# **Differential Topology Notes**

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Notes for the Spring 2021 graduate section of Differential Topology (Math 382D) at UT Austin, taught by Dr. Freed. Source files: https://git.simonxiang.xyz/math\_notes/files.html

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Lecture 1

## April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

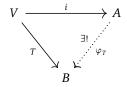
A lot of definitions, here's the new ones:

**Definition 1.1.** A **subalgebra** of an algebra A is a linear subspace  $A' \subseteq A$  containing 1 such that  $a'_1 a'_2 \in A'$  for all  $a'_1, a'_2 \in A'$ . A **2-sided ideal**  $I \subseteq A$  is a linear subspace such that AI = I and IA = I. A  $\mathbb{Z}$ -**grading** of an algebra A is a direct sum decomposition  $A = \bigoplus_{k \in \mathbb{Z}} A^k$  such that  $A^{k_1} A^{k_2} \subseteq A^{k_1 + k_2}$  for all  $k_1, k_2 \in \mathbb{Z}$ . If A is a  $\mathbb{Z}$ -graded algebra and  $a \in A^k, k \in \mathbb{Z}^{>0}$ , then a is **decomposable** if it is expressible as a product  $a = a_1 \cdots a_k$  for  $a_1, \cdots, a_k \in A^1$ . If not, a is **indecomposable**.

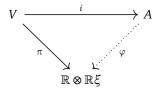
#### 1.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra  $\bigotimes V$ , thought of as the "free algebra generated by V".

**Definition 1.2.** Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map  $i: V \to A$  such that for all (B, T) of an algebra B and a linear map  $T: V \to B$  such that  $\varphi_T$  is a homomorphism of algebras.



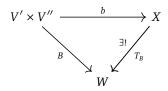
(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If  $(\xi \neq 0) \in V$  and  $i(\xi) = 0$ , set  $B = \mathbb{R} \oplus \mathbb{R} \xi$  and define  $\xi^2 = 0$ .



Note that  $\pi|_{\mathbb{R}\xi} = \text{id}$ . But  $\xi = \pi(\xi) = \varphi_1(\xi) = 0$ , a contradiction. Furthermore, A has a canonical  $\mathbb{Z}$ -grading.  $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$ ,  $T_\lambda : V \to V$  is scalar multiplication,  $\varphi_\lambda : A \to A$  is a homomorphism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

**Definition 1.3.** Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map  $b: V \times V'' \to X$  such that for all (W, B),



We denote  $X = V' \otimes V''$ , and  $b(\xi', \xi'') = \xi' \otimes \xi'', \xi' \in V', \xi'' \in V''$ .

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If S' is a basis of V', S" a basis of V", then  $S' \times S''$  is a basis of  $V' \otimes V''$ , where

$$S' \times S'' \cong \{ \xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S'' \}.$$

Note that  $\bigotimes$  is "commutative" and "associative" with unit  $\mathbb{R}$ , so

$$\mathbb{R} \otimes V \to V$$

$$V_1 \otimes V_2 \to V_2 \otimes V_1$$

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),$$

forming what we call a **symmetric monoidal category**. We write  $\otimes^1 V = V$ ,  $\otimes^2 V = V \otimes V$ ,  $\otimes^3 V = V \otimes V \otimes V$  and so on. We also write  $\otimes^0 V = \mathbb{R}$ , and sometimes replace  $\otimes^n V$  with  $V^{\otimes n}$ .

#### 1.2 Existence of tensor algebras

Let V be a vector space, and  $A = \bigoplus_{k=0}^{\infty} \otimes^k V$ . Let  $i: V \hookrightarrow A$  be the inclusion into  $\otimes' V = V$ .

**Claim.** (A, i) is a tensor algebra over V.

To see this, note that

$$\xi_1 \otimes \cdots \otimes \xi_k) \cdot_A \eta_1 \otimes \cdots \otimes \eta_\ell = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that  $A = \otimes' V$  is not commutative.

