Complex Analysis Homework

Math 361

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Homework 1 (8/27/20)

Section 2: Problems $1, 4, 10$. Let P represent the ordered set of problems under the $<$ relation (note that $<$ is a strict total ordering), e.g. $\{1, 4, 10\}$ for Homework 1. We accept the Axiom of Choice: then problem numbers in this LaTeX document are represented by the inverse image $f^{-1}(p)$ of some $p \in P$, where $f\colon \mathbb{N} \to P$ is the natural order surjection (f is not injective unless we restrict its domain to the subset $A_n \subset \mathbb{N}$, where $A_n = \{1, 2, ..., n\}$, $n = |P|$). We have $1 \mapsto p_1$, where p_1 is the least element of P (which exists by the Well-Ordering Theorem, if you view P as a non-empty subset of the set of all problems $\mathscr{P}$). Similarly, $2 \mapsto p_2$, where p_2 is the next element such that $p_2 > p_1$ but for every $p \in P$ not equal to p_1 or p_2, $p > p_2$. Continuing on, we map elements of $\mathbb{N}$ onto P in this way. For example, even though I may be working on the question $4 \in P$, in reality it is denoted in the LaTeX document by question $2 \in \mathbb{N}$, since $f^{-1}(4) = 2$ (that is, problem 4 is the second problem in the list).

Problem 1 (Question 1)**.** Verify that

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$;

(b) $(2, -3)(-2, 1) = (-1, 8)$;

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$

Solution. The solutions follow from some computations.

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = (\sqrt{2} - i - i + i^2\sqrt{2}) = \sqrt{2} - 2i - \sqrt{2} = -2i$.

(b) $(2, -3)(-2, 1) = ((2 \cdot -2) - (1 \cdot -3), (-3 \cdot -2) + (2 \cdot 1)) = (-4 + 3, 6 + 2) = (-1, 8)$.

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (9 + 1, 3 - 3)\left(\frac{1}{5}, \frac{1}{10}\right) = (10, 0)\left(\frac{1}{5}, \frac{1}{10}\right) = (2 - 0, 0 + 1) = (2, 1)$.

$\blacksquare$

Problem 2 (Question 2, not assigned. Safe to ignore)**.** Show that

(a) $\operatorname{Re}(iz) = -\operatorname{Im} z$;

(b) $\operatorname{Im}(iz) = \operatorname{Re} z$.

Solution. The solutions follow from some algebraic manipulation.

(a) Let $z \in \mathbb{C}$, then $z = a + bi$ for $a, b \in \mathbb{R}$. Note that $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$. Then $\operatorname{Re}(iz) = \operatorname{Re}(i(a + bi)) = \operatorname{Re}(ia + i^2b) = \operatorname{Re}(-b + ia) = -b = \operatorname{Im} z$.

(b) Let $z \in \mathbb{C}$, then $\operatorname{Im}(iz) = \operatorname{Im}(i(a + bi)) = \operatorname{Im}(ia + i^2b) = \operatorname{Im}(-b + ia) = a = \operatorname{Re} z$.

$\blacksquare$

Problem 3 (Question 4)**.** Verify that $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

Solution. Let $z = 1 + i$. Then $z^2 - 2z + 2 = (1+i)^2 - 2(1+i) + 2 = (1 + 2i - 1) - 2 - 2i + 2 = 2i - 2i = 0$.

Now let $z = 1 - i$. Then $z^2 - 2z + 2 = (1-i)^2 - 2(1-i) + 2 = (1 - 2i - 1) - 2 + 2i + 2 = -2i + 2i = 0$.

Note that this is just an example of that fact that conjugate elements are defined as both being solutions to the minimal polynomial of an algebraic element over a field. ■

Problem 4 (Question 10)**.** Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Then show that the additive inverse of $z = x + iy \in \mathbb{C}$ can be written as $-z = -x - iy$ without ambiguity.

Solution. We have $-(iy) = -((0,1) \cdot (y,0)) = -(0 - 0, y + 0) = -(0, y) = (0, -y)$. We also have $(-i)y = (0, -1) \cdot (y, 0) = (0 - 0, -y + 0) = (0, -y)$. We conclude that $-(iy) = (-i)y$.

