Gromov's Norm and Bounded Cohomology Lecture Notes

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Lecture notes for the Spring 2022 graduate section of a topics course called "Gromov's norm and bounded cohomology" (Math 392C) at UT Austin, taught by Dr. Chen. These notes were taken live in class (and so they may contain many errors). Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

1 First day of class

This is a topics course on Gromov's simplicial norm and bounded cohomology. There will be optional problems to discuss online or over email. Office hours are tentatively Wednesdays 3:30 to 4:30, and Friday 12:30 to 1:30 (also by appointment). There may be specific topics that we won't have time to go into, so there may be a presentation at the end of the course. Grades are based off participation. For the first few weeks we are meeting over zoom, and we will decide how to give the lectures afterward.

It's time to talk about math!

1.1 Motivation

There are two notions popularized by Gromov's paper todo:references, one being Gromov's simplicial norm and a dual theory called bounded cohomology. We start by introducing the simplicial norm, then we will talk about bounded cohomology, and later we will use bounded cohomology to prove more intricate things about the simplical norm.

What is the general idea? Suppose we have a topological space X (thought of as a closed manifold), and we have homology groups $H_n(X;\mathbb{R})$ (typically $\mathbb{R}$-vector spaces). We want to equip these spaces with a (semi-)norm $\|\cdot\|_1$ to measure the size of a homology class $\sigma \in H_n(X;\mathbb{R})$. If $X = M^n$ is an oriented connected closed manifold, we know that $H_n(M;\mathbb{R}) \simeq \mathbb{R}$ generated by a fundamental class $[M]$. If this space is equipped with a norm, we can talk about the norm of the fundamental class $\|[M]\|_1$ which should measure the "volume" of M. We usually think about the volume form or a Riemannian metric when talking about volume, which depends on the metric. If there is a natural way to induce a norm, this will be a topological invariant.

How could volume be a topological invariant? There is some evidence that this is true.

- One piece of evidence is **Gauss-Bonnet**; if $M = S_g$ (surface of genus g), then $-2\pi \cdot \chi(S_g) = \text{area}(S_g)$. In general we need to specify a metric, but the magic of Gauss-Bonnet is that $\text{area}(S_g)$ is the same for any hyperbolic metric on S_g.
- Another piece of evidence is **Mostow's rigidity**; if M^n for $n \geq 3$ admits some hyperbolic metric (the quotient of the n-dimensional hyperbolic plane mod some cocompact lattice), it is a theorem of Mostows that the hyperbolic metric is unique, so volume is "well-defined" or only depends on the topology.

In both cases, we introduce hyperbolic geometry and say "it doesn't depend on the hyperbolic metric", which isn't purely topological. What Gromov did was introduce a purely topological definition, then showed that it agrees with the hyperbolic geometry.

A fun exercise from algebraic topology. If S, S' are both conected closed surfaces, say $g(S') > g(S)$. Can you find a map $f : S \to S'$ with $\deg(f) \neq 0$? There are several ways to map higher genus to lower genus surfaces (collapse, double cover), but can you map the lower genus surface into the higher genus surface? Intuition says we cannot do this because area increases. The simplest solution uses Gromov's simplicial norm.

1.2 Defining Gromov's simplicial norm

Let X be a topological space, $H_n(X;\mathbb{R})$ be the singular homology of X. We should have a chain complex $C_n(X;\mathbb{R})$, the space of all real n-chains with basis $S_n(X)$, where $S_n(X)$ is the set of all singular n-simplices. Here $C \in S_n(X)$ is a map $C : \Delta^n \to X$. With a chosen basis, we can talk about all kinds of norms; $C_n(X;\mathbb{R})$ comes with an ℓ^1-norm, and a chain $C \in C_n(X;\mathbb{R})$ is a unique expression $C = \sum_{i=1}^k \lambda_i C_i$ with $\lambda_i \in \mathbb{R}$. Then $|C|_1 = \sum_{i=1}^k |\lambda_i|$ is a norm at the chain level. To get homology, we have a sequence

$$C_{n+1}(X;\mathbb{R}) \xrightarrow{\partial_{n+1}} C_n(X;\mathbb{R}) \xrightarrow{\partial_n} C_{n-1}(X;\mathbb{R}).$$

Then we have *boundaries* $B_n := \operatorname{im} \partial_{n+2}$ and *cycles* $Z_n := \ker \partial_n$. The relation is $\partial^2 = 0$, so $B_n \subseteq Z_n \subseteq (C_n, |\cdot|_1)$. Recall from functional analysis that a normed quotient vector space comes with a quotient norm of its own. Here $H_n(X;\mathbb{R}) = Z_n/B_n$ has an induced semi-norm $\|\cdot\|_1$. Concretly, given a homology class σ and the origin, the quotient norm is the infimum over all cycles.

Definition 1.1. We define **Gromov's simplicial norm** as $\|\sigma\|_1 = \inf\limits_{\substack{[C]=\sigma \\ C\in Z_n}} |C|_1$, where $|C|_1$ is the number of simplices in the cycle C.

This can be thought of as the "smallest" distance to the origin todo:figure. In words, the seminorm $\|\sigma\|_1$ is the infimal number of simplices to represent σ as a cycle. Some properties:

Proposition 1.1 (Functoriality)**.** *For $f : X \to Y$ (implied to be continuous), then the induced map $f_* \colon H_n(X) \to H_n(Y)$ ($\mathbb{R}$-coefficients are implied) is non-increasing with respect to $\|\cdot\|_1$, i.e., for any $\sigma \in H_n(X)$, $\|f_*\sigma\|_1 \le \|\sigma\|_1$.*

Proof. Let $\sigma = [c]$ for a cycle c, $c = \sum \lambda_i c_i$, $\lambda_i \in \mathbb{R}$, $c_i \in S_n(X)$. Then we have the push-forward $f_* c = \sum \lambda_i f_* c_i = \sum \lambda_i (f \circ c_i)$.

$$\begin{tikzcd} \Delta^n \arrow[r, "c_i"] \arrow[rr, bend right, "f_* c_i"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

So $f_*\sigma = [f_* c]$. So $\|f_*\sigma\|_1 \le \|f_* c\|_1 \le \sum |\lambda_i| = |c|_1$. The left hand size doesn't depend on c, so take the infimum over c with $[c] = \sigma$. This implies that $\|f_*\sigma\|_1 \le \|\sigma\|_1$. ⊠

Corollary 1.1 (Invariance)**.** *If $f : X \to Y$ is a homotopy equivalence, then $f_* \colon H_n(X) \to H_n(Y)$ is an isometric (preserving $\|\cdot\|_1$) isomorphism. More generally, let $f : X \to Y$. If there is a $g \colon Y \to X$ such that $g_* f_* = \mathrm{id}_{H_n(X)}$, then f_* is an isometric embedding.*

Proof. If f is a homotopy equivalence, then we have a homotopy inverse $g \colon Y \to X$ with the property that $(f \circ g) \simeq \mathrm{id}_Y$, $(g \circ f) \simeq \mathrm{id}_X$. In particular these maps induce identity maps $f_* g_* = \mathrm{id}_{H_n(Y)}$, $g_* f_* = \mathrm{id}_{H_n(X)}$. Then the isomorphism follows from applying the second part twice.

To show the second part, all we need to do is show that f_* preserves the norm (injectivity is already shown), or $\|f_*\sigma\|_1 = \|\sigma\|_1$. To prove the reverse inequality, we need to show that $\|\sigma\|_1 \le \|f_*\sigma\|_1$ for every $\sigma \in H_n(X)$. By assumption $g_* f_*$ is the identity, so

$$\|\sigma\|_1 = \|(g_* f_*)\sigma\|_1 = \|g_*(f_*\sigma)\|_1 \le \|f_*\sigma\|_1$$

where the last inequality follows by the functoriality of g. ⊠

This implies that the norm $\|\cdot\|_1$ is a homotopy invariant. Next time we will continue this line and talk about simplicial volume. This gets us a way to estimate degree and be able to do some examples. As homework, how does $\|\cdot\|_1$ behave on $H_0(X;\mathbb{R})$?

2 Simplicial volume

Recall that last time we introduced the simplicial norm. We took a topological space X and considered the singular homology group $H_n(X;\mathbb{R}) := Z_n(X)/B_n(X)$. Given every homology class $\sigma \in H_n(X;\mathbb{R})$, we can represent it by cycles, aand every cycle is represented by simplices. So we minimize the number of simplices, and so $\|\sigma\|_1 = \inf_{[C]=\sigma, C\in Z_n(X)} |C|_1$. What do we mean by semi-norm? The conditions are non-negativity ($\|\sigma\|_1 \ge 0$), linearity ($\|\lambda\sigma\|_1 = |\lambda| \cdot \|\sigma\|_1$, and the triangle inequality ($\|\sigma_1 + \sigma_2\|_1 \le \|\sigma_1\|_1 + \|\sigma_2\|_1$). The only difference is that you can have non-trivial elements with zero norm.

2.1 Rational coefficients

What happens if we use rational coefficients instead of real coefficients? If we are able to represent some homology class σ by a rational cycle, all we need to preserve is that the subspace B_n is a rational subspace. This is because the boundary map $\partial_{n+1}\colon C_{n+1} \to C_n$ is rational (todo:see figure).

Lemma 2.1. *If $\sigma \in H_n(X;\mathbb{Q})$, then $\|\sigma\|_1 = \inf_{[C]=\sigma, C\in Z_n(X;\mathbb{Q})} |C|_1$. In general, for $\sigma \in H_n(X;\mathbb{R})$, for every $\varepsilon > 0$ there exists a $\sigma' \in H_n(X;\mathbb{Q})$ such that $\|\sigma - \sigma'\|_1 < \varepsilon$. In other words, rational homology classes are dense in real homology classes.*

A detailed proof is in the notes todo:references. Sometimes we would like to work with rational cycles, so up to scaling we are working with integral cycles. This leads to literally counting simplices, which can be useful. We could also consider the *relative homology classes*; if $A \subseteq X$, then $H_n(X,A;\mathbb{R})$ leads to another simplicial norm $\|\cdot\|_1$. Many times rational homology classes can be represented by manifolds (in degrees two or three), which leads to integral classes being more geometric in nature.

2.2 Simplicial volume

Let M be an orientable, connected, and compact manifold with boundary ∂M. What we know from algebraic topology is that since M is compact and connected, $H_n(M,\partial M;\mathbb{Z}) \simeq \mathbb{Z}$. The orientation gives us a choice of generator $[M]$, which is the fundamental class. A more geometric description is this: suppose we can triangulate our manifold. Then using our given orientation we can take the formal sum of all the simplices involved in the triangulation, and so the fundamental class is the sum of the top simplices. The simplicial volume is then $\|[M]\|_1$; we denote this by $\|M\|_1 = \|[M]\|_1$.

Remark 2.1. Some remarks;

- The choice of orientation doesn't matter because the two choices differ by a negative sign, and the norm ignores negative signs. So we don't have to choose an orientation. If M is non-orientable, consider the orientable double cover N and divide by two ($\|M\|_1 = \|N\|_1/2$).
- If $M = \amalg_{i=1}^k M_i$, then $\|M\|_1 = \sum_{i=1}^k \|M_i\|_1 = \left\|\sum_{i=1}^k [M_i]\right\|_1$.
- If we have a map $f : M^n \to N^n$ where M^n and N^n are both degree n occ[1] manifolds, then $f_*[M] \in H_n(N;\mathbb{Z}) = \langle [N]\rangle$, which implies $f_*[M] = \deg(f)\cdot[N]$, which subsequently implies that $\deg(f) \in \mathbb{Z}$. Here orientation matters.

