# Gromov's Norm and Bounded Cohomology Lecture Notes

Simon Xiang

Lecture notes for the Spring 2022 graduate section of a topics course called "Gromov's norm and bounded cohomology" (Math 392C) at UT Austin, taught by Dr. Chen. These notes were taken live in class (and so they may contain many errors). Source files: https://git.simonxiang.xyz/math_notes/files.html

## Contents

# 1 Straightening

**Theorem 1.1.** *Let $M^n$ be a hyperbolic closed orientable manifold. Then $\|M\|_1 \geq \mathrm{vol}(M)/v^n$, where $v_n = \sup \mathrm{vol}(\Delta^n) < \infty$, $v_2 = \pi$, $\Delta^n$ hyperbolic.*

**Lemma 1.1.** $v_2 = \pi$.

*Proof.* There are two proofs.

(1) A hyperbolic triangle with angles $A, B, C$ has the formula area $= \pi - (A + B + C)$. This implies that sup $= \pi$.

(2) For any hyperbolic triangle, area$(\Delta) <$ area$(\Delta')$ where $\Delta'$ is an ideal hyperbolic triangle. We have shown that area$(\Delta')$ is $\pi$, so sup $= \pi$. Why can we always bound a triangle by an ideal triangle? Take an arbitrary hyperbolic triangle and a point in its interior, then take geodesics toward the boundary. This results in an ideal triangle strictly containing the original. This argument works in higher dimensions as well. ⊠

About *straightening*; consider the linear map str: $C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$ on the basis $C\colon \Delta^k \to M$. This lifts to a map $\widetilde{C}\colon \Delta^k \to \mathbb{H}^n$.

$$\begin{tikzcd} & \mathbb{H}^n \arrow[d, "p"] \\ \Delta^k \arrow[ur, "\widetilde{C}"] \arrow[r] & M \end{tikzcd}$$

Recall $\widetilde{\mathrm{str}}\colon C_k(\mathbb{H}^n;\mathbb{R}) \to C_k(\mathbb{H}^n;\mathbb{R})$.

(1) For every $g \in \mathrm{Isom}(\mathbb{H}^n)$, $g \cdot \widetilde{\mathrm{str}}(\widetilde{C}) = \widetilde{\mathrm{str}}(g \cdot \widetilde{C})$. Then define $\mathrm{str}(C) = p \cdot \widetilde{\mathrm{str}}(\widetilde{c})$.

(2) Straightening commutes with the boundary, or $\partial \cdot \widetilde{\mathrm{str}} = \widetilde{\mathrm{str}} \cdot \partial$.

One might object that there are different choices for the lift, but thesea ll differ by a transformation, and property (1) says we can move the $g$ outside the list. Translating and projecting is the same as projecting onto the translation, so this is well-defined. A nice property is that the boundary also commutes with the straightening downstairs, or $\partial \cdot \mathrm{str} = \mathrm{str} \cdot \partial$, where $\partial\colon C_{k+1}(M;\mathbb{R}) \to C_k(M;\mathbb{R})$. Any linear map commuting with the boundary induces a map on the homology, so str induces a map $\mathrm{str}_*\colon H_k(M;\mathbb{R}) \to H_k(M;\mathbb{R})$.

**Lemma 1.2.** $\mathrm{str}_* = \mathrm{id}_{H_k(M;\mathbb{R})}$. *In other words, straightening does not change homology classes.*

*Proof.* When we do straightening, the new things are homotopic to the stuff we had earlier. Take a linear homotopy on $\mathbb{R}^{n+1}$, then project to $\mathbb{H}^n$. This implies that $\mathrm{str}_* = \mathrm{id}$. ⊠

**Lemma 1.3.** $|\mathrm{str}_* c|_1 \leq |c|_1$.

**Corollary 1.1.** *For every $\sigma \in H_k(M;\mathbb{R})$, we have $\|\sigma\|_1 = \inf_{[c]=\sigma} |c|_1$ where $c$ is straight.*

This is the key takeaway from the straightening operation; to calculate the norm on homology classes, all we have to do is take it over straightened classes.

*Proof of Theorem 1.1.* Suppose $[M]$ is represented by some cycle $c = \sum \lambda_i c_i$. By Corollary 1.1, we may assume that each $c_i$ is a straight hyperbolic $n$-simplex. This implies $\mathrm{vol}(c_i) \leq v_n$. Let vol be the volume form on $M$, representing a cohomology class dual to the fundamental class. Then

$$\mathrm{vol}(M) = \int_M \mathrm{vol} = \langle [M], \mathrm{vol}\rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol} \right\rangle = \sum \lambda_i \langle c_i, \mathrm{vol}\rangle = \sum \lambda_i \mathrm{vol}(c_i) \leq \sum |\lambda_i| \cdot \mathrm{vol}(c_i) \leq v_n \cdot \sum |\lambda_i| = v_n \cdot |c|_1.$$

This implies $|c|_1 \geq \frac{\mathrm{vol}(M)}{v_n}$. Since $c$ is arbitrary, $\|M\|_1 \geq \frac{\mathrm{vol}(M)}{v_n}$. In the case $n = 2$, $\mathrm{vol}(M) = -2\pi\chi(M)$, $v_2 = \pi$, which implies $\|M\|_1 \geq -2\chi(M)$. ⊠

**Remark 1.1.** The volume form is not a bounded function on all singular simplices. However, it is bounded on straight simplices. We have $\mathrm{vol} \circ \mathrm{str}$ bounded, representing the volume class. This leads to bounded cohomology, asking which cocycles can be bounded.

