

Miscellaneous Notes on Linear Algebra

Simon Xiang

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Who ever suffered from learning too much linear algebra? These notes will seek to fill in my linear algebra gaps. New inclusion: these notes will also cover any miscellaneous material I should have learned in my undergraduate analysis, abstract algebra, topology, or whatever classes but didn't. Source files: https://git.simonxiang.xyz/math_notes/files.html

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Lecture 1

Miscellaneous Topics

TODO: affine spaces, inverse function, change of variables for multiple integrals (spivak 34,67) or tu appendix, rank, nullity, binomial theorem, freed's thing, maybe topology bases, subspace/product, tychonoff, convergnece, etc

1.1 The Inverse Function Theorem

Inner-Product Spaces

What is an inner product?? Let's find out.

2.1 Inner Products

The length of a vector x is the **norm** of x , denoted $\|x\|$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we have $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$. Note that the norm is not linear. For $x, y \in \mathbb{R}^n$, the **dot product** of x and y , denoted $x \cdot y$, is defined by $x \cdot y = x_1 y_1 + \dots + x_n y_n$. Note that this is a number, not a vector. Clearly $x \cdot x = \|x\|^2$ for all $x \in \mathbb{R}^n$, which implies $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$ ($x \cdot x = 0$ only if x is the zero vector). The map that sends $x \in \mathbb{R}^n$ to $x \cdot y$ in \mathbb{R} for fixed y is linear since \mathbb{R} is a field. The dot product is also commutative, since \mathbb{R} is.

Inner products generalize dot products. Recall that $|\lambda|^2 = \lambda \bar{\lambda}$ for $\lambda \in \mathbb{C}$. For $z \in \mathbb{C}^n$, we define the norm of z by $\|z\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$. We take the modulus of z_i since we want the result to be nonnegative. Note that $\|z\|^2 = z_1 \bar{z}_1 + \dots + z_n \bar{z}_n$. We want to think of $\|z\|^2$ as the inner product of z with itself, like in \mathbb{R}^n . This suggests we define the inner product of $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ with z as $w_1 \bar{z}_1 + \dots + w_n \bar{z}_n$. We expect the inner product of w with z equal the complex conjugate of the inner product of z with w . With this motivation in mind, let us define inner products.

Definition 2.1 (Inner product). An **inner product** on an F -vector space V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ such that

- (i) $\langle v, v \rangle \geq 0$ for all $v \in V$; (**positivity**)
- (ii) $\langle v, v \rangle = 0$ iff $v = 0$; (**definiteness**)
- (iii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$; (**additivity in first slot**)
- (iv) $\langle av, w \rangle = a \langle v, w \rangle$ for all $a \in F$ and all $v, w \in V$; (**homogeneity in first slot**)
- (v) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$. (**conjugate symmetry**).

For real numbers, condition (v) simply becomes $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$. An **inner product space** is a vector space V along with an inner product on V .

Example 2.1. The most important example is the **Euclidian inner product** on \mathbf{F}^n (Axler uses \mathbf{F} to denote either \mathbb{C} or \mathbb{R}). We define an inner product on \mathbf{F}^n by

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1 \bar{z}_1 + \dots + w_n \bar{z}_n.$$

An example of another inner product on \mathbf{F}^n is defined by $\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = c_1 w_1 \bar{z}_1 + \dots + c_n w_n \bar{z}_n$ for c_i positive constants. The case where $c_i = 1$ for all i is simply the standard Euclidian inner product.

Example 2.2. Consider the vector space $\mathcal{P}_m(\mathbf{F})$, the polynomial ring over \mathbf{F} of polynomials with degree at most m . We can define an inner product on $\mathcal{P}_m(\mathbf{F})$ by

$$\langle p, q \rangle = \int_0^1 p(x) \overline{q(x)} dx.$$

For fixed $w \in V$, the function that takes v to $\langle v, w \rangle$ is a linear map $V \rightarrow \mathbf{F}$. So $\langle 0, w \rangle = 0$, and by condition (v) $\langle w, 0 \rangle = 0$ as well. Furthermore, $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ and $\langle u, av \rangle = \bar{a} \langle u, v \rangle$ hold as well: This second condition is known as conjugate homogeneity in the second slot.

