Differential Geometry Notes

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Lecture 1

Curves and Surfaces

1.1 Curves

A curve $\mathscr{C} := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$. Curves in $\mathbb{R}^3$ are defined similarly. These are called **level curves**.

Definition 1.1. A **parametrized curve** in $\mathbb{R}^n$ is a map $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ for some α, β with $-\infty \le \alpha \le \beta \le \infty$. A parametrized curve whose image is contained in a level curve $\mathscr{C}$ is called a **parametrization** of $\mathscr{C}$.

Example 1.1. We parametrize the parabola. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the components γ_1 and γ_2 of γ must satisfy $\gamma_2(t) = \gamma_1(t)^2$ for all $t \in (\alpha,\beta)$. The parametrization $\gamma\colon (-\infty,\infty) \to \mathbb{R}^2$, $\gamma(t) = (t,t^2)$ works, as well as $\gamma(t) = (t^3, t^6)$, $\gamma(t) = (2t, 4t^2)$, and so on.

For the circle $x^2 + y^2 = 1$, we could try $x = t$, but that only hits half of S^1. What satisfies $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$? $\gamma_1(t) = \cos t$ and $\gamma_2(t) = \sin t$ do. The interval $(-\infty,\infty)$ is overkill since the map has infinite degree.

Example 1.2. Consider the *astroid* $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in \mathbb{R}$. Since $\cos^2 t + \sin^2 t = 1$ for all t, then $x = \cos^3 t,\ y = \sin^3 t$ satisfy $x^{2/3} + y^{2/3} = 1$.

A function $f : (\alpha,\beta) \to \mathbb{R}$ is **smooth** if $\frac{d^n f}{dt^n}$ exists for all $n \ge 1$ and $t \in (\alpha,\beta)$. Smoothness is preserved under addition, multiplication, composition, etc. You differentiate vector valued functions componentwise, and we denote $d\gamma/dt$ by $\dot\gamma(t)$, $d^2\gamma/dt^2$ by $\ddot\gamma(t)$, etc.

Definition 1.2. If γ is a parametrized curve, then $\dot\gamma(t)$ is the **tangent vector** of γ at the point $\gamma(t)$.

Proposition 1.1. *If the tangent vector of a parametrized curve is constant, then the image of the curve is a straight line.*

Proof. If $\dot\gamma(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is constant, then

$$\gamma(t) = \int \frac{d\gamma}{dt}\,dt = \int \mathbf{a}\,dt = t\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is another constant vector. ⊠

Example 1.3. The **limaçon** is the parametrized curve $\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$, $t \in \mathbb{R}$. There's a self intersection at the origin, the tangent vector is $\dot\gamma(t) = (-\sin t - 2\sin 2t, \cos t + 2\cos 2t)$. This is well defined, but takes two different values at $t = 2\pi/3$ and $t = 4\pi/3$.

1.2 Arc Length

The length of a straight line segment between two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is $||\mathbf{u} - \mathbf{v}||$, given the standard norm/inner product/metric/blah on $\mathbb{R}^n$.

Definition 1.3. The **arc-length** of a curve γ starting at $\gamma(t_0)$ is the function $s(t)$ given by

$$s(t) = \int_{t_0}^{t} ||\dot\gamma(u)||\,du.$$

Example 1.4. For a **logarithmic spiral** $\gamma(t) = (e^{kt}\cos t, e^{kt}\sin t)$, we have $\dot\gamma = (e^{kt}(k\cos t - \sin t), e^{kt}(k\sin t + \cos t))$, so $||\dot\gamma||^2 = e^{2kt}(k\cos t - \sin t)^2 + e^{2kt}(k\sin t + \cos t)^2 = (k^2+1)e^{2kt}$. Then the arc length of γ starting at $\gamma(0) = (1,0)$ is

$$s = \int_0^t \sqrt{k^2+1}\,e^{ku}\,du = \frac{\sqrt{k^2+1}}{k}(e^{kt} - 1).$$

Note that the arc-length is differentiable, that is,

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du = \|\dot{\gamma}(t)\|.$$

Definition 1.4. If $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, its **speed** at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be a **unit-speed** curve if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha, \beta)$.

