# Complex Analysis Notes

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These are some notes I took to complement my complex analysis course freshman year, since I was generally unhappy with our (very applied) textbook. Some also miscellaneous topics I choose to study about in complex analysis might go here. If you have questions, comments, corrections, etc, feel free to email them to me at simonxiang@utexas.edu.

## Contents

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Lecture 1

# The Basics

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I want to do some real math! These notes will follow Stein and Shakarchi §1.2.

## 1.1 Continuous functions

We've already seen the standard epsilon-delta definition of continuity. An equivalent definition is the sequential definition, that is, for every sequence $\{z_1, z_2, \cdots\} \subseteq \Omega \subseteq \mathbb{C}$ such that $\lim z_n = z_0$, then $f$ is continuous at $z_0$ if $\lim f(z_n) = f(z_0)$. Since the notions for convergence of complex numbers and $\mathbb{R}^2$ is the same, $f$ of $z = x + iy$ is continuous iff it's continuously viewed as a function of two real variables $x$ and $y$. If $f$ is continuous, then the real valued function defined by $z \mapsto |f(z)|$ is clearly continuous (by the triangle inequality).

We say $f$ attains a **maximum** at the point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in \Omega.$$

The definition of a minimum is what you think it is.

**Theorem 1.1.** *A continuous function on a compact set $\Omega$ is bounded and attains a maximum and minimum on $\Omega$.*

*Proof.* Same as the any one you'd find in a Real Analysis course. ⊠

## 1.2 Holomorphic functions

Let's talk about the good stuff. Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then $f$ is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, such that the quotient is well-defined. This limit is called the **derivative of** $f$ and $z_0$, and is denoted by

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Note that $h$ approaches $0$ from any direction. The function $f$ is said to be **holomorphic on** $\Omega$ if $f$ is holomorphic at every point of $\Omega$. If $C \subseteq \mathbb{C}$ is closed, then $f$ is **holomorphic on** $C$ if $f$ is holomorphic on some open set containing $C$. Finally, if $f$ is holomorphic on $\mathbb{C}$ then $f$ is said to be **entire**. Sometimes the terms **regular** or **complex differentiable** are used in place of holomorphic, but holomorphic functions are much much nicer than real values differentiable functions. Furthermore, every holomorphic function is analytic (that is, it has a power series expansion near every point), and so we also use the term **analytic** to refer to holomorphic functions. Once again, things are not as nice in Real Analysis, with infinitely differentiable real valued functions not having power series expansions.

**Example 1.1.** We have any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ entire, and $f(z) = \frac{1}{z}$ holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$. However, $f(z) = \frac{\bar{z}}{z}$ is not entire, as $\frac{f(z_0+h)-f(z_0)}{h} = \frac{\bar{h}}{h}$, which has no limit as $h \to 0$.

If we write the definition of a holomorphic function as $f$ being holomorphic at $z_0 \in \mathbb{C}$ iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h),$$

where $\psi$ is a function defined for all "small" $h$, and $\lim_{h \to 0} \psi(h) = 0$, we can see that $f$ is holomorphic implies $f$ is continuous (clearly $a = f'(z_0)$). The basic stuff, distribution over addition, product rule, quotient rule, chain rule, yada yada all apply.

## 1.3   Cauchy-Riemann equations and the Jacobian

OK, here's the difference between real and complex valued functions again. In terms of real variables, $f(z) = \bar{z}$ corresponds to $F \colon (x, y) \to (x, -y)$, which is differentiable in the real sense, its derivative at a point being the map corresponding to its Jacobian. In fact, $F$ is linear and is equal to its derivative at any point, and is therefore infinitely differentiable. So a complex valued function having a real derivative need not imply the complex valued function is holomorphic. We can associate complex valued functions $f = u + iv$ to the mapping $F(x, y) = (u(x, y), v(x, y))$, where $F \colon \mathbb{R}^2 \to \mathbb{R}^2$.

