Algebraic Geometry Notes

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The name of "*Algebraic Geometry*" has always been alluring to me, dangling like bait on a fish hook, a mountain of prerequisites and problems awaiting on the surface above. There are notes for independent reading in Algebraic Geometry, following *The Rising Sea: Foundations of Algebraic Geometry* by Ravi Vakil. The reason I follow this book as opposed to a classical literature like Hartshorne is simply because it's funnier— usually humorous or entertaining works tend to be more readable (counterexample: Hatcher). Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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> *A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of eath or hard marl, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet finally it surrounds the resistant substance.*
>
> —*Alexander Grothendieck*

Section 1

Category Theory

Before we start, here are a handful of entertaining quotes from the introduction.

> *"Algebraic geometry 'seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to wake over all the rest of mathematics!' "*
>
> *"Do not be seduced by the lotus eaters into infatuation with untethered abstraction."*
>
> *"This I promise: if I use the word 'topoi', you can shoot me.*
>
> *"Finally, if you attempt to read this without working through a significant number of exercises, I will come to your house and pummel you with* Éléments de Géométrie Algébrique *until you beg for mercy."*
>
> —Albert Einsten

And one more:

> *"Should you just be an algebraist or a geometer?" is like saying "Would you rather be deaf or blind?"*
>
> —M. Atiyah

In this case, I'll choose to be both deaf *and* blind! Sorry, but I must include this last quote (I swear!).

> In an ideal world, people would learn this material over many years, after having backgroudn courses in commutative algebra, algebraic topology, differential geometry, complex analysis, homological algebra, number theory, and French literature.

Exercise: describe a groupoid that isn't a group. This is kind of cheating, but the fundamental groupoid.

Exercise: If $\mathscr{A}$ is an object in a category $\mathscr{C}$, show that the invertible elements of $\mathrm{Mor}(\mathscr{A}, \mathscr{A})$ form a group (called the **automorphism group of** $\mathscr{A}$, denoted by $\mathrm{Aut}(\mathscr{A})$). What are the automorphism groups of the category of sets and vector spaces over a field k? Show that two isomorphic objects have isomorphic automorphism groups. In the category of sets, the automorphism group is the set of bijections of a set onto itself (S_n for a set with n elements), and for the category of vector spaces, since isomorphism are bijective linear transformations (isomorphic in a group/ring sense), automorphisms are dual to the notion of an automorphism group in group theory. To show the invertible elements of $\mathrm{Mor}(\mathscr{A}, \mathscr{A})$ form a group, let