Lemma 2.2. *If $f : M^n \to N^n$ with M,N occ, then $\deg(f)\cdot\|N\|1 \le \|M\|_1$. Moreover, if f is a finite covering map, then equality holds.*

Proof. Functoriality tells us that $\|f_*[M]\|_1 \le \|M\|_1$. By definition, $\|\deg(f)\cdot[N]\|_1$ which is equal to $|\deg(f)|\cdot\|N\|_1$. So the inequality follows easily from functoriality. To prove equality, we want to lift our triangulation upstairs to find the fundamental class of M. Our covering provides a way to do this; if f is a covering map, suppose $[N] = [c]$, where $c = \sum \lambda_i c_i$. Each c_i lifts to some $\widetilde{c}_i^{\,j}$, where $j = 1,2,\cdots,d$. Then $\widetilde{c} = \sum_i \sum_{j=1}^d \lambda_i \widetilde{c}_i^{\,j}$, and $f_*[\widetilde{c}] = \left[\sum d\lambda_i c_i\right] = d[c] = d[N] = \pm f_*[M]$. We conclude that $[\widetilde{c}] = \pm[M]$. Explicitly, $|\widetilde{c}|_1 = d\cdot\sum_i |\lambda_i| = d\cdot|c|_1$. Since c is arbitrary, $\|M\|_1 \le d\cdot\|N\|_1 = |\deg(f)|\cdot\|N\|_1$, and functorality tells us that equality holds. ⊠

From here, we can deduce many examples.

Corollary 2.1. *If an occ manifold M admits a map $f : M \to M$ with $|\deg(f)| > 1$, then $\|M\|_1 = 0$.*

[1] Orientable, connected, closed. We will use these assumptions often.

Proof. We have $\|M\|_1 < \deg(f) \cdot \|M\|_1 \le \|M\|_1$ if $\|M\|$ is positive, which is a contradiction. So $\|M\|_1 = 0$. The intuition is that if a manifold covers itself several times, we cannot make sense of volume. ⊠

Example 2.1. Some examples;

(1) The circle S^1 has a self map of degree n for every integer n. and so $\|S^1\|_1 = 0$.
(2) We can generalize this by taking products, and so $T^n = (S^1)^n$ satisfies $\|T^n\|_1 = 0$.
(3) Another way to generalize this is to take higher dimensional spheres, constructing the degree map in a similar way. So $\|S^n\|_1 = 0$ when $n \geq 0$.
(4) If $M = S^1 \times N$ (product of some manifold with a circle), we can keep the second factor by the identity and take the degree map for the circle, which results in a self map of non-trivial degree. So $\|M\|_1 = 0$.

What is a non-trivial example? We'll probably cover this next time.

Now we can go back to the simplicial norm using our understanding of simplicial volume.

Lemma 2.3. *If $\sigma \in H_n(X;\mathbb{R})$ such that there exists a map $f : S^n \to X$ where $f_*[S^n] = \sigma$, then $\|\sigma\|_1 = 0$.*

Proof. This again follows by functoriality; $\|\sigma\|_1 = \|f_*[S^n]\|_1 \le \|[S^n]\|_1 = 0$. ⊠

Corollary 2.2. *The norm $\|\cdot\|_1$ vanishes on H_1.*

3 ok

We prove the corollary from last time.

Proof of Corollary 2.2. We approximate by rational homology classes. It suffices to show that $\|\sigma\|_1 = 0$ for any *rational* σ, which implies there is a $c = \sum \lambda_i c_i,\ \lambda_i \in \mathbb{Q}$ such that $[c] = \sigma$. This subsequently implies there is a $N \in \mathbb{Z}_+$ such that $N \cdot c = \sum \mu_i c_i,\ \mu_i \in \mathbb{Z}$ is an integral cycle. This reduces the problem to integral cycles.

Claim. *For any integral 1-cycle $\sum \mu_i c_i,\ \mu_i \in \mathbb{Z}$ in X, there is a map $f : \amalg_{j=1}^k S_i^1 \to X$ such that $f_*\left(\sum_{j=1}^k [S_i^1]\right) = \left[\sum_i \mu_i c_i\right]$.*

Given this claim, $\|\sum \mu_i c_i\|_1 = \left\|\sum_{j=1}^k f_*[S_i^1]\right\|_1 \le \sum_{j=1}^k \|f_*[S_i^1]\|_1 = 0$, which implies that $\left\|\left[\sum \mu_i c_i\right]\right\|_1 = 0$ for any integral cycle.

We are missing a proof of the claim.

Proof of claim. Gluing μ_is into a circle? Not sure what happened here. The fact that $\partial\left(\sum \mu_i c_i\right) = 0$ implies there is a pairing on the set of end points of this collection of segments. So this is a closed 1-manifold, equal to $\amalg_j S_j^1$. ⊠

This completes the overall proof. ⊠

Remark 3.1. In dimension 2, an integral homology class is represented by $f : \amalg_{j=1}^k S_j \to X$ such that $\sigma = f_* \sum_{j=1}^k [S_j]$, where the S_j are closed connected oriented surfaces. The situation is more complicated in higher dimensions.

Now we compute the volume of surfaces. We already know that $\|S^2\|_1 = \|T^2\|_1 = 0$.

Theorem 3.1. *Let S be an occ surface. Then the simplicial volume of S is $\|S\|_1 = -2\chi(S)$ if S has genus two or above.*

Proof. Today we prove that $\|S\|_1 \leq -2\chi(S)$. With a hyperbolic metric, $\text{area}(S) = -2\pi \cdot \chi(S) = \pi\|S\|_1$, where π is the area of the ideal hyperbolic triangle. By definition, $\|S\|_1 = \inf_{[c]=[S]} |c|_1 \leq |c|_1$. Recall that if S has a triangulation with f triangles, then we set a cycle with f triangles representing $[S]$, which implies $\|[S]\|_1 = f$. Also recall taht $\chi(S) = v - e + f$, and $3f = 2e$. Together these implies that $\chi(S) = v - \frac{3f}{2} + f = v - \frac{f}{2}$. So $2\chi(S) = 2v - f$, which implies that $2v - 2\chi(S)$. We may choose $v = 1$, then $\|S\|_1 \leq 2 - 2\chi(S)$. How do we get ride of the two? Let $f : S' \to S$, then $\chi(S') = d\chi(S)$, so $\|S'\|`_1 = d\|S\|_1$. So $d\|S\|_1 \leq 2 - 2d\chi(S)$, and $\|S\|_1 \leq \frac{2}{d} - 2\chi(S)$. Since d can be arbitrarily large, take $d \to \infty$ and $\|S\|_1 \leq -2\chi(S)$.

The reversed inequality uses “straightening”, which replaces an arbitarary singular cycle by one only involving hyperbolic simplices. Then use hyperbolic volume to get the bound. ⊠

4 Hyperbolic geometry

Last time we did some computations to compute the volume of an arbitrary surface. We need hyperbolic volume to get the lower bound.

4.1 Review of hyperbolic geometry

The space that is relevant is $\mathbb{H}^n$, which is n-dimensional hyperbolic space. We can define $\mathbb{H}^n$ as the unique simply connected complete[2] Riemannian manifold with constant sectional curvature -1. Such a space is unique by the classification of spaces. We go through three models of this space.

(1) **Hyperboloid model**: In this model, $\mathbb{H}^n$ is a subspace of $\mathbb{R}^{n+1}$. Fix a bilinear form $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$, which looks like a standard inner product for the first n coordinates, but the last has a negative sign. Representing this as a matrix with x, y vectors, $\langle x, y\rangle = x^T B y$, where

$$B = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

If we let x and y be equal, we can think of this as a projective form. Let $S = \{x \mid \langle x, x\rangle = -1\}$, or $x_1^2 + x_2^2 + \cdots x_n^2 - x_{n+1}^2 = -1$. This results in a hyperboloid. Let S_+ be the component with $x_{n+1} > 0$. $\langle \cdot, \cdot \rangle$ restricts to an inner product on $T_pS = p^\perp$ for any $p \in S$. So this is precisely the Riemannian structure on the manifold. With this metric, S_+ is complete with curvature -1. So we can think of $S_+ \simeq \mathbb{H}^n$ as a model for $\mathbb{H}^n$.

The isometry group $\text{Isom}(\mathbb{H}^n)$ tells us the symmetry of this geometry, and gives us a way to understand the behavior under this geometry. Let $\text{O}(n, 1)$ denote the group of linear transformations preserving $\langle \cdot, \cdot \rangle$. In particular this preserves S, but we are only taking the upper sheet, so we ignore elements that swap the two components, denoted $\text{O}^+(n, 1)$ (an index two subgroup). This turns out to be exactly $\text{Isom}(\mathbb{H}^n)$. If we want to preserve orientation, this becomes $\text{SO}^+(n, 1)$.

Let us talk about (totally) geodesic subspaces of dimension k. These are subspaces intersected with the half we care about, or $V \cap S^+$, where V is a linear subspace of $\dim k + 1$. These subspaces are “linear” by taking intersections with linear subspaces. todo:lots of stuff

(2) **Poincaré disk model**: This model helps us visualize the “boundary at infinity”. This is a conformal model, which means locally the metric looks like a scaling of the Euclidian metric. Here $\mathbb{D}^n \subseteq \mathbb{R}^n$ denotes the unit

[2]Synonymous for bounded and closed are compact, or all geodesics extend to infinite geodesics.

open disk, and our metric is given by $\frac{4 \cdot ds^2}{(1-\|x\|^2)^2}$. It looks like the Euclidian metric near the origin, but as we get farther (closer to the boundary) the metric scales a lot. The choice of scaling ensures the curvature is -1, and this is a way to visualize the boundary at ∞, denoted $\partial \mathbb{H}^n$. Topologically, this is identified as $\partial \mathbb{D}^n$. We can topologize $\mathbb{H}^n \cup \partial \mathbb{H}^n$ as $\overline{\mathbb{D}^n}$ (the *closed* unit ball). Furthermore,

- Any isometry on $\mathbb{H}^n$ extends to a homeomorphism on the closure $\overline{\mathbb{H}^n}$.
- Any *geodesic* in this model has two distinct points on $\partial \mathbb{H}^n$, where geodesics are circular arcs perpendicular to $\partial \mathbb{D}^n$.

Isometries are Möbius transformations preserving the unit disk $\mathbb{D}^n$. What are Möbius transformations? In the two dimensional case, these are like fractional linear maps on the complex plane/sphere. We can think of these as maps that preserve angles, like Euclidian singularities, translations, and inversions. In particular, we can generate all these Möbius transformation by inversions, which map round spheres/hyperplanes to round spheres/hyperplanes.

For example, consider a round sphere with radius r centered at a point p in $\mathbb{R}^2 \cup \{\infty\}$. Then $p \longleftrightarrow \infty$, where $d \cdot d' = r^2$ todo:figure

Refractions can be thought of as inversions $S(p, r)$ with respect to hyperplanes with infinite radius. The general form is $f(x) = \lambda \cdot Ai(x) + b$ where $A \in \mathrm{O}(n), \lambda > 0, b \in \mathbb{R}^n, i$ is inversion or the identity.