Proposition 1.2. *Let $\mathbf{n}(t)$ be a unit vector that is a smooth function of parameter t. Then the dot product $\dot{\mathbf{n}}(t)\cdot\mathbf{n}(t) = 0$ for all t, i.e., $\dot{\mathbf{n}}(t)$ is zero or orthogonal to $\mathbf{n}(t)$ for all t. If γ is a unit-speed curve, then $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.*

Proof. We differentiate $\mathbf{n} \cdot \mathbf{n} = 1$ to get $\dot{\mathbf{n}} \cdot \mathbf{n} + \mathbf{n} \cdot \dot{\mathbf{n}} = 0$, so $\dot{\mathbf{n}} \cdot \mathbf{n} = 0$. ⊠

1.3 Reparametrization

A parametrized curve $\widetilde{\gamma}\colon (\widetilde{\alpha}, \widetilde{\beta}) \to \mathbb{R}^n$ is a **reparametrization** of a parametrized curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi\colon (\widetilde{\alpha}, \widetilde{\beta}) \to (\alpha, \beta)$ (the *reparametrization map*) such that the inverse map $\phi^{-1}\colon (\alpha, \beta) \to (\widetilde{\alpha}, \widetilde{\beta})$ is also smooth and $\widetilde{\gamma}(\widetilde{t}) = \gamma(\phi(\widetilde{t}))$ for all $\widetilde{t} \in (\widetilde{\alpha}, \widetilde{\beta})$.

Note that since ϕ has a smooth inverse, γ is a reparametrization of $\widetilde{\gamma}$, since $\widetilde{\gamma}(\phi^{-1}(t)) = \gamma(\phi(\phi^{-1}(t))) = \gamma(t)$ for all $t \in (\alpha, \beta)$.

Example 1.5. We can reparametrize the circle as $\widetilde{\gamma}(t) = (\sin t, \cos t)$. To show this, we want to find a reparametrization map ϕ such that $(\cos\phi(t), \sin\phi(t)) = (\sin t, \cos t)$. $\phi(t) = \pi/2 - t$ works.

Definition 1.5. A point $\gamma(t)$ of a parametrized curve γ is called a **regular point** if $\dot{\gamma}(t) \neq \mathbf{0}$; otherwise $\gamma(t)$ is a **singular point** of γ. A curve is **regular** if all of its points are regular.

Proposition 1.3. *Any reparametrization of a regular curve is regular.*

Proof. Suppose $\widetilde{\gamma}$ is a reparametrization of γ, let $t = \phi(\widetilde{t})$ and $\psi = \phi^{-1}$ such that $\widetilde{t} = \psi(t)$. Differentiating both sides of $\phi(\psi(t)) = t$ WRT t gives $\frac{d\phi}{d\widetilde{t}}\frac{d\psi}{dt} = 1$. So $d\phi/d\widetilde{t}$ is never zero. Since $\widetilde{\gamma}(\widetilde{t}) = \gamma(\phi(\widetilde{t}))$, differentiating again gives $\frac{d\widetilde{\gamma}}{d\widetilde{t}} = \frac{d\gamma}{dt}\frac{d\phi}{d\widetilde{t}}$, so $d\widetilde{\gamma}/d\widetilde{t}$ is never zero, if $d\gamma/dt$ is never zero. ⊠

Proposition 1.4. *If $\gamma(t)$ is regular, then s is a smooth function of t.*

Proof. Recall that $\frac{ds}{dt} = \|\dot{\gamma}(t)\| = \sqrt{\dot{u}^2 + \dot{v}^2}$. Since $f(x) = \sqrt{x}$ is *smooth* on $(0, \infty)$, along with u and v, and $\dot{u}^2 + \dot{v}^2 > 0$ for all t (since γ is regular), s itself is also smooth. ⊠

Proposition 1.5. *A parametrized curve has a unit-speed reparametrization iff it is regular.*

