# Complex Analysis Lecture Notes

Simon Xiang

These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`. I was also unhappy with the textbook, so some supplementary notes from different texts are found at the bottom of the document.

## Contents

# §1 August 27, 2020

## §1.1 Basic Properties of Complex Numbers

We talk about functions $f\colon \mathbb{C} \to \mathbb{C}$ that map variables $z \mapsto f(z)$. This course is "not a very hard course" (it's a fun course!). Holomorphic functions have very nice properties automatically that real valued differentiable functions simply don't have.

**Definition 1.1** (Complex Addition)**.** We define complex numbers as ordered pairs $z = (x, y)$ where $x, y \in \mathbb{R}$, with the binary operation of complex addition being defined as

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$

where $+$ denotes addition on the reals.

Once we define multiplication and additive/multiplicative inverses, we will have (almost) formed the field $\mathbb{C}$.

**Definition 1.2** (Complex Multiplication)**.** For $x, y \in \mathbb{C}$, we have

$$(x_1, y_1)(x_2)(y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Note: for $a \in \mathbb{R}$, we define

$$a(x, y) = (ax, ay).$$

Recall $(a, 0)(x, y) = (ax, ay)$. So one can understand that $a \in \mathbb{R}$ is simply the real analog of $(a, 0)$ (or simply, $\mathrm{Re}(a, 0) = a \in \mathbb{R}$).

How do we define multiplication of a complex number by a real number? We can think of the reals acting (in a group sense) on the complex numbers, with the operation being the standard multiplication.

**Example 1.1.** Take $(1, 0)(x, y) = (x, y)$. So $1(x, y) = (x, y)$ (where $1 \in \mathbb{R}$).

**Example 1.2** (Complex Addition is Commmutative)**.** We have already defined the sum of two complex numbers $z_1 + z_2$ as $z_3 = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$. Since addition is commutative on the real numbers, we have

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1,$$

so complex addition is commutative.

Claim: multiplication of complex numbers is commutative. You can verify this at home.

**Theorem 1.1** (Distributive Law)**.** *We have*

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$

*for* $z_1, z_2, z_3 \in \mathbb{C}$.

*Proof.* This follows from the fact that $\mathbb{C}$ has a ring structure. $\boxtimes$

## §1.2 Real and Imaginary Parts

**Definition 1.3.** If $z = (x, y)$, then $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$. Furthermore, we can associate a complex number with a point in the plane in many ways:

(insert figure 1 later)

## §1.3 Complex Numbers in the Plane

Point: the plane is just a plane. The plane doesn't have to have a coordinate system (coordinate axes don't have to be perpendicular). Any coordinate system is "useful" for adding complex numbers. For example, you can interpret complex addition as simply vector addition in the plane (no need for orthogonal axes!).

**Definition 1.4** (Additive Inverse)**.** We have

$$-(x, y) = (-1)(x, y) = (-x, -y).$$

So $(x, y) + [-(x, y)] = (0, 0)$.

Note: $(x, y)(0, 1) = (-y, x)$, a *rotation* of $(x, y)$ by $90°$. Another note: We have $(x, y) \in \mathbb{C} \cong x + iy$ and $i = (0, 1)$. So

$$(x, y) \cong x + iy \cong (x, 0) + (0, 1)(y, 0).$$

## §2 September 1, 2020

### §2.1 Units and Zero Divisors in the Complex Numbers

Recall from last time: A complex number can be defined as $(x, y) = x + iy$, where $x, y \in \mathbb{R}$. Addition is easy: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + y_1) + i(y_1 + y_2)$. In particular, $(0, 0) = 0 + i \cdot 0 = 0$. For multiplication, assume $i^2 = -1$. Then

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 + iy_1x_2 + iy_2x_1 + i^2y_1y_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1).\end{aligned}$$

On division: what does it mean to divide complex numbers? We say the multiplicative unit of a complex number (wrt the ring $\mathbb{C}$ ) as the unique $\frac{1}{z} = z^{-1}$ s.t. $z \cdot z^{-1} = z^{-1} \cdot z = (1, 0) \in \mathbb{C}$ (the unity of $\mathbb{C}$). Assume $(x, y)(x, y)^{-1} = (1, 0)$. Then do $u$ and $v$ exist such that the system of equations

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases}$$

holds? Yes, iff the determinant $\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$ is non zero.

**Definition 2.1** (Complex Conjugate)**.** We have $(x, -y)$ the complex conjugate of the complex number $z = (x, y)$, denoted $\bar{z}$.

We show that $\mathbb{C}$ has no zero divisors and is therefore an integral domain. WLOG, assume there exists $z_1, z_2$ such that $z_1 \neq 0$, $z_1z_2 = 0$: then we have $z_1^{-1}$ exists. So $z_1^{-1}z_1z_2 = 1z_2 = 0$, therefore $z_2 = 0$. For example: the group $\mathrm{GL}_n(\mathbb{R})$ is not an integral domain, since we have zero divisors (two matrices that when multipled equal zero).

### §2.2 Polar Coordinate Notation

**Definition 2.2** (Polar Coordinates)**.** Think of $(x, y)$ as rectangular coordinates in the $xy$-plane, and consider the *polar coordinate* notation $z = [r, \theta]$, where $r = \sqrt{x^2 + y^2} = |z|$ (modulus of $z$), and $\theta = \arctan(\frac{y}{x})$. So $[r, \theta] = (r\cos\theta, r\sin\theta)$.

