Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1.1 Problem 1

Problem. *An n-dimensional* manifold with boundary *means a Hausdorff space M, such that every $x \in M$ has a neighborhood U such that the pair (U, x) is homeomorphic to either $(\mathbb{R}^n, 0)$ or $\big(\mathbb{R}^{n-1} \times [0, \infty), 0\big)$, where in both cases 0 means $(0, \ldots, 0)$. We call x an interior or boundary point according to which of these holds. Note that this is* not *the usual use of "interior" and "boundary" from point-set topology. The set of boundary points is written ∂M.*

Assume that ∂M is compact. Prove that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence.

You may use without proof the fact that no point can be both an interior and a boundary point. Also, the 2-dimensional case is enough to give a complete understanding. Finally, a hint: chain together a sequence of homotopies, being careful that the result makes sense and is continuous.

Remarks: informally, I think of $M \setminus \partial M$ as a sort of deformation-retract of M. But it is easy to see that if $\partial M \neq \emptyset$ then M does not actually deformation retract to $M \setminus \partial M$. Also, without the extra hypotheses, the only solution I know uses something you probably have not seen: topological dimension, which lets you build an open cover with good overlap properties.

Solution. (Thank you for changing the problem to assume ∂M is compact!) We want to show the inclusion $M \setminus \partial M \hookrightarrow M$ is a homotopy equivalence. Let $\mathcal{G}_\alpha$ be an open cover of ∂M. Then we can reduce $\mathcal{G}_\alpha$ to a finite subcover $\bigcup_{i=1}^n G_i$ of ∂M. Consider the inclusion $M \setminus \partial M \hookrightarrow (M \setminus \partial M) \cup G_1$, we'll denote this as ι. We'll also denote the set $(M \setminus \partial M) \cup G_1$ as MG_1 from now on (sorry for the confusing notation, but I couldn't think of anything better). We can find a quotient map

$$f \colon MG_1 \to MG_1/(MG_1 \cap \partial M)$$

by identifying all $x \in MG_1 \cap \partial M$ with an $x_0 \in M \setminus \partial M \subseteq MG_1$ (which exists since G_1 is part of a cover of ∂M). Then the quotient space is homeomorphic to $M \setminus \partial M$. Furthermore, the homotopies $H_1 \colon M \setminus \partial M \times I \to M \setminus \partial M$, $H_2 \colon MG_1 \times I \to MG_1$ given by $H_1(x,t) = (1-t)(f \circ \iota)(x) + tx$, $H_2(x,t) = (1-t)(\iota \circ f)(x) + tx$ satisfy the conditions for homotopy equialence, that is,

$$\begin{aligned} H_1(x,0) &= (f \circ \iota)(x), & H_2(x,0) &= (\iota \circ f)(x), \\ H_1(x,1) &= \mathrm{id}_{M \setminus \partial M}, & H_2(x,1) &= \mathrm{id}_{MG_1}. \end{aligned}$$

So the inclusion $M \setminus \partial M \hookrightarrow MG_1$ is a homotopy equivalence.

Now we generalize this to show MG_i has the same homotopy type as $MG_i \cup G_{i+1}$. Consider the inclusion $\iota\colon MG_i \hookrightarrow MG_i \cup G_{i+1}$. We find a quotient map

$$f\colon MG_i \cup MG_{i+1} \to MG_i \cup MG_{i+1}/\left((MG_i \cup MG_{i+1}) \setminus MG_i\right)$$

by identifying all $x \in MG_i \cup G_{i+1}$ with an $x_0 \in (MG_i \cup MG_{i+1}) \cap \partial M$ (which exists since the G_i are part of a cover of ∂M). Then the quotient space

$$MG_i \cup MG_{i+1}/\left((MG_i \cup MG_{i+1}) \setminus MG_i\right)$$

is homeomorphic to MG_i. Furthermore, the homotopies $H_1\colon MG_i \times I \to MG_i$, $H_2\colon MG_i \cup MG_{i+1} \times I \to MG_i \cup MG_{i+1}$ given by $H_1(x,t) = (1-t)(f \circ \iota)(x) + tx$, $H_2(x,t) = (1-t)(\iota \circ f)(x) + tx$ satisfy the conditions for homotopy equialence, that is,

$$\begin{aligned} H_1(x,0) &= (f \circ \iota)(x), & H_2(x,0) &= (\iota \circ f)(x), \\ H_1(x,1) &= \mathrm{id}_{MG_i}, & H_2(x,1) &= \mathrm{id}_{MG_i \cup MG_{i+1}}. \end{aligned}$$

