De Rham Cohomology and Characteristic Classes Notes

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I'm excited to say that I'm participating in the Directed Reading Program (DRP) this semester, mentored by Arun Debray! (Read more here: web.ma.utexas.edu/users/drp). This semester, I'm following a book called *From Calculus to Cohomology: De Rham cohomology and characteristics classes* by Madsen and Tornehave.

These are the full version of the notes, taken to help me learn the material. I plan on summarizing my results in a sort of exposition style to put on the DRP website, as well as a beamer presentation for the symposium. I plan on having all three files hosted on my website somewhere, probably around here: https://git.simonxiang.xyz/math_notes/files.html

PREREQUISITES

The reader should be familiar with multivariable calculus and linear algebra at the minimum, as well as basic group theory (up to the first isomorphism theorem). Some things that are helpful but not necessary include:

- Basic analysis, including open and closed sets, and the inverse function theorem.
- Point-set topology would be very nice.
- Algebraic topology would be very helpful, but I assume no knowledge of cohomology.

In general, these notes will be taken like you know what open sets are, properties of connected spaces, what a commutative diagram is, stuff like that (because they were taken to help me learn the material). But for the condensed paper, I plan on introducing everything I need (besides stuff from calculus and linear algebra), so they can be somewhat self contained.

Contents

Lecture 1

Preliminary Material

1.1 Calculus

Question. Let $f: U \to \mathbb{R}^2$ be a smooth function, where $U \subseteq \mathbb{R}^2$ is open. Is there a smooth function $F: U \to \mathbb{R}$ such that $\partial_{x_1} F = f_1,\ \partial_{x_2} F = f_2$, where $f = (f_1, f_2)$? Note that this implies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. Is this a sufficient condition to show the existence of F?

Example 1.1. Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$, where

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right)$$

Now

$$\begin{aligned}
\partial_{x_2} f_1 &= \frac{-(x_1^2 + x_2^2) + 2x_2^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, \\
\partial_{x_1} f_2 &= \frac{(x_1^2 + x_2^2) - 2x_1^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}.
\end{aligned}$$

So f satisfies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. However, we have no $F: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$: assume there was such an F, then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0.$$

But

$$\frac{d}{d\theta} F(\cos\theta, \sin\theta) = \frac{dF}{dx}(-\sin\theta) + \frac{\partial F}{\partial y}\cos\theta = -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta = 1$$

by the chain rule, a contradiction. So we have procured a counterexample.

Definition 1.1 (Star-shaped)**.** A subset $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to $x_0 \in X$ if the line segment $\{tx_0 + (1-t)x \mid t \in [0,1]\}$ is contained in X for all $x \in X$.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^2$ be open and star-shaped. Then for any smooth function $(f_1, f_2): U \to \mathbb{R}^2$ satisfying $\partial_{x_2} f_1 = \partial_{x_1} f_2$, there exists a smooth function $F: U \to \mathbb{R}$ such that $\partial_{x_1} F = f_2,\ \partial_{x_2} F = f_1$.*

Proof. Messy. ⊠

1.2 Sneak peek of cohomology

Say $U \subseteq R^2$ is open, then let $C^\infty(U, \mathbb{R}^k)$ be the vector space of smooth functions $\phi: U \to \mathbb{R}^k$. Define the **gradient** and **curl** functions[1] $\text{grad}: C^\infty(U,\mathbb{R}) \to C^\infty(U, \mathbb{R}^2),\ \text{curl}: C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$ by

$$\text{grad}(\phi) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \text{curl}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}.$$

Note that the curl of the gradient is zero, or $\text{curl} \circ \text{grad} = 0$. So the kernel of the curl contains the image of the gradient, since mapping im(grad) by curl gives zero. Since curl and grad are linear, both ker(curl) and im(grad) are (infinite-dimensional) vector spaces, furthermore, im(grad) is a subspace of ker(curl). So we can consider the quotient space (since vector spaces are abelian groups) $H_1(U) = \ker(\text{curl}) / \text{im}(\text{grad})$. This is a sneak peek of the *cohomology* groups (in this case, vector spaces) assigned to a space. Somehow the cohomology groups tend to be finite-dimensional.

[1] The book uses *rotation* instead of curl, but I think this is the standard notation.

$$\begin{tikzcd} \overset{\ker(\text{grad})}{H^0(U)} \arrow[r] & \overset{\ker(\text{curl})/\operatorname{im}(\text{grad})}{H^1(U)=0} \arrow[r] & H^2(U) \\ C^\infty(U,\mathbb{R}) \arrow[u] \arrow[r, "\text{grad}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^2)} \arrow[u] \arrow[r, "\text{curl}"] & C^\infty(U,\mathbb{R}) \arrow[u] \end{tikzcd}$$

Figure 1: The commutative diagram of gradient and curl for U star-shaped.

