Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

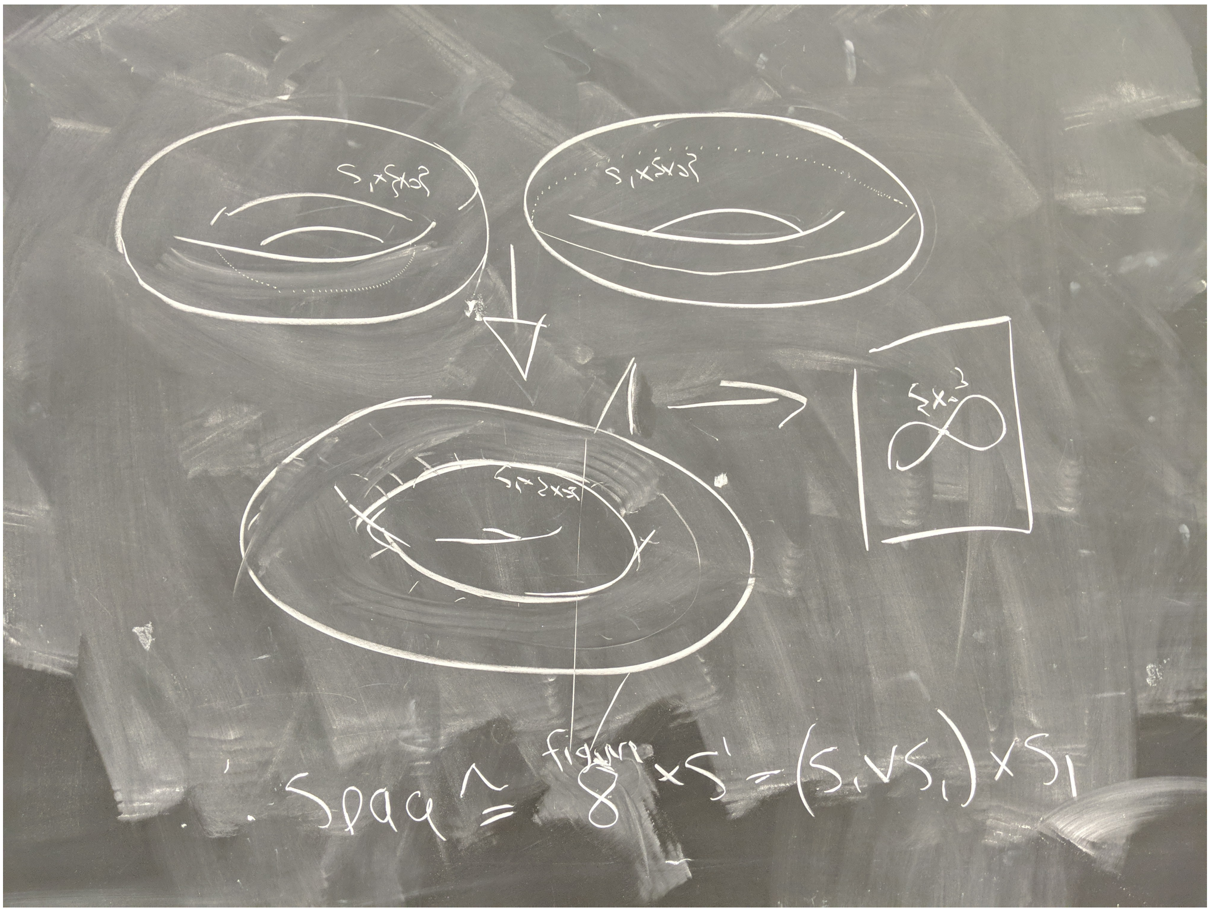
§1 September 14, 2020: Homework 3

Hatcher Section 1.2 (p. 52): 8, 11, 13,
Hatcher Section 1.3 (p. 79): 1, 4,
Assigned problem parts (a) and (b).

§1.1 Problem 8 Section 1.2

Problem. *Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.*

Solution. To see what this group looks like, if we take an "outer" and "inner" circle of the respective tori and glue them together, we get a two hulahoops stacked and glued inside each other. This is homeomorphic to the figure 8 cross the circle (it looks like a figure 8 if you cut it open), so this space is homeomorphic to $(S^1 \vee S^1) \times S^1$.



Formally, we can look at this in terms of presentations and relations: say π_1 of a torus is given by the presentation $\langle a, b \mid a^{-1}b^{-1}ab = [a,b] \rangle$ ($[a,b]$ denotes the commutator of a and b). Then we can similarly represent the second torus' fundamental group by $\langle c, d \mid [c,d] \rangle$. Note that π_1 of the intersection is equal to $\pi_1(S^1) = \langle a \rangle$. What this identification does is glue the second circle of this first torus to the second circle of the second torus, such that the homotopy class $[b] = [d]$. Then by van Kampens, we have π_1 of the two tori joined at a circle as