5 More on hyperbolic geometry

Finally, onto the last model.

(3) **Upper-half space model**: Here, let $H = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with metric ds^2/x_n^2. Geodesics are straight lines pointing upward, or circular arcs with endpoints perpendicular to the boundary plane. The first case is a special case of the second one, where we only have one point at the boundary plane $\mathbb{R}^{n-1}$. Here, $\partial \mathbb{H}^n = \{x_n = 0\} \cup \{\infty\}$, thought of as the one-point compactification of $\mathbb{R}^{n-1}$, in other words, the sphere S^{n-1}. Isometries are Möbius transformations preserving H. These include scaling, rotation, translation, and inversions.

Let us talk about the classification of isometries. Let $n = 2$, then $\mathbb{R}^2 \simeq \mathbb{C}$. So $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$. For $n = 3$, $\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}_2(\mathbb{C})$.

Example 5.1. Let us talk about the area of an ideal triangle (in $\mathbb{H}^2$). One way to compute the area is using Gauss-Bonnet. Ideal triangles are determined by the location of the vertices (x, y, z)– we want to show that another ideal triangle (x', y', z') has the same area as (x, y, z) up to isometry.

(1) $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on $\partial \mathbb{H}^2$, so we may assume that $x = x'$. todo:missed some things

5.1 Classification of isometries

Okay, now we can move onto the classification of isometries. We have $\mathrm{Isom}(\mathbb{H}^n)$ acting on $\mathbb{H}^n \simeq B^n$, by the Brouwer fixed point theorem a fixed point exists.

Theorem 5.1. *A non-identity element $g \in \mathrm{Isom}^+(\mathbb{H}^n)$ falls into three classes:*

*(1) g is **elliptic**: g fixes some point inside $\mathbb{H}^n$. Up to conjugation, $g \in \mathrm{SO}(n)$ (in the ball model).*

*(2) g is **parabolic**: g has no fixed point in $\mathbb{H}^n$ and has a unique fixed point in $\partial \mathbb{H}^n$. Up to conjugation, g acts on the upper half space by translation.*

(3) *g is **hyperbolic**: g has not fixed point in $\mathbb{H}^n$ and has more than one fixed point (actually two points) in $\partial\mathbb{H}^n$. Up to conjugation, $g(x) = \lambda \cdot A(x), \lambda > 0$. This preserves a unique geodesic, called* $\operatorname{axis}(g)$*.*

Okay, so why did we spend so much time talking about hyperbolic geometry? We want to show that $\|S\|_1 = -2\chi(S)$. We proved that $\|S\|_1 \leq -2\chi(S)$, but we need to show that $\|S\|_1 \geq -2\chi(S)$. We will actually do something more– instead of two dimensions, next time we will try to prove this for an arbitrary closed orientable hyperbolic manifold M^n of dimension greater n than than two. In other words, we want to show $\|M\|_1 \geq \frac{\operatorname{vol}(M)}{v_n}$, where $v_n = \sup \operatorname{vol}(\Delta^n) < \infty$, Δ^n hyperbolic.

6 Straightening

Theorem 6.1. *Let M^n be a hyperbolic closed orientable manifold. Then $\|M\|_1 \geq \operatorname{vol}(M)/v^n$, where $v_n = \sup \operatorname{vol}(\Delta^n) < \infty$, $v_2 = \pi$, Δ^n hyperbolic.*

Lemma 6.1. $v_2 = \pi$.

Proof. There are two proofs.

(1) A hyperbolic triangle with angles A, B, C has the formula area $= \pi - (A + B + C)$. This implies that $\sup = \pi$.

(2) For any hyperbolic triangle, $\operatorname{area}(\Delta) < \operatorname{area}(\Delta')$ where Δ' is an ideal hyperbolic triangle. We have shown that $\operatorname{area}(\Delta')$ is π, so $\sup = \pi$. Why can we always bound a triangle by an ideal triangle? Take an arbitrary hyperbolic triangle and a point in its interior, then take geodesics toward the boundary. This results in an ideal triangle strictly containing the original. This argument works in higher dimensions as well. ⊠

About *straightening*; consider the linear map $\operatorname{str}\colon C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$ on the basis $C\colon \Delta^k \to M$. This lifts to a map $\widetilde{C}\colon \Delta^k \to \mathbb{H}^n$.

$$\begin{tikzcd} & \mathbb{H}^n \arrow[d, "p"] \\ \Delta^k \arrow[ur, "\widetilde{C}"] \arrow[r] & M \end{tikzcd}$$

Recall $\widetilde{\operatorname{str}}\colon C_k(\mathbb{H}^n;\mathbb{R}) \to C_k(\mathbb{H}^n;\mathbb{R})$.

(1) For every $g \in \operatorname{Isom}(\mathbb{H}^n)$, $g \cdot \widetilde{\operatorname{str}}(\widetilde{C}) = \widetilde{\operatorname{str}}(g \cdot \widetilde{C})$. Then define $\operatorname{str}(C) = p \cdot \widetilde{\operatorname{str}}(\widetilde{c})$.

(2) Straightening commutes with the boundary, or $\partial \cdot \widetilde{\operatorname{str}} = \widetilde{\operatorname{str}} \cdot \partial$.

One might object that there are different choices for the lift, but thesea ll differ by a transformation, and property (1) says we can move the g outside the list. Translating and projecting is the same as projecting onto the translation, so this is well-defined. A nice property is that the boundary also commutes with the straightening downstairs, or $\partial \cdot \operatorname{str} = \operatorname{str} \cdot \partial$, where $\partial : C_{k+1}(M;\mathbb{R}) \to C_k(M;\mathbb{R})$. Any linear map commuting with the boundary induces a map on the homology, so str induces a map $\operatorname{str}_*\colon H_k(M;\mathbb{R}) \to H_k(M;\mathbb{R})$.

Lemma 6.2. $\operatorname{str}_* = \operatorname{id}_{H_k(M;\mathbb{R})}$. *In other words, straightening does not change homology classes.*

Proof. When we do straightening, the new things are homotopic to the stuff we had earlier. Take a linear homotopy on $\mathbb{R}^{n+1}$, then project to $\mathbb{H}^n$. This implies that $\operatorname{str}_* = \operatorname{id}$. ⊠

Lemma 6.3. $|\operatorname{str}_* c|_1 \leq |c|_1$.

Corollary 6.1. *For every $\sigma \in H_k(M;\mathbb{R})$, we have $\|\sigma\|_1 = \inf_{[c]=\sigma} |c|_1$ where c is straight.*

This is the key takeaway from the straightening operation; to calculate the norm on homology classes, all we have to do is take it over straightened classes.

Proof of Theorem 6.1. Suppose $[M]$ is represented by some cycle $c = \sum \lambda_i c_i$. By Corollary 6.1, we may assume that each c_i is a straight hyperbolic n-simplex. This implies $\mathrm{vol}(c_i) \le v_n$. Let vol be the volume form on M, representing a cohomology class dual to the fundamental class. Then

$$\mathrm{vol}(M) = \int_M \mathrm{vol} = \langle [M], \mathrm{vol}\rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol}\right\rangle = \sum \lambda_i \langle c_i, \mathrm{vol}\rangle = \sum \lambda_i \mathrm{vol}(c_i) \le \sum |\lambda_i| \cdot \mathrm{vol}(c_i) \le v_n \cdot \sum |\lambda_i| = v_n \cdot |c|_1.$$

This implies $|c|_1 \ge \frac{\mathrm{vol}(M)}{v_n}$. Since c is arbitrary, $\|M\|_1 \ge \frac{\mathrm{vol}(M)}{v_n}$. In the case $n = 2$, $\mathrm{vol}(M) = -2\pi\chi(M)$, $v_2 = \pi$, which implies $\|M\|_1 \ge -2\chi(M)$. ⊠

Remark 6.1. The volume form is not a bounded function on all singular simplices. However, it is bounded on straight simplices. We have $\mathrm{vol} \circ \mathrm{str}$ bounded, representing the volume class. This leads to bounded cohomology, asking which cocycles can be bounded.

Remark 6.2. It is not crucial to do this for something exactly hyperbolic. We can do a similar thing for M negatively curved and closed, the argument helps us show that $\|M\|_1 > 0$.

Conjecture (Gromov). *Let M be closed, non-positively curved with negative Ricci curvature. Then $\|M\|_1 > 0$.*

Proposition 6.1. *If M is closed hyperbolic, then $\|\cdot\|_1$ is an honest norm (not just a semi-norm) on $H_k(M;\mathbb{R})$ for every $k \ge 2$, i.e. $\sigma \ne 0 \in H_k(M;\mathbb{R})$ implies $\|\sigma\|_1 > 0$.*

Proof. There is a pairing $H_k(M;\mathbb{R}) \times H^k(M;\mathbb{R}) \to \mathbb{R}$ which is non-singular. In other words, $\sigma \in H_k(M;\mathbb{R})$ corresponds to $\sigma^* \in H^k(M;\mathbb{R})$ such that $\langle \sigma, \sigma^* \rangle \ne 0$. More concretely, we can think of $H^k(M;\mathbb{R})$ as de Rham cohomology where the classes are differential forms, and integrate the k-forms on something k-dimensional. Represent σ^* by some differential form ω. Let $\|\mathrm{vol}\|_\infty := \sup |\omega_p(v_1, \cdots, v_k)|$ where we take the supremum over $p \in M$, the $v_1, \cdots, v_k \in T_p(M)$ are orthogonal, and $\|v_i\| = 1$. By compactness $\sup |\omega_p(v_1, \cdots, v_k)| < \infty$; then we can do the same pairing argument where $|\langle \sigma, \sigma^* \rangle| = \left|\left\langle \sum \lambda_i c_i, \omega \right\rangle\right| \le \sum |\lambda_i| \cdot (\|\mathrm{vol}\|_\infty \cdot \mathrm{vol}(c_i))$. Then $\left.\frac{\omega}{\|\mathrm{vol}\|_\infty}\right|_{c_i} \le \mathrm{vol}|_{c_i}$,
todo:unfinished ⊠

The point is we can do a similar argument for other cohomology classes as well. This argument is a powerful tool; we can do pairing with differential forms to get a lower bound. We saw that everything vanishes on H_1, why doesn't this argument work? This is because $v_1 = \infty$ (not-bounded).

Next time we will continue and give a proof that v_n is a finite number.

7 ok

Lemma 7.1. *We have $v_n = \sup_{\Delta^n} \mathrm{vol}(\Delta^n) \le \pi/(n-1)!$ for all $n \ge 2$, where Δ^n is a hyperbolic simplex.*

Proof. We already have $v_2 = \pi$. It suffices to look at ideal simplices. We show $v_n \le \frac{v_{n-1}}{n-1}$ for all $n \ge 3$, which paired with induction implies the bound. Let Δ^n be an arbitrary ideal simplex. Let s map a circle into a half-sphere that sits in $S^{n-1} \subseteq \mathbb{R}^n$. Then for $x \in \mathbb{D}^{n-1}, x \mapsto (x, h(x)), \|x\|^2 + |h(x)|^2 = 1$. So $h(x)^2 = 1 - \|x\|^2$, which implies $h(x) = \sqrt{1 - \|x\|^2}$.

(1) $\Delta^n = \{(x, y) \mid x \in \tau_0, y \ge h(x)\}$,

(2) $\tau = S(\tau_0)$.