So the subsequent inclusion maps $MG_i \hookrightarrow MG_i \cup MG_{i+1}$ are all inclusion maps, and preserve homotopy type. Let $i = 1$, then $M \setminus \partial M$ has the same homotopy type as $\bigcup_{i=1}^{n} (M \setminus \partial M) \cup G_i$, but all the G_i form an open cover of ∂M. So

$$\bigcup_{i=1}^{n} G_i = \partial M \implies \bigcup_{i=1}^{n} (M \setminus \partial M) \cup G_i = M \setminus \partial M \cup \partial M = M,$$

and we are done. ■

§1.2 Problem 2

Problem (A "bad" group action). *Let $X = \mathbb{R}^2 \setminus \{0\}$ where 0 is the origin. Let G be the group of homeomorphisms of X generated by the transformation $(x,y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.*

(a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).

(b) Prove that G's action on X satisfies the hypothesis of the theorem from class about $\pi_1(X/G) \cong G$, namely: every $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for every $g \in G \setminus \{1\}$.

(c) Prove that Y is a manifold, except for the fact that it is not *Hausdorff.*

(When working on a theorem involving a group action, if I wonder whether some hypothesis can be omitted, checking it for this single example usually reveals the answer.)

Solution. (The condition for an orbit to be discrete comes from the Wikipedia page for a discrete group.)

(a) Let $(x,y) \in \mathbb{R}^2 \setminus \{0\}$, $G(x,y)$ be the orbit of (x,y). We want to show that the singleton containing the identity $\{(x,y)\}$ is open, a sufficient condition for the orbit to be discrete. We know the next two subsets of the orbit "closest" to $\{(x,y)\}$ are the singletons $\{(2x, y/2)\}$ and $\{(x/2, 2y)\}$, generated by the given homeomorphism and its inverse. Let $\varepsilon = 1/4 \min\{x, y\}$. So take an open set $B((x, y), \varepsilon)$ around (x, y): this doesn't intersect the other two sets, and

$$B((x, y), \varepsilon) \cap \{(x, y)\} = \{(x, y)\}.$$

Therefore $\{(x, y)\}$ is open, and orbits in this group action are discrete.

(b) The neighborhood $B((x, y), \varepsilon)$ from the previous part does the trick: the minimum possible distance from one singleton subset to another is $\min\{x, y\}/2$, so taking $\varepsilon = 1/4 \min\{x, y\}$ ensures that two open $g(B((x, y), \varepsilon))$'s won't intersect (given $g \in G, g \neq 1$). This fulfills the condition from class.

(c) (Not entirely sure about this one): We want to show that Y is a manifold, but *not* Hausdorff. G acts freely (fixing only identity), smoothly, and properly (not sure how to show these two conditions, but I *think* they're true), so since $\mathbb{R}^2 \setminus \{0\}$ is a manifold, the quotient space Y is a manifold. Forgot how to formalize this, but to show Y is not Hausdorff, take the orbits of a rational and irrational "right next to each other" (by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$). The rational orbit contains entirely rational points but the irrational orbit contains entirely irrational points (since rational times irrational is irrational) so these two orbits are distinct. Then any open set containing the rational orbit (and therefore base point) must also contain the irrational base point by the definition of open sets in $\mathbb{R}$, so we have found two points that can't be separated by open sets, and we are done.

