# Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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# §1 September 14, 2020: Homework 3

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## §1.1 Problem 1 Section 1.2

**Problem.** *Show that the free product $G * H$ of nontrivial groups $G$ and $H$ has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of $G$ and $H$.*

*Solution.* Assume the center of the free product $G * H$ of nontrivial groups is nontrivial, that is, there exists a $z \in Z(G * H)$ such that $zw = wz$ for all $w \in G * H$. WLOG, take a nontrivial reduced word $w \in G * H$ (we can do this because $G$ and $H$ are nontrivial) that ends in $h \in H$. If $z$ ends in $g \in G$, we are done, since $zw$ will end in $h$ while $wz$ will end in $g$, contradicting the fact that $z$ lies in $Z(G * H)$. If $z$ ends in $h$, then if the ending letter of $w$ is of the form $h^n$ while the ending letter of $z$ is of the form $h^m$, $zw$ will end in $h^n$ while $wz$ will end in $h^{n+m}$, implying that $zw \neq wz$, a contradiction.

Next, we'll show the only elements of $G * H$ of finite-order are the conjugates of finite-order elements of $G$ and $H$. We'll do this proof by cases.

**Case 1:** WLOG, $w \in G * H$ starts with $g \in G$ and ends with $h \in H$. Clearly this doesn't terminate as $(g_i \cdots h_i) \cdot (g_i \cdots h_i)$ can't reduce down at $h_i \cdot g_i$, so this product will just keep growing longer with each multiplication.

**Case 2:** WLOG, $w \in G * H$ starts and ends with $g$, but the ending term isn't the inverse of the beginning term. That is, $w = g^n \cdots g^m$, but $n + m \neq 0$. So $w^2 = (g^n \cdots g^m) \cdot (g^n \cdots g^m) = g^n \cdots g^{n+m} \cdots g^m$. Since $g^{n+m}$ can't simplify, none of the other terms can, so this product will just keep growing longer to infinity like the previous one.

**Case 3:** WLOG, $w \in G * H$ starts with $g$ and ends with $g^{-1}$, but is not the conjugate of some element. In the previous cases, the possibility of $w$ being a conjugate wasn't even there, but now we can consider it (in the next case). For now, assume it isn't: then $(g \cdots g^{-1}) \cdot (g \cdots g^{-1})$ will reduce to $g \cdots 1 \cdots g$, but the middle dots won't reduce because $w$ isn't the conjugate of some element. (Note that $w$ starting with $g$ and ending with $g^{-1}$ is just a special case of $w$ being a conjugate of some element with the $\omega \in G * H$ set to $g$. This time, the WLOG also includes if $w$ starts with $g^{-1}$ and ends with $g$).

