De Rham Cohomology and Characteristic Classes Notes

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I'm excited to say that I'm participating in the Directed Reading Program (DRP) this semester, mentored by Arun Debray! (Read more here: web.ma.utexas.edu/users/drp). This semester, I'm following a book called *From Calculus to Cohomology: De Rham cohomology and characteristics classes* by Madsen and Tornehave.

These are the full version of the notes, taken to help me learn the material. I plan on summarizing my results in a sort of exposition style to put on the DRP website, as well as a beamer presentation for the symposium. I plan on having all three files hosted on my website somewhere, probably around here: https://git.simonxiang.xyz/math_notes/files.html

Prerequisites

The reader should be familiar with multivariable calculus and linear algebra at the minimum.

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Preliminary Material

1.1 Calculus

Question. Let $f : U \to \mathbb{R}^2$ be a smooth function, where $U \subseteq \mathbb{R}^2$ is open. Is there a smooth function $F : U \to \mathbb{R}$ such that $\partial_{x_1} F = f_1$, $\partial_{x_2} F = f_2$, where $f = (f_1, f_2)$? Note that this implies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. Is this a sufficient condition to show the existence of F?

Example 1.1. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right)$$

Now

$$\begin{aligned}
\partial_{x_2} f_1 &= \frac{-(x_1^2 + x_2^2) + 2x_2^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, \\
\partial_{x_1} f_2 &= \frac{(x_1^2 + x_2^2) - 2x_1^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}.
\end{aligned}$$

So f satisfies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. However, we have no $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$: assume there was such an F, then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0.$$

But

$$\frac{d}{d\theta} F(\cos\theta, \sin\theta) = \frac{dF}{dx}(-\sin\theta) + \frac{\partial F}{\partial y}\cos\theta = -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta = 1$$

by the chain rule, a contradiction. So we have procured a counterexample.

Definition 1.1 (Star-shaped)**.** A subset $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to $x_0 \in X$ if the line segment $\{tx_0 + (1-t)x \mid t \in [0,1]\}$ is contained in X for all $x \in X$.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^2$ be open and star-shaped. Then any smooth function $(f_1, f_2) \colon U \to R^2$ satisfying $\partial_{x_2} f_1 = \partial_{x_1} f_2$, there exists a smooth function $F : U \to \mathbb{R}$ such that $\partial_{x_1} F = f_1$, $\partial_{x_2} F = f_2$.*

Proof. Messy. ⊠

Say $U \subseteq R^2$ is open, then let $C^\infty(U, \mathbb{R}^k)$ be the vector space of smooth functions $\phi : U \to \mathbb{R}^k$. Define the **gradient** and **curl** functions[1] $\mathrm{grad} \colon C^\infty(U,\mathbb{R}) \to C^\infty(U, \mathbb{R}^2)$, $\mathrm{curl} \colon C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$ by

$$\mathrm{grad}(\phi) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \mathrm{curl}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}.$$

Note that the curl of the gradient is zero, or $\mathrm{curl} \circ \mathrm{grad} = 0$. So the kernel of the curl contains the image of the gradient, since mapping im(grad) by curl gives zero. Since curl and grad are linear, both ker(curl) and im(grad) are (infinite-dimensional) vector spaces, furthermore, im(grad) is a subspace of ker(curl). So we can consider the quotient space

[1] The book uses *rotation* instead of curl, but I think this is the standard notation.