Differential Topology Notes

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Contents

Part I
Class Notes

Lecture 1

January 19, 2021

> "Differential topology is a subset of geometry, which is a subset of math. Broadly, math is about space and numbers, and this is more on the space side. This isn't a partition, however."

1.1 Smooth Manifolds

Our main object of study is the smooth manifold, which is broadly a space on which you can do calculus. All these spaces look the same locally, the difference is in the global structure. We want to know how to do calculus on flat space first, which means doing calculus on open sets $U \subseteq \mathbb{A}^n = \{(x^1, \cdots, x^n) \mid x^i \in \mathbb{R}\}$, where $\mathbb{A}^n$ is affine n-space. Broadly, this means functions $f : U' \to U$, n functions of m variables, which are smooth (C^∞).

Smooth manifolds patch together these open sets, or a collection $\{U_\alpha\}\alpha \in A$. By patching together, we mean X is a smooth manifold, and a surjective map from this collection onto X. We go from one piece (chart) to the other by transition maps. Atlases will include some border towns when you're crossing over. The information must correspond, which is the idea of a diffeomorphism.

Example 1.1. Our first example will be two copies of the affine line $\mathbb{A}^1_x, \mathbb{A}^1_y$ projecting onto the circle $S^1 = \{\lambda \in \mathbb{C} \,\big|\, |\lambda| = 1\}$. If we define

$$\lambda = \begin{cases} e^{2i\tan^{-1} x} \\ e^{2i\left(\frac{\pi}{2} - \tan^{-1} y\right)} \end{cases}$$

we see that $0_x \mapsto 1$ and $0_y \mapsto -1$. So we patch $\mathbb{A}^1_x \setminus \{0\} \xrightarrow{f} \mathbb{A}^1_y \setminus \{0\}$ by the map $x \mapsto 1/x$.

Example 1.2. Another example is glueing two affine planes together by stereographic projection on a sphere. Work out what the transition function is in your free time.

Example 1.3. Take an affine plane $\mathbb{A}^2$ and a line ℓ in it, or the manifold X of affine lines through the plane.

Let's talk about the correspondence of manifolds and functions. For $f : X \to Y \ni c$, we can make shapes from functions like so:

(1) The image $f(X) \subseteq Y$,

(2) The fiber of f at c, $f^{-1}(c) \subseteq X$, and the inverse image $f^{-1}(z) \subseteq X$,

(3) The graph $\Gamma_f \subseteq X \times Y$.

1.2 Local-to-global and Classification theorems

Another idea is local vs global structure. An example of local structure is the inverse function theorem. Classification is also a big issue, it's good to know that manifolds are topologizable metric spaces. So we can talk about things like compactness and connectedness.

(1) Our only example in dimension 1 is the circle S^1.

(2) In dimension 2, we have the genus n-surfaces, the Klein bottle, projective space, etc.

(3) In dimension 3, if we add a simply-connected hypothesis this becomes the classic Poincaré (no longer!) conjecture.

(4) In dimension 4, it's a zoo

(5) In dimensions greater than 5, we have more wiggle room with the extra dimensions, so we can apply techniques from algebraic topology which are more effective with this wiggle room.

How do we classify functions? For smooth manifolds we consider manifolds up to diffeomorphism. For functions $f : S^1 \to S^1$, we can give it a nice topology (say the compact-open topology) and look at the path components, or classify the maps up to homotopy. In this case, homotopies are maps $F : [0,1] \times S^1 \to S^1$, classifying these are a kind of global property. A type of map from the circle to the circle is $f_n(\lambda) = \lambda^n$ for $n \in \mathbb{Z}$, these maps have winding numbers. In a homotopy, a path in the interval can wind around and intersect itself several times, but always evens out (points being born and dying). An important concept is the orientations, knowing which way things are facing. This is the first example of what's called *intersection theory*, which is what we use to make invariants.

Back to smooth manifolds. They arise in many places, including:

(1) Moduli spaces of geometric objects

(2) Solutions to (nonlinear) differential equations

This finishes the survey of the course. Now let's begin the actual content.

1.3 Topological Manifolds

Definition 1.1. Let X be a topological space.

(i) X is **locally Euclidian** if for all $x \in X$ there exists an open subset $U_x \subseteq X$ and a homeomorphism $U \to U'$ where $U' \subseteq \mathbb{A}^n$ for some $n \in \mathbb{Z}^{\geq 0}$.

(ii) X is a **topological manifold** if X is locally Euclidian, Hausdorff, and second countable.

