Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 380C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't TEX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1 Category Theory

Today we talk about abstract nonsense! These notes will follow Evan Chen's Napkin §60 and May's "A Concise Course in Algebraic Topology" §2.

§1.1 Motivation

Why do we talk about categories? Categories rise from objects (sets, groups, topologies) and maps between them (bijections, isomorphisms, homeomorphisms). Algebraic topology speaks of maps from topologies to groups, which makes maps between categories a suitable tool for us.

Example 1.1. Here are some examples of morphisms between objects:

- A bijective homomorphism between two groups G and H is an isomorphism. What also works is two group homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ which are mutual inverses, that is $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.
- Metric (or topological) spaces X and Y are isomorphic if there exists a continuous bijection $f\colon X \to Y$ such that f^{-1} is also continuous.
- Vector spaces V and W are isomorphic if there is a bijection $T\colon V \to W$ that's a linear map (aka, T and T^{-1} are linear maps).
- Rings R and S are isomorphic if there is a bijective ring homomorphism ϕ (or two mutually inverse ring homomorphism).

§1.2 Categories

Definition 1.1 (Category)**.** A *category* $\mathcal{A}$ consists of

- A class of *objects* , denoted $\mathrm{obj}(\mathcal{A})$.
- For any two objects $A_1, A_2 \in \mathrm{obj}(\mathcal{A})$, a class of *arrows* (also called *morphisms* or *maps* between them). Let's denote the set of arrows by $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \mathrm{obj}(\mathcal{A})$, if $f\colon A_1 \to A_2$ is an arrow and $g\colon A_2 \to A_3$ is an arrow, we can compose the two arrows to get $h = g \circ f\colon A_1 \to A_3$ an arrow, represented in the *commutative diagram* below:

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[dr, dotted, "h"] & A_2 \arrow[d, "g"] \\ & A_3 \end{tikzcd}$$

  The composition operation can be denoted as a function

$$\circ\colon \mathrm{Hom}_{\mathcal{A}}(A_2, A_3) \times \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$$

  for any three objects A_1, A_2, A_3. Composition must be associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$. In the diagram above, we say h *factors* through A_2.

- Every object $A \in \mathrm{obj}_{\mathcal{A}}$ has a special *identity arrow* $\mathrm{id}_{\mathcal{A}}$. The identity arrow has the expected properties $\mathrm{id}_{\mathcal{A}} \circ f = f$ and $f \circ \mathrm{id}_{\mathcal{A}} = f$.

Note. We can't use the word "set" to describe the class of objects because of some weird logic thing (there is no set of all sets). But you can think of a class as a set.

From now on, $A \in \mathcal{A}$ is the same as $A \in \mathrm{obj}(\mathcal{A})$. A category is *small* if it has a set of objects, and *locally small* if $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathcal{A}$.

Example 1.2 (Basic Categories)**.** Here are some basic examples of categories:

- We have the category of groups Grp.
  - The objects of Grp are groups.
  - The arrows of Grp are group homomorphisms.
  - The composition of Grp is function composition.
- Describe the category CRing (of commutative rings) in a similar way.
- Consider the category Top of topological spaces, whose arrows are continuous maps between spaces.

- Also consider the category Top_* of topological spaces with a distinguished basepoint, that is, a pair (X, x_0), $x_0 \in X$. Arrows are continuous maps $f\colon X \to Y$ with $f(x_0) = y_0$.

- Similarly, the category of (possibly infinite-dimensional) vector spaces over a field k Vect_k has linear maps for arrows. There is even a category FDVect_k of finite-dimensional vector spaces.

- Finally, we have a category Set of sets, arrows denote any map between sets.

Definition 1.2 (Isomorphism)**.** An arrow $A_1 \xrightarrow{f} A_2$ is an *isomorphism* if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \mathrm{id}_{A_2}$ and $g \circ f = \mathrm{id}_{A_1}$. We say A_1 and A_2 are *isomorphic*, denoted $A_1 \cong A_2$.