**Example 2.1** (Multiplication with Polar Coordinates)**.** We have

$$[r_1, \theta_1][r_2, \theta_2] = (r_1\cos\theta_1, r_1\sin\theta_1)(r_2\cos\theta_2, r_2\sin\theta_2).$$

Then

$$\begin{gathered}(r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) = \\ r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right] + ir_1r_2\left[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right] = \\ r_1r_2\cos(\theta_1 + \theta_2) + r_1r_2 i\sin(\theta_1 + \theta_2) = \\ [r_1r_2, \theta_1 + \theta_2].\end{gathered}$$

**Example 2.2.** Assume that a complex number $z = (x, y)$ is nonzero. Then

$$\frac{1}{(x, y)} = \frac{1(x, -y)}{(x, y)(x, -y)} = \frac{(x, -y)}{x^2 + y^2}.$$

### §2.3 On the Norm (Modulus) of a Complex Number

**Example 2.3.** Some properties of the modulus (norm) $|z|$:

1. $|z_1z_2| = |z_1||z_2|$,
2. $\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = \left|z_1 \cdot \frac{\bar{z_2}}{|z_2|^2}\right| = |z_1|\frac{|z_2|}{|z_2|^2} = \frac{|z_1|}{|z_2|}$ (clearly $|\bar{z_2}| = |z_2|$),
3. $|z_1 + z_2| \leq |z_1| + |z_2|$ ($\mathbb{C}$ is a metric space, so the triangle inequality holds),
4. $|z_1 + z_2| \geq ||z_1| - |z_2||$ (reverse triangle inequality).

We prove the Reverse Triangle Inequality.

*Proof.* We have $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|$, so $|z_1 + z_2| \geq |z_1| - |z_2|$. A similar argument holds for $z_2$. ☒

Think of the polar angle as only well defined for multiples of $2\pi$. Define the argument (angle) as $\mathrm{Arg} = -\pi < \theta \leq \pi$ (what??). So $\mathrm{Arg}(1, 1) = \frac{\pi}{4}$, $\mathrm{Arg}(-1, 0) = \pi$. OTOH, we would have $\arg(1, 1) = \frac{\pi}{4} + 2\pi n$.

## §2.4 Euler's Formula

**Theorem 2.1** (Euler's Formula)**.** *We claim*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

*Proof.* Try using Maclaurin series. $\boxtimes$

This suggests $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We proved this when we showed $[r_1, \theta_1][r_2, \theta_2] = [r_1r_2, \theta_1 + \theta_2]$.

The reason why Dr. Radin says to "forget about Euler" is because he's trying to make a semi-rigorous (or self-contained) construction of the complex numbers. I think it's fine to rely on intuition from other courses, this isn't Real Analysis (nowhere near as rigorous). If we truly were to construct the field $\mathbb{C}$, we would have to cover polynomial rings and the fields generated by PID's quotient irreducible polynomials, then show that $\mathbb{C} \simeq \mathbb{R}[x]/\langle x^2 + 1\rangle$ (and show that this new field is algebraically closed too!). Of course this isn't feasible. So let's just think of this as Euler's Formula, and not some weird definition!

Back to math: using our newfound formula, we can simply say $\arg z = \theta$ such that $z = re^{i\theta}$ for any $z \in \mathbb{C}$. Similarly, $\operatorname{Arg} z$ is just $\theta$ restricted to the interval $(-\pi, \pi]$.

**Example 2.4.** If $z = re^{i\theta}$ nonzero, then what is the polar form of $\frac{1}{z}$? It must be

$$\frac{1}{r}e^{-i\theta}.$$

**Example 2.5.** We've seen that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Then

$$e^{i\theta_1}\left(e^{i\theta_2}e^{i\theta_3}\right) = e^{i\theta_1}e^{i(\theta_2+\theta_3)} = e^{i(\theta_1+\theta_2+\theta_3)}.$$

So $(\cos\theta + i\sin\theta)^m = \cos(m\theta) + i\sin(m\theta)$. This is known as *de Moivre's formula* .

## §3 September 3, 2020

### §3.1 Fractional Powers

Let $z_0 \in \mathbb{C}$, and define the fractional power $(z_0)^{\frac{1}{m}}$ for $m \geq 2$. This is a complex number such that

$$\left[(z_0)^{\frac{1}{m}}\right]^m = z_0.$$

This many not be unique. To determine the value of the fractional power $(z_0)^{\frac{1}{m}}$, let $z_0 = r_0 e^{i\theta_0}$, $r_0 = |z_0|$, $\theta_0 \in \operatorname{Arg} z_0$. Then

$$(z_0)^{\frac{1}{m}} = (r_0)^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}.$$

**Example 3.1.** In polar form, $z_0 = i = e^{i\frac{\pi}{2}}$. We want $i^{\frac{1}{6}}$, one value is $e^{i\frac{\pi}{12}}$. Also,

$$e^{i\frac{\left[\frac{\pi}{2}+2\pi\right]}{6}} = e^{i\left[\frac{\pi}{12}+\frac{\pi}{3}\right]} = e^{i\frac{5\pi}{12}}.$$

In general, $i = e^{i\left[\frac{\pi}{2}+2\pi m\right]}$, so $e^{i\left[\frac{\pi}{12}+\frac{m\pi}{3}\right]}$ is a value of $i^{\frac{1}{6}}$ for any $m$. In particular, consider the choices $m = 0, 1, .., 5$. Then

(insert figure later- it has to do with roots of unity on the circle group tho)

This method gives all possible $n$-th roots. In particular, in the circle group $U_1$, each "walk" is equal to a multiplication of $\zeta$.

We will eventually generalize the fractional power $z_0^{p/q}$ to $z_0^w$. Yada yada no exponentials allowed reeee. If you're going to formalize do it right or don't do it at all. Half baked rigor is about as useful as a potato (at least a potato can feed your family).

