Geometric Group Theory

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1 Groups

1.1 An introduction

Geometric group theory is about the relation between algebraic and geometric properties of a group. Specifically:

- Can groups be viewed as geometric objects?
- If so, are the geometric and algebraic properties related?
- Which geometric objects can a group act on and how does the geometry relate to the algebra?

We do this by associating group-valued invariants with geometrical objects like isometry groups or the fundamental group. One of the central insights is that this process can be somewhat reversed.

- Associate a geometric object with the group inquestion.
- Take geometric invariants and apply these to the aforementioned geometric objects.

1.2 Group theory review

Definition 1.1 (Group). A **group** is a set G together with a binary operation $G \times G \to G$ satisfying the following axioms:

(a) *Associativity:* For all $g_1, g_2, g_3 \in G$, we have

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3.$$

(b) *Identity*: There exists an *identity* $e \in G$ for $\cdot$ such that for every $g \in G$,

$$e \cdot g = g = g \cdot e.$$

It follows that identities are necessarily unique.

(c) *Inverses:* For every $g \in G$ there exists an *inverse element* $g^{-1} \in G$ wrt $\cdot$ such that

$$g \cdot g^{-1} = e = g^{-1} \cdot g.$$

A group is **abelian** if $g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g_1, g_2 \in G$.

Definition 1.2 (Subgroup). Let G be a group with respect to $\cdot$. A subset $H \subseteq G$ is a **subgroup** if H is a group wrt the restriction of $\cdot$ to $H \times H \subseteq G \times G$. The **index** $[G : H]$ of a subgroup is the cardinality of $\{g \cdot H \mid g \in G\}$; here $g \cdot H = \{g \cdot h \mid h \in H\}$.

Definition 1.3 (Group homomorphism/isomorphism). Let G, H be groups.

- A map $\varphi \colon G \to H$ is a **group homomorphism** if $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$ for all $g_1, g_2 \in G$. It follows that homomorphisms maps identities to identities.
- A homomorphism is an **isomorphism** if there exists some homomorphism $\psi \colon H \to G$ such that $\varphi \cdot \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$.

1.3 Automorphism groups

Example 1.1. Let X be a set. Then the set S_X of all bijections $X \to X$ is a group wrt composition, the **symmetric group** over X. If $n \in \mathbb{N}$, then we abbreviate $S_n := S_{\{1,\cdots,n\}}$. In general, S_X is nonabelian.

Proposition 1.1 (Cayley's theorem)**.** *Every group is isomorphic to a subgroup of some symmetric group.*

Proof. Let G be a group. For $g \in G$ we define the map $f_g : G \to G$, $x \mapsto g \cdot x$. For all $g, h \in G$ we have $f_g \circ f_h = f_{g \cdot h}$. Therefore, looking at $f_{g^{-1}}$ shows that $f_g : G \to G$ is a bijection for all $g \in G$. It follows that

$$f : G \to S_G, \quad g \mapsto f_g$$

is a group homomorphism, which is injective (trivial kernel or cancellation). So f induces an isomorphism $G \cong \operatorname{im} f \subset S_G$. ⊠

Example 1.2 (Automorphism groups)**.** Let G be a group. Then the set $\operatorname{Aut}(G)$ of group isomorphisms $G \to G$ is a group wrt composition, the **automorphism group** of G. Clearly $\operatorname{Aut}(G)$ is a subgroup of S_G.

Example 1.3 (Isometry groups/Symmetry groups)**.** Let X be a metric space. The set $\operatorname{Isom}(X)$ of all isometries of type $X \to X$ forms a group wrt composition (a subgroup of S_X). For example, the dihedral groups naturally occur as symmetry groups of regular polygons.

Example 1.4 (Matrix groups)**.** Let k be a commutative ring with unit, and let V be a k-module. Then the set $\operatorname{Aut}(V)$ of all k-linear isomorphisms forms a group with respect to composition. In particular, the set $\mathrm{GL}(n,k)$ of invertible $(n \times n)$-matrices over k is a group for every $n \in \mathbb{N}$. Similarly, $\mathrm{SL}(n,k)$ is also a group.

Example 1.5 (Galois groups)**.** Let $K \subset L$ be a Galois extension of fields. Then the set

$$\mathrm{Gal}(L/K) := \{\sigma \in \operatorname{Aut}(L) \,\big|\, \sigma|_K = \mathrm{id}_K\}$$

of field automorphisms of L fixing K is a group wrt composition of the so-called **Galois group** of the extension L/K.

Example 1.6 (Deck transformation groups)**.** Let $\pi : X \to Y$ be a covering map of topological spaces. Then the set

$$\{f \in \operatorname{map}(X,X) \mid f \text{ is a homeomorphism with } \pi \circ f = \pi\}$$

of **deck transformations** forms a group wrt composition.

These are all examples of the fact that if X is an object in a category C, then the set $\operatorname{Aut}_C(X)$ of C-isomorphisms of type $X \to X$ is a group with respect to composition in C. It should also be easy to see that if G is a group, there exists a category C and an object $X \in \operatorname{Ob}(C)$ such that $\operatorname{Aut}_C(X) \cong G$ by considering the category with one object.

Example 1.7. Using categorical language, for X a set we have $S_X \cong \operatorname{Aut}_{\mathsf{Set}}(X)$. In the algebraic categories $\mathsf{Grp}, \mathsf{Ab}, \mathsf{Vect}_{\mathbb{R}}, {}_R\mathsf{Mod}$, objects are isomorphic categorically iff they are isomorphic algebraically, and the definitions of automorphism groups coincide. In $\mathsf{Met}_{\mathsf{Isom}}$, objects are isomorphic iff they're isometric, and automorphism groups are isometry groups. In Top isomorphisms are homeomorphisms, and automorphism groups are self-homeomorphisms.

1.4 Normal subgroups and quotients

Definition 1.4 (Normal subgroup)**.** Let G be a group. A subgroup N of G is **normal** if it is conjugation invariant, i.e., if $g \cdot n \cdot g^{-1} \in N$ holds for all $n \in N$and all $g \in G$. If N is a normal subgroup of G, then we write $N \trianglelefteq G$.

Example 1.8.
- All subgroups of abelian groups are abelian and normal.
- Let $\tau = (1\,2) \in S_3$, then $\{\mathrm{id}, \tau\}$ is a subgroup of S_3 but not normal. OTOH, $\{\mathrm{id}, \sigma, \sigma^2\}$ generated by $\sigma = (1\,2\,3)$ is normal.
- Kernels of group homomorphisms are normal in the domain group, conversely, every normal subgroup is the kernel of a certain group homomorphism (namely the canonical projection to the quotient).