Recall that $F(x, y)$ is differentiable at a $P_0 = (x_0, y_0) \in \mathbb{R}^2$ if there exists a linear transformation $J \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \to 0 \quad \text{as } |H| \to 0, H \in \mathbb{R}^2.$$

We could also write

$$F(P_0 + H) - F(P_0) = J(H) + |H|\Psi(H),$$

where $|\Psi(H)| \to 0$ as $|H| \to 0$. Very similar to what just happened above. Then the transformation $J$ is unique and called the **derivative** of $F$ at $P_0$. Given that $F$ is differentiable and the partial derivatives exist, we have

$$J = J_F(x, y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

**Note.**   In the case of complex differentiation, the derivative is a complex number $f'(z_0)$, but for real derivatives, it's a matrix.

However, there is a way to link these two notions. Let's consider the limit of $h \in \mathbb{R}$, that is, $h = h_1 + ih_2$ with $h_2 = 0$. Then if $z_0 = x_0 + iy_0$, we have

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1}$$
$$= \frac{\partial f}{\partial x}(z_0).$$

Similarly, for $h \in \mathbb{C} \setminus \mathbb{R}$, say $h = ih_2$ ($h$ is purely imaginary), we have

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2}$$
$$= \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

So if $f$ is holomorphic, then we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Now $f = u + iv$, so $\partial f/\partial x = \partial u/\partial x + i\partial v/\partial x$, and similarly $\partial f/\partial y = \partial u/\partial y + i\partial v/\partial y$. Separate the real and imaginary parts and note that $1/i = -i$, then we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y}\right) + \frac{1}{i} \cdot i\left(\frac{\partial v}{\partial y}\right) \implies$$
$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y},$$

which implies the following nontrivial relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

The relations described in Equation (1) are known as the **Cauchy-Riemann** equations, which link real and complex analysis. Here we state the converse of the Cauchy-Riemann theorems in an important theorem.

**Theorem 1.2.** *Suppose $f = u + iv$ is a complex-valued function on an open set $\Omega$. If $u$ and $v$ are continuously differentiable and satisfy the Cauchy-Riemann equations for all $\omega \in \Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \partial f / \partial z$.*

*Proof.* Recall that $u(x+h_1, y+h_2) - u(x,y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \psi_1(h)$ and $v(x+h_1, y+h_2) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \psi_2(h)$, where $\psi_j(h) \to 0$ (for $j \in \{1,2\}$) as $|h|$ tends to 0, and $h = h_1 + ih_2$. Then by the Cauchy-Riemann equations we have

$$ f(z+h) - f(z) = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + |h| \psi(h), $$

where $\psi(h) = \psi_1(h) + \psi_2(h) \to 0$, as $|h| \to 0$. Therefore $f$ is holomorphic and $f'(z) = 2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$.     ⊠

## 1.4 Power series

I was wondering when we were going to cover these? The prime example of a power series is the complex **exponential** function, defined for $z \in \mathbb{C}$ by

$$ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. $$

When $z$ is real, this is the same as the usual exponential function, and the series converges absolutely for all $z \in \mathbb{C}$. To see this, note that $\left| \frac{z^n}{n!} \right| = \frac{|z|^n}{n!}$, so compare $|e^z|$ to $\sum |z|^n / n! = e^z < \infty$. This shows that the series defining $e^z$ is uniformly convergent in every disk in $\mathbb{C}$. We'll prove that $e^z$ is entire, and that its derivative can be found by termwise differentiation of the power series expansion. Hence

$$ (e^z)' = \sum_{n=0}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z $$

for $m = n - 1$, which shows that $e^z$ is its own derivative. In contrast, the geometric series $\sum_{n=0}^{\infty} z^n$ converges absolutely only in the disk $|z| < 1$, and the sum is equal to $\frac{1}{1-z}$, which is holomorphic in the punctured plane $\mathbb{C} \setminus \{1\}$. The proof is the same as the one for real variables: observe that

$$ \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}, $$

then note that if $|z| < 1$ we have $\lim_{N \to \infty} z^{N+1} = 0$.

**Definition 1.1** (Power series). In general, a **power series** is an expansion of the form

$$ \sum_{n=0}^{\infty} a_n z^n, $$

where $a_n \in \mathbb{C}$.

Testing for absolute convergence involves looking at the sum $\sum_{n=0}^{\infty} |a_n||z|^n$, and if $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for some $z_0$, then it converges for all $z$ in the disk $|z| \le |z_0|$. Now we prove a theorem that there always exists an open disk (possibly empty)[1] on which the power series converges absolutely.

