Miscellaneous Notes on Linear Algebra

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April 23, 2021

Who ever suffered from learning too much linear algebra? These notes will seek to fill in my linear algebra gaps. New inclusion: these notes will also cover any miscellaneous material I should have learned in my undergraduate analysis, abstract algebra, topology, or whatever classes but didn't. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

Basic linear algebra

Here we review things like how to multiply matrices.

1.1 Basics

A set of vectors $\{v^i\}$ **linearly independent** if $\sum_i c_i v^i = 0$ implies $c_i = 0$ for all i. A **basis** is a linearly independent *spanning set*, that is, for a basis $\{e_i\}$, every vector $v \in V$ can be written as a linear combination $v = \sum_i v^i e_i$. A map $T\colon V \to W$ is **linear** (or a **homomorphism**) if for $v^1, v^2 \in V$ and $a_1, a_2 \in \mathbb{F}$, $T(a_1 v^1 + a_2 v^2) = a_1 T(v^1) + a_2 T(v^2)$. For $U := \{u^1, u^2, \cdots\}$ a finite subset of vectors in V, any map $T\colon U \to W$ induces a linear map $T\colon V \to W$ by the rule

$$T\left(\sum_i a_i u^i\right) := \sum_i a_i T(u^i).$$

The original map is said to have been **extended by linearity**[1]. The set of $v \in V$ such that $Tv = 0$[2] is the **kernel** of T, and $\dim\ker T$ is called the **nullity** of T. The **rank** of T is defined as $\dim\operatorname{im} T$. If T is bijective then it is an **isomorphism**, where V and W are said to be **isomorphic**. A linear map from a space to itself is an **endomorphism**, and a self-bijection is an **automorphism**.

Consider the short exact sequence

$$0 \longrightarrow \ker T \overset{\iota}{\hookrightarrow} V \xrightarrow{T} W \longrightarrow 0$$

for $T\colon V \to W$ surjective.

Theorem 1.1. *For the short exact sequence above, there exists a linear map $S\colon W \to V$ such that $T \circ S = 1$. We say the exact sequence* ***splits****.*

To see this, by surjectivity each basis element of W gets mapped onto by some element in V. Extend the inverse map by linearity, then this new map S satisfies $T \circ S = 1$. This map S is called a **section** of T.

Rank-Nullity Theorem. *For the short exact sequence above, let S be a section of T. Then*

$$V = \ker T \oplus S(W).$$

In particular, $\dim V = \dim\ker T + \dim S(W)$.

Proof. By the first isomorphism theorem, we have the short exact sequence $0 \to \ker T \hookrightarrow V \to \operatorname{im} T \to 0$. Then since $V \to \ker T$ is a retract, apply the splitting lemma to get that the middle map is an isomorphism in the diagram below.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \ker T \arrow[r, hook] \arrow[d] & V \arrow[r, "T"] \arrow[d, "\text{iso}"] & \operatorname{im} T \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker T \arrow[r] & \ker T \oplus \operatorname{im} T \arrow[r] & \operatorname{im} T \arrow[r] & 0
\end{tikzcd}$$

The rank nullity theorem follows. ⊠

[1] Doesn't this only work when U is a spanning set for V?

[2] We use the notation $T(v) := Tv$ from now on.

1.2 Fiddling with indices (without explanation)

For an endomorphism $T\colon V \to V$ with a basis $\{e_i\}$ of V, we can construct an $n \times n$ matrix whose entries T^i_j are given by

$$Te_j = \sum_i e_i T^i_j.$$

We write (T^i_j) or $\mathbf{T}$ to indicate the matrix with entries T^i_j. The map $T \to \mathbf{T}$ is a **representation** of T in the basis $\{e_i\}$. A different basis leads to a different matrix, but they represent the same endomorphism. Here's how I visualize the indices (with $j = 3$ as an example):

$$T\left(e_j\right) = T\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = T_{13}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + T_{23}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + T_{33}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + T_{43}\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + T_{53}\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \sum_i e_i T^i_j.$$

