Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 380C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1 Category Theory

Today we talk about abstract nonsense! These notes will follow Evan Chen's Napkin §60 and May's "A Concise Course in Algebraic Topology" §2.

§1.1 Motivation

Why do we talk about categories? Categories rise from objects (sets, groups, topologies) and maps between them (bijections, isomorphisms, homeomorphisms). Algebraic topology speaks of maps from topologies to groups, which makes maps between categories a suitable tool for us.

Example 1.1. Here are some examples of morphisms between objects:

- A bijective homomorphism between two groups G and H is an isomorphism. What also works is two group homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ which are mutual inverses, that is $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.
- Metric (or topological) spaces X and Y are isomorphic if there exists a continuous bijection $f\colon X \to Y$ such that f^{-1} is also continuous.
- Vector spaces V and W are isomorphic if there is a bijection $T\colon V \to W$ that's a linear map (aka, T and T^{-1} are linear maps).
- Rings R and S are isomorphic if there is a bijective ring homomorphism ϕ (or two mutually inverse ring homomorphism).

§1.2 Categories

Definition 1.1 (Category). A *category* $\mathcal{A}$ consists of

- A class of *objects* , denoted $\mathrm{obj}(\mathcal{A})$.
- For any two objects $A_1, A_2 \in \mathrm{obj}(\mathcal{A})$, a class of *arrows* (also called *morphisms* or *maps* between them). Let's denote the set of arrows by $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \mathrm{obj}(\mathcal{A})$, if $f\colon A_1 \to A_2$ is an arrow and $g\colon A_2 \to A_3$ is an arrow, we can compose the two arrows to get $h = g \circ f\colon A_1 \to A_3$ an arrow, represented in the *commutative diagram* below:

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[dr, dotted, "h"] & A_2 \arrow[d, "g"] \\ & A_3 \end{tikzcd}$$

The composition operation can be denoted as a function

$$\circ\colon \mathrm{Hom}_{\mathcal{A}}(A_2, A_3) \times \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$$

for any three objects A_1, A_2, A_3. Composition must be associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$. In the diagram above, we say h *factors* through A_2.

- Every object $A \in \text{obj}_\mathcal{A}$ has a special *identity arrow* $\text{id}_\mathcal{A}$. The identity arrow has the expected properties $\text{id}_\mathcal{A} \circ f = f$ and $f \circ \text{id}_\mathcal{A} = f$.

Note. We can't use the word "set" to describe the class of objects because of some weird logic thing (there is no set of all sets). But you can think of a class as a set.

From now on, $A \in \mathcal{A}$ is the same as $A \in \text{obj}(\mathcal{A})$. A category is *small* if it has a set of objects, and *locally small* if $\text{Hom}_\mathcal{A}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathcal{A}$.

Example 1.2 (Basic Categories)**.** Here are some basic examples of categories:

- We have the category of groups Grp.
  - The objects of Grp are groups.
  - The arrows of Grp are group homomorphisms.
  - The composition of Grp is function composition.
- Describe the category CRing (of commutative rings) in a similar way.
- Consider the category Top of topological spaces, whose arrows are continuous maps between spaces.
- Also consider the category Top_* of topological spaces with a distinguished basepoint, that is, a pair (X, x_0), $x_0 \in X$. Arrows are continuous maps $f : X \to Y$ with $f(x_0) = y_0$.
- Similarly, the category of (possibly infinite-dimensional) vector spaces over a field k Vect_k has linear maps for arrows. There is even a category FDVect_k of finite-dimensional vector spaces.
- Finally, we have a category Set of sets, arrows denote any map between sets.

§1.3 Functors

Motivation: maps between categories, objects rising from other objects.

Example 1.3 (Basic Functors)**.** Here are some basic examples of functors:

- Given an algebraic structure (group, field, vector space) we can take its underlying set S: this is a functor from $\mathsf{Grp} \to \mathsf{Set}$ (or whatever you want to start with).
- If we have a set S, if we consider the vector space with basis S we get a functor $\mathsf{Set} \to \mathsf{Vect}$.
- Taking the power set of a set S gives a functor $\mathsf{Set} \to \mathsf{Set}$.

- Given a locally small category $\mathcal{A}$, we can take a pair of objects (A_1, A_2) and obtain a set $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$. This turns out to be a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$.

Finally, the most important example (WRT this course):

- In algebraic topology, we build groups like $H_1(X)$, $\pi_1(X)$ associated to topological spaces. All these group constructions are functors $\mathsf{Top} \to \mathsf{Grp}$.

Definition 1.2 (Functors)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* F takes every object of $\mathcal{A}$ to an object of $\mathcal{B}$. In addition, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. Refer to the commutative diagram:

$$\begin{tikzcd}
 & A_1 \arrow[d, "f"'] & B_1 = F(A_1) \arrow[d, "F(f)"] & \\
\mathcal{A} \ni & {} \arrow[r, dotted, "F"] & {} & \in \mathcal{B} \\
 & A_2 & B_2 = F(A_2) &
\end{tikzcd}$$

Functors also satisfy the following requirements:

- Identity arrows get sent to identity arrows, that is, for each identity arrow id_A, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

- Functors respect composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3$ are arrows in $\mathcal{A}$, then $F(g \circ f) = F(g) \circ F(f)$.