Remark 1.1. In the category Set, $X \cong Y \iff |X| = |Y|$.

In other fields, we can tell a lot about the structure of an object by looking at maps between them. In category theory, we *only* look arrows, and ignore what the objects themselves are.

Example 1.3 (Posets are Categories)**.** Let $\mathcal{P}$ be a poset. Then we can construct a category P for it as follows:

- The objects of P are elements of $\mathcal{P}$.

- We define the arrows of P as follows:
  - For every object $p \in P$, we add an identity arrow id_p, and
  - For any pair of distinct objects $p \leq q$, we add a single arrow $p \to q$.

  There are no other arrows.

- We compose arrows in the only way possible, examining the order of the first and last object.

Here's a figure depicting the category of a poset $\mathcal{P}$ on four objects $\{a, b, c, d\}$ with $a \leq b$ and $a \leq c \leq d$.

$$
\begin{tikzcd}
 & {[a]} \arrow[loop above, "\mathrm{id}_a"] \arrow[dl, "a\leq b"'] \arrow[dr, "a\leq c"] \arrow[dd, "a\leq d"] & \\
{[b]} \arrow[loop left, "\mathrm{id}_b"] & & {[c]} \arrow[loop right, "\mathrm{id}_c"] \arrow[dl, "c\leq d"] \\
 & {[d]} \arrow[loop below, "\mathrm{id}_d"] &
\end{tikzcd}
$$

Note that no two distinct objects of a poset are isomorphic.

This shows that categories don't have to refer to just structure preserving maps between sets (these are called "concrete categories".

Example 1.4 (Groups as a category with one object)**.** A group G can be thought of as a category $\mathcal{G}$ with one object $*$, all of whos arrows are isomorphisms.

If the universe were structured differently and kids learned category theory before groups, symmetries transforming X into itself would be a natural extension of categories that transform X into other objects, a special case in which all the maps are invertible. Alas, this is not the right timeline.

Example 1.5 (Deriving Categories)**.** We can make categories from other categories!

(a) Given a category $\mathcal{A}$, we can construct the *opposite category* A^{op}, which is the same as $\mathcal{A}$ but with all the arrows reversed.

(b) Given categories $\mathcal{A}$ and $\mathcal{B}$, we can construct the *product category* $\mathcal{A} \times \mathcal{B}$ as follows: the objects are pairs (A, B) for $A \in \mathcal{A}$, $B \in \mathcal{B}$, and the arrows from (A_1, B_1) to (A_2, B_2) are pairs

$$\left(A_1 \xrightarrow{f} A_2, B_1 \xrightarrow{g} B_2\right).$$

The composition is just pairwise composition, and the identity is the pair of identity functions on A and B.

§1.3 Special objects in categories

Some categories have things called *initial objects*. For example the empty set $\varnothing$, the trivial group, the empty space, initial element in a poset, etc. More interestingly: the initial object of CRing is the ring $\mathbb{Z}$.

Definition 1.3 (Initial object)**.** An *initial object* of $\mathcal{A}$ is an object $A_{\text{init}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{init}}$), there is exactly one arrow from A_{init} to A.

The yang to this yin is the *terminal object*:

Definition 1.4 (Terminal object)**.** A *terminal object* of $\mathcal{A}$ is an object $A_{\text{final}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{final}}$, there is exactly one arrow from A to A_{final}.

For example, the terminal object of Set is $\{*\}$, Grp is $\{1\}$, CRing is the zero ring, Top is the single point space, and a poset its maximal element (if one exists).

§1.4 Monic and epic maps

Injectivity and surjectivity don't really make sense when talking about categories. todo

§1.5 Functors

Motivation: maps between categories, objects rising from other objects.