The splitting happens because that's how matrix multiplication is defined. For $v = \sum_i v^i e_i \in V$, we have

$$v' := Tv = \sum_j v^j Te_j = \sum_{ij} v^j e_i T^i_j = \sum_i \left(\sum_j T^i_j v^j\right) e_i = \sum_i v^{i'} e_i,$$

so the components of v' are related to the components of v by the rule $v^{i'} = \sum_j T^i_j v^j$. It is time to introduce Einstein summation notation, where flipping the indices means an implicit sum. So our equation above becomes

$$v' := Tv = v^j Te_j = v^j e_i T^i_j = T^i_j v^j e_i = v^{i'} e_i \implies v^{i'} = T^i_j v^j.$$

For S and T two endomorphisms of V, if $ST := S \circ T$, matrix multiplication is defined as $ST_{ij} = \sum_k S_{ik} T_{kj}$. In Einstein summation notation, this is notated $ST^i_j = S^i_k T^k_j$.

Note. Indices are confusing. From Wikipedia, some mnemonics: the *up*per indices go *up* to down, *l*ower indices go *l*eft to right. Covariant tensors are row vectors with lower indices (but they sum over an upper index). The lower index indicates which *column* you are in, hence why the indeed go left to right. Similarly, the upper index indicates which *row* you are in. This is the picture to keep in mind:

$$\alpha = \begin{pmatrix} & \alpha & \end{pmatrix}, \quad v = \begin{pmatrix} \\ v \\ \\ \end{pmatrix}, \qquad \phi^j = \begin{pmatrix}0 & 0 & 1 & 0 & 0\end{pmatrix}, \quad e_i = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}.$$

$$\begin{pmatrix} & \alpha & \end{pmatrix}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = \alpha_i, \qquad \begin{pmatrix}0 & 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix} \\ v \\ \\ \end{pmatrix} = v^j.$$

Note that the only things you should be looking at are ϕ^j and e_i, since they're the actual vectors, while α_j and v^i are coordinate functions with flipped indices so we can sum over them. If you think of a covector $\alpha = \begin{pmatrix}w_1 & w_2 & \cdots\end{pmatrix}$, you can see why we say they have *lower* indices. However, when you write the implicit sum $\alpha = \alpha_j \phi^j$, the ϕ^j (which are covectors) have an upper index because that's what we're summing over: the actual entries have lower indices. For multi-index sums like $v^j e_i T^i_j$, we sum left to right.

The **row rank** (resp **column rank**) of a matrix T is the maximum number of LI rows (resp columns) when considered as vectors in $\mathbb{R}^n$. These concepts are equal, and we call this the **rank** of T, denoted $\operatorname{rank} T$. If $\operatorname{rank} T = n$, then T has **maximal rank**, otherwise T is **rank deficient**. For $\{e_i\}$ and $\{e'_i\}$ two bases of V, we can write $e'_j = e_j A^i_j$ for some nonsingular A, called the **change of basis matrix**. If $v = v^i e_i = v^{i'} e'_i$, then

$$v^{j'} e'_j = v^{j'} e_i A^i_j = A^i_j v^{j'} e_i = v^i e_i.$$

So $v_j = A^i_j \{v_j\}'$, or $v'_j = (A^{-1})^i_j v^j$. We write $\langle v, f\rangle$ or $\langle f, v\rangle$ to denote $f(v)$. Then for $\{\phi^j\}$ a dual basis for $\{e_i\}$, we have $\langle e_i, \phi^j\rangle = \delta^j_i$. For $\{\phi^{i'}\}$ a dual basis corresponding to $\{e'_i\}$, write $\phi^{j'} = \phi^i B^j_i$. Then

$$\delta^j_i = \langle e'_i, \phi^{j'}\rangle = \langle e_k A^k_i, \phi^\ell B^j_\ell\rangle = A^k_i B^j_\ell \langle e_k, \phi^\ell\rangle = A^k_i B^j_\ell \delta^\ell_k = A^k_i B^j_k.$$