Now Theorem 1.1 becomes the statement "$H^1(U) = 0$ whenever $U \subseteq \mathbb{R}^2$ is star-shaped". To see this, note that ker(curl) consists of precisely the functions $\phi\colon U \to \mathbb{R}^2$ such that $\partial_{x_2}\phi_1 = \partial_{x_1}\phi_2$, and if the image of grad are such functions ϕ (since ker(curl) = im(grad)), then there must exist an $F \in C^\infty(U,\mathbb{R})$ mapping onto $\phi = (f_1, f_2)$, where $\partial_{x_1} F = f_2$, $\partial_{x_2} = f_1$.

We know that $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$, since Example 1.1 details a function in ker(curl) that doesn't get mapped onto by im(grad). We will see later that $H^1(\mathbb{R}^2 \setminus \{0\})$ is 1-dimensional as a vector space, and that $H^1(\mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}) \cong \mathbb{R}^k$. So the dimension of the cohomology groups gives us data about how many "holes" a space has. We will introduce cochain complexes and coboundaries later, but for now let us define $H^0(U) = \ker(\text{grad})$ analagously. This is well-defined for open sets $U \subseteq \mathbb{R}^k$ for $k \geq 1$, for

$$\text{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right).$$

Theorem 1.2. *An open set $U \subseteq \mathbb{R}^k$ is connected iff $H^0(U) = \mathbb{R}$.*

Proof. If $f \in \ker(\text{grad})$ (so $\text{grad}(f) = 0$), then f is locally constant, that is, every $x_0 \in U$ has a neighborhood $V(x_0)$ such that $f(x) = f(x_0)$ for $x \in V(x_0)$. This makes sense because having zero derivative geometrically means "zero rate of change", so the function will be constant if we "zoom in close enough". To see this, apply the mean value theorem to the closure of a neighborhood around x_0, say $[a,b] \subseteq U$. Then $f'(c) = \frac{f(b)-f(a)}{b-a}$, and since $f'(c) = 0$, $f(b) - f(a)$. Since the derivative is zero everywhere, this implies the image of the neighborhood (and then x_0) is constant. This generalizes to multiple variables by parametrizing by one variable.

Now suppose U is connected. Then locally constant functions are actually constant, since for $x_0 \in U$, the set

$$\{x \in U \mid f(x) = f(x_0)\} = f^{-1}(f(x_0))$$

is closed since it's the preimage of a closed set by the continuity of f, and open since f is locally constant (every neightborhood has apoint). So since this set is nonempty, by connectedness this is all of U, and $H^0(U) = \mathbb{R}$.

Conversely, if U is not connected, then we have a smooth, surjective function $f : U \to \{0,1\}$ defined by taking all but one of the connected components to 0, and the other to 1. Since f is locally constant, $\text{grad}(f) = 0$, so $\dim H^0(U) > 1$. We can easily extend this to show $\dim H^0(U) > 1$ by replacing $\{0,1\}$ with $\{1, \cdots, n\}$, where n is the number of connected components of U. ⊠

Now let's move on to functions of three variables. Let $U \subseteq \mathbb{R}^3$ be open. Define

$$\begin{aligned} \text{grad}\colon C^\infty(U,\mathbb{R}) \to C^\infty(U,\mathbb{R}^3), &\quad f \mapsto \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right), \\ \text{curl}\colon C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}^3), &\quad (f_1,f_2,f_3) \mapsto \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial f_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right), \\ \text{div}\colon C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}), &\quad (f_1,f_2,f_3) \mapsto \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}. \end{aligned}$$

Note that $\text{curl} \circ \text{grad} = 0$, and $\text{div} \circ \text{curl} = 0$. Most textbooks leave this as an exercise but let's work this out in detail.

$$\text{curl}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2}, \frac{\partial^2 f}{\partial x_3 \partial x_1} - \frac{\partial^2 f}{\partial x_1 \partial x_3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1}\right) = 0,$$

$$\text{div}\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0.$$

The first equality is true because mixed partial derivatives commute, and the second because the first component in the expression for curl has no part containing x_1. So $\frac{\partial}{\partial x_1}\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) = \frac{\partial^2 f_3}{\partial x_1 \partial x_2} - \frac{\partial^2 f_2}{\partial x_1 \partial x_3} = 0$, and so on.