Remark 1.1. At each $x \in X$, the dimension n is well-defined. So we have a function $\dim : X \to \mathbb{Z}^{\geq 0}$. If the dimension is constant, then we say such a manifold is an n-manifold. But this doesn't always have to be the case.

Remark 1.2. A topological manifold has a metrizable topology.

Example 1.4. Here we give some examples and nonexamples of topological manifolds.

(1) Consider $\mathbb{A}^1$ and S^2, then $X = \mathbb{A}^1 \amalg S^2$ is a topological manifold. It has two components with dimension 1 and 2, respectively.

(2) A nonexample is a circle with a line through it, since it's not locally Euclidian at the intersection point.

(3) Another nonexample is $\mathbb{A}^1 \cup \mathbb{A}^1 / \sim$ under the identification that glues every point together that isn't zero. So it's a line with a double point, each of which has an interval as an open point. Therefore we can't separate these points, and so this space is not Hausdorff.

(4) $\mathbb{A}^1_{\text{discrete}}$ is an uncountable set, so this is not second countable.

Remark 1.3. We do not study topological manifolds in this class. But smooth manifolds are topological manifolds with extra structure. In dimensions 1,2,3, they are the same, that is, every topological manifold admits a smooth structure.

In dimension four, TOP $\neq$ DIFF. For example, $\mathbb{A}^4$ has infinitely many unique smooth structures. In dimension seven, S^7 has 28 smooth structures. Milnor went on to classify smooth structures of spheres in all dimensions.

Lecture 2

January 21, 2021

2.1 Charts

Definition 2.1. Let X be a topological manifold.

(i) An n-dimensional **chart** on X is a pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to \mathbb{A}^n$ is continuous such that ϕ is a homeomorphism onto $\phi(U)$.

(ii) Charts $(U, \phi), (V, \psi)$ are C^∞**-related** if $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$ is a C^∞ map as its inverse. This map is sometimes called the overlap between the charts or the transition function. We already know $\psi \circ \phi^{-1}$ is a bijection/homeomorphism, and we just need smoothness.

Example 2.1. Not all charts are C^∞ related. Let $X = \mathbb{R}$ and $U = V = \mathbb{R}$, $\phi(x) = x, \psi(x) = x^3$. Composing one direction sends $x \mapsto x^3$, while $y \mapsto y^{1/3}$ is not C^∞. These are perfectly valid charts, but not C^∞-related, they are in one direction but not in the other.

Example 2.2. Take $S^2 \subseteq \mathbb{A}^3$, and consider $U = \{x > 0\}$, $\phi(x, y, z) = (y, z)$, projecting onto the yz-plane. Given any point in this disc, we can solve for x^+ given by the equation $x^2 + y^2 + z^2 = 1$. Similarly, let $V = \{y > 0\}$ and $\psi(x, y, z) = (x, z)$. If we use α, β for xz coordinates and u, v for yz-coordinates, the transition map can be expressed on the domain of intersection as $\alpha = \sqrt{1 - u^2 - v^2}$ and $\beta = v$, where the inverse is also smooth.

2.2 Calculus on Affine Space

There are two arenas where we do calculus, $\mathbb{R}^n = \{(\xi^1, \cdots, \xi^n) \mid \xi^i \in \mathbb{R}\}$ as a vector space, and $\mathbb{A}^n = \{(x^1, \cdots, x^n) \mid x^i \in \mathbb{R}\}$ the affine space of points. As sets these are the same. We have some extra data on $\mathbb{R}^n$: first the zero vector $0 \in \mathbb{R}^n$, addition $+ \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and multiplication $\cdot \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

The affine space $\mathbb{A}^n$ has one additional operation, $+ \colon \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n$. This takes in a point and a vector, and displaces the point by the vector. So affine space has a vector space of translations, and for V a vector space we have A *affine over* V. Another way to say this is that V acts on A by translations, where the action is *simply transitive*. This means that given two points, we have the existence of a unique vector that takes one point to the other.

Let V, W be vector spaces, A, B be affine over V, W. For $U \subseteq A$ open, let $f : U \to B$. Then for $p \in U, \xi \in V$, we have the **directional derivative** as the map

$$\xi f(p) = \lim_{t \to 0} \frac{f(p + t\xi) - f(p)}{t}.$$

Of course, this may or may not exist.