**Theorem 1.3.** *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that*

   *(i) If $|z| < R$ the series converges absolutely.*

   *(ii) If $|z| > R$ the series diverges.*

*Moreover, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$, then $R$ is given by Hadamard's formula*

$$ \frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}. $$

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[1]Wait, this theorem is pretty lame then.

We say that $R$ is the **radius of convergence** of the power series, and the region $|z| < R$ is the **disc of convergence** . In particular, $R = \infty$ for exp, and $R = 1$ for the geometric series.

*Proof.* Let $L = 1/R$, and suppose that $L \neq 0, \infty$[2]. If $|z| < R$, choose a small $\varepsilon > 0$ such that

$$(L + \varepsilon)|z| = \frac{1 + \varepsilon R}{R} \cdot \text{``}R\text{''} = r < 1.$$

By the definition of $L$, we have that $|a_n|^{\frac{1}{n}} \leq L + \varepsilon$ for all large $n$, therefore

$$|a_n||z|^n \leq \{(L + \varepsilon)|z|\}^n = r^n.$$

Comparing this with the geometric series $\sum r^n$ shows that $\sum a_n z^n$ converges. If $|z| > R$, then a similar argument shows that there exists a sequence in terms of the series whose absolute value goes to infinity, so the series diverges. ⊠

**Remark 1.1.** Things are much more delicate when we deal with the boundary of the disk of convergence, where $|z| = R$. Here, we can have either convergence or divergence.

More examples include the **trigonometric functions**, defined by

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

which agree with the usual cosine and sine of a real argument when $z \in \mathbb{R}$. A simple calculation yields

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

These are the **Euler formulas** for cosine and sine.

**Theorem 1.4.** *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. The derivative of $f$ is also a power series obtained by differentiating term by term the series for $f$, that is,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

*Moreover, $f'$ has the same radius of convergence as $f$.*

*Proof.* To show that $f'$ has the same radius of convergence as $f$, refer to Hadamard's formula. Indeed, $\lim_{n\to\infty} n^{1/n} = 1$, and therefore

$$\limsup |a_n|^{\frac{1}{n}} = \limsup |n a_n|^{\frac{1}{n}},$$

so $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence, and therefore so do $\sum a_n x^n$ and $\sum n a_n z^{n-1}$. Now for the first assertion, we need to show that

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

gives a description of $f'$. Let $R$ be the radius of convergence of $f$, and suppose that $|z_0| < r < R$. Write $f(z) = S_N(z) + E_N(z)$, where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Some crazy stuff happened, and we'll omit the rest of the proof. ⊠

**Corollary 1.1.** *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

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[2]These two cases are exercises for the reader.

**Definition 1.2** (Power series not centered at the origin)**.** We can define a power series centered at $z_0 \in \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The disk of convergence is now centered at $z_0$ and the radius is still given by Hadamard's formula. In fact, if $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f$ is simply obtained by translating $g$, where $f(z) = g(w)$, $w = z - z_0$. So everything we know about $g$ works for $f$, in particular, by the chain rule we have

$$f'(z) = g'(w) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

A function $f$ defined on an open set $\Omega$ is **analytic** (or has a **power series expansion**) if at a point $z_0 \in \Omega$ there exists a power series $\sum a_n (z - z_0)^n$ centered at $z_0$, with positive radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z$ in a nbd of $z_0$. If $f$ has a power series expansion for all $z_0 \in \Omega$, then $f$ is **analytic on $\Omega$**. By Theorem 1.4, an analytic function on $\Omega$ is also holomorphic there. Next chapter we prove a deep theorem that every holomorphic function is analytic, which is why we use the terms holomorphic and analytic interchangeably.

## 1.5   Integration along curves

**Definition 1.3** (Parametrized curve)**.** A **parametrized curve** is a function $z(t) \colon [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. We say a parametrized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for $t \in [a, b]$.