Normalness implies well definedness because

$$(aN)(bN) = a(Nb)N \underset{\text{normal}}{=} a(bN)N = (ab)NN = (ab)N.$$

Alternatively, say $a, \overline{a}, b, \overline{b} \in G$ with $aN = \overline{a}N$, $bN = \overline{b}N$, $n, m \in N$ with $\overline{a} = an$, $\overline{b} = bm$. Then

$$(\overline{a}\overline{b})N = (an \cdot bm)N = (a(bnb^{-1})bm)N \underset{\text{normal}}{=} (abnm)N = (ab)N.$$

Example 1.9. Let G be a group. An automorphism $\varphi\colon G \to G$ is an **inner automorphism** of G if φ is given by conjugation by an element of G, i.e., if there is an element $g \in G$ such that for all $h \in G$, $\varphi(h) = ghg^{-1}$. Then $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. To see this, let $\varphi_g \in \mathrm{Inn}(G)$ denote conjugation by g. Then for $\psi \in \mathrm{Aut}(G)$, $\psi\varphi_g\psi^{-1}$ is conjugation by $\psi(g)$, since

$$\psi\varphi_g\psi^{-1}(g') = \psi\varphi_g(\psi^{-1}(g')) = \psi\left(g\psi^{-1}(g')g^{-1}\right) = \psi(g)g'\psi(g^{-1}).$$

Therefore $\psi\varphi_g\psi^{-1} = \varphi_{\psi(g)} \in \mathrm{Inn}(G)$, and we are done. Define the **outer automorphism group** as $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

1.5 Generating sets

Definition 1.5. Let G be a group and $S \subseteq G$. The subgroup **generated** by S in G is the smallest subgroup $\langle S\rangle_G$ of G containing S. A group is **finitely generated** if it contains a finite subset that generates it. Explicitly, we have

$$\langle S\rangle_G = \bigcap\{H \mid H \subset G \text{ is a subgroup with } S \subset H\} = \{s_1^{\varepsilon 1} \cdots s_n^{\varepsilon_n} \mid n \in \mathbb{N}, s_1, \cdots, s_n \in S, \varepsilon_1, \cdots, \varepsilon_n \in \{\pm 1\}\}$$

Example 1.10. If G is a group, then G generates G. The trivial group is generated by $\emptyset$. $\{1\}$ generates $\mathbb{Z}$ as well as $\{2, 3\}$, but $\{2\}, \{3\}$ don't. S_X is finitely generated iff X is finite.

1.6 Free groups

Every vector space admits LI generating sets that are as free as possible, i.e. no relations between them. However most group do not.

Definition 1.6 (Free groups, universal property)**.** Let S be a set. A group F is **freely generated** by S if F has the following universal property: For any group G and any map $\varphi\colon S \to G$ there is a unique group homomorphism $\overline{\varphi}\colon F \to G$ extending φ:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook] & G \\ F \arrow[ur, dotted, "\overline{\varphi}"'] & \end{tikzcd}$$

A group is **free** if it contains a free generating set.

Example 1.11. $\mathbb{Z}$ is freely generated by $\{1\}$, but is not freely generated by $\{2,3\}$ or $\{2\}$ or $\{3\}$.

Proposition 1.2. *Let S be a set. Then, up to canonical isomorphism, there is at most one group freely generated by S.*

Proof. Let F, F' be two groups freely generated by S, $\varphi\colon S \hookrightarrow F, \varphi'\colon S \hookrightarrow F'$. Because F is freely generated by S, the existence part of the universal property guaranteed the existence of a group homomorphism $\overline{\varphi}'\colon F \to F'$ such that $\overline{\varphi}' \circ \varphi = \varphi'$. Analogously, there is a group homomorphism $\overline{\varphi}\colon F' \to F$ satisfying $\overline{\varphi} \circ \varphi' = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[d, hook, "\varphi"'] & F' \\ F \arrow[ur, dotted, "\overline{\varphi}'"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi'"'] & F \\ F' \arrow[ur, dotted, "\overline{\varphi}"] & \end{tikzcd}$$

Composing $\overline{\varphi} \circ \overline{\varphi}'\colon F \to F$, the universal property tells us that this must be id_F since it also fits into the diagram. These isomorphism are canonical since they induce the identity on S, and are the only isomorphisms extending the identity on S. ⊠

Theorem 1.1. *Let S be a set. Then there exists a group freely generated by S.*

Proof. todo: ⊠

Corollary 1.1. *Let F be a free group, and let S be a free generating set of F. Then S generates F.*

Proposition 1.3. *Let F be a free group.*

(1) Let $S \subseteq F$ be a free generating set of F and let S' be a generating set of F. Then $|S'| \geq |S|$.

(2) In particular, all generating sets of F have the same cardinality, called the rank *of F.*

Definition 1.7. Let $n \in \mathbb{N}$ and let $S = \{x_1, \cdots, x_n\}$. We write F_n for "the" group freely generated by S, and call F_n the **free group of rank** n.

Note that while subspaces of vector spaces cannot have bigger dimension than the ambient space, free groups of rank 2 contain subgroups isomorphic to free groups of higher rank, even free subgroups of (countably) infinite rank.

Corollary 1.2. *A group is finitely generated iff it is the quotient of a finitely generated free group, i.e., a group G is finitely generated iff there exists a finitely generated free group F and a surjection $F \to G$.*

Proof. Quotients of finitely generated groups are finitely generated. Conversely, let G be finitely generated by $S \subseteq G$, and let F be the free group generated by S. Using the universal property of F we have a group homomorphism $\pi\colon F \to G$ that is the identity on S. Because S generates G and S lies in the image of π, it follows that $\operatorname{im}\pi = G$. ⊠

This essentially tells us that all groups can be written in presentation form.

1.7 Generators and relations

Definition 1.8. Let G be a group and let $S \subseteq G$ be a subet. The **normal subgroup of** G **generated by** S is the smallest (wrt inclusion) normal subgroup of G containing S, denoted by $\langle S \rangle_G^{\trianglelefteq}$.

Remark 1.1. Explicitly, let G be a group and $S \subseteq G$. Then $\langle S \rangle_G^{\trianglelefteq}$ always exists and can be described as follows:

$$\langle S \rangle_G^{\trianglelefteq} = \bigcap \{H \mid H \subseteq G \text{ is a normal subgroup with } S \subseteq H\} = \{g_1 s_1^{\varepsilon 1} g_1^{-1} \cdots g_n s_n^{\varepsilon_n} g_n^{-1} \mid n \in \mathbb{N}, s_i \in S, \varepsilon \in \{\pm 1\}, g_i \in G\}.$$

Example 1.12. All subgroups of abelian groups are normal and $\langle S\rangle_G^{\trianglelefteq} = \langle S\rangle_G$. Considering S_3 and $\tau = (1\ 2) \in S_3$, we have $\langle \tau\rangle_{S_3} = \{\mathrm{id}_{1,2,3}, \tau\}$ and $\langle \tau\rangle_{S_3}^{\trianglelefteq} = S_3$.

If G is a group and $N \trianglelefteq G$, in general it is difficult to determine the minimal number of elements of $S \subseteq G$ that satisfies $\langle S\rangle_G^{\trianglelefteq} = N$. Let A^* denote the set of words.

Definition 1.9. Let S be a set, and let $R \subseteq (S \cup S^{-1})^*$ be a subset; let $F(S)$ be the free group generated by S. Then the group

$$\langle S \mid R\rangle := F(S)/\langle R\rangle_{F(S)}^{\trianglelefteq}$$

is said to be **generated by** S **with the relations** R; if G is a group with $G \cong \langle S \mid R\rangle$, then $\langle S \mid R\rangle$ is a **presentation** of G.

Example 1.13. For all $n \in \mathbb{N}$, we have $\langle x \mid x^n\rangle \cong \mathbb{Z}/n$ (universal property or explicit construction). We have $\langle x, y \mid |xyx^{-1}y^{-1}\rangle \cong \mathbb{Z}^2$ (universal property).