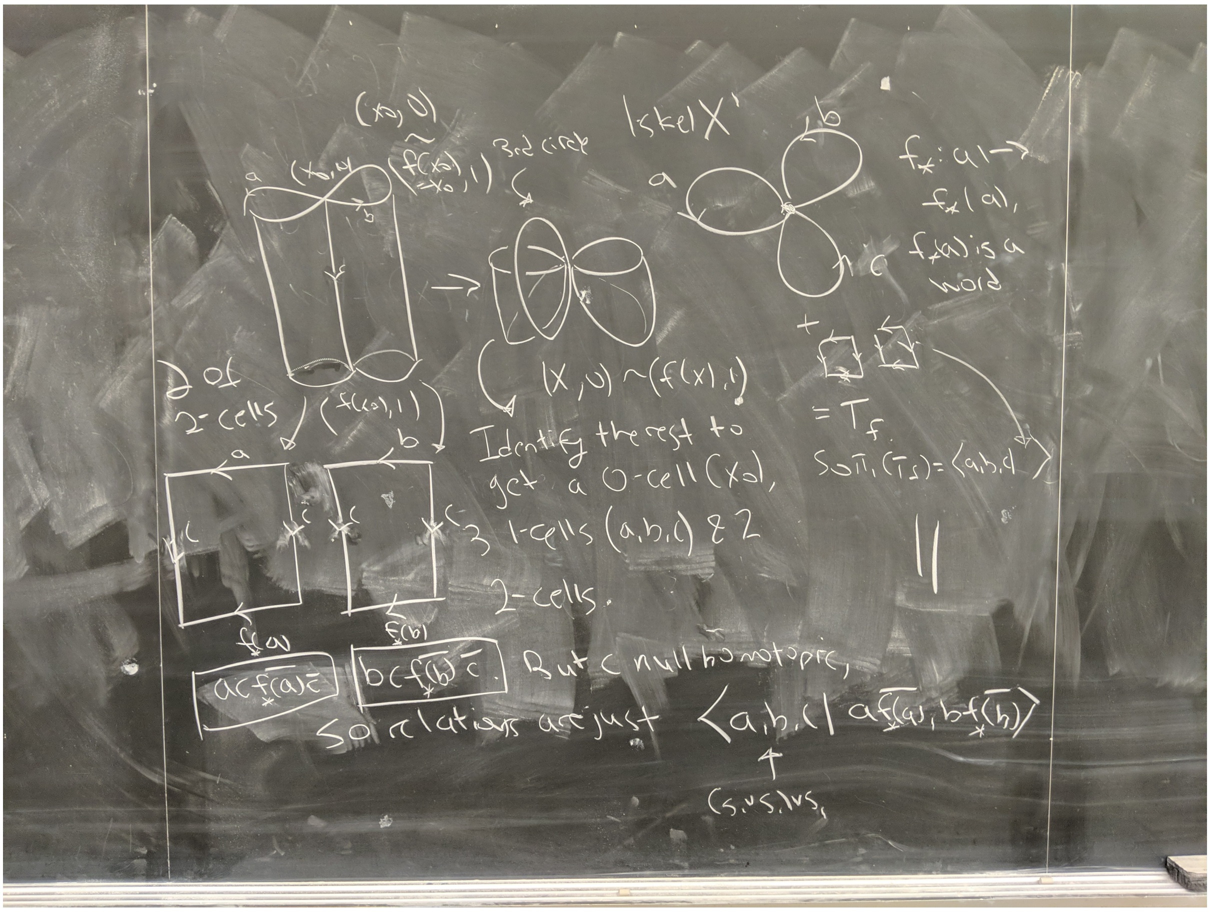
$$\langle a, b, c \mid [a,b], [c,b] \rangle.$$

Since b commutes with a and c but a and c don't commute, this is isomorphic to $(\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$, which is our expected result if we think of this space as $(S^1 \vee S^1) \times S^1$. ■

§1.2 Problem 11

Problem. *The* mapping torus T_f *of a map* $f\colon X \to X$ *is the quotient of* $X \times I$ *obtained by identifying each point* $(x, 0)$ *with* $(f(x), 1)$*. In the case* $X = S^1 \vee S^1$ *with* f *basepoint-preserving, compute a presentation for* $\pi_1(T_f)$ *in terms of the induced map* $f_*\colon \pi_1(X) \to \pi_1(X)$*. Do the same when* $X = S^1 \times S^1$*. [One way to do this is to regard* T_f *as built from* $X \vee S^1$ *by attaching cells.]*

Remark: In the most important case, f_* *is an isomorphism. In this case, observe that* $\pi_1(T_f)$ *is a semidirect of* $\pi_1(X)$ *by* $\mathbb{Z}$*.*



Solution. See the figure above for the mapping torus of $S^1 \vee S^1$ (the top half of the figure): it consists of two loops a, b and also $f_*(a), f_*(b)$, where $f_*(a), f_*(b)$ are just words in $\pi_1(X, f(x_0) = \pi_1(X)$ on the letters $\{a, b\}$ (we add in c to keep track of the 1-skeleton later). We take Hatcher's hint and view the mapping torus T_f as the cell 1-skeleton $X^1 = (S^1 \vee S^1) \vee S^1$ (a 0-cell, the basepoint, with three 1-cells, the loops, attached) with two 2-cells attached. Then by a theorem[1], we can present $\pi_1(M_f)$ by relations on the boundaries of the 2-cells, with generators being the loops formed by the 1-cells. Then $S^1 \vee S^1 \vee S^1$ consists of three loops, say a, b, and c, with two 2-cells attached. See the figure (bottom left portion) to see how their relations are drawn. The relations are $ac\overline{f_*(a)}\bar{c}$ and $bc\overline{f_*(b)}\bar{c}$: but c is nullhomotopic, so they simply become $a\overline{f_*(a)}$ and $b\overline{f_*(b)}$. So by the theorem, we have

$$\pi_1(M_f) = \langle a, b, c \mid a\overline{f_*(a)}, b\overline{f_*(b)} \rangle.$$

For $X = S^1 \times S^1$, you can make a torus out of $S^1 \vee S^1$ by attaching a 2-cell, which has the presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid [a, b] \rangle$. So just add that 2-cell plus the other two 2-cells (with the same relations as above), and a 3-cell because of the mapping cylinder (which we will promptly ignore since $3 > 2$). This space is a little bit harder to visualize, but we can find the generators in the same way as last time, except most of the work is done: just add on the relations from the torus. Since the 1-skeleton is still $S^1 \vee S^1 \vee S^1$, we have a presentation for M_f given by

$$\pi_1(M_f) = \langle a, b, c \mid a\overline{f_*(a)}, b\overline{f_*(b)}, [a, b] \rangle.$$

