Complex Analysis Lecture Notes

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These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`.

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§1 August 27, 2020

§1.1 Basic Properties of Complex Numbers

We talk about functions $f : \mathbb{C} \to \mathbb{C}$ that map variables $z \mapsto f(z)$. This course is "not a very hard course" (it's a fun course!). Holomorphic functions have very nice properties automatically that real valued differentiable functions simply don't have.

Definition 1.1 (Complex Addition)**.** We define complex numbers as ordered pairs $z = (x, y)$ where $x, y \in \mathbb{R}$, with the binary operation of complex addition being defined as

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$

where $+$ denotes addition on the reals.

Once we define multiplication and additive/multiplicative inverses, we will have (almost) formed the field $\mathbb{C}$.

Definition 1.2 (Complex Multiplication)**.** For $x, y \in \mathbb{C}$, we have

$$(x_1, y_1)(x_2)(y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Note: for $a \in \mathbb{R}$, we define

$$a(x, y) = (ax, ay).$$

Recall $(a, 0)(x, y) = (ax, ay)$. So one can understand that $a \in \mathbb{R}$ is simply the real analog of $(a, 0)$ (or simply, $\mathrm{Re}(a, 0) = a \in \mathbb{R}$).

How do we define multiplication of a complex number by a real number? We can think of the reals acting (in a group sense) on the complex numbers, with the operation being the standard multiplication.

Example 1.1. Take $(1, 0)(x, y) = (x, y)$. So $1(x, y) = (x, y)$ (where $1 \in \mathbb{R}$).

Example 1.2 (Complex Addition is Commmutative)**.** We have already defined the sum of two complex numbers $z_1 + z_2$ as $z_3 = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$. Since addition is commutative on the real numbers, we have

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1,$$

so complex addition is commutative.

Claim: multiplication of complex numbers is commutative. You can verify this at home.

Theorem 1.1 (Distributive Law)**.** *We have*

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$

for $z_1, z_2, z_3 \in \mathbb{C}$.

Proof. This follows from the fact that $\mathbb{C}$ has a ring structure. ☒

§1.2 Real and Imaginary Parts

Definition 1.3. If $z = (x, y)$, then $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Furthermore, we can associate a complex number with a point in the plane in many ways:

(insert figure 1 later)

§1.3 Complex Numbers in the Plane

Point: the plane is just a plane. The plane doesn't have to have a coordinate system (coordinate axes don't have to be perpendicular). Any coordinate system is "useful" for adding complex numbers. For example, you can interpret complex addition as simply vector addition in the plane (no need for orthogonal axes!).

Definition 1.4 (Additive Inverse)**.** We have

$$-(x, y) = (-1)(x, y) = (-x, -y).$$

So $(x, y) + [-(x, y)] = (0, 0)$.

Note: $(x, y)(0, 1) = (-y, x)$, a *rotation* of (x, y) by $90°$. Another note: We have $(x, y) \in \mathbb{C} \cong x + iy$ and $i = (0, 1)$. So

$$(x, y) \cong x + iy \cong (x, 0) + (0, 1)(y, 0).$$

§2 September 1, 2020

§2.1 Units and Zero Divisors in the Complex Numbers

Recall from last time: A complex number can be defined as $(x, y) = x + iy$, where $x, y \in \mathbb{R}$. Addition is easy: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + y_1) + i(y_1 + y_2)$. In particular, $(0, 0) = 0 + i \cdot 0 = 0$. For multiplication, assume $i^2 = -1$. Then

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 + iy_1x_2 + iy_2x_1 + i^2y_1y_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1).\end{aligned}$$

On division: what does it mean to divide complex numbers? We say the multiplicative unit of a complex number (wrt the ring $\mathbb{C}$) as the unique $\frac{1}{z} = z^{-1}$ s.t. $z \cdot z^{-1} = z^{-1} \cdot z = (1, 0) \in \mathbb{C}$ (the unity of $\mathbb{C}$). Assume $(x, y)(x, y)^{-1} = (1, 0)$. Then do u and v exist such that the system of equations

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases}$$

holds? Yes, iff the determinant $\left|\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right| = x^2 + y^2$ is non zero.

Definition 2.1 (Complex Conjugate)**.** We have $(x, -y)$ the complex conjugate of the complex number $z = (x, y)$, denoted $\bar{z}$.

We show that $\mathbb{C}$ has no zero divisors and is therefore an integral domain. WLOG, assume there exists z_1, z_2 such that $z_1 \neq 0$, $z_1z_2 = 0$: then we have z_1^{-1} exists. So $z_1^{-1}z_1z_2 = 1z_2 = 0$, therefore $z_2 = 0$. For example: the group $\mathrm{GL}_n(\mathbb{R})$ is not an integral domain, since we have zero divisors (two matrices that when multipled equal zero).

§2.2 Polar Coordinate Notation

Definition 2.2 (Polar Coordinates)**.** Think of (x, y) as rectangular coordinates in the xy-plane, and consider the *polar coordinate* notation $z = [r, \theta]$, where $r = \sqrt{x^2 + y^2} = |z|$ (modulus of z), and $\theta = \arctan(\frac{y}{x})$. So $[r, \theta] = (r\cos\theta, r\sin\theta)$.

