Abstract Algebra Lecture Notes

Simon Xiang

Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html`.

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§1 September 14, 2020

§1.1 Class Introductions (not math)

Zoom classes suck: time for brief introductions. About Dr. Ciperiani: Number Theory, Elliptic Curves, Princeton, Albania, Smith $\implies$ France, Colombia, MSRI, UT! Swimming and Traveling, two kids (2 and 6).

CJ (Christopher Izzo), undergrad, math major, algebraic geometry (number theory)? Wyoming, cooking, hiking, dog

Ella

ALlie Embry (Indiani) outside is nice

Zhou Fang (Umich) first year PHD (representation theory)

Jacob Wood (sophomore), dallas, on campus

auditing dallas, freshman, dorm, running and reading/dog

Luis (from berkeley)

Luke T Timmerman (also a freshman) yay

Xiantao Chen (physics phd)

Elijah (sophomore)

I've omitted the rest of the personal introductions for privacy, but all my class mates are very interesting and cool people.

§1.2 Sylow Theory

Last time: Sylow Theorems. Let p be a prime.

Theorem 1.1. *Let G be a group of order $p^r m$ where $p \nmid m, r \in \mathbb{N}$. Then*

1. *Sylow p-subgroups exist,*
2. *They are all conjugate,*
3. *Every p-subgroup of G lies in some Sylow p-subgroup of G,*
4. *Let $n_p :=$ the number of Sylow p-subgroups of G, P be a Sylow p-subgroup. Then $n_p = [G : N_G(P)]$, where $N_G(P)$ is the normalizer of P in G. In particular, $n_p \mid m = [G : P]$.*
5. *$n_p \equiv 1 \pmod p$,*
6. *$n_p = 1$ if and only if $P \trianglelefteq G$.*

We introduce our key lemma:

Lemma 1.1. *Let P denote any maximal p-subgroup of G, $N = N_G(P)$. If Q is any p-subgroup of N, then $Q \subseteq P$. Consequently, $p \nmid [N : P]$.*

Proof. Consider the map
$$\pi\colon N \to \overline{N} = N/P.$$

Then $\pi(Q) = \overline{Q}$. Q a p-subgroup $\implies$ $|\overline{Q}| = p^m$. $\pi^{-1}(\overline{Q}) = QP \supseteq P$, $|QP| = |\overline{Q}| \cdot |P| = p^m \cdot |P| \implies QP = P$, P is maximal. So QP is a p-subgroup, $p^m = 1$, $p \mid [\overline{N} : P] \implies p \mid |N/P| \implies$ by Cauchy's Lemma that there exists a $g \in N/P$ such that $\operatorname{ord} g = p$. Take $\pi^{-1}\langle g\rangle = \langle P, g\rangle$, $|\pi^{-1}\langle g\rangle| = p|P|$. $\pi\colon \langle P, g\rangle \to \langle g\rangle$, $\ker \pi|_{\langle P,g\rangle} = P$.

$$O \to \underset{\ker \pi}{P} \to \langle P, g\rangle \xrightarrow{\pi} \underset{\operatorname{im} \pi}{\langle g\rangle} \to O$$

implies

$$kP, g > | = |\operatorname{im} \pi| \cdot |\ker \pi| = p|P|,$$

$\langle P, g\rangle \not\supseteq P$, since P is maximal. ☒

Now let's prove the theorem.

Proof. Let P be a maximal p-subgroup of G. We have

$$X = \{gPg^{-1} \mid g \in\}$$

Observe that

1. $|X| = [G : N_G(P)]$,
2. Every element of X is a maximal p-subgroup of G, $gPg^{-1} \not\subseteq Q$ a p-subgroup $\implies P \not\subseteq g^{-1}Qg$ which is false since P is a maximal p-subgroup,

Fine. (One of Dr. Ciperiani's (lovingly) idiosyncracies). We have P acting on X by conjugation. The only fixed point of X under the action of P is P, ie, $X^P = P$.

Claim. If $gPg^{-1} \in X^p \iff P \subseteq N_G(gPg^{-1})$, ie, $h(gPg^{-1})h^{-1} = gPg^{-1}$ for all $h \in P$, then (??) $P = gPg^{-1}$.

The first claim said that $|X^P| = |\{p\}| = 1$. Nontrivial orbits of X under the action of P have size dividing $|P|$. This implies that the size is equal to p^k for some $k \in \mathbb{N}$. $|X| = |X^P| + \sum$ sizes of distinct larger orbits, all of which are powers of p. Since $X^P| = 1$, we have $|X| \equiv 1 \pmod{p}$. Whoops, we're a little overtime. We have one more claim to prove before completing the proof of the Sylow Theorems, then we will be done. ☒