Proof. Suppose a parametrized curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ has a unit-speed reparametrization $\widetilde{\gamma}$, with a reparametrization map ϕ. Letting $t = \phi(\widetilde{t})$, we have $\widetilde{\gamma}(\widetilde{t}) = \gamma(t)$ and so

$$\frac{d\widetilde{\gamma}}{d\widetilde{t}} = \frac{d\gamma}{dt}\frac{dt}{d\widetilde{t}} \implies \left\|\frac{d\widetilde{\gamma}}{d\widetilde{t}}\right\| = \left\|\frac{d\gamma}{dt}\right\| \left|\frac{dt}{d\widetilde{t}}\right|.$$

Since $\widetilde{\gamma}$ is unit speed, $\|d\widetilde{\gamma}/d\widetilde{t}\| = 1$, so $d\gamma/dt$ cannot be zero. ⊠

Lecture 2

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We start by talking about curves in space. Differential geometry is about infinitesmal stuff, tangent lines, things like that. Curvature is about approximating things by the radius of a circle, it's pretty intuitive. After curves, we get into surfaces. Geodesics are like the shortest way to connect two points, a locally length-minimizing curve. We have extrinsic and intrisic curvature, which depend and don't depend on embeddings. The natural next step after curves and surfaces is Riemannian geometry (woohoo).

2.1 Curves

We have two kinds of curves: level curves and parametrized curves. A **parametrized curve** is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for example, $\gamma(t) = (t^2, t^3)$. My take on open vs closed intervals: paths take one point to another, while curves describe a, well, curve in $\mathbb{R}^2$. They don't necessarily have to start somewhere or end somewhere, and aren't necessarily compact of course.

A **level curve** is (informally) something of the form $f^{-1}(x_0)$ where $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $x_0 \in \mathbb{R}^{n-1}$. We usually study the special case $n = 2$.

Example 2.1. Precisely, $f^{-1}(x_0) = \{y \in \mathbb{R}^n \mid f(y) = x_0\}$. Take $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^3 + y^3 - 3xy$, this is called the *Foliom of Descartes*.

Usually in this course we study parametrized curves, since they're easy to compute arc length ($\int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt$ and curvature. Meanwhile, level curves are good for applications, as they arise naturally as graphs of functions. If $\gamma : \mathbb{R} \to \mathbb{R}^n$, we have $\gamma(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t))$, where $\gamma_1 : \mathbb{R} \to \mathbb{R}$, $\gamma_2 : \mathbb{R} \to \mathbb{R}$, and so on. Then the derivative is given by the n-tuple

$$\dot{\gamma} = \gamma' = \frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \cdots, \frac{d\gamma_n}{dt}\right).$$

We say γ is **smooth** if $\frac{d^n \gamma}{dt^n}$ exists for all $n \geq 0$. We don't really care about curves that aren't smooth.

2.2 Tangent Vectors

We have $\gamma'(t)$ the **tangent vector** at time t. The **tangent line** at time t is $\{\gamma(t) + u\gamma'(t) \mid u \in \mathbb{R}\}$, the direction is much more important than the magnitude (speed). The **speed** of γ at time t is $\|\gamma'(t)\|$. The **arc length** of γ from time t_0 is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du.$$

Integrating over speed gives distance traveled, which is arc length.

Example 2.2. If $\gamma(t) = (t^2, t^3)$, the length from zero to one is $s(1) = \int_0^1 \sqrt{4u^2 + 9u^4} \, du = \int_0^1 u\sqrt{4 + 9u^2} \, du = \frac{(4+9u^2)^{3/2}}{27}\Big|_0^1 = \text{blah}$.

Note that we can differentiate dot products. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\lambda : \mathbb{R} \to \mathbb{R}^n$. Then $\phi : \mathbb{R} \to \mathbb{R}, \phi(t) = \langle \gamma(t), \lambda(t) \rangle$. How do you compute $\frac{d\phi}{dt}$? It's the product rule, $\frac{d\phi}{dt} = \frac{d\gamma}{dt} \cdot \lambda + \gamma \cdot \frac{d\lambda}{dt}$.