Example 1.14. Let $n \in \mathbb{N}_{\geq 3}$ and let $X_n \subseteq \mathbb{R}^2$ be a regular n-gon (inheriting the induced Euclidian metric). Then

$$\mathrm{Isom}(X_n) \cong \langle s, t \mid s^n, t^2, tst^{-1} = s^{-1}\rangle =: D_n.$$

Geometrically s correspondings to a $2\pi/n$ rotation about the center and t reflects across the diameter passing through one of the vertices.

todo:thompson's group

Example 1.15. For $m, n \in \mathbb{N}_{>0}$, the **Baumslag-Solitar group** $\mathrm{BS}(m,n)$ is defined via the presentation

$$\mathrm{BS}(m,n) := \langle a, b \mid ba^m b^{-1} = a^n\rangle.$$

For instance, $\mathrm{BS}(1,1) \cong \mathbb{Z}^2$. The family of Baumslag-Solitar groups contain many intriguing examples of groups. For example, $\mathrm{BS}(2,3)$ is a group with only two generators and one relation that is *non-Hopfian*, i.e., there exists a surjective group homomorphism $\mathrm{BS}(2,3) \to \mathrm{BS}(2,3)$ that is *not* an isomorphism, namely the homomorphism given by

$$\mathrm{BS}(2,3) \mapsto \mathrm{BS}(2,3), \quad a \mapsto a^2, \quad b \mapsto b.$$

Proving that this homomorphism is not injective requires more advanced techniques.

Example 1.16. The group

$$G := \langle x, y \mid xyx^{-1} = y^2, yxy^{-1} = x^2\rangle$$

is trivial. Write $x = xyx^{-1}xy^{-1}$, using the relation $xyx^{-1} = y^2$ we have $x = y^2xy^{-1} = yyxy^{-1}$. Using the relation $yxy^{-1} = x^2$ we have $x = yx^2$, so $x = y^{-1}$. So $y^{-2} = x^2 = yxy^{-1} = yy^{-1}y^{-1} = y^{-1}$, and $x = y = e$.

Definition 1.10. A group G is **finitely presented** if there exists a finite generating set S and a finite set $R \subseteq (S \cup S^{-1})^*$ of relations that $G \cong \langle S \mid R\rangle$.

Clearly any finitely presented group is finitely generated. The converse is not always true.

Example 1.17. The group

$$\langle s, t \mid \{[t^n st^{-n}, t^m st^{-m}] \mid n, m \in \mathbb{Z}\}\rangle$$

is finitely generated but not finitely presented.

1.8 New groups of old

Definition 1.11. Let I be a set and $(G_i)_{i \in I}$ be a family of groups. The **direct product group** $\prod_{i \in I} G_i$ of $(G_i)_{i \in I}$ is the group whose underlying set is the cartesian product $\prod_{i \in I}$ and whose composition is componentwise:

$$\prod_{i \in I} G_i \times \prod_{i \in I} G_i \to \prod_{i \in I} G_i, \quad \big((g_i)_{i \in I}, (h_i)_{i \in I}\big) \mapsto (g_i \cdot h_i)_{i \in I}.$$

Definition 1.12. Let Q and N be groups. An **extension** of Q by N is an exact sequence

$$I \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$$

of groups. Note that this implies $Q \cong G/\operatorname{im}(N)$.

Definition 1.13. Let N and Q be groups, and $\varphi \colon Q \to \operatorname{Aut}(N)$ be a group homomorphism (i.e., Q acts on N via φ). The **semi-direct product** of Q by N with respect to φ is the group $N \rtimes_\varphi Q$ whose underlying set is the cartesian product $N \times Q$ and whose composition is

$$(N \rtimes_\varphi Q) \times (N \rtimes_\varphi Q) \to (N \rtimes_\varphi Q), \quad ((n_1, q_1), (n_2, q_2)) \mapsto (n_1 \cdot \varphi(q_1)(n_2), q_1 \cdot q_2)$$

Remark 1.2. A group extension $1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$ **splits** if there exists a group homomorphism $s \colon Q \to G$ such that $\pi \circ s = \mathrm{id}_Q$. If $\varphi \colon Q \to \operatorname{Aut}(N)$ is a homomorphism, then

$$1 \to N \xrightarrow{i} N \rtimes_\varphi Q \xrightarrow{\pi} Q \to 1$$

is a split extension with the split given by $Q \to N \rtimes_\varphi Q$, $q \mapsto (e, q)$, since π is just projection onto Q. Conversely, for a split extension, the extension is a semi-direct product of the quotient by the kernel. Let $1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$ be a group extension splitting by $s \colon Q \to G$. Then

$$N \rtimes_\varphi Q \rightleftarrows G, \quad (n, q) \mapsto n \cdot s(q), \quad g \mapsto (g \cdot (s \circ \pi(g))^{-1}, \pi(g))$$

are well-defined mutually inverse group homomorphisms, where $\varphi \colon Q \to \operatorname{Aut}(N), q \mapsto \big(n \mapsto s(q) \cdot n \cdot s(q)^{-1}\big)$. However, there are group extensions that do *not* split, for example the extension

$$1 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 1$$

does not split because there is no non-trivial homomorphism from the torsion group $\mathbb{Z}/2$ to $\mathbb{Z}$.

Example 1.18. Some examples of semi-direct products:

- If N, Q are groups and $\varphi \colon Q \to \operatorname{Aut}(N)$ is the trivial homomorphism, then the identity map yields and isomorphism $N \rtimes_\varphi Q \cong NxQ$.
- Let $n \in \mathbb{N}_{\geq 3}$. Then the dihedral group $D_n = \langle s, t \mid s^n, t^2, tst^{-1} = s^{-1} \rangle$ is a semi-direct product

  $$D_n \leftrightarrows \mathbb{Z}/n \rtimes_\varphi \mathbb{Z}/2, \quad s \mapsto ([1], 0), \quad t \mapsto (0, [1])$$

  where $\varphi \colon \mathbb{Z}/2 \to \operatorname{Aut}(\mathbb{Z}/n)$ is given by multiplication by -1. Similarly, the infinite dihedral group $D_\infty = \langle s, t \mid t^2, tst^{-1} = s^{-1} \rangle \cong \operatorname{Isom}(\mathbb{Z})$ can be written as a semi-direct product of $\mathbb{Z}/2$ by $\mathbb{Z}$ with respect to multiplication by -1.
- Semi-direct products of $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$ lead to interesting examples of groups provided the automorphism $\varphi(1) \in \mathrm{GL}(n, \mathbb{Z}) \subseteq \mathrm{GL}(n, \mathbb{R})$ is chosen suitably, e.g., if $\varphi(1)$ has interesting eigenvalues.

- Let G be a group. Then the **lamplighter group over** G is the semi-direct product group $\left(\prod_{\mathbb{Z}} G\right) \rtimes_{\varphi} \mathbb{Z}$, where $\mathbb{Z}$ acts on the product $\prod_{\mathbb{Z}} G$ by shifting the factors:

$$\varphi\colon \mathbb{Z} \to \operatorname{Aut}\left(\prod_{\mathbb{Z}} G\right), \quad z \mapsto \left((g_n)_{n\in\mathbb{Z}} \mapsto (g_{n+z})_{n\in\mathbb{Z}}\right).$$

- More generally, the **wreath product** of two groups G and H is the semi-direct product $\left(\prod_{H} G\right) \rtimes_{\varphi} H$, where φ is the shift action of H on $\prod_H G$. The wreath product of G and H is denoted by $G \wr H$.

1.9 Free products and free amalgamated products

We now described a construction that "glues" two groups along a common subgroup, analogous to the universal property of pushouts in category theory.

