Riemannian Geometry Notes

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Notes for the Spring 2021 graduate section of Riemannian Geometry (Math 392C) at UT Austin, taught by Dr. Sadun. The course somewhat follows *Introduction to Riemannian Manifolds* (2^{nd} edition), by Lee. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

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1.1 Geodesics as paths with no acceleration

Recall our three notions of a geodesic.

(1) A geometric path that "locally" minimizes distance,
(2) A parametrized path $\gamma(t)$ that has stationary energy (where $E(\gamma)=\int_0^T g(\dot\gamma,\dot\gamma)\,dt$),
(3) A path with no acceleration; $\nabla_{\dot\gamma}\dot\gamma=0$.

We have already related (1) and (2), and now we want to show (2) and (3) are equivalent, with the Levi-Civita connection. In $\mathbb{R}^n$, $g_{ij}=\delta^i_j$, so $\partial_i g_{jk}=0$. Consider $\gamma^{\text{new}}(t):=\gamma(t)+\delta\gamma(t)$, then we are interested in making $E(\gamma^{\text{new}}-E(\gamma)=\int(\ )\,\delta\gamma+O(\delta\gamma^2)$. This is the same process as taking derivatives: for f a multivariable function, we have $f(x+dx)=f(x)+L(dx)+O(dx^2)$, and $L(dx)$ is called the derivative df. Then taking $\lim_{dx\to 0}(f(x+dx)-f(x)-L(dx))/|dx|$ is precisely what a derivative is. If $f(x+dx)$ is stationary, then $df=\frac{\partial f}{\partial x^i}dx^i=0$, so we set $\delta E=0$.

$$E(\gamma)=\int_0^T g_{ij}\dot\gamma^i\dot\gamma^j\,dt, \tag{1}$$

$$\delta E=\int_0^T 2g_{ij}\dot\gamma^i(\delta\dot\gamma^j)\,dt \tag{2}$$

$$=\underbrace{2g_{ij}\dot\gamma^i\delta\gamma^j\Big|_0^T}_{=0}-\int 2g_{ij}\ddot\gamma^i\delta\gamma^j\,dt \tag{3}$$

From (1) to (2), for a general manifold the g_{ij} will change, since g_{ij} becomes a function of time. But here it stays the same. We get to (3) by integration by parts, and the first component vanishes because $\gamma(0)=p,\ \gamma(T)=q$, so $\delta\gamma(0)=0,\ \delta\gamma(T)=0$. In general, we have

$$0=-2\int g_{ij}\ddot\gamma^i(\delta\gamma^j)\,dt.$$

The only way this integral is zero for all possible values of $\delta\gamma^j$ is if $g_{ij}\ddot\gamma^i$ is zero. Then $g_{ij}\ddot\gamma^i=0$, so a geodesic in $\mathbb{R}^n$ is something with zero acceleration. For a general manifold, since g_{ij} is no longer constant we get a new metric and other terms, which look like

$$g_{k\ell}\ddot\gamma^\ell+(\text{derivatives of } g)\dot\gamma^i\dot\gamma^j=0.$$

Working out these terms is homework, they turn out to be precisely Γ^k_{ij}. For an arbitrary Riemannian manifold, without loss of generality we can assume that we're working in a single chart, which locally is just $\mathbb{R}^n$ with a funny metric.

1.2 Parallel transport

Suppose V is a vector field and γ is a curve. What is $\nabla_{\dot\gamma}V$? (Here ∇ is an arbitrary connection.) Since $V=v^i(x)e_i$, then

$$\begin{aligned}\nabla_{\dot\gamma}V=\nabla_{\dot\gamma}(v^ie_i)&=\dot\gamma(v^i)e_i+v^i\nabla_{\dot\gamma}e_i\\&=\dot\gamma^j(\partial_jv^i)e_i+v^i\dot\gamma^j\nabla_{e_j}e_i\\&=\dot\gamma^j(\partial_jv^i)e_i+\Gamma^k_{ij}v^i\dot\gamma^je_k\\&=\dot v^ie_i+\Gamma^k_{ij}v^i\dot\gamma^je_k.\end{aligned}$$

Definition 1.1. We say V is **parallel** along the path γ if and only if $\nabla_{\dot\gamma} V = 0$.

todo:figure?

