Abstract Algebra Lecture Notes

Math 380C

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1 Lecture 1

Unfortunately, I couldn't attend.

2 Lecture 2: Quotient Groups (8/28/20)

Lemma 2.1. *Let $H \subseteq G$, $\langle G, \cdot \rangle$ a group and $H \neq \emptyset$. Then H is a subgroup of G if and only if $h_1 h_2 \in H \implies h_1 h_2^{-1} \in H$.*

Proof. For all $h_2 \in H$, $h_2^{-1} \in H$ since H is a group. H is closed under multiplication implies $h_1 h_2^{-1} \in H$ for all $h_1, h_2 \in H$. Conversely, assume that $h_1 h_2 \in H \implies h_1 h_2^{-1} \in H$. Then for $h \in H$, $hh^{-1} \in H$ so $1 \in H$. Now that we know $1 \in H$, then for $h \in H$ we have $1 \cdot h \in H \implies h^{-1} \cdot 1 \in H$, so H is closed under inverses. Finally, associativity follows from the fact that $H \subseteq G \implies \forall h \in H, h \in G$ where G is a group, and we are done. $\boxtimes$

Definition 2.1 (Normal Subgroup)**.** A *subgroup* H of G is normal if $gHg^{-1} = H$ for all $g \in G$.

Example 2.1. Let G be abelian: then every subgroup is normal since $ghg^{-1} = gg^{-1}h = h$ for all $g \in G, h \in H$.

Example 2.2. Take $G = S_3$. Then the subgroup $\langle (123) \rangle$ is normal. However, the subgroup $\langle (1,2) \rangle$ is not normal, since $(13)(12)(13)^{-1} = (23) \notin \langle (12) \rangle$.

Example 2.3. Take $\mathrm{SL}_n\mathbb{R} \subseteq \mathrm{GL}_n\mathbb{R}$, where $\mathrm{SL}_n\mathbb{R}$ is the set of matrices with $\det(A) = 1$ for $A \in \mathrm{SL}_n\mathbb{R}$. We know $\mathrm{SL}_n\mathbb{R}$ forms a subgroup. Question: is $\mathrm{SL}_n\mathbb{R}$ normal? Answer: yes.

$$\det(ABA^{-1}) = \det(A)\det(B)\det(A^{-1}) = \det(A)\det(A)^{-1}\det(B) = \det(B).$$

Proposition 2.1. Let H, K be subgroups of G, then $H \cap K$ is a subgroup of G. You can verify this in your free time.

Note: is $H \cup K$ a subgroup? No!

Definition 2.2 (Product Groups)**.** Let G, H be groups. We define the *direct product* $G \times H$ with the group operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. The identity is just $(1_G, 1_H)$ where 1_G and 1_H denotes the respective identities for G and H. Finally, the inverse is similarly defined as (g_1^{-1}, h_1^{-1}) where g_1^{-1} and h_1^{-1} are the respective inverses for $g_1 \in G$, $h_1 \in H$.

Some examples of product groups include $\mathbb{Z} \times \mathbb{Z}$ ($\mathbb{Z}$ denotes $\langle \mathbb{Z}, + \rangle$), and $\mathbb{Z} \times \langle \mathbb{R} \setminus \{0\}, \cdot \rangle$

Example 2.4 (Quotient Groups)**.** Let $n \in \mathbb{Z}$, for example $\langle \mathbb{Z}/n\mathbb{Z}, + \rangle$, equivalence relations: modulo n. $a, b \in \mathbb{Z}, a \equiv b \mod n \iff n \mid (a - b)$. Equivalence classes: $a + n\mathbb{Z} = \{a + nk \mid k \in \mathbb{Z}\}$. Notation: $\bar{a} = a + n\mathbb{Z} = [a]$. Our set $\mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} \mid a \in \mathbb{Z}\} = \{a + n\mathbb{Z} \mid a = 0, ..., n-1\}$. $(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z}$, so this is a group operation. In this case, the identity is just $0 + n\mathbb{Z} = n\mathbb{Z}$. We have the inverse of $(a + n\mathbb{Z})$ equal to $(a + n\mathbb{Z})^{-1} = -a + n\mathbb{Z}$.

Remark: $\langle \mathbb{Z}/n\mathbb{Z}, +\rangle$ is a quotient of the group $\langle \mathbb{Z}, +\rangle$ by the subgroup $\langle n\mathbb{Z}, +\rangle$. $\langle 1\rangle = \mathbb{Z}, \langle 1 + n\mathbb{Z}\rangle = \langle \mathbb{Z}/n\mathbb{Z}\rangle$.

Quotient groups in general: G a group, H a **normal** subgroup.

Definition 2.3 (Cosets)**.** Left cosets: $gH = \{gh \mid h \in H\}$. Right cosets: $Hg = \{hg \mid h \in H\}$. G/H - set of left cosets. $H \setminus G$ - set of right cosets.

Observe: Left and right cosets are in bijection with one another. $gH \mapsto Hg$, $gh \mapsto g^{-1}(gh)g = hg$. You can verify that this is a bijection. Let $g_1, g_2 \in G$, what map maps $g_1H \to g_2H$? $g_1h \mapsto (g_2g_1^{-1})g_1h = g_2h$.

Note. We have

$$\bigcup_{g \in G} gH = G.$$

Also: $g_1H \cap g_2H$ is either $\emptyset$ or they are equal. (Equivalence relation).

Proposition 2.2. If G is finite and H a subgroup of G, then $|H| \mid |G|$.

Proof. By the statement above,

$$G = \bigcup_{i=1}^{n} g_iH$$

since G is finite for $n \in \mathbb{N}$. Note that this is a disjoint union. So

$$|G| = \sum_{i=1}^{n} |g_iH| = n \cdot |H| \implies |H| \mid |G|.$$

$\boxtimes$

Quotient group: G a group, H a normal subgroup, $G/H = \{gH \mid g \in G\}$. The multiplication is defined as $g_1H \cdot g_2H = g_1g_2H$. You can verify this operation is well defined (given that H is normal).