Definition 1.14. Let A be a group, and let $\alpha_1\colon A \to G_1$ and $\alpha_2\colon A \to G_2$ be group homomorphisms. A group G together with homomorphisms $\beta_1\colon G_1 \to G$ and $\beta_2\colon G_2 \to G$ satisfying $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an **amalgamated free product** of G_1 and G_2 over A (wrt α_1 and α_2) if the following universal property is satisfied: for any group H and any two group homomorphisms $\varphi_1\colon G_1 \to H$ and $\varphi_2\colon G_2 \to H$ with $\varphi_1 \circ \alpha_1 = \varphi_2 \circ \alpha_2$ there is exactly one homomorphism $\varphi\colon G \to H$ of groups with $\varphi \circ \beta_1 = \varphi \circ \beta_2$.

$$\begin{tikzcd}
& G_1 \arrow[dr, "\beta_1"] \arrow[drrr, bend left, "\varphi_1"] & & & \\
A \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & G \arrow[rr, dotted, "\varphi"] & & H \\
& G_2 \arrow[ur, "\beta_2"'] \arrow[urrr, bend right, "\varphi_2"'] & & &
\end{tikzcd}$$

Such a free product with amalgamation is denoted by $G_1 *_A G_2$. If A is the trivial group, we write $G_1 * G_2 := G_1 *_A G_2$ and call $G_1 * G_2$ the **free product** of G_1 and G_2.

Example 1.19. Free groups can be viewed as free products of several copies of $\mathbb{Z}$; e.g., the free group of rank 2 is $\mathbb{Z} * \mathbb{Z}$.

$D_\infty \cong \operatorname{Isom}(\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/2 * \mathbb{Z}/2$, for instance, reflection at 0 and reflection at $1/2$ provide generators of D_∞ corresponding to the obvious generators of $\mathbb{Z}/2 * \mathbb{Z}/2$.

The matrix group $\mathrm{SL}(2,\mathbb{Z})$ is isomorphic to the free amalgamated product $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$.

Free amalgamated products occur naturally in topology: by van Kampen, π_1 of a pointed space glued together of several components is a free amalgamated product of π_1 of the components over π_1 of the intersection.

Theorem 1.2. *All free products with amalgmation exist and are unique up to canonical isomorphism.*

Definition 1.15 (HNN-extension)**.** Let G be a group, let $A, B \subseteq G$ be two subgroups, and let $\vartheta\colon A \to B$ be an isomorphism. Then the **HNN-extension of** G **with respect to** ϑ is the group

$$G *_{\vartheta} := \left\langle \left\{x_g \mid g \in G\right\} \amalg \{t\} \mid \{t^{-1} x_a t = x_{\vartheta(a)} \mid a \in A\} \cup R_G \right\rangle$$

where

$$R_G := \{x_g x_h x_k^{-1} \mid g,h,k \in G \text{ with } g \cdot h = k \text{ in } G\}.$$

Using an HNN-extension, we can force two given subgroups to be conjugate. Topologically they arise as fundamental groups of mapping tori of maps that are injective on the level of fundamental groups.

2 Groups to geometry I: Cayley graphs

2.1 Review of graph notation

Unless otherwise stated, graphs are undirected and simple.

Definition 2.1. A **graph** is a pair $G = (V, E)$ of disjoint sets where E is a set of subses of V that contain exactly two elements, i.e.,

$$E \subseteq V^{[2]} := \{e \mid e \subseteq V, |e| = 2\}.$$

The elements of V are the **vertices**, the elements of E are the **edges** of G.

Graphs are a different POV on relations, and graphs are used to model relations. Classical graph theory has many applications, notably in anything involving networks (computer science).

Definition 2.2. Let (V, E) be a graph.

- We say two vertices $v, v' \in V$ are **neighbors** or **adjacent** if they are joined by an edge, i.e., if $\{v, v'\} \in E$.
- The number of neighbours of a vertex is the **degree** of this vertex.
- A **complete** graph has all vertices neighbors of each other.

Definition 2.3. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. The graphs G and G' are **isomorphic** if there is a **graph isomorphism** between G and G', i.e., a bijection $f : V \to V'$ such that for all $v, w \in V$ we have $\{v, w\} \in E$ if and only if $\{f(v), f(w)\} \in E'$; i.e., isomorphic graphs only differ in the labels of the vertices.

Definition 2.4. Let $G = (V, E)$ be a graph.

- Let $n \in \mathbb{N} \cup \{\infty\}$. A **path** in G of length n is a sequence $v_0, \cdots, v_n$ of different vertices $v_0, \cdots, v_n \in V$ with the property that $\{v_j, v_{j+1}\} \in E$ holds for all $j \in \{0, \cdots, n-1\}$; if $n < \infty$, then we say that this path **connects** the vertices v_0 and v_n.
- The graph G is called **connected** if any tow of its vertices can be connected by a path in G.
- Let $n \in \mathbb{N}_{>2}$. A **cycle** in G of length n is a sequence $v_0, \cdots, v_{n-1}$ of different vertices $v_0, \cdots, v_{n-1} \in V$ with $\{v_{n-1}, v_0\} \in E$ and moreover $\{v_j, v_{j+1}\} \in E$ for all $j \in \{0, \cdots, n-2\}$.

Definition 2.5. A **tree** is a connected graph containing no cycles. A graph containing no cycles is a **forest** (so a tree is a connected forest).

Proposition 2.1. *A graph is a tree iff for every pair of vertices there exists exactly one path connecting these vertices.*

Proof. Let G be a graph such that for every pair of vertices can be connected by exactly one path, so G is connected. Assume that G contains a cycle $v_0, \cdots, v_{n-1}$; because $n > 2$, the two paths v_0, v_{n-1} and $v_0, \cdots, v_{n-1}$ are different. So G is a tree.

Conversely, let G be a tree, so G is connected. Assume that there exists two vertices that can be connectd by differing paths p, p'. We can construct a cycle from these two paths, a contradiction. So every two vertices can be connected by exactly one path in G. ⊠

2.2 Cayley graphs

Definition 2.6. Let G be a group and let $S \subseteq G$ be a generating set of G. Then the **Cayley graph** of G with respect to S is the graph $\mathrm{Cay}(G, S)$ whose

- set of vertices is G, and whose
- set of edges is

$$\{\{g, g \cdot s\} \mid g \in G, s \in (S \cup S^{-1}) \setminus \{e\}\}.$$

I.e., two vertices in a Cayley graph are adjacent iff they differ by an element of the generating set in question.

Example 2.1. Some examples:

- Consider the Cayley graphs of $\mathbb{Z}$ wrt $\{1\}$ and $\{2, 3\}$. From "far away" these have the same global structure (the real line). In more technical terms, these graphs are quasi-isometric wrt to the corresponding word metrics.
- The Cayley graph of $\mathbb{Z}^2$ wrt $\{(1,0),(0,1)\}$ looks like the integer lattice in $\mathbb{R}^2$, when viewed far away it looks like the Euclidean plane.
- The Cayley graph of the cyclic group $\mathbb{Z}/6$ looks like a cycle graph.
- Let $\tau = (1\ 2) \in S_3, \sigma = (1\ 2\ 3) \in S_3$. The Cayley graph of S_3 wrt $\{\tau, \sigma\}$ looks like two triangles. The Cayley graph of S_3 is a complete graph on six vertices; similarly, $\mathrm{Cay}(\mathbb{Z}/6, \mathbb{Z}/6)$. In particular, we see that non-isomorphic groups may have isomorphic Ccayley graphs wrt certain generating sets. However, the isomorphism type of any Cayley graph of a finitely generated abelian group is still rigid enough to remember the size of the torsion part.
- The Cayley graph of a free group wrt to a free generating set is a tree.

