Riemannian Geometry Notes

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What is Riemannian geometry?? Consider $\{(x, y) \mid x^2 + y^2 < 1\}$, this is a coordinate chart. This doesn't tell us anything about the geometry of the surface, since $z = \sqrt{1 - x^2 - y^2}$ and $z = x^2 - y^2$ have the same coordinates. Geometry is extra data on a surface, besides the topological data. Our basic extra ingredient is the inner product, which allows us to talk about length, which tells us all sorts of things about the manifold.

1.1 Introduction to curvature

> "In fact, with the inherited inner product on $\mathbb{R}^2$, this curved string is *not* curved!"

The map sending a curved string to a straight line is an isometry. So there are two different notions of curvature: extrinsic curvature talks about to what extent is something bent relative to something else, and intrinsic talks about what happens if you look locally. Locally, all 1-manifolds are isometric to $\mathbb{R}$.

Now to 2-manifolds. We have the circle, half-sphere, saddle, and half-cylinder discussed earlier. We can't really tell the difference between a cylinder and the plane, they're isometric. Extrinsically they're different, but intrinsically they're the same. The sphere is different— a circle has circumference $2\pi R$, that is, the circumference is the set of points of distance R away. What is the circumference of a sphere? Use spherical coordinates: a circle of radius R is a latitude line, and the length is 2π times the distance of the great circle cut out at R, or $2\pi\sin(R)$ if you draw out the angles. Approximately, $2\pi\sin(R) \approx 2\pi(R - \frac{R^3}{6})$. So circles are too small!

This means that spheres are in some sense, sphere shaped. Another thing is the area: the area of the cap is $A = \int 2\pi\sin(R)dR = 2\pi(1 - \cos(R)) = 2\pi(\frac{R^2}{2} - \frac{R^4}{4!} + \cdots) = \pi R^2 - \frac{\pi R^4}{12} + \cdots$. So circumferences and areas are a little bit too small. If we worked in $z = x^2 + y^2$, we would find that circumferences and areas would be a little bit too big. This is why you can't flatten and orange peel, or an accurate scaled map of the world preserving angles and area.

1.2 Dual Space

Suppose V is an n-dimensional vector space, with basis $\mathscr{E} = \{e_1, \cdots, e_n\}$. Then we can write any $v \in V$ as the sum $\sum v^i e_i$. So we have a natural correspondence between vectors v and coordinates $\{v^i\}$ where $v^i \in \mathbb{R}^n$. From now on, we use $v^i e_i$ to denote $\sum v^i e_i$, this is called Einstein summation notation.

The dual space $V^* = \mathrm{Hom}(V, \mathbb{R})$ is the space of linear functionals from V into the base field, the reals. One element of V^* is the function that assigns each vector to its ith coordinate $\phi^i(v) = v^i$. So we have a nice set of transformations $\{\phi^1, \phi^2, \cdots, \phi^n\}$.

Claim. The set $\{\phi^1, \phi^2, \cdots, \phi^n\}$ forms an n-dimensional basis for V^*, called the *dual basis* to $\mathscr{E}$.

Proof. Let $\alpha = \alpha_i\phi^i$. Suppose $\alpha = 0$, then for all $v, \alpha(v) = 0$. So $\alpha(e_j) = 0$, and $\alpha_i\phi^i(e_j) = 0$. Now $\phi^i(e_j) = \delta^i_j$, so $\alpha_i\delta^i_j = 0$ and therefore $\alpha_j = 0$. Now define $\alpha_j = \alpha(e_j)$. Applying this to a vector v gives $(\alpha_j\phi^j)(v) = \alpha_j(\phi^j(v)) = \alpha_j v^j$. Then $\alpha(v) = \alpha(v^j e_j) = v^j\alpha(e_j) = v^j\alpha_j$, and these are equal. We conclude that $\alpha = \alpha_j v^j$. ⊠

Summary: write arbitrary elements of V as $v^i e_i$, and the dual space as $\alpha = \alpha_j\phi^j$. We have $\phi^j(v) = v^j, \alpha(e_i) = \alpha_i$, and $\alpha(v) = \alpha_i v^i$. This is why we call V^* the dual space: pairing elements together in either order gives a number. On this vein, $V^{**} = V$, where the basis $\{e_1, \cdots, e_n\}$ is dual to $\{\phi^1, \cdots, \phi^n\}$. V is the space of contravariant vectors, while V^* is the space of covectors, which are covariant in a categorical sense.