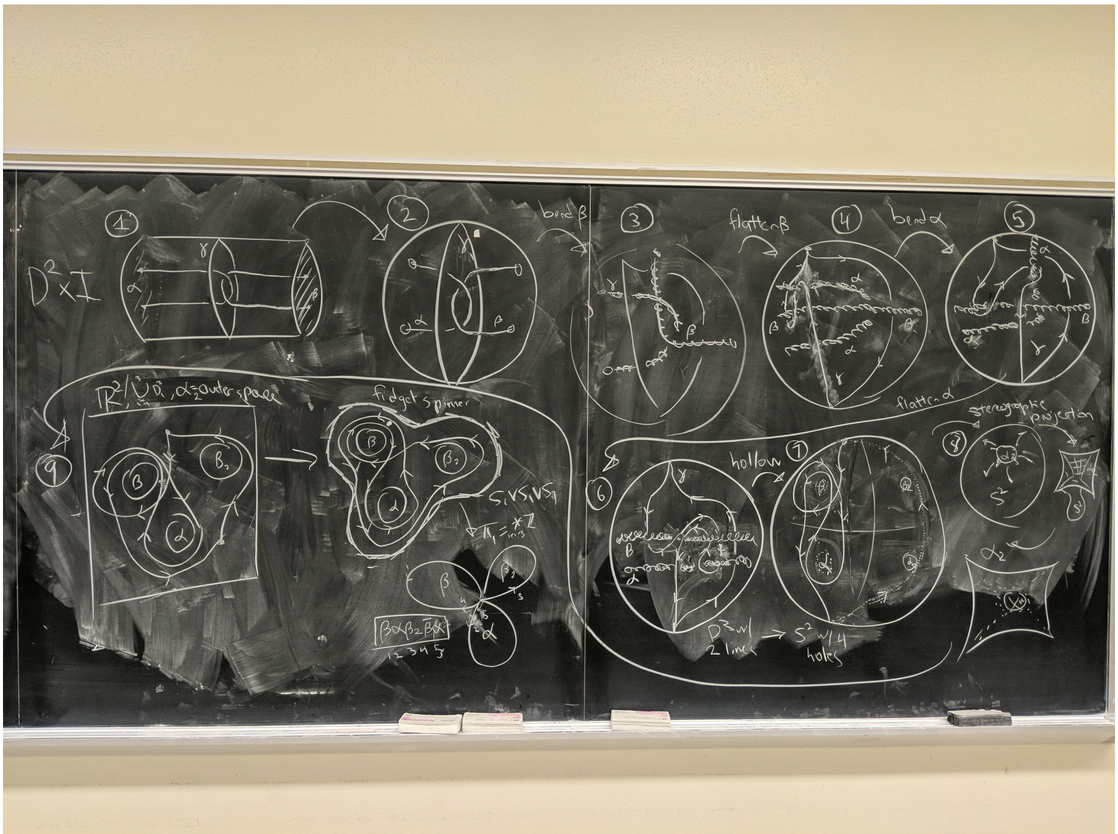
**Case 4:** $w \in G * H$ is the conjugate of some element of $G * H$. We'll show that the only element that allows $w$ to terminate are elements of finite-order in $G$ or $H$. We have $w = \omega a \omega^{-1}$ for $\omega, a \in G * H$. Each multiplication of $w$ gives $w^2 = (\omega a \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^2 \omega^{-1})$, $w^2 = (\omega a^2 \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^3 \omega^{-1})$, and so on. Therefore $w^n = \omega a^n \omega^{-1}$. If $a$ is a word (as in $a$ is not just an element of $G$ or $H$), it must be the conjugate of another word, which must be the conjugate of another word, and so on. We can't repeat this forever, so $a$ can't be a word. Now assume $a$ is an element of $G$ or $H$. Then if $a$ has infinite order in either of these groups, we have no $n$ satisfying $a^n = 1$, which subsequently means $w = \omega a^n \omega^{-1}$ will never terminate. If $a$ has finite order in $G$ or $H$, then there exists an $n$ such that $a^n = 1$. Then at that $n$, $w^n = \omega a^n \omega^{-1} = \omega \cdot 1 \cdot \omega^{-1} = 1$. Therefore $w$ has order $n$ if and only if $w$ is the conjugate of some element of finite order in $G$ or $H$, and we are done. ■

## §1.2 Problem 10

**Problem.** *Consider two arcs $\alpha$ and $\beta$ embedded in $D^2 \times I$ as shown in the figure. The loop $\gamma$ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of $\gamma$ in the complement of $\alpha \cup \beta$.*

*Hint (from Dr. Allcock): this can be done directly with Van Kampen's, but it becomes easier if you manipulate $(D^2 \times I) \setminus (\alpha \cup \beta)$ first, being careful not to change the homotopy type, and carrying along the loop $\gamma$.*

*Solution.* See the figure below: if the image is not clear/zoomed in enough, I'll be happy to email some more to you (don't want to clutter my document too much).




We start off with $D^2 \times I$ minus two "ropes", then we deform it into a sphere, and undo the knots to get a sphere with two lines missing, taking much care to keep track of where $\gamma$ is. (We denote the ropes $\alpha$ and $\beta$ with a bunch of swirls at this stage to differentiate it from the loop $\gamma$, and because the classroom had no colored chalk). Then, we hollow out the sphere $D^3 \setminus \{\text{two lines}\}$ to get $S^2$ with four holes, still keeping track of $\gamma$. Then we use a technique similar to the homework last week (lines through the origin) of taking one of the four holes and blowing it up to get the plane with three holes in it. We contract that to the fidget spinner, and denote the holes with $\beta, \beta_2$, and $\alpha$. This has the homotopy type of $S^1 \vee S^1 \vee S^1$, and so $\pi_1$ of this space $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The loop $\gamma$ corresponds to a loop around $\beta, \alpha, \beta_2, \overline{\beta}$, and $\alpha$ in that order. So $\pi_1(D^2 \times I \setminus \{\alpha \cap \beta\}) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, and since $\gamma$ corresponds to the loop $\beta\alpha\beta_2\overline{\beta}\alpha \neq 1$, we have $\gamma$ not nullhomotopic in this space. ■