More precisely, these are covariant functors. A contravariant functor F reverses the direction of arrows, so that F sends $f \colon A_1 \to A_2$ to $F(f) \colon F(A_2) \to F(A_1)$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ instead. A category $\mathcal{A}$ has an opposite category $\mathcal{A}^{\mathrm{op}}$ with the same objects and with $\mathcal{A}^{\mathrm{op}}(A_1, A_2) = \mathcal{A}(A_2, A_1)$. A contravariant functor $F \colon \mathcal{A} \to \mathcal{B}$ is just a covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Example 1.4. We have already talked about *free* and *forgetful* functors in Example 1.3: the forgetful functors are functors from spaces to sets (the underlying set of a group) and free functors are from sets to spaces (the basis set forming a vector space).

- Another example of a forgetful functor is a functor $\mathsf{CRing} \to \mathsf{Grp}$ by sending a ring R to its abelian group $(R, +)$.

- Another example of a free functor is a functor $\mathsf{Set} \to \mathsf{Grp}$ by taking the free group generated by a set S (who would have known this is free?)

Here is a cool example: functors preserve isomorphism. If two groups are isomorphic, then they must have the same cardinality. In the language of category theory, this can be expressed as such: if $G \cong H$ in Grp and $U \colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor, then $U(G) \cong U(H)$. We can generalize this to *any* functor and category!

Theorem 1.1. *If $A_1 \cong A_2$ are isomorphic objects in $\mathcal{A}$ and $F\colon \mathcal{A} \to \mathcal{B}$ is a functor then*

$$F(A_1) \cong F(A_2).$$

Proof. Let's go diagram chasing!

$$\begin{tikzcd}
\mathcal{A} \ni & A_1 \arrow[d, shift right, "f"'] \arrow[r, dotted, "F"] & B_1 = F(A_1) \arrow[d, shift right, "F(f)"'] & \in \mathcal{B} \\
& A_2 \arrow[u, shift right, "g"'] & B_2 = F(A_2) \arrow[u, shift right, "F(g)"'] &
\end{tikzcd}$$

The main idea of the proof follows from the fact that functors preserve composition and the identity map. ☒

This is very very useful for us (people who are doing algebraic topology) because functors will preserve isomorphism between spaces (we get that homotopic spaces have isomorphic fundamental groups).

Note. As a meme (or not really, but it's still funny), we can construct the category Cat whose objects are categories and arrows are functors.

§1.4 Natural Transformations

We talked about maps between objects which led to categories, and then maps between categories which lead to functors. Now let's talk about maps between functors, the natural transformation: this is actually not too strange (recall the homotopy, a "deformation" from a map to another map).

In this case, we also want to pull a map (functor) G to another map G by composing a bunch of arrows in the target space $\mathcal{B}$.

Definition 1.3 (Natural Transformations)**.** Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be two functors. A *natural transformation* $\alpha\colon F \to G$ denoted

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{B} \end{tikzcd}$$

consists of, for each $A \in \mathcal{A}$ an arrow $\alpha_A \in \mathrm{Hom}_{\mathcal{B}}(F(A), G(A))$, which is called the component of α at A. Pictorially, it looks like this:

$$\begin{tikzcd} & F(A) \in \mathcal{B} \arrow[dd, "\alpha_A"] \\ \mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\ & G(A) \in \mathcal{B} \end{tikzcd}$$

The α_A are subject to the "naturality" requirement such that for any $A_1 \xrightarrow{f} A_2$, the following diagram commutes:

$$\begin{tikzcd} F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\ G(A_1) \arrow[r, "G(f)"'] & G(A_2) \end{tikzcd}$$

The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (like in a homotopy from f to g the point $f(t)$ gets deformed to the point $g(t)$ continuously). Think of f representing the homotopy and the basepoints being $F(A_1), G(A_1)$ to $F(A_2), G(A_2)$.

Natural transformations can be composed. Take two natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$. Consider the following commutative diagram:

$$\begin{tikzcd} & F(A) \arrow[d, "\alpha_A"] \\ \mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[r, dotted, "G"] \arrow[dr, dotted, "H"'] & G(A) \arrow[d, "\beta_A"] \\ & H(A) \end{tikzcd}$$

We can also construct inverses: suppose α is a natural transformation such that α_A is an isomorphism for each A. Then we construct an inverse arrow β_A in the following way:

$$\begin{tikzcd}
 & F(A) \in \mathcal{B} \arrow[dd, shift right, "\alpha_A"'] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\
 & G(A) \in \mathcal{B} \arrow[uu, shift right, "\beta_A"']
\end{tikzcd}$$

We say α is a *natural isomorphism*. Then $F(A) \cong G(A)$ *naturally* in A (and β is an isomorphism too!) We write $F \cong G$ to show that the functors are naturally isomorphic.