Suppose we have a new basis $\widetilde{\mathscr{E}} = \{\widetilde{e}_1, \cdots, \widetilde{e}_n\}$. There must be some change of basis matrix, that is, $\widetilde{e}_i = A^j_i e_j$, similarly we have a new dual basis with $\widetilde{\phi}^i = B^i_j\phi^j$. If we have a vector $v = v^i e_i = \widetilde{v}^i\widetilde{e}_i$, we have $\widetilde{v}^i = \widetilde{\phi}^i(v) =$

$B^i_j\phi^j(v) = B^i_j v^j$. When your basis vectors get big, your coordinates get small, so they transform oppositely. This is the origin of the term contravariant. We claim that $A^j_i B^k_j = \delta^k_i = B^j_i A^k_j$. To see this, note that $\delta^k_i = \widetilde{\phi}^k(\widetilde{e}_i) = \widetilde{\phi}^k(A^j_i e_j) = \phi^\ell B^k_\ell(A^j_i e_j) = B^k_\ell A^j_i \delta^\ell_j = B^k_j A^j_i = A^j_i B^k_j$.

We can kinda visualize vectors as columns and covectors as rows. Then e_i is a column with 1 in the ith while ϕ^j is a row in the jth slot. Applying α to e_i gives α_i, while applying ϕ^j to v gives v^i.

1.3 Tensors

A (k,ℓ) tensor eats k-vectors and ℓ-covectors, and pops out a number. These should be multilinear in each slot. In terms of coordinates, $T(v,w) = T(v^ie_i, w^je_j) = v^iw^iT(e_i,e_j)$. We define $T_{ij} = T(e_i,e_j)$, so $T(v,w)$ becomes $T_{ij}v^iw^j$. This is a $(2,0)$-tensor. What is a $(0,2)$-tensor? This is essentially a matrix, where $S(v,\alpha) = S^j_j v^i\alpha_j$. If you don't give a tensor enough information, it turns into a tensor of lower rank, which works the same in matrix multiplication. If you have a doubly covariant tensor, it turns vectors into covectors. If you have a doubly contravariant tensor, it turns covectors into vectors. Next time, we'll talk about the most important doubly covariant tensor, the inner product.

Lecture 2

January 21, 2021

2.1 A Basis for Tensors

Let's continue the algebra from yesterday. Recall a tensor takes two two vectors as input, denoted $T(v,w) = T(v^ie_i, w^je_j) = v^iw^jT(e_i,e_j) = \left(\sum_{i,j}\right) T_{ij}v^iw^{ij}$. If $V = \mathbb{R}^2$, what's an example of a covariant 2-tensor? The standard inner product $\langle v \mid w\rangle = v^1w^1 + v^2w^2$ works. Using the notation $g(v,w)$, we have $g_{11} = g(e_1,e_1) = 1, g_{12} = 0, g_{21} = 0, g_{22} = 1$. In general, for an inner product $g_{ij} = g_{ij}$, and if you think of as a matrix, this will be a positive definite matrix, or $g_{ij}v^iv^j \geq 0$ if $v \neq 0$ (all eigenvectors are positive). For example, the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with positive eigenvectors} \quad \begin{bmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

works. Another interesting tensor is the determinant $A(v,w) = v^1w^2 - v^2w^1 = -A(w,v)$. In this case, $A_{12} = -A_{21} = 1$ and $A_{11} = A_{22} = 0$.