Define $H^0(U)$, $H^1(U)$ as earlier and set $H^2(U) = \ker(\mathrm{div})/\operatorname{im}(\mathrm{curl})$.

Theorem 1.3. *For an open star-shaped set in $\mathbb{R}^3$ we have $H^0(U) = \mathbb{R}$, $H^1(U) = 0$, and $H^2(U) = 0$.*

$$\begin{tikzcd}
\overset{\ker(\mathrm{grad})}{H^0(U)} \arrow[r] & \overset{\ker(\mathrm{curl})/\operatorname{im}(\mathrm{grad})}{H^1(U) = 0} \arrow[r] & \overset{\ker(\mathrm{div})/\operatorname{im}(\mathrm{curl})}{H^2(U) = 0} \arrow[r] & H^3(U) \\
C^\infty(U,\mathbb{R}) \arrow[u] \arrow[r, "\mathrm{grad}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^3)} \arrow[u] \arrow[r, "\mathrm{curl}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^3)} \arrow[u] \arrow[r, "\mathrm{div}"] & C^\infty(U,\mathbb{R}) \arrow[u]
\end{tikzcd}$$

Figure 2: The updated commutative diagram for U star-shaped, now with divergence.

Proof. Since U is star-shaped by assumption (and therefore connected), we immediately have $H^0(U) = \mathbb{R}$ and $H^1(U) = 0$ by our previous theorems. We want to show that $H^2(U) = 0$, or $\ker(\mathrm{div}) = \operatorname{im}(\mathrm{curl})$. Since $\mathrm{div}(\operatorname{im}(\mathrm{curl})) = 0$, we have $\operatorname{im}(\mathrm{curl}) \subseteq \ker(\mathrm{div})$. So the goal has been reduce to showing that $\ker(\mathrm{div}) \subseteq \operatorname{im}(\mathrm{curl})$. To accomplish this, it suffices to exhibit a smooth function $U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ such that the curl of this function is equal to some chosen element of $\ker(\mathrm{div})$.

Assume U is star-shaped with respect to 0, and let $F : U \to \mathbb{R}^3$ such that $\mathrm{div}\, F = 0$. Consider $G : U \to \mathbb{R}^3$ defined by

$$G(\mathbf{x}) = \int_0^1 (F(t\mathbf{x}) \times t\mathbf{x})\, dt.$$

Then if $\mathbf{x} = (x_1, x_2, x_3)$, $F = (f_1, f_2, f_3)$, we have

$$\begin{aligned}
\mathrm{curl}(F(t\mathbf{x}) \times t\mathbf{x}) &= \mathrm{curl}((f_1(tx_1), f_2(tx_2), f_3(tx_3)) \times (tx_1, tx_2, tx_3)) \\
&= \mathrm{curl}(f_2(tx_2)tx_3 - f_3(tx_3)tx_2, f_3(tx_3)tx_1 - f_1(tx_1)tx_3, f_1(tx_1)tx_2 - f_2(tx_2)tx_1) \\
&= \left(\left(tf_1(tx_1) - tx_1 \frac{\partial f_2}{\partial x_2}(tx_2)\right) - \left(tx_1 \frac{\partial f_3}{\partial x_3}(tx_3) - tf_1(tx_1)\right), \cdots\right) \\
&= \left(2tf_1(tx_1) - tx_1\left(\frac{\partial f_2}{\partial x_2}(tx_1) + \frac{\partial f_3}{\partial x_3}(tx_3)\right), \cdots\right) \\
&= 2tF(t\mathbf{x}) + ?? \\
&= \frac{d}{dt}(t^2 F(t\mathbf{x})).
\end{aligned}$$

Therefore

$$\mathrm{curl}\, G(\mathbf{x}) = \int_0^1 \frac{d}{dt}(t^2 F(t\mathbf{x}))\, dt = F(\mathbf{x}). \qquad \boxtimes$$

Example 1.2. If U is not star-shaped then we can have nontrivial first and second cohomology groups. Consider $f : (\mathbb{R}^3 \setminus S^1) \to \mathbb{R}^3$ by

$$f(x_1, x_2, x_3) = \left(\frac{-2x_1x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{-2x_2x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{x_1^2 + x_2^2 - 1}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}\right).$$