### §3.2 Point Set Topology

Why are we studying abstract nonsense? We need topology to define limits of complex numbers. We will eventually define a derivative as a quotient of deltas, eg

$$\frac{\Delta f}{\Delta z} \to \frac{df}{dz} \quad \text{as } \Delta z \to 0.$$

We'll talk about open and closed sets and accumulation points and such (basic things needed for limits). Consider

$$\widetilde{S} = \{z \mid |z| \leq 1 \text{ and } |z| \neq 1 \text{ if } \operatorname{Re} z < 0\}.$$

**Definition 3.1** (Open Ball)**.** We define an open ball

$$B(z_0, \epsilon) = \{z \mid |z - z_0| < \epsilon\}.$$

### §3.3 Interior, Closure, Boundary

**Definition 3.2** (Interior Point)**.** We have an *interior point* a point in a set such that there exists an open ball centered at the point entirely contained in the set. We define the set of all interior points of a set $X$ as $\operatorname{Int}(X)$.

Note that $\operatorname{Int}(\widetilde{S}) = \{z \mid |z| < 1.\}$

**Definition 3.3** (Exterior Point)**.** A point $z_0$ is an exterior point of $S$ if there exists a ball

$$B(z_0, \epsilon) \subseteq S^c,$$

ie, $z_0 \in \operatorname{Int}(S^c)$.

**Definition 3.4** (Boundary Point)**.** A point $z_0$ is a boundary point of $S$ if for ball $B(z_0, \epsilon)$ centered at $z_0$, $B(z_0, \epsilon) \cap S \neq \emptyset$ and $B(z_0, \epsilon) \cap S^c \neq \emptyset$. We define the *boundary* of a set $S$ as the set of all boundary points, denoted $\partial S$.

Basic things: points can't be both in the interior and exterior, boundary and interior, etc etc.

**Theorem 3.1.** *For any set $S$, $\operatorname{Int}(S)$ , $\operatorname{Ext}(S)$ , and $\partial S$ form a partition of $S$.*

We will use $S^\circ$ to denote the interior and $(S^c)^\circ$ to denote the exterior of a set from now on.

**Example 3.2.** $\partial\widetilde{S} = \{z \mid |z| = 1\}$.

**Example 3.3.** We have the unit circle $S = \{z \mid |z| = 1\} \cup zi$ (where $zi$ is a point). $S^\circ = \emptyset$, $zi \in \partial S$, any point on the rim $\in \partial S$, so $\partial S = S$. By our previous theorem, $(S^c)^\circ = S^c$. (Who even studies the exterior of a set??)

## §3.4 Open and Closed Sets

From now on a set refers to a subset of $\mathbb{C}$.

**Definition 3.5** (Open Sets)**.** A set is open if it contains none of its boundary. Alternatively, a set is open iff $S = S^\circ$.

**Example 3.4.** $\mathbb{C}$ is open (and closed)! Furthermore, $\partial\mathbb{C} = \emptyset$ (which is an alternate condition for a set to be clopen). Note that $\emptyset$ is also both open and closed, since $\partial\emptyset = \emptyset$. This also makes sense if we look at it from the interior perspective (no interior points in $\emptyset$, every point has an open ball in $\mathbb{C}$).

**Definition 3.6** (Closed Sets)**.** A set is closed if it contains all of its boundary. (What do you mean not the complement of open???)

**Theorem 3.2.** *$S$ is closed $\iff$ $S^c$ is open.*

*Proof.* Immediate. In general topology, we define open sets this way. ⊠

**Example 3.5.** Like I said earlier, both $\mathbb{C}$ and $\emptyset$ are closed. In general topology, we define both $S, \emptyset \in \tau$, since they're complements of course they're both open and closed. Exercise: prove that no other sets are both open and closed.

**Definition 3.7** (Closure)**.** The closure $\bar{S}$ of $S$ is the union

$$S \cup \partial S.$$

Clearly $\bar{S}$ is always closed (by our definition).

**Theorem 3.3.** *$S^\circ$ is open for any $S$.*

Doesn't this follow from the definition too??

## §3.5 Jank Connectedness

**Definition 3.8** (Path-connectedness)**.** A set $S$ is path-connected if every pair of points $z_1, z_2 \in S$ is connected by a continuous path in $S$.

Every path-connected set is connected (can be written as the union of two disjoint sets). Something about polygonal paths?? Dr. Radin is right, this is most definitely not standard. Is this what physicists do to topology?

Now he's talking about the Topologist's sine curve (the classic counterexample). This is a counterexample to the (false) idea that connected implies path-connected by exhibiting a set that is connected but not path-connected (but we haven't even talked about the standard definition of connectedness yet!).

## §4 September 8, 2020

### §4.1 Accumulation Points

**Definition 4.1.** A connected open set is a *domain*.

**Definition 4.2.** A *region* is a domain that contains none, some, or all of its boundary.

**Definition 4.3** (Bounded Set)**.** A set $S$ is bounded if

$$S \subseteq B(x_0, \epsilon).$$

for some $x_0 \in \mathbb{C}, \epsilon > 0$.

**Definition 4.4** (Accumulation Points)**.** $z_0$ is an accumulation point of $S$ if for all balls $B(z_0, \frac{1}{m})$ centered at $z_0$, we have

$$B(z_0, \frac{1}{m}) \setminus \{z_0\} \cap S \neq \emptyset.$$

**Example 4.1.** Let $S = \mathbb{Q}$. Then $\frac{1}{2}, \sqrt{2}$ etc are accumulation points of $S$ (this relies on the fact that $\mathbb{Q}$ is *dense* in $\mathbb{R}$). This example shows that accumulation points don't have to be in the set themselves.