Remark 2.1. Some elementary properties of Cayley graphs:

(1) Cayley graphs are connect as each vertex g can be reached from the vertex of e by walking along the edges corresponding to a presenttaion of minimal length of g in terms of the given generators.
(2) Cayley graphs are regular in the sense that every vertex has the same number $|(S \cup S^{-1}) \setminus \{e\}|$ of neighbors.
(3) A Cayley graph is locally finite iff the generating set is finite; a graph is said to be **locally finite** if each vertex has only finitely many neighbors.

Remark 2.2. They are higher dimensional analogues of Cayley graphs in topology; associated with a presentation of a group, these is the *Cayley complex*, a 2-dimensional object. More generally, every group admits a *classifying space* whose fundamental group is the given group, and higher dimensional homotopy groups are trivial. These spaces allow us to model group theory in topology and play an important role in the study of group cohomology. Note that Cayley graphs and complexes require additional data on the group (generating sets and presentations) and, viewed as combinatorial objects, are only functorial with resepct to maps/homomorphism respecting this additional data.

2.3 Cayley graphs of free groups

A combinatorical characterisation of free groups can be given in terms of trees:

Theorem 2.1. *Let F be a free group, freely generated by $S \subseteq F$. Then the corresponding Cayley graph* $\mathrm{Cay}(F, S)$ *is a tree.*

Example 2.2. The converse is *not* true in general.

- The Cayley graph $\mathrm{Cay}(\mathbb{Z}/2, [1])$ consists of two vertices joined by an edge; clearly this graph is a tree, but the group $\mathbb{Z}/2$ is not free.

- The Cayley graph $\mathrm{Cay}(\mathbb{Z}, \{-1, 1\}$ coincides with $\mathrm{Cay}(\mathbb{Z}, \{1\})$, which is a tree. But $\{-1, 1\}$ is not a free generating set of $\mathbb{Z}$.

However these are the only two things that can go wrong.

Theorem 2.2. *Let G be a group and let $S \subseteq G$ be a generating set satisfying $s \cdot t \neq e$ for all $s, t \in S$. If the Cayley graph $\mathrm{Cay}(G, S)$ is a tree, then S is a free generating set of G.*

2.4 Free groups and reduced words

Definition 2.7. Let S be a set, and let $(S \cup \overline{S})^*$ a be the set of words over S and formal inverses of elements of S.

- Let $n \in \mathbb{N}$ and let $s_1, \cdots, s_n \in S \cup \overline{S}$. The word $s_1 \cdots s_n$ is **reduced** if

$$s_{j+1} \neq \overline{s_j} \quad \text{and} \quad \overline{s_{j+1}} \neq s_j$$

  holds for all $j \in \{1, \cdots, n-1\}$. (In particular, ε is reduced).
- We write $F_{\mathrm{red}}(S)$ for the set of all reduced words in $(S \cup \overline{S})^*$.

Proposition 2.2. *Let S be a set. The set $F_{\mathrm{red}}(S)$ of reduced words over $S \cup \overline{S}$ forms a group wrt the composition $F_{\mathrm{red}}(S) \times F_{\mathrm{red}}(S) \to F_{\mathrm{red}}(S)$ given by*

$$(s_1 \cdots s_n, s_{n+1} \cdots s_m) \mapsto (s_1 \cdots s_{n-r} s_{n+1+r} \cdots s_{n+m})$$

where $s_1 \cdots s_n, s_{n+1} \cdots s_m \in F_{\mathrm{red}}(S)$, and

$$r := \max\{k \in \{0, \cdots, \min(n, m-1)\} \mid \forall_{j \in \{0, \cdots, k-1\}}\ s_{n-j} = \overline{s_{n+1+j}} \ \vee\ \overline{s_{n-j}} = s_{n+1+j}\}.$$

Furthermore, the group $F_{\mathrm{red}}(S)$ is freely generated by S.tubhub day of or close to gametim

Corollary 2.1. *Let S be a set. Any element of $F(S) = (S \cup \overline{S})^* / \sim$ can be represented by exactly one reduced word over $S \cup \overline{S}$.*

Corollary 2.2. *The word problem in free groups wrt to free generating sets is solvable; consider and compare reduced words.*

Remark 2.3. Using the same method proof, one can describe free products $G_1 * G_2$ of groups G_1 and G_2 by reduced words; in this case, one calls a word

$$g_1 \cdots g_n \in (G_1 \sqcup G_2)^*$$

with $n \in \mathbb{N}$ and $g_1, \cdots, g_n \in G_1 \sqcup G_2$ **reduced** if for all $j \in \{1, \cdots, n-1\}$, either $g_j \in G_1 \setminus \{0\}$ and $g_{j+1} \in G_2 \setminus \{e\}$, or $g_j \in G_2 \setminus \{e\}$ and $g_{j+1} \in G_1 \setminus \{e\}$.

2.5 Free groups to trees

Proof that Cayley graphs of free groups are trees. Suppose the group F is freely generated by S. Then F is isomorphic to $F_{\mathrm{red}}(S)$ via an isomorphism that is the identity on S; WLOG we can assume that F is $F_{\mathrm{red}}(S)$. Because S generates F, the graph $\mathrm{Cay}(F, S)$ is connected. Assume that $\mathrm{Cay}(F, S)$ contains a cycle $g_0, \cdots, g_{n-1}$ of length n with $n \geq 3$; in particular, the elements $g_0, \cdots, g_{n-1}$ are distinct, and

$$s_{j+1} := g_{j+1} \cdot g_j^{-1} \in S \cup S^{-1}$$

for all $j \in \{0, \cdots, n-2\}$, as well as $s_n := g_0 \cdot g_{n-1}^{-1} \in S \cup S^{-1}$. Because the vertices are distinct, the word $s_0 \cdots s_{n-1}$ is reduced; on the other hand, we obtain

$$s_n \cdots s_1 = g_0 \cdot g_{n-1}^{-1} \cdot \cdots \cdot g_2 \cdot g_1^{-1} * g_1 \cdot g_0^{-1} = e = \varepsilon$$

in $F = F_{\mathrm{red}}(S)$, a contradiction. Therefore $\mathrm{Cay}(F, S)$ cannot contain any cycles, and is a tree. ⊠

2.6 Trees to free groups

3 Problems

3.1 Groups

Here we write up some problems.

Exercise 3.1 (Rationals are not finitely generated)**.** To see that $\mathbb{Q}$ is not finitely generated, assume that $\{a_i/b_i\}_{i\in\{1,\cdots,n\}}$ generates $\mathbb{Q}$ for $n\in\mathbb{N}$, $a_i, b_i \in \mathbb{Z}$. Let $q\in\mathbb{Q}$, then $q=\sum_{i\in\{1,\cdots,n\}} c_i\left(\frac{a_i}{b_i}\right)=\frac{\ell}{\prod_{i\in\{1,\cdots,n\}} b_i}$ for coefficients $c_i\in\mathbb{Z}$, $\ell\in\mathbb{Z}$ (the value of ℓ is irrelevant). Choose p a prime number such that $p\nmid\prod_{i\in\{1,\cdots,n\}} b_i$, and set $q=1/p$. This implies that

$$\ell p=\prod_{i\in\{1,\cdots,n\}} b_i,$$

which contradicts the fact that $p\nmid\prod_{i\in\{1,\cdots,n\}} b_i$.

