380C PROBLEM SET 6

DUE WEDNESDAY, OCTOBER 13TH

Let k be a commutative ring. A *(resp. commutative) k-algebra* is a (resp. commutative) ring A equipped with a ring homomorphism $i = i_A : k \to A$. Morphisms $\phi : A \to B$ of k-algebras are maps of rings such that $\phi \circ i_A = i_B$; isomorphisms of k-algebras are bijective k-algebra morphisms, or equivalently, morphisms with inverses.

This problem set assumes some familiarity with basic linear algebra.

Problem 1. Let k be a field and let A be a k-algebra with defining homomorphism $i : k \to A$. Consider A as a vector space over k using the addition on A and the scalar multiplication $\lambda \cdot f := i(\lambda)f$ for $\lambda \in k$ and $f \in A$.

Suppose A is an integral domain and finite-dimensional as a k-vector space. Show that A is a field.

Problem 2. Suppose k is a field such that $2 \neq 0$ in k. Suppose K/k is field extension (i.e., K is a k-algebra that is a field) of degree 2 (i.e., K is 2-dimensional as a k-vector space).

(a) Show that there exists an element $d \in k^\times$ such that K is isomorphic to $k[t]/(t^2 - d))$ as a k-algebra.
(b) Show that there is a unique non-identity isomorphism $\sigma : K \to K$ of k-algebras. Observe that $\sigma^2 = \mathrm{id}_K$.
(c) Suppose $d_1, d_2 \in k$ are two possible choices of d as above. Show that $d_1/d_2 \in (k^\times)^2$, i.e., this ratio is the square of some element in k.