If we write $A^T := A^j_i$, we can write the result above as $A^T B = I$, equivalently $B = (A^T)^{-1} = (A^{-1})^T$, the **contragredient matrix** of A. For $f \in V^*$ a covector, under a change of basis we have

$$f' = f'_j \phi^{j'} = f'_j \phi^i B^j_i = B^j_i f'_j \phi^{i'} = f_i \phi^i = f, \quad \Longrightarrow \quad f_i = B^j_i f'_j, \quad f'_i = (B^{-1})^j_i f_j.$$

Rewriting in terms of A, we have

$$\phi^{i'} = \phi^j B^i_j = (B^T)^i_j \phi^j = (A^{-1})^i_j \phi^j, \quad f'_j = (B^{-1})^i_j f_i = (A^T)^i_j f_i = f_i A^i_j.$$

1.3 Upstairs or downstairs?

Let's talk about what just happened. If we use standard notation, the symbol a_j is ambiguous: are they components of vectors, covectors, or neither? How can we tell? We can't, we can only guess (you can tell when they're paired with the corresponding basis elements e_i or ϕ^i, but sometimes those are omitted for brevity). Introducing up down indices allows us to differentiate the two.

Under a change of basis, the components of a covector transform like basis vectors, while the components of a vector transform like cobasis vectors. We say the components of a covector transform **covariantly** (with the basis vectors), while the components of a vector transform **contravariantly** (against the basis vectors). Because of this, we write e_i for a basis vector as normal, but we use a raised index ϕ^i to denote the basis covectors. Then vector components are written with upstairs (contravariant) indices and covector components are written with downstairs (covariant indices).

Writing $v = v^i e_i$ and $f = f_i \phi^i$ allows us to quickly pair the up indices and down indices to see what is being summed. When this happens, we say the indices have been **contracted**. Avoid things like $a_i = b^i$. To summarize our results, we have $\langle e_j, \phi^j\rangle = \delta^j_i$, $e'_j = e_i A^i_j$, $v'^i = (A^{-1}) i_j v^j$. This notation also leads to much pedanticism and confusion as you may have already noticed. Introducing the shorthand

$$\mathbf{A} = \begin{pmatrix} A^i_j \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \vdots \\ \theta^n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix}$$

helps, since the above equations become $\mathbf{e}' = \mathbf{eA}$, $\mathbf{v}' = \mathbf{A}^{-1}\mathbf{v}$, $\theta' = \mathbf{A}^{-1}\theta$, $\mathbf{f}' = \mathbf{fA}$. The invariance of v and f under a change of basis become easy to see, for example $v' = \mathbf{e}'\mathbf{v}' = \mathbf{eAA}^{-1}\mathbf{v} = \mathbf{ev} = v$.

1.4 Inner product spaces

This is a more mature treatment of the material later in this paper thing. Let $\mathbb{F}$ be a subfield of $\mathbb{C}$, and V and $\mathbb{F}$-vector space. A **sesquilinear form** on V is a map $g \colon V \times V \to \mathbb{F}$ satisfying the following properties: for all $u, v, w \in V$ and $a, b \in \mathbb{F}$, the map g is

(i) **linear on the second entry**: $g \colon (u, av + bw) \to = ag(u, v) + bg(u, w)$, and

(ii) **Hermitian**: $g(v, u) = \overline{g(u, v)}$.

These two properties imply that g is **antilinear** on the first entry, that is, $g(au+bv,w)=\overline{a}g(u,w)+\overline{b}g(v,w)$. If $\mathbb{F}$ is a real field (subfield of $\mathbb{R}$), then this just says that g is a **symmetric bilinear form**. If a sequilinear form g is **nongenerate**, where $g(u,v)=0$ for all v implies $u=0$, then g is an **inner product.** A space equipped with an inner product is an **inner product space**.

Note that $g(u,u)$ is real by Hermiticity. If $g(u,u)\geq 0$ (resp $g(u,u)\leq 0$), then g is **nonnegative definite** (resp **nonpositive definite**). If $g(u,u)=0$ implies that $u=0$, then g is **positive definite** (resp **negative definite**).