**Remark 1.2.** It is not crucial to do this for something exactly hyperbolic. We can do a similar thing for $M$ negatively curved and closed, the argument helps us show that $\|M\|_1 > 0$.

**Conjecture (Gromov).** *Let $M$ be closed, non-positively curved with negative Ricci curvature. Then $\|M\|_1 > 0$.*

**Proposition 1.1.** *If $M$ is closed hyperbolic, then $\|\cdot\|_1$ is an honest norm (not just a semi-norm) on $H_k(M;\mathbb{R})$ for every $k \geq 2$, i.e. $\sigma \neq 0 \in H_k(M;\mathbb{R})$ implies $\|\sigma\|_1 > 0$.*

*Proof.* There is a pairing $H_k(M;\mathbb{R}) \times H^k(M;\mathbb{R}) \to \mathbb{R}$ which is non-singular. In other words, $\sigma \in H_k(M;\mathbb{R})$ corresponds to $\sigma^* \in H^k(M;\mathbb{R})$ such that $\langle \sigma, \sigma^* \rangle \neq 0$. More concretely, we can think of $H^k(M;\mathbb{R})$ as de Rham cohomology where the classes are differential forms, and integrate the $k$-forms on something $k$-dimensional. Represent $\sigma^*$ by some differential form $\omega$. Let $\|\mathrm{vol}\|_\infty := \sup |\omega_p(v_1, \cdots, v_k)|$ where we take the supremum over $p \in M$, the $v_1, \cdots, v_k \in T_p(M)$ are orthogonal, and $\|v_i\| = 1$. By compactness $\sup |\omega_p(v_1, \cdots, v_k)| < \infty$; then we can do the same pairing argument where $|\langle \sigma, \sigma^* \rangle| = \left|\left\langle \sum \lambda_i c_i, \omega \right\rangle\right| \leq \sum |\lambda_i| \cdot (\|\mathrm{vol}\|_\infty \cdot \mathrm{vol}(c_i))$. Then $\left.\frac{\omega}{\|\mathrm{vol}\|_\infty}\right|_{c_i} \leq \mathrm{vol}|_{c_i}$,
todo:unfinished ⊠

The point is we can do a similar argument for other cohomology classes as well. This argument is a powerful tool; we can do pairing with differential forms to get a lower bound. We saw that everything vanishes on $H_1$, why doesn't this argument work? This is because $v_1 = \infty$ (not-bounded).

Next time we will continue and give a proof that $v_n$ is a finite number.

# 2 ok

**Lemma 2.1.** *We have $v_n = \sup_{\Delta^n} \mathrm{vol}(\Delta^n) \leq \pi/(n-1)!$ for all $n \geq 2$, where $\Delta^n$ is a hyperbolic simplex.*

*Proof.* We already have $v_2 = \pi$. It suffices to look at ideal simplices. We show $v_n \leq \frac{v_{n-1}}{n-1}$ for all $n \geq 3$, which paired with induction implies the bound. Let $\Delta^n$ be an arbitrary ideal simplex. Let $s$ map a circle into a half-sphere that sits in $S^{n-1} \subseteq \mathbb{R}^n$. Then for $x \in \mathbb{D}^{n-1}, x \mapsto (x, h(x)), \|x\|^2 + |h(x)|^2 = 1$. So $h(x)^2 = 1 - \|x\|^2$, which implies $h(x) = \sqrt{1 - \|x\|^2}$.

(1) $\Delta^n = \{(x,y) \mid x \in \tau_0, y \geq h(x)\}$,

(2) $\tau = S(\tau_0)$.

We know the metric $ds^2/y$ is a Euclidian metric scaled by the last coordinate. So

$$\mathrm{vol}(\Delta^n) = \int_{\Delta^n} \frac{dx\,dy}{y^n} = \int_{\tau_0} \int_{h(x)}^\infty \frac{dy}{y^n}\,dx.$$

We can directly compute the inner integral since we are working in the upper half space model; $\int y^{-n} = \frac{y^{-n+1}}{-n+1}$, so this integral becomes $\frac{1}{n-1} \int_{\tau_0} \frac{1}{h(x)^{n-1}}\,dx \overset{\text{key}}{\leq} \frac{1}{n-1} \cdot \mathrm{vol}(\tau) \leq \frac{v_{n-1}}{n-1}$. Now it remains to show the key inequality. By definition, $\mathrm{vol}(\tau) = \int_{\tau_0} s^*\mathrm{vol}|_\tau$. We now compare the integrals point by point to see that one dominates the other. The pullback $s^*\mathrm{vol}$ is a 2-form on the unit disk, so we evaluate it on $(e_1, e_2)$. This is equal to the volume of the pushforward, or $\mathrm{vol}(s_* e_1, s_* e_2) > \mathrm{vol}|_\tau$ (std basis). What is the standard basis? Whatever it is, the restriction to $\tau$ adds one more vector in the orthonormal direction to make it an orthonormal basis, then evaluate. So this becomes $\frac{1}{h(x)^{n-1}}$. ⊠

**Remark 2.1.** A theorem of Haagerup-Munkholm shows that $v_n$ is uniquely achieved by the *regular* ideal $n$-simplex. So $\operatorname{sym}(\Delta^n) = S_{n+1}$.

This finishes our estimate. In the two-dimensional case, we have the computation of the volume of surfaces. Let us go back to our theorem.