The natural next question is: what is the space of covariant 2-tensors on $\mathbb{R}^2$? This is a vector space, what would the basis be? It's denoted $\phi^i \otimes \phi^j(e_k,e_\ell)$. Given two tensors $S(v_1,\cdots,v_n)$ and $T(w_1,\cdots,w_m)$, we define $S\otimes T(v_1,\cdots,v_n,w_1,\cdots,w_m) := S(v_1,\cdots,v_n)T(w_1,\cdots,w_m)$. So $\phi^i\otimes\phi^j(e_k,e_\ell) =$?? missed this portion. So $T = T_{ij}\phi^i\otimes\phi^j$, and $T(e_k,e_\ell)$.

Consider the inner product g, we can think of it as taking a vector and turning it into a covector. We have $g(v,\cdot)(v) \neq 0$, so g induces a map onto the dual space $g : V \to V^*$, which is an isomorphism. So $\alpha = g(v,\cdot), \alpha_j = g_{ij}v^i$. You can think of this as lowering the indices. The matrix that *raises* the indices is the inverse matrix, denoted g^{ij}, where $g^{ij}g_{jk} = \delta^i_k$.

Now that we have a basis for 2-tensors, what about a basis for $\tau^{k,\ell}$ (space of tensors)? It's going to be

$$\{\phi^{i_1}\otimes\phi^{i_2}\otimes\cdots\otimes\phi^{i_k}\otimes e_{j_1}\otimes\cdots\otimes_{j_\ell}\}.$$

So $T = T^{j_1\cdots j_\ell}_{i_1\cdots i_k}\phi^{i_1}\otimes\cdots\otimes e_{j_\ell} = T(e_{i_1},\cdots,e_k,\phi^{j_1},\cdots,\phi^{j_\ell})$ this may be wrong

2.2 Trace of a Matrix

Now let's talk about the trace of a matrix, defined as $\mathrm{Tr}(M) = M^i_i$. If we change our basis by $\widetilde{e}_i = A^j_ie_j$, recall that this gives rise to a dual basis $\widetilde{\phi}^i = B^i_j\phi^j$. In our new basis, $\widetilde{M}^i_j = B^i_kA^\ell_jM^k_\ell = B^i_kM^k_\ell A^\ell_j$, this is kind of like PMP^{-1}

that we did in linear algebra. But this doesn't really work with tensors, what about $2,3,4,5$-tensors? That's why we're doing it this way.

We have the trace $\text{Tr} = \widetilde{M}_i^i = B_k^i M_\ell^k A_i^\ell$. But $B_k^i A_i^\ell = \delta_k^\ell$, since these two are inverse matrices. So $\text{Tr} = \delta_\ell^k M_\ell^k = M_k^k$, which is the old definition of trace. We can apply this to higher rank tensors: suppose we have a tensor that takes in a covector and two vectors, denoted $T(\alpha, v, w)$. Given a tensor $S(w) = T(\phi^i, e_i, w)$, so $S_i = T_{ji}^j$. What happens if you change basis, that is, is $T(\phi^i, e_i, w) = T(\widetilde{\phi}^i, \widetilde{e}_i, w)$?

$$\langle \textit{breakout rooms} \rangle$$

For fixed w, let's define a new tensor, $U(\alpha, v) = T(\alpha, v, w)$. Let's take the trace of this, we've already shown that the trace of a $(1,1)$-tensor doesn't depend on basis. So the w comes along for the ride. This is a slick solution, since $\text{Tr}\, U$ is well defined.

2.3 Tangent Vectors

Enough about about tensors. Now let's talk about manifolds. Usually the vector space at a point p we care about is the tangent space $T_p(M)$ at that point. What's a tangent vector? In $\mathbb{R}^n$, consider a point p: what is a tangent vector there?