Theorem 2.1. *If $\xi f(p)$ exists for all $\xi \in \mathbb{R}^n$, $p \in U$, and if each ξf is a continuous function of p, then for each $p \in U$, $\xi \mapsto \xi f(p)$ is a* linear *function of $\xi \in V$. This is called the* ***differential****, denoted $df_p \colon V \to W$. So $p + \xi \mapsto f(p) + df_p(\xi)$ is the best affine approximation of f at p.*

A conceptual approach to the differential is that $df_p \colon V \to W$ is the unique linear map such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\xi \in V$, $\|\xi\| < \delta$ implies $p + \xi \in U$ and $\|f(p + \xi) - f(p) - df_p(\xi)\| \le \varepsilon \|\xi\|$.

Example 2.3. Let $V = \mathbb{R}^n, A = \mathbb{A}^n_{x^1, \cdots, x^n}$, $W = \mathbb{R}^m, B = \mathbb{A}^m_{y^1, \cdots, y^m}$. Let

$$f = \begin{cases} y^1 = y^1(x^1, \cdots, x^n) \\ y^2 = y^2(x^1, \cdots, x^n) \\ \qquad \vdots \\ y^m = y^m(x^1, \cdots, y^n) \end{cases}$$

, $x^i : \mathbb{A}^n \to \mathbb{R}, (x^1, \cdots, x^n) \mapsto x^i$. Then dx^i_p is independent of $p \in \mathbb{A}^n$, $dx^i : \mathbb{R}^n \to \mathbb{R}$ linear, $dx^i \in R^{n*} : (\xi^1, \cdots, \xi^n) \mapsto \xi^i$. Then $dx^1, \cdots, dx^n$ is a *basis* of $\mathbb{R}^{n*}$, and the dual of the dual is $\mathbb{R}^n$, so we also have a basis for $\mathbb{R}^n$ by $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$. We usually think of these as matrices, for example

$$\frac{\partial}{\partial x_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad dx^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j.$$

Since the differential $df_p : \mathbb{R}^n \to \mathbb{R}^n$ is linear, $df_p \left(\frac{\partial}{\partial x^j} \right) = A^i_j \frac{\partial}{\partial y^i}$. Here we use up-down indices and sum over i. We have $A^i_j = \frac{\partial y^i}{\partial x^j}$, the partial derivative. So

$$df_p \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}, \quad df_p = \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \otimes dx^j.$$

Definition 2.2. Given our standard setup, $f : U \to B$ is C^∞ if the iterated directional derivatives

$$\xi_1 \xi_2 \cdots \xi_k f : U \to W$$

exist and are continuous for all $k \in \mathbb{Z}^{>0}$, $\xi_1, \cdots, \xi_k \in V$.

Example 2.4. If f is C^∞, then for all $\xi_1, \xi_2 \in V$, $\xi_1 \xi_2 f = \xi_2 \xi_1 f$.

Example 2.5. For example,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

This idea of second derivates being symmetric functions will come in handy later.

2.3 Smooth Manifolds (for real this time)

Definition 2.3. Let X be a topological manifold.

(i) An **atlas** on X is a collection $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ such that

   (a) The charts cover X, that is, $\bigcup_{\alpha \in A} U\alpha = X$,

   (b) For all $\alpha_1, \alpha_2 \in A$, $(U_{\alpha_1}, \phi_{\alpha_1})$ and $(U_{\alpha_1}, \phi_{\alpha_2})$ are C^∞ related.

(ii) An atlas is a **differentiable structure** on X if in addition

   (c) $\mathscr{A}$ is maximal: if (U, ϕ) is a chart which is C^∞-related to all $(U_\alpha, \phi_\alpha) \in \mathscr{A}$, then $(U, \phi) \in \mathscr{A}$.

(iii) A **smooth manifold** is a pair $(X, \mathscr{A})$ of a topological manifold and a differentiable structure.

Remark 2.1. Any atlas $\mathscr{A}$ is contained in a unique differentiable structure, given by

$$\overline{\mathscr{A}} = \{(U, \phi) \text{ charts} \mid (U, \phi) \text{is } C^\infty\text{-related to all } (U_\alpha, \phi_\alpha) \in \mathscr{A}\}.$$

Remark 2.2. We have an atlas $\mathscr{A}$ on S^2 with $|\mathscr{A}| = 6$ at the beginning of lecture, and there exists an $\mathscr{A}^1$ on S^2 with $|\mathscr{A}^1| = 2$. But there exists no $\mathscr{A}^n$ in S^2 with $|\mathscr{A}^n| = 1$.

Lecture 3

January 26, 2021

3.1 Examples of Smooth Manifolds

Let $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ be an atlas with $x_\alpha \colon U_\alpha \to A_\alpha$. Then we have a surjection

$$\coprod_{\alpha \in A} x(U_\alpha) \xrightarrow{\coprod_{\alpha \in A} x_\alpha^{-1}} M.$$

Is this disjoint union a manifold? It's clearly locally Euclidian and Hausdorff, but whether or not it's second countable depends on the indexing set A.