**Theorem 2.1.** *If $S$ is an orientable closed connected surface with genus $g \geq 1$, then $\|S\|_1 = -2\chi(S)$.*

Note that this also holds in the genus 1 case since both the volume and the Euler characteristic are zero (admits a self map). If $S$ is orientable, closed, and connected, then denote

$$\chi^-(S) = \begin{cases} \chi(S) & g \geq 1 \\ 0 & g \leq 1. \end{cases}$$

With this notation, then $\|S\|_1 = -2\chi^-(S)$. If $S$ is orientable closed (without assuming connectedness), then $\chi^-(S) = \sum_{\Sigma \subseteq S} \chi^-(\Sigma)$, where the $\Sigma$ are components of $S$. Equivalently this is $\sum \chi$ over aspherical components. Then $\|S\|_1 = -2\chi^-(S)$.

**Recall.** Our problem: $\deg(S, S') = \{\deg(f) \mid f : S \to S'\}$, $S, S'$ occ.

**Proposition 2.1.** *If $S, S'$ occ with $g \geq 1$, then* $\deg(S, S') = \{d \in \mathbb{Z} \mid |d\chi(S')| \leq |\chi(S)|\}$.

*Proof.* By the degree inequality, we have $f : S \to S'$, $|\deg(f)| \cdot \|S'\|_1 \leq \|S\|_1$. It suffices to show that every $d$ satisfies $|d \cdot \chi(S')| \leq |\chi(S)|$. We can find $f : S \to S'$ with $\deg(f) = d$. We can compose with a self-homeomorphism to flip the degree, and degree 0 is just a constant map, so focus on $d$ positive. We have $d \cdot |\chi(S')| \leq |\chi(S)|$ by symmetry. We take two kinds of maps. ⊠

# 3 Measure homology

Recap:

- We used the straightening argument to show that the volume of a hyperbolic manifold $\|M\|_1 \geq \operatorname{vol}(M)/v_n$ for $n \geq 2$, $v_2 = \pi$.
- We also know an upper bound when studying triangulations on surfaces, eg for $n = 2$, $\|S\|_1 = \operatorname{vol}(S)/\pi = -2\chi(S)$ if the genus $\geq 1$.

An application is to compute the set of possible degrees $\{\deg f \mid f : S \to S'\}$, and the claim is that this is equal to the set $\{d \mid |\chi(S)| \geq |d\chi(S')|\}$. To make this work exactly, we need $S, S'$ closed oriented, not $S^2$.

*Proof.* First we prove the inclusion $\{\deg f \mid f : S \to S'\} \subseteq \{d \mid |\chi(S)| \geq |d\chi(S')|\}$. Let $f : S \to S'$. By the degree inequality, $|d| \cdot \|S'\|_1 \leq \|S\|_1$, which is just saying that $|d| \cdot |\chi(S')| \leq |\chi(S)|$ with $d = \deg f$.

The other direction is done by constructing maps between surfaces. First, exclude some silly cases; we may assume $d > 0$ and satisfies the inequality. There are two kinds of maps. To construct a map $S \to S'$ with degree one, we construct an intermediary covering surface $\Sigma$ for $S'$, where $S \to \Sigma$ is surjective and the covering $\Sigma \to S'$ has degree $d$. Let $\Sigma$ be a degree $d$ cover of $S'$. The nice thing is that $|\chi(\Sigma)| = |d\chi(S')| \leq |\chi(S)|$, so the genus $g(\Sigma) \leq g(S)$. We have $|\chi(\Sigma)| = 2g - 2$, and the map $S \to \Sigma$ “pinches” any extra genus out. To make sure this is true, look at the preimage of a regular value. ⊠

Here is a theorem that possibly could be due to Thurston.

**Gromov's proportionality.** *Let $M$ be a oriented closed connected hyperbolic manifold, then $\|M\|_1 = \mathrm{vol}(M)/v_n$, where $v_n = \sup \mathrm{vol}(\Delta^n)$ for $\Delta^n$ a hyperbolic simplex.*

*Proof.* It suffices to show $\|M\|_1 \le \mathrm{vol}(M)/v_n$ by our earlier inequality. The easiest way to prove an upper bound is the following; by definition it is an infimum of simplices. What we need to do is find a nice class representing this fundamental class, where the number of simplices represents the number $\mathrm{vol}(M)/v_n$. To do this, construct cycles representing $[M]$ with the number of simplices approximately optimal, or $\mathrm{vol}(M)/v_n$.

Our strategy is to construct $c = \sum \lambda_i c_i$ with

(1) $\lambda_i > 0$,

(2) $c_i$ a straight hyperbolic (with consistent orientation) simplex with $\mathrm{vol}(c_i) > v_n - \varepsilon$

(3) $c$ is a cycle.

Why do these three conditions show our theorem? By (3), $[c] = \lambda[M]$. Pairing, we get

$$\lambda \mathrm{vol}(M) = \langle \lambda[M], \mathrm{vol}\rangle \implies \langle c, \mathrm{vol}\rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol}\right\rangle = \sum \lambda_i \mathrm{vol}(c_i) \ge \left(\sum \lambda_i\right)(v_n - \varepsilon).$$

Since $\lambda > 0, \sum \lambda_i = |c|_1$, which means $[c/\lambda] = [M], |c/\lambda|_1 = \sum \lambda_i/\lambda \le \lambda \mathrm{vol}/\lambda(v_n - \varepsilon) = \mathrm{vol}(M)/(v_n - \varepsilon)$. By letting $\varepsilon \to 0$, we get our desired upper bound. This is cool but how do we construct a cycle with these properties? Recall the surface case where we triangulated very large covers and projected down, but it's not always clear if we can do this for manifolds. We have $(\ell_1)^* = \ell_\infty$, but $(\ell_\infty)^* \supsetneq \ell_1$.