Given $V(\gamma(t))$, does there exist a parallel V? We want $0 = \dot v^k e_k + \Gamma^k_{ij} v^i \dot\gamma^j e_k$. In other words, we want $\dot v^k = -\Gamma^k_{ij} v^i \dot\gamma^j$. This is just a linear first order differential equation, since we want to solve for the derivative of v in terms of v! As long as the data is smooth (Γ is a smooth function of position, γ^j a smooth function of time), then by the Existence-Uniqess theorem for differential equations we have a unique solution. Since the ODE is linear, the solution gives a linear map from T_pM to $T_{\gamma(t)}M$, which we call **parallel transport**. Without a connection the tangent spaces are just vector spaces and we wouldn't know what to do, but given a connection, we can "drag" one vector to another along the curve, in particular we can send frames to frames this way.

todo:figure about transporting frames?

A natural question is "does parallel transport depend on γ?" In $\mathbb{R}^n$, it doesn't matter what path you transport by, you always get the same result, since $\dot v^k = 0$ which implies v^k is a constant. However on another manifold, it might depend on which path you take.

Example 1.1. In S^2, transporting a vector along two different paths to the north pole gives a different result depending on what path you take, because there's a singularity at the north pole.

todo:figure

Curvature means that things twist when we go along loops. Whenever the answer depends on path, then going along a closed loop does not give the identity. However, we do get a linear transformation $T_pM \to T_pM$. We claim this linear transformation is an *isometry*. Note that since our connection is metric,

$$\frac{d}{dt} g(v,v) = \dot\gamma(g(v,v)) = g(\nabla_g v, v) + g(v, \nabla_g v) = g(0,v) + g(v,0) = 0.$$

Similarly, under parallel transport the derivative of two vectors doesn't change. Going around a loop gives you a rotation of your tangent space. On a two dimensional surface, all rotations commute, since the rotation group is abelian. For a closed loop, you can "chop up" the interior of the closed curve and see how much rotation you get from each bit, then sum. So the rotation from parallel transport by γ a closed loop is $\int \kappa\, dA$ for some function K. This function κ is called the **Gauss curvature**. On S^n it's zero, in $\mathbb{H}$ it's negative 1, and on $\mathbb{R}^n$ its zero. So if you want to know the curvature at a point, just go along an infinitesmal path, and we get that κ is a function of $g, \Gamma, \partial\Gamma$. (Sneak peek of future content.)

In short, if you what a connection is, you know what parallel transport is. And if you know what parallel transport is, you know what curvature is. If the connection happens to be the Levi-Civita connection, then Γ and $\partial\Gamma$ come from the metric, so we get formulas for curvature in terms of the metric.

Example 1.2. Let's think about $S^3 \subseteq \mathbb{R}^4$, the space of quaternions. That is, $x = x^1 + x^2 i + x^3 j + x^4 k$, where $i^2 = j^2 = k^2 = ijk = -1$. Then $ij = k, jk = i, ki = j$, but $ji = -k, kj = -i, ik = -j$. This is a nonabelian extension of the complex numbers. Multiplication of the unit quaternions forms a group of order eight. Given the vector field $e_1(x) = ix, e_2(x) = jx, e_3(x) = kx$, we claim all these vectors are tangent to S^3 and form an orthonormal basis for the tangent space. This gives us a frame, but we don't have a coordinate system. We could also consider the alternate frame $\tilde e_1(x) = xi, \tilde e_2(x) = xj, \tilde e_3(x) = xk$.

Let us define some interesting connections. One is by defining $\nabla^L_{e_i} e_j = 0$. In the homework we show this connection is metric but has torsion. Another connection says that $\nabla^K_{\tilde e_i} \tilde e_j = 0$. This is also metric but not symmetric. A third connection is defined by $\nabla^M = (\nabla^L + \nabla^R)/2$, which will turn out to be metric and symmetric, so it's the Levi-Civita connection. Curvature in three dimensions is a bit more subtle since rotations don't commute. Eventually we get to curvature in higher dimensions, which is a tensor with four indices, a big mess.

1.3 A coordinate free approach to the fundamental theorem

Sometimes we don't have coordinates, and just want to talk about things in terms of frames or vector fields. The point is that frames e_i may not be coordinates $\partial/\partial x^i$. What are $\nabla_{e_i} e_j, [e_i, e_j]$, and $g(e_i, e_j)$? In a coordinate frame, the brackets are zero since mixed partials commute. However, this doesn't always hold for arbitrary frames. It would be nice if we had a formula for $\nabla_{e_i} e_j$ in terms of $[e_i, e_j]$ and $g(e_i, e_j)$. In full generality, we could talk about $g(\nabla_X Y, Z)$, $[X, Y]$ etc, $g(X, Y)$ etc, for X, Y, Z vector fields. The fundamental theorem says that there exists a unique metric is, so we can figure out $g(\nabla_X Y, Z)$ in terms of the other data. But there are other formulas for the Levi-civita connection, which we'll talk more about next time. This expression is called **Koszul's formula**.