1. An arrow, add them head to tail.

2. An element of $\mathbb{R}^n$, a list of n numbers. This is pretty much an arrow, just take the coefficients and impose them on the standard basis.

3. The equivalence class of curves $\gamma(t)$ with $\gamma(0) = p$.

4. Velocity. Consider all possible parametrized curves through a point, and identify all curves with the same velocity at time zero. So we mean the equivalence class of curves $\gamma(t)$ with $\gamma(0) = p$. The beauty of this third definition is it makes sense on any manifold. So we can sonsider the curves going through this point, and take them up to equivalence.

5. Directional Derivative. For $f : \mathbb{R}^n \to \mathbb{R}$, we can take $\frac{df \cdot \gamma(t)}{dt}\Big|_{t=0} = \frac{d\gamma^i}{dt}\Big|_{t=0} \frac{\partial f}{\partial x^i}\Big|_p$. The partial derivaties $\{\partial_1, \cdots, \partial_n\}$ gives a basis for this vector space.

6. Derivations. A derivation at p is a map $D \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ with the following properties:

   (1) $D(af + bg) = aD(f) + bD(g)$,

   (2) $D(fg) = f(p)D(g) + D(f)g(p)$.

   The partial derivatives $\partial_i|_p$ are derivations.

Claim. We have $\{\partial_i\}$ a basis for the set of derivations at p, denoted $\mathscr{D}_p(\mathbb{R}^n)$.

This idea of thinking about tangent vectors as derivations carries over very nicely to abstract manifolds, which is why we care. We want to show a couple of things:

1. To show $D(\text{constant}) = 0$, note that $D(cf) = cD(f) = cD(f) + f(p)D(c)$, so we need D of a constant to be zero.

2. If $f(p) = g(p) = 0$, then $D(fg) = 0$, this follows from the product rule.

3. Taylor series, we have $f(x) = f(p) + \partial_i f(p)(x^i - p^i) +$ higher order terms. Then $D(f) = 0 + \partial_1 f(p) D(x^i) + 0 = D^i \partial_1 f(p)$, where $D = D^i \partial_i$.

This works for analytic functions. There is some cheating going on, but we don't need the entire Taylor series for the most part. So every derivation is a linear combination of partial derivatives. From now on, think of vectors as a combination of partial derivatives, or inducing curves along a vector field.

A vector field $V^i(x)\frac{\partial}{\partial x^i}$ is a bunch of coefficients in combination with the basis vectors of partial derivatives. How do we change coordinates here? We do this by the chain rule, that is,

$$\text{for } \{x\} \longleftrightarrow \{y\} \quad \text{we have} \quad \frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}.$$

Let's try this in $\mathbb{R}^2$ with $(r,\theta) \leftrightarrow (x,y)$. For $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$ and $\widetilde{e}_1 = \frac{\partial}{\partial r}, \widetilde{e}_2\frac{\partial}{\partial \theta}$, recall that $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, and $x = r\cos\theta$, $y = r\sin\theta$. So

$$\frac{\partial y}{\partial r} = \cos\theta = \frac{x}{\sqrt{x^2+y^2}},\ \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{\sqrt{x^2+y^2}},\ \frac{\partial x}{\partial \theta} = -r\sin\theta = -y,\ \frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

So now we can convert between the two bases, that is, we have the change of basis matrices

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix}.$$

Lecture 3

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What is the tangent vector of a point on an arbitrary manifold, like $\mathbb{C}\mathrm{P}^2$? Arrows? No. Equivalence class of curves? This works for manifolds. Consider a curve through p, and another curve that looks the same locally (we have to apply the fact that manifolds look locally like $\mathbb{R}^n$). Derivations also work perfectly.