We measure homology; earlier we said $C_n(M;\mathbb{R}) = \mathrm{span}_{\mathbb{R}} S_n(M)$, where $S_n(M) = \mathrm{Maps}(\Delta^n, M)$. We equip this with the compact open topology to make this into a space. Let $\mathcal{C}_n(M;\mathbb{R})$ be signed measures on $S_n(M)$ with compact support and bounded total variation. Let $\nu = \nu_+ - \nu_-, |\nu_+(S_n(M)) + \nu_-(S_n(M))$, $C_n(M;\mathbb{R}) \hookrightarrow \mathcal{C}_n(M;\mathbb{R})$, $\sum \lambda_i c_i \mapsto \sum \lambda_i \Delta_{c_i}$ which is norm-preserving. We also have $\partial\colon \mathcal{C}_{n+1}(M;\mathbb{R}) \to \mathcal{C}_n(M;\mathbb{R})$. This leads to another homology theory, called the **measure homology**. ⊠

**Theorem 3.1** (Zastrow, Hansen)**.** *For CW complexes, measure homology is isomorphic to singular homology. Furthermore, this is isometrically isomorphic (Löh, 2006).*

The "isometrically" says that we can alternatively define the Gromov norm this way. This construction is called "smearing", because we put this these simplices everywhere (smearing) to construct a cycle. Next time we approximate this smearing cycle with an honest cycle. The idea is that we fix $\Delta\colon \Delta^n \to \mathbb{H}^n$ such that $\mathrm{vol}(\Delta) > v_n - \varepsilon$. We have $\mathrm{Isom}^+(\mathbb{H}^n)$ acting on $\mathbb{H}^n$, $\mathrm{Isom}^+(\mathbb{H}^n) \cdot \Delta$. By Haar, this is locally finite and $\mathrm{Isom}^+(\mathbb{H}^n)-$invariant. Identify two copies if they differ by some $g \in \pi_1(M)$.

# 4 Bounded cohomology

Like we said last time, we shift to the dual theory of bounded cohomology. The Gromov norm is like an $\ell^1$-norm, so there should be a dual $\ell^\infty$ norm. For an $\ell^\infty$-norm to make sense, we need bounds, hence the "bounded" in bounded cohomology. First we talk about ordinary group homology and cohomology, then bounded cohomology of groups, then bounded cohomology of spaces and explain why we only need them for groups. We use a topological point of view.

Let $G$ be a group. We would like to talk about the homology and cohomology groups $H_*(G;R), H^*(G;R)$ for some ring $R$ (usually $\mathbb{Z}$ or $\mathbb{R}$). From the topological POV we use the $K(G,1)$ space, or Eilenberg-Maclane space, or classifying space of $G$ as a discrete group.

## 4.1 Eilenberg-Maclance spaces

**Definition 4.1.** We say $X$, a connected CW complex is a $K(G,1)$ space if we have the following:

- $\pi_1(X) = G$,
- $X$ is **aspherical**, or $\pi_n(X) = 0$ for all $n > 1$. Equivalently, the universal cover $\widetilde{X}$ is contractible.

There are two facts; first, they exist. The second is a universal property of sorts, which implies that they are unique up to homotopy equivalence.

**Lemma 4.1.** *Let $X$ be a $K(G,1)$ space and $Y$ some connected CW complex. Any homomorphism $\varphi\colon \pi_1(Y,y) \to \pi_1(X,x)$ is realized by some map $f : (Y,y) \to (X,x)$ (i.e. $f_* = \varphi$) and $f$ is unique up to homotopy.*

This corresponds to the fact that maps between spaces induce maps on $\pi_1$; here maps between $\pi_1$'s induce a unique map on the space, given that the target is a $K(G,1)$ space.

**Corollary 4.1.** *If $X$ and $Y$ are both $K(G,1)$, then any isomorphism $\pi_1(X,x) \to \pi_1(Y,y)$ is induced by a homotopy equivalence $f : X \to Y$.*

*Proof.* Realize $\varphi$ by $f : (X,x) \to (Y,y)$, $\varphi^{-1}$ by $g\colon (Y,y) \to (X,x)$. The composition $g \circ f : (X,x) \to (X,x)$, and $(g \circ f)_* = \varphi^{-1} \circ \varphi = \mathrm{id}_{\pi_1 X}$. Not only do we have existence we have uniqueness, which implies $g \circ f \simeq \mathrm{id}_X$. Similarly, flipping $g$ and $f$, $f \circ g \simeq \mathrm{id}_Y$. Therefore $g$ is a homotopy inverse. ⊠

**Definition 4.2.** Given a ring $R$, define the homology and cohomology of a group $G$ as $H_*(G;R) := H_*(K(G,1);R)$ and $H^*(G;R) := H^*(K(G,1);R)$.

**Example 4.1.** Some examples.

- $H_0(G;R) = H_0(K(G,1);R) = R$, since $G$ is connected.
- $H_1(G;\mathbb{Z}) = H_1(K(G,1);\mathbb{Z}) = \mathrm{Ab}(\pi_1(K(G,1))) = \mathrm{Ab}(G)$. Similarly, $H^1(G;R) = \mathrm{Hom}(G,R)$.
- What is $H_*(\mathbb{Z};\mathbb{Z})$? Let $X = S^1$ which is a $K(\mathbb{Z},1)$ space. Then

$$H_k(\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & k > 1. \end{cases}$$

  There is a notion of the geometric dimension of $G$, which is the smallest dimension of $X$ for $X$ a $K(G,1)$ space. Formally, this is defined as $\mathrm{gd}(G) := \min\{\dim X \mid X \text{ is } K(G,1)\}$. There is a dual notion $\mathrm{cd}(G) = \min\{K \mid H_n(G;R) = 0 \text{ for all } n > k, \text{ all } R\}$. Here we need to take twisted coefficients (comes with a $G$-action), the important this is that the cohomological dimension is always less than or equal to the homological dimension, or $\mathrm{cd}(G) \leq \mathrm{gd}(G)$.