By some calculation we have $\mathrm{curl}(f) = 0$. So f defines some element $[f] \in H^1(U)$. To show $[f]$ is nontrivial, we integrate along a curve $\gamma \subseteq U$ "linked" to the missing S^1. Define $\gamma(t) = \left(\sqrt{1 + \cos t}, 0, \sin t\right)$ for $t \in [-\pi, \pi]$. Assume $\mathrm{grad}(F) = f$ for some function F. One one hand, we have

$$\int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{d}{dt} F(\gamma(t))\, dt = F(\gamma(\pi - \varepsilon)) - F(\gamma(-\pi + \varepsilon)) \to 0 \quad \text{for } \varepsilon \to 0,$$

and on the other hand we have by the chain rule

$$\frac{d}{dt}F(\gamma(t)) = f_1(\gamma(t)) \cdot \gamma_1'(t) + \cdots = \sin^2 t + 0 + \cos^2 t = 1.$$

So the integral also converges to 2π, a contradiction.

Example 1.3. Let $U \subseteq \mathbb{R}^k$ be open and $X: U \to \mathbb{R}^k$ be smooth (X is a smooth vector field). The **energy** $A_\gamma(X)$ of X along a smooth curve $\gamma: [a,b] \to U$ is defined by

$$A_\gamma(X) = \int_a^b \langle X \circ \gamma(t), \gamma'(t)\rangle \, dt.$$

If $X = \text{grad}(\Phi)$ and $\Phi_{\gamma(a)} = \Phi_{\gamma(b)}$, then the energy of X is zero, since

$$\langle X \circ \gamma(t), \gamma'(t)\rangle = \frac{d}{dt}\Phi(\gamma(t)).$$

1.3 The alternating algebra

Let V be a real vector space. A map $f : \overbrace{V \times V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R}$ is **k-linear** (or multilinear) if f is linear in each factor.

Definition 1.2. A k-linear map $\omega: V^k \to \mathbb{R}$ is **alternating** if $\omega(\xi_1, \cdots, \xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. Denote the vector space of alternating k-linear maps as $A_k(V)$.

Note that $A_k(V) = 0$ if $k > \dim V$, since two vectors in the domain have to be linearly dependent. Recall that $\text{sgn}: S_k \to \{\pm 1\}$ is a homomorphism, since $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma) \circ \text{sgn}(\tau)$.

Lemma 1.1. *If $\omega \in A_k(V)$ and $\sigma \in S_k$, then*

$$\omega(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}) = \text{sgn}(\sigma)\omega(\xi_1, \cdots, \xi_k).$$

Proof. It is sufficient to show this is true for $\sigma = (i,j)$. Let $\omega_{i,j}(\xi, \xi') = \omega(\xi_1, \cdots, \xi, \cdots, \xi', \cdots, \xi_k)$, where ξ and ξ' occur at positions i, j respectively. The remaining $\xi_\nu \in V$ are arbitrary fixed vectors. Now $\omega_{i,j} \in A_2(V)$ since $\omega \in A_k(V)$, so $\omega_{i,j}(\xi_i + \xi_j, \xi_i + \xi_j) = 0$. By bilinearity, we have $\omega_{i,j}(\xi_i, \xi_j) + \omega_{i,j}(\xi_j, \xi_i) = 0$, and so $\omega_{i,j}(\xi_i, \xi_j) = -\omega_{i,j}(\xi_j, \xi_i) = \text{sgn}(\sigma)\omega_{i,j}(\xi_j, \xi_i)$. ⊠

Example 1.4. If $V = \mathbb{R}^k$ and $\xi_i = (\xi_{i1}, \cdots, \xi_{ik})$, the determinant function $(\xi_1, \cdots, \xi_k) \mapsto \det(\xi_{ij})$ is alternating.

Definition 1.3. A **(p, q)-shuffle** σ is a permutation in S_{p+q} such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. Denote the set of all (p,q)-shuffles by $S_{(p,q)}$. Since a (p,q)-shuffle is uniquely determined by the set $\{\sigma(1), \cdots \sigma(p)\}$, to form $S_{(p,q)}$ we choose subsets of order p from S_{p+q}. So $|S_{(p,q)}| = \binom{p+q}{p}$.

Definition 1.4. For $\omega_1 \in A_p(V)$ and $\omega_2 \in A_q(V)$, define

$$(\omega_1 \wedge \omega_2)(\xi_1, \cdots, \xi_{p+q}) = \sum_{\sigma \in S_{(p,q)}} \text{sgn}(\sigma)\omega_1\left(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}\right)\omega_2\left(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}\right).$$

Note that $\omega_1 \wedge \omega_2$ is $(p+q)$-linear. This product is called the **exterior product** or **wedge product**.