We know the metric ds^2/y is a Euclidian metric scaled by the last coordinate. So

$$\operatorname{vol}(\Delta^n) = \int_{\Delta^n} \frac{dx\,dy}{y^n} = \int_{\tau_0} \int_{h(x)}^{\infty} \frac{dy}{y^n}\,dx.$$

We can directly compute the inner integral since we are working in the upper half space model; $\int y^{-n} = \frac{y^{-n+1}}{-n+1}$, so this integral becomes $\frac{1}{n-1} \int_{\tau_0} \frac{1}{h(x)^{n-1}}\,dx \overset{\text{key}}{\leq} \frac{1}{n-1} \cdot \operatorname{vol}(\tau) \leq \frac{v_{n-1}}{n-1}$. Now it remains to show the key inequality. By definition, $\operatorname{vol}(\tau) = \int_{\tau_0} s^* \operatorname{vol}|_\tau$. We now compare the integrals point by point to see that one dominates the other. The pullback $s^*\operatorname{vol}$ is a 2-form on the unit disk, so we evaluate it on (e_1, e_2). This is equal to the volume of the pushforward, or $\operatorname{vol}(s_* e_1, s_* e_2) > \operatorname{vol}|_\tau$ (std basis). What is the standard basis? Whatever it is, the restriction to τ adds one more vector in the orthonormal direction to make it an orthonormal basis, then evaluate. So this becomes $\frac{1}{h(x)^{n-1}}$. ⊠

Remark 7.1. A theorem of Haagerup-Munkholm shows that v_n is uniquely achieved by the *regular* ideal n-simplex. So $\operatorname{sym}(\Delta^n) = S_{n+1}$.

This finishes our estimate. In the two-dimensional case, we have the computation of the volume of surfaces. Let us go back to our theorem.

Theorem 7.1. *If S is an orientable closed connected surface with genus $g \geq 1$, then $\|S\|_1 = -2\chi(S)$.*

Note that this also holds in the genus 1 case since both the volume and the Euler characteristic are zero (admits a self map). If S is orientable, closed, and connected, then denote

$$\chi^-(S) = \begin{cases} \chi(S) & g \geq 1 \\ 0 & g \leq 1. \end{cases}$$

With this notation, then $\|S\|_1 = -2\chi^-(S)$. If S is orientable closed (without assuming connectedness), then $\chi^-(S) = \sum_{\Sigma \subseteq S} \chi^-(\Sigma)$, where the Σ are components of S. Equivalently this is $\sum \chi$ over aspherical components. Then $\|S\|_1 = -2\chi^-(S)$.

Recall. Our problem: $\deg(S, S') = \{\deg(f) \mid f : S \to S'\}$, S, S' occ.

Proposition 7.1. *If S, S' occ with $g \geq 1$, then* $\deg(S, S') = \{d \in \mathbb{Z} \mid |d\chi(S')| \leq |\chi(S)|\}$.

Proof. By the degree inequality, we have $f : S \to S'$, $|\deg(f)| \cdot \|S'\|_1 \leq \|S\|_1$. It suffices to show that every d satisfies $|d \cdot \chi(S')| \leq |\chi(S)|$. We can find $f : S \to S'$ with $\deg(f) = d$. We can compose with a self-homeomorphism to flip the degree, and degree 0 is just a constant map, so focus on d positive. We have $d \cdot |\chi(S')| \leq |\chi(S)|$ by symmetry. We take two kinds of maps. ⊠

8 Measure homology

Recap:

- We used the straightening argument to show that the volume of a hyperbolic manifold $\|M\|_1 \geq \operatorname{vol}(M)/v_n$ for $n \geq 2$, $v_2 = \pi$.
- We also know an upper bound when studying triangulations on surfaces, eg for $n = 2$, $\|S\|_1 = \operatorname{vol}(S)/\pi = -2\chi(S)$ if the genus ≥ 1.

An application is to compute the set of possible degrees $\{\deg f \mid f : S \to S'\}$, and the claim is that this is equal to the set $\{d \mid |\chi(S)| \geq |d\chi(S')|\}$. To make this work exactly, we need S, S' closed oriented, not S^2.

Proof. First we prove the inclusion $\{\deg f \mid f : S \to S'\} \subseteq \{d \mid |\chi(S)| \geq |d\chi(S')|\}$. Let $f : S \to S'$. By the degree inequality, $|d| \cdot \|S'\|_1 \leq \|S\|_1$, which is just saying that $|d| \cdot |\chi(S')| \leq |\chi(S)|$ with $d = \deg f$.

The other direction is done by constructing maps between surfaces. First, exclude some silly cases; we may assume $d > 0$ and satisfies the inequality. There are two kinds of maps. To construct a map $S \to S'$ with degree one, we construct an intermediary covering surface Σ for S', where $S \to \Sigma$ is surjective and the covering $\Sigma \to S'$ has degree d. Let Σ be a degree d cover of S'. The nice thing is that $|\chi(\Sigma)| = |d\chi(S')| \leq |\chi(S)|$, so the genus $g(\Sigma) \leq g(S)$. We have $|\chi(\Sigma)| = 2g-2$, and the map $S \to \Sigma$ "pinches" any extra genus out. To make sure this is true, look at the preimage of a regular value. ⊠

Here is a theorem that possibly could be due to Thurston.

Gromov's proportionality. *Let M be a oriented closed connected hyperbolic manifold, then $\|M\|_1 = \mathrm{vol}(M)/v_n$, where $v_n = \sup \mathrm{vol}(\Delta^n)$ for Δ^n a hyperbolic simplex.*

Proof. It suffices to show $\|M\|_1 \leq \mathrm{vol}(M)/v_n$ by our earlier inequality. The easiest way to prove an upper bound is the following; by definition it is an infimum of simplices. What we need to do is find a nice class representing this fundamental class, where the number of simplices represents the number $\mathrm{vol}(M)/v_n$. To do this, construct cycles representing $[M]$ with the number of simplices approximately optimal, or $\mathrm{vol}(M)/v_n$.

Our strategy is to construct $c = \sum \lambda_i c_i$ with

(1) $\lambda_i > 0$,

(2) c_i a straight hyperbolic (with consistent orientation) simplex with $\mathrm{vol}(c_i) > v_n - \varepsilon$

(3) c is a cycle.

Why do these three conditions show our theorem? By (3), $[c] = \lambda[M]$. Pairing, we get

$$\lambda \mathrm{vol}(M) = \langle \lambda[M], \mathrm{vol}\rangle \implies \langle c, \mathrm{vol}\rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol}\right\rangle = \sum \lambda_i \mathrm{vol}(c_i) \geq \left(\sum \lambda_i\right)(v_n - \varepsilon).$$

Since $\lambda > 0, \sum \lambda_i = |c|_1$, which means $[c/\lambda] = [M], |c/\lambda|_1 = \sum \lambda_i/\lambda \leq \lambda \mathrm{vol}/\lambda(v_n - \varepsilon) = \mathrm{vol}(M)/(v_n - \varepsilon)$. By letting $\varepsilon \to 0$, we get our desired upper bound. This is cool but how do we construct a cycle with these properties? Recall the surface case where we triangulated very large covers and projected down, but it's not always clear if we can do this for manifolds. We have $(\ell_1)^* = \ell_\infty$, but $(\ell_\infty)^* \supsetneq \ell_1$.

We measure homology; earlier we said $C_n(M;\mathbb{R}) = \mathrm{span}_{\mathbb{R}} S_n(M)$, where $S_n(M) = \mathrm{Maps}(\Delta^n, M)$. We equip this with the compact open topology to make this into a space. Let $\mathcal{C}_n(M;\mathbb{R})$ be signed measures on $S_n(M)$ with compact support and bounded total variation. Let $\nu = \nu_+ - \nu_-, |\nu_+(S_n(M)) + \nu_-(S_n(M))$, $C_n(M;\mathbb{R}) \hookrightarrow \mathcal{C}_n(M;\mathbb{R})$, $\sum \lambda_i c_i \mapsto \sum \lambda_i \Delta_{c_i}$ which is norm-preserving. We also have $\partial : \mathcal{C}_{n+1}(M;\mathbb{R}) \to \mathcal{C}_n(M;\mathbb{R})$. This leads to another homology theory, called the **measure homology**. ⊠

Theorem 8.1 (Zastrow, Hansen)**.** *For CW complexes, measure homology is isomorphic to singular homology. Furthermore, this is isometrically isomorphic (Löh, 2006).*

The "isometrically" says that we can alternatively define the Gromov norm this way. This construction is called "smearing", because we put this these simplices everywhere (smearing) to construct a cycle. Next time we approximate this smearing cycle with an honest cycle. The idea is that we fix $\Delta \colon \Delta^n \to \mathbb{H}^n$ such that $\mathrm{vol}(\Delta) > v_n - \varepsilon$. We have $\mathrm{Isom}^+(\mathbb{H}^n)$ acting on $\mathbb{H}^n$, $\mathrm{Isom}^+(\mathbb{H}^n) \cdot \Delta$. By Haar, this is locally finite and $\mathrm{Isom}^+(\mathbb{H}^n)-$invariant. Identify two copies if they differ by some $g \in \pi_1(M)$.

9 Bounded cohomology

Like we said last time, we shift to the dual theory of bounded cohomology. The Gromov norm is like an ℓ^1-norm, so there should be a dual ℓ^∞ norm. For an ℓ^∞-norm to make sense, we need bounds, hence the "bounded" in bounded cohomology. First we talk about ordinary group homology and cohomology, then bounded cohomology of groups, then bounded cohomology of spaces and explain why we only need them for groups. We use a topological point of view.

Let G be a group. We would like to talk about the homology and cohomology groups $H_*(G;R), H^*(G;R)$ for some ring R (usually $\mathbb{Z}$ or $\mathbb{R}$). From the topological POV we use the $K(G,1)$ space, or Eilenberg-Maclane space, or classifying space of G as a discrete group.

9.1 Eilenberg-Maclance spaces

Definition 9.1. We say X, a connected CW complex is a $K(G,1)$ space if we have the following:

- $\pi_1(X) = G$,
- X is **aspherical**, or $\pi_n(X) = 0$ for all $n > 1$. Equivalently, the universal cover $\widetilde{X}$ is contractible.

There are two facts; first, they exist. The second is a universal property of sorts, which implies that they are unique up to homotopy equivalence.

Lemma 9.1. *Let X be a $K(G,1)$ space and Y some connected CW complex. Any homomorphism $\varphi\colon \pi_1(Y,y) \to \pi_1(X,x)$ is realized by some map $f : (Y,y) \to (X,x)$ (i.e. $f_* = \varphi$) and f is unique up to homotopy.*

This corresponds to the fact that maps between spaces induce maps on π_1; here maps between π_1's induce a unique map on the space, given that the target is a $K(G,1)$ space.

Corollary 9.1. *If X and Y are both $K(G,1)$, then any isomorphism $\pi_1(X,x) \to \pi_1(Y,y)$ is induced by a homotopy equivalence $f : X \to Y$.*

Proof. Realize φ by $f : (X,x) \to (Y,y)$, φ^{-1} by $g : (Y,y) \to (X,x)$. The composition $g \circ f : (X,x) \to (X,x)$, and $(g \circ f)_* = \varphi^{-1} \circ \varphi = \mathrm{id}_{\pi_1 X}$. Not only do we have existence we have uniqueness, which implies $g \circ f \simeq \mathrm{id}_X$. Similarly, flipping g and f, $f \circ g \simeq \mathrm{id}_Y$. Therefore g is a homotopy inverse. ⊠

Definition 9.2. Given a ring R, define the homology and cohomology of a group G as $H_*(G;R) := H_*(K(G,1);R)$ and $H^*(G;R) := H^*(K(G,1);R)$.