Example 1.5. If $F\colon \mathsf{Set} \to \mathsf{Grp}$ is the free functor that sends a set to the free group on such set and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor sending a free group to its generating set, then we have a natural inclusion of $S \hookrightarrow UF(S)$. THe functors F and U are left and right adjoint to each other, in the sense that we have a natural isomorphism

$$\mathsf{Grp}(F(S), A) \cong \mathsf{Set}(S, U(A))$$

for a set S and an abelian group A. This expresses the "universal property" of free objects: a map of sets $S \to U(A)$ extends uniquely to a homomorphism of groups $F(S) \to A$.

Definition 1.4. Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there are functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $FG \to \mathrm{Id}$ and $GF \to \mathrm{Id}$, where the Id are the respective identity functors.

§1.5 Homotopy Categories and Homotopy Equivalence

Todo

§2 Free Groups

Not to be confused with free *abelian* groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39 and Hatcher §1.2.

§2.1 Words and Reduced Words

Let A_i be a set of elements (not necessarily finite). We say A is an *alphabet* and think of the $a_i \in A$ as *letters*. Symbols of the form a_i^n are *syllables* and *words* are a finite string of syllables. We denote the *empty word* 1 as the word with no syllables.

Example 2.1. Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to a_i^{m+1} (*elementary contractions*) or replacing a_i^0 by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a *reduced word*.

Example 2.2. The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

§2.2 Free Groups

Denote the set of all reduced words from our alphabet A as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words w_1 and w_2 in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of w_2 onto w_1.

Example 2.3. If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet A. Man, the weather outside today is nice.

Definition 2.1 (Free Group)**.** The group $F[A]$ described above is the *free group generated by A*.

Sometimes we have a group G and a generating set $A = \{a_i \mid\in I\}$, and we want to know whether or not G is *free* on $\{a_i\}$, that is, G is the free group generated by $\{a_i\}$.

Definition 2.2 (Free Generators)**.** If G is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi\colon G \to F[A]$ such that $\phi(a_i) = a_i$, then G is *free on A*, and the a_i are *free generators of G*. A group is *free* if it is free on some nonempty set A.

Oh you'll be free... free indeed...

Example 2.4. $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on n letters" as opposed to the "free group on n generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

Example 2.5. $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

Theorem 2.1. *If G is free on A and B, then A and B have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

Proof. Refer "to the literature". ☒

Definition 2.3 (Rank)**.** If G is free on A, then the number of letters in A is the *rank of the free group G*.

Theorem 2.2. *Two free groups are isomorphic if and only if they have the same rank.*

Proof. Immediate. ☒

Theorem 2.3. *A nontrivial proper subgroup of a free group is free.*

Proof. Back "to the literature". ☒

Example 2.6. Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

§2.3 Homomorphisms of Free Groups

Theorem 2.4. *Let G be generated by $A = \{a_i \mid \in I\}$ and let G' be any group. If $a_i{}'$ for $i \in I$ are any elements in G' not necessarily distinct, then there is at most one homomorphism $\phi\colon G \to G'$ such that $\phi(a_i) = a_i{}'$. If G is free on A, then there is exactly one such homomorphism.*

Proof. Let ϕ be a homomorphism from G into G'such that $\phi(a_i) = a_i{}'$. Then any $x \in G$ can be written as a finite product of the generators a_i, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the a_i not necessarily distinct. Since ϕ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i{}'$.

Now suppose that G is free on A, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi \colon G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in G, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ☒

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

Corollary 2.1. *Every group G' is a homomorphic image of a free group G.*

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 2.4 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i{}'$. Clearly the image of G under ψ is all of G'. ☒

Only the free group on one letter is abelian.

§2.4 Free Products of Groups

Definition 2.4 (Free Products)**.** As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of w_1 and the beginning of w_2 belong to the same G_α? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of a and b are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, \ldots$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha \colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi \colon *_\alpha G_\alpha \to H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under φ.

Example 2.7. For a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

§3 Van Kampen's Theorem

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups. These notes follow Hatcher §1.2.

§3.1 The van Kampen Theorem

Let's take a space X and say it's the union of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha \colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi \colon *_\alpha \, \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is often onto but in general, we can expect Φ to have a nontrivial kernel.

For if $i_{\alpha\beta} \colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both of these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Van Kampen says under fairly broad hypotheses that this determines all of Φ.

Theorem 3.1. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi \colon *_\alpha \, (A_\alpha) \to \pi_1(X)$$

is onto. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N.$$

Example 3.1 (Wedge Sums)**.** I like the visual of the wedge sum but the terminology of the smash product. We define the wedge sum $\bigvee_\alpha X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ as the disjoint union $\amalg_\alpha X_\alpha$ with all the basepoints x_α identified to a single point.