Abstract Algebra Lecture Notes

Math 380C

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August 28, 2020

1 Lecture 2: Basic Group Theory (8/28/20)

Lemma 1. *Let $H \subset G$, $\langle G, \cdot \rangle$ a group and $H \neq \emptyset$. Then $H \leq G \iff h_1 h_2^{-1} \in H$ if $h_1 h_2 \in H$.*

Proof. For all $h_2 \in H$, $h_2^{-1} \in H$ since H is a group. H is closed under $\cdot$ implies $h_1 h_2^{-1} \in H$ for all $h_1, h_2 \in H$. Finish this later. $\boxtimes$

Definition 1 (Subgroup)**.** A *subgroup* H of G is normal if $gHg^{-1} = H$ for all $g \in G$.

Example 1. Let G be abelian: then every subgroup is normal since $ghg^{-1} = gg^{-1}h = h$ for all $g \in G, h \in H$.

Example 2. Take $G = S_3$. Then the subgroup $\langle (1, 2, 3) \rangle$ is normal, the subgroup $\langle (1, 2) \rangle$ is not normal, since $(13)(12)(13)^{-1} = (23) \notin \langle (12) \rangle$.

Example 3. Take $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$, where $SL_n(\mathbb{R})$ is the set of matrices such that $det(A) = 1$ for $A \in SL_n(\mathbb{R})$. $SL_n(\mathbb{R})$ forms a subgroup. Question: is it normal? Answer: Yes.

$$det(ABA^{-1}) = finishlater$$

Proposition: Let H, K be subgroups of G, then $H \cap K$ is a subgroup of G.
Note: is $H \cup K$ a subgroup? No!

Definition 2 (Product Groups)**.** Let G, H be groups. We define the *direct product* $G \times H$ with the group operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. Identity, Inverses. Ex: $\mathbb{Z}^2$

Example 4 (Quotient Groups)**.** Let $n \in \mathbb{Z}$, for example $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, +\right)$, equivalence relations: modulo n. $a, b \in \mathbb{Z}, a \equiv b \mod n \iff n \mid (a - b)$. Equivalence classes: $a + n\mathbb{Z} = \{a + nk \mid k \in \mathbb{Z}\}$. Notation: $\bar{a} = a + n\mathbb{Z} = [a]$. Our set $\frac{\mathbb{Z}}{n\mathbb{Z}} = \{a + n\mathbb{Z} \mid a \in \mathbb{Z}\} = \{a + n\mathbb{Z} \mid a = 0, ..., n-1\}$. $(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z}$, so this is a group operation. In this case, the identity is just $0 + n\mathbb{Z} = n\mathbb{Z}$. We have the inverse of $(a + n\mathbb{Z})$ equal to $(a + n\mathbb{Z})^{-1} = -a + n\mathbb{Z}$.

Remark: $(\mathbb{Z}/n\mathbb{Z}, +)$ is a quotient of the group $(\mathbb{Z}, +)$ by the subgroup $(n\mathbb{Z}, +)$. $< 1 >= \mathbb{Z}, < 1 + n\mathbb{Z} >=< \mathbb{Z}/n\mathbb{Z} >$.
Quotient groups in general: G a group, H a **normal** subgroup.

Definition 3 (Cosets)**.** Left cosets: $gH = \{gh \mid h \in H\}$. Right cosets: $Hg = \{hg \mid h \in H\}$. G/H - set of left cosets. HnG - set of right cosets.

Observe: Left and right cosets are in bijection with one another. $gH \mapsto Hg$, $gh \mapsto g^{-1}(gh)g = hg$. You can verify that this is a bijection. Let $g_1, g_2 \in G$, what map maps $g_1 H \to g_2 H$? $g_1 h \mapsto (g_2 g_1^{-1}) g_1 h = g_2 h$.
Note: $\bigcup gH = G, g \in G$.
Also: $g_1 H \cap g_2 H$ is either $\emptyset$ or they are equal. (Equivalence relation).
Proposition: If G is finite and H a subgroup of G, then $|H| \mid |G|$.

Proof. By the (also),

$$G = \bigcup_{i=1}^{n} giH$$

since G is a finite $n \in \mathbb{N}$. Disjoin union. So

$$|G| = \sum_{i=1}^{n} |g_i H| = n \cdot |G| \implies |H| \mid |G|.$$

$\boxtimes$

Quotient group: G a group, H a normal subgroup, $G/H = \{gH \mid g \in G\}$. $g_1 H \cdot g_2 H = g_1 g_2 H$.