Example 1.1 (The **Lorentizan inner product** on $\mathbb{R}^n$)**.** Let $u=(u_0,u_1,\cdots,u_{n-1})$ and $v=(v_0,v_1,\cdots,v_{n-1})$, and define

$$g(u,v):=-u_0v_0+\sum_{i=1}^{n-1}u_iv_i.$$

The vector space $\mathbb{R}^n$ equipped with this inner product is denoted $\mathbb{M}^n$ and is called **Minkowski space** (or **Minkowski spacetime**). Note that while the Lorentzian inner product is an indeed an inner product, it is not positive definite.

A set $\{v_i\}$ of vectors is **orthogonal** if $g(v_i,v_j)=0$ for $i\neq j$, and is **orthonormal** if $g(v_i,v_j)=\pm\delta_{ij}$. A vector v satisfying $g(v,v)=\pm 1$ is a **unit vector**.

Theorem 1.2. *Every inner product space has an orthonormal basis.*

First proof of Theorem 1.2. We use induction on $k=\dim V$. If todo:some algebra ⊠

Second proof of Theorem 1.2. todo:grammian, spectral theorem, diagonalization, sylvester's law of inertia ⊠

todo:the reisz lemma

1.5 The tensor product

What are tensors? Define a new vector product called the **tensor product**, denoted by $v\otimes w$[3]. The product is a **tensor of order 2** or a **second-order tensor** or a **2-tensor**. The tensor product is *noncommutative* in general, and we form higher order tensors by repeated iteration. Order-0 tensors are scalars, while order-1 tensors are vectors. In older literature $v\otimes w$ becomes vw and is called a *dyadic* product.

The set $\mathscr{T}^r$ of order r tensors forms a natural vector space: for S and T order r tensors, $aT+bS$ is another order r tensor. We write $\mathscr{T}^r:=V\otimes V\otimes\cdots\otimes V=V^{\otimes r}$. The set $\mathscr{T}=\bigcup_r\mathscr{T}^r$ forms an **algebra**, basically a ringed vector space satisfying homogeneity. The multiplication says that for R a tensor of order r and S an s-tensor, then $R\otimes S$ is an $(r+s)$-tensor. Let us write the (graded) algebra conditions in tensor language:

(1) **left distributivity**: $R\otimes(S+T)=R\otimes S+R\otimes T$,
(2) **right distributivity**: $(S+T)\otimes R=S\otimes R+T\otimes R$,
(3) **homogeneity**: $T\otimes(aS)=(aT)\otimes S=a(T\otimes S)$.

A tensor also has components in some basis. For e_i a basis of $\mathbb{R}^n$, the canonical basis for $\mathbb{R}^n\otimes\mathbb{R}^m$ is given by the nm elements of $\{e_i\otimes e_j\}$ as i varies over n and j varies over m. A general second-order tensor on $\mathbb{R}^n$ is a linear combination of these basis vectors of the form $T=\sum_{ij}T^{ij}e_i\otimes e_j=T^{ij}e_i\otimes e_j$. Usually the basis is understood, so T^{ij} is called a tensor, when it actually gives the components of a tensor with respect to some basis. To find the components of $v\otimes w$, observe that

$$v\otimes w=v^ie_i\otimes w^je_j=v^iw^i(e_i\otimes e_j).$$

Example 1.2. Given a rigid body consisting of a bunch of point masses m_α at positions $\mathbf{r}_\alpha=(x_{\alpha,1},x_{\alpha,2},x_{\alpha,3})$, its **inertia tensor** is given by

$$I_{ij}=\sum_\alpha m_\alpha(r_\alpha^2\delta_{ij}-x_{\alpha,i},x_{\alpha,j}),$$

where $r_\alpha^2=\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha$. There is a lot of sloppiness going on with indices and denoting components as tensors.

[3]These are actually defined by a *universal property* in category theory, but let's brush over the details.