**Theorem 4.1.** *We have $S$ is closed if and only if $S$ contains all of its accumulation points, the set of which is denoted $S'$. Furthermore, the closure of $S$ denoted $\bar{S}$ is equal to $S \cup S'$.*

*Proof.* $\implies$ Accumulation points are either in the boundary of $S$ or in $S$ itself. Since $S$ is closed, we have $S' \subseteq S$. $\impliedby$ If $z_0 \in \partial S \cap S^c$ it would be an accumulation point of $S$, a contradiction. So $\partial S \subseteq S \implies S$ is closed. (I'll try to write a better proof later). ⊠

A quick summary of basic p-set topology:

1. $S$ is open $\iff S = S^\circ$,
2. $S$ is closed $\iff S^c$ is open,
3. $S$ is open $\iff S$ contains none of $\partial S$,
4. $S$ is closed $\iff S$ contains all of $\partial S$,
5. $S$ is closed $\iff S$ contains all of $S'$.

### §4.2 Limits

Consider a map $f\colon \operatorname{Dom}(f) \to \mathbb{C}$, $\operatorname{Ran}(f) \subseteq \mathbb{C}$ (I prefer the notation $f\colon X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$, and $\operatorname{Ran}(f) = f[X]$). The fact that $f$ is well defined on $X$ holds because define $X$ to be a set on which $f$ is well defined, duh).

We want to talk about whether a function is continuous or not. Intuitively, a function is continuous if points in the image being "close" together imply that points in the preimage are also "close" together (the preimage of an open set is open).

**Definition 4.5** (Epsilon Delta Limits)**.** For $z_0$ an accumulation point of some subset $X$ of $\mathbb{C}$ (a region), $\lim_{z \to z_0} f(z)$ exists and has a value of $L \iff$ for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon,$$

where $z \in X$. The modulus is just a distance metric: so the epsilon delta definition is the same as what I said earlier, if points are close to each other in the codomain ($|f(z) - L| < \epsilon$), then such points are close to each other in the domain ($0 < |z - z_0| < \delta$).

Some notes: the limit is only defined when $z_0$ is an accumulation point. This why accumulation points are also sometimes referred to as *limit points*.

### §4.3 Continuity

**Definition 4.6** (Continuity)**.** $f$ is continuous at $z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$. $f$ is said to be continuous on a set $X$ if for all $x \in X$, $f$ is continuous at $x$.

We want to *analyze* a function $f(z)$, let $z = (x, y)$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, $u(x, y) = \operatorname{Re} f$ and $v(x, y) = \operatorname{Im} f$.

**Theorem 4.2.** *We have*

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \lim_{z \to z_0} \operatorname{Re} f(z) \to \operatorname{Re} L \\ \lim_{z \to zo} \operatorname{Im} f(z) \to \operatorname{Im} L. \end{cases}$$

*Proof.* Homework. $\boxtimes$

**Theorem 4.3.** *Let $f\colon X \to \mathbb{C}, g\colon Y \to \mathbb{C}$. For an accumulation point $z_0$ of $X \cap Y$, if $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then (excuse the abuse of notation)*

1. $\lim(f+g) = L + M$,
2. $\lim fg = LM$,
3. $\lim \frac{f}{g} = \frac{L}{M}$ *if* $M \neq 0$.

*Proof.* Same as the ones you'd find in any analysis course. $\boxtimes$

Continuity of sums, products, and quotients of functions follow from the above theorem. Now we turn our attention to the composition of functions.

**Theorem 4.4.** *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$. Let $z_0$ be an accumulation point of $X$. Then if $f$ is continuous at $z_0$ and $g$ is continous at $f(z_0)$, we have $f \circ g$ continuous at $z_0$.*

**Example 4.2.** $f(z) = |z^m|$ for a fixed $m$ is equal to $(g \circ h)(z)$ where $h(z) = z^m$ and $g(w) = |w|$. Both $h$ and $g$ are continuous on $\mathbb{C}$, so $|z^m|$ is also continuous everywhere.

**Example 4.3.** The identity map is continuous. This is trivial (let $\delta = \epsilon$). It follows that maps of the form $z^n$ is continuous for some positive integer $n$.

**Corollary 4.1.** *Functions of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

*where $p(z)$ and $g(z)$ are polynomials are continuous given $g(z) \neq 0$.*

**Example 4.4.** Let $f(z) = \frac{z}{\bar{z}}$, $z \neq 0$. Consider $z = x + iy$ near 0 with $x \neq 0, y = 0$, then $f(z) = 1$. If $x = 0, y \neq 0$ then $f(z) = -1$. Therefore $\lim_{z\to z_0} \frac{z}{\bar{z}}$ does not exist (standard technique for proving multivariate limits don't exist).

## §5 September 10, 2020

### §5.1 More on Continuity

Last time we talked about the function $\frac{z}{\bar{z}}$. What if we define the domain as $\mathbb{C} \setminus \{0\}$? Does $\lim_{z \to z_0} \frac{z}{\bar{z}}$ exist? (AKA: is $\frac{z}{\bar{z}}$ continuous on its domain?)

**Theorem 5.1.** *Let $f : \mathbb{C} \to \mathbb{C}$ be defined as $f = u + iv$. If $f$ is continuous at $z_0$, then*

1. *$\overline{f} = u - iv$ is continuous at $z_0$. We can also write $\overline{f}$ as $g \circ f$ where $g(w) = \overline{w}$.*
2. *$\frac{f + \overline{f}}{2} = \mathrm{Re}(f)$ is continuous at $z_0$.*
3. *$\frac{f - \overline{f}}{2i} = \mathrm{Im}(f)$ is continuous at $z_0$.*

*Proof.* We prove that $f(z) = \bar{z}$ is continuous at any $z_0$. Given $\varepsilon > 0$, consider

$$|f(z) - f(z_0)| = |\bar{z} - \overline{z_0}|.$$

We need a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |\bar{z} - \overline{z_0}| < \varepsilon.$$

Claim: If $S = \varepsilon$, $|\bar{z} - \overline{z_0}| = |\overline{(z - z_0)}| = |z - z_0| = \delta = \varepsilon$. This is easy to see, so we are done. ⊠

**Note.** To show that

$$\lim_{z \to z_0} f(z) = L,$$

we consider neighborhoods (open sets around $L$), or the set of $z$ such that $|f(z) - L| < \varepsilon$ (equivalently, the $z$ such that $f(z) \in B(L, \varepsilon)$). Also, $\lim_{z \to z_0} f(z) - L \iff \lim_{z \to z_0}(f(z) - L) = 0 \iff \lim_{z - z_0 \to 0}(f(z) - L) = 0$.