Example 1.6 (Basic Functors)**.** Here are some basic examples of functors:

- Given an algebraic structure (group, field, vector space) we can take its underlying set S: this is a functor from $\mathsf{Grp} \to \mathsf{Set}$ (or whatever you want to start with).
- If we have a set S, if we consider the vector space with basis S we get a functor $\mathsf{Set} \to \mathsf{Vect}$.
- Taking the power set of a set S gives a functor $\mathsf{Set} \to \mathsf{Set}$.
- Given a locally small category $\mathcal{A}$, we can take a pair of objects (A_1, A_2) and obtain a set $\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)$. This turns out to be a functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$.

Finally, the most important example (WRT this course):

- In algebraic topology, we build groups like $H_1(X)$, $\pi_1(X)$ associated to topological spaces. All these group constructions are functors $\mathsf{Top} \to \mathsf{Grp}$.

Definition 1.5 (Functors)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* F takes every object of $\mathcal{A}$ to an object of $\mathcal{B}$. In addition, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. Refer to the commutative diagram:

$$\begin{tikzcd}
 & A_1 \arrow[d, "f"'] & B_1 = F(A_1) \arrow[d, "F(f)"] & \\
\mathcal{A} \ni & A_2 & B_2 = F(A_2) & \in \mathcal{B}
\end{tikzcd}$$

$$f \overset{F}{\dashrightarrow} F(f)$$

Functors also satisfy the following requirements:

- Identity arrows get sent to identity arrows, that is, for each identity arrow id_A, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
- Functors respect composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3$ are arrows in $\mathcal{A}$, then $F(g \circ f) = F(g) \circ F(f)$.

More precisely, these are covariant functors. A contravariant functor F reverses the direction of arrows, so that F sends $f\colon A_1 \to A_2$ to $F(f)\colon F(A_2) \to F(A_1)$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ instead. A category $\mathcal{A}$ has an opposite category $\mathcal{A}^{\text{op}}$ with the same objects and with $\mathcal{A}^{\text{op}}(A_1, A_2) = \mathcal{A}(A_2, A_1)$. A contravariant functor $F\colon \mathcal{A} \to \mathcal{B}$ is just a covariant functor $\mathcal{A}^{\text{op}} \to \mathcal{B}$.

Example 1.7. We have already talked about *free* and *forgetful* functors in Example 1.3: the forgetful functors are functors from spaces to sets (the underlying set of a group) and free functors are from sets to spaces (the basis set forming a vector space).

- Another example of a forgetful functor is a functor $\mathsf{CRing} \to \mathsf{Grp}$ by sending a ring R to its abelian group $(R, +)$.

- Another example of a free functor is a functor Set $\to$ Grp by taking the free group generated by a set S (who would have known this is free?)

Here is a cool example: functors preserve isomorphism. If two groups are isomorphic, then they must have the same cardinality. In the language of category theory, this can be expressed as such: if $G \cong H$ in Grp and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor, then $U(G) \cong U(H)$. We can generalize this to *any* functor and category!

Theorem 1.1. *If $A_1 \cong A_2$ are isomorphic objects in $\mathcal{A}$ and $F\colon \mathcal{A} \to \mathcal{B}$ is a functor then*

$$F(A_1) \cong F(A_2).$$

Proof. Let's go diagram chasing!

$$\begin{tikzcd} & A_1 \arrow[d, shift right, "f"'] \arrow[rr, dotted, "F", start anchor={[yshift=-30pt]}, end anchor={[yshift=-30pt]}] & & B_1 = F(A_1) \arrow[d, shift right, "F(f)"'] & \\ \mathcal{A} \ni & A_2 \arrow[u, shift right, "g"'] & & B_2 = F(A_2) \arrow[u, shift right, "F(g)"'] & \in \mathcal{B} \end{tikzcd}$$

The main idea of the proof follows from the fact that functors preserve composition and the identity map. ⊠

This is very very useful for us (people who are doing algebraic topology) because functors will preserve isomorphism between spaces (we get that homotopic spaces have isomorphic fundamental groups).