#### 1.3 The Exterior Algebra

We want to impose todo:come back

Lecture 2

## April 8, 2020

todo:see notes on chapter 21, multivariate analysis

#### 2.1 Exterior algebra of a direct sum

**Definition 2.1.** Let V be a vector space. An **exterior algebra** (E, j) over V is an algebra E and a linear map  $j: V \to E$  satisfying  $j(\xi)^2 = 0$  for all  $\xi \in V$  such that for all pairs (B, T) consisting of an algebra E and a linear map E: E and E satisfying E and E such that E and E such that E and E such that E and E are E and E such that E and E are E are E and E are E and E are E are E are E and E are E are E are E are E are E are E and E are E and E are E are E are E are E are E and E are E and E are E are

Let  $L_1, L_2$  be linear, and  $\bigwedge^* (L_1 \oplus L_2 = V)$ .

Lecture 3

## April 15, 2021

todo:is this lecture 24??

**Theorem 3.1.** Let X be a smooth manifold. Then there exists a unique  $d: \Omega^*(X) \to \Omega^{*+1}(X)$  satisfying

(i) Linearity,

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- (ii) The Liebniz rule,
- (iii)  $d^2 = 0$ ,
- (iv)  $d|_{\Omega^0(X)}$  is the usual differential.

*Proof.* Let  $\{(U_i, x_i)\}_{i \in I}$  be an open cover of X by charts. Let  $\{\rho_i\}_{i \in I}$  be a partition of unity, where  $\operatorname{Supp} \rho_i \subseteq U_i$ . If  $\alpha \in \Omega^*(X)$ , then  $\alpha = \sum_i \rho_i \alpha_i$ , where  $\operatorname{supp}(\rho_i \alpha) \subseteq U_i$ . Define  $d\alpha = \sum_i d(\rho_i \alpha)$ , where we compute  $x_i(U_i) \subseteq A_i$ ,  $\operatorname{supp} d(\rho_i \alpha)$  (note that d increases support).

For this to be a good definition, we need to show that this is well-defined. say  $\{(V_a, y_a)_{a \in A} \text{ is another atlas, } \{\sigma_a\}_{a \in A} \text{ a partition of unity. Then}$ 

$$\sum_{i} d(\rho_{i}\alpha) = \sum_{i} \sum_{a} d(\rho_{i}\sigma_{a}\alpha)$$
$$= \sum_{a} \sum_{i} d(\sigma_{a}\rho_{i}\alpha)$$
$$= \sum_{a} d(\sigma_{a}\alpha).$$

Note that supp  $\rho_i \sigma_a \alpha \subseteq U_i \cap V_a$ . Something about d commuting with pullback, the first is defined on  $x_i(U_i \cap V_a)$ , the second on  $y_a(U_i \cap V_a)$ , and the final on  $y_a(V_a)$ . todo:this, plus something about transition maps

#### 3.1 Orientation

We have all seen Riemann integration on the line, and hopefully you have learned how to integrate in  $\mathbb{R}^n$ , and perhaps Lebesgue integration. We do not focus on the analytic aspects, but the geometric aspects, which allows us to integrate on manifolds. Unfortunately we do not have a fixed vector space, giving a fixed Lebesgue measure, so we have to start from the beginning. Let's talk about orientation.

Recall that if L is a real line (1-dimensional vector space), then an **orientation** of L is an element of  $\pi_0(L \setminus \{0\})$ .

**Definition 3.1.** If *V* is a finite dimensional real vector space, then an **orientation** of *V* is an orientation of det *V*. A **basis** of *V* is an isomorphism  $b: \mathbb{R}^n \to V$  if  $\dim V = n$ .