Example 9.1. Some examples.

- $H_0(G;R) = H_0(K(G,1);R) = R$, since G is connected.
- $H_1(G;\mathbb{Z}) = H_1(K(G,1);\mathbb{Z}) = \mathrm{Ab}(\pi_1(K(G,1))) = \mathrm{Ab}(G)$. Similarly, $H^1(G;R) = \mathrm{Hom}(G,R)$.
- What is $H_*(\mathbb{Z};\mathbb{Z})$? Let $X = S^1$ which is a $K(\mathbb{Z},1)$ space. Then

$$H_k(\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & k > 1. \end{cases}$$

  There is a notion of the geometric dimension of G, which is the smallest dimension of X for X a $K(G,1)$ space. Formally, this is defined as $\mathrm{gd}(G) := \min\{\dim X \mid X \text{ is } K(G,1)\}$. There is a dual notion $\mathrm{cd}(G) = \min\{K \mid H_n(G;R) = 0 \text{ for all } n > k,\ \text{all } R\}$. Here we need to take twisted coefficients (comes with a G-action), the important this is that the cohomological dimension is always less than or equal to the homological dimension, or $\mathrm{cd}(G) \leq \mathrm{gd}(G)$.

9.2 Manifolds as $K(G, 1)$ spaces

Every finitely presented group can be represented as π_1 of some closed 4-manifold. What if we want to realized a $K(G, 1)$ space as a manifold? The question is, given G, is there a manifold that is a $K(G, 1)$ space? In other words, is there an aspherical manifold with $\pi_1 = G$?

Example 9.2. For $G = F_2$ this manifold exists, for example the inside of the 2-torus (handlebody with body removed), the punctured torus with boundary removed, or the thrice punctured sphere with boundary removed.

What if we look for M closed? Then $H_n(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for M^n closed. If we further require that M^n is closed orientable, then $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ generated by the fundamental class.

Lemma 9.2. *(1) If M^n is closed, aspherical with $\pi_1(M) = G$, then $H_n(G; \mathbb{Z}/2) \simeq 2) \cong \mathbb{Z}/2, H_k(G; R) = 0$, $k > n$.*
(2) If M^n is orientable in addition, $H_n(G; \mathbb{Z}) \cong \mathbb{Z}$, etc.

Example 9.3. Continuing our example, for $G = F_r$, $X = S^1 \vee S^1$, $H_k(G; \mathbb{Z}) = 0$ for $k > 1$, $\mathbb{Z}$ for $k = 0$, and $\mathbb{Z}^r$ for $k = 1$ (r is the rank). This leads to the following corollary.

Corollary 9.2. *We cannot realize $K(F_r, 1)$ as a closed manifold when $r > 1$.*

For $G = \mathbb{Z}/n$, $H_k(\mathbb{Z}/n; \mathbb{Z}) = \mathbb{Z}$ for $k = 0$, $\mathbb{Z}/n$ when k is odd, and 0 when $k > 0$, even. In particular, the cohomological dimension of G is infinite, so the geometrical dimension of G is infinite as well, so there is no finite dimensional $K(G, 1)$. In particular, it cannot be a manifold.

Proposition 9.1. *Let G be a finite group, G acting on $\mathbb{R}^n$. Then this action is not free.*

Proof. Suppose the action of G is free. Let $H = \mathbb{Z}/m$ be a cyclic subgroup (take the powers of an element of G), then H acts on $\mathbb{R}^n$ freely. Then the projection $\mathbb{R}^n \to \mathbb{R}^n/H$ to the quotient is a covering map, since H acts freely and is properly discontinuous. Define $X := \mathbb{R}^n/H$. Then X is a $K(\mathbb{Z}/m, 1)$ space since $\mathbb{R}^n$ is contractible, a contradiction. ⊠

10 Co-Hopfian groups and group homology

10.1 Co-Hopfian groups

Definition 10.1. A group G is **co-Hopfian** if every *injective* $G \hookrightarrow G$ is an isomorphism.

Example 10.1. Some examples:

(1) Finite groups are co-Hopfian.
(2) $\mathbb{Z}$ is not co-Hopfian by the map $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.
(3) $\mathbb{Z}^n$ is not co-Hopfian.
(4) F_n is not co-Hopfian. For example, consider $F_2 = \langle a, b \rangle$ with the self-map $a \mapsto a, b \mapsto b^2$. If we think of F_2 as $\mathbb{Z} * \mathbb{Z}$, then the first $\mathbb{Z}$ maps to itself by the identity and the second maps to itself by squaring.
(5) Let $G = H * K$ (free product), if K is not co-Hopfian, then G is not co-Hopfian by the same logic as above.

Lemma 10.1. *Let M be a complete Riemannian manifold such that sectional curvature is non-positive. Then M is aspherical.*

Proof. The proof is by Cartan-Hamadard, which tells us that the exponential map $T_pM \to M$ is a covering. todo:missed this proof ⊠

Lemma 10.2. *Let M, N be connected aspherical n-manifolds. Suppose that $\pi_1 M \simeq \pi_1 N$. Then M and N have the same compactness.*

Proof. Note that M is closed iff $H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2$. We have $G = \pi_1 M = \pi_1 N$, and N is closed iff $H_n(N;\mathbb{Z}/2) = \mathbb{Z}/2$, which is true, so we are done. ⊠

Lemma 10.3. *If M is a closed connected aspherical manifold, then any subgroup H of $\pi_1 M$ with $H \simeq \pi_1 M$ must have finite index.*

Proof. Let $G = \pi_1 M \geq H$ a subgroup. We have an isomorphism $f : G \to H$. Let $\widetilde{M}$ be the covering space corresponding to H. Since M is aspherical, M is $K(G,1)$, which implies that $\widetilde{M}$ is aspherical and $K(H,1)$. So f can be realized as a homotopy equivalence $\varphi \colon M \to \widetilde{M}$. M is compact implies that $\widetilde{M}$ is compact, so π is a *finite* cover. Finite covers correspond to finite index subgroups, therefore H has finite index. ⊠

Lemma 10.4. *Let M be a closed, connected, aspherical, orientable manifold. If $\pi_1 M$ is* not *co-Hopfian, then there is a self-map $f : M \to M$ with $|\deg f| > 1$. So $\|M\|_1 = 0$.*

Proof. Let $G = \pi_1 M, h \colon G \hookrightarrow G, H = \operatorname{im} h$. By the lemma above, we get π a finite cover, φ a homotopy equivalence..

$$\begin{tikzcd} & H \arrow[d, hook] \\ G \arrow[ur, "\overset{h}{\cong}"] \arrow[r, "h"'] & G \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{M} \arrow[d, "\pi"] \\ M \arrow[ur, "\varphi"] \arrow[r, "f"'] & M \end{tikzcd}$$

We have $n = \dim M$. The map $\varphi \colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(M;\mathbb{Z})$ an isomorphism, $[M] \mapsto \pm[M]$, so $\deg(\varphi) = \pm 1$. Then we have

$$|\deg f| = |\deg \varphi| \cdot |\deg \pi| = |\deg \pi| > 1$$

if H is proper, where $\deg \pi$ is the index of H in G. By the degree inequality, we get $\|M\| \leq |\deg f| \cdot \|M\|$ which implies $\|M\|_1 = 0$. ⊠

Corollary 10.1. *If M is closed with negative sectional curvature, then $\pi_1 M$ is co-Hopfian.*

Proof. Negative curvature implies M is aspherical, which also implies $\|M\|_1 = 0$. By the lemma above, $\pi_1 M$ is co-Hopfian. ⊠

Example 10.2. For S occ, $g > 1$, then $\pi_1(S)$ is co-Hopfian.

This ends the detour, and we will go back to the algebraic definition of group homology and group cohomology.

10.2 The bar complex

Let $C_n(G;R)$ be the free R-module with basis consisting of n-tuples $(g_1, \cdots, g_n) \in G^n$. In the same way, define co-chains as the dual $C^n(G;R) = \operatorname{Hom}(C_n(G;R), R)$. The differential $\partial : C_n(G;R) \to C_{n-1}(G;R)$ looks a little strange:

$$\partial(g_1, \cdots, g_n) = (g_2, \cdots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n) + (-1)^n (g_1, \cdots, g_n).$$

For example, when $n = 2$, $\partial(g_1, g_2) = g_2 - g_1 g_2 + g_1$. We can think of this as a map to $\mathbb{Z}$, which is zero when this is a homomorphism. For $n = 3$, we have $\partial(g_1, g_2, g_3) = (g_2, g_3) + (g_1 g_2, g_3) + (g_1, g_2 g_3) - (g_1, g_2)$. This corresponds to the four faces of a tetrahedron. The key thing is that $\partial^2 = 0$, and the corresponding homology $H_n(G;R) = \ker \partial / \operatorname{im} \partial$ is the group homology. Similarly, $H^n(G;R)$ is the group cohomology; we will use this model to explain bounded cohomology. What we will do next time is build a particular $K(G,1)$ space which corresponds to our formala, and from there we will define bounded cohomology.

11 Bounded cohomology

todo:was late, missed definition of $\delta_n \colon C^n(G;R) \to C^{n+1}(G;R)$. bounded functions, R is the size. sup norm over simplices. bounded funtion is still bounded over coboundary. hence we get bounded cohomology $H_b^n(G;R) = Z_b^n/B_b^n$. equipped with nan inudced norm $\sigma \in H_b^n(G;R)$. $\|\alpha A\|_\infty = \inf_{[f]=\sigma, f\in Z_b^n G;R} |f|_\infty$.

Definition 11.1. There is a comparison map $c\colon H_b^n(G;R) \to H^n(G;R)$ induced by $C_b^n(G;R) \to C^n(G;R)$.

Some questions:

(1) Given $\sigma \in H^n(G;R)$, is there a $\sigma \in \operatorname{im}\sigma$. Related question: is c surjective? Furthermore, if $\sigma_b \mapsto \sigma$ by c, is there a natural σ_b? Does it carry any extra info?
(2) What is $\ker(c)$? What do they correspond to?

Example 11.1. Some examples in lower degrees.

- For degree $n = 0$, the only 0-cochain is a constant function, which are always bounded. The 0-cochain is also a 0-cycle, there is nothing interesting; so $H_b^0(G;R) = H^0(G;R) = R$.
- For degree $n = 1$, let $f \in C^1(G;R)$, i.e. $f: G \to R$. The coboundary $(\delta f)(g,h) = f(g) + f(h) - f(gh)$. We have $f \in Z^1(G;R)$ iff $f: G \to R$ is a homomorphism (coboundary zero). If f is bounded, consider $|f(g^n)| = |nf(g)| \le C$, so $|f(g)| = 0$. Therefore if f is a homomorphism then $f = 0$, so $Z_b^1(G;R) = 0$, which implies $H_b^1(G;R) = 0$. In general, the comparison map is not an isomorphism.

A general idea to obtain $\sigma \in \ker c$. Consider $C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta^2} C^{n+1}$. For $f \in C^{n-1}$, we know $\delta^2 = 0$, so $\delta f \in Z^n$. As a cohomology class it is trivial, but it might be a nontrivial bounded cohomology class. If δf is bounded, then $[\delta f] \in Z_b^n$. This gives us a class in H_b^n. Furthermore, $[\delta f] \in \ker c$.