Note. As a meme (or not really, but it's still funny), we can construct the category Cat whose objects are categories and arrows are functors.

§1.6 Homotopy Categories and Homotopy Equivalence

Let Top_* be the category of pointed topological spaces. Then the fundamental group gives a functor $\mathsf{Top}_* \to \mathsf{Grp}$. When we have a suitable relation of homotopy between maps in a category $\mathcal{C}$, we define the homotopy category $\mathsf{Ho}(\mathcal{C})$ to be the category sharing the same objects as $\mathcal{C}$, but morphisms the homotopy classes of maps. On Top_*, we require homotopies to map basepoint to basepoint, and we get the homotopy category hTop_* of pointed spaces.

Homotopy equivalences in $\mathcal{C}$ are isomorphisms in $\mathsf{Ho}(\mathcal{C})$. More concretely, recall that a map $f\colon X \to Y$ is a homotopy equivalence if there is a map $g\colon Y \to X$ such that both $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In the language of category theory, we can obtain the analogous notion of a pointed homotopy equivalence. Functors carry isomorphisms to isomorphisms, so then the pointed homotopy equivalence will induce an isomorphism of fundamental groups. This also holds, but less obviously, for the category of non pointed homotopy equivalences.

Theorem 1.2. *If $f\colon X \to Y$ is a homotopy equivalence, then*

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is an isomorphism for all $x \in X$.

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. By our homotopy invariance diagram, we see that the composites

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x))$$

and

$$\pi_1(Y, y) \xrightarrow{g_*} \pi_1(X, g(y)) \xrightarrow{f_*} \pi_1(Y, (f \circ g)(y))$$

are isomorphisms determined by paths between basepoints given by chosen homotopies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Then in each displayed composite, the first map is a monomorphism and the second is an epimorphism. Taking $y = f(x)$ in the second composite, we see that the second map in the first composite is an isomorphism. Therefore so is the first map, and we are done. ⊠

A space X is said to be contractible if it is homotopy equivalent to a point.

Corollary 1.1. *The fundamental group of a contractible space is zero.*

§1.7 Natural Transformations

We talked about maps between objects which led to categories, and then maps between categories which lead to functors. Now let's talk about maps between functors, the natural transformation: this is actually not too strange (recall the homotopy, a "deformation" from a map to another map).

In this case, we also want to pull a map (functor) F to another map G by composing a bunch of arrows in the target space $\mathcal{B}$.

Definition 1.6 (Natural Transformations)**.** Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be two functors. A *natural transformation* $\alpha\colon F \to G$ denoted

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B}
\end{tikzcd}$$

consists of, for each $A \in \mathcal{A}$ an arrow $\alpha_A \in \mathrm{Hom}_{\mathcal{B}}(F(A), G(A))$, which is called the component of α at A. Pictorially, it looks like this:

$$\begin{tikzcd}
& F(A) \in \mathcal{B} \arrow[dd, "\alpha_A"] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\
& G(A) \in \mathcal{B}
\end{tikzcd}$$

The α_A are subject to the "naturality" requirement such that for any $A_1 \xrightarrow{f} A_2$, the following diagram commutes:

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\
G(A_1) \arrow[r, "G(f)"'] & G(A_2)
\end{tikzcd}$$

The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (like in a homotopy from f to g the point $f(t)$ gets deformed to the point $g(t)$ continuously). Think of f representing the homotopy and the basepoints being $F(A_1), G(A_1)$ to $F(A_2), G(A_2)$.

Natural transformations can be composed. Take two natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$. Consider the following commutative diagram:

$$\begin{tikzcd}
& F(A) \arrow[d, "\alpha_A"] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[r, dotted, "G"] \arrow[dr, dotted, "H"'] & G(A) \arrow[d, "\beta_A"] \\
& H(A)
\end{tikzcd}$$

We can also construct inverses: suppose α is a natural transformation such that α_A is an isomorphism for each A. Then we construct an inverse arrow β_A in the following way:

$$\begin{tikzcd}
& F(A) \in \mathcal{B} \arrow[dd, shift right, "\alpha_A"'] \\
\mathcal{A} \ni A \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] & \\
& G(A) \in \mathcal{B} \arrow[uu, shift right, "\beta_A"']
\end{tikzcd}$$

We say α is a *natural isomorphism*. Then $F(A) \cong G(A)$ *naturally* in A (and β is an isomorphism too!) We write $F \cong G$ to show that the functors are naturally isomorphic.

