathcal{O} \subseteq X_i$ is the set of all left cosets of some subgroup $H \subseteq G$ of order i.

(b) Show that $|X_{p^r}| = m \mod p$.

(Hint, though there are other ways to do it too: consider $\mathbb{Z}/p^r \simeq m\mathbb{Z}/n \subseteq \mathbb{Z}/n$ acting on the set Y of subsets of $\mathbb{Z}/n$ of order p^r and apply the p-group congruence lemma to obtain a binomial coefficient congruence.)

(c) Show that there is an orbit for the action of G on X_{p^r} with order coprime to p. Deduce that G admits a p-Sylow subgroup.

Problem 5. Let G be a finite group and let p be a prime number. Suppose $G \subseteq H$ and $H_p \subseteq H$ is a p-Sylow subgroup of H. Show that there exists an element $h \in H$ such that $\mathrm{Ad}_h(H_p) \cap G$ is a p-Sylow subgroup of G. Deduce that the first Sylow theorem follows once it is known for symmetric groups (we will give a direct construction in that case later in the course).

Problem 6. Let X be a set with a G-action. For $x, y \in X$ lying in the same orbit, show that the stabilizers of x and y in G are conjugate subgroups of G.