At the endpoints $t = a$ and $t = b$, we interpret $z'(a)$ and $z'(b)$ as the one-sided limits

$$z'(a) = \lim_{h \to 0, h > 0} \frac{z(a + h) - z(a)}{h} \quad \text{and} \quad \lim_{h \to 0, h < 0} \frac{z(b + h) - z(b)}{h}.$$

We call these the right-hand derivative of $z(t)$ at $a$ and the left-hand derivative of $z(t)$ at $b$, respectively[3]. Similarly, we say the parametrized curve is **piecewise-smooth** if $z$ is continuous on $[a, b]$, and there exist points $\{a_0, a_1, \cdots, a_n\}$ such that

$$a = a_0 < a_1 < \cdots < a_n = b,$$

where $z(t)$ is smooth on $[a_k, a_{k+1}]$ for $1 \le k \le n$. This differs from the standard definition of a smooth curve in that the right and left hand derivatives at $a_k$ may differ for some $1 \le k \le n-1$. Why don't we just say parametrizations are just paths (like in the definition of path-connectedness)? That would save me a lot of typing, because the word "path" is much shorter than the word "parametrization"[4].

We say two parametrizations $z \colon [a, b] \to \mathbb{C}$ and $\widetilde{z} \colon [c, d] \to \mathbb{C}$ are **equivalent** if there exists a continuouously differentiable bijection $s \mapsto t(s)$ from $[c, d]$ to $[a, b]$ such that $t'(s) > 0$ and $\widetilde{z}(x) = z(t(s))$. The fact that we require the derivative to be positive says that the orientation is preserved: as $s$ walks on the path from $c$ to $d$, $t(s)$ walks on the path from $a$ to $b$. The family of all parametrizations that are equivalent to $z(t)$ determine a **smooth curve** $\gamma \subseteq \mathbb{C}$, which is the image of $[a, b]$ under $z$ with the given orientation. We can define a curve $\gamma^-$ obtained from $\gamma$ by reversing the orientation, for example consider the parametrization $z^- \colon [a, b] \to \mathbb{R}^2$ defined by

$$z^-(t) = z(b + a - t).$$

We can also define a **piecewise-smooth curve** in the same way: let $z(a)$ and $z(b)$ be the end-points of the curve (independent of parametrization). Then $\gamma$ begins at $z(a)$ and ends at $z(b)$. A curve is **closed** if $z(a) = z(b)$ for any parameterization (it forms a loop), and is **simple** if it isn't self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$ (of course, we make an exception if it's closed). From now on, we'll call a piecewise-smooth curve a **curve** for brevity, since we don't really care if it's not piecewise-smooth.

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[3]Intuitively, shouldn't it be the other way around? Like left-hand derivative corresponds to $a$, since it's on the "left-hand side" of the interval...

[4]Thank you for the observation, very cool.

**Example 1.2.** The standard example of a curve is a circle. Consider the circle

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

The **positive orientation** (counterclockwise) is given by $z(t) = z_0 + re^{it}$ while the **negative orientation** is given by $z(t) = z_0 + re^{-it}$, for $t \in [0, 2\pi]$. When we talk about circles $C$, we'll usually be talking about the positively oriented circle.

$\sim\!\diamond\!\sim$

Now let's talk about integration along curves! A key theorem says that if a complex valued function is holomorphic in the interior of a closed circle $\gamma$, then

$$\int_\gamma f(z)\, dz = 0.$$

A version of this theorem is called *Cauchy's Theorem*, which we'll talk about later. Given a smooth curve $\gamma \subseteq \mathbb{C}$ parametrized by $z : [a, b] \to \mathbb{C}$ and $f$ a continuous function on $\gamma$, we define the **integral of $f$ along $\gamma$** by

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt.$$

How do we know this doesn't depend on the parametrization of $\gamma$? Say $\widetilde{z}$ is an equivalent parametrization of $z$, then

$$\int_a^b f(z(t))z'(t)\, dt = \int_c^d f(z(t(s)))z'(t(s))t'(s)\, ds = \int_c^d f(\widetilde{z}(s))\widetilde{z}'(s)\, ds,$$

proving that $\int_\gamma f(z)\, dz$ is well-defined. Now if $\gamma$ is piecewise-smooth, given a piecewise-smooth parametrization $z(t)$ we have

$$\int_\gamma f(z)\, dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\, dt.$$

We can also define the **length** of the smooth curve $\gamma$ as $\mathrm{length}(\gamma) = \int_a^b |z'(t)|\, dt$. Apply the same arguments as before to get that $\mathrm{length}(\gamma)$ is parametrization independent and that if $\gamma$ is piecewise smooth, $\mathrm{length}(\gamma)$ is the sum of the lengths of the smoooth components.