### §5.2 Limits near Infinity

Infinity is not a complex number!! Consider the limits

$$\lim_{z \to \infty} f(z)$$

and

$$\lim_{z \to z_0} f(z) = \infty.$$

To define these, we use neighborhoods of "$\infty$ ". There is no notion of "$\pm\infty$" in the complex numbers. The definition is similar to the one you encountered in Real Analysis: $z$ is "large" if $|z| > R$ for all $R \in \mathbb{R}$.

**Definition 5.1** (Limits at Infinity)**.** For $z_0 \in \mathbb{C}$ we say

$$\lim_{z \to z_0} f(z) = \infty$$

if given some $R > 0$, $R \in \mathbb{R}$, there exists some $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z)| > R.$$

**Example 5.1.** We have $\lim_{z \to 0}(\frac{1}{z}) = \infty$ since given $R > 0$, there exists a $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|\frac{1}{z}| > R$, namely, $\delta = \frac{1}{R}$, because

$$|z| < \frac{1}{R} \implies \frac{1}{|z|} > R \iff \left|\frac{1}{z}\right| > R.$$

**Definition 5.2** (Limits to Infinity)**.** We say $\lim_{z \to \infty} f(z) = L$, $L \in \mathbb{C}$ if and only if for all $\varepsilon > 0$, there exists some $R > 0$ such that

$$|z| > R \implies |f(z) - L| < \varepsilon.$$

**Example 5.2.** We have $\lim_{z \to \infty} \frac{1}{z} = 0$, let $\varepsilon > 0$, $R = \frac{1}{\varepsilon}$. Then $|f(z) - L| = \left|\frac{1}{z}\right|$, so

$$|z| > R \implies |z| > \frac{1}{\varepsilon} \implies \varepsilon > \frac{1}{|z|} = \left|\frac{1}{z}\right|,$$

and we are done.

**Definition 5.3.** Finally, we say

$$\lim_{z \to \infty} f(z) = \infty$$

if (for $R_1, R_2 \in \mathbb{C}$) given some $R_1 > 0$, there exists an $R_2 > 0$ such that

$$|z| > R_2 \implies |f(z)| > R_1.$$

**Example 5.3.** We have $\lim_{z \to \infty} z^2 = \infty$ since $|z^2| > R$ whenever $|z| > \sqrt{R}$.

## §5.3 Derivates

We are finally ready to define the derivative of a function (the good stuff). Given a function $f\colon X \to \mathbb{C}$, we will only define the derivative of $f$ at a point $z \in X^\circ$. Recall that $X^\circ = \{z \in X \mid B(z,\gamma) \subseteq X\}$ for some $\gamma > 0$.

**Definition 5.4** (Complex Derivative)**.** A function $f\colon X \to \mathbb{C}$ is said to be *differentiable* at $z_0 \in X^\circ$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$ (so limits to infinity are not allowed. We will examine these "poles" later in the course). If the limit exists, we denote this limit as $f'(z_0)$.

**Example 5.4.** Let $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto 7$. We claim that $f'(z) = 0$ for all $z$, since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{7-7}{z - z_0} = 0.$$

We only look at the points $z$ "near" (accumulation points) $z_0$, so we don't have to worry about the case where $z = z_0$. So given $\varepsilon > 0$,

$$|z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \varepsilon$$

for any $\delta > 0$.

**Example 5.5.** Let $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z$. We claim $f'(z) = 1$ since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z - z_0}{z - z_0} = 1$$

for any $z \neq 0$. This limit is one since

$$\left| \frac{f(z) - f(z_0}{z - z_0} - 1 \right| = \left| \frac{z - z_0}{z - z_0} \right| = 0.$$

**Example 5.6.** Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. We will show $f'(z_0) = 2z_0$. We want to find a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - 2z_0 \right| < \varepsilon.$$

So

$$\left| \frac{z^2 - z_0^2}{z - z_0} - 2z_0 \right| = |(z + z_0) - 2z_0| = |z - z_0| < \varepsilon$$

if $|z - z_0| < \delta$ with $\delta = \varepsilon$. There aren't any limit signs because we directly invoked the epsilon-delta definition.

**Example 5.7.** Consider $f(z) = |z|$ (maps will map $\mathbb{C} \to \mathbb{C}$ unless otherwise stated from now on). We have showed $f$ is continuous for all $z$, but $f$ isn't differentiable at 0. Use the technique at the end of the last example (write out the piecewise definition of the absolute value and show that the limits don't agree).

What about $z_0 \neq 0$? Is $f\colon \mathbb{C} \setminus \{0\} \to C$ differentiable? Let $z_0 \in C \setminus \{0\}$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z| - |z_0|}{z - z_0} = \frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}}.$$

We let $z$ get close to $z_0$ in two different ways. First, assume $r = r_0$ but $\theta \neq \theta_0$ (vary the angle, but all having length $r$). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{0}{r(e^{i\theta} - e^{i\theta_0})} = 0.$$

Next, assume $r \neq r_0$ but $\theta = \theta_0$ (points on a line with angle $\theta$, vary the length). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{r - r_0}{e^{i\theta}(r - r_0)} = e^{-i\theta} \neq 0.$$

So $f$ is nowhere differentiable.

