# Complex Analysis Lecture Notes

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These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`.

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## §1.1 Accumulation Points

**Definition 1.1.** A connected open set is a *domain.*

**Definition 1.2.** A *region* is a domain that contains none, some, or all of its boundary.

**Definition 1.3** (Bounded Set)**.** A set $S$ is bounded if

$$S \subseteq B(x_0, \epsilon).$$

for some $x_0 \in \mathbb{C}$, $\epsilon > 0$.

**Definition 1.4** (Accumulation Points)**.** $z_0$ is an accumulation point of $S$ if for all balls $B(z_0, \frac{1}{m})$ centered at $z_0$, we have

$$B(z_0, \frac{1}{m}) \setminus \{z_0\} \cap S \neq \emptyset.$$

**Example 1.1.** Let $S = \mathbb{Q}$. Then $\frac{1}{2}, \sqrt{2}$ etc are accumulation points of $S$ (this relies on the fact that $\mathbb{Q}$ is *dense* in $\mathbb{R}$). This example shows that accumulation points don't have to be in the set themselves.

**Theorem 1.1.** *We have $S$ is closed if and only if $S$ contains all of its accumulation points, the set of which is denoted $S'$. Furthermore, the closure of $S$ denoted $\bar{S}$ is equal to $S \cup S'$.*

*Proof.* $\implies$ Accumulation points are either in the boundary of $S$ or in $S$ itself. Since $S$ is closed, we have $S' \subseteq S$.
$\impliedby$ If $z_0 \in \partial S \cap S^c$ it would be an accumulation point of $S$, a contradiction. So $\partial S \subseteq S \implies S$ is closed. (I'll try to write a better proof later). ⊠

A quick summary of basic p-set topology:

1. $S$ is open $\iff S = S^\circ$,
2. $S$ is closed $\iff S^c$ is open,
3. $S$ is open $\iff S$ contains none of $\partial S$,
4. $S$ is closed $\iff S$ contains all of $\partial S$,
5. $S$ is closed $\iff S$ contains all of $S'$.

## §1.2 Limits

Consider a map $f\colon \operatorname{Dom}(f) \to \mathbb{C}$, $\operatorname{Ran}(f) \subseteq \mathbb{C}$ (I prefer the notation $f\colon X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$, and $\operatorname{Ran}(f) = f[X]$. The fact that $f$ is well defined on $X$ holds because define $X$ to be a set on which $f$ is well defined, duh).

We want to talk about whether a function is continuous or not. Intuitively, a function is continuous if points in the image being "close" together imply that points in the preimage are also "close" together (the preimage of an open set is open).

**Definition 1.5** (Epsilon Delta Limits)**.** For $z_0$ an accumulation point of some subset $X$ of $\mathbb{C}$ (a region), $\lim_{z\to z_0} f(z)$ exists and has a value of $L \iff$ for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon,$$

where $z \in X$. The modulus is just a distance metric: so the epsilon delta definition is the same as what I said earlier, if points are close to each other in the codomain ($|f(z)-L| < \epsilon$), then such points are close to each other in the domain ($0 < |z-z_0| < \delta$).

Some notes: the limit is only defined when $z_0$ is an accumulation point. This why accumulation points are also sometimes referred to as *limit points*.

## §1.3 Continuity

**Definition 1.6** (Continuity)**.** $f$ is continuous at $z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$. $f$ is said to be continuous on a set $X$ if for all $x \in X$, $f$ is continuous at $x$.

We want to *analyze* a function $f(z)$, let $z = (x, y)$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$. $u(x, y) = \operatorname{Re} f$ and $v(x, y) = \operatorname{Im} f$.

**Theorem 1.2.** *We have*

$$\lim_{z\to z_0} f(z) = L \iff \begin{cases} \lim_{z\to z_0} \operatorname{Re} f(z) \to \operatorname{Re} L \\ \lim_{z\to zo} \operatorname{Im} f(z) \to \operatorname{Im} L. \end{cases}$$

*Proof.* Homework. ⊠

**Theorem 1.3.** *Let $f\colon X \to \mathbb{C}, g\colon Y \to \mathbb{C}$. For an accumulation point $z_0$ of $X \cap Y$, if $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then (excuse the abuse of notation)*

1. $\lim(f + g) = L + M$,
2. $\lim fg = LM$,
3. $\lim \frac{f}{g} = \frac{L}{M}$ *if* $M \neq 0$.

*Proof.* Same as the ones you'd find in any analysis course. ⊠

Continuity of sums, products, and quotients of functions follow from the above theorem. Now we turn our attention to the composition of functions.

**Theorem 1.4.** *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$. Let $z_0$ be an accumulation point of $X$. Then if $f$ is continuous at $z_0$ and $g$ is continous at $f(z_0)$, we have $f \circ g$ continuous at $z_0$.*

**Example 1.2.** $f(z) = |z^m|$ for a fixed $m$ is equal to $(g \circ h)(z)$ where $h(z) = z^m$ and $g(w) = |w|$. Both $h$ and $g$ are continuous on $\mathbb{C}$, so $|z^m|$ is also continuous everywhere.

**Example 1.3.** The identity map is continuous. This is trivial (let $\delta = \epsilon$). It follows that maps of the form $z^n$ is continuous for some positive integer $n$.

**Corollary 1.1.** *Functions of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

*where $p(z)$ and $g(z)$ are polynomials are continuous given $g(z) \neq 0$.*

**Example 1.4.** Let $f(z) = \frac{z}{\bar{z}}$, $z \neq 0$. Consider $z = x + iy$ near 0 with $x \neq 0, y = 0$, then $f(z) = 1$. If $x = 0, y \neq 0$ then $f(z) = -1$. Therefore $\lim_{z \to z_0} \frac{z}{\bar{z}}$ does not exist (standard technique for proving multivariate limits don't exist).