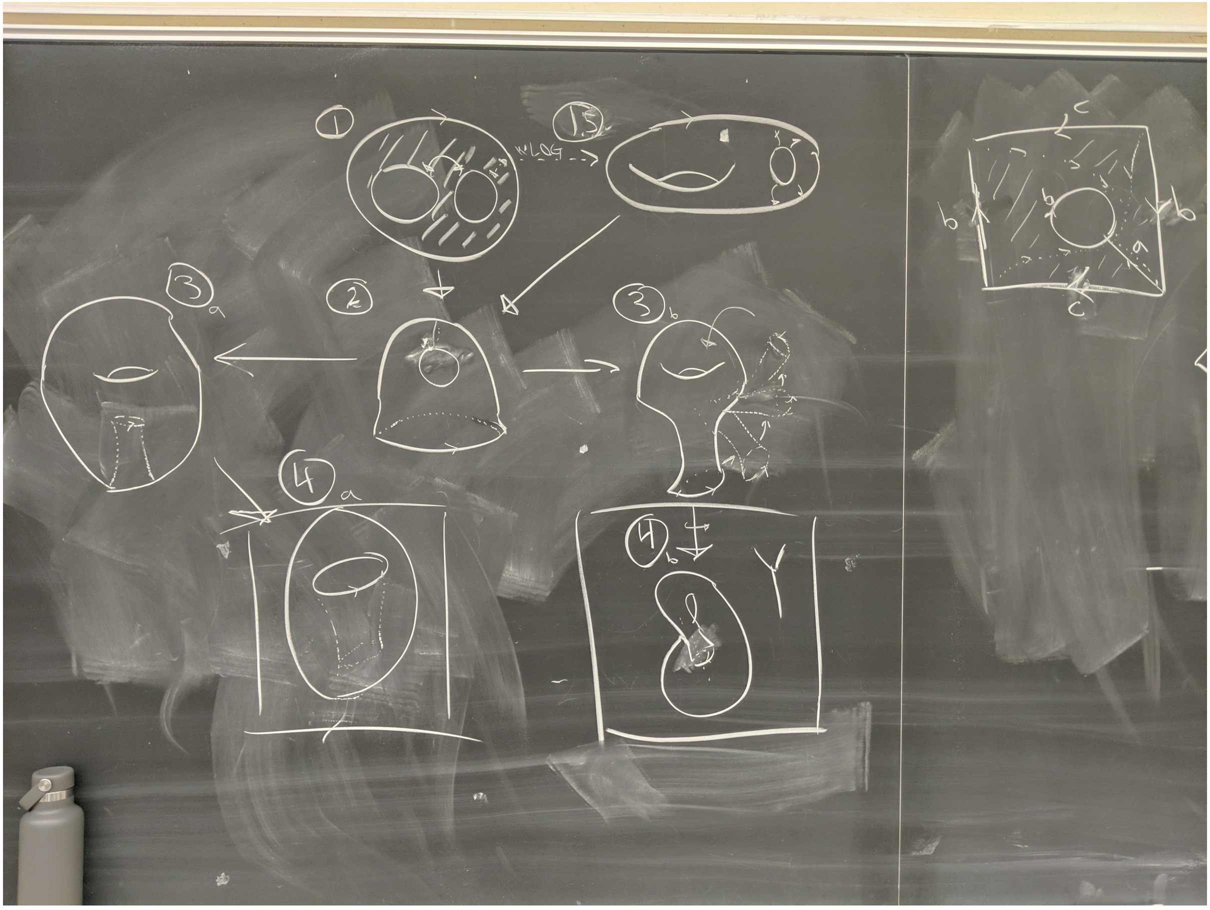
■

[1] Proposition 1.26 of Hatcher is probably equivalent to this theorem, but I found this formulation (specifically, Theorem 1.24) to be the most adapted and useful for our scenario: it gives a presentation for $\pi_1(X)$ (what we want) in terms of loops made by the boundaries of the 2-cells and generators by the 1-cells (what we have).

§1.3 Problem 13

Problem. *The space Y in the preceding exercise (Klein bottle embedded in $\mathbb{R}^3$ with a deleted open disk at the circle of self-intersection) can be obtained from a disk with two holes by identifying its three boundary circles. There are only two essentially different ways of identifying the three boundary circles. Show that the other way yields a space Z with $\pi_1(Z)$ not isomorphic to $\pi_1(Y)$. [Abelianize the fundamental groups to show they are not isomorphic.]*

Solution. See the figure below:



Some notes about the identification: no matter what, the first move will be the same, as seen how Step 1.5 goes to Step 2 just like Step 1. Then at Step 2, we can either elongate the disk to form a trunk, then glue it onto the disk from the top: this gives us our Klein bottle with deleted disk. Or, we can push the disk inward and identify it from the inside: this gives us our new space, the "inverted trunk Klein bottle"[2]. Now we have $\pi_1(Y) = \langle a, b, c \mid aba^{-1}b^{-1}cb^{-1}c^{-1}\rangle$ by the previous exercise in Hatcher: see the top right of the figure for the fundamental polygon and how we draw it. Then the fundamental group of our new space $\pi_1(Z)$ is simply this space with the same glueing but the relation a that goes to the circle b drawn the opposite direction: so

$$\pi_1(Z) = \langle a, b, c \mid a^{-1}ba^{-1}b^{-1}cb^{-1}c^{-1}\rangle.$$

But the abelianization of $\pi_1(Y)$ is $\langle a, b, c \mid b^{-1}\rangle = \langle a, c\rangle = \mathbb{Z} * \mathbb{Z}$, the free group on two generators, while the abelianization of $\pi_1(Z)$ is $\langle a, b, c \mid a^{-2}b^{-1}\rangle = \langle a, b, c \mid ba^2\rangle$. c is free, but $\langle a, b \mid ba^2\rangle$ is not isomorphic to the free group on one generator (we want a free product $\mathbb{Z} * \mathbb{Z}$, and one of our generators is free. So we want the other one free too.) So the abelianization of $\pi_1(Z)$ is not equal to the abelianization of $\pi_1(Y)$, and we are done. (Note that if $\varepsilon = 1$ as in the previous problem in Hatcher, we would have the relation a^2b instead, which is still not isomorphic to the free group on one generator). ■

[2] So I was thinking about how to do this problem, and attempted several other methods, one was taking the third circle after Step 2 and stretching it out to envelop the rest of the space, and identifying it from the top. A little story: I was trying to work it out on the blackboard to no avail. So I left for dinner and by the time I came back, to my dismay, the door was locked. Naturally, I went back to my dorm, disappointed because I had nowhere to visualize things— but then I had a genius idea: a disk minus two circles is homeomorphic to a pair of pants! So I got a pair of jeans, stuck one leg into the other to form a torus with a hole at the waist, inverted the waist band onto the torus hole, and saw two holes in the side that I didn't think of on the blackboard. And then my roommate walked in on me sitting on my bed fiddling with a pair of jeans, so I told him I was doing advanced mathematics. Anyways, thanks for reading (I never figured out what that space looked like in the end, so I just went with the inverted trunk).

