An introduction to de Rham cohomology

How algebra and calculus relate to topology

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Motivation

Question

Does there exist a function that is the gradient of some other function? More precisely, when does there exists an $F\colon U \to \mathbb{R}^2$ for some open $U \subseteq \mathbb{R}^2$ that satisfies

$$\frac{\partial F}{\partial x} = f_1, \quad \frac{\partial F}{\partial y} = f_2 \quad \text{for a vector field } f = (f_1, f_2)?$$

(You could also think of this question as asking when vector fields have potential.)

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Answer

It depends on the topology of U!

Some vector calculus

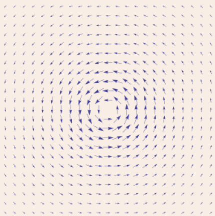
Note that $\frac{\partial F}{\partial x} = f_1$, $\frac{\partial F}{\partial y} = f_2$ implies $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. Is this condition sufficient to show F is the gradient of some other function?

Example

The vector field $f(x, y) = (y, x)$ satisfies $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 1$, and the gradient of $F = xy$ is f.

Example

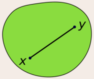
However, consider $f(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. This vector field cannot be conservative, which you can show by integrating around a closed loop.



Div, grad, curl

Proposition

The condition $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$ is sufficient for the existence of a conservative vector field F if U looks like a ball (convex).



Definition

Define $C^\infty(U, \mathbb{R}) \overset{\text{grad}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{curl}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{div}}{\to} C^\infty(U, \mathbb{R})$,

$$\begin{aligned} \text{grad}\colon f &\mapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right), \\ \text{curl}\colon (f_1, f_2, f_3) &\mapsto \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right), \\ \text{div}\colon (f_1, f_2, f_3) &\mapsto \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}. \end{aligned}$$

Sneak peek of de Rham cohomology

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This implies that $\operatorname{im}(\operatorname{grad}) \subseteq \ker(\operatorname{curl})$, so we can form the *quotient space* $H^1(U) := \ker(\operatorname{curl}) / \operatorname{im}(\operatorname{grad})$.

Definition

The space $H^1(U)$ is the **1st de Rham cohomology group** of U.

Now the aforementioned proposition is equivalent to saying $H^1(U) = 0$ whenever U is convex. Here's why:

- All we need to show is that $\ker(\operatorname{curl}) \subseteq \operatorname{im}(\operatorname{grad})$.

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- In two dimensions, $f \in \ker(\operatorname{curl})$ means $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$.
- So $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, and we know what to do from here!

More on de Rham cohomology

Proposition

$\mathrm{div} \circ \mathrm{curl} = 0$.

Definition

Since the composition $\mathrm{div} \circ \mathrm{curl}$ is zero, we can also form the **2nd de Rham cohomology group** $H^2(U) := \ker(\mathrm{div}) / \operatorname{im}(\mathrm{curl})$. To fit with the theme, define $H^0(U) = \ker(\mathrm{grad})$.

It turns out de Rham cohomology measures the amount of "holes" in a space. Since $\mathbb{R}^3$ is "completely solid", there should be no nontrivial de Rham cohomology. The following theorem demonstrates this.

A quick summary

Theorem

For a convex open set $U \subseteq \mathbb{R}^3$, we have $H^0(U) = \mathbb{R}$, $H^1(U) = 0$, and $H^2(U) = 0$.

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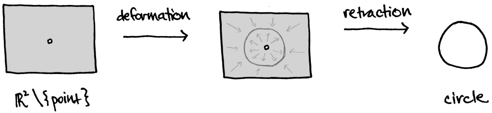

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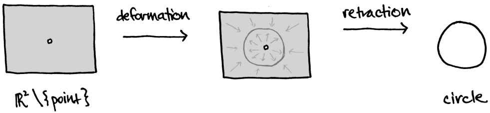

Note: This is not an actual commutative diagram!

$$\begin{tikzcd}
\overset{\ker(\text{grad})}{H^0(U)} \arrow[r] & \overset{\ker(\text{curl})/\operatorname{im}(\text{grad})}{H^1(U) = 0} \arrow[r] & \overset{\ker(\text{div})/\operatorname{im}(\text{curl})}{H^2(U) = 0} \arrow[r] & H^3(U) \\
C^\infty(U, \mathbb{R}) \arrow[u] \arrow[r, "\text{grad}"] & \underset{\text{exact}}{C^\infty(U, \mathbb{R}^3)} \arrow[u] \arrow[r, "\text{curl}"] & \underset{\text{exact}}{C^\infty(U, \mathbb{R}^3)} \arrow[u] \arrow[r, "\text{div}"] & C^\infty(U, \mathbb{R}) \arrow[u]
\end{tikzcd}$$

The de Rham complex

Let's shift gears to a more abstract setting.