**Remark 3.1.** Let  $\mathcal{O}(V)$  be the set of bases of V. The group  $\operatorname{GL}_n\mathbb{R}=\{g:\mathbb{R}^n\stackrel{\cong}{\to}\mathbb{R}^n\}$  acts simply transitively on  $\mathcal{O}(V)$ .\(^1\) This is a right action  $\operatorname{GL}_n\mathbb{R}$ , or a torsor. Then  $\det\colon\operatorname{GL}_n\mathbb{R}\to\mathbb{R}^{\neq 0}$  is an isomorphism on  $\pi_0$ . Introduce  $\mathcal{O}(V)\to\det V\setminus\{0\},\ e_1,\cdots,e_n\mapsto e_1\wedge\cdots\wedge e_n$ . An orientation partitions  $\mathcal{O}(V)$  into  $\mathscr{B}^\pm(V)$ . If  $T\colon V'\to V$ , then  $\dim V'=\dim V$  if T is an isomorphism. Then  $\det T\colon\det V'\to\det V^2$  is an isomorphism, and T is orientation preserving (resp reversing) if T(O')=0 (resp  $T(O')\neq O$ ). (Here O denotes the orientation of a space.)

**Definition 3.2.** Let V be a finite dimensional real vector space. A nonzero element of  $\text{Det }V^*$  is a **volume form**. For  $\xi_1, \dots, \xi_k \in V$ ,  $(\xi_1, \dots, \xi_k) = \{t^i \xi_i \mid 0 \le \to i \le 1\} \subseteq \text{span}\{\xi_i\}$ , the vectors are **nondegenerate** if the  $\xi_1, \dots, \xi_k$  are LI iff  $\xi_1 \land \dots \land \xi_k \ne 0$  in  $\bigwedge^k V$ . If  $e_1, \dots, e_n$  is a basis of V, define

$$vol(//(e_1, \cdots, e_n)) = \|\langle \omega, e_1 \wedge \cdots \wedge e_n \rangle\|.$$

**Proposition 3.1.** If  $e'_1, \dots, e'_n$  is another basis, and  $e'_i = T^i_i e_i$  for  $T^i_i \in \mathbb{R}$ , then

$$\text{vol} / / (e'_1, \dots, e'_n) = (\det T) \text{vol} / / (e_1, \dots, e_n).$$

**Remark 3.2.** *Ratios* of volume are defined without a volume form. A k-form  $\alpha \in \bigwedge^k V_6 *$  induces a notion of volume on all k-dimensional subspaces  $W \subseteq V$  such that  $\alpha|_W \neq 0$ . On  $\mathbb{R}^n$  we take  $\omega = e^1 \wedge \cdots \wedge e^n \in \operatorname{Det} \mathbb{R}^{n^*}$ .

todo:?? canonical double cover, orientation bundle, homology

**Definition 3.3.** An orientation of X is a section of  $\pi_0^{\text{vert}}(\text{Det }TX\setminus 0)\to X$ . A **volume form** on X is a nonvanishing  $\omega\in\Omega^n(X)$  if  $\dim X=n$ .

<sup>&</sup>lt;sup>1</sup>Apparently in physics, left vs right actions form the idea of passive vs active actions or something like that. This is a right action.

<sup>&</sup>lt;sup>2</sup>Confused on usage of det and Det

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**Example 3.1.** If  $X = S^1$ , then we have two double covers up to isomorphism. If  $X = \mathbb{R}P^2$ , then  $D^2 \subseteq \mathbb{A}^2$  todo:something happen, so the orientation double cover has total space  $S^2$ , and  $\mathbb{R}P^2$  is not orientable.

**Definition 3.4.** Suppose *X* is an oriented manifold. A standard chart  $(U, x), x : U \to \mathbb{A}^n$  is **oriented** if  $\frac{\partial}{\partial x^1}\Big|_p$ ,  $\cdots$ ,  $\frac{\partial}{\partial x^n}\Big|_p$  is an oriented basis of  $T_pX$  for all  $p \in U$ .

If (U, x), (V, y) are oriented charts, then  $\det d(y \circ x^{-1}) > 0$ . Look forward to integration.