§1.4 Problem 1 Section 1.3

Problem. *For a covering space $p\colon \widetilde{X} \to X$ and a subspace $A \subseteq X$, let $\widetilde{A} = p^{-1}(A)$. Show that the restriction $p\colon \widetilde{A} \to A$ is a covering space.*

Proof. Let $a \in A$, then $U_a \cap A$ is open in A by the subspace topology. We WTS that $p^{-1}(U_a \cap A)$ is a disjoint union of open sets, each of which maps homeomorphically onto $U_a \cap A$ by p. Now

$$p^{-1}(U_a \cap A) = p^{-1}(U) \cap p^{-1}(A) = (\amalg V_\alpha) \cap \widetilde{A} = \amalg \left(V_\alpha \cap \widetilde{A} \right),$$

where $\amalg V_\alpha$ denotes the disjoint union of sheets of U_a[3]. All that remains is to show each $V_\alpha \cap \widetilde{A}$ is homeomorphic to U_a by p: take $p(V_\alpha \cap \widetilde{A})$. Then this is equivalent to $p(V_\alpha) \cap p(\widetilde{A})$ [4]. Now $p(V_\alpha)$ is equal to U_a by the definition of each V_α (that is, p is a homeomorphism). We also have $p(\widetilde{A}) = p(p^{-1}(A)) = A$[5], so $p(V_\alpha \cap \widetilde{A}) = U_a \cap A$, and we are done. ⊠

§1.5 Problem 4

Problem. *Let $p\colon \widetilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff iff X is compact Hausdorff.*

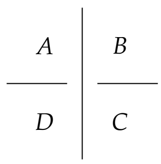
Proof. ($\Longrightarrow$) Assume $\widetilde{X}$ is compact Hausdorff. Then $p(\widetilde{X}) = X$ is compact since p is continuous. Now to show X is Hausdorff: we construct U_x and U_y open sets containing x and y. We know $p^{-1}(x)$ and $p^{-1}(y)$ are finite sets of points (by assumption), so we just apply the Hausdorff property until exhaustion, leaving us with two disjoint open covers of $p^{-1}(x)$ and $p^{-1}(y)$. We claim these evenly cover some neighborhood of x and some neighborhood of y, respectively: just take the neighborhood U_x to be contained in the smallest of the open sets in the cover of $p^{-1}(x)$, and similarly construct U_y, these sets being non-empty because if $a \in U_x \cap U_y$, then $p^{-1}(a)$ is in both the disjoint covers (by the Hausdorff condition of $\widetilde{X}$), a contradiction.

($\Longleftarrow$) Assume X is compact Hausdorff. Then to show $\widetilde{X}$ is Hausdorff, we can separate two points $p(x)$ and $p(y)$ in X ($x, y \in \widetilde{X}$) with open sets U_x, U_y (assuming $p(x) \neq p(y)$). Then $p^{-1}(U_x) \cap p^{-1}(U_y) = \varnothing$, because if not, then for $a \in p^{-1}(U_x) \cap p^{-1}(U_y)$, we would have $p(a) \in U_x \cap U_y$, contradicting the fact that $U_x \cap U_y = \varnothing$. Now if $p(x) = p(y)$, then we can find an open set U_{xy} that satisfies the covering space condition since $\widetilde{X}$ covers X. Since the unions are disjoint, we either have $x = y$ or x and y on different sheets. We conclude $\widetilde{X}$ is Hausdorff.

Now we show $\widetilde{X}$ is compact. Since X is compact, we can take open sets $\bigcup_\alpha U_\alpha$ around all $x \in X$ such that every U_α is evenly covered (since $\widetilde{X}$ covers X): then this is an open cover of X. By the compactness of X, we have a finite subcover $\bigcup_{i=1}^n U_i$ of X. Then since each disjoint open set in $p^{-1}(U_i)$ is a subset of some open set in $\widetilde{X}$, $\bigcup_{i=1}^n p^{-1}(U_i)$ covers $\widetilde{X}$ (since each $p^{-1}(U_i)$ is nonempty), furthermore, they will also form a subcover of open covers of $\widetilde{X}$ (by the first part of the sentence). To show this subcover is finite, note that each $p^{-1}(U_i)$ is a union of finite sets by assumption. So a finite union of finite unions is still a finite union[6], and we are done. ⊠

§1.6 Problem (Dehn presentation)

Problem. *The Dehn presentation of a $\pi_1(\mathbb{R}^3 - L)$, for L a link, is different from the Wirtinger presentation from class (see also Hatcher' §1.2#22). Write p for the projection $\mathbb{R}^3 \to \mathbb{R}^2$. Suppose the projection of L has the usual form for a link diagram. Then $p(L)$ divides $\mathbb{R}^2$ into components. The Dehn presentation has one generator for each bounded region. (We take the identity element as the "generator" associated to the unbounded region.) There is one relation for each crossing. Suppose the regions involved at that crossing are A, B, C, D as in the picture. The relation is $AB^{-1}CD^{-1} = 1$.*



Prove that this presentation gives $\pi_1(\mathbb{R}^3 - L)$.