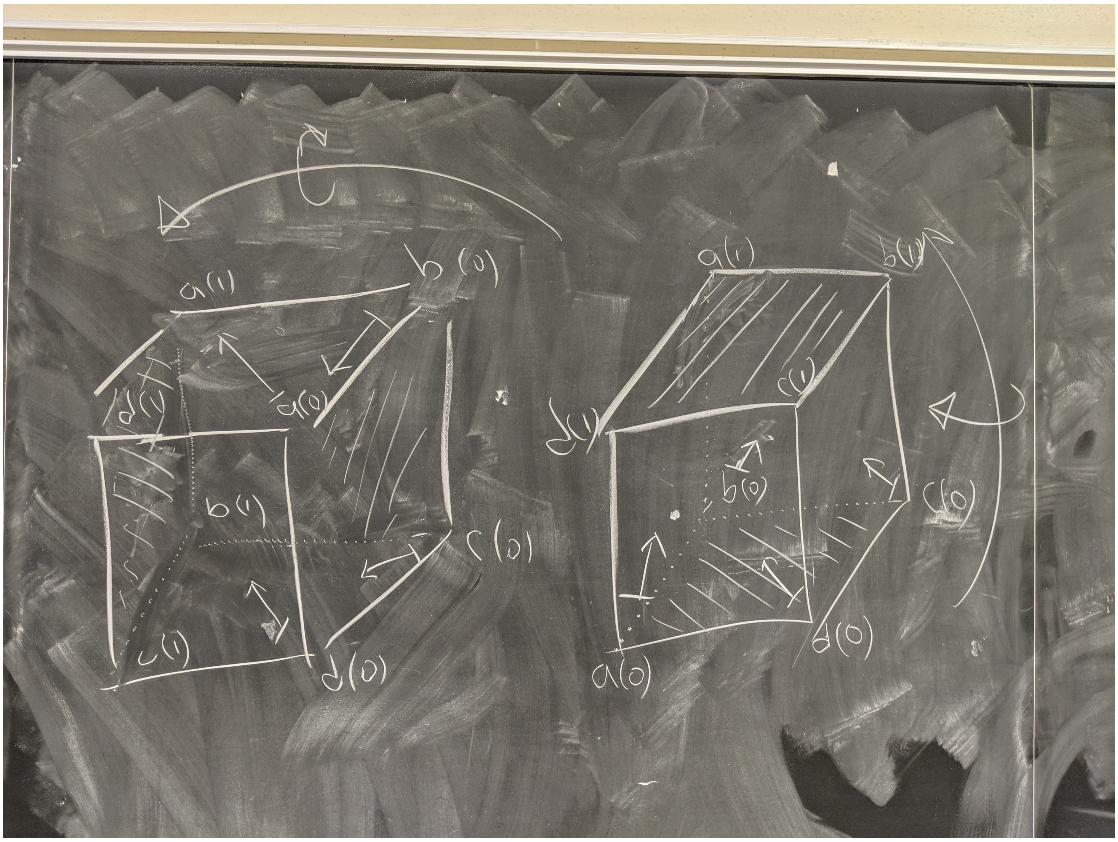
## §1.3 Problem 14

**Problem.** *Consider the quotient space of a cube $I^3$ obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.*

*Solution.* Recall a group presentation for the quaternion group is given by

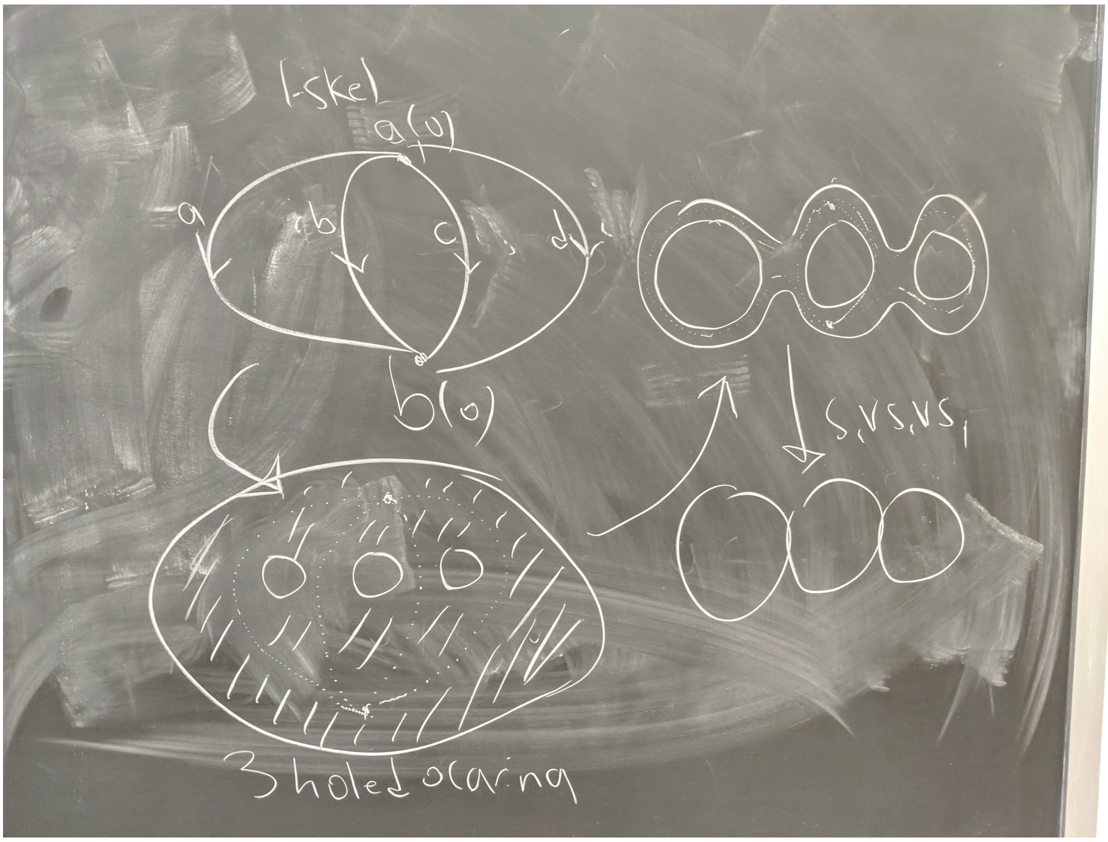
$$\langle -1, i, j, k \mid i^2 = j^2 = k^2 = -1, (-1)^2 = 1 \rangle.$$