Remark 1.1. Often you also see the exterior product defined as

$$\omega_1 \wedge \omega_2(\xi_1, \cdots, \xi_{p+q}) := \frac{1}{p!q!}\sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma)\omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}).$$

This definition compensates for the $|S_p| = p!$ and $|S_q| = q!$ repetitions by dividing by them in the coefficient.

Lemma 1.2. *If $\omega_1 \in A_p(V)$ and $\omega_2 \in A_q(V)$, then $\omega_1 \wedge \omega_2 \in A_{p+q}(V)$.*

Proof. We show that $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{p+q}) = 0$ when $\xi_1 = \xi_2$. Let $S_{12} = \{\sigma \in S_{(p,q)} \mid \sigma(1) = 1, \sigma(p+1) = 2\}$, $S_{21} = \{\sigma \in S_{(p,q)} \mid \sigma(1) = 2, \sigma(p+1) = 1\}$, and $S_0 = S_{(p,q)}$ todo:algebra proof ⊠

Lemma 1.3. *A k-linear map ω is alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ for all k-tuples with $\xi_i = \xi_{i+1}$ for some $1 \le i \le k-1$.*

Proof. Recall that S_k is generated by the transpositions $(i, i+1)$, and so by Lemma 1.1, we have e

$$\omega(\xi_1, \cdots, \xi_i, \xi_{i+1}, \cdots, \xi_k) = -\omega(\xi_1, \cdots, \xi_{i+1}, \xi_i, \cdots, \xi_k).$$

Then Lemma 1.1 holds for all $\sigma \in S_k$, so ω is alternating.[2] ⊠

Lemma 1.4. *The exterior product is anticommutative. That is, for $\omega_1 \in A_p(V)$ and $\omega_2 \in A_q(V)$, we have $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$.*

Proof. Define $\tau \in S_{p+q}$ to be the permutation

$$\tau = \begin{pmatrix} 1 & \cdots & q & q+1 & \cdots & q+p \\ p+1 & \cdots & p+q & 1 & \cdots & p \end{pmatrix}.$$

For any $\xi_1, \cdots, \xi_{p+q} \in V$,

$$\begin{aligned} \omega_1 \wedge \omega_2(\xi_1, \cdots, \xi_{p+q}) &= \sum_{\sigma \in S_{p+q}} (\operatorname{sgn}\sigma) f(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}) g(\xi_{\sigma(p+1)}, \cdots, \xi_{p\sigma(p+q)}) \\ &= \sum_{\sigma \in S_{p+q}} (\operatorname{sgn}\sigma) f(\xi_{\sigma\tau(q+1)}, \cdots, \xi_{\sigma\tau(q+p)}) g(\xi_{\sigma\tau(1)}, \cdots, \xi_{\sigma\tau(q)}) \\ &= (\operatorname{sgn}\tau) \sum_{\sigma \in S_{p+q}} (\operatorname{sgn}\sigma\tau) g(\xi_{\sigma\tau(1)}, \cdots, \xi_{\sigma\tau(q)}) f(\xi_{\sigma\tau(q+1)}, \cdots, \xi_{\sigma\tau(q+p)}) \\ &= (\operatorname{sgn}\tau) A(g \otimes f)(\xi_1, \cdots, \xi_{p+q}). \end{aligned}$$

todo:adapt this proof to the (p,q)-shuffle definition ⊠

Lemma 1.5. *The exterior product is associative. That is, for $\omega_1 \in A_p(V)$, $\omega_2 \in A_q(V)$, $\omega_3 \in A_r(V)$, we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

Proof. todo:spending less time on the algebra to get to the good stuff ⊠

On top of making sure $A_k(V)$ is closed under multiplication and being associative, the exterior product is also associative and satisfies homogeneity, making it $A_k(V)$ into an algebra. What's an algebra? An $\mathbb{R}$**-algebra** A is a real vector space with an associative bilinear map $\mu\colon A \times A \to A$. The algebra is **unitary** if there exists a unit element (say 1) such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$.

Definition 1.5.

(i) A **graded** $\mathbb{R}$**-algebra** A_* is a sequence of vector spaces A_k, $k = 0, 1, \cdots$ and bilinear maps $\mu\colon A_k \times A_\ell \to A_{k+\ell}$ which are associative.

(ii) The graded algebra A_* is **connected** if there exists a unit element $1 \in A_0$, and the map $\varepsilon\colon \mathbb{R} \to A_0$, $r \mapsto r \cdot 1$ is an isomorphism.