Here is the meaning of the generators. (You should be drawing pictures like mad throughout this problem.) We think of the link as lying close to $\mathbb{R}^2$, and the basepoint far above it. A generator goes down from the basepoint, through the corresponding region, and then comes back up through the unbounded region. (This is why the "generator" associated to the outside region is the identity.)

Here is an approach to proving that the presentation is legit. We start by building a 2-complex X containing L. Its 1-skeleton is the union of L and one vertical segment for each crossing, joining the upper strand to the lower strand at that crossing. Then

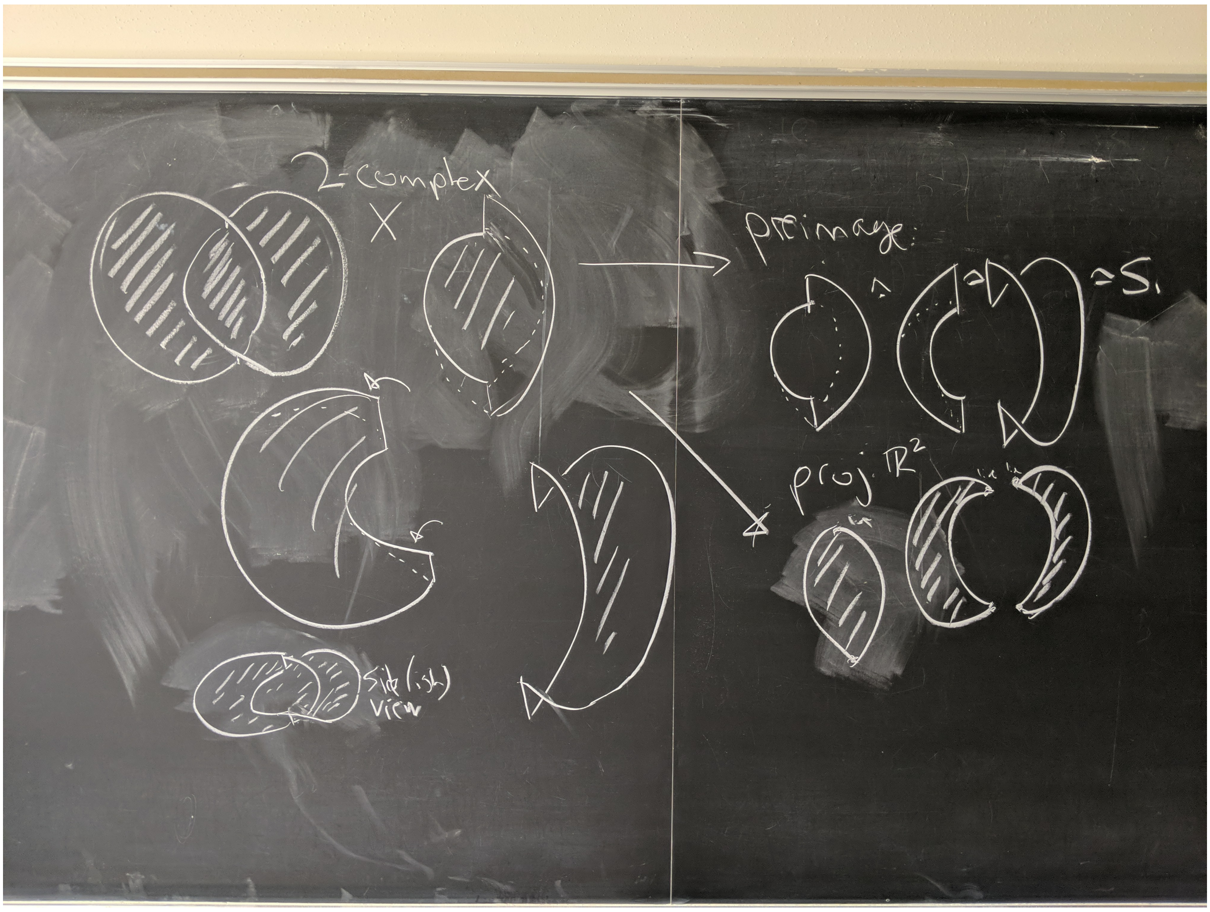
[3] $p^{-1}(U_a \cap A) = p^{-1}(U_a) \cap p^{-1}(A)$ because inverses behave nicely with respect to everything.
[4] This isn't true for every set and function, it only holds if f is 1-1. But it's true in this case.
[5] This, however, holds for every function and every set.
[6] We can do this finitely many times!

you add one 2-cell for each region. For a particular (closed) region, consider its preimage in L. You get a subset of L. Take the union of this with the vertical segments corresponding to the corners of the region. This gives a circle in the 1-skeleton, along which you attach a disk. You can see the disk inside $\mathbb{R}^3$; it projects bijectively to its image in $\mathbb{R}^2$, except that each vertical segment projects to a corner of the region. (You should be drawing pictures like mad throughout this problem.)

Work out $\pi_1(\mathbb{R}^3 - L)$ by starting with $\mathbb{R}^3 - X$ and adding stuff until we have $\mathbb{R}^3 - L$, using VK at each stage. To start, show $\mathbb{R}^3 - X$ is simply connected. Then take the union of $\mathbb{R}^2 - X$ with the interior of one of the 2-cells of X. Use Van Kampen's theorem to show that that this adjoins a generator. (You should be drawing pictures like mad throughout this problem.) Repeating this for the other regions yields the generators in the presentation. This gives $\mathbb{R}^3 - X^{(1)}$. Next add in the interior of one of the vertical segments. Use VK's theorem to show that this imposes one relation, namely the one associated to that crossing in the link diagram. (You should be drawing pictures like mad throughout this problem.) Repeating this gives $\mathbb{R}^3 - K$, with known generators and relations for its fundamental group.

This approach is "dual" to the usual approach of "1-cells are generators, 2-cells are relations". To use the usual approach, one can find a sort of "dual" complex to X in $\mathbb{R}^3$, to which $\mathbb{R}^3 - L$ deformation-retracts. But I have an easier time seeing the relation by the approach above, because they are represented by very small loops around the vertical segments.