Exercise 3.2 (Outer and inner automorphism groups)**.** To show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, let $\varphi_g\in\mathrm{Inn}(G)$ denote conjugation by g. Then for $\psi\in\mathrm{Aut}(G)$, $\psi\varphi_g\psi^{-1}$ is conjugation by $\psi(g)$, since

$$\psi\varphi_g\psi^{-1}(g')=\psi\varphi_g(\psi^{-1}(g'))=\psi\left(g\psi^{-1}(g')g^{-1}\right)=\psi(g)g'\psi(g^{-1}).$$

for $g'\in G$. Therefore $\psi\varphi_g\psi^{-1}=\varphi_{\psi(g)}\in\mathrm{Inn}(G)$, and we are done.

To determine $\mathrm{Out}(\mathbb{Z})$, there are only two automorphisms of $\mathbb{Z}$; $1\mapsto 1$ and $1\mapsto -1$. The automorphism $1\mapsto -1$ is not inner, so $\mathrm{Out}(\mathbb{Z})\cong\mathbb{Z}/2$.

For $\mathbb{Z}/n$ in general, $\mathrm{Aut}(\mathbb{Z}/n)\cong(\mathbb{Z}/n)^\times$ through the automorphisms $x\mapsto x^a$ for $x\in\mathbb{Z}/n, a\in(\mathbb{Z}/n)^\times$. We specify units because if a and n are not relatively prime, $x\mapsto x^a$ fails to be an automorphism since there exists a prime divisor p of n, implying the existence of an order p element, which is not allowed. There are no inner automorphisms so $\mathrm{Out}(\mathbb{Z}/2016)\cong\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/6$, and $\mathrm{Out}(\mathbb{Z}/2017)\cong\mathbb{Z}/2016$.

Exercise 3.3 (Isometry groups of the unit square)**.** Let ρ denote rotation by 90 degrees, μ denote reflection over the x-axis. Then $D_4=\{\rho,\rho^2,\rho^3,\mu,\rho\mu,\rho^2\mu,\rho^3\mu,\mathrm{id}\}$. Equivalently we have D_4 generated by $\{\rho,\mu\mid\rho^4=1,\mu^2=1,\mu\rho\mu^{-1}=\rho^{-1}\}$. This is not abelian while $\mathbb{Z}/8$ is.

Exercise 3.4 (More isometry groups)**.** Yes. For $n=2$, consider the isosceles triangle. For $n=3$ and above, consider the n-gon divided into n isosceles triangles with vertex at the center. Cut out an arrow out of each all with the same orientation. Then these subsets of $\mathbb{R}^2$ have rotational symmetry with isometry group $\mathbb{Z}/n$, but no reflectional or translational symmetry.

Exercise 3.5 (Even more isometry groups)**.** No. We have that $\mathrm{Isom}(\mathbb{R}^n)\cong\mathbb{R}^n\rtimes O(n)$; $\mathbb{R}^n$ represents the translations, and $O(n)$ is the set of distance preserving transformations (reflections and rotations) of $\mathbb{R}^n$ fixing the origin. Let ρ be a transfomration and μ be a translation. Then $\rho^{-1}\mu\rho$ is a translation. Here $O(n)$ acts on $\mathbb{R}^n$ in the natural way (as linear transformations are automorphisms). When composing isometries, the translation component should first apply the rotation of the first component before applying the translation of the second, encoded in the relation

$$(v_1^n,o_1)\cdot(v_2^n,o_2)=(v_1^n o_1(v_2^n)),o_1o_2)$$

which implies that $\mathrm{Isom}(\mathbb{R}^n)\cong\mathbb{R}^n\rtimes O(n)$. todo: $\mathbb{Z}$ acts on $\mathbb{Q}$? $\mathbb{Q}/\mathbb{Z}$? Assume

Exercise 3.6 (Unfree groups)**.** Finite groups cannot be free. $\mathbb{Z}^2$ cannot be free as it has the relation $[x,y]$.

Using the universal properties, if $\mathbb{Z}/2017$ were free with generating set S, let $\varphi\colon S\to\mathbb{Z}/2$ map an element to $[1]\in\mathbb{Z}/2$ and the rest to $[0]$. There is no non-trivial homomorphism $\varphi\colon\mathbb{Z}/2017\to\mathbb{Z}/2$, so $\mathbb{Z}/2017$ cannot be free. An analogous argument applies with $\mathbb{Z}^2$ and $\mathbb{R}$.

Exercise 3.7 (Rank of free groups)**.** In the infinite case: say $|S'| < |S|$, where S' generates (not freely) F. Then we have a surjection $F_{S'} \to F_S$, and passing to $(F_{S'})_{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $(F)_{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$ gives a surjection from a vector space of rank $|S'|$ onto one of rank $|S|$, a contradiction.

Now let S, S' be free generating sets for F. Since free generating sets are also generating sets, we have $|S| \geq |S'|$ and $|S'| \geq S$, which implies $|S| = |S'|$. From this we conclude that all free generating sets of a free group have the same cardinality.

To show that the free group generated by two elements contains a subgroup not generated by two elements, consider the subgroup generated by elements $\{xyx^{-1}, x^2yxy^{-2}, x^3yx^{-3}\}$. This group has rank 3 by construction and elements that compose non-trivially, and we are done.

Exercise 3.8 (Generators and relations, examples)**.** (1) We have $\langle x, y \mid xyx^{-1}y^{-1}\rangle$ isomorphic to $\mathbb{Z}^2$, since the commutator relation makes the generators commute and words reduce to $x^n y^m$ for $n, m \in \mathbb{Z}$, precisely $\mathbb{Z}^2$.

(2) These groups are isomorphic since $t = t^{-1}$ by the relation $t^2 = 0$.

(3) Abelianize to get $\langle x, y \mid x^{2014} = y^{2014}\rangle$, which is a non-trivial relation. So this group is non-trivial.

(4) For $\langle x, y \mid xyx = yxy\rangle$, note that $xyxy = x^2yx = yxy^2$, and abelianizing this relation $x^2yx = yxy^2$ gives a non-trivial relation $x^2 = y^2$ (abelianizing the original group also leads to $\langle x, y \mid x = y\rangle \cong \mathbb{Z}$).

Exercise 3.9 (Positive relations)**.** We want to show that for every presentation $\langle S \mid R\rangle$ there is a positive relation $\langle S' \mid R'\rangle$ with $\langle S' \mid R'\rangle \cong \langle S \mid R\rangle$ and $|S'| \leq |S| + 1$, $|R'| \leq |R| + 1$.

$$\langle a, b, c, d \mid ab^{-1}c^{-1}dab, ba^{-1}b^{-1}cda\rangle$$

Add a new generator s' and the relation that (deemed unimportant exercise)

Exercise 3.10 (Abelianization)**.** Some problems about abelianization:

(1) To see that $[G, G]$ is normal, let $[g, h] \in [G, G]$, where $[g, h] := ghg^{-1}h^{-1}$. Let $\ell \in G$. Then

$$\ell[g,h]\ell^{-1} = \ell[g,h]\ell^{-1}[g,h]^{-1}[g,h] = (\ell,[g,h])[g,h]$$

which lies in $[G, G]$ as a product of commutators. So $[G, G]$ is normal. The quotient $G_{\text{ab}} := G/[G, G]$ is abelian because setting a commutator $[g, h] = 0$ makes the elements g, h commute, e.g. $gh = hg$. Since the commutator subgroup contains commutators spanning every pair (g, h) for $g, h \in G$, every element of G commutes with every other element and G_{ab} is indeed abelian.

(2) Let H be abelian and $\varphi\colon G \to H$ be a homomorphism. Then $\varphi(gh) = \varphi(g)\varphi(h) = \varphi(h)\varphi(g) = \varphi(hg)$, so $[g, h] \in \ker\varphi$ for all $g, h \in G$. This implies that $[G, G] \subseteq \ker f$, and from here apply the fundamental homomorphism theorem to get our universal property.

