Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html`.

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§1 August 26, 2020

§1.1 Oops

Unfortunately, I couldn't attend Lecture 1.

§2 August 28, 2020

§2.1 Subgroups and Normal Subgroups

Lemma 2.1. *Let $H \subseteq G$, $\langle G, \cdot \rangle$ a group and $H \neq \emptyset$. Then H is a subgroup of G if and only if $h_1 h_2 \in H \implies h_1 h_2^{-1} \in H$.*

Proof. For all $h_2 \in H$, $h_2^{-1} \in H$ since H is a group. H is closed under multiplication implies $h_1 h_2^{-1} \in H$ for all $h_1, h_2 \in H$. Conversely, assume that $h_1 h_2 \in H \implies h_1 h_2^{-1} \in H$. Then for $h \in H$, $hh^{-1} \in H$ so $1 \in H$. Now that we know $1 \in H$, then for $h \in H$ we have $1 \cdot h \in H \implies h^{-1} \cdot 1 \in H$, so H is closed under inverses. Finally, associativity follows from the fact that $H \subseteq G \implies \forall h \in H, h \in G$ where G is a group, and we are done. $\boxtimes$

Definition 2.1 (Normal Subgroup)**.** A *subgroup* H of G is normal if $gHg^{-1} = H$ for all $g \in G$.

Example 2.1. Let G be abelian: then every subgroup is normal since $ghg^{-1} = gg^{-1}h = h$ for all $g \in G, h \in H$.

Example 2.2. Take $G = S_3$. Then the subgroup $\langle (123) \rangle$ is normal. However, the subgroup $\langle (1,2) \rangle$ is not normal, since $(13)(12)(13)^{-1} = (23) \notin \langle (12) \rangle$.

Example 2.3. Take $\mathrm{SL}_n\mathbb{R} \subseteq \mathrm{GL}_n\mathbb{R}$, where $\mathrm{SL}_n\mathbb{R}$ is the set of matrices with $\det(A) = 1$ for $A \in \mathrm{SL}_n\mathbb{R}$. We know $\mathrm{SL}_n\mathbb{R}$ forms a subgroup. Question: is $\mathrm{SL}_n\mathbb{R}$ normal? Answer: yes.

$$\det(ABA^{-1}) = \det(A)\det(B)\det(A^{-1}) = \det(A)\det(A)^{-1}\det(B) = \det(B).$$

Proposition 2.1. Let H, K be subgroups of G, then $H \cap K$ is a subgroup of G. You can verify this in your free time.

Note: is $H \cup K$ a subgroup? No!

§2.2 Product and Quotient Groups

Definition 2.2 (Product Groups)**.** Let G, H be groups. We define the *direct product* $G \times H$ with the group operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. The identity is just $(1_G, 1_H)$ where 1_G and 1_H denotes the respective identities for G and H. Finally, the inverse is similarly defined as (g_1^{-1}, h_1^{-1}) where g_1^{-1} and h_1^{-1} are the respective inverses for $g_1 \in G$, $h_1 \in H$.

Some examples of product groups include $\mathbb{Z}\times\mathbb{Z}$ ($\mathbb{Z}$ denotes $\langle \mathbb{Z}, +\rangle$), and $\mathbb{Z}\times\langle \mathbb{R}\setminus\{0\}, \cdot\rangle$

Example 2.4 (Quotient Groups)**.** Let $n \in \mathbb{Z}$, for example $\langle \mathbb{Z}/n\mathbb{Z}, +\rangle$, equivalence relations: modulo n. $a, b \in \mathbb{Z}, a \equiv b \mod n \iff n \mid (a-b)$. Equivalence classes: $a + n\mathbb{Z} = \{a + nk \mid k \in \mathbb{Z}\}$. Notation: $\bar{a} = a + n\mathbb{Z} = [a]$. Our set $\mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} \mid a \in \mathbb{Z}\} = \{a + n\mathbb{Z} \mid a = 0, ..., n-1\}$. $(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a+b) + n\mathbb{Z}$, so this is a group operation. In this case, the identity is just $0 + n\mathbb{Z} = n\mathbb{Z}$. We have the inverse of $(a + n\mathbb{Z})$ equal to $(a + n\mathbb{Z})^{-1} = -a + n\mathbb{Z}$.

Remark: $\langle \mathbb{Z}/n\mathbb{Z}, +\rangle$ is a quotient of the group $\langle \mathbb{Z}, +\rangle$ by the subgroup $\langle n\mathbb{Z}, +\rangle$. $\langle 1\rangle = \mathbb{Z}, \langle 1 + n\mathbb{Z}\rangle = \langle \mathbb{Z}/n\mathbb{Z}\rangle$.

Quotient groups in general: G a group, H a **normal** subgroup.

§2.3 Left and Right Cosets

Definition 2.3 (Cosets)**.** Left cosets: $gH = \{gh \mid h \in H\}$. Right cosets: $Hg = \{hg \mid h \in H\}$. G/H - set of left cosets. $H \setminus G$ - set of right cosets.