Lecture 2

March 11, 2021

2.1 General formulas for the Levi-Civita connection

Consider $g(Y,Z)$ which is just a function, so we can take its directional derivative $X(g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. (Denote $\langle X, Y\rangle = g(X,Y)$). So

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

is the equation for being metric. For a symmetric connection, we have

$$[X,Y] = \nabla_X Y - \nabla_Y X.$$

In a coordinate basis $[X,Y] = 0$, so $\Gamma_{ij}^k = \Gamma_{ji}^k$. Now just by changing the names of the vector fields we have $Y\langle X, Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle$ and $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$. Then

$$\begin{aligned}
+X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
+Y\langle X,Z\rangle &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\
-Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \\
&\implies \\
X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle &= \langle \nabla_X Y + \nabla_Y X, Z\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle + \langle \nabla_Y Z - \nabla_Z Y, X\rangle \\
&= 2\langle \nabla_X Y, Z\rangle - \langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle \\
&\implies
\end{aligned}$$

$$\boxed{\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle\right).}$$

This equation is called **Koszul's formula**, which is the most general formula for the Levi-Civita connection. There are some bases we care about:

(1) If $X = \partial_i, Y = \partial_j, K = \partial_k$, then $\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$ since all the bracket terms are zero. This is the form we're familiar with of the Levi-Civita connection, and the most common one as well.

(2) Say we have an orthonormal basis $\{E_i\}$, where $X = E_i, Y = E_j, Z = E_k$. Then $[E_i, E_j] = c_{ij}^k E_k$, where the coefficients c_{ij}^k tell you to what extent are the brackets nonzero. If $c_{ijk} = \langle [E_i, E_j], E_k\rangle$, then $\Gamma_{ijk} = \Gamma_{ij}^k = \frac{1}{2}(c_{ijk} - c_{ikj} - c_{kji})$.

(3) Given a general frame, we have $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) + \frac{1}{2}(c_{ijk} - c_{ikj} - c_{jki})$.

2.2 Return to geodesics

Recall that a geodesic satisfies $\nabla_{\dot{x}}\dot{x} = 0$, and $\ddot{x}^k + \Gamma_{jk}^k \dot{x}^i \dot{x}^j = 0$. Say we have a point p and a vector v_0, we want to talk about a geodesic starting at p with velocity v_0. We want to convert this second order differential equation of n variables into a first order ODE of two variables. Say $v(t) = \dot{x}(t)$, then what is $\dot{v}$? Recall $\nabla_v v = 0 \iff \dot{v}^k + \Gamma_{jk}^k(x) v^i v^j = 0$, so

$$\begin{aligned}
\dot{v}^k &= -\Gamma_{ij}^k(x) v^i v^j, \\
\dot{x}^k &= v^k.
\end{aligned}$$

These are our two desired differential equations. Then we have unique solutions since the Γ's are smooth functions of x (the particular smooth function is given by the Levi-Civita connection). In other words, there exists a unique solution $x(0) = p, v(0) = v_0$ for a short time (locally).

For an arbitrary amount of time, consider the line going through the origin in $\mathbb{R}^2 \setminus \{0\}$, which fails near the origin. Another example is the open interval $M = (a, b), g = dx^2$; after a certain amount of time you fall of the edge of the world. If somebody gives you a manifold with a starting point and velocity, we want to run a geodesic for as long as possible in the $\pm t$ direction. This motivates the following definition.

Definition 2.1. A geodesic is **maximal** if it can't be extended.

Is there a maximal geodesic? Given a starting point, take the union of the geodesics on an interval. This is unique, and we can't extend this any farther because this implies a bigger interval. So given any point and any starting vector, there is always a unique maximal geodesic.

2.3 The exponential map

Say we have a manifold M, $p \in M$, and the tangent space T_pM. Then the **exponential map** is defined as $\exp_p(v) = \gamma(1)$, where γ is a geodesic with $\gamma(0) = p, \dot{\gamma}(0) = v$. Is this defined on all of T_pM? This is defined for all sufficiently small v (for large v you may fall of the edge of the world). You can think of $\exp_p(sv) = \gamma_{sv}(1) = \gamma_v(s)$. For every p there exists a small neighborhood that looks like $\mathbb{R}^n$, so exp is well defined on the tangent space in that neighborhood. So think of the exponential map as the map