Example 1.8. If $F\colon \mathsf{Set} \to \mathsf{Grp}$ is the free functor that sends a set to the free group on such set and $U\colon \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor sending a free group to its generating set, then we have a natural inclusion of $S \hookrightarrow UF(S)$. The functors F and U are left and right adjoint to each other, in the sense that we have a natural isomorphism

$$\mathsf{Grp}(F(S), A) \cong \mathsf{Set}(S, U(A))$$

for a set S and an abelian group A. This expresses the "universal property" of free objects: a map of sets $S \to U(A)$ extends uniquely to a homomorphism of groups $F(S) \to A$.

Definition 1.7. Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there are functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $FG \to \mathrm{Id}$ and $GF \to \mathrm{Id}$, where the Id are the respective identity functors.

§1.8 The Yoneda lemma

Definition 1.8 (The functor category)**.** The *functor category* of two categories $\mathcal{A}$ and $\mathcal{B}$, denoted $[\mathcal{A}, \mathcal{B}]$ is defined as follows:

- The objects of $[\mathcal{A}, \mathcal{B}]$ are (covariant) functors $F\colon \mathcal{A} \to \mathcal{B}$, and
- The morphisms are natural transformations $\alpha\colon F \to G$.

todo

§2 Free Groups

Not to be confused with free *abelian* groups. Whether or not we can count is uncertain, but can we even spell? These notes will follow Fraleigh §39,40 and Hatcher §1.2.

§2.1 Words and Reduced Words

Let A_i be a set of elements (not necessarily finite). We say A is an *alphabet* and think of the $a_i \in A$ as *letters*. Symbols of the form a_i^n are *syllables* and *words* are a finite string of syllables. We denote the *empty word* 1 as the word with no syllables.

Example 2.1. Let $A = \{a_1, a_2, a_3\}$. Then

$$a_1 a_3^{-4} a_2^2 a_3,\ a_2^3 a_2^{-1} a_3 a_1, \text{ and } a_3^2$$

are all words (given that $a_i^1 = a_i$).

We can reduce $a_i^m a_i^n$ to a_i^{m+1} (*elementary contractions*) or replacing a_i^0 by 1 (dropping something out of the word). Using a finite number of elementary contractions, we get something called a *reduced word*.

Example 2.2. The reduced word of $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-7}$ is $a_2^2 a_3 a_1^{-5}$.

Is it obvious or not that the reduced form of a word is unique? Does it stay the same rel elementary contractions? Apparently you have to be a great mathematician to know.

§2.2 Free Groups

Denote the set of all reduced words from our alphabet A as $F[A]$. We give $F[A]$ a group structure in the natural way: for two words w_1 and w_2 in $F[A]$, let $w_1 \cdot w_2$ be the result by string concatenation of w_2 onto w_1.

Example 2.3. If $w_1 = a_2^3 a_1^{-5} a_3^2$ and $w_2 = a_3^{-2} a_1^2 a_3 a_2^{-2}$, then $w_1 \cdot w_2 = a_2^3 a_1^{-3} a_3 a_2^{-2}$.

"It would seem obvious" that this indeed forms a group on the alphabet A. Man, the weather outside today is nice.

Definition 2.1 (Free Group)**.** The group $F[A]$ described above is the *free group generated by* A.

Sometimes we have a group G and a generating set $A = \{a_i \mid\in I\}$, and we want to know whether or not G is *free* on $\{a_i\}$, that is, G is the free group generated by $\{a_i\}$.