Definition

Define Ω^* to be the algebra generated by $dx_1, \cdots, dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0, \\ dx_i dx_j = -dx_j dx_i, \ i \neq j. \end{cases}$$

Then a **differential form** is an element of $\Omega^*(U)$, formally defined as $C^\infty(U, \mathbb{R}) \otimes_{\mathbb{R}} \Omega^*$, where $U \subseteq \mathbb{R}^n$ is open. This algebra has a natural grading $\Omega^*(U) = \bigoplus_{q=0}^{n} \Omega^q(U)$.

Example

Concretely, a form $\omega \in \Omega^q(U)$ can be written uniquely as $\sum f_I dx_I$, where I denotes a strictly increasing sequence of length $q < n$.

The exterior derivative

Definition

Define a *differential operator* $d\colon \Omega^q(U) \to \Omega^{q+1}(U)$ by the following properties:

1. if $f \in \Omega^0(U)$, then $df = \sum \frac{\partial f}{\partial x^i} dx^i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

This operator is called **exterior differentiation.**

Definition

The algebra $\Omega^*(U)$ paired with the differential operator d is called the **de Rham complex** on U.

Example

Let $\omega = xy\, dx \in \Omega^1(\mathbb{R})$ be a 1-form. Then
$d\omega = \left(\frac{\partial(xy)}{\partial x} dx + \frac{\partial(xy)}{\partial y} dy\right) dx = y\, dx\, dx + x\, dy\, dx = -x\, dx\, dy.$
We will see more exterior derivative calculations very soon.

How the exterior derivative generalizes calculus

In $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are one dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are both three dimensional. Then identify

$$\underset{f}{\{\text{functions}\}} \simeq \underset{f}{\{\text{0-forms}\}} \simeq \underset{f\,dx\,dy\,dz}{\{\text{3-forms }\}},$$

$$\underset{X=(f_1,f_2,f_3)}{\{\text{vector fields}\}} \simeq \underset{f_1dx+f_2dy+f_3dz}{\{\text{1-forms}\}} \simeq \underset{f_1dy\,dz-f_2dx\,dz+f_3dx\,dy}{\{\text{2-forms }\}}.$$

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So,

- Taking the exterior derivative of a 0-form (function) gives $\boxed{\textbf{gradient},}$
- The exterior derivative of a 1-form (vector field) is $\boxed{\textbf{curl},}$
- And the exterior derivative of a 2-form is $\boxed{\textbf{divergence}.}$

Example

Consider a 2-form defined by $f_1 dy\, dz - f_2 dx\, dz + f_3 dx\, dy$. Then

$$\begin{aligned} d(\text{2-form}) &= \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz\right) dy\, dz + \cdots \\ &= \frac{\partial f_1}{\partial x} dx\, dy\, dz + 0 + 0 + \cdots \\ &= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz. \end{aligned}$$

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This is precisely $\boxed{\textit{divergence}!}$ Similarly, we also have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \boxed{\text{grad},}$$

$$d(\text{1-form}) = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\, dy + \cdots = \boxed{\text{curl}\,.}$$

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So the exterior derivative generalizes all the previous notions of derivatives from calculus!

The derivative of the derivative is zero

Proposition

$d^2 = 0$, where d denotes the exterior derivative.

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Proof.

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0$$

since mixed partials commute, while the $dx_j dx_i$ anti-commute (the property that $dx_j dx_i = -dx_i dx_j$). $\boxtimes$

This generalizes the previous ideas $\operatorname{curl} \circ \operatorname{grad} = 0,\ \operatorname{div} \circ \operatorname{curl} = 0$, which allowed us to define $H^1(U)$ and $H^2(U)$. Since d is defined in all dimensions, we can define a more general de Rham cohomology group!