$$\exp_p : \text{Nbd of } 0 \in T_pM \to \text{Nbd of } p \in M.$$

Given a compact manifold we can extend this, so geodesics run forever and the exponential map is defined for all t. What is $d\exp_p|_{v=0}$? This is the identity, since $d\exp_p|_{v=0} : T_0(\text{Nbd of } 0 \in T_pM) \to T_pM$ which is the same as a map $T_pM \to T_pM$. At $v = 0$, this is saying "given an infinitesmally small vector, where do you wind up?" This is the same thing as saying "how fast are you moving at time zero if you have a large vector?", since how far you wind up with an infinitesmally small vector is how far you wind up with an ordinary geodesic in a short amount of time. So this asks how geodesics at small time, which is just the derivative of a geodesic at 0, which is just v!

The nice thing about the identity is that it's invertible. Suppose we have two manifolds $M, N, f : M \to N, p \in M$, $q = f(p) \in N$, $df : T_pM \to T_qN$. If df_p is invertible, then f is a local diffeomorphism, so $f|_V$ is a diffeomorphism $U \to V$ (where U, V are neighborhoods of p, q). So $\exp_p$ is a diffeomorphism, since it takes neighborhoods to neighborhoods. Let $r = \sup\{\text{radii } \mid \exp_p \text{ is a diffeomorphism on } B_p\}$. We say r is the **injectivity radius** at p.

Example 2.1. Consider M the torus, if we draw it as a rectangle, say it has width L_1 and height L_2, $L_2 > L_1$. Then the injectivity radius at p is $L_2/2$, since any points past $L_2/2$ wrap around. The exponential map is a local diffeomorphism, but it fails to be injective; this is why it's called the *injectivity* radius. On a sphere with p the north pole, our injectivity radius is π (since the circumference is 2π).

You might think the injectivity radius is about the topology, since $H_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$ is nontrivial and we have a cycle to wrap around. It turns out the injectivity radius isn't just about the topology, since S^2 has no interesting first homology group. A handwavy way to think about the injectivity radius is the biggest radius such that a neighborhood of size r around p looks like a ball.

If we have orthonormal coordinates for T_pM, this induces coordinates on T_qN by the exponential map. Then $g_{ij}(p) = \delta_{ij}$, or even stronger, we have $g_{ij}(\exp_p(v)) = \delta_{ij} + O(v^2)$. Consider a ball of radius $\varepsilon < r$ in T_pM, for $p, q \in U_p$ (for U_p a neighborhood around p in M 0. Is there a geodesic connecting p and q? Sure there is, since exp is a diffeomorphism. The geodesic is also locally unique. If $q = \exp_p(v)$, then what is the distance from p to q? It's the inf of all lengths; not all geodesics globally minimize length, but a length minizming curve is certainly a geodesic. This is equivalent to the energy concept, that the variational equations for energy give us geodesics WRT the Levi-Civita connection. Since the shortest path gives us a geodesic, and there exists exactly one geodesic, then the geodesic must be the shortest path. So the distance from p to q is the magnitude of v.

In other words, the distance function from p is just the magnitude function in T_pM. If we use polar coordinates on T_pM, the metric in the radial direction $g_{rr} = 1$, so the metric will always be dr^2+something.

2.4 Tubular neighborhoods

Suppose we have a Riemannian manifold M and a submanifold N. The tubular neighborhood theorem states that if N is compact, the set of all points within ε of N is a tubular neighborhood, and is diffeomorphic to a neighborhood of the zero section of the normal bundle. If N is not compact, then ε is not globally chosen (it varies from point to point). What does a neighborhood of $p \in N$ look like? A neighborhood of p in the big space looks like a neighborhood of the origin in T_pM. But $T_pM = T_pN \oplus N_pN$, so we parametrize by $(p, v) \to \exp_p(v)$.

Lecture 3

March 23, 2021

3.1 Calculus of variations

Suppose we have $z(t)$, where $z(0) = 0, z(2) = 0$. We want to minimize the integral $\mathscr{L} = \int_0^2 \frac{1}{2}\dot{z}^2 - 32z\,dt$. We will do this two ways: one is the "sloppy" version with our usual notation for the calculus of variations, and one by a mathematically precise method. The usual notation has a precise meaning behind it, but it just seems sloppy.

Lecture 4

April 6, 2021

In two dimensions, parallel transport around an orientation preserving loop is a rotation, which you can parametrize by just a number θ. You can sum up the rotations around tiny little loops. Define K to be the rotation per unit block on an area A bounded by a loop γ, then define the size of rotation by $P_\gamma = \iint_A K\,d\text{Area}$.

4.1 Gauss-Bonnet Theorem