### §5.4 Product, Quotient, and Chain Rules

To get $f'(z)$ for $f(z) = z^m$, we want a formula. Time for induction!

**Theorem 5.2.** *If $f'(z_0)$ and $g'(z_0)$ exist for two functions $f$ and $g$, then so do the derivatives*

1. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$,
2. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,
3. $(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$ *provided* $g(z_0) \neq 0$.

**Theorem 5.3.** *If $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$ then $f \circ g$ is differentiable at $z_0$ and*

$$(f \circ g)'(z_0) = f'[g(z_0)]g'(z_0).$$

**Note** (Leibniz Rule)**.** Suppose we have $f_1, f_2, ..., fn$ functions all differentiable at $z_0$. Then

$$(f_1 f_2 f_3 \cdots f_n)'(z_0) = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + f_1 f_2 f_3' f_4 \cdots f_n + \cdots .$$

In particular, $(z^n)' = n(z' z^{n-1}) = nz^{n-1}$ (just take $f_i = f$ and it becomes clear that this is true).

## §6 September 15, 2020

I could be studying fundamental groups right now, but instead I'm sitting here verifying limits and derivatives by hand. Why?? OK so Gradescope is a meme. Anything new?

Everything so far has been awfully boring. But now it gets interesting. Finally, I've been waiting for this.

### §6.1 Cauchy-Riemann Equations

Suppose we have a function $f: \mathbb{C} \to \mathbb{C}$, write it as $f(z) = u(x,y) = iv(x,y)$. Assume $f'(z_0)$ exists, and is equal to

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(u(x,y)+iv(x,y)) - (u(x_0,y_0)+iv(x_0,y_0))}{(x+iy)-(x_0+iy_0)}.$$

Then we rewrite this to get

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)+i\left[v(x,y)-v(x_0,y_0)\right]}{(x-x_0)+i(y-y_0)}.$$

Consider two special ways $(x,y)$ can be "near" $(x_0,y_0)$. First, let $x = x_0$ but $y \neq y_0$. Then the quotient becomes

$$\frac{u(x_0,y)-u(x_0,y_0)+i[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)} =$$
$$\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}.$$

Then the limit is equal to

$$\frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0).$$

Now let $y = y_0$ but $x \neq x_0$. Then the quotient becomes

$$\frac{u(x,y_0)-u(x_0,y_0)+i\left[v(x,y_0)-v(x_0,y_0)\right]}{x-x_0} = \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + \frac{i\left[v(x,y_0)-v(x_0,y_0)\right]}{x-x_0},$$

so the limit $f'(z_0)$ is equal to

$$\frac{\partial u(x_0,y_0)}{\partial x} + i\frac{\partial v}{\partial x}(x_0,y_0).$$

Why are we doing this? It's because if the limit exists, it should be the same whichever direction you approach it from, so you can derive some cool equalities.

The two equations must agree, so

$$\frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$

Examine the real and imaginary parts, so we have

$$\frac{\partial v}{\partial y}(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0). \tag{1}$$

These are known as the *Cauchy-Riemann Equations*. Furthermore,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).$$

What does this tell us? If your function is differentiable at a point, then we have a way to compute the derivative at that point. The converse does not hold, that is, if the Cauchy-Riemann equations hold this doesn't necessarily guarantee the existence of a derivative at that point.

**Example 6.1.** Recall that the function $f(z) = |z|$ is nowhere differentiable. However, consider $g(z) = |z|^2 = x^2 + y^2 = u(x,y) + iv(x,y)$, where $v(x,y) = 0$ and $u(x,y) = x^2 + y^2$. Let's check to see if this function satisfies the Cauchy-Riemann equations. $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$. Does $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$? Only if $2x = 0 \implies x = 0$. Does $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$? Only if $2y = 0 \implies y = 0$. So this function only satisfies the Cauchy-Riemann equations at the origin, which implies that the function is nowhere differentiable at any other point ($g'(z_0)$ does not exists if $z_0 \neq 0$). Now it does satisfy the CR equations at 0, but we don't have the existence of the derivative guaranteed.

Let's check the case if $z_0 = 0$.

$$\frac{g(z)-g(0)}{z-0} = \frac{|z|^2-0}{z-0} = \frac{\bar{z}z}{z} = \bar{z}.$$

Does $\lim_{z\to 0}\bar{z}$ exist? Yes, and it's equal to 0. So $g'(0)$ exists and is equal to 0.

## §6.2 Weak Converse of CR Equations

Let's talk about the opposite of the CR equations.

**Theorem 6.1** (Weak Converse of Cauchy-Riemann Equations)**.** *Let $f = u + iv$ be defined on a neighborhood of $z_0 = x_0 + iy_0$. Suppose the partial derivatives of $u$ and $v$ exist in that neighborhood, and are continuous at $z_0$. Furthermore, suppose the functions $u$ and $v$ satisfy the CR-equations at $z_0$. Then $f'(z_0)$ exists.*

**Note.** We claim the hypotheses hold for $|z|^2 : u(x, y) = u^2 + y^2$, $v(x, y) = 0$. Oops, I went to the restroom here. I don't think I missed anything interesting though.

Now the next topic is very important.

**Example 6.2.** Let $f(x, y) = e^x(\cos(y) + i \sin(y)) = e^x \cos(y) + ie^x \sin(v)$. Note: $u$ and $v$ are *nice*. Let's compute the CR equations: $\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} = e^x \cos(y)$. We also have $\frac{\partial u}{\partial y} = -e^x \sin(y) = -\frac{\partial v}{\partial x} = -e^x \sin(y)$. Then $f$ is differentiable everywhere, furthermore,

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos(y) + ie^x \sin(y) = e^x(\cos(y) + i \sin(y)) = f.$$

So $f$ is equal to its derivative everywhere. This is probably the single most important function in the entire course.