Closed and exact forms

Definition

If $d\omega = 0$, then ω is a **closed** form, while if $\omega = d\tau$ for some form τ, we say ω is an **exact** form. Precisely, $\ker d$ consists of all the closed forms, while $\operatorname{im} d$ are the exact forms.

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Definition

The q-th **de Rham cohomology** of U is the space

$$H^q_{DR}(U) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

Since $d^2 = 0$, $\operatorname{im} d \subseteq \ker d$ trivially. Now the question at the beginning of the talk reduces to "can we find a nontrivial closed form on U"? The generalized de Rham cohomology measures to what extent we can do this, by collapsing the trivial solutions to zero.

The cohomology of $\mathbb{R}^n$

Example

Let us compute the cohomology of $\mathbb{R}^1$. $\ker d$ in $\Omega^0(\mathbb{R}^1)$ consists of constant functions, so $H^0(\mathbb{R}^1) = \mathbb{R}$. Every 1-form $\omega = g(x)dx$ is exact, since $d\int_0^x g(u)\,du = \omega$; this implies $H^1(\mathbb{R}^1) = 0$, since we mod out by the entire space. Succinctly, we have

$$H^q(\mathbb{R}^1) = \begin{cases} \mathbb{R}, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$$

More generally, it is true that

$$H^*(\mathbb{R}^n) \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases}$$

This result is called the *Poincaré lemma.*

The Mayer-Vietoris sequence

A map between spaces induces a map on forms, formally stated below:

Remark

Note that a smooth map $f: X \to Y$ induces a **pullback** $f^*: \Omega^0(Y) \to \Omega^0(X), g \mapsto g \circ f$, which naturally extends to a pullback on forms $f^*: \Omega^*(X) \to \Omega^*(Y)$

$$f^* \left(\sum g_I \, dy_{i_1} \cdots dy_{i_q} \right) = \sum (g_I \circ f) \, d(y_{i_1} \circ f) \cdots d(y_{i_q} \circ f)$$

which commutes with d. So assigning the complexes Ω^* to a sequence of maps is a **contravariant functor**.

Suppose our space $X = U \cup V$ for U, V open. Then we have a sequence of inclusions

$$X \leftarrow U \amalg V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

where $U \amalg V$ is the (set-theoretic) disjoint union, and ∂_0, ∂_1 denote inclusions in V, U respectively.

The Mayer-Vietoris sequence

Applying the contravariant functor Ω^* to the sequence of inclusions gives

$$\Omega^*(X) \to \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

Take the difference of the maps to get the **Mayer-Vietoris sequence**

$$0 \longrightarrow \Omega^*(X) \longrightarrow \underset{(\omega,\tau)}{\Omega^*(U) \oplus \Omega^*(V)} \underset{\mapsto}{\longrightarrow} \underset{\tau-\omega}{\Omega^*(U \cap V)} \longrightarrow 0,$$

which turns out to be exact. This induces a long exact sequence on cohomology

$$\cdots \to H^q(X) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(X) \to \cdots$$

So how do we actually use this weird algebraic construction?

The de Rham cohomology of the punctured plane

Example

For $X = \mathbb{R}^2 \setminus \{0\}$, we can cover it with two open sets U, V, whose intersection $U \cap V$ is just two solid chunks of $\mathbb{R}^2$. So

$$H^0(U) \oplus H^0(V) = H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R},$$
$$H^1(U) = H^1(V) = H^1(U \cap V) = 0.$$

Clearly H^2 and above of X, U, V, etc are all zero. Our long exact sequence from Mayer-Vietoris becomes

$$H^0(X) \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1(X) \to 0 \to \cdots$$

Since $\delta\colon (\omega, \tau) \mapsto (\tau - \omega, \tau - \omega)$, $\operatorname{im} \delta$ is 1-dimensional, and so is $\ker \delta$. Then by the first isomorphism theorem, $H^0(X)/0 \cong \ker \delta = \mathbb{R}$, and $H^1(X) \cong \mathbb{R} \oplus \mathbb{R}/(\ker d^* = \operatorname{im} \delta) = \mathbb{R}$. This shows that $\mathbb{R}^2 \setminus \{0\}$ has a nontrivial first cohomology, like expected!

Thank you!

Thank you for listening to my talk, and a special thank you to Arun Debray for mentoring me and answering all my dumb questions! These slides and detailed notes can be found on my website: `https://simonxiang.xyz/math`