Refer to the figure to see what's being identified to what: in the left side, I identified the right face with the left (0 and 1 just denoting what goes where, probably should have used subscripts) and in the right side of the figure I identified the bottom face with the top: together they describe the identification space of $I^3$.




The cube $I^3$ has the cell complex structure of eight 0-cells, twelve 1-cells, six 2-cells, and one 3-cell. Applying the first identification will identity eight 0-cells with four, and the second will identify four with two. Similarly, for 1-cells we identify twelve 1-cells onto eight (by mashing four together) and eight onto four (by mashing another four together). Finally, the identification will identify two 2-cells with each other, bringing us from four 2-cells to 3 2-cells. So $I^3/\sim$ has the desired cell complex structure.

We examine the 1-skeleton: See the figure to see why the resultant structure and $S^1 \vee S^1 \vee S^1$ are homotopy equivalent.




Therefore $\pi_1(X^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. We have the loops $a \cdot \overline{b}, a \cdot \overline{c}, a \cdot \overline{d}$. Let $i, j$, and $k$ denote the homotopy classes of these loops (to make it feel more like the quaternions).

We have three 2-cells attached, being $f_1, f_2$, and $f_3$. It can be seen that $f_1 = [a\overline{b}c\overline{d}]$, $f_2 = [c\overline{b}d\overline{a}]$, $f_3 = [d\overline{b}\overline{a}c]$ by following the lines. We can perform some calculations to rewrite these in a form we like. We have

- $f_1 = [a\overline{b}c\overline{d}] = [a\overline{b}c\overline{a}a\overline{d}] = [ij^{-1}k]$.
- $f_2 = [c\overline{b}d\overline{a}] = [c\overline{a}a\overline{b}d\overline{a}] = [j^{-1}ik^{-1}]$.
- $f_3 = [d\overline{b}\overline{a}c] = [d\overline{a}a\overline{b}\overline{a}c] = k^{-1}ij$.