## 4.2 Manifolds as $K(G,1)$ spaces

Every finitely presented group can be represented as $\pi_1$ of some closed 4-manifold. What if we want to realized a $K(G,1)$ space as a manifold? The question is, given $G$, is there a manifold that is a $K(G,1)$ space? In other words, is there an aspherical manifold with $\pi_1 = G$?

**Example 4.2.** For $G = F_2$ this manifold exists, for example the inside of the 2-torus (handlebody with body removed), the punctured torus with boundary removed, or the thrice punctured sphere with boundary removed.

What if we look for $M$ closed? Then $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $M^n$ closed. If we further require that $M^n$ is closed orientable, then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ generated by the fundamental class.

**Lemma 4.2.** *(1) If $M^n$ is closed, aspherical with $\pi_1(M) = G$, then $H_n(G;\mathbb{Z}/2) \simeq 2) \cong \mathbb{Z}/2, H_k(G;R) = 0$, $k > n$.*
*(2) If $M^n$ is orientable in addition, $H_n(G;\mathbb{Z}) \cong \mathbb{Z}$, etc.*

**Example 4.3.** Continuing our example, for $G = F_r$, $X = S^1 \vee S^1$, $H_k(G;\mathbb{Z}) = 0$ for $k > 1$, $\mathbb{Z}$ for $k = 0$, and $\mathbb{Z}^r$ for $k = 1$ ($r$ is the rank). This leads to the following corollary.

**Corollary 4.2.** *We cannot realize $K(F_r, 1)$ as a closed manifold when $r > 1$.*

For $G = \mathbb{Z}/n$, $H_k(\mathbb{Z}/n;\mathbb{Z}) = \mathbb{Z}$ for $k = 0$, $\mathbb{Z}/n$ when $k$ is odd, and $0$ when $k > 0$, even. In particular, the cohomological dimension of $G$ is infinite, so the geometrical dimension of $G$ is infinite as well, so there is no finite dimensional $K(G,1)$. In particular, it cannot be a manifold.

**Proposition 4.1.** *Let $G$ be a finite group, $G$ acting on $\mathbb{R}^n$. Then this action is not free.*

*Proof.* Suppose the action of $G$ is free. Let $H = \mathbb{Z}/m$ be a cyclic subgroup (take the powers of an element of $G$), then $H$ acts on $\mathbb{R}^n$ freely. Then the projection $\mathbb{R}^n \to \mathbb{R}^n/H$ to the quotient is a covering map, since $H$ acts freely and is properly discontinuous. Define $X := \mathbb{R}^n/H$. Then $X$ is a $K(\mathbb{Z}/m, 1)$ space since $\mathbb{R}^n$ is contractible, a contradiction. ⊠

# 5 Co-Hopfian groups and group homology

## 5.1 Co-Hopfian groups

**Definition 5.1.** A group $G$ is **co-Hopfian** if every *injective* $G \hookrightarrow G$ is an isomorphism.

**Example 5.1.** Some examples:

(1) Finite groups are co-Hopfian.
(2) $\mathbb{Z}$ is not co-Hopfian by the map $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.
(3) $\mathbb{Z}^n$ is not co-Hopfian.
(4) $F_n$ is not co-Hopfian. For example, consider $F_2 = \langle a, b\rangle$ with the self-map $a \mapsto a, b \mapsto b^2$. If we think of $F_2$ as $\mathbb{Z} * \mathbb{Z}$, then the first $\mathbb{Z}$ maps to itself by the identity and the second maps to itself by squaring.
(5) Let $G = H * K$ (free product), if $K$ is not co-Hopfian, then $G$ is not co-Hopfian by the same logic as above.

**Lemma 5.1.** *Let $M$ be a complete Riemannian manifold such that sectional curvature is non-positive. Then $M$ is aspherical.*

*Proof.* The proof is by Cartan-Hamadard, which tells us that the exponential map $T_pM \to M$ is a covering. todo:missed this proof ⊠

**Lemma 5.2.** *Let $M, N$ be connected aspherical n-manifolds. Suppose that $\pi_1 M \simeq \pi_1 N$. Then $M$ and $N$ have the same compactness.*

*Proof.* Note that $M$ is closed iff $H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2$. We have $G = \pi_1 M = \pi_1 N$, and $N$ is closed iff $H_n(N;\mathbb{Z}/2) = \mathbb{Z}/2$, which is true, so we are done. ⊠

**Lemma 5.3.** *If $M$ is a closed connected aspherical manifold, then any subgroup $H$ of $\pi_1 M$ with $H \simeq \pi_1 M$ must have finite index.*

*Proof.* Let $G = \pi_1 M \geq H$ a subgroup. We have an isomorphism $f : G \to H$. Let $\widetilde{M}$ be the covering space corresponding to $H$. Since $M$ is aspherical, $M$ is $K(G,1)$, which implies that $\widetilde{M}$ is aspherical and $K(H,1)$. So $f$ can be realized as a homotopy equivalence $\varphi\colon M \to \widetilde{M}$. $M$ is compact implies that $\widetilde{M}$ is compact, so $\pi$ is a *finite* cover. Finite covers correspond to finite index subgroups, therefore $H$ has finite index. ⊠

**Lemma 5.4.** *Let $M$ be a closed, connected, aspherical, orientable manifold. If $\pi_1 M$ is* not *co-Hopfian, then there is a self-map $f : M \to M$ with $|\deg f| > 1$. So $\|M\|_1 = 0$.*