■

§1.3 Problem 9 Chapter 0

Problem. *Show that a retract of a contractible space is contractible.*

Solution. Let A be a retract of a contractible space X. Then there exists a homotopy of X onto a point, and a retract of X onto A: denote this retract with f, and the homotopy encoding the contraction as $H : X \times I \to X$, where $f|_A = \operatorname{id} A$, $H(x, 0) = \operatorname{id}_X$, $H(x, 1) = \{x_0\}$. Consider the homotopy $H' = f \circ H|_A$ from $A \times I \to A$. Then this homotopy is continuous since f and H are continuous and shows that id_A is nullhomotopic since $H'(x, 0) = \operatorname{id}_A, H'(x, 1) = \{x_0\}$. Therefore A is contractible. ■

§1.4 Problem 20

Problem. *Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.*

Solution. We can contract the intersecting disk of the Klein bottle to itself, so the resulting structure resembles S^2/S^0 (the sphere with two points identified), which is homotopy equivalent to $S^1 \vee S^2$ by Example 0.8. Counting the boundary of the intersecting disk itself, this forms another S^1 identified with the rest of the bottle (now $S^1 \vee S^2$) at a point. Therefore the self-intersecting Klein bottle in $\mathbb{R}^3$ has the homotopy type of $S^1 \vee S^1 \vee S^2$. ■

§1.5 Problem 17 Section 1.1

Problem. *Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$ (whoops, attempted this one last week).*

Solution. Informal idea: Take the first S^1 and twist it like a pretzel n times. Then fold these loops onto the second S^1, a retraction. This is an infinite family of nonhomotopic retractions.

Formal idea: We can express $S^1 \vee S^1$ as a unit circles centered at $(-2, 0)$ and the origin (the left and right unit circles, respectively) wedged together at the point $(-1, 0)$. Then we have an infinite family of retractions $R = \{r_n \mid n \in \mathbb{Z}\}$, where each r_n is defined as

$$r_n\colon (\cos\theta - 2, \sin\theta) \mapsto (e^{ni\theta}).$$

(Is there a way to write the left side in terms of e? My algebra is lacking). These retractions are the identity on the right circle and are non-homotopic because they correspond to different elements of the fundamental group $\pi_1(S^1) = \mathbb{Z}$. ■

§1.6 Problem 18

Problem. *Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:*

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

Solution. Let X be a space obtained from a path connected subspace A by attaching a cell e^n with $n \geq 2$. We know $\pi_1(e^n)$ is trivial (Proposition 1.14). Let $e^n = A_\alpha$ and $A = A_\beta$ in Lemma 1.15. Then $X = A_\alpha \cup A_\beta$. We also have $A_\alpha \cap A_\beta = e^n \cap A = \partial e^n$ which is path connected— let $x_0 \in \partial e^n$. Applying Lemma 1.15, every loop in X at x_0 is homotopic to a product of loops, but since $\pi_1(e^n)$ is trivial every loop in e^n is nullhomotopic. Therefore every loop in X is homotopic to a loop in A, and so the inclusion map induces a surjection on $\pi_1(X)$, and we are done.

(a) We want to show the wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$. We attach the 1-cell S^1 to the 2-cell S^2: then by the thing we proved earlier, we have a surjection $\pi_1(S^1) \to \pi_1(S^1 \vee S^2)$ induced by the inclusion map $S^1 \hookrightarrow S^1 \vee S^2$ (let's denote this map ι. Also note that S^1 is path-connected). We use a nice theorem from group theory: apply the Fundamental Homomorphism Theorem to get

$$\pi_1(S^1 \vee S^2) \simeq \pi_1(S^1)/\ker\iota.$$

Since $\pi_1(S^1) = \mathbb{Z}$, we have $\pi_1(S^1 \vee S^2)$ isomorphic to a quotient group of Z. Furthermore, $\pi_1(S^1 \vee S^2)$ is infinite because each loop winds around n times for n an integer (see problem 17), so $\pi_1(S^1 \vee S^2) \simeq \mathbb{Z}$.

(b) X is obtained from the 1-skeleton X^1 (which is path-connected since X is path-connected) by attaching the e^n cells (for $n \geq 2$). Then it follows from the thing above that the inclusion $X^1 \hookrightarrow X$ induces a surjection $\pi_1(X^1) \to \pi_1(X^2)$. Each subsequent inclusion induces another surjection $\pi_1(X^i) \to \pi_1(X^{i+1})$, so chaining the inclusions together yields a surjection from $\pi_1(X^1) \to \pi_1(X)$.