Then by Proposition 1.26, we have

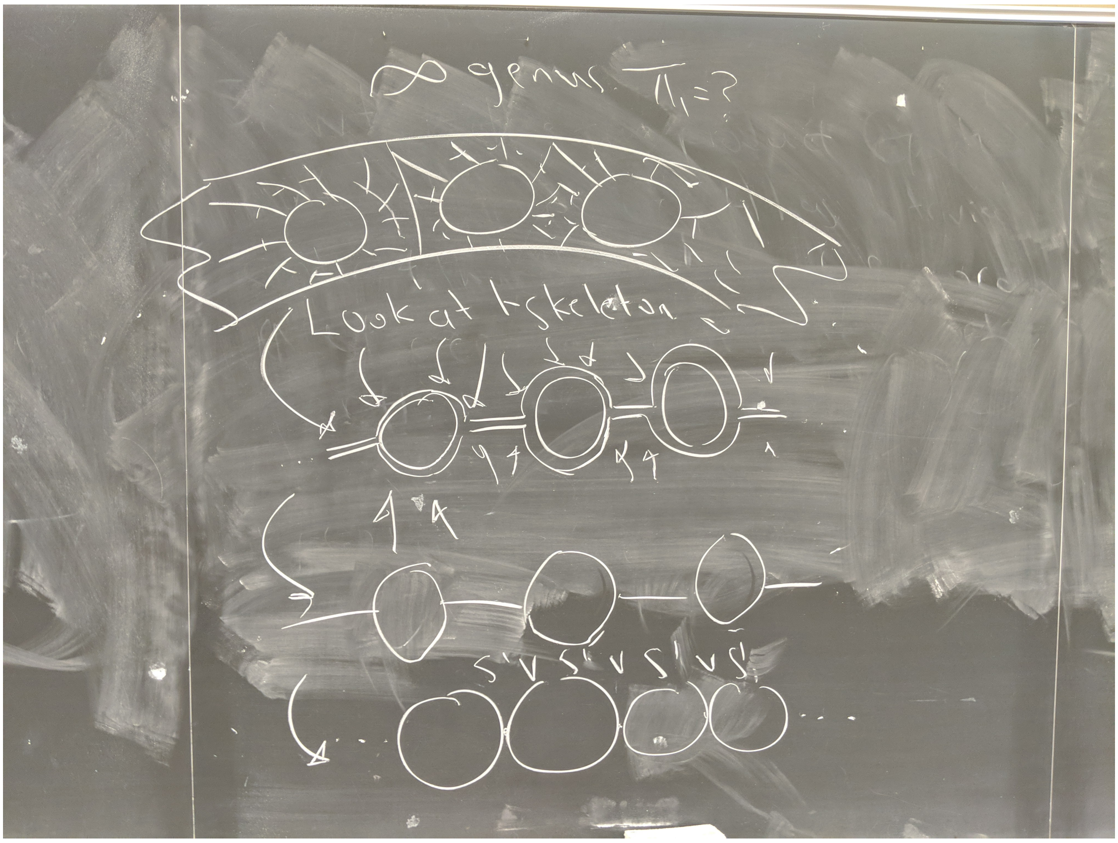
$$\pi_1(X) \simeq \pi_1(X_2) \simeq= \langle i, j, k \mid ij^{-1}k = j^{-1}ik^{-1} = k^{-1}ij = 1 \rangle.$$

This is just the quaternion group in disguise: we have $ki = j, i = kj, ij = k$. So $i^2 = (ik)j = j^2$, $j^2 = k(ij) = k^2$. Therefore $i^2 = k^2 = j^2$, denote this element as $-1$. Finally, all we have to do is show that $(-1)^2 = 1$ (by the group presentation given at the beginning). We have $(-1)^2 = i^2j^2 = i(ij)j = (k^{-1}k)ikj = k^{-1}(ki)kj = k^{-1}j(kj) = k^{-1}(ji) = k^{-1}k = 1$, and we are done. ■

## §1.4 Problem 16

**Problem.** *Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.*

*Solution.* I don't have a way to explicitly describe the deformation retraction (since I don't have a way to describe the surface), but this is an image I drew showing that the surface has the same homotopy type as an infinite wedge of $S^1$'s, denoted $S^1 \vee S^1 \vee \cdots$.




Note that even though the circles are side by side, we can slide them together and glue at a single point. So by Hatcher Example 1.21, the fundamental group of this surface of infinite genus is just

$$\pi_1(\text{surface}) = \pi_1(S^1 \vee S^1 \vee \cdots) = *_\alpha \pi_1(S^1) = \mathbb{Z} * \mathbb{Z} * \cdots .$$

Then $\pi_1$ is just the free group on an infinite number of generators (one for each $\mathbb{Z}$), and we are done (although a free product may not always be free, a free product of free groups is, and $\mathbb{Z} \simeq F_1$). ■

## §1.5 Problem 21

**Problem.** *Show that the join $X * Y$ of two nonempty spaces $X$ and $Y$ is simply-connected if $X$ is path-connected.*

*Hint (from Dr. Allcock): If you are not comfortable with the join of spaces then wrap your mind around the following examples in order:*

1. *Join of two points*
2. *Join of a point and an interval*
3. *Join of a point and a circle*
4. *Join of 2 copies of the interval*
5. *Join of a circle and an interval*
6. *Join of two circles (doesn't embed in $\mathbb{R}^3$, but still understandable).*

*That might be enough: if not, work out examples using the figure* 8 *or $S^2$.*

*Solution.* First we show the join $X * Y$ of two spaces is path-connected. Let $(x_0, 0, y_0) \in X * Y = X \times I \times Y$ (I like putting $I$ in the middle because it captures the idea of one space joining with the interval, and on the other side pops out the second space. The definition is from ncatlab). We have a path $\alpha\colon [0,1] \to X$, $\alpha(0) = x_0, \alpha(1) = x$ since $X$ is path-connected. Then we have a path $f(\alpha(t), it, y)$ for $i \in I$ connecting any $(x, i, y)$ to $(x_0, 0, y_0)$ (this works for $y$ because at $t = 0$ just let $y = y_0$, then see what happens to $y$ naturally and that will be a path, because of the construction of $X * Y$).