1.6 Two ways to view general tensors

1: As an element of the tensor product space

We have been excluding covectors from the fun. A **tensor of type** $(\mathbf{r},\mathbf{s})$ is an element of the tensor product space

$$T^r_s = \overbrace{V \otimes V \otimes \cdots \otimes V}^{r\text{ times}} \otimes \overbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}^{s\text{ times}} = V^{\otimes r} \otimes (V^*)^{\otimes s}.$$

An r-tensor previously is now a tensor of type $(r,0)$. This space of all tensors forms a **multigraded algebra**, that is, multiplying a (r,s)-tensor and a (p,q)-tensor gives a tensor of type $(r+p,s+q)$. For a basis $\{e_i\}$ of V and dual basis $\{\phi^i\}$ of V^*, a basis for $\mathscr{T}^r_s$ is given by

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \phi^{j_1} \otimes \phi^{j_2} \otimes \cdots \otimes \phi^{j_s},$$

where the indices run from 1 to $\dim V$. A general tensor of type (r,s) is a linear combination

$$T = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \phi^{j_1} \otimes \phi^{j_2} \otimes \cdots \otimes \phi^{j_s},$$

with an implicit sum over $i_1 \cdots i_r$, $j_1 \cdots j_s$. From before, we can see that upstairs indices transform contravariantly, while downstairs indices transform covariantly.

$$T^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s} = T^{i_1 \cdots i_r}_{j_1 \cdots j_s} (A^{-1})^{i'_1}_{i_1} \cdots (A^{-1})^{i'_r}_{i'_r} A^{j_1}_{j'_1} \cdots A^{j_s}_{j'_s}.$$

2: As a multilinear functional on the dual space

Consider the space of multilinear maps $\widetilde{\mathscr{T}}^r_s$. Recall the **natural pairing**, where $\langle f, v\rangle = \langle v, f\rangle$ denotes $f(v)$. We can view the tensor $e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \phi^{j_1} \otimes \cdots \otimes \phi^{j_s}$ as a multilinear map on the space $(V^*)^{\times r} \times V^{\times s}$ that acts according to the rule

$$\begin{aligned}
&(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \phi^{j_1} \otimes \cdots \otimes \phi^{j_s})(\phi^{k_1}, \cdots, \phi^{k_r}, e_{\ell_1}, \cdots, e_{\ell_s}) \\
&\quad = \langle e_{i_1}, \phi^{k_1}\rangle \cdots \langle e_{i_r}, \phi^{k_r}\rangle \langle \phi^{j_1}, e_{\ell_1}\rangle \cdots \langle \phi^{j_s}, e_{\ell_s}\rangle \\
&\quad = \delta^{k_1}_{i_1} \cdots \delta^{k_r}_{i_r} \delta^{j_1}_{\ell_1} \cdots \delta^{j_s}_{\ell_s}.
\end{aligned}$$

If we view the tensor product this way, we have

$$\begin{aligned}
&T(\phi^{k_1}, \cdots, \phi^{k_r}, e_{\ell_1}, \cdots, e_{\ell_s}) \\
&\quad = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \times (e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \phi^{j_1} \otimes \phi^{j_2} \otimes \cdots \otimes \phi^{j_s})(\phi^{k_1}, \cdots, \phi^{k_r}, e_{\ell_1}, \cdots, e_{\ell_s}) \\
&\quad = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \delta^{k_1}_{i_1} \cdots \delta^{k_r}_{i_r} \delta^{j_1}_{\ell_1} \cdots \delta^{j_s}_{\ell_s} \\
&\quad = T^{k_1 k_2 \cdots k_r}_{\ell_1 \ell_2 \cdots \ell_s}.
\end{aligned}$$

This gives an isomorphism between $\widetilde{\mathscr{T}}^r_s$ and $\mathscr{T}^r_s$. In essence, you can choose to view tensors *passively* as elements of a certain vector space (the tensor product space), or *actively* as multilinear functionals on the dual space. They are two sides of the same coin, so we can interchange the notations as we please.

Lecture 2

Miscellaneous topics

TODO: affine spaces, inverse function, change of variables for multiple integrals (spivak 34,67) or tu appendix, rank, nullity, binomial theorem, freed's thing, maybe topology bases, subspace/product, tychonoff, convergnece, etc

2.1 The Inverse Function Theorem

Lecture 3

Inner-Product Spaces

What is an inner product?? Let's find out.