(3) To view $(\cdot)_{\text{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ as a functor, let $(G)_{\text{ab}} = G_{\text{ab}} \in \text{Ob}(\mathsf{Ab})$ for $G \in \text{Ob}(\mathsf{Grp})$. For $\varphi\colon G \to H, \varphi \in \text{Mor}(\mathsf{Grp})$, compose with the projection π_H to get $\overline{\varphi}\colon G \to H_{\text{ab}}, \overline{\varphi} := \pi_H \circ \varphi$. This yields a homomorphism $G \to H_{\text{ab}}$, and lift this by the universal property of G_{ab} to a map $\varphi_{\text{ab}}\colon G_{\text{ab}} \to H_{\text{ab}}$.

$$\begin{tikzcd}
 & H \arrow[d, "\pi_H"] \\
G \arrow[ur, "\varphi"] \arrow[r, "\overline{\varphi}"] \arrow[d, "\pi_G"] & H_{\text{ab}} \\
G_{\text{ab}} \arrow[ur, dotted, "\varphi_{\text{ab}}"'] &
\end{tikzcd}$$

Then define $(\varphi)_{\mathrm{ab}} := \varphi_{\mathrm{ab}}$ as in the above construction. To show that this is indeed a functor, it is clear that $(1_{\mathsf{Grp}})_{\mathrm{ab}} = 1_{\mathsf{Ab}}$ since Grp and Ab have the same unit, which is $1 = 1_{\mathrm{ab}}$. For $\theta\colon G \to H, \varphi\colon H \to K$, the fact that $(\varphi \circ \theta)_{\mathrm{ab}} = \varphi_{\mathrm{ab}} \circ \theta_{\mathrm{ab}}$ is a natural consequence of the universal property.

$$\begin{tikzcd}
 & & K \arrow[d, "\pi_K"] \\
 & H \arrow[ur, "\varphi"] \arrow[r, "\overline{\varphi}"] \arrow[d, "\pi_H"] & K_{\mathrm{ab}} \\
G \arrow[ur, "\theta"] \arrow[r, "\overline{\theta}"] \arrow[d, "\pi_G"] & H_{\mathrm{ab}} \arrow[ur, dotted, "\varphi_{\mathrm{ab}}"'] & \\
G_{\mathrm{ab}} \arrow[ur, dotted, "\theta_{\mathrm{ab}}"'] & &
\end{tikzcd}$$

This shows that $(\cdot)_{\mathrm{ab}}$ is indeed a functor $\mathsf{Grp} \to \mathsf{Ab}$.

(4) F_{ab} is the free abelian group inheriting the same basis as the free group, since we mod out by all commutators.

(5) Abelianization mods out all commutators so this is true as well.

Exercise 3.11 (The infinite dihedral group)**.** An isometry of $\mathbb{Z}$ is multiplication by -1 which squares to itself. Another isometry is addition by one, and composing these isometries under the rule that multiplying by -1, adding, and multipliying again by -1 is the same thing as subtraction gives all the isometries of $\mathbb{Z}$. This data is precisely encoded in the group $D_\infty := \langle s, t \mid t^2, tst = s^{-1}\rangle \cong \mathrm{Isom}(\mathbb{Z})$.

Exercise 3.12 (Baumslag-Solitar groups)**.** (1) We have $\mathrm{BS}(1,1) = \langle a, b \mid bab^{-1} = a\rangle = \langle a, b \mid [b,a]\rangle \cong \mathbb{Z}^2$.

(2) To see that $\mathrm{BS}(m,n)$ is infinite, consider the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}$. Note that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & 1 \end{pmatrix}$. Then if we consider the map $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}$, multiplying matrices gives us

$$\begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

precisely encoding the Baumslag-Solitar relation that $ba^mb^{-1} = a^n$. There are an infinite amount of matrices generated from these two matrices by repeated multiplication of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, in other words, the infinite set $\left\{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}\right\}_{i \in \mathbb{N}\setminus\{0\}}$ is a subset of $\mathrm{BS}(m,n)$.

(3) To see that $\mathrm{BS}(m,n)$ is not cyclic, abelianize to get $\mathrm{BS}(m,n)_{\mathrm{ab}} \cong \langle a, b \mid a^{m-n}\rangle$. Here b freely generates words with no relations, so there is no single generator for $\mathrm{BS}(m,n)_{\mathrm{ab}}$, and therefore $\mathrm{BS}(m,n)$. We conclude that $\mathrm{BS}(m,n)$ is not cyclic.

(4) We want to show that

$$\varphi\colon \mathrm{BS}(2,3) \to \mathrm{BS}(2,3), \quad a \mapsto a^2, b \mapsto b$$

defines a well-defined surjective group homomorphism. Under φ the relation becomes $ba^4b^{-1} = a^6$, which adds no additional data. The map φ is surjective because 2 divides 2 but not 3, so any $a^2 \in \mathrm{im}(\varphi)$ gets represented (doesn't get cancelled by a^3 relation). Now to see that $[bab^{-1}, a] \in \ker\varphi$, write $[bab^{-1}, a] = bab^{-1}aba^{-1}b^{-1}a^{-1}$, then

$$\varphi(bab^{-1}aba^{-1}b^{-1}a^{-1}) = (ba^2b^{-1})a^2(ba^{-2}b^{-1})a^{-2} = a^3a^2a^{-3}a^{-2} = 1$$

and $[bab^{-1}, a] \in \ker\varphi$. So φ is a non-injective surjection $\mathrm{BS}(2,3) \to \mathrm{BS}(2,3)$, and $\mathrm{BS}(2,3)$ is non-Hopfian.

Exercise 3.13 (A normal form for $\mathrm{BS}(1,2)$)**.** todo: arbitrary word?

Exercise 3.14 (Surface groups)**.** First let $n = m$, then the groups G_n and G_m have the same generators and relations, and are therefore isomorphic. Now suppose $n \neq m$. Abelianization cancels all the relations, and we are left with two free groups of different sizes ($G_n = F_{2n}, G_m = F_{2m}$) which are not isomorphic by assumption. So $G_n \cong G_m$ iff $n = m$. G_n is only abelian for $n = 1$, since for $n = 2$ the pair of generators a_2, b_1 do not commute with each other (since $[a_2, b_1]$ is not a relation)todo:fix this part .

Exercise 3.15 (Coxeter groups)**.** (1) If $j, k \in \{1, \cdots, n\}$ with $j \neq k$ and $m_{jk} = 2$, we want to show that the corresponding elements $\bar{s}_j, \bar{s}_k$ commute in W. We have the relations

$$\bar{s}_j^2 = 1, \quad \bar{s}_k^2 = 1, \quad (\bar{s}_j \bar{s}_k)^2 = 1,$$

which implies that $\bar{s}_j = \overline{s_j}^{-1}, \bar{s}_k = \overline{s_k}^{-1}$, and so

$$(\bar{s}_j \bar{s}_k)^2 = \bar{s}_j \bar{s}_k \bar{s}_j \bar{s}_k = \bar{s}_j \bar{s}_k \overline{s_j}^{-1} \overline{s_k}^{-1} = [\bar{s}_j, \bar{s}_k] = 1.$$

This implies that $\bar{s}_j$ and $\bar{s}_k$ commute.