*Proof.* Let $G = \pi_1 M, h\colon G \hookrightarrow G, H = \operatorname{im} h$. By the lemma above, we get $\pi$ a finite cover, $\varphi$ a homotopy equivalence..

$$\begin{tikzcd} & H \arrow[d, hook] \\ G \arrow[ur, "\overset{h}{\cong}"] \arrow[r, "h"'] & G \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{M} \arrow[d, "\pi"] \\ M \arrow[ur, "\varphi"] \arrow[r, "f"'] & M \end{tikzcd}$$

We have $n = \dim M$. The map $\varphi\colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(M;\mathbb{Z})$ an isomorphism, $[M] \mapsto \pm[M]$, so $\deg(\varphi) = \pm 1$. Then we have

$$|\deg f| = |\deg \varphi| \cdot |\deg \pi| = |\deg \pi| > 1$$

if $H$ is proper, where $\deg \pi$ is the index of $H$ in $G$. By the degree inequality, we get $\|M\| \leq |\deg f| \cdot \|M\|$ which implies $\|M\|_1 = 0$. ⊠

**Corollary 5.1.** *If $M$ is closed with negative sectional curvature, then $\pi_1 M$ is co-Hopfian.*

*Proof.* Negative curvature implies $M$ is aspherical, which also implies $\|M\|_1 = 0$. By the lemma above, $\pi_1 M$ is co-Hopfian. ⊠

**Example 5.2.** For $S$ occ, $g > 1$, then $\pi_1(S)$ is co-Hopfian.

This ends the detour, and we will go back to the algebraic definition of group homology and group cohomology.

## 5.2 The bar complex

Let $C_n(G;R)$ be the free $R$-module with basis consisting of $n$-tuples $(g_1, \cdots, g_n) \in G^n$. In the same way, define co-chains as the dual $C^n(G;R) = \operatorname{Hom}(C_n(G;R), R)$. The differential $\partial : C_n(G;R) \to C_{n-1}(G;R)$ looks a little strange:

$$\partial(g_1, \cdots, g_n) = (g_2, \cdots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n) + (-1)^n (g_1, \cdots, g_n).$$

For example, when $n = 2$, $\partial(g_1, g_2) = g_2 - g_1 g_2 + g_1$. We can think of this as a map to $\mathbb{Z}$, which is zero when this is a homomorphism. For $n = 3$, we have $\partial(g_1, g_2, g_3) = (g_2, g_3) + (g_1 g_2, g_3) + (g_1, g_2 g_3) - (g_1, g_2)$. This corresponds to the four faces of a tetrahedron. The key thing is that $\partial^2 = 0$, and the corresponding homology $H_n(G;R) = \ker \partial / \operatorname{im} \partial$ is the group homology. Similarly, $H^n(G;R)$ is the group cohomology; we will use this model to explain bounded cohomology. What we will do next time is build a particular $K(G,1)$ space which corresponds to our formala, and from there we will define bounded cohomology.

# 6 Bounded cohomology

todo:was late, missed definition of $\delta_n\colon C^n(G;R) \to C^{n+1}(G;R)$. bounded functions, $R$ is the size. sup norm over simplices. bounded funtion is still bounded over coboundary. hence we get bounded cohomology $H_b^n(G;R) = Z_b^n / B_b^n$. equipped with nan inudced norm $\sigma \in H_b^n(G;R)$. $\|\alpha A\|_\infty = \inf_{[f]=\sigma, f \in Z_b^n G;R} |f|_\infty$.

**Definition 6.1.** There is a comparison map $c\colon H_b^n(G;R) \to H^n(G;R)$ induced by $C_b^n(G;R) \to C^n(G;R)$.

Some questions:

(1) Given $\sigma \in H^n(G;R)$, is there a $\sigma \in \operatorname{im}\sigma$. Related question: is $c$ surjective? Furthermore, if $\sigma_b \mapsto \sigma$ by $c$, is there a natural $\sigma_b$? Does it carry any extra info?
(2) What is $\ker(c)$? What do they correspond to?

**Example 6.1.** Some examples in lower degrees.

- For degree $n = 0$, the only 0-cochain is a constant function, which are always bounded. The 0-cochain is also a 0-cycle, there is nothing interesting; so $H_b^0(G;R) = H^0(G;R) = R$.
- For degree $n = 1$, let $f \in C^1(G;R)$, i.e. $f : G \to R$. The coboundary $(\delta f)(g,h) = f(g) + f(h) - f(gh)$. We have $f \in Z^1(G;R)$ iff $f : G \to R$ is a homomorphism (coboundary zero). If $f$ is bounded, consider $|f(g^n)| = |nf(g)| \le C$, so $|f(g)| = 0$. Therefore if $f$ is a homomorphism then $f = 0$, so $Z_b^1(G;R) = 0$, which implies $H_b^1(G;R) = 0$. In general, the comparison map is not an isomorphism.

A general idea to obtain $\sigma \in \ker c$. Consider $C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta^2} C^{n+1}$. For $f \in C^{n-1}$, we know $\delta^2 = 0$, so $\delta f \in Z^n$. As a cohomology class it is trivial, but it might be a nontrivial bounded cohomology class. If $\delta f$ is bounded, then $[\delta f] \in Z_b^n$. This gives us a class in $H_b^n$. Furthermore, $[\delta f] \in \ker c$.

- When $n = 2$, $f \in C^1(G;\mathbb{R})$ is a function $G \to \mathbb{R}$. We hope that $(\delta f) = f(g) + f(h) - f(gh)$ is bounded. This leads to the following definition.