Observe: Left and right cosets are in bijection with one another. $gH \mapsto Hg$, $gh \mapsto g^{-1}(gh)g = hg$. You can verify that this is a bijection. Let $g_1, g_2 \in G$, what map maps $g_1H \to g_2H$? $g_1h \mapsto (g_2g_1^{-1})g_1h = g_2h$.

Note. We have

$$\bigcup_{g\in G} gH = G.$$

Also: $g_1H \cap g_2H$ is either $\emptyset$ or they are equal. (Equivalence relation).

§2.4 Lagrange's Theorem

Proposition 2.2. If G is finite and H a subgroup of G, then $|H| \mid |G|$.

Proof. By the statement above,

$$G = \bigcup_{i=1}^{n} g_iH$$

since G is finite for $n \in \mathbb{N}$. Note that this is a disjoint union. So

$$|G| = \sum_{i=1}^{n} |g_iH| = n \cdot |H| \implies |H| \mid |G|.$$

⊠

Quotient group: G a group, H a normal subgroup, $G/H = \{gH \mid g \in G\}$. The multiplication is defined as $g_1H \cdot g_2H = g_1g_2H$. You can verify this operation is well defined (given that H is normal).

§3 August 31, 2020

§3.1 The Dihedral Group

Example 3.1 (Dihedral Group)**.** Consider the free group $G = \langle g, \tau \rangle$ and the normal subgroup H_n of G generated by

$$g^n, \tau^2, \tau g \tau^{-1} g.$$

The dihedral group $D_{2n} = G/H_n$ (sometimes denoted D_n), is it automatically normal? What about conjugating by powers of g?

Observe that $\langle g \rangle \simeq \langle gH_n \rangle \subseteq D_{2n}$. $\langle g \rangle$ has order n and is normal (convince yourselves of this). τ has order 2 and so does $\langle \tau g^i \rangle$ for any i. Are these subgroups normal? (Yes sometimes, no some other times).

Consider the following: $2\mathbb{Z} \trianglelefteq \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \{2\mathbb{Z}, 1 + 2Z\}$, $\langle (123) \rangle \trianglelefteq S_3 = {}^{S_3}/_{\langle (123) \rangle} = \{1_{S_3}, (\bar{12})\}$, $\mathbb{R}^+ \setminus \{0\} \trianglelefteq \mathbb{R} \setminus \{0\} \to {}^{\mathbb{R} \setminus \{0\}}/_{\mathbb{R}^+ \setminus \{0\}} = \{\bar{1}, \bar{-1}\}$. What distinguishes these groups (they all have order two)?

§3.2 Group Homomorphisms, Isomorphisms, and Automorphisms

Definition 3.1 (Homomorphisms)**.** Let G, H be two groups. A map $\phi\colon G \to H$ is a homomorphism if

$$\phi(g_1 g_2) = \phi(g_1) \cdot \phi(g_2).$$

Definition 3.2 (Isomorphism)**.** A map ϕ is an isomorphism is ϕ if a homomorphism and a bijection. If $\phi\colon G \to H$ is an isomorphism then we write $G \simeq H$.

Definition 3.3 (Automorphism)**.** We have ϕ an automorphism if ϕ is an isomorphism from G onto itself, that is, $G = H$.

§3.3 The First Homomorphism Theorem

Remark 3.1. Let $\phi\colon G \to H$ be a group homomorphism. Then

1. $\phi(1_G) = 1_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$,
2. $\phi(G) = \operatorname{im}\phi$ is a subgroup of H,
3. $\ker\phi = \{g \in G \mid \phi(g) = 1H\}$ is a normal subgroup of G,
4. ϕ is injective $\iff$ $\ker\phi = \{1_G\}$,
5. If G is finite then $|G| = |\ker\phi| \cdot |\operatorname{im}\phi|$.

Theorem 3.1. *Let $\phi\colon G \to H$ be a group homomorphism. Then $\bar{\phi}\colon G/\ker\phi \to \operatorname{im}\phi$ is an isomorphism.*

Proof. Left as an exercise to the reader (verify that $\bar{\phi}$ is well-defined, injective, surjective, and a homomorphism). $\boxtimes$

Example 3.2. Recall the groups ${}^{\mathbb{Z}}/_{2\mathbb{Z}} = \langle 1+2\mathbb{Z} \rangle$, ${}^{S_3}/_{\langle (123) \rangle} = \langle (12)\langle (123) \rangle \rangle$, ${}^{\mathbb{R} \setminus \{0\}}/_{\mathbb{R}^+ \setminus \{0\}} = \langle (-1)\mathbb{R}^+ \setminus \{0\} \rangle$. Then we have isomorphisms onto all of them, so they are the same.

Remark 3.2. Product of groups $\iff$ quotient groups. H, K be groups. $G = H \times K$, $H \simeq H \times \{1_K\} \trianglelefteq G$. ??? $H, K \trianglelefteq G$, $H \cap K = \{1_G\}, HK = G \implies G \simeq H \times K$ (prove this). ${}^{G}/_{H} \simeq K$ and ${}^{G}/_{K} = H$: relax any of the implications, and the isomorphisms will fail.