(iii) The graded algebra A_* is **commutative** (resp **anti-commutative**) if $\mu(a, b) = (-1)^{k\ell}\mu(b, a)$ for $a \in A_k$ and $b \in A_\ell$.

Elements in A_k are said to have degree k.

Note that $A_k(V)$ is a real vector space since

$$\begin{aligned} (\omega_1 + \omega_2)(\xi_1, \cdots, \xi_k) &= \omega_1(\xi_1, \cdots, \xi_k) + \omega_2(\xi_1, \cdots, \xi_k), \\ (\lambda\omega) = (\xi_1, \cdots, \xi_k) &= \lambda\omega(\xi_1, \cdots \xi_k), \quad \lambda \in \mathbb{R}. \end{aligned}$$

[2] Isn't this lemma true by definition?

Theorem 1.4. *$A_*(V)$ with the exterior product is an anti-commutative and connected graded algebra.*

Proof. Set $A_0(V) = \mathbb{R}$, since maps that take no vectors and output a scalar are just scalars themselves. Expand the product to $A_0(V) \times A_p(V)$ using the vector space structure. We have seen above that the exterior product is closed, associative, distributive, and anticommutative. ⊠

$A_*(V)$ is the **exterior algebra** or **alternating algebra** associated with V. Elements of $A_1(V)$ are called **1-forms**.

Lemma 1.6. *For 1−forms $\omega_1, \cdots, \omega_p \in A_1(V)$, we have*

$$(\omega_1 \wedge \cdots \wedge \omega_p)(\xi_1, \cdots, \xi_p) = \det \begin{pmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) & \cdots & \omega_1(\xi_p) \\ \omega_2(\xi_1) & \omega_2(\xi_2) & \cdots & \omega_2(\xi_p) \\ \vdots & \vdots & & \vdots \\ \omega_p(\xi_1) & \omega_p(\xi_2) & \cdots & \omega_p(\xi_p) \end{pmatrix}.$$

Proof. We use induction on p. If $p = 2$, then the two elements $(12), (21)$ of S_2 are $(1,1)$-shuffles. So $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_1(\xi_2)\omega_2(\xi_1) = \det\left(\begin{smallmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) \\ \omega_2(\xi_1) & \omega_2(\xi_2) \end{smallmatrix}\right)$. Now

$$\omega_1 \wedge (\omega_2 \wedge \cdots \wedge \omega_p)(\xi_1, \cdots, \xi_p) = \sum_{j=1}^{p} (-1)\omega_1(\xi_j)(\omega_2 \wedge \cdots \wedge \omega_p)\big(\xi_1, \cdots, \hat{\xi_j}, \cdots, \xi_p\big).$$

Expanding the determinant along the first row gives our result. ⊠

This lemma shows that if the 1-forms $\omega_1, \cdots, \omega_p$ are linearly independent, then $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$. This is an equivalence: we can choose elements $\xi_i \in V$ with $\omega_i(\xi_j) = 0$ for $i \neq j$ and $\omega_j(\xi_j) = 1$, which implies that $\det(\omega_i(\xi_j)) = 1$. Conversely, if the ω_i were linearly dependent, we could write $\omega_p = \sum_{i=1}^{p-1} r_i \omega_i$. So the determinant in the previous lemma would have two equal rows and be zero. To summarize:

Lemma 1.7. *For 1-forms $\omega_1, \cdots, \omega_p$ on V, we have $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$ iff they are linearly independent.*

Theorem 1.5. *For $\{e_i\}$ a basis of V and $\{\phi_i\}$ the dual basis of $A_1(V)$ (as i varies over n), we have*

$$\{\phi_{\sigma_1} \wedge \phi_{\sigma(2)} \wedge \cdots \wedge \phi_{\sigma(p)}\}_{\sigma \in S_{(p,n-p)}}$$

a basis of $A_p(V)$. In particular, $\dim A_p(V) = \binom{\dim V}{p}$.

Proof. todo:less time on algebra ⊠

This tells us that $A_n(V) \cong \mathbb{R}$ if $n = \dim V$ (since they're both one dimensional real vector spaces, $\binom{n}{n} = 1$) and $A_p(V) = 0$ for $p > n$ (since two factors will be the same). A linear map $f : V \to W$ induces the linear map

$$A_p(f) \colon A_p(W) \to A_p(V)$$

by setting $A_p(f)(\omega(\xi_1, \cdots, \xi_p)) = \omega(f(\xi_1), \cdots, f(\xi_p))$. We have $A_p(g \circ f) = A_p(f) \circ A_p(g)$, and $A_p(\mathrm{id}) = \mathrm{id}$. This is equivalent to saying that $A_p(-)$ is a **contravariant functor**. For $\dim V = n$, $f : V \to V$ linear, the induced map $A_n(f) \colon A_n(V) \to A_n(V)$ is a linear endormorphism of a 1-dimensional vector space, and is therefore just scalar multiplication. It follows from the theorem below that this scalar is $\det f$.