**Definition 6.2.** A function $\varphi : G \to \mathbb{R}$ is a **quasimorphism** if $D(\varphi) := \sup_{g,h\in G} |\varphi(g) + \varphi(h) - \varphi(gh)| < \infty$. The number $D(\varphi)$ is called the **defect** of $\varphi$. We say $\varphi$ is **homogeneous** if $\varphi(g^n) = n\varphi(g)$ for all $n \in \mathbb{Z}$, $g \in G$.

**Example 6.2.** Some examples:

(1) Homomorphisms are homogeneous quasimorphisms.
(2) Bounded functions are quasimorphisms, but never homogeneous.
(3) Quasimorphisms form a real linear space: you can multiply by a scalar and add them.

Denote the space of all quasimorphisms by $\hat{Q}(G)$, and there is a containment $H^1(G;\mathbb{R}) \subseteq Q(G) \subseteq \hat{Q}(G) \subseteq C_b^1(G;\mathbb{R})$ where $Q(G)$ denotes the linear subspace of homogeneous quasimorphisms, and $C_b^1(G;\mathbb{R})$ are bounded functions.

**Proposition 6.1.** *The space of all quasimorphisms decomposes as $\hat{Q}(G) = Q(G) \oplus C_b^1(G;\mathbb{R})$.*

We will prove this later!

**Definition 6.3** (Homogenization)**.** Take an arbitrary $\varphi \in \hat{Q}(G)$, then define the homogenization $\overline{\varphi}$ by $\overline{\varphi}(g) = \lim_{n\to+\infty} \frac{\varphi(g^n)}{n}$.

Often times we need check whether this limit exists. It turns out $\overline{\varphi}$ is well-defined, homogeneous, and a quasimorphism. To see well-definedness, we quote the following definition from analysis.

**Lemma 6.1.** *Consider $\{a_n\}$ a sub-additive sequence, which means $a_{m+n} \le a_m + a_n$ for all $m, n \ge 1$. Then $\lim_{n\to+\infty} a^n/n = \inf_{n\ge1} a^n/n$. In particular, the limit exists if $a_n/n$ is bounded below.*

*Proof.* We have
$$\overline{\lim} \frac{a_n}{n} \ge \underline{\lim} \frac{a_n}{n} \ge \inf_{n\ge1} \frac{a_n}{n}$$

by definition. We want tos how that $\inf_{n\geq 1}\frac{a_n}{n} \geq \overline{\lim}\frac{a_n}{n}$. Fix $m \leq 1$, $n = qm + r$, $0 < r \leq m$ by the division algorithm. If $n > m$, then $a_n \leq a_m + a_{n-m} \leq \cdots \leq qa_m + a_r$. To bound $a_r$, introduce $B = \max_{0<r\leq m} a_r$. Then this is less than or equal to $qa_m + B$. By the inequality above,

$$\frac{a_n}{n} \leq \frac{aq_m + B}{qm + r} = \frac{a_m + \frac{B}{q}}{m + \frac{r}{q}} \xrightarrow{n\to\infty} \frac{a_m}{m}.$$

So $\overline{\lim} a_n/n \leq a_m/m$. Taking $m$ arbitrary, this implies $\overline{\lim} a_n/n \leq \inf_{m\geq 1} a_m/m$. ⊠

We will see the rest next lecture.

# 7 More on quasimorphisms

Last time we had the comparison map $c\colon H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})$, and we were trying to understand what is $\ker c$. We had quasimorphisms $\varphi\colon G \to \mathbb{R}$ with the property that $|\varphi(gh) - \varphi(g) - \varphi(h)| \leq D(\varphi)$ for all $g, h$. This leads to a vector space $\hat{Q}(G)$ of all quasimorphisms, with a homogeneous subspace $Q(G)$ and another subspace $H^1(G;\mathbb{R})$. Another containment is the bounded functions $C_b^1(G;\mathbb{R}) \supseteq \hat{Q}(G)$. We were trying to show that $\hat{Q}(G) = C_b^1(G;\mathbb{R}) \oplus Q(G)$, and we have already shown how to get the homogeneous component (from homogenization).

Let $\varphi_+ = \varphi + D(\varphi)$. Then $\varphi(gh) - D(\varphi) \leq \varphi(g) + \varphi(h) \leq \varphi(gh) + D(\varphi)$. So

$$\begin{aligned}
\varphi_+(g) + \varphi_+(h) &= \varphi(g) + \varphi(h) + 2D(\varphi) \\
&\geq (\varphi(gh) - D(\varphi)) + 2D(\varphi) \\
&= \varphi(gh) + D(\varphi) \\
&= \varphi_+(gh),
\end{aligned}$$

which implies $\varphi_+(g^n)$ is subadditive. Similarly, define $\varphi_- = \varphi - D(\varphi)$, doing an analogous calculation shows that $\varphi_-(g^n)$ is sup-additive, so $\varphi_-(g^{m+n}) \geq \varphi_-(g^m) + \varphi_-(g^n)$. Now we have bounds on both sides in a sense. Explicitly, we have $\varphi_-(g) \leq \varphi(g) \leq \varphi_+(g)$. Therefore

$$\varphi_-(g) \leq \frac{\varphi - (g^n)}{n} \leq \frac{\varphi_+(g^n)}{n} \leq \varphi_+(g).$$

The sup-additive sequence has an upper bound and the sub-additive sequence has a lower bound. Taking the limit, by our analysis lemma, we have

$$\sup \frac{\varphi_-(g^n)}{n} = \lim \frac{\varphi_-(g^n)}{n} = \lim_{n\to+\infty} \frac{\varphi_+(g^n)}{n} = \inf \frac{\varphi + (g^n)}{n}.$$

