# Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 380C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TeX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

## Contents

# §1 Free Groups

Not to be confused with free *abelian* groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39 and Hatcher §1.2.

## §1.1 Words and Reduced Words

Let $A_i$ be a set of elements (not necessarily finite). We say $A$ is an *alphabet* and think of the $a_i \in A$ as *letters*. Symbols of the form $a_i^n$ are *syllables* and *words* are a finite string of syllables. We denote the *empty word* 1 as the word with no syllables.

**Example 1.1.** Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to $a_i^{m+1}$ (*elementary contractions*) or replacing $a_i^0$ by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a *reduced word*.

**Example 1.2.** The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

## §1.2 Free Groups

Denote the set of all reduced words from our alphabet $A$ as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words $w_1$ and $w_2$ in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of $w_2$ onto $w_1$.

**Example 1.3.** If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet $A$. Man, the weather outside today is nice.

**Definition 1.1** (Free Group)**.** The group $F[A]$ described above is the *free group generated by $A$*.

Sometimes we have a group $G$ and a generating set $A = \{a_i \mid \in I\}$, and we want to know whether or not $G$ is *free* on $\{a_i\}$, that is, $G$ is the free group generated by $\{a_i\}$.

**Definition 1.2** (Free Generators)**.** If $G$ is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi \colon G \to F[A]$ such that $\phi(a_i) = a_i$, then $G$ is *free on $A$*, and the $a_i$ are *free generators of $G$*. A group is *free* if it is free on some nonempty set $A$.

Oh you'll be free... free indeed...

**Example 1.4.** $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on $n$ letters" as opposed to the "free group on $n$ generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

**Example 1.5.** $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

**Theorem 1.1.** *If $G$ is free on $A$ and $B$, then $A$ and $B$ have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

*Proof.* Refer "to the literature". ☒

**Definition 1.3** (Rank)**.** If $G$ is free on $A$, then the number of letters in $A$ is the *rank of the free group $G$*.

**Theorem 1.2.** *Two free groups are isomorphic if and only if they have the same rank.*

*Proof.* Immediate. ☒

**Theorem 1.3.** *A nontrivial proper subgroup of a free group is free.*

*Proof.* Back "to the literature". ☒

**Example 1.6.** Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The $y_k$ for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

## §1.3 Homomorphisms of Free Groups

**Theorem 1.4.** *Let $G$ be generated by $A = \{a_i \mid \in I\}$ and let $G'$ be any group. If $a_i{}'$ for $i \in I$ are any elements in $G'$ not necessarily distinct, then there is at most one homomorphism $\phi \colon G \to G'$ such that $\phi(a_i) = a_i{}'$. If $G$ is free on $A$, then there is exactly one such homomorphism.*

*Proof.* Let $\phi$ be a homomorphism from $G$ into $G'$such that $\phi(a_i) = a_i{}'$. Then any $x \in G$ can be written as a finite product of the generators $a_i$, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the $a_i$ not necessarily distinct. Since $\phi$ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i{}'$.

Now suppose that $G$ is free on $A$, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi\colon G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in $G$, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements $x$ and $y$ in $G$, so $\psi$ is indeed a homomorphism. ⊠

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

**Corollary 1.1.** *Every group $G'$ is a homomorphic image of a free group $G$.*

*Proof.* Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid\in I\}$ be a set with the same number of elements as $G'$. Let $G = F[A]$. Then by Theorem 1.4 there exists a homomorphism $\psi$ mapping $G$ into $G'$ such that $\psi(a_i) = a_i{}'$. Clearly the image of $G$ under $\psi$ is all of $G'$. ⊠

Only the free group on one letter is abelian.

## §1.4 Free Products of Groups

**Definition 1.4** (Free Products)**.** As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter $g_i$ belongs to a group $G_{\alpha_i}$ and is not the identity element of $G_{\alpha_i}$, and adjacent letters $g_i$ and $g_{i+1}$ belong to different groups $G_\alpha$, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of $w_1$ and the beginning of $w_2$ belong to the same $G_\alpha$? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of $a$ and $b$ are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, \ldots$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha\colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi\colon *_\alpha G_\alpha \to H$. Namely, the value of $\varphi$ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define $\varphi$ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under $\varphi$.

**Example 1.7.** For a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.