# Some Problems

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I'm gonna try to type up my solutions to some problems here. They may or may not be correct.

## Contents



Lecture 1

# Euclidian Spaces

## 1.1 Smooth Functions on Euclidian Space

**Problem.** *Find a function $h\colon \mathbb{R} \to \mathbb{R}$ that is $C^2$ but not $C^3$ at $x = 0$.*

*Solution.* Take $h(x) = x^{5/2}$. Then $h''(x) = \frac{15}{4}\sqrt{x}$, but $h'''(x) = \frac{15}{8}x^{-1/2}$ for $x \neq 0$ and undefined at zero. Therefore $h$ is $C^2$ but not $C^3$. ■

**Problem.** *Define $f(x)$ on $\mathbb{R}$ by*

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0; \\ 0 & \text{for } x \leq 0. \end{cases}$$

(a) *Show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in $y$.*

(b) *Prove that $f$ is $C^\infty$ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.*

*Solution.*

(a) We use induction on $k$. For $k = 1$, we have $f'(x) = \frac{1}{x^2}e^{-1/x}$. In this case, $p_2(y) = y^2$, and so $p_2(1/x) = \frac{1}{x^2}$. Now assume $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in $y$. Then by the product rule,

$$f^{(k+1)}(x) = p'_{2k}(1/x)e^{-1/x} + \frac{1}{x^2}e^{-1/x}p_{2k}(1/x) = e^{-1/x}\left(p'_{2k}(1/x) + \frac{1}{x^2}p_{2k}(1/x)\right).$$

For the sum in the right expression, $p'_{2k}(1/x) + \frac{1}{x^2}p_{2k}(1/x)$ has degree $2(k+1)$: to see this, note that $p'_{2k}$ has degree $2k-1$, so we can forget about it. If $p_{2k}(1/x) = a_{2k}\left(\frac{1}{x}\right)^{2k} + b_{2k-1}\left(\frac{1}{x}\right)^{2k-1} + \cdots$ for constants $a_{2k}, b_{2k}, \cdots$, we have $\frac{1}{x^2}p_{2k} = a_{2k}\frac{1}{x^{2k+2}} + b_{2k}\frac{1}{x^{2k+1}} + \cdots = a_{2k}\frac{1}{x^{2(k+1)}} + \cdots$. So this polynomial has degree $2(k+1)$. Therefore $f^{(k)}(x)$ is of the form $p_{2(k+1)}(1/x)e^{-1/x}$ for some polynomial $p_{2(k+1)}$ of degree $2(k+1)$, and we are done.

(b) Our strategy is to show that $f^{(k)}(x) = 0$ for $x < 0$, $f^{(k)} = 0$, and $\lim_{x\to 0} f^{(k)}(x) = 0$ for $x > 0$. These conditions ensure that $f$ is smooth and the $k$th derivative vanishes at zero. First, note that $f^{(k)}(x) = 0$ for $x < 0$ by definition. To show $\lim_{x\to 0} f^{(k)}(x) = 0$ for $x > 0$, recall that $f^{(k)}(x) = p_{2k}(1/x)e^{-\frac{1}{x}}$. Using the genius substitution $u = \frac{1}{x}$, we can rewrite this limit as $\lim_{u\to\infty} \frac{p_{2k}}{e^u}$. From here, apply L' Hôpital's rule $2k$ times to get our desired result.

Finally, we show $f^{(k)}(0) = 0$. We do this by induction on $k$. The base case is true by definition. Assume $f^{(k)}(0) = 0$. Then $f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(h)-f^{(k)}(0)}{h-0} = \lim_{h\to 0} \frac{f^{(k)}(h)}{h} = \lim_{h\to 0} \frac{p_{2k}(1/h)e^{-1/h}}{h}$. Once again, make the substitution $u = 1/h$ to get $f^{(k+1)}(0) = \lim_{u\to\infty} \frac{up_{2k}(u)}{e^u} = 0$ by $2k+1$ applications of L' Hôpital's rule. ■

**Problem.** *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ be open subsets. A $C^\infty$ map $F\colon U \to V$ is called a **diffeomorphism** if it is bijective and has a $C^\infty$ inverse $F^{-1}\colon V \to U$.*

*(a) Show that the function $f : (-\pi/2, \pi/2) \to \mathbb{R},\ f(x) = \tan x$ is a diffeomorphism.*

*(b) Find a linear function $h\colon (a, b) \to (-1, 1)$, thus proving that any two finite open intervals are diffeomorphic.*

*Then the composition $f \circ h\colon (a, b) \to \mathbb{R}$ is then a diffeomorphism of an open interval to $\mathbb{R}$.*

*Solution.*

(a) We want to show that $\tan x$ is a smooth bijection and has a smooth inverse. Let $\tan(a) = \tan(b)$, then these numbers are associated to the same angle in $(-\pi/2, \pi/2)$, similarly, every real number is mapped onto by an angle in $(-\pi/2, \pi/2)$. For smoothness, note that $\tan'(x) = \sec^2(x)$, $\tan''(x) = 2\sec^2(x)\tan(x)$. From here you can see that the remaining derivatives are all products of sec and tan, which are both defined on $(-\pi/2, \pi/2)$ (since cos never hits zero on this interval). So $\tan x$ is smooth.

The $C^\infty$ inverse has to be $\arctan\colon (-\pi/2, \pi/2) \to \mathbb{R}$, there are no better candidates. We have $\arctan \circ \tan(x) = \mathrm{id}_{\mathbb{R}}$ by definition, so arctan is an inverse: to see smoothness, note that $\arctan'(x) = \frac{1}{1+x^2}$, $\arctan''(x) = -\frac{2x}{(1+x^2)^2}$, and so on. These functions are all continuous on $(-\pi/2, \pi/2)$, and so arctan is a smooth inverse for tan. Therefore $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism.

(b) Consider the function with its graph being a line segment joining $(a, 1)$ to $(b, -1)$.

■