**Proposition 1.1.** *Let $\gamma$ be a curve, and $f, g$ be functions. Then*

    *1. Integration is a linear operation, that is, for $\alpha, \beta \in \mathbb{C}$ we have*

$$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

    *2. For $\gamma^-$ the curve representing the reverse orientation of $\gamma$, we have*

$$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz.$$

    *3. The following inequality holds:*

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \mathrm{length}(\gamma)$$

*Proof.* Basically, do it yourself.

    1. Follows from the definition and linearity of the Riemann integral.

2. Exercise for the reader.

3. Note that

$$\left| \int_{\gamma} f(z)\,dz \right| \le \sup_{t\in[a,b]} |f(z(t))| \int_{a}^{b} |z'(t)|\,dt \le \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma).$$

⊠

Suppose $f$ is define on some open set $\Omega \in \mathbb{C}$. A **primitive** for $f$ on $\Omega$ is a function $F$ that's holomorphic on $\Omega$ such that $F'(z) = f(z)$ for all $z \in \Omega$. Now let's look at the Fundamental Theorem of Calculus again.

**Theorem 1.5.** *If a continuous function $f$ has a primitive $F$ in $\Omega$, and $\gamma$ is a curve in $\Omega$ starting at $\omega_1$ and ending at $\omega_2$, then*

$$\int_{\gamma} f(z)\,dz = F(\omega_2) - F(\omega_1).$$

*Proof.* Good thing we have the FTC from Real Analysis. If $z(t)\colon [a,b] \to \mathbb{C}$ is a parametrization of $\gamma$ with $z(a) = \omega_1$ and $z(b) = \omega_2$, then

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \int_{a}^{b} f(z(t))z'(t)\,dt \\
&= \int_{a}^{b} F'(z(t))z'(t)\,dt \\
&= \frac{d}{dt}F(z(t))dt \\
&= F(z(b)) - F(z(a)).
\end{aligned}$$

Clearly we're done if $\gamma$ is smooth. If $\gamma$ is only piecewise-smooth, then

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\
&= F(z(a_n)) - F(z(a_0)) \\
&= F(z(b)) - F(z(a)).
\end{aligned}$$

⊠

**Corollary 1.2.** *If $\gamma$ is a closed curve in some open $\gamma \in \mathbb{C}$ and $f$ is continuous and has a primitive in $\Omega$, then*

$$\int_{\gamma} f(z)\,dz = 0.$$

*Proof.* Note that if $\gamma$ is a closed curve, then $\omega_1 = \omega_2$. ⊠

**Example 1.3.** We can use this corollary to show that functions don't have primitives. For example, $f(z) = 1/z$ doesn't have a primitive in $\mathbb{C} \setminus \{0\}$, since if $C$ is the unit circle parametrized by $z(t) = e^{it}$ when $0 \le t \le 2\pi$, we have

$$\int_{C} f(z)\,dz = \int_{0}^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i \ne 0.$$

**Corollary 1.3.** *If $f$ is holomorphic on a region $\Omega$ and $f' = 0$, then $f$ is a constant function.*

*Proof.* Let $\omega_0 \in \Omega$. We WTS that $f(\omega) = f(\omega_0)$ for all $\omega \in \Omega$: since $\Omega$ is connected we can find a curve (path) joining $\omega_0$ and $\omega$. Clearly $f$ is a primitive for $f'$. Then

$$\int_{\gamma} f'(z)\,dz = f(\omega) - f(\omega_0) = 0$$

by assumption, which implies that $f(\omega) = f(\omega_0)$, finishing the proof. ⊠

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┌─ Lecture 2 ─────────────────────────────────────────────────────────┐

# Cauchy's Theorem and Its Applications

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These will follow Stein and Shakarchi §2.