3.1 Basis for a Tangent Space

We know what a basis is for tensors, which comes from a basis for the dual space. What's the basis for the dual space of tangent vectors? Suppose we have a function $f : M \to \mathbb{R}$. Then $df(V) := v(f)$. Looking at $\mathbb{R}^n$, we have $(df)\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial f}{\partial x^1}$, and similarly $(df)\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i} = \partial_j f$. So $(df) = \frac{\partial f}{\partial x^j}\phi^j$. Note that we have $(dx^i)\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$. So on a manifold with coordinates $x^1, \cdots, x^n$, the basis for $T_p(M) = \{\frac{\partial}{\partial x^i}\}$. The basis for the dual space $T_p^*(M) = \{dx^i\}$.

We deal with a particular 2-covariant tensor all the time, which is the metric. If we have the notion of an inner product at every point, with a metric $g = g_{ij}dx^i \otimes dx^j$, where $g_{ij}^{(x)} = g(\partial_i, \partial_j)_x = \langle \partial_i, \partial_j \rangle_x$. We will spend a ridiculous amount of time talking about this tensor at a point x. Let's play around with the manifold of the upper hemisphere, given by $\{(x,y,z) \mid x^2+y^2+z^2 = 1, z > 0\}$. Some possibilities for coordinates:

1. x, y, where $z = \sqrt{1-x^2-y^2}$,
2. θ, ϕ, where θ measures the angle from the north pole, and ϕ measures the longitude. So $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$.

How do we find a metric? We know $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = \frac{-x(dy/dt)-y(dy/dt)}{\sqrt{1-x^2-y^2}}$. So we get the vectors $(1,0,-\frac{x}{z}), (0,1,-\frac{y}{z})$. So $g_{11} = 1+\frac{x^2}{z^2} = 1+\frac{x^2}{1-x^2=y^2} = \frac{1-y^2}{1-x^2-y^2}$, $g_{12} = \frac{xy}{z^2} = \frac{xy}{1-x^2-y^2}$, $g_{21} = g_{12}, g_{22} = \frac{1-x^2}{1-x^2-y^2}$.

Now lets move onto spherical coordinates. This is a bunch of trig derivatives, look at the recorded lecture in your free time.

Lecture 4

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4.1 Metrics

How to make a metric?? One option is to live inside a space with an existing metric, particularly a surface in $\mathbb{R}^3$ or something like that. (Missed a big chunk of discussion). Given two manifolds (M_1, g_1) and (M_2, g_2) with coordinates $(x^1, \cdots, x^n)$, $(y^1, \cdots, y^m)$, we have $g_1 = (g_1)_{ij} dx^i \otimes dx^j$, $g_2 = (g_2)_{k\ell} dy^k \otimes dy^\ell$. Then we can define the product manifold $M_1 \times M_2, (p, q)$, with coordinates $x^1, \cdots, x^n, y^1, \cdots, y^m)$, $T_{(p,q)}(M_1 \times M_2) = T_p M_1 \times T_q M_2$, missed some more stuff.

Example 4.1. The torus is the product of two circles, $\mathbb{T}^2 = S^1_{R_1} \times S^1_{R_2}$. The metric is given by $g_1 = R_1^2 d\theta^2 + R_2^2 d\theta^2$.

Definition 4.1. Suppose we have a function $f : M_1 \to \mathbb{R}^+$. Then $M_1 \times_{f_2} M_2 = (M_1 \times M_2, g)$, $g_{(p,q)} = (g_1)_{ij} dx^i \otimes dx^j + (f(p))^2 (g_2)_{k\ell} dy^k \otimes dy^\ell$. This is a warped product.

Another way is by quotients by discrete actions. Consider $\mathbb{R}/L\mathbb{Z}$, or the set $\{x \in \mathbb{R}\}/\sim$, $x \sim (x + nL)$. If G acts freely on (X, g) by isometries, then X/G inherits a metric from X. In this case, it's a circle. When would this fail?

Example 4.2. Suppose $X = (\mathbb{R}^+, dx^2)$, and $G = \mathbb{Z}$, $(n, x) \to z^n x$. Another example is $X = \mathbb{R}^2$, $G = \mathbb{Z}_2$.