Definition 2.2 (Free Generators)**.** If G is a group with a set $A = \{a_i\}$ of generators and is isomorphic to $F[A]$ under a map $\phi\colon G \to F[A]$ such that $\phi(a_i) = a_i$, then G is *free on* A, and the a_i are *free generators of* G. A group is *free* if it is free on some nonempty set A.

Oh you'll be free... free indeed...

Example 2.4. $\mathbb{Z}$ is the free group on one generator.

I wish we would call it the "free group on n letters" as opposed to the "free group on n generators", which is lame, to be consistent with the whole "mathematicians don't know how to spell" theme.

Example 2.5. $\mathbb{Z}$ is the free group on one letter.

Much better. Time for theorem spam.

Theorem 2.1. *If G is free on A and B, then A and B have the same order, that is, any two sets of free generators of a free group have the same cardinality.*

Proof. Refer "to the literature". ⊠

Definition 2.3 (Rank)**.** If G is free on A, then the number of letters in A is the *rank of the free group* G.

Theorem 2.2. *Two free groups are isomorphic if and only if they have the same rank.*

Proof. Immediate. ⊠

Theorem 2.3. *A nontrivial proper subgroup of a free group is free.*

Proof. Back "to the literature". ⊠

Example 2.6. Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. So the rank of the free subgroup of a free group can be much greater than the rank of the whole group.

§2.3 Homomorphisms of Free Groups

Theorem 2.4. *Let G be generated by $A = \{a_i \mid \in I\}$ and let G' be any group. If $a_i{}'$ for $i \in I$ are any elements in G' not necessarily distinct, then there is at most one homomorphism $\phi\colon G \to G'$ such that $\phi(a_i) = a_i{}'$. If G is free on A, then there is exactly one such homomorphism.*

Proof. Let ϕ be a homomorphism from G into G'such that $\phi(a_i) = a_i{}'$. Then any $x \in G$ can be written as a finite product of the generators a_i, denoted

$$x = \prod_j a_{i_j}^{n^j},$$

the a_i not necessarily distinct. Since ϕ is a homomorphism, we have

$$\phi(x) = \prod_j \phi\left(a_{i_j}^{n_j}\right) = \prod_j \left(a'_{i_j}\right)^{n_j},$$

so a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i{}'$.

Now suppose that G is free on A, that is, $G = F[A]$. For

$$x = \prod_j a_{i_j} \in G,$$

define $\psi\colon G \to G'$ by

$$\psi(x) = \prod_j \left(a'_{i_j}\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words. Since the rules for computation involving exponents are formally the same as those involving exponents in G, it can be seen that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ⊠

Note that this theorem states that a group homomorphism is completely determined by its value on each element of a generating set: eg, a homomorphism of a cyclic group is completely determined by its value on any single generator.

Corollary 2.1. *Every group G' is a homomorphic image of a free group G.*

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 2.4 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i{}'$. Clearly the image of G under ψ is all of G'. ⊠

Only the free group on one letter is abelian.

§2.4 Free Products of Groups

Definition 2.4 (Free Products)**.** As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$.

Basically, reduced words with alternating letters from different groups. The group operation is concatenation: what if the end of w_1 and the beginning of w_2 belong to the same G_α? Merge them into a syllable: what if they merge into the identity, and so the next two letters are from the same alphabet? Merge again, and repeat forever. Eventually we'll get a reduced word.

How to prove this is associative? Relate it to a subgroup of the symmetric group, it takes care of a lot of work. So we have the free product $\mathbb{Z} * \mathbb{Z}$, which is also free. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is *not* a free group: since $a^2 = e = b^2$, powers of a and b are not needed. So $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the alternating words $a, b, ab, ba, aba, bab, abab, ...$ together with the empty word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha\colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi\colon *_\alpha G_\alpha \to H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducting an unreduced product in $*_\alpha G_\alpha$ goes not affect its image under φ.