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Last time we talked about some good stuff: open sets, holomorphic functions, integration along curves. Our first cool theorem relates the three, which is (you guessed it) Cauchy's theorem. It loosely states that if $f$ is holomorphic on an open set $\Omega$ and $\gamma \subseteq \Omega$ is a closed curve whose interior is also contained in $\Gamma$, then

$$\int_\gamma f(z)\,dz = 0.$$

Cool stuff happens, including the calculus of residues. Right now we restrict ourselves to easy curves (toy contours) for simplicity, but we'll deal with the general case soon. We'll also get to Cauchy's integral formula, which says that for $f$ holomorphic in an open set containing a circle $C$ and its interior, then for all $z \in C$,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

From here, by differentiating we'll see that as long as the first derivative exists, all of them do! From there, we'll get

- The property of "analytic continuation" (finally), namely that a holomorphic function is determined by its restriction to any open subset of its domain, which follows from the fact that holomorphic functions are analytic.

- Liouville's theorem, which can prove the fundamental theorem of algebra.

- Morera's theorem, which gives a simple integral characterization of holomorphic functions, preserved under uniform limits.

## 2.1   Goursat's theorem

We know that if $f$ has a primitive in $\Omega$ open, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve $\gamma \subseteq \Omega$. We also can show that if the relation above holds, then a primitive will exist.

**Theorem 2.1** (Goursat's theorem). *If $\Omega$ is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in $\Omega$, then*

$$\int_T f(z)\,dz = 0,$$

*given $f$ is holomorphic in $\Omega$.*

*Proof.* Given a triangle $T^{(0)}$ with a fixed positive orientation, consider its barycentric subdivision yielding the similar triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$, defined to be the same orientation as $T^{(0)}$. So

$$\int_{T^{(0)}} f(z)\,dz = \int_{T_1^{(1)}} f(z)\,dz + \int_{T_2^{(1)}} f(z)\,dz + \int_{T_3^{(1)}} f(z)\,dz + \int_{T_4^{(1)}} f(z)\,dz. \tag{2}$$

Then there exists a $j$ such that

$$\left| \int_{T^{(0)}} f(z)\, dz \right| \leq 4 \left| \int_{T_j^{(1)}} f(z)\, dz \right|,$$

because if that weren't the case, it would contradict Equation (2). Denote such $T_j^{(1)}$ by $T^{(1)}$: note that $d^{(1)} = \frac{1}{2} d^{(0)}$ and $p^{(1)} = \frac{1}{2} p^{(0)}$, where $d^{(i)}$ and $p^{(i)}$ are the diameters and perimeters of the $T^{(i)}$'th triangle, respectively. Continuing on, we have a sequences of triangle $T^{(0)}, T^{(1)}, \cdots, T^{(n)}, \cdots$ such that

$$\left| \int_{T^{(0)}} f(z)\, dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)\, dz \right|, \quad d^{(n)} = 2^{-n} d^0, \; p^{(n)} = 2^{-n} p^{(0)}.$$

Denote the *solid* closed triangle with boundary by $\mathscr{T}^{(n)}$, and note that our construction yields a sequence of nested compact sets

$$\mathscr{T}^{(0)} \supset \mathscr{T}^{(1)} \supset \cdots \supset \mathscr{T}^{(n)} \supset \cdots$$

whose diameter approaches zero. By the NIP, there exists a unique $z_0 \in \bigcup_{i=1} \mathscr{T}^{(i)}$. Now $f$ is holomorphic at $z_0$, so we write $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$, where $\psi(z) \to 0$ as $z \to z_0$. Since $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, then we integrate and get

$$\int_{T^{(n)}} f(z)\, dz = \int_{T^{(n)}} \psi(z)(z - z_0)\, dz.$$

Now $z_0 \in \overline{\mathscr{T}^{(n)}}$ and $z \in \partial \mathscr{T}^{(n)} \implies |z - z_0| \leq d^{(n)}$. Recall that $\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \mathrm{length}(\gamma)$. So we have the estimate

$$\left| \int_{T^{(n)}} f(z)\, dz \right| \leq \varepsilon_n d^{(n)} p^{(n)},$$

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$. Therefore we have

$$\left| \int_{T^{(n)}} f(z)\, dz \right| \leq \varepsilon_n 4^{-n} d^{(0)} p^{(0)} \implies \left| \int_{T^{(0)}} f(z)\, dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)\, dz \right| \leq \varepsilon_n d^{(0)} p^{(0)}.$$

Letting $n \to \infty$ concludes the proof, since $\varepsilon_n \to 0$.                                                                 ⊠

Attempt to sum up the proof in easy to read words: take a triangle and subdivide it naturally into four more triangles. Choose one of them such that it follows a nice inequality, and continue subdiving until infinity. Since $\mathbb{C}$ is complete, we can find a point in all their closures: rewriting the formula for a derivative, we can apply the inequalities and get our result.