Theorem 1.6. *The characteristic polynomial of a linear endomorphism $f : V \to V$ is given by*

$$\det(f - t) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(A_{n-i}(f))\, t^i.$$

Proof. todo:algebra ⊠

Lecture 2

De Rham Cohomology

We finally get to the goods.

2.1 The exterior derivative

Let U denote an open set in $\mathbb{R}^n$, $\{e_1, \cdots, e_n\}$ the standard basis and $\{\phi_1, \cdots, \phi_n\}$ the dual basis of $A_1(\mathbb{R}^n)$ (or the basis for the dual space to $\mathbb{R}^n$).

Definition 2.1. A **differential p-form** on U is a smooth map $\omega\colon U \to A_p(\mathbb{R}^n)$. The vector space of all such maps is denoted by $\Omega^p(U)$.

If $p = 0$, then $A_0(\mathbb{R}^n) = \mathbb{R}$, and $\Omega^0(U)$ is just the set of smooth functions on $U, C^\infty(U, \mathbb{R})$. The derivative of a smooth map $\omega\colon U \to A_p(\mathbb{R}^n)$ is denoted $D\omega$, and is the linear map

$$D_x\omega\colon \mathbb{R}^n \to A_p(\mathbb{R}^n), \quad e_i \mapsto \frac{d}{dt}\omega(x + te_i)_{t=0} = \frac{\partial \omega}{\partial x_i}(x).$$

Let $I = (i_1, \cdots, i_p)$, and write ϕ_I[3] for $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$. Then we have the basis ϕ_I for $A_p(\mathbb{R}^n)$ as I runs over all sequences of length $p \leq n$. So every $\omega \in \Omega^p(U)$ can be written in the form $\omega(w) = \sum \omega_I(x)\phi_I$, where the ω_I are smooth real-valued functions of $x \in U$. The differential $D_x\omega$ is the linear map

$$D_x\omega(e_j) = \sum_I \frac{\partial \omega_I}{\partial x_j}(x)\phi_I, \quad j = 1, \cdots, n.$$

The function $x \mapsto D_x\omega$ is a smooth map from U to $\mathrm{Hom}(\mathbb{R}^n, A_p(\mathbb{R}^n))$. todo:why? how exactly? difference between thsi and derivative?

Definition 2.2. The **exterior differential** $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is the linear operator

$$d_x\omega(\xi_1, \cdots, \xi_{p+1}) = \sum_{\ell=1}^{p+1} (-1)^{\ell-1} D_x\omega(\xi_\ell)(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1})$$

where $(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \xi_{\ell-1}, \xi_{\ell+1}, \cdots, \xi_{p+1})$. todo:what?

The result lies in $\Omega^{p+1}(U)$ by Lemma 1.3. If $\xi_i = \xi_{i+1}$, then

$$\begin{aligned}
&\sum_{\ell=1}^{p+1} (-1)^{\ell-1} D_x\omega(\xi_\ell)(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1}) \\
=&(-1)^{i-1} D_x\omega(\xi_i)(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) \\
&+ (-1)^{i} D_x\omega(\xi_{i+1})(\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1}) \\
=&0.
\end{aligned}$$

In the second step, the rest of the terms cancel out by properties of the exterior product, since they all contain both ξ_i and ξ_{i+1}. The final term also cancels out since $(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})$.

Example 2.1. Let $x_i : U \to \mathbb{R}$ be ith projection. Then $dx_1 \in \Omega^1(U)$ is the constant map $dx_i : x \to \phi_i$, which follows from the definition of the differential. In general, for $f \in \Omega^0(U)$, we have

$$d_x f(\zeta) = \frac{\partial f}{\partial x_1}(x)\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}(x)\zeta^n.$$

[3] It slightly annoys me that indices aren't in the right place, but I don't want to make any mistakes deviating too far from the book, so they stay at the bottom for covectors.

