Complex Analysis Lecture Notes

Simon Xiang

These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`. I was also unhappy with the textbook, so some supplementary notes from different texts are found at the bottom of the document.

Contents

§1 September 24, 2020

Last time: hyperbolic trig functions. We have some functions and theorems to deal with them, like rational functions p/q, trig functions $\cos z, \sin z, \tan z$, and hyperbolic trig functions $\cosh z, \sinh z$, the exponential function $\exp(z)$, etc. Where these functions are defined they are analytic.

§1.1 Logarithmic functions

Let f be a function. Then $f(z) = w \iff z = f^{-1}(w)$. This is only a function if f is onto. Consider the case where $f(z) = e^z$ and $e^z = w$.

Definition 1.1 (Logarithm)**.** We define the functional inverse of the exponentatial function as the *logarithm*, that is,

$$\log w = \{z \mid e^z = w\},$$

that is, $\log = \exp^{-1}$. Suppose $z = x + iy$, so $e^z = e^{x+iy} = e^x(\cos x + i \sin y) = w$. Write w in polar form, so $w = |w|e^{i \arg w}$. What values of z give rise to this? We want $e^x = |w|$, $e^{iy} = e^{i \arg w}$. $e^x = |w| \iff x = \ln |w|$, so $y \in \arg w$. Therefore we have

$$\log w = \ln |w| + i \arg w$$

for $w \neq 0$. So this function is multivariate.

We want to do calculus with this function. We are not terribly interested in multivalued things (actual functions lmao). One way to get an actual function is to define

$$\operatorname{Log} w = \ln |w| + i \operatorname{Arg}(w)$$

for $w \neq 0$. But that's not enough: we also want functions to be differentiable, etc. Note that this function Log wouldn't be continuous on the negative real axis. If $w \in S^1$, then ... I stopped paying attention here. Because of the discontinuity (which I was not paying attention for), we define $\operatorname{Log}(w) = \ln |w| + i \operatorname{Arg} w$ only for w not a negative real number.

Claim. Log satisfies the CR equations. Verify this in your free time.

Note that $\frac{d}{dz} \operatorname{Log} z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{1}{r} + 0) = \frac{1}{z}$ on its domain. This is very cool, thank you logarithm. This is useful, but we need other ways to get "honest" functions from $\log w = \ln |w| + i \arg w$.

§1.2 Branches of the logarithm(todo)

aa hayaku ouchi ni kaeritai

Recall the identity that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, with addition being componentwise for the infinite sets. So $\log(z_1 z_2) = \ln |z_1 z_2| + i \arg(z_1 z_2) = \ln |z_1| + \ln |z_2| + i \arg z_1 + i \arg z_2 = \log z_1 + \log z_2$.

Where are we headed? To the next class of functions z^α and beyond.