3.1 Inner Products

The length of a vector x is the **norm** of x, denoted $\|x\|$. If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we have $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Note that the norm is not linear. For $x, y \in \mathbb{R}^n$, the **dot product** of x and y, denoted $x \cdot y$, is defined by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. Note that this is a number, not a vector. Clearly $x \cdot x = \|x\|^2$ for all $x \in \mathbb{R}^n$, which implies $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$ ($x \cdot x = 0$ only if x is the zero vector). The map that sends $x \in \mathbb{R}^n$ to $x \cdot y$ in $\mathbb{R}$ for fixed y is linear since $\mathbb{R}$ is a field. The dot product is also commutative, since $\mathbb{R}$ is.

Inner products generalize dot products. Recall that $|\lambda|^2 = \lambda\overline{\lambda}$ for $\lambda \in \mathbb{C}$. For $z \in \mathbb{C}^n$, we define the norm of z by $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. We take the modulus of z_i since we want the result to be nonnegative. Note that $\|z\|^2 = z_1\overline{z_1} + \cdots z_n\overline{z_n}$. We want to think of $\|z\|^2$ as the inner product of z with itself, like in $\mathbb{R}^n$. This suggests we define the inner product of $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$ with z as $w_1\overline{z_1} + \cdots + w_n\overline{z_n}$. We expect the inner product of w with z equal the complex conjugate of the inner product of z with w. With this motivation in mind, let us define inner products.

Definition 3.1 (Inner product)**.** An **inner product** on an F-vector space V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in F$ such that

(i) $\langle v, v\rangle \geq 0$ for all $v \in V$; (**positivity**)

(ii) $\langle v, v\rangle = 0$ iff $v = 0$; (**definiteness**)

(iii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$; (**additivity in first slot**)

(iv) $\langle av, w\rangle = a\langle v, w\rangle$ for all $a \in F$ and all $v, w, \in V$; (**homogeneity in first slot**)

(v) $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$. (**conjugate symmetry**).

For real numbers, condition (v) simply becomes $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$. An **inner product space** is a vector space V along with an inner product on V.

Example 3.1. The most important example is the **Euclidian inner product** on $\mathbf{F}^n$ (Axler uses $\mathbf{F}$ to denote either $\mathbb{C}$ or $\mathbb{R}$). We define an inner product on $\mathbf{F}^n$ by

$$\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n)\rangle = w_1\overline{z_1} + \cdots w_n\overline{z_n}.$$

An example of another inner product on $\mathbf{F}^n$ is defined by $\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n)\rangle = c_1 w_1\overline{z_1} + \cdots + c_n w_n\overline{z_n}$ for c_i positive constants. The case where $c_i = 1$ for all i is simply the standard Euclidian inner product.

Example 3.2. Consider the vector space $\mathscr{P}_m(\mathbf{F})$, the polynomial ring over $\mathbf{F}$ of polynomials with degree at most m. We can define an inner product on $\mathscr{P}_m(\mathbf{F})$ by

$$\langle p, q\rangle = \int_0^1 p(x)\overline{q(x)}\,dx.$$

For fixed $w \in V$, the function that takes v to $\langle v, w\rangle$ is a linear map $V \to \mathbf{F}$. So $\langle 0, w\rangle = 0$, and by condition (v) $\langle w, 0\rangle = 0$ as well. Furthermore, $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ and $\langle u, av\rangle = \overline{a}\langle u, v\rangle$ hold as well: This second condition is known as conjugate homogeneity in the second slot.

3.2 Norms

For $v \in V$, we define the **norm** of v, denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v\rangle}$. For example, if $p \in \mathscr{P}_m(\mathbf{F})$, then $\|p\| = \sqrt{\int_0^1 |p(x)|^2\,dx}$. Some properties: $\|v\| = 0$ iff $v = 0$, and $\|av\| = |a|\|v\|$. To see this, note that $\|av\|^2 = \langle av, av\rangle = a\langle v, av\rangle = a\overline{a}\langle v, v\rangle = |a|^2\|v\|^2$, taking square roots gives us our result. This illustrates a general idea: working with norms squared is easier than working directly with norms.