Example 2.7. For a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

§2.5 Group Presentations

Apparently, I never took group theory. Let's talk about group presentations! Motivation: form a group by giving generators and having them follow certain relations. We want the group as free (free indeed) as possible on these generators.

Example 2.8. Suppose G has generators x and y and is *free except for the relation* $xy = yx$, or $xyx^{-1}y^{-1} = 1$. This makes sure G is abelian, and so G is isomorphic to $F[\{x, y\}]$ modulo its commutator subgroup, the smallest normal subgroup containing $xyx^{-1}y^{-1}$. This is because any normal subgroup containing $xyx^{-1}y^{-1}$ gives rise to an abelian factor group and thus contians the commutator subgroup (by a previous theorem).

This example illustrates what we want: let $F[A]$ be a free group and we want a new group as free as possible, with certain equations satisfied. We can always write these equations with the RHS equal to 1, so we consider the equations to be $r_i = 1$ for $i \in I$, where $r_i \in F[A]$. If $r_i = 1$, then

$$x(r_i^n)x^{-1} = 1$$

for any $x \in F[A]$, $n \in \mathbb{Z}$. Any product of elements equal to 1 again equals 1, so any finite product of the form

$$\prod_j x_j \left(r_{i_j}^{n_j}\right) x_j^{-1}$$

where r_{i_j} need not be distinct equals 1 in the new group. It can be seen that the set of all these finite products is a normal subgroup R of $F[A]$. Then any group that looks like $F[A]$ given $r_i = 1$ also has $r = 1$ for all $r \in R$. But $F[A]/R$ looks like $F[A]$, except that R has been collapsed to form the identity 1. Hence the group we are after is (at least isomorphic to) $F[A]/R$. We can view this group as described by the generating set A and the set $\{r_i \mid i \in I\}$, abbreviated $\{r_i\}$.

Definition 2.5 (Group Presentations)**.** Let A be a set and $\{r_i\} \subseteq F[A]$. Let R be the least normal subgroup of $F[A]$ containing the r_i. An isomorphism ϕ of $F[A]/R$ onto a group G is a *presentation of G*. The sets A and $\{r_i\}$ give a *group presentation*. The set A is the set of *generators for the presentation* and each r_i is a *relator*. Each $r \in R$ is a *consequence of* $\{r_i\}$. An equation $r_i = 1$ is a *relation*. A *finite presentation* is one in which both A and $\{r_i\}$ are finite sets.

Refer back to Example 2.1: $\{x, y\}$ is our set of generators and $xyx^{-1}y^{-1}$ is the only relator. The equation $xyx^{-1}y^{-1} = 1$ or $xy = yx$ is a relation— this was an example of a finite presentation.

§3 Van Kampen's Theorem

OK guys, let's decompose big spaces into smaller ones and compute their fundamental groups. These notes follow Hatcher §1.2.

§3.1 The van Kampen Theorem

Let's take a space X and say it's the union of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is often onto but in general, we can expect Φ to have a nontrivial kernel.

For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both of these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Van Kampen says under fairly broad hypotheses that this determines all of Φ.

Theorem 3.1. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi\colon *_\alpha (A_\alpha) \to \pi_1(X)$$

is onto. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N.$$

Example 3.1 (Wedge Sums)**.** I like the visual of the wedge sum but the terminology of the smash product. If only we could keep the `\vee` symbol ($\vee$) and say we "smash the spaces together" at a point.

We define the wedge sum $\bigvee_\alpha X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ as the disjoint union $\amalg_\alpha X_\alpha$ with all the basepoints x_α identified to a single point. If each $x_\alpha$is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Then by van Kampens theorem,

$$\Phi\colon *_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$$

is an isomorphism, provided each X_α is path-connected, hence also each A_α. Therefore for a wedge sum of circles, $\pi_1(\bigvee_\alpha S^1_\alpha)$ is a free group, the free product of copies of $\mathbb{Z}$.