- When $n = 2$, $f \in C^1(G;\mathbb{R})$ is a function $G \to \mathbb{R}$. We hope that $(\delta f) = f(g) + f(h) - f(gh)$ is bounded. This leads to the following definition.

Definition 11.2. A function $\varphi\colon G \to \mathbb{R}$ is a **quasimorphism** if $D(\varphi) := \sup_{g,h\in G} |\varphi(g) + \varphi(h) - \varphi(gh)| < \infty$. The number $D(\varphi)$ is called the **defect** of φ. We say φ is **homogeneous** if $\varphi(g^n) = n\varphi(g)$ for all $n \in \mathbb{Z}$, $g \in G$.

Example 11.2. Some examples:

(1) Homomorphisms are homogeneous quasimorphisms.
(2) Bounded functions are quasimorphisms, but never homogeneous.
(3) Quasimorphisms form a real linear space: you can multiply by a scalar and add them.

Denote the space of all quasimorphisms by $\hat{Q}(G)$, and there is a containment $H^1(G;\mathbb{R}) \subseteq Q(G) \subseteq \hat{Q}(G) \subseteq C_b^1(G;\mathbb{R})$ where $Q(G)$ denotes the linear subspace of homogeneous quasimorphisms, and $C_b^1(G;\mathbb{R})$ are bounded functions.

Proposition 11.1. *The space of all quasimorphisms decomposes as* $\hat{Q}(G) = Q(G) \oplus C_b^1(G;\mathbb{R})$.

We will prove this later!

Definition 11.3 (Homogenization)**.** Take an arbitrary $\varphi \in \hat{Q}(G)$, then define the homogenization $\overline{\varphi}$ by $\overline{\varphi}(g) = \lim_{n\to+\infty} \frac{\varphi(g^n)}{n}$.

Often times we need check whether this limit exists. It turns out $\overline{\varphi}$ is well-defined, homogeneous, and a quasimorphism. To see well-definedness, we quote the following definition from analysis.

Lemma 11.1. *Consider $\{a_n\}$ a sub-additive sequence, which means $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$. Then $\lim_{n\to+\infty} a^n/n = \inf_{n\geq 1} a^n/n$. In particular, the limit exists if a_n/n is bounded below.*

Proof. We have
$$\overline{\lim}\frac{a_n}{n} \geq \underline{\lim}\frac{a_n}{n} \geq \inf_{n\geq 1}\frac{a_n}{n}$$
by definition. We want tos how that $\inf_{n\geq 1}\frac{a_n}{n} \geq \overline{\lim}\frac{a_n}{n}$. Fix $m \leq 1$, $n = qm + r$, $0 < r \leq m$ by the division algorithm. If $n > m$, then $a_n \leq a_m + a_{n-m} \leq \cdots \leq qa_m + a_r$. To bound a_r, introduce $B = \max_{0<r\leq m} a_r$. Then this is less than or equal to $qa_m + B$. By the inequality above,

$$\frac{a_n}{n} \leq \frac{aq_m + B}{qm + r} = \frac{a_m + \frac{B}{q}}{m + \frac{r}{q}} \xrightarrow{n\to\infty} \frac{a_m}{m}.$$

So $\overline{\lim} a_n/n \leq a_m/m$. Taking m arbitrary, this implies $\overline{\lim} a_n/n \leq \inf_{m\geq 1} a_m/m$. ⊠

We will see the rest next lecture.

12 More on quasimorphisms

Last time we had the comparison map $c\colon H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})$, and we were trying to understand what is $\ker c$. We had quasimorphisms $\varphi\colon G \to \mathbb{R}$ with the property that $|\varphi(gh) - \varphi(g) - \varphi(h)| \leq D(\varphi)$ for all g, h. This leads to a vector space $\hat{Q}(G)$ of all quasimorphisms, with a homogeneous subspace $Q(G)$ and another subspace $H^1(G;\mathbb{R})$. Another containment is the bounded functions $C_b^1(G;\mathbb{R}) \supseteq \hat{Q}(G)$. We were trying to show that $\hat{Q}(G) = C_b^1(G;\mathbb{R}) \oplus Q(G)$, and we have already shown how to get the homogeneous component (from homogenization).

Let $\varphi_+ = \varphi + D(\varphi)$. Then $\varphi(gh) - D(\varphi) \leq \varphi(g) + \varphi(h) \leq \varphi(gh) + D(\varphi)$. So

$$\begin{aligned}\varphi_+(g) + \varphi_+(h) &= \varphi(g) + \varphi(h) + 2D(\varphi)\\ &\geq (\varphi(gh) - D(\varphi)) + 2D(\varphi)\\ &= \varphi(gh) + D(\varphi)\\ &= \varphi_+(gh),\end{aligned}$$

which implies $\varphi_+(g^n)$ is subadditive. Similarly, define $\varphi_- = \varphi - D(\varphi)$, doing an analogous calculation shows that $\varphi_-(g^n)$ is sup-additive, so $\varphi_-(g^{m+n}) \geq \varphi_-(g^m) + \varphi_-(g^n)$. Now we have bounds on both sides in a sense. Explicitly, we have $\varphi_-(g) \leq \varphi(g) \leq \varphi_+(g)$. Therefore

$$\varphi_-(g) \leq \frac{\varphi - (g^n)}{n} \leq \frac{\varphi_+(g^n)}{n} \leq \varphi_+(g).$$

The sup-additive sequence has an upper bound and the sub-additive sequence has a lower bound. Taking the limit, by our analysis lemma, we have

$$\sup\frac{\varphi_-(g^n)}{n} = \lim\frac{\varphi_-(g^n)}{n} = \lim_{n\to+\infty}\frac{\varphi_+(g^n)}{n} = \inf\frac{\varphi + (g^n)}{n}.$$

So $\varphi_+(g^n) = \varphi_-(g^n) + 2D(\varphi)$. By the squeeze lemma, $\sup\frac{\varphi_-(g^n)}{n} = \lim\frac{\varphi^n(g)}{n} = \overline{\varphi}(g)$. So the limit exists, and we actually have two new descriptions of our limit. On one hand, $\overline{\varphi}(g) = \inf_{n\geq 1}\frac{\varphi_+(g^n)}{n} \leq \varphi_+(g) = \varphi(g) + D(\varphi)$ (for $n = 1$). On the other hand, $\overline{\varphi}(g) = \sup_{n\geq 1}\frac{\varphi_-(g^n)}{n} \geq \varphi_-(g) = \varphi(g) - D(\varphi)$. Together, $|\overline{\varphi}(g) - \varphi(g)| \leq D(\varphi)$ for every g. So $\overline{\varphi}(g)$ is only a bounded distance away from $\varphi(g)$, which implies $\overline{\varphi}$ itself is a quasimorphism (sum of two quasimorphisms). The last thing to show is that $\overline{\varphi}$ is a homogeneous quasimorphism. This follows by the limit definition. We have

$$\overline{\varphi}(g^k) = \lim_{n\to\infty} \frac{\varphi(g^{kn})}{n} = k \lim_{n\to\infty} \frac{\varphi(g^{kn})}{kn} = k\overline{\varphi}(g).$$

However, this only works for $k \in \mathbb{Z}_+$. If $k = 0$, $\overline{\varphi}(\mathrm{id}) = \lim \frac{\varphi(\mathrm{id}^n)}{n} = 0$. If k is negative, then $\overline{\varphi}(g^{-k}) = k\overline{\varphi}(g^{-1})$, so if $\overline{\varphi}(g^{-1}) = -\overline{\varphi}(g)$ then we are good. To see why adding them up gives $|\varphi(g^n) + \varphi(g^{-n}) - \varphi(\mathrm{id})| \le D(\varphi)$, which implies $\varphi(g^n) + \varphi(g^{-n})$ is bounded.

Lemma 12.1. *$\overline{\varphi}$ is a well-defined homogeneous quasimorphism, moreover, we have an explicit bound $|\overline{\varphi} - \varphi|_\infty = \sup_g |\overline{\varphi}(g) - \varphi(g)| \le D(\varphi)$.*

This summarizes our previous discussion.

Remark 12.1. Recall that $D(\varphi + \psi) \le D(\varphi) + D(\psi)$, and for f bounded we have $D(f) \le 3|f|_\infty$ by the triangle inequality. In our case, $D(\overline{\varphi}) \le D(\overline{\varphi} - \varphi) + D(\varphi) \le 3D(\varphi) + D(\varphi) = 4D(\varphi)$. Actually, we have a better bound $D(\overline{\varphi}) \le 2D(\varphi)$. But we may not need this fact.

Proposition 12.1. $\hat{Q}(G) = Q(G) \oplus C_b^1(G)$.

Proof. We have

$$\varphi = \underbrace{\overline{\varphi}}_{\in Q(G)} + \underbrace{(\varphi - \overline{\varphi})}_{\in C_b^1(G)},$$

which means $\hat{Q}(G) = Q(G) + C_b^1(G)$. To show $Q(G) \cap C_b^1(G) = 0$, let $\varphi \in Q(G) \cap C_b^1(G)$. Then $|\varphi(g)| = |\varphi(g^n)/n| \le |\varphi|_\infty / n \xrightarrow{n\to\infty} 0$, and we are done. ⊠

Proposition 12.2. *We have an exact sequence*

$$0 \to H^1(G) \to Q(G) \xrightarrow{\delta} H_b^2(G;\mathbb{R}) \xrightarrow{c} H^2(G;\mathbb{R}).$$

Proof. Exactness at the first entry means that $H^1(G) \to Q(G)$ is injective, which is true since it is defined as in inclusion. So we have to show two things:

(1) $\ker\delta = H^1(G)$
(2) $\operatorname{im}\delta = \ker c$.

For (1), let $\varphi \in Q(G)$. Then $(\delta\varphi)(g,h) = \varphi(g) + \varphi(h) - \varphi(gh)$. So $\delta\varphi = 0 \iff \varphi$ is a homomorphism, i.e. $\varphi \in H^1(G)$. For (2), the easier direction is to show $\operatorname{im}\delta \subseteq \ker c$. This is by definition, since $\delta^2 = 0$. So $\delta\varphi$ is a coboundary in the ordinary sense, i.e. $[\delta\varphi = 0$ in $H^2(G;\mathbb{R})$.

The harder part is to show $\operatorname{im}\delta \supseteq \ker c$. Let $\alpha \in H_b^2(G;\mathbb{R})$ such that $c(\alpha) = 0 \in H^2(G;\mathbb{R})$, or $\alpha = \delta\varphi$, $\varphi: G \to \mathbb{R}, \varphi \in C^1(G;\mathbb{R})$. Since α is a bounded class, $\delta\varphi$ is bounded which implies φ is a quasimorphism. To show that this is homogeneous, $\varphi - \overline{\varphi}$ is bounded, so $[\delta\varphi] = [\delta\overline{\varphi}] = \alpha \in H_b^2(G;\mathbb{R})$. Then $\overline{\varphi} \in Q(G)$, which implies $\alpha \in \operatorname{im}\delta$. ⊠

Corollary 12.1. $\ker c = Q(G)/H^1(G;\mathbb{R}) = \operatorname{im}\delta$.