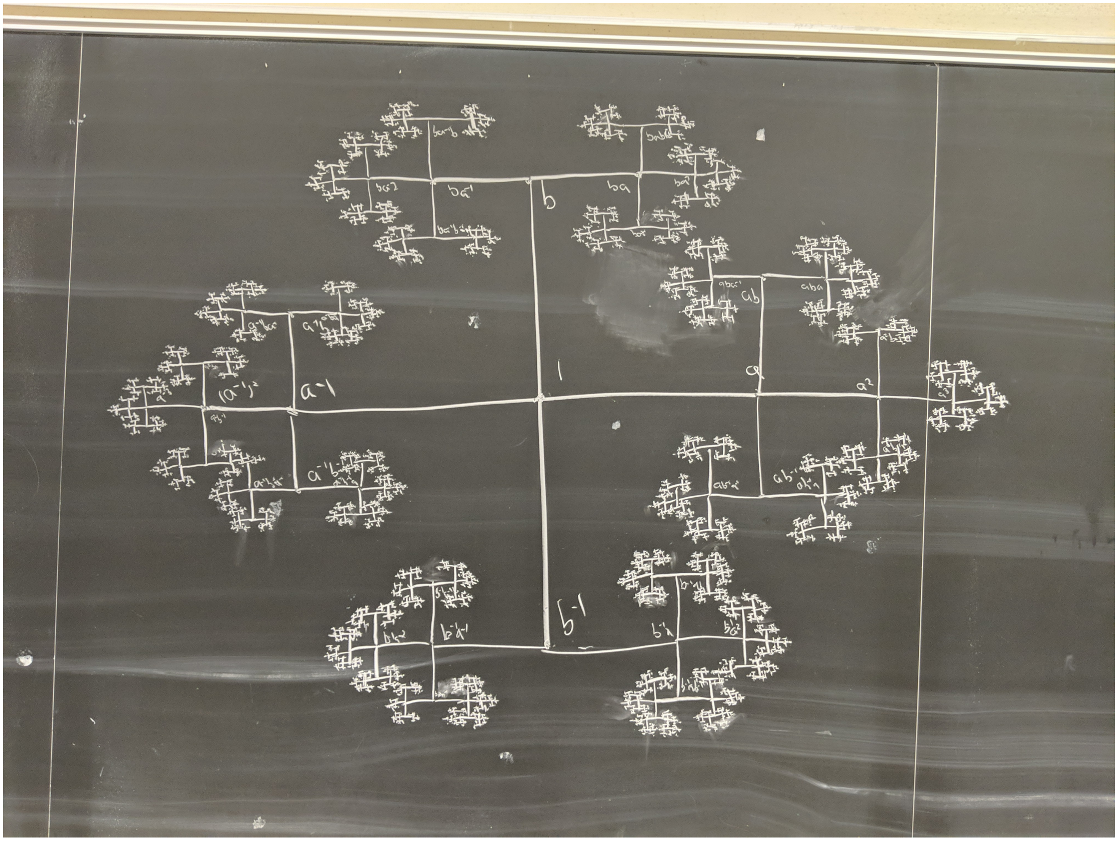
Assume $Y$ is path-connected. Now we split up $X * Y$ into two spaces $A$ and $B$, one where $I$ becomes $[0, \frac{2}{3})$ and another where $I$ becomes $(\frac{1}{3}, 0]$ (this is just a natural way to split the space in half with a nonempty intersection): these subsets are both open. Then $A$ deformation retracts onto $X$, $B$ onto $Y$, $A \cap B = X \times (\frac{1}{3}, \frac{2}{3}) \times Y$ onto $X \times Y$. Then since $X \times Y$ is path-connected by assumption, $\pi_1(A \cap B) \cong \pi_1(X) \times \pi_1(Y)$, $\pi_1(A) \cong \pi_1(X)$, $\pi_1(B) \cong \pi_1(Y)$. Then the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$ induce projections $\pi_1(X) \times \pi_1(Y) \to \pi_1(X)$ and $\pi_1(X) \times \pi_1(Y) \to \pi_1(Y)$, respectively. Then the normal subgroup described by van Kampens is simply the free product $\pi_1(X) * \pi_1(Y)$, since it's free on $ab^{-1}$ for $a \in \pi_1(X)$, $b \in \pi_1(Y)$. So $\pi_1(X * Y) \simeq \pi_1(X) * \pi(Y)/N = \pi_1(X) * \pi_1(Y) \implies \pi_1(X * Y) = 1$. (Man, the conflicting notation for free product and topological join is getting confusing about now).