We will eventually denote this function as $\exp(z)$. Notee: if we use Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$, then

$$\exp(z) = e^x e^{iy} = e^{x+iy} = e^z.$$

But we have to make sure we can add the exponents first.

## §6.3 CR Equations in Polar Coordinates

Before discussing this further, consider polar coordinates for $z$. For any function $g$, write $f(z) = u(r, \theta) + iv(r, \theta)$. Then after the change of coordinates we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

where $x = r\cos\theta$, $y = r\sin\theta$. We have $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$. Use these to get

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

and

$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta.$$

If $f$ is differentiable, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Plugging these into the CR equations, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, ??stopmovingthepage$$

Next time: no. We'll do it next time. We have a test in 2 weeks BTW.

## §7 September 17, 2020

### §7.1 CR Equations (cont)

Last time: Cauchy Riemann equations for $f = u + iv = u(x,y) + iv(x,y)$. They are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Also discussed this for polar coordinates, yada yada.

**Example 7.1.** Let $f(z) = \frac{1}{z^4}$.

$$\frac{1}{z^4} = \frac{1}{r^4 e^{i4\theta}} = \frac{1}{r^4} e^{-i4\theta} = \frac{1}{r^4}[\cos(4\theta) - i\sin(4\theta)].$$

So

$$\begin{aligned} f' &= e^{-i\theta}\left(-\frac{4}{r^5}\cos(4\theta) + i\frac{4}{r^5}\sin(4\theta)\right) \\ &= -\frac{4}{r^5} e^{-i\theta}(e^{-i4\theta}) \\ &= -\frac{4}{r^5} e^{-5i\theta} = -\frac{4}{r^5 e^{5i\theta}} \\ &= -\frac{4}{z^5}, \end{aligned}$$

which is a known formula for the derivative.

### §7.2 Analytic Functions

**Definition 7.1** (Analytic)**.** A function $f$ is said to be analytic at $z_0$ if $f$ is differentiable at all $z$ in some ball centered at $z_0$. $f$ is said to be analytic on a set $S$ if for all $z_0 \in S$, $f$ is analytic at $z_0$.

**Example 7.2.** Let $f(z) = z^m$, $m \in \mathbb{N}$. Then $f'(z_0) = mz_0^{m-1}$ for all $z_0$. So such $f$ are analytic in $\mathbb{C}$. If $m \in \mathbb{Z} \setminus \mathbb{N}$, this formula still holds for $z_0$ nonzero. So $f$ is analytic on the punctured plane $\mathbb{C} \setminus \{0\}$.

**Definition 7.2** (Entire function)**.** We say a function $f$ is *entire* if $f$ is analytic on $\mathbb{C}$. For example, $f(z) = z^3$ is entire. More generally, all polynomials are entire.

**Theorem 7.1.** *If $f'(z) = 0$ for all $z \in D$ a domain, then $f$ is constant in $D$. (Is this weak Liouville's Theorem?)*

*Proof.* We will show that for any pair $z_1, z_2 \in D$, $f(z_1) = f(z_2)$. Let $z_1, z_2 \in D$, then there is some finite set of straight lines connecting $z_1$ and $z_2$ (what is this definition reeee). Consider $f$ on a segment $z = z(t)$, $0 \le t \le 1$. Then $F(t) = f[z(t)]$, $0 \le t \le 1$ which is equal to $u[x(t), y(t)] + iv[x(t), y(t)]$ So

$$\frac{dF}{dt} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\right].$$

By assumption, $\frac{df}{dz} = 0$ in $D$. We can write this in two ways: $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0$. So $\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial x}{\partial y}$ in $D$, and so $\frac{dF}{dt} = 0$, $0 \le t \le 1$.

Consider $\mathrm{Re}[F(t)] = u[x(t), y(t)]$. It follows that $\frac{d}{dt}u(x(t), y(t)) = 0$, $0 \le t \le 1$. It follows that $u(x(t), y(t))$ is constant, since

$$\int_0^1 \frac{d}{dt} u(x(t), y(t))\, dt = 0.$$

Similarly, $V(x(1), y(1)) - V(x(0), y(0)) = 0$. So $F(t) = f(z(t))$ has the same values at $z_a$ and $z_b$. (What?? Why??) Once we get Louiville's theorem we will get a less bad proof. This proof was big bad. ⊠

We will show later that if $f = u + iv$ is analytic at $z_0$, then $u(x,y)$ and $v(x,y)$ have partial derivatives of all orders in a neighborhood of $z_0$. If a function is analytic, then it is infinitely differentiable: what?? Complex analysis is crazy.

## §7.3 Harmonic Equations

**Definition 7.3** (Harmonic)**.** If $u_{xx} + u_{yy} = 0$, $u$ is *harmonic* in that nbd of $z_0$, similarly for $v$. Laplace equation.

**Definition 7.4.** We say $v$ is a *harmonic conjugate* of $u$ in some region $D$ if $u$ and $v$ are harmonic in $D$, and $u, v$ satisfy the CR equations.