**Corollary 2.1.** *If $f$ is holomorphic in an open set $\Omega$ that contains a rectangle $R$ and its interior, then*

$$\int_R f(z)\, dz.$$

*Proof.* Immediate by constructing the $n$-gon with $n - 2$ simplices.                                                                 ⊠

## 2.2   Local existence of primitives and Cauchy's theorem in a disk

Ah, so I was complaining about the jank-connectedness definition by polygonal paths earlier in my notes. I see why they do it that way now.

**Theorem 2.2.** *A holomorphic function in an open disc has a primitive in that disk.*

*Proof.* WLOG, let $D$ be a disk centered at the origin, $z \in D$ be a point. Consider a curve from $0$ to $z$ made by joining straight line segments, starting by joining $0$ to $\mathrm{Re}\, z$ then from $\mathrm{Re}\, z$ to $z$, with the orientation you'd expect. We'll

denote this polygonal path by $\gamma_z$. Define the unique function $F(z) = \int_{\gamma_z} f(w)\,dw$: we claim $F$ is holomorphic in $D$ and $F'(z) = f(z)$. Let $z, h \in \mathbb{C}$ and choose $h > 0$ such that $z + h \in D$. Consider

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw.$$

Draw some lines and make a geometric argument that $F(z+h) - F(z) = \int_\eta f(w)\,dw$, where $\eta$ is the straight line segment from $z$ to $z+h$. Then since $f$ is continuous at $z$ we have $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. Therefore

$$F(z+h) - F(z) = \int_\eta f(z)\,dw + \int_\eta \psi(w)\,dw = f(z)\int_\eta dw + \int_\eta \psi(w)\,dw.$$

On one hand, 1 has $w$ as a primitive, so the first integral is just $h$. On the other hand, we have the estimate

$$\left| \int_\eta \psi(w)\,dw \right| \le \sup_{w \in \eta} |\psi(w)| |h|.$$

Since the supremum goes to zero as $h \to 0$, we conclude that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

$\boxtimes$

Basically, every holomorphic function has a local primitive. It's crucial that this holds not only for arbitrary disks, but other sets as well.

**Theorem 2.3** (Cauchy's theorem for a disc)**.** *If $f$ is holomorphic in a disk, then*

$$\int_\gamma f(z)\,dz = 0$$

*for any closed curve $\gamma$ in that disc.*

*Proof.* Since $f$ has a primitive, apply Corollary 1.2. $\boxtimes$

**Corollary 2.2.** *Suppose $f$ is holomorphic in an open set containing the circle $C$ and its interior. Then*

$$\int_C f(z)\,dz = 0.$$

*Proof.* Let $D$ be the disk with boundary circle $C$, then there exists a slightly larger disc $D'$ containing $D$ such that $f$ is holomorphic on $D'$: apply Cauchy's theorem on $D'$ to conclude that $\int_C f(z)\,dz = 0$. $\boxtimes$

Basically, **toy contours** are curves for which the notion of interior is clear. For a toy countour $\gamma$ we easily have

$$\int_\gamma f(z)\,dz - 0,$$

if $f$ is holomorphic in an open set containing $\gamma$ and its interior. You might ask what about general curves? Maybe we'll talk about Jordan's theorem for pieceweise-smooth curves, which states that a simple closed piecewise-smooth curve has a well defined interior that is simply connected.

## 2.3   Evaluating some integrals

I was just thinking about the Fourier transform today! What a coincidence— also, this is the second book in a three part series, so I'm already expected to know what this is and what it means, yay.