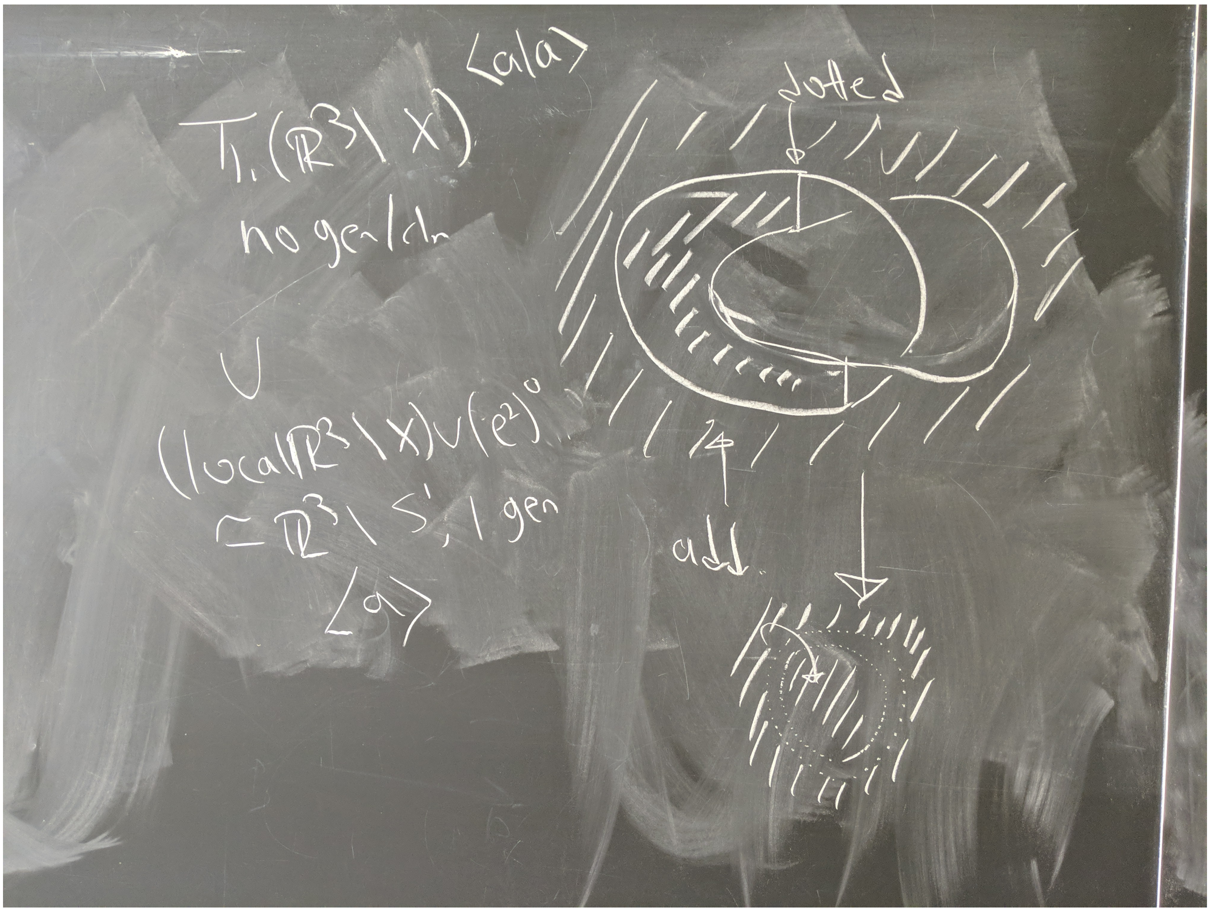
Proof. We want to show that the Dehn presentation gives $\pi_1(\mathbb{R}^3 \setminus L)$. We'll work out stuff on the board with the simplest link I could think of, the Hopf link consisting of two circles linked nontrivially exactly once. Let L_2 denote the Hopf link for convenience. Below is the 2-complex of L_2.



The 1-skeleton of L_2 decomposes into three components, each of which is a circle, as seen in the top right of the figure. In general, the 1-skeleton of a link L will decompose into n circles. Adding 2-cells, we get three closed disks that embed in $\mathbb{R}^3$, as seen in the middle of the figure. Each disk is linked to the other by "half" of the circle, the points in $\mathbb{R}^2$ where the vertical lines are drawn. In general, attaching 2-cells to the 1-skeleton of L will get n closed disks, joined at the points where the vertical lines are.

Now to show $\mathbb{R}^3 \setminus X$ is simply connected: first let's look X when it denotes the 2-complex of L_2. Then this is three closed disks glued together, which has the homotopy type as one closed disks, which is contractible. So $\mathbb{R}^3 \setminus X$ is homotopic to $\mathbb{R}^3 \setminus \{0\}$, which is simply connected. In general, if X is the 2-complex described above of any link L, $\mathbb{R}^3 \setminus X$ has the same homotopy type as $\mathbb{R}^3 \setminus (\bigcup_{i=1}^n \{x_i\})$, where the $\{x_i\}$'s are just distinct singletons, and n is the number of unlinks the link has. However, this is still simply connected since the singletons are disjoint. So $\mathbb{R}^3 \setminus X$ is simply connected for any link.