Next time we will talk about many different kinds of quasimorphisms. todo:previous lecture

13 Quasimorphisms on free groups

We talk about Rolli's construction. Consider $F_2 = \langle a, b\rangle$. Our first construction lies in $\ell^\infty(\mathbb{Z}_+) = \{\mathbb{R}\text{-valued bounded funtions on } \mathbb{Z}_+\}$, or bounded sequences. So $\sup_{n\geq 1} |f(n)|$ must be finite, and denote this $|f|_\infty$. Given $f, g \in \ell^\infty(\mathbb{Z}_+)$, define $\varphi_{f,g}$ on F_2. Let $w \in F_2$. Then $w = a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}$, ${}_i, n_i \in \mathbb{Z}$, with $m_i, n_i \neq 0$ except for m_1 or n_k. Then our quasimorphism is defined by

$$\varphi_{f,g}(w) = \sum_{i=1}^{k} f(m_i) + \sum_{j=1}^{k} g(n_i)$$

where we extend f, g to odd functions on $\mathbb{Z}$. An odd function satisfies $f(-n) = -f(n), f(0) = 0$; odd functions are purely determined by their positive part, and similarly you can uniquely extend functions defined on the positive integers to an odd function. We need to check that this is a quasimorphism.

Lemma 13.1. *$\varphi_{f,g}$ is a quasimorphism, with defect $D(\varphi_{f,g}) \leq 3\max\{|f|_\infty, |g|_\infty\}$.*

Proof. Let $u, v \in F_2$. If $G = A * B$ with $u = a_1b_1\cdots a_kb_k$, $v = a_1', b_1'\cdots a_\ell' b_\ell'$, when multiplying together you get $a_1b_1\cdots a_kb_ka_1', b_1'\cdots a_\ell' b_\ell'$ by concatenation. If $b_k = \mathrm{id}$, and $a_k^{-1} = a_1'$, and so on, things might cancel. For example. if $u = xay^{-1}, v = ya'z^{-1}$, then $uv = xaa'z^{-1}$. Now let $a = a^m, a' = a^n, m, n \neq 0, m+n \neq 0$, $a \cdot a' = a^{m+n}$. Let $a', a'' \in A, a' \cdot a'' \neq \mathrm{id}$. Then $\varphi_{f,g}(uv) - \varphi_{f,g}(n) - \varphi_{f,g}(v) = f(m+n) - f(m) - f(n)$ todo:this ⊠

Lemma 13.2. *$\overline{\varphi}_{f,g}(w) = \varphi_{f,g}(v)$, where $w = uvuv^{-1}$ is cyclically reduced.*

For example, let $w = a^{10}b^{12}a^{-2}b^3a^5b^{-12}a^{-10}$. todo:?

Lemma 13.3. *We have a linear embedding $\ell^\infty(\mathbb{Z}_+) \to Q(F_2), f \mapsto \overline{\varphi}_{f,f}$.*

Proof. $\overline{\varphi}_{f,f} \neq 0$ if $f \neq 0$. Then $\overline{\varphi}_{f,f}(a^n, b^n) = \varphi_{f,f}(a^nb^n) = f(n) + f(n) = 2f(n)$. $n \in \mathbb{Z}_+$ is arbitrary so $\overline{\varphi}_{f,f} = 0$ implies $f = 0$. ⊠

Theorem 13.1. *$Q(F_2)$ and $H_b^2(F_2;\mathbb{R})$ are infinite dimensional.*

Proof. $\dim Q(F_2) = \infty$ since $\dim \ell^\infty(\mathbb{Z}_+) = \infty$. Then

$$Q(F_2)/H^1(F_2) = \ker(H_b^2(F_2;\mathbb{R}) \to H_2(F_2)\mathbb{R}) = H_b^2(F_2;\mathbb{R})$$

which implies $H_b^2(F_2;\mathbb{R})$ is infinite dimensional. ⊠

Corollary 13.1. *If $G \twoheadrightarrow F_2$, then $Q(G)$ is ∞-dimensional.*

Proof. Let $h^*: Q(F_2) \hookrightarrow Q(G)$ be the pullback, composing gives our result. ⊠

Example 13.1. F_n for $n \geq 2$ has the surjective map $\pi_1(S_g) \twoheadrightarrow F_g$, $g \geq 2$. We deduce all these groups have infinite-dimensional quasimorphisms.

14 Mostow Rigidity

hisashiburi.

Lemma 14.1. *$\partial\widetilde{\varphi}$ preserves $\mathfrak{V}$.*

Why is this true? $\widetilde{\varphi}$ is a lift of φ, a homotopy equivalence between $M \to N$. Since simplicial volume is a homotopy invariant, Gromov's proportionality tells us $\|M\|_1 = \mathrm{vol}(M)/v_n$ is equal to $\|N\|_1 = \mathrm{vol}(N)/v_n$. Recall that v_n is uniquely achieved by the volume of regular ideal simplices.

Proof. Suppose there exists a regular ideal simplex Δ_r such that $\partial\widetilde{\varphi}(V(\Delta_r)) = V(\widetilde{\Delta})$ where $\widetilde{\Delta}$ is not regular. Then

$$\mathrm{vol}\left(\widetilde{\Delta}\right) + 2\varepsilon < \mathrm{vol}(\Delta_r) = v_n.$$

Take an approximation of Δ_r by a finite straight simplex Δ, then $\mathrm{vol}(\Delta') > v_n - \varepsilon$. By our smearing argument, we can represent the fundamental class of M as a summation, where $[M] = \sum \lambda_i c_i, \lambda_i > 0$. Each c_i is the projection of some Δ'. Sending this through th emap φ, this sends $\varphi\colon [M] \to [N] = \sum \lambda_i \mathrm{str}(\varphi_* c_i)$. Now $\mathrm{vol}(\mathrm{str}(\varphi_* c_i))$ differes from $\mathrm{vol}(\widetilde{\Delta})$ by a factor of ε. Computing the volume of N we get a contradiction:

$$\mathrm{vol}(N) = \langle [N], \mathrm{vol}\rangle = \sum \lambda_i \mathrm{vol}(\mathrm{str}(\varphi_* c_i)) < \sum \lambda_i (v_n - \varepsilon) = \left(\sum \lambda_i\right)(v_n - \varepsilon) \leq (\|M\|_1 + \delta)(v_n - \varepsilon) = (\|N\|_1 + \delta)(v_n - \varepsilon)$$

ε is fixed as the distance between the volumes. We choose δ very very small, such that $(\|N\|_1 + \delta)(v_n - \varepsilon)$ is approximately $\|N\|_1 (v_n - \varepsilon) < \|N\|_1 \cdot v_n$. If we have a map on the boundary taking the vertex set of any regular ideal simplex to another regular ideal simplex, then this has to be the boundary map of an isometry.

Proposition 14.1. *For $n \geq 3$, $h\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ a homeomorphism such that h preserves $\mathfrak{V}$, then $h = \partial F$ for some $F\colon \mathbb{H}^n \to \mathbb{H}^n$ a hyperbolic isometry.*

Note that any two regular ideal simplices Δ_1, Δ_2 differ by some hyperbolic isometry, since equilateral triangles differ by some Euclidian similary ($\mathrm{stab}(\infty)$). By composing with some $F \in \mathrm{Isom}(\mathbb{H}^n)$, we may assume h fixes the vertices of some regular ideal simplex. The goal is to have $h = \mathrm{id}_{\partial\mathbb{H}^n}$. $\mathrm{Fix}(h)$ contains a dense set in $\partial\mathbb{H}^n$. Then the convex hull $\mathrm{co}(v_0, v_1, v_2, v_3)$ is regular, as well as $\mathrm{co}(v_0, v_1', v_2, v_3)$. The point is that there are exactly two regular simplices containing these points, so they must be the only two regular simplices containing v_0, v_2, v_3. Therefore v_1' must be fixed.

The inversion takes $v_1' \to m_1$ the midpoint of $[v_2, v_3]$, which implies that $m_1 \in \mathrm{Fix}(h)$. Continuing this process gives us the fact that $\mathrm{Fix}(h)$ contains a dense set of $\partial\mathbb{H}^n$.

Now to conclude the proof. By our proposition, $\partial\widetilde{\varphi} = \partial F$ which is π_1-equivariant, $F \in \mathrm{Isom}(\mathbb{H}^n)$, which implies F is π_1-equivariant. So F induces an isometry $f : M \to N$. We also have an equivariant homotopy H between $\widetilde{\varphi}$ and F. The map $\{H(x,t) \mid 0 \leq t \leq 1\}$ (the geodesic from $\widetilde{\varphi}(x)$ to $F(x)$), which means H induces a homotopy between φ and f. But f is an isometry, and we are done. This concludes Mostow rigidity. ⊠

15 Consequences of Mostow rigidity

Recall we have a homotopy equivalence of manifolds $\varphi : M \to N$ and an π_1-equivariant map $\widetilde{\varphi}(\widetilde{M} = \mathbb{H}^n) \to (\widetilde{N} = \mathbb{H}^n)$ with boundary $\partial\widetilde{\varphi}\colon \partial\mathbb{H}^n \to \mathbb{H}^n$ a self-homeomorphism. Then this boundary map $\partial\widetilde{\varphi}$ must be ∂F for some isometry F, which implies $f : M \to N$ is an isometry $f \simeq \varphi$. This can be proved in different ways:

(1) $\partial\varphi$ is a quasi-conformal ergordicity means constant distortion. Here $\pi_1 M$ is ergodic on $\partial\mathbb{H}^n$, which implies $\partial\widetilde{\varphi} = \partial F$ is conformal.

(2) We can view $M = \mathbb{H}^n/\Gamma$, where $\Gamma \cong \pi_1 M$ acting on $\mathbb{H}^n$ by isometry. Then $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$, which is a discrete subgroup (lattice acting by isometries). This is a so called **cocompact lattice**. Furthermore since the quotient is a manifold this lattice is also torsion free. Let $M = \mathbb{H}^n/\Gamma_1, N = \mathbb{H}^n/\Gamma_2$. Any $\varphi : \Gamma_1 \xrightarrow{\cong} \Gamma_2$ is realized by conjugation in $F \in \mathrm{Isom}(\mathbb{H}^n)$, $\varphi(\gamma) = F\gamma F^{-1}$. The reason for this is basically lifting of universal covers.

$$\begin{tikzcd}
\mathbb{H}^n \arrow[r, "\widetilde{\varphi}=F"] \arrow[d] & \mathbb{H}^n \arrow[d] \\
\mathbb{H}^n/\Gamma_1 = M \arrow[r, "\varphi"] & \mathbb{H}^n/\Gamma_2 = N
\end{tikzcd}$$

Here φ is a homotopy equivalence, but Mostow rigidity tells us that we can choose φ to be an isometry. Then for $\gamma \in \Gamma_1$, this leads to a diagram

$$\begin{tikzcd} \mathbb{H}^n \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathbb{H}^n \arrow[d, "\partial(\gamma)=F\gamma F^{-1}"] \\ \mathbb{H}^n \arrow[r, "F"'] & \mathbb{H}^n \end{tikzcd}$$

There are higher analogues for Lie groups, one of them being Margul's super rigidity. It essentially says a linear representation of a higher dimensional lattice comes from a representation of the underlying Lie group.

Some consequences.

Theorem 15.1. *For $M = \mathbb{H}^n/\Gamma$, the following groups are isomorphic and finite if $n \geq 3$.*

(1) $\mathrm{Isom}(M)$.