To show that we can write the additive inverse of $z = x + iy \in \mathbb{C}$ (denoted by $-z$) as $-z = -x - iy$ without ambiguity: Our first possibility is that $-x - iy$ refers to $-x + (-(iy))$ (denoted $-x - (iy)$ from now on). Then $-z + z = (-x - (iy)) + (x + (iy)) = (-x + x) + (-(iy) + (iy))$. Clearly $-x$ and $-(iy)$ are the additive inverses of x and (iy) respectively, so this sum is equal to zero plus zero which is just zero. The second possibility is that $-x - iy$ refers to $-x + ((-i)y)$, in which case we have previously shown that $(-i)y = -(iy)$, so this sum is equal to $-x - (iy)$, and we are done. ■

Homework 2 (8/28/20)

Section 3: Problems 2,4,7.

Problem 5 (Question 2)**.** Show that

$$\frac{1}{1/z} = z,$$

where $z \neq 0$.

Solution. We know that $z^{-1} = 1/z$ exists and is equal to

$$\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

since z is non-zero. Continuing on, we have $(1/z)^{-1} = \frac{1}{1/z}$ exists ($z \neq 0$), and with a simple application of the previous formula is equal to

$$\left(\left(\frac{\left(\frac{x}{x^2+y^2}\right)}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2}\right), \left(\frac{-\left(\frac{-y}{x^2+y^2}\right)}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2}\right)\right).$$

This may look intimidating, but we can easily reduce this to

$$\left(\frac{\frac{x}{x^2+y^2}}{\left(\frac{x^2+(-y)^2}{(x^2+y^2)^2}\right)}, \frac{\frac{-(-y)}{x^2+y^2}}{\left(\frac{x^2+(-y)^2}{(x^2+y^2)^2}\right)}\right),$$

which once again simplifies to

$$\left(\frac{\frac{x}{x^2+y^2}}{\frac{x^2+y^2}{(x^2+y^2)^2}}, \frac{\frac{y}{x^2+y^2}}{\frac{x^2+y^2}{(x^2+y^2)^2}}\right) = \left(\frac{\frac{x}{x^2+y^2}}{\frac{1}{x^2+y^2}}, \frac{\frac{y}{x^2+y^2}}{\frac{1}{x^2+y^2}}\right) = (x, y) = z.$$

■

Problem 6 (Question 4)**.** Prove that if $z_1z_2z_3 = 0$, then at least one of the three factors is equal to zero.

Proof. Let $z_1z_2z_3 = (z_1z_2)z_3 = 0$. Then either (z_1z_2) or z_3 is zero (proof from the book): WLOG, assume that $(z_1z_2)z_3 = 0$ and $(z_1z_2) \neq 0$. Since the complex numbers form a field, we have $(z_1z_2) \in \mathbb{C}$ so $(z_1z_2)^{-1} \in \mathbb{C}$, and $z \cdot 0 = 0$ for all $z \in \mathbb{C}$. So

$$\begin{aligned} z_3 &= z_3 \cdot 1 \\ &= z_3 \left((z_1z_2)(z_1z_2)^{-1}\right) \\ &= \left((z_1z_2)^{-1}(z_1z_2)z_3\right) \\ &= (z_1z_2)^{-1}\left((z_1z_2)z_3\right) \\ &= (z_1z_2)^{-1} \cdot 0 \\ &= 0. \end{aligned}$$

If z_3 is zero, then we are done. If $(z_1 z_2)$ is zero, then we apply the same logic again to conclude that either z_1 or z_2 is zero. So either way, one of the factors z_1, z_2, or z_3 must be zero, and we are done (note that you can prove that this holds for any number of factors by induction). ⊠

Problem 7 (Question 7)**.** Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

Proof. We have

$$\begin{aligned}
z(z_1 + z_2 + z_3) &= z((z_1 + z_2) + z_3) \text{ by Associativity of Addition} \\
&= z(z_1 + z_2) + zz_3 \text{ by the Distributive Law} \\
&= (zz_1 + zz_2) + zz_3 \text{ by the Distributive Law} \\
&= zz_1 + zz_2 + zz_3 \ \text{ by Associativity of Addition,}
\end{aligned}$$

completing the proof. ⊠