Example 3.1. Here are some examples of manifolds.

(1) We have $M = \emptyset$ a smooth manifold. This qualifies as a smooth manifold of any dimension, even negative one. This can be useful.

(2) An affine space on V a vector space is a smooth manifold (it has an atlas with a single chart and the identity map).

(3) If $n \geq 0$, $S^n \subseteq \mathbb{A}^{n+1}$ is a smooth manifold.

(4) We can also construct new manifolds from old.

   (a) If $\{M_\alpha\}_{\alpha \in A}$, where M_α are smooth manifolds and A is countable, then $\coprod_{x \in A} M_{\alpha A}$ is a smooth manifold.

   (b) Given $\{M_\alpha\}_{\alpha \in A}$, the Cartesian product product of manifold is also a manifold. For example, the torus $S^1 \times S^1$ is also a smooth manifold.

   (c) Let M be a smooth manifold. Then $N \subseteq M$ open means that N is also a smooth manifold, with the subspace topology. For example, $\mathrm{GL}_n(\mathbb{R}) \subseteq M_n\mathbb{R}$ as an n^2-dimensional vector space, so this forms a smooth manifold. This forms an open subset, which can be realized as the inverse image of an open set $(\mathbb{R} \setminus \{0\})$ under a continuous map, the determinant.

(5) Let V be a real vector space with positive dimension n, and $k \in \{0, 1, \cdots, n\}$. Then we define the **Grassmannian** $\mathrm{Gr}_k(V)$ as the set of $W \subseteq V$ subspaces of dimension k. For example, if $V = \mathbb{A}^2$ and $k = 1$, then this is $\mathbb{R}\mathrm{P}^1$. In general, $\mathrm{Gr}_1(V) = \mathbb{P}V$ which is projective space.

   To think about how to construct an atlas, let $w' \in \mathrm{Gr}_k(V)$. Then $w' \oplus w'' = V$ (dimension k and $n-k$). Consider $\psi \colon \mathrm{Hom}(w', w'') \to \mathrm{Gr}_k(V)$, $L \mapsto \Gamma_L$, the graph of L. This is an injective map, and $U_{w''} := \operatorname{im} \psi = \{W \in \mathrm{Gr}_k(V) \mid w \cap w'' = 0\}$ (can't be vertical). Then ψ^{-1} is a chart with values in the vector space $\mathrm{Hom}(w', w'')$, and image $\psi = U_{w''}$ only depends on w''. It can be given an *affine* space structure.

3.2 Functions on Smooth Manifolds

Say we have spaces A, B, C with $U \subseteq A, V \subseteq B$ open, and $f : U \to B$, $g : V \to C$. Since $f(U) \subseteq V$, $g \circ f : U \to C$.

Theorem 3.1. *If f, g are C^∞, then $g \circ f$ is C^∞. Furthermore, we have $d(g \circ f)_p = dg_{f(p)} \circ df_p$, where $df_p \colon V \to W$, $dg_{f(p)} \colon W \to X$, $d(g \circ f)_p \colon V \to X$.*

What does it mean for a map $f : M \to N$ between topological spaces to be smooth? If p is a point, pick a chart (U_α, x_α) containing p and another chart (V_β, y_β) containing $f(p)$.

Definition 3.1. A function f is C^∞ at $p \in M$ if for some charts (U_α, x_α) about p and (V_β, y_β) about $f(p)$, the function

$$y_\beta \circ f \circ x_\alpha^{-1} \colon x_\alpha(U_\alpha) \to y_\beta(V_\beta)$$

is C^∞.

Lemma 3.1. *If the condition above is true for one choice of chart, then it is true for all choices of charts.*

This relies on the fact that a composition of smooth maps is smooth and the chain rule. Explicitly, say f is smooth. Then $(y\beta \circ f \circ x_\alpha^{-1})$ is C^∞, and we compose with the transition function $(x_\alpha \circ x_{\alpha'}^{-1})$ to change charts. But this is a composition of C^∞ maps, which is also C^∞. Similarly, changing charts in the codomain gives the composition $(y_{\beta'} \circ y_\beta^{-1}) \circ (y_\beta \circ f \circ x_\alpha^{-1})$, which is C^∞.

