Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1.1 Problem 5 Section 1.3

Problem. *Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $\widetilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\widetilde{X}$. Deduce that X has no simply-connected covering space.*

Solution. Let $\mathcal{G}$ be an open cover of $L := [0,1] \times \{0\}$, the left edge of X. Then since X is compact (and so is L), we have a finite subcover $\bigcup_{i=1}^n G_i$ of L. Our strategy: construct evenly covered neighborhoods from the finite subcovers that evenly cover the space. Choose a point, say $x = (0,0) \in G_1$. Then we have an evenly covered neighborhood U_x around x such that $p^{-1}(U_x)$ is a disjoint union of open sets that map homeomorphically to U_x. Ideally, our job would be really easy if the finite subcover was in the form $\amalg_{i=1}^n G_i$, so we could just take the union of every evenly covered open set around every x, finishing the construction. However, L is connected so there must be points at which the open sets in the cover interset: say $G_i \cap G_{i+1}$ is nonempty for some $1 \le i \le n$, that is, there exists some $x_0 \in G_i \cap G_{i+1}$. We can find an evenly covered neighborhood $U_{x_0} \subseteq G_i \cap G_{i+1}$ containing x_0 by simply taking the intersection of the evenly covered neighborhoods of x_0 in G_i and G_{i+1}, respectively. Then this set is evenly covered (each disjoint copy of the neighborhood in G_i intersect the disjoint copy of the neighborhood in G_{i+1} is just a copy of $G_i \cap G_{i+1}$, and will map homeomorphically onto it, still disjoint). Now that we've taken care of every point, we formally construct the neighborhood by taking unions of everything: let $U_{(0,0)}$ be the evenly covered neighborhood of $(0,0)$, and U_2 be the evenly covered neighborhood of some other point $x_2 \in G_1$. Then their union evenly covers $\{(0,0), x_2\}$: repeat until we have an evenly covered neighborhood of G_1. Now repeat for every G_i: in the case of intersection, take the union with the evenly covered neighborhood(s) of $G_i \cap G_{i+1}$ for all $x \in G_i \cap G_{i+1}$. This concludes our construction of an evenly covered neighborhood of L.

Now it's not too difficult to see that this evenly covered neighborhood lifts homeomorphically to $\widetilde{X}$: since each sheet must contain a rectangle with all but a finite amount of lines in the box, it maps homeomorphically onto X. To see that X has no simply connected covering space, every single covering space $\widetilde{X}$ has sheets that have nontrivial loops (take any loop starting at $(0,0)$, then going to $(0,1)$, then $(1/n, 1)$ and finally $(1/n, 0)$ for some $n \in \mathbb{N}$: then this is a nontrivial loop). Therefore $\pi_1(\widetilde{X})$ cannot possibly be trivial, and so X has no simply connected covering space. ■

§1.2 Problem 9

Problem. *Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]*

Solution. See the commutative diagram below:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dotted, "f'"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

For this diagram to commute, we need the lift f' to exist: the lifting criterion tells us this happens when X is path-connected and locally path-connected (which we have by assumption), and when $f_*(\pi_1(X)) \subseteq p_*(\pi_1(S^1))$. But we have $\pi_1(X)$ finite by assumption, so $\pi_1(X) \subseteq \pi_1(S^1) = \mathbb{Z}$ and the lift f' exists by the lifting criterion. So the diagram commutes, and $p \circ f' = f$. Since $\mathbb{R}$ is contractible, take a homotopy from $\mathrm{id}_{\mathbb{R}}$ to the constant map. Then composing the lift f' with this homotopy gives a new homotopy from f to the constant map, and we are done. ■

§1.3 Problem 14

Problem. *Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.*

Solution. First, recall that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, since $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/n\mathbb{Z}$ for $n \ge 2$. So by van Kampens, we have $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which we can denote with the presentation $\langle a, b \mid a^2, b^2 \rangle$. By the fundamental theorem of Galois theory (not really), the task of classifying all covering spaces has been reduced to classifying subgroups of $\langle a, b \mid a^2, b^2 \rangle$ and seeing what covering they correspond to. Now the words in this group look like

$$a, ab, aba, abab, ababa, \cdots$$

and so on. ■

§1.4 Problem 16

Problem. *Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.*

§1.5 Problem 20

Problem. *Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.*

§1.6 Problem 31

Problem. *Show that the normal covering spaces of $S^1 \vee S^1$ are precisely the graphs that are Cayley graphs of groups with two generators. More generally, the normal covering spaces of the wedge sum of n circles are the Cayley graphs of groups with n generators.*

§1.7 Problem 6 Section 1.A

Problem. *Let F be the free group on two generators and let F' be its commutator subgroup. Find a set of free generators for F' by considering the covering space of the graph $S^1 \vee S^1$ corresponding to F'.*

§1.8 Problem 7

Problem. *If F is a finitely generated free group and N is a nontrivial normal subgroup of infinite index, show, using covering spaces, that N is not finitely presented.*