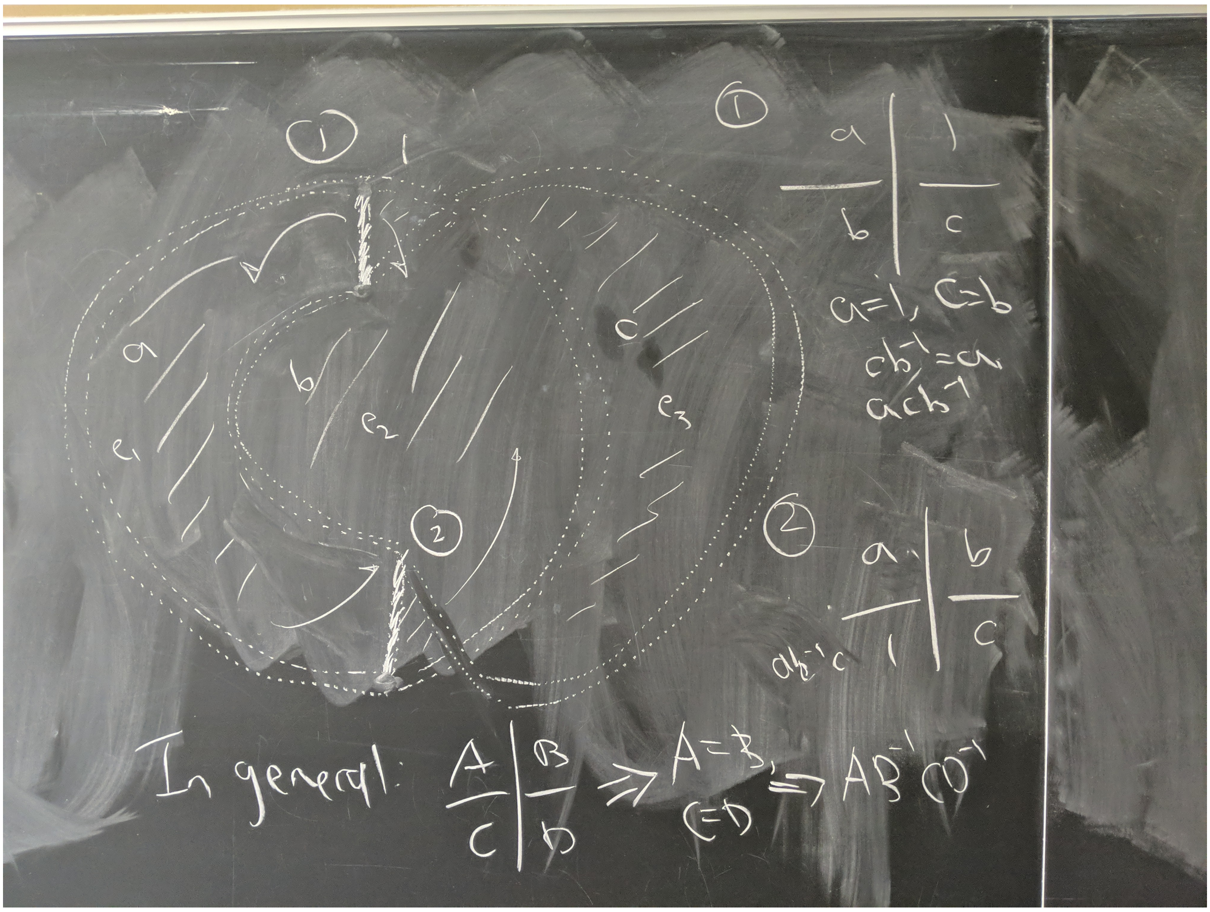
Now we take the union of $\mathbb{R}^3 \setminus X$[7] and the open interior of a 2-cell. In the case of L_2, we start by adding the interior of the left most 2-cell, let's denote this $(e^2)^\circ$. See the figure below:



We can apply van Kampens to $(\mathbb{R}^3 \setminus X) \cup (e^2)^\circ$ by expressing it as a union of a local subspace of $\mathbb{R}^3$ containing X^c union $(e^2)^\circ$ (denoted $(V \setminus X) \cup (e^2)^\circ$) and $\mathbb{R}^3 \setminus X$. We know that $\mathbb{R}^3 \setminus X$ is simply connected, so it only presents a useless generator and relation (we can write it as $\langle a \mid a \rangle$). Now $(V \setminus X) \cup (e^2)^\circ$ has the homotopy type of $\mathbb{R}^3 \setminus S^1$, which has a generator for the deleted circle that is free. So this adjoins a generator. We don't have to do any work to generalize this to L. Similarly, we can repeat this for the other regions to get $\pi_1((\mathbb{R}^3 \setminus X) \cup (\bigcup_{i=1}^{n} (e_i^2)^\circ)) = \langle g_1, \cdots, g_n \rangle$, where n is the number of 2-cells in X, and $(e_i^2)^\circ$ denotes each 2-cell[8]. So this shows us that adjoining the interior of 2-cells will give us our generators.

[7] Here Dr. Allcock says $\mathbb{R}^2 \setminus X$, but I think that's a typo, so I'm going to go with $\mathbb{R}^3 \setminus X$ because it seems a lot easier to set up van Kampen's that way. Or am I missing something?

[8] In hindsight, I realize this is the same as $\mathbb{R}^3 \setminus X^1$, but I don't want to admit it.



The figure above shows $\mathbb{R}^3 \setminus X^1$, where X^1 is the 1-skeleton of L_2, except with the vertical lines added in at the crossings. Now adding in the vertical line "connects" segments that were "separated" by the upper link of crossing. So at crossing 1 in the figure, it joins the generator a of the 2-cell e_2 with the "useless" generator (denoted 1) of $\mathbb{R}^3 \setminus X$, the 2-complex of L_2. It also adjoins the 2-cells e_2 and e_3, giving us the relations $a = 1$, $b = c$. Simplifying this, we get the relation acb^{-1}, which is the same as the one obtain from the Dehn presentation. For crossing 2, we have $e_1 = e_2$, $1 = e_3$, so we get the relation $ab^{-1}c$. Generalizing this to any L, each crossing will merge adjacent cells separated by the top link, that is, for a general crossing

A	B
D	C

where A, B, C, D are generators of the 2-cells e_i^2, we get that $A = B$ and $C = D$, which gives the relation $AB^{-1}CD^{-1}$ as expected. Repeating this for every crossing gives $\mathbb{R}^3 \setminus L$, and we are done. ⊠