Example 3.2. Let $f: S^2 \to S^2$ be the antipodal map. This is just the restriction of an affine map $f: \mathbb{A}^3 \to \mathbb{A}^3, (x,y,z) \mapsto (-x,-y,-z)$. If $p = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $U_\alpha = \{x > 0\}$, $f(p) = (-1/\sqrt{2}, -1/\sqrt{2}, 0)$, $V_\beta = \{y < 0\}$. Then

$$y_\beta \circ f \circ x_\alpha^{-1}(u,v) = \left(-\sqrt{1-u^2-v^2}, -v\right).$$

Lecture 4

January 28, 2021

4.1 The Tangent Space

come back for ntoes

Part II

Guillemin and Pollack

Lecture 5

Chapter 1: Manifolds and Smooth Maps

Introduction

These are supplementary notes, following the classic text on differential topology Guillemin and Pollack. Here are some things we should know before starting:

A cover $\{V_\beta\}$ is a **refinement** of another cover $\{U_\alpha\}$ if every set V_β is contained in at least one U_α. Since $\mathbb{R}^n$ is second-countable, every open cover $\{U_\alpha\}$ in $\mathbb{R}^n$ has a countable refinement. For a quick proof, take the collection of all open balls contained in some U_α with rational radii, centered at points with rational coordinates.

If $X \subseteq \mathbb{R}^n$, then $V \subseteq X$ is **(relatively) open** in X if it can be written as the intersection of X with an open subset of $\mathbb{R}^n$, or $V = \widetilde{V} \cap X$, where $\widetilde{V}$ is open in $\mathbb{R}^n$. If $Z \subseteq X$, we can also speak of open covers of Z in X, meaning coverings of Z by relatively open subsets of X. Every such cover of Z may be written as the intersection of X with a covering of Z by open subsets of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable, every open cover of Z relative to X has a countable refinement. To see this, given $\{U_\alpha\}$ relatively open in X, write $U_\alpha = \widetilde{U}_\alpha \cap X$. Then let $\widetilde{V}_\beta$ be a countable refinement of $\{\widetilde{U}_\alpha\}$ in $\mathbb{R}^n$, and define $V_\beta = \widetilde{V}_\beta \cap X$.

A mapping $f: U \to \mathbb{R}^m$ of an open $U \subseteq \mathbb{R}^n$ is called *smooth* if f has continuous partial derivatives of all orders. If the domain of f is not open, we usually cannot speak of partial derivatives (for the concept to work, we need to be able to find a neighborhood around each point). So we generalize this definition a little. A map $f: X \to \mathbb{R}^m$ defined on an arbitrary X in $\mathbb{R}^n$ is **smooth** if it can be locally extended to a smooth map on open sets, that is, if around each $x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a smooth map $F: U \to \mathbb{R}^m$ such that F equals f on $U \cap X$.

A smooth map $f: X \to Y$ of two subsets of Euclidian spaces is a **diffeomorphism** if it is a bijection, and the inverse map $f^{-1}: Y \to X$ is also smooth. In this course, diffeomorphic sets are intrinsically equivalent.

5.1 Tangent Space and the Differential

These are actually supplementary notes handed out by Dr. Freed, not from G&P.

Definition 5.1. Let $\{V_\alpha\}_{\alpha\in A}$ be a collection of vector spaces indexed by a set A. Then the **direct product** $\prod_{\alpha\in A} V_\alpha$ is the Cartesian product of the sets V_α with componentwise addition and scalar multiplication. It is a vector space, possibly infinite dimensional. An element of the direct product is denoted $\xi = \{\xi_\alpha\} \in \prod_{\alpha\in A} V_\alpha$; the α-component of ξ is ξ_α. The sum is defined by $(\xi+\eta)_\alpha = \xi_\alpha + \eta_\alpha$, or $\{\xi_\alpha\} + \{\eta_\alpha\} = \{\xi_\alpha + \eta_\alpha\}$.

Let X be a smooth manifold with atlas $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha\in A}$. For $p \in X$, let $A_p \subseteq A$ be the set of indices $\alpha \in A$ such that $p \in U_\alpha$ and set $\mathscr{A}_p = \{(U_\alpha, x_\alpha)\}_{\alpha\in A}$. Suppose the dimension of X at p is n.

Definition 5.2. The **tangent space** T_pX is the subspace of the direct product $\prod_{\alpha\in A_p} \mathbb{R}^n_\alpha$ consisting of vectors $\xi = \{\xi_\alpha\}$ such that

$$\xi_\beta = d(x_\beta \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha)$$

for all $\alpha, \beta \in A_p$. Here $\mathbb{R}^n_\alpha$ denotes the vector space $\mathbb{R}^n$ thought of as displacements in the codomain of the coordinate map $x_\alpha \colon U_\alpha \to \mathbb{A}^n$.