Now let $Y$ be a union of path components $Y_i$. Let $A$ be the portion of $X * Y$ from $[0, \frac{1}{3})$, and $C_\alpha = A \cap (X * Y_i)$ open sets covering $X * Y$. Each $\cap_\alpha C_\alpha$ is path-connected since the intersections just deform onto $X$ path-connected, and so each $C_\alpha$ is simply-connected. Therefore $X * Y$ is simply-connected by van Kampen's. ■

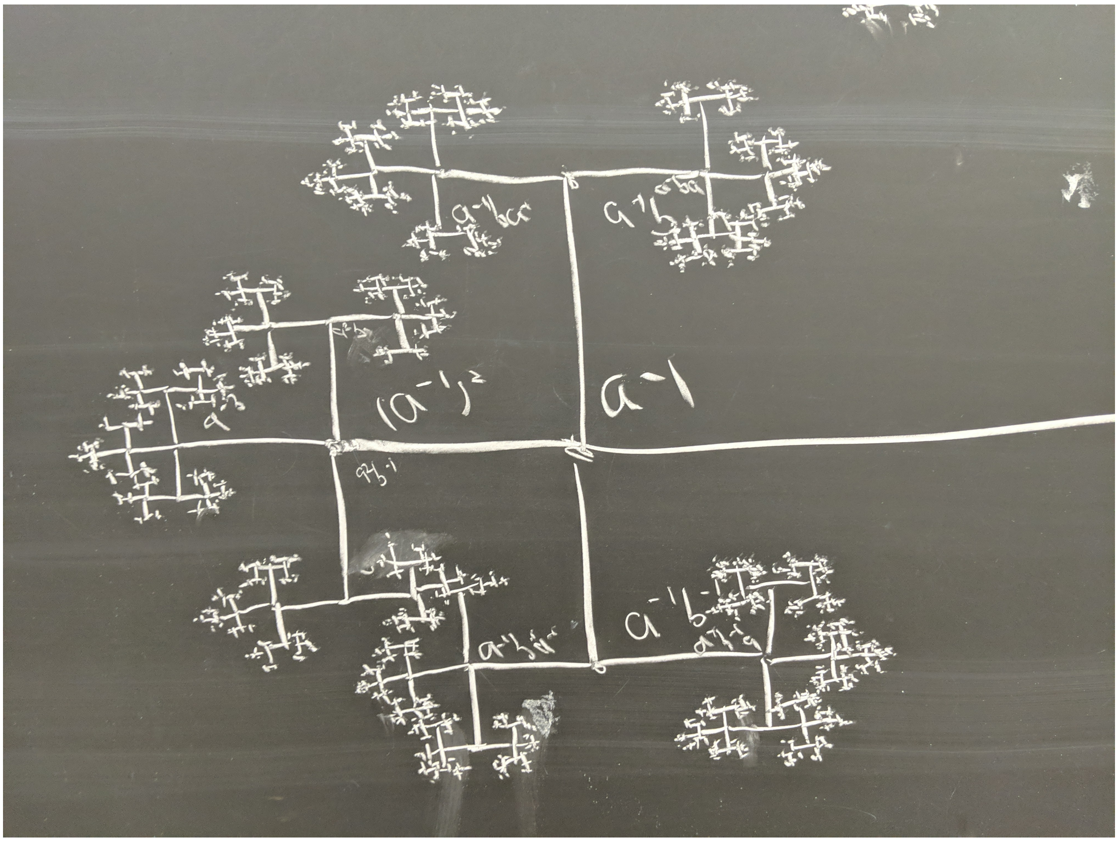
## §1.6 Problem 30 Section 1.3

**Problem.** *Draw the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.*

*Solution.* Below is the Cayley graph: we start with 1 in the center, and multiplication by $b, a, b^{-1}, a^{-1}$ is denoted by a line to the north, east, south, and west respectively. Only the first four iterations of points are labeled for brevity. It looks like a fractal/tree, similar to the Cayley graph of $F_2$, the top half of every other iteration missing because $b^2 = (b^{-1})^2 = 1$.