**Example 2.1.** If $\xi \in \mathbb{R}$, then

$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx.$$

This shows that $e - \pi x^2$ is its own Fourier transform. If $\xi = 0$, then this formula just becomes the Gaussian integral, which we know to be

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = \sqrt{\frac{\pi}{\pi}} = 1.$$

Assume $\xi > 0$, and consider $f(z) = e^{-\pi z^2}$ an entire function. Oh my gosh no I'm not finishing this proof.

**Example 2.2.** Another classical example is

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} \, dx = \frac{\pi}{2}.$$

Consider $f(z) = (1 - e^{iz})/z^2$, and integrate over the indented semicircle in the upper half of the annulus in the positive plane, with $\gamma_\varepsilon^+$ and $\gamma_R^+$ denoting the semicircles of radii $\varepsilon$ and $R$ with negative and positive orientations, respectively (where $R > \varepsilon$). Then by Cauchy's theorem, we have

$$\int_{-R}^{-\varepsilon} \frac{1 - e^{ix}}{x^2} \, dx + \int_{\gamma_\varepsilon^+} \frac{1 - e^{iz}}{z^2} \, dz + \int_{\varepsilon}^{R} \frac{1 - e^{ix}}{x^2} \, dx + \int_{\gamma_R^+} \frac{1 - e^{iz}}{z^2} \, dz = 0.$$

Let $R \to \infty$, and note that

$$\left| \frac{1 - e^{iz}}{z^2} \right| \le \frac{2}{|z|^2},$$

so the integral over $\gamma_R^+$ goes to zero. Therefore

$$\int_{|x| \ge \varepsilon} \frac{1 - e^{ix}}{x^2} \, dx = - \int_{\gamma_\varepsilon^+} \frac{1 - e^{iz}}{z^2} \, dz.$$

Next, note that $f(z) = \frac{-iz}{z^2} + E(z)$, where $E(z)$ is bounded at $z \to 0$, while on $\gamma_\varepsilon^+$ we have $z = \varepsilon e^{i\theta}$ and $dz = i\varepsilon e^{i\theta} \, d\theta$. Therefore

$$\int_{\gamma_\varepsilon^+} \frac{1 - e^{iz}}{z^2} \, dz \to \int_{\pi}^{0} (-ii) \, d\theta = -\pi \quad \text{as } \varepsilon \to 0.$$

Taking real parts yields

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx = \pi.$$

Since the integrand is even, we are done.

## 2.4   Cauchy's integral formula (todo skipped some stuff)

This is nice because we can learn about the behavior of functions on big sets by looking at small sets.

**Theorem 2.4** (Cauchy's integral formula)**.** *Suppose $f$ is holomorphic in an open set that contains the closure of a disk $D$. If $C$ denotes the boundary circle of the disk with positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for any point } z \in D.$$

*Proof.*  keyhole                                                                                                      ⊠

**Corollary 2.3.** *If $f$ is holomorphic in an open set $\Omega$, then $f$ has infinitely many complex derivatives in $\Omega$. Moreover, if $C \subseteq \Omega$ is a circle whose interior is also contained in $\Omega$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

*for all $z \in C^\circ$.*

*Proof.*  induction, base case is CIF                                                                                 ⊠

There are hereby known as the **Cauchy integral formulas**.

**Theorem 2.5** (Liouville's theorem)**.** *If $f$ is entire and bounded, then $f$ is constant.*

*Proof.*  If suffices to prove that $f' = 0$, since $\mathbb{C}$ is connected, then we can apply Corollary 1.3.                ⊠

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> Lecture 3

# Residues, Meromorphic Functions, and Poles

These will follow Stein and Shakarchi §3.

⤳◇⤞

In order of increasing severity, we have the following types of singularities:

- Removable singularities

- Poles

- Essential singularities

The first type are harmless since you can extend a function to be holomorphic at its removable singularities. At essential singularities, functions tend to oscillate and grow faster than powers, which makes them hard to understand. Poles are deeply connected with the calculus of residues, which we will see soon. We know for $f$ holomorphic in an open set containing a curve $\Gamma$ and its interior, we have $\int_\Gamma f(z)\, dz = 0$. What if $f$ has a pole in the middle? Recall that for $f(z) = \frac{1}{z}$ and $C$ a circle centered at 0, then

$$\int_C \frac{dz}{z} = 2\pi i.$$

This turns out to be the key ingredient.