2.2 Norms

For $v \in V$, we define the **norm** of v , denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v \rangle}$. For example, if $p \in \mathcal{P}_m(\mathbf{F})$, then $\|p\| = \sqrt{\int_0^1 |p(x)|^2 dx}$. Some properties: $\|v\| = 0$ iff $v = 0$, and $\|av\| = |a|\|v\|$. To see this, note that $\|av\|^2 = \langle av, av \rangle = a \langle v, av \rangle = a \bar{a} \langle v, v \rangle = |a|^2 \|v\|^2$, taking square roots gives us our result. This illustrates a general idea: working with norms squared is easier than working directly with norms.

Two vectors $u, v \in V$ are **orthogonal** if $\langle u, v \rangle = 0$. The zero vector is orthogonal to every vector, and the only vector orthogonal to itself. Assume $V = \mathbb{R}^2$, now let us state a 2500 year old theorem.

Pythagorean Theorem. *If u, v are orthogonal vectors in V , then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. Exercise. □

Suppose $u, v \in V$. We want to write u as a scalar multiple of v plus a vector w orthogonal to v . Let $a \in \mathbf{F}$ be a scalar, then $u = av + (u - av)$. We need to choose a such that v is orthogonal to $u - av$, in other words, we want $0 = \langle u - av, v \rangle = \langle u, v \rangle - a\|v\|^2$. So we should choose $a = \langle u, v \rangle / \|v\|^2$ (where $v \neq 0$). Then

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right).$$

Cauchy-Schwarz Inequality. *If $u, v \in V$, then*

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

This inequality is an equality iff one of u, v is a scalar multiple of the other.

Proof. Let $u, v \in V$, and assume $v \neq 0$. Consider $u = \frac{\langle u, v \rangle}{\|v\|^2} v + w$, where w is orthogonal to v . By the Pythagorean theorem, we have

$$\|u\|^2 = \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 = \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}.$$

Multiply both sides, take a square root, and we are done. This is an equality iff $w = 0$, but this is true iff u is a multiple of v . □

Triangle Inequality. *If $u, v \in V$, then*

$$\|u + v\| \leq \|u\| + \|v\|.$$

This is an equality iff one of u, v is a nonnegative multiple of the other.

Proof. Let $u, v \in V$. Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} = \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v \rangle \leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2.$$

The inequality step follows from Cauchy-Schwarz, where $2\operatorname{Re}\langle u, v \rangle \leq 2|\langle u, v \rangle|$. Taking square roots gives the triangle inequality. This is an equality iff the two inequalities above are equalities, which is true iff $\langle u, v \rangle = \|u\|\|v\|$. □

Parallelogram Equality. *If $u, v \in V$, then*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof. Exercise. □

2.3 Orthonormal Bases

A list (e_1, \dots, e_m) of vectors in V is orthonormal if $\langle e_j, e_k \rangle = 0$ when $j \neq k$ and equals 1 when $j = k$, for $j, k \in \{1, \dots, m\}$. Orthonormal lists are nice.

Proposition 2.1. *If (e_1, \dots, e_m) is an orthonormal list of vectors in V , then*

$$\|a_1 e_1 + \dots + a_m e_m\|^2 = |a_1|^2 + \dots + |a_m|^2$$

for all $a_1, \dots, a_m \in \mathbb{F}$.

Proof. Since each e_j has norm 1, this follows from repeated applications of the Pythagorean theorem. \square

Corollary 2.1. *Every orthonormal list of vectors is linearly independent.*

An **orthonormal basis** of V is an orthonormal list of vectors in V that forms a basis for V . The standard basis is a good example. If we find an orthonormal list of length $\dim V$, then this is automatically an orthonormal basis of V (since they must be LI). In general, given a basis (e_1, \dots, e_n) of V and a vector $v \in V$, we know there is some choice of scalars a_1, \dots, a_n such that $v = a_1 e_1 + \dots + a_n e_n$, but finding the a_j 's can be difficult. This is not the case for an orthonormal basis.

Theorem 2.1. *Suppose (e_1, \dots, e_n) is an orthonormal basis of V . Then*

$$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$$

for every $v \in V$.

Proof. Let $v \in V$. Since (e_1, \dots, e_n) is a basis of V , there exist scalars a_1, \dots, a_n such that $v = a_1 e_1 + \dots + a_n e_n$. Taking the inner product of both sides with e_j , we get $\langle v, e_j \rangle = a_j$. The second part follows from the first proposition and our previous result. \square