So $\varphi_+(g^n) = \varphi_-(g^n) + 2D(\varphi)$. By the squeeze lemma, $\sup \frac{\varphi_-(g^n)}{n} = \lim \frac{\varphi^n(g)}{n} = \overline{\varphi}(g)$. So the limit exists, and we actually have two new descriptions of our limit. On one hand, $\overline{\varphi}(g) = \inf_{n\geq 1} \frac{\varphi_+(g^n)}{n} \leq \varphi_+(g) = \varphi(g) + D(\varphi)$ (for $n = 1$). On the other hand, $\overline{\varphi}(g) = \sup_{n\geq 1} \frac{\varphi-(g^n)}{n} \geq \varphi_-(g) = \varphi(g) - D(\varphi)$. Together, $|\overline{\varphi}(g) - \varphi(g)| \leq D(\varphi)$ for every $g$. So $\overline{\varphi}(g)$ is only a bounded distance away from $\varphi(g)$, which implies $\overline{\varphi}$ itself is a quasimorphism (sum of two quasimorphisms). The last thing to show is that $\overline{\varphi}$ is a homogeneous quasimorphism. This follows by the limit definition. We have

$$\overline{\varphi}(g^k) = \lim_{n\to\infty} \frac{\varphi(g^{kn})}{n} = k \lim_{n\to\infty} \frac{\varphi(g^{kn})}{kn} = k\overline{\varphi}(g).$$

However, this only works for $k \in \mathbb{Z}_+$. If $k = 0$, $\overline{\varphi}(\mathrm{id}) = \lim \frac{\varphi(\mathrm{id}^n)}{n} = 0$. If $k$ is negative, then $\overline{\varphi}(g^{-k}) = k\overline{\varphi}(g^{-1})$, so if $\overline{\varphi}(g^{-1}) = -\overline{\varphi}(g)$ then we are good. To see why adding them up gives $|\varphi(g^n) + \varphi(g^{-n}) - \varphi(\mathrm{id})| \leq D(\varphi)$, which implies $\varphi(g^n) + \varphi(g^{-n})$ is bounded.

**Lemma 7.1.** *$\overline{\varphi}$ is a well-defined homogeneous quasimorphism, moreover, we have an explicit bound $|\overline{\varphi}-\varphi|_\infty = \sup_g |\overline{\varphi}(g)-\varphi(g)| \leq D(\varphi)$.*

This summarizes our previous discussion.

**Remark 7.1.** Recall that $D(\varphi+\psi) \leq D(\varphi)+D(\psi)$, and for $f$ bounded we have $D(f) \leq 3|f|_\infty$ by the triangle inequality. In our case, $D(\overline{\varphi}) \leq D(\overline{\varphi}-\varphi)+D(\varphi) \leq 3D(\varphi)+D(\varphi) = 4D(\varphi)$. Actually, we have a better bound $D(\overline{\varphi}) \leq 2D(\varphi)$. But we may not need this fact.

**Proposition 7.1.** $\hat{Q}(G) = Q(G) \oplus C_b^1(G)$.

*Proof.* We have

$$\varphi = \underbrace{\overline{\varphi}}_{\in Q(G)} + \underbrace{(\varphi-\overline{\varphi})}_{\in C_b^1(G)},$$

which means $\hat{Q}(G) = Q(G)+C_b^1(G)$. To show $Q(G)\cap C_b^1(G) = 0$, let $\varphi \in Q(G)\cap C_b^1(G)$. Then $|\varphi(g)| = |\varphi(g^n)/n| \leq |\varphi|_\infty/n \xrightarrow{n\to\infty} 0$, and we are done. ⊠

**Proposition 7.2.** *We have an exact sequence*

$$0 \to H^1(G) \to Q(G) \xrightarrow{\delta} H_b^2(G;\mathbb{R}) \xrightarrow{c} H^2(G;\mathbb{R}).$$

*Proof.* Exactness at the first entry means that $H^1(G) \to Q(G)$ is injective, which is true since it is defined as in inclusion. So we have to show two things:

(1) $\ker\delta = H^1(G)$

(2) $\operatorname{im}\delta = \ker c$.

For (1), let $\varphi \in Q(G)$. Then $(\delta\varphi)(g,h) = \varphi(g)+\varphi(h)-\varphi(gh)$. So $\delta\varphi = 0 \iff \varphi$ is a homomorphism, i.e. $\varphi \in H^1(G)$. For (2), the easier direction is to show $\operatorname{im}\delta \subseteq \ker c$. This is by definition, since $\delta^2 = 0$. So $\delta\varphi$ is a coboundary in the ordinary sense, i.e. $[\delta\varphi = 0$ in $H^2(G;\mathbb{R})$.

The harder part is to show $\operatorname{im}\delta \supseteq \ker c$. Let $\alpha \in H_b^2(G;\mathbb{R})$ such that $c(\alpha) = 0 \in H^2(G;\mathbb{R})$, or $\alpha = \delta\varphi$, $\varphi\colon G \to \mathbb{R}, \varphi \in C^1(G;\mathbb{R})$. Since $\alpha$ is a bounded class, $\delta\varphi$ is bounded which implies $\varphi$ is a quasimorphism. To show that this is homogeneous, $\varphi-\overline{\varphi}$ is bounded, so $[\delta\varphi] = [\delta\overline{\varphi}] = \alpha \in H_b^2(G;\mathbb{R})$. Then $\overline{\varphi} \in Q(G)$, which implies $\alpha \in \operatorname{im}\delta$. ⊠

**Corollary 7.1.** $\ker c = Q(G)/H^1(G;\mathbb{R}) = \operatorname{im}\delta$.

Next time we will talk about many different kinds of quasimorphisms.