**Note.** For some analytic function $f$ its true that $\overline{f(z)} = f(z)$.

**Theorem 7.2** (The Reflection Principle)**.** *"Apparently this is famous, but I've never used it" -Dr. Radin*
*Suppose $f$ is analytic in a domain $D$ which is symmetric WRT the x-axis. Then for all $z \in D$,*

$$\overline{f(z)} = f(\overline{z}) \iff f \text{ is real on the segment of the x-axis in } D.$$

**Note.** I've been taking less notes because I'm simultaneously doing my weekly Differential Equations quiz while TEXing notes. Just wanted to say that

## §8 September 22, 2020

Last time: for any $z = (x, y) \in \mathbb{C}$, $\exp(z) = e^x[\cos y + i \sin y] = e^x e^{iy}$. $e^z = \text{Exp}(z) = \exp(z)$. We showed this function is differentiable on $\mathbb{C}$ and that its derivative is itself.

The product of complex numbers $e^{z_1}e^{z_2} = (e^{x_1}e^{iy_1})(e^{x_2}e^{iy_2})$. Since multiplication is commutative, we have $(e^{x_1}e^{x_2})(e^{iy_1}e^{iy_2}) = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{z_1+z_2}$. This follows from our defintions, its not an assumption.

**Corollary 8.1.** *$e^z e^{-z} = e^0 = 1$. So $e^{-z} = \frac{1}{e^z}$. Also, for $m = 1, 2, \cdots$ $(e^z)^m = e^{mz}$. This also holds for negative integers. Finally, by our differentiation rules,*

$$\frac{d}{dz}e^{az^n} = naz^{n-1}e^{az^n}.$$

So far we've covered how to differentiate polynomials (or more generally, rational functions), and now we've added $e^z$ to our arsenal. Let's introduce some more basic functions to our list. Why do we differentiate? This is a course in functions of a complex variable, differentiating them, integrating the, etc. (I wish we covered analytic continuity). The next set of functions are trig functions.

### §8.1 Trig functions

Recall that $e^{ix} = \cos x + i \sin x$, $e^{-ix} = \cos x - i \sin x$, so $\frac{e^{ix}+e^{-x}}{2} = \cos x$, $\frac{e^{ix}-e^{-x}}{2} = \sin x$. (Not sure if I got the definitions right). We can extend these to the complex plane, we define $\cos z = (e^{iz} + e^{-iz})/2$, $\sin z = (e^{iz} - e^{-iz})/2i$ for all $z$[1]. So $\frac{d}{dz} \cos z = (ie^{iz} - ie^{-iz})/2 = i^2 \sin z = -\sin z$. Similarly, $\frac{d}{dz} \sin z = (ie^{iz} + ie^{-iz})/2i = \cos z$. So these formulas agree with their real analog. We write the definitions again for clarity:

**Definition 8.1.** We define the trigonometric functions $\sin z$ and $\cos z$ on $\mathbb{C}$ as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Now by our new definitions of trig functions,

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = e^{iz}.$$

From our definition,

$$\sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{zi}.$$

We claim that this is equal to $\sin z_1 \cos z_2 + \cos z_2 \sin z_1$. This is just a bunch of tedious manual labor. I don't really want to type this out, but here I am. We have this equal to

$$\begin{aligned} &\left(\frac{e^{iz_1} - e^{iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right) \\ = \frac{1}{4i}&\left[e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)} + e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(z_1+z_2)} - e^{i(-z_1-z_2)}\right] \\ &= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{i(-z_1-z_2)}\right] \\ &= \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{i(-z_1-z_2)}\right] \\ &= \sin(z_1 + z_2). \end{aligned}$$

I think I may have typed the second (long) equation incorrectly, but I am not in the mood for going back and double checking this. Manual labor should be reserved for homework (and even then, I am still unwilling to do it).

We have a special case: $\sin(z + 2\pi) = \sin(z)\cos(2\pi) + \cos(z)\sin(2\pi) = \sin(z)$. Clearly this generalizes to $\sin(z + 2\pi n)$ for $n \in \mathbb{Z}$. So sin is periodic.

**Definition 8.2** (Tangent)**.** Let

$$\tan z = \frac{\sin z}{\cos z}.$$

Note that this isn't defined at $\cos z = 0$: when does this happen? I stopped taking notes here for a little bit.

[1] Note that $e^{iz}$ is differentiable, and $\frac{d}{dz}e^{iz} = ie^{iz}$.

## §8.2 Hyperbolic trig functions

Let's look at another class of functions: the cool dudes, hyperbolic trig functions (sinh is pronounced "sinch", cosh is pronounced "coush", etc). This gives me good memories of my first Calculus class with Dr. Neal Brand at UNT.

**Definition 8.3.** We define the hyperbolic trig functions $\cosh z$ and $\sinh z$ as

$$\cosh z = \frac{e^z + e^{-z}}{z}, \quad \sinh z = \frac{e^z - e^{-z}}{z}.$$

Similarly, $\tanh z$ is defined as

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

## §9 Holomorphic Functions

I want to do some real math! These notes will follow Stein and Shakarchi §1.2.

### §9.1 Continuous functions

We've already seen the standard epsilon-delta definition of continuity. An equivalent definition is the sequential definition, that is, for every sequence $\{z_1, z_2, \cdots\} \subseteq \Omega \subseteq \mathbb{C}$ such that $\lim z_n = z_0$, then $f$ is continuous at $z_0$ if $\lim f(z_n) = f(z_0)$. Since the notions for convergence of complex numbers and $\mathbb{R}^2$ is the same, $f$ of $z = x + iy$ is continuous iff it's continuously viewed as a function of two real variables $x$ and $y$. If $f$ is continuous, then the real valued function defined by $z \mapsto |f(z)|$ is clearly continuous (by the triangle inequality).

We say $f$ attains a *maximum* at the point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in \Omega.$$

The definition of a minimum is what you think it is.

**Theorem 9.1.** *A continuous function on a compact set $\Omega$ is bounded and attains a maximum and minimum on $\Omega$.*

*Proof.* Same as the any one you'd find in a Real Analysis course. ☒

### §9.2 Holomorphic functions

Let's talk about the good stuff.