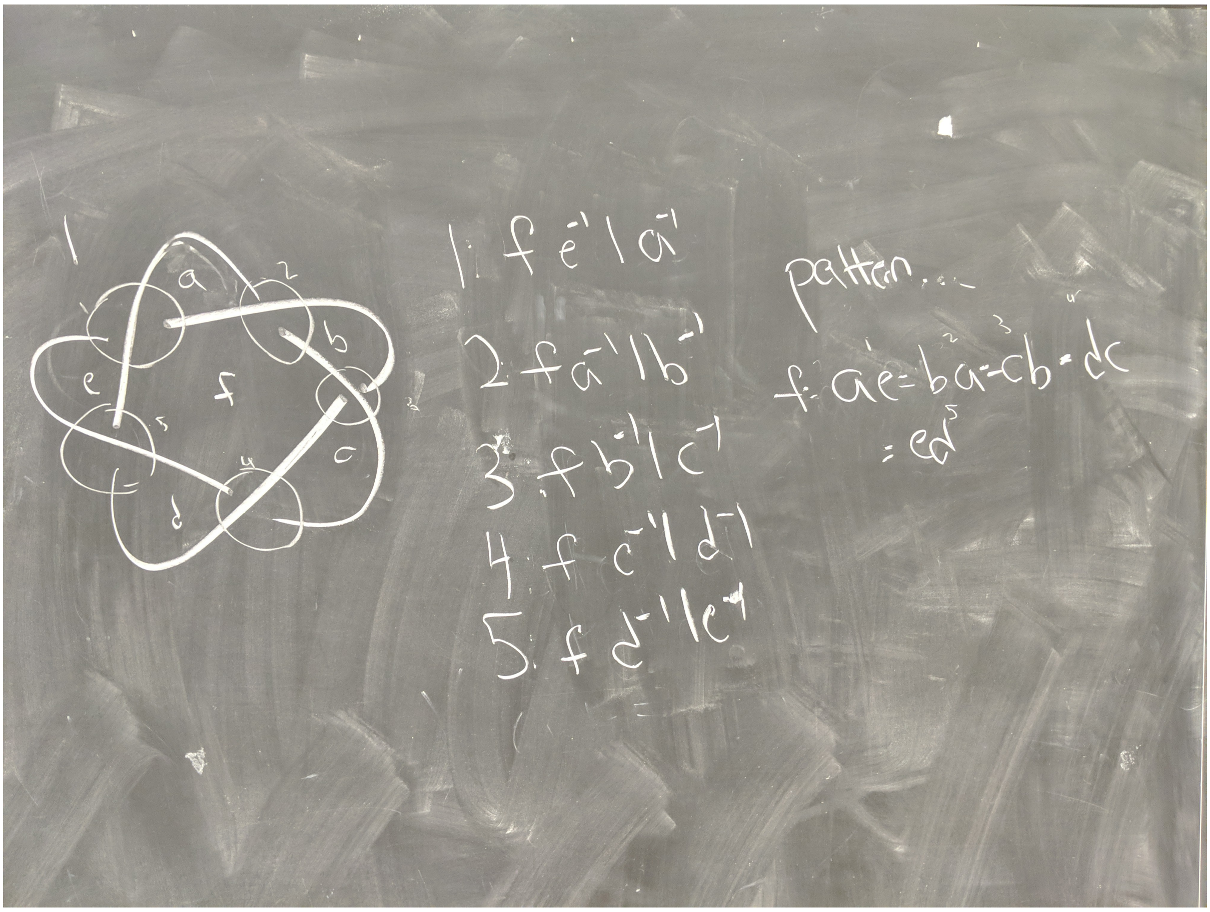
§1.7 Problem (Dehn presentation example)

Problem. *Example of the Dehn presentation:*

(a) *Use the Dehn presentation to present $\pi_1(\mathbb{R}^3 - K)$ where K is the $(2, n)$ torus knot (in particular, n is odd).*

(b) *You'll get a presentation with $n + 1$ generators. Simplify it down to 2 generators. How similar is the result to the elegant presentation $\langle x, y | x^2 = y^n \rangle$ from the book?*

Solution. See the figure for the presentation of the $(2, 5)$ torus knot.



This helps us see a pattern (that isn't as apparent with $n = 3$): we have the presentation of the $(2,5)$ torus knot as $\langle a, b, c, d, e \mid ae, ba, cb, dc, ed \rangle$, or alternatively, $\langle a, b, c, d, e, f \mid aef^{-1}, baf^{-1}, cbf^{-1}, dcf^{-1}, edf^{-1} \rangle$. In particular, we have the Dehn presentation for a $(2, n)$ torus knot for n odd given by

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle g_1, \cdots, g_n, f \mid g_1 g_n f^{-1}, g_2 g_1 f^{-1}, \cdots, g_{i+1} g_i f^{-1}, \cdots, g_n g_{n-1} f^{-1} \rangle,$$

where $1 \leq i \leq n$, n is odd, the g_i are all generators given by the bounded regions from the Dehn presentation, and f is the "center" region/face. Now for the second part of this question: we can simply this down to two generators. Let $y = g_n g_{n-1} \cdots g_1$. Then

$$y^2 = \left(\overbrace{(g_n g_{n-1}) \cdots (g_3 g_2)}^{\text{even}} g_1 \right) \left(g_n \overbrace{(g_{n-1} g_{n-2}) \cdots (g_2 g_1)}^{\text{even}} \right).$$

The term y has n "components". Since $n = 2k + 1$ for some $k \in \mathbb{N}$[9], we can split off all the terms besides g_1 and g_n respectively into pairs, and each of those simply reduce to $\overbrace{f \cdots f}^{k \text{ times}}$. So we have

$$y^2 = (f^k)(g_1 g_n)(f^k),$$

but $g_1 g_n = f$ by the first relation in our presentation. Therefore $y^2 = f^{2k+1} = f^n$ is a relation that expresses all data of the presentation, so we have the more elegant presentation in the form of two generators as

$$\langle f, y \mid f^n = y \rangle.$$

■

[9] Hopefully n is positive, or else things would get weird.