(2) N_Γ/Γ *(the normalizer of Γ), where $N_\Gamma = \{F \in \mathrm{Isom}(\mathbb{H}^n) \mid F\Gamma F^{-1} = \Gamma\}$. In other words, this is the largest subgroup of the ambient group such that Γ is a normal subgroup.*

(3) $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$.

(4) $\mathrm{MCG}(M) = \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$.

Proof. ($\mathrm{Isom}(M) \cong N_\Gamma/\Gamma$): This is a standard process that doesn't have much to do with Mostow rigidity. For $f \in \mathrm{Isom}$ acting on M, $\Gamma\colon \mathbb{H}^n \to M$, we can lift f to the universal cover F which implies $F\Gamma F^{-1} = \Gamma$. Conversely, if f is a normalizer, then $\gamma \mapsto F\gamma F^{-1}$ is an isometry $\Gamma \cong \Gamma$. Here $\Gamma = \ker(N_\Gamma \to \mathrm{Isom}(M))$, F a life of id iff $F \in \Gamma$ (the deck group). This is a general argument, we could even replace $\mathbb{H}^n$ with some other geometric group.

($N_\Gamma/\Gamma \cong \mathrm{Out}(\Gamma)$): One direction is clear. For $F \in N_\Gamma$, $\varphi_\Gamma\colon \Gamma \to \Gamma, \gamma \mapsto F\gamma F^{-1}$ if $F \in \Gamma$ implies $\partial_F \in \mathrm{Inn}(F)$.

$$\begin{tikzcd} N_\Gamma \arrow[r, "F\mapsto\varphi_F"] \arrow[dr, "h"'] & \mathrm{Aut}(\Gamma) \arrow[d] \\ & \mathrm{Out}(\Gamma) \end{tikzcd}$$

Here $\ker(h) \supseteq \Gamma$. We need to show that h is surjective (Mostow rigidity) and $\ker h = \Gamma$. Pick $\varphi \in \mathrm{Aut}(\Gamma)$, then $\varphi_F = \varphi$ up to conjugation (different lifts differ by conjugation). Suppose $\varphi_F = \varphi_\gamma$ for $\gamma \in P, F \in N_\Gamma$. Then $\varphi F = \mathrm{id}, FgF^{-1} = g$ for all $g \in F$. This implies F commutes with all $g \in \Gamma$, and $F = \mathrm{id}$. Here $\mathrm{Fix}(g) = \mathrm{Fix}(FgF^{-1}) = F(\mathrm{Fix}(G))$, attracting/repelling implies F fixes g^+ and g^-. Conjugate g by $\gamma \in P$, then $g^+ \mapsto \gamma(g^+)$ (fixed by F) by varying γ.

$$\begin{tikzcd} \mathrm{Isom}(M) \arrow[r] \arrow[dr, hook, "\cong"] \arrow[ddr, "\cong"'] & \mathrm{Homeo}(M) \arrow[d] \\ & \mathrm{MCG}(M) \arrow[d] \\ & \mathrm{Out}(M) \end{tikzcd}$$

Mostow rigidity tells us the red map is surjective, which implies the blue map is an isomorphism. ⊠

Next time we will explain that the isometry group of a closed hyperbolic manifold is finite. Then $\mathrm{Isom}(M)$ acts on M acting on FM (compact), so orbits are discrete, and subsequently finite. Then we need to show the stabilizers are trivial, which is true because isometries cannot fix the frame (otherwise it is trivial).

16 ok

17 ok

Last day of class today wow. Last time we talked about the bounded Euler class associated to a group action on the circle. One is the that the bounded Euler class is trivial iff the action has a global fixed point. The idea is that we can use that point to lift the action. The other thing is that this Euler class captures the action completely up to semiconjugal equivalence.

Example 17.1. Consider $G = \mathbb{Z}$. Pick $f \in T = \mathrm{Homeo}^+(S^1)$. Let $\rho : \mathbb{Z} \to T$, then $\rho(n) = f^n$. This gives us an action. Something nice is that $\mathbb{Z}$ is amenable, so $H_b^2(\mathbb{Z};\mathbb{R}) = 0$ as with all amenable groups. We mentioned earlier there is an exact sequence relating these;

$$0 \to \mathrm{Hom}(\mathbb{Z}, S^1) \to H_b^2(\mathbb{Z};\mathbb{Z}) \to H_b^2(\mathbb{Z};\mathbb{R}) = 0$$

Therefore $\mathrm{Hom}(\mathbb{Z}, S^1) \simeq H_b^2(\mathbb{Z};\mathbb{Z})$, and our Euler class $\mathrm{eu}_b^{\mathbb{Z}}(\rho)$ lives here, so it should correspond to some homomorphism into the circle. It turns out that $\varphi(n) = n \cdot \mathrm{rot}(f) = \mathrm{rot}(f^n)$. Going back to the context of Ghys' theorem, we start with an arbitrary homomorphism and try to understand its action. It says it corresponds to its rotation number, but the rotation number corresponds to another rigid action, giving an Euler class, and this map tells you that this Euler class is equal to the original Euler class, and Ghys' theorem says they are the same action up to semi-conjugacy. In summary, $\mathrm{eu}_b^{\mathbb{Z}}(\rho') = \delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho)$, and by Ghys we have $\rho' \sim \rho$.

Proposition 17.1. *An action ρ of a group G on S^1 has the bounded Euler class with real coefficients $\mathrm{eu}_b^{\mathbb{R}}(\rho) = 0$ iff the action is semiconjugate to an action by rigid rotations. In this case, the rotation number $\mathrm{rot}_\rho : G \to S^1$, $g \mapsto \mathrm{rot}(\rho(g))$ is a homomorphism.*

Proof. We have

$$0 \to \mathrm{Hom}(G, S^1) \to H_b^2(G;\mathbb{Z}) \to H_b^2(G;\mathbb{R})$$

with $H_b^2(G;\mathbb{Z}) \ni \mathrm{eu}_b^{\mathbb{Z}}(\rho) \mapsto \mathrm{eu}_b^{\mathbb{R}}(\rho) = 0$. Then there exists a φ iff $\delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho)$. So $\varphi : G \to S^1$ acting on S^1 by rigid rotations, leading to an action of ρ' at G on S^1 by rigid rotations. Recall the snake lemma: nvm let's skip that. We check that $\delta\varphi = \mathrm{eu}_b^{\mathbb{Z}}(\rho')$. By definition this is equal to $\mathrm{eu}_b^{\mathbb{Z}}(\rho)$, and applying Ghys' theorem these two are related by semiconjugacy, explaining the first part.

For the second statement, for rigid rotations the rotation number is exactly the angle of rotation. So $\mathrm{rot}_\rho(g) = \mathrm{rot}(\rho(g)) = \mathrm{rot}(\rho'(g)) = \varphi(g)$. So $\mathrm{rot}_p = \varphi$, which is a homomorphism. ⊠

Note that the ordinary Euler class $\mathrm{eu}(\rho) \in H^2(\mathbb{Z};\mathbb{Z}) = H^2(S^1;\mathbb{Z}) = 0$. This reflects the fact that we can life $f \in T$ to some $\widetilde{f} \in \hat{T}$. The point is that the ordinary Euler class carries no information when we restrict to $\mathbb{Z}$ classes, but the *bounded* Euler class still carries data about the rotation.

Theorem 17.1 (Hirsch-Thurston)**.** *If G is amenable, then any action of G on S^1 is semi-conjugate to an action by rigid rotations. In particular, the rotation number is a homomorphism.*

Proposition 17.2. *Any finite subgroup of $T = \mathrm{Homeo}^+(S^1)$ is cyclic.*

Proof. Let $G \subseteq T$ be finite. We want to deduce that this is cyclic. Finite implies amenable, so we have a homomorphism $\mathrm{rot} : G \to S^1$. If G is finite, this homomorphism has to be injective. We have $\mathrm{rot}(G) = 0$ iff g has a fixed point on S^1. If $g \neq \mathrm{id}$, the only case where g has a fixed point is that if it fixed everything. So g acts by translation on complementary intervals. Then $G \simeq \mathrm{rot}(G) \subseteq S^1$ implies $\mathrm{rot}(G) \simeq G$ is cyclic. ⊠

Corollary 17.1. *A group G cannot act faithfully on S^1 if it has a finite subgroup that is not cyclic.*

Example 17.2. The mapping class group of a surface $\mathrm{Mod}(S)$ cannot act faithfully on S^1. The reason is that these always contain a Klein-4 group $K \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. Realize the Klein-4 group as a group of rotations in $\mathbb{R}^3$. We can rotate around each axis by π, so they are order two rotations. We can actually write these as matrices:

$$a = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Then $ab = c, a^2 = b^2 = c^2 = \mathrm{id}$. This tells us that $\{\mathrm{id}, a, b, c\} \simeq K$, where a, b are the two generators. What does this have to do with surfaces and the mapping class group? Embed $S \hookrightarrow \mathbb{R}^3$ "symmetrically" such that K embeds into the homeomorphism group $\mathrm{Homeo}^+(S)$, and inject K into the mapping class group $\mathrm{Mod}(S)$.

In contrast, there is another mapping class group of homeomorphisms preserving a basepoint $\mathrm{Mod}(S,p)$ acts faithfully on S^1. Consider $S \setminus p$ and put a hyperbolic structure on it. Then we have S^1 formed from unique rays coming out of this cusp. Another way to see this is that $\mathrm{Mod}(S,p)$ acts on $\pi_1(S,p)$ by isomorphisms, which is quasi-isomorphic to $\mathbb{H}^2$. So $\mathrm{Mod}(S,p)$ acts on $\partial \pi_1(S,p) \simeq \partial \mathbb{H}^2 = S^1$.

Consider another cocycle representing $2\mathrm{eu}_b^{\mathbb{Z}}$. We can talk about an oriented triple $\mathrm{Or}(x,y,z)$, remove x and this unwraps into an interval $[x, \cdots, y, \cdots, z, \cdots, x]$ (positive orientation). Then for G acting on S^1, take $g, h, k \in G, x \in G$. This defines a function on triples of elements of G given by

$$\mathrm{Or}(g,h,k) = \mathrm{Or}(\rho(g)x, \rho(h)x, \rho(k)x)$$

which is G invariant. Then $\mathrm{Or}(\ell g, \ell h, \ell k) = \mathrm{Or}(g,h,k)$, turning out to be a cocycle. This defines a 2-cocycle in homogeneous coordinates, representing an $[\mathrm{Or}] \in H_b^2(G;\mathbb{Z})$.

Theorem 17.2 (Thurston)**.** *This 2-cocycle describes twice the Euler class, or* $[\mathrm{Or}] = 2 \cdot \mathrm{eu}_b^{\mathbb{Z}}(p)$.

Corollary 17.2. $\|\mathrm{eu}_b^{\mathbb{R}}\|_\infty \leq 1/2$*, leading to Milnor-Wood.*

Theorem 17.3 (Ghys)**.** *If G is countable, $\alpha \in H_b^2(G;\mathbb{Z})$, α can be represented by a cocycle with values in* $\{0,1\}$ *iff* $\alpha = \mathrm{eu}_b^{\mathbb{Z}}(\rho), \rho : G \to S^1$.

Proof. One direction is by definition, the other is by Ghys. The idea is that the choice of $\{0,1\}$ determines whether the triple has positive or negative orientation, and we can use this to get a "circular order" on G. This leads to an action of G on S^1. ⊠