Lemma 2.1. *If $\omega(x) = f(x)\phi_I$, then $d_x\omega = d_x f \wedge \phi_I$.*

Proof. Note that

$$D_x\omega(\zeta) = (D_x f)(\zeta)\phi_I = \left(\frac{\partial f}{\partial x_1}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n\right)\phi_I = d_x f(\zeta)\phi_I.$$

So by the definition of the exterior derivative, we have

$$\begin{aligned} d_x\omega(\xi_1, \cdots, \xi_{p+1}) &= \sum_{k=1}^{p+1}(-1)^{k-1} d_x f(\xi_k)\phi_I(\xi_1, \cdots, \hat{\xi}_k, \cdots, \xi_{p+1}) \\ &= [d_x f \wedge \phi_I](\xi_1, \cdots, \xi_{p+1}). \end{aligned}$$

⊠

todo:this entire proof? For $\phi_I \in A_p(\mathbb{R}^n)$, we have $\phi_k \wedge \phi_I = 0$ if $k \in I$, and $(-1)^r \phi_J$ if $k \notin I$, where r is determined by $i_r < k < i_{r+1}$ and $J = (i_1, \cdots, i_r, k, \cdots, i_p)$.

Lemma 2.2. *For $p \geq 0$, the composition $\Omega^p(U) \to \Omega^{p+1}(U) \to \Omega^{p+2}(U)$ is identically zero.*

Proof. Let $\omega = f\phi_I$. Then $d\omega = df \wedge \phi_I = \frac{\partial f}{\partial x_1}\phi_1 \wedge \phi_I + \cdots + \frac{\partial f}{\partial x_n}\phi_n \wedge \phi_I$. todo:alternating terms?? does I denote one sequence or several? Since $\phi_i \wedge \phi_i = 0$ and $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$, we have

$$\begin{aligned} d^2\omega &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\phi_i \wedge (\phi_j \wedge \phi_I) \\ &= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_j}\right)\phi_i \wedge \phi_j \wedge \phi_I = 0. \end{aligned}$$

⊠

The exterior product on $A_*(\mathbb{R}^n)$ induces an exterior product on $\Omega^*(U)$ by defining $(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)$. The exterior product of a p-form and q-form is a $(p+q)$-form, so it induces a bilinear map $\wedge : \Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U)$. Then for $f \in C^\infty(U, \mathbb{R})$, we have $(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge f\omega_2$. Note that $f \wedge \omega = f\omega$ when $f \in \Omega^0(U)$ and $\omega \in \Omega^p(U)$.

Lemma 2.3. *For $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^q(U)$,*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2.$$

Proof. It suffices to show this holds for $\omega_1 = f\phi_I$ and $\omega_2 = g\phi_J$. Since $\omega_1 \wedge \omega_2 = fg\phi_I \wedge \phi_J$, we have

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d(fg) \wedge \phi_I \wedge \phi_J = ((df)g + f\,dg) \wedge \phi_I \wedge \phi_J \\ &= df\,g \wedge \phi_I \wedge \phi_J + f\,dg \wedge \phi_I \wedge \phi_J \\ &= df \wedge \phi_I \wedge g\phi_J + (-1)^p f\phi_I \wedge dg \wedge \phi_J \\ &= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2. \end{aligned}$$

⊠

2.2 Finally, de Rham cohomology

In short, we have a new anti-commutative algebra $\Omega^*(U)$ with a *differential* (or boundary) operator

$$d : \Omega^*(U) \to \Omega^{*+1}(U), \quad d \circ d = 0,$$

and d is a *derivation* (since $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + d\omega_2 \wedge \omega_1$ by Lemma 2.3 and anticommutativity). Then $(\Omega^*(U), d)$ is an *commutative differential graded algebra*[4], called the **de Rham complex** of U.

Theorem 2.1. *There is precisely one linear operator $d : \Omega^p \to \Omega^{p+1}(U)$, $p = 0, 1, \cdots$, such that*

(i) $f \in \Omega^0(U)$, $df = \frac{\partial f}{\partial x_1}\phi_1 + \cdots + \frac{\partial f}{\partial x_n}\phi_n$,
(ii) $d \circ d = 0$,
(iii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$ *if* $\omega_1 \in \Omega^p(U)$.

Proof. We know that the exterior differential d satisfies these properties. To show uniqueness, say d' satisfies these properties: we will show it has to be the exterior derivative. (i) tells us that $d = d'$ on $\Omega^0(U)$ (since it characterizes smooth functions on U), in particular $d'x_i = dx_i = \phi_i$. Since $d' \circ d' = 0$, then $d'\phi_i = d'(d'(x_i)) = 0$. ⊠

[4] help