(2) To see that $\langle s_1, s_2, s_3 \mid (s_1 s_2)^2, (s_1 s_3)^2, (s_2 s_3)^2 \rangle \cong (\mathbb{Z}/2)^3$, map $s_1 \mapsto (s_1 s_2), s_2 \mapsto (s_2 s_3)$, and $s_3 \mapsto (s_1 s_3)$.

(3) The isometry group of a regular n-gon D_n can be viewed as a Coxeter group as follows; consider the presentation $D_n = \langle s, t \mid s^n, t^2, (st)^2 \rangle$. Send $s \mapsto st$ and $t \mapsto t$, then this presentation becomes $\langle st, t \mid (st)^2, t^2, (t(st))^{-n} \rangle$ since $(st)^2 = 1$ implies

$$s = t^{-1} s^{-1} t^{-1} = (tst)^{-1} = (t(st))^{-1}, \quad s^n = (t(st))^{-n}.$$

Then this is a Coxeter group of rank two with Coxeter matrix $\left(\begin{smallmatrix} 1 & -n \\ -n & 1 \end{smallmatrix}\right)$.

Exercise 3.16. braid

Exercise 3.17. a finitely generated group that is not finitely presented

Exercise 3.18 (Special pushouts)**.** (1) Yes, remove the relations of A to get $G * 1 \subseteq G * A$.

(2) For $\varphi : A \to G$, $G *_A 1$ is just $G / \operatorname{im} \varphi$ since we adjoin relations $\varphi(a) = 1$ for $a \in A$.

(3) For $\varphi : A \to G$, $\mathrm{id}_A : A \to A$, we have $G *_A A \cong (G * A)/\langle \varphi(a) = a \rangle$.

Exercise 3.19 (The infinite dihedral group strikes back)**.** To see that $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$, using the presentations $\langle s, t \mid t^2, tst = s^{-1} \rangle$ for D_∞ and $\langle a, b \mid a^2, b^2 \rangle$ for $\mathbb{Z}/2 * \mathbb{Z}/2$, consider the map $(ts) \mapsto a, t \mapsto b$. Since $tst = s^{-1}$ is the same relation as $(ts)^2$, this map yields an isomorphism $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

Now we want to show that $\mathrm{Isom}(\mathbb{Z}) \cong \mathbb{Z} \rtimes_\varphi \mathbb{Z}/2$, where $\varphi : \mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z})$ is multiplication by -1. Consider the exact sequence

$$1 \to \mathbb{Z} \to D_\infty \to \mathbb{Z}/2 \to 1,$$

this splits by $\mathbb{Z}/2 \to D_\infty$ where $[1] \mapsto ([0], [1])$. So $D_\infty \cong \mathbb{Z} \rtimes_\varphi \mathbb{Z}/2$ by $\varphi : \mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z})$, $[1] \mapsto (1 \mapsto 1 \cdot n \cdot -1 = -n)$, which is precisely the automorphism given.

Exercise 3.20 (Heisenberg group)**.** Say the extension

$$1 \to \mathbb{Z} \xrightarrow{i} H \xrightarrow{\pi} \mathbb{Z}^2 \to 1$$

splits, that is, there exists a map φ

$$(0,1) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad (1,0) \mapsto \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix}$$

such that projecting gives back the generators of $\mathbb{Z}^2$. Then we must have $x = 0, y = 1$ and $x' = 1, y' = 0$, and $z, z' = 0$ (maps take identities to identities). So

$$\varphi((0,1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi((1,0)) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It should be the case that $\varphi(1,1) = \varphi((0,1)+(1,0)) = \varphi((1,0)+(0,1)) = \varphi((0,1))\varphi((1,0)) = \varphi((1,0))\varphi((0,1))$. todo: However,

$$\varphi((0,1))\varphi((1,0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

while

$$\varphi((1,0))\varphi((0,1)) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

These two matrices are not equal, and we conclude that φ must not exist and the extension $1 \to \mathbb{Z} \xrightarrow{i} H \xrightarrow{\pi} \mathbb{Z}^2 \to 1$ does not split. Now to see that $\langle x, y, z \mid [x,z], [y,z], [x,y] = z \rangle$ is a presentation of H, consider the maps

$$x \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can verify these satisfy the commutator relations using a computer or by hand.

Exercise 3.21 (Equivalence of extensions)**.** We say two extensions $1 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 1, 1 \to K \xrightarrow{i'} G' \xrightarrow{\pi'} H \to 1$, are equivalent if there exists an isomorphism $T : G \to G'$ making the following diagram commute:

$$\begin{tikzcd}
 & & G \arrow[dd, "T"] \arrow[dr, "\pi"] & & \\
1 \arrow[r] & K \arrow[ur, "i"] \arrow[dr, "i'"'] & & H \arrow[r] & 1 \\
 & & G' \arrow[ur, "\pi'"'] & &
\end{tikzcd}$$

The short five lemma tells us that in fact, a homomorphism is enough to make the diagram commute. Let us give three pairwise non-equivalent group extensions of the following type:

$$1 \to \mathbb{Z} \xrightarrow{?} ? \xrightarrow{?} \mathbb{Z}/3 \to 1.$$

First consider the extension

$$1 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3 \to 1.$$

Then consider the extension

$$1 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \rtimes_{\varphi_n} \mathbb{Z}/3 \xrightarrow{\pi} \mathbb{Z}/3 \to 1.$$

Finally consider the extension

$$1 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}/3 \xrightarrow{\pi} \mathbb{Z}/3 \to 1.$$

Exercise 3.22. lamplighter

Exercise 3.23. big centre

Exercise 3.24 (Baumslag-Solitar groups and HNN-extensions)**.** We can view Baumslag-Solitar groups $\mathrm{BS}(m,n)$ as the HNN-extension of $\mathbb{Z}$ with $A = m\mathbb{Z}, B = n\mathbb{Z}$, with $\vartheta\colon A \to B$ defined by $\vartheta(m) = n$. Then in the definition of HNN-extensions

$$\mathbb{Z}*_{\vartheta} := \left\langle x \amalg \{t\} \mid \{t^{-1}x_a t = x_{\vartheta(a)} \mid a \in A\} \cup R_G \right\rangle$$

where

$$R_G := \{x_g x_h x_k^{-1} \mid g,h,k \in \mathbb{Z} \text{ with } g \cdot h = k \text{ in } \mathbb{Z}\},$$

the relations R_G all cancel since $\mathbb{Z}$ has no relations. This leaves us with the group

$$\mathbb{Z}*_{\vartheta} = \langle x, t \mid t^{-1}x^m t = x^n \rangle,$$

which looks like $\mathrm{BS}(m,n)$ when you send $x \mapsto a, t \mapsto b$ given the following definition of Baumslag-Solitar groups:

$$\mathrm{BS}(m,n) := \langle a, b \mid ba^m b^{-1} = a^n \rangle.$$

Exercise 3.25 (Ascending HNN-extensions)**.** If $\vartheta = \mathrm{id}_G$, then the relation $t^{-1}x_a t = x_{\vartheta(a)}$ for $a \in G$ becomes $x_a t = x_a t$, which is redundant. So $G*_{\vartheta} \cong G * \mathbb{Z}$ generated by the extra generator t.

Now let $\vartheta \in \mathrm{Aut}(G)$, the twisting of $\varphi\colon G \to \mathrm{Aut}(G)$, $1 \mapsto \vartheta$ is represented by the relation $t^{-1}x_a t = x_{\vartheta(a)}$, so the semidirect has the relations of G as well as this extra generator and twisting relation. This is precisely $G*_{\vartheta}$.

3.2 Cayley graphs

Exercise 3.26 (Petersen graph)**.**