■

§1.7 Problem 20

Problem. *Suppose $f_t\colon X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]*

Solution. Let $f_t(x_0) = h$. We have f_0, f_1 equal to id_X, so f_{0*}, f_{1*} are also the identity map. By Lemma 1.19, composing $\varphi_{1*} = f_{1*}$ (the identity) and β_h gives $\varphi_{0*} = f_{0*}$ (also the identity), which implies β_h is also the identity map. But β_h is the change-of-basepoint map, defined as $\beta_h[f] = [hf\overline{h}]$. So if β_h is the identity,

$$\beta_h[f] = f = [hf\overline{h}] \implies [fh] = [hf]$$

for all f. Recall the center of the fundental group is defined as

$$Z[\pi_1(X, x_0)] = \{[h] \in \pi_1(X, x_0) \mid \forall [f] \in \pi_1(X, x_0), [hf] = [fh]\}.$$

Then $h = f_t(x_0)$ is in the center of $\pi_1(X, x_0)$. ■

§1.8 Problem 2 Section 1.2

Problem. *Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.*

Solution. Intuition: A bunch of simply connected open sets linked together, although some two maybe not appear to be connected, the third will connected with another "structure", all the way down. So you can draw paths through the whole thing/retract it all to a point.

Formal idea: We show that $\pi_1(X)$ is trivial by Van Kampen's theorem, proving that X is simply-connected (since X is path-connected). We want to find a basepoint to use for Van Kampen's, then the fact that there is a basepoint in all the X_n's ($\bigcap_{i=n} X_i \neq \emptyset$) can be realized by a sort of "reverse" induction: If $\bigcap_{i=1}^{n-1} X_i \neq \emptyset$, then $\bigcap_{i=1}^{n} X_i = \emptyset$, this would contradict our hypothesis by choosing $i = n-2, j = n-1, k = n$ (since the intersect with X_k would be empty). We can repeat this process all the way down to three sets, which is nonempty by assumption. Apply Van Kampen's theorem to get

$$\pi_1(X) \simeq *_i \pi_1(X_i)/N.$$

But all the $\pi_1(X_i)$'s are trivial, so $\pi_1(X)$ must be trivial, and we are done.

■

§1.9 Problem 4

Problem. *Let $X \subseteq \mathbb{R}^3$ be the finite union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.*

Solution. Informal idea: Retract space minus a line to the square with two sides glued together, then compress this into a circle. So when $n = 1$, $\pi_1(\mathbb{R}^3 \setminus X) = \mathbb{Z}$. Now if we take two points, this deformation retracts onto S^1 minus four points (like cutting a cross through $\mathbb{R}^3$ given $n = 2$). The fundamental group of this spaces is the free group on $2n - 1$ generators.

(somewhat) Formal idea: We have $\pi_1(\mathbb{R}^3 \setminus X) = \mathbb{Z}$ (or the F_1, the free group on one generator) when $n = 1$. We can retract $\mathbb{R}^3 \setminus X$ onto S^1 minus $2n$ points for n the number of lines through the origin (let's call this set $S^1 - 2n$). We want to find $\pi_1(S^1 - 2n)$: then the problem will be solved, since $\mathbb{R}^3 \setminus X$ deformation retracts onto $S^1 - 2n$. (Not sure how to do this formally): Take one of the holes in the sphere and use it to "flatten out" the sphere onto the plane. Then $\pi_1(\text{new space}) = *\mathbb{Z}^{2n-1}$ or F_{2n-1}, the free group on $2n - 1$ generators since we took $2n$ holes and got ride of one of them to "flatten" the sphere onto the plane, resulting in $2n - 1$ holes (so $\pi_1(\text{new space}) = F_{2n-1}$). Therefore since $\mathbb{R}^3 \setminus X$ deformation retracts onto this new space, $\pi_1(\mathbb{R}^3 \setminus X) = F_{2n-1}$. ■