Two vectors $u, v \in V$ are **orthogonal** if $\langle u, v\rangle = 0$. The zero vector is orthogonal to every vector, and the only vector orthogonal to itself. Assume $V = \mathbb{R}^2$, now let us state a 2500 year old theorem.

Pythagorean Theorem. *If u, v are orthogonal vectors in V, then*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. Exercise. ⊠

Suppose $u, v \in V$. We want to write u as a scalar multiple of v plus a vector w orthogonal to v. Let $a \in \mathbf{F}$ be a scalar, then $u = av + (u - av)$. We need to choose a such that v is orthogonal to $u - av$, in other words, we want $0 = \langle u - av, v\rangle = \langle u, v\rangle - a\|v\|^2$. So we should choose $a = \langle u, v\rangle / \|v\|^2$ (where $v \neq 0$). Then

$$u = \frac{\langle u, v\rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v\rangle}{\|v\|^2} v\right).$$

Cauchy-Schwarz Inequality. *If $u, v \in V$, then*

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality iff one of u, v is a scalar multiple of the other.

Proof. Let $u, v \in V$, and assume $v \neq 0$. Consider $u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$, where w is orthogonal to v. By the Pythagorean theorem, we have

$$\|u\|^2 = \left\|\frac{\langle u, v\rangle}{\|v\|^2} v\right\|^2 + \|w\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Multiply both sides, take a square root, and we are done. This is an equality iff $w = 0$, but this is true iff u is a multiple of v. ⊠

Triangle Inequality. *If $u, v \in V$, then*

$$\|u+v\| \leq \|u\| + \|v\|.$$

This is an equality iff one of u, v is a nonnegative multiple of the other.

Proof. Let $u, v \in V$. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} = \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \leq \|u\|^2 + \|v^2\| + 2\|u\|\|v\| = (\|u\| + \|v\|)^2.$$

The inequality step frollows from Cauchy-Schwartz, where $2\operatorname{Re}\langle u, v\rangle \leq 2|\langle u, v\rangle|$. Taking square roots gives the triangle inequality. This is an equality iff the two inequalities above are equalities, which is true iff $\langle u, v\rangle = \|u\|\|v\|$. ⊠

Parallelogram Equality. *If $u, v \in V$, then*

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

Proof. Exercise. ⊠

3.3 Orthonormal Bases

A list $(e_1, \cdots, e_m)$ of vectors in V is orthonormal if $\langle e_j, e_k \rangle = 0$ when $j \neq k$ and equals 1 when $j = k$, for $j, k \in \{1, \cdots, m\}$. Orthonormal lists are nice.

Proposition 3.1. *If $(e_1, \cdots, e_m)$ is an orthonormal list of vectors in V, then*

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \cdots, a_m \in \mathbf{F}$.

Proof. Since each e_j has norm 1, this follows from repeated applications of the Pythagorean theorem. ⊠

Corollary 3.1. *Every orthonormal list of vectors is linearly independent.*

An **orthonormal basis** of V is an orthonormal list of vectors in V that forms a basis for V. The standard basis is a good example. If we find an orthonormal list of length $\dim V$, then this is automatically an orthonormal basis of V (since they must be LI). In general, given a basis $(e_1, \cdots, e_n)$ of V and a vector $v \in V$, we know there is some choice of scalars $a_1, \cdots, a_m$ such that $v = a_1 e_1 + \cdots a_n e_n$, but finding the a_j's can be difficult. This is not the case for an orthonormal basis.

Theorem 3.1. *Suppose $(e_1, \cdots, e_n)$ is an orthonormal basis of V. Then*

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$$

for every $v \in V$.

Proof. Let $v \in V$. Since $(e_1, \cdots, e_n)$ is a basis of V, there exist scalars $a_1, \cdots, a_n$ such that $v = a_1 e_1 + \cdots + a_n e_n$. Taking the inner product of both sides with e_j, we get $\langle v, e_j \rangle = a_j$. The second part follows from the first proposition and our previous result. ⊠