Here's a zoomed in pictures of the most detailed node:

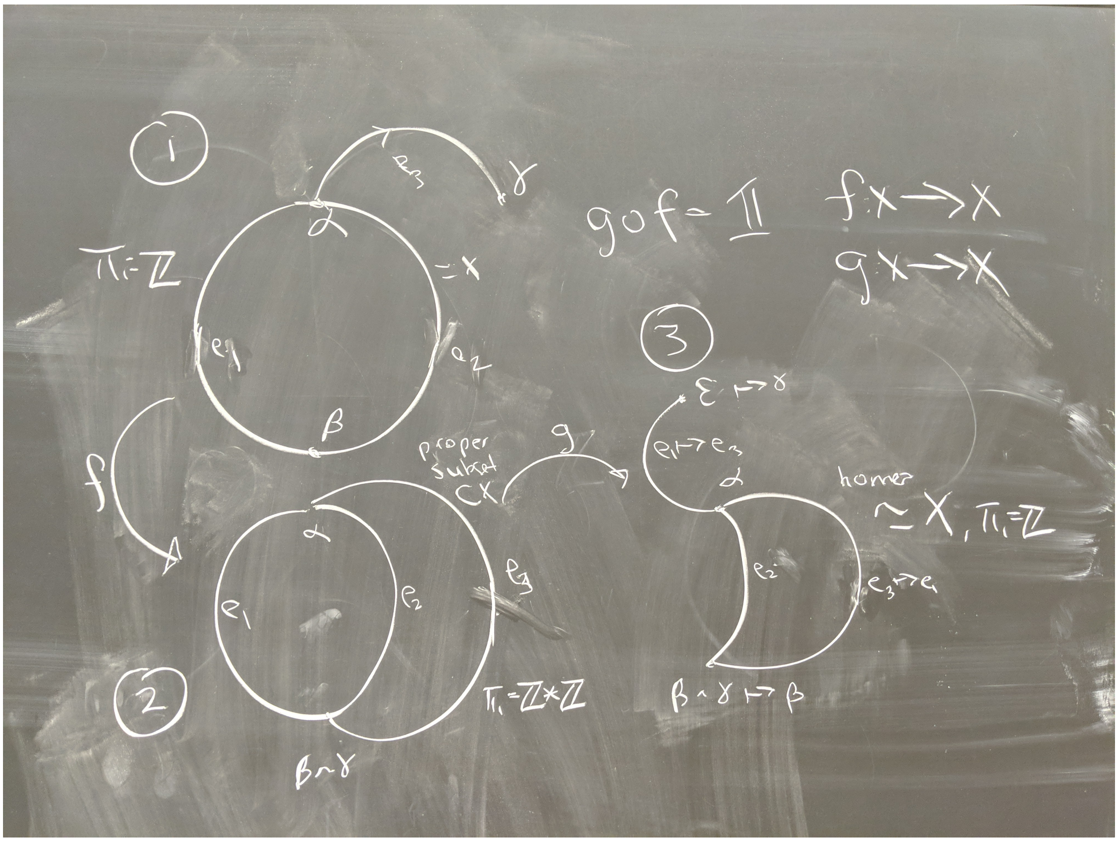



(This took a lot more time than I thought it would...) ∎

## §1.7 Problem 5 Section 1.A

**Problem.** *Construct a connected graph $X$ and maps $f, g\colon X \to X$ such that $fg = \mathbb{1}$ but $f$ and $g$ do not induce isomorphisms on $\pi_1$. [Note that $f_* g_* = \mathbb{1}$ implies that $f_*$ is surjective and $g_*$ is injective.]*

*Solution.* We construct a graph $X$ that looks like a cherry with three 0-cells $\alpha, \beta, \gamma$ and three 1-cells $e_1, e_2, e_3$ arranged as shown in the figure.




It is clear that $X$ is connected, and $\pi_1(X) = \mathbb{Z}$. $f\colon X \to X$ identifies $\beta$ with $\gamma$, so the stem of the cherry ($e_3$) becomes a new circle. Note that $f$ is onto, but not 1-1. The image of $f$ denoted $f[X]$ is a proper subset of $X$, and its fundamental group $\pi_1(f[X])$is equal to $\mathbb{Z} * \mathbb{Z}$. $g\colon X \to X$ takes $e_1$ and rips it out to make it the new stem. More precisely, $g$ takes $e_1$ and identifies its endpoint with the top point, let's call it $\varepsilon$. $g$ is 1-1, but not onto. Although the labels are now different, we can easily remap them to show that $g[f[X]]$ is homeomorphic to $X$. So $g \circ f = \mathbb{1}$, but $f$ doesn't induce an isomorphism (since $\pi_1(f[X]) = \mathbb{Z} * \mathbb{Z}$) and neither does $g$ (since $g[X]$ becomes two detached stems, implying that $\pi_1(g[X])$ is trivial), so we are done.

■