Example 2.1 (Multiplication with Polar Coordinates)**.** We have

$$[r_1, \theta_1][r_2, \theta_2] = (r_1\cos\theta_1, r_1\sin\theta_1)(r_2\cos\theta_2, r_2\sin\theta_2).$$

Then

$$\begin{gathered}(r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) = \\ r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right] + ir_1r_2\left[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right] = \\ r_1r_2\cos(\theta_1 + \theta_2) + r_1r_2i\sin(\theta_1 + \theta_2) = \\ [r_1r_2, \theta_1 + \theta_2].\end{gathered}$$

Example 2.2. Assume that a complex number $z = (x, y)$ is nonzero. Then

$$\frac{1}{(x, y)} = \frac{1(x, -y)}{(x, y)(x, -y)} = \frac{(x, -y)}{x^2 + y^2}.$$

§2.3 On the Norm (Modulus) of a Complex Number

Example 2.3. Some properties of the modulus (norm) $|z|$:

1. $|z_1 z_2| = |z_1||z_2|$,
2. $\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = \left|z_1 \cdot \frac{\bar{z_2}}{|z_2|^2}\right| = |z_1|\frac{|z_2|}{|z_2|^2} = \frac{|z_1|}{|z_2|}$ (clearly $|\bar{z_2}| = |z_2|$),
3. $|z_1 + z_2| \leq |z_1| + |z_2|$ ($\mathbb{C}$ is a metric space, so the triangle inequality holds),
4. $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$ (reverse triangle inequality).

We prove the Reverse Triangle Inequality.

Proof. We have $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|$, so $|z_1 + z_2| \geq |z_1| - |z_2|$. A similar argument holds for z_2. ⊠

Think of the polar angle as only well defined for multiples of 2π. Define the argument (angle) as $\mathrm{Arg} = -\pi < \theta \leq \pi$ (what??). So $\mathrm{Arg}(1,1) = \frac{\pi}{4}$, $\mathrm{Arg}(-1,0) = \pi$. OTOH, we would have $\arg(1,1) = \frac{\pi}{4} + 2\pi n$.

§2.4 Euler's Formula

Theorem 2.1 (Euler's Formula)**.** *We claim*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Try using Maclaurin series. ⊠

This suggests $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We proved this when we showed $[r_1, \theta_1][r_2, \theta_2] = [r_1 r_2, \theta_1 + \theta_2]$.

The reason why Dr. Radin says to "forget about Euler" is because he's trying to make a semi-rigorous (or self-contained) construction of the complex numbers. I think it's fine to rely on intuition from other courses, this isn't Real Analysis (nowhere near as rigorous). If we truly were to construct the field $\mathbb{C}$, we would have to cover polynomial rings and the fields generated by PID's quotient irreducible polynomials, then show that $\mathbb{C} \simeq \mathbb{R}[x]/\langle x^2 + 1\rangle$ (and show that this new field is algebraically closed too!). Of course this isn't feasible. So let's just think of this as Euler's Formula, and not some weird definition!

Back to math: using our newfound formula, we can simply say $\arg z = \theta$ such that $z = re^{i\theta}$ for any $z \in \mathbb{C}$. Similarly, $\mathrm{Arg}\, z$ is just θ restricted to the interval $(-\pi, \pi]$.

Example 2.4. If $z = re^{i\theta}$ nonzero, then what is the polar form of $\frac{1}{z}$? It must be

$$\frac{1}{r}e^{-i\theta}.$$

Example 2.5. We've seen that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Then

$$e^{i\theta_1}\left(e^{i\theta_2}e^{i\theta_3}\right) = e^{i\theta_1}e^{i(\theta_2+\theta_3)} = e^{i(\theta_1+\theta_2+\theta_3)}.$$

So $(\cos\theta + i\sin\theta)^m = \cos(m\theta) + i\sin(m\theta)$. This is known as *de Moivre's formula* .

§3 September 3, 2020

§3.1 Fractional Powers

Let $z_0 \in \mathbb{C}$, and define the fractional power $(z_0)^{\frac{1}{m}}$ for $m \geq 2$. This is a complex number such that

$$\left[(z_0)^{\frac{1}{m}}\right]^m = z_0.$$

This many not be unique. To determine the value of the fractional power $(z_0)^{\frac{1}{m}}$, let $z_0 = r_0 e^{i\theta_0}$, $r_0 = |z_0|$, $\theta_0 \in \operatorname{Arg} z_0$. Then

$$(z_0)^{\frac{1}{m}} = (r_0)^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}.$$

Example 3.1. In polar form, $z_0 = i = e^{i\frac{\pi}{2}}$. We want $i^{\frac{1}{6}}$, one value is $e^{i\frac{\pi}{12}}$. Also,

$$e^{i\frac{\left[\frac{\pi}{2}+2\pi\right]}{6}} = e^{i\left[\frac{\pi}{12}+\frac{\pi}{3}\right]} = e^{i\frac{5\pi}{12}}.$$

In general, $i = e^{i\left[\frac{\pi}{2}+2\pi m\right]}$, so $e^{i\left[\frac{\pi}{12}+\frac{m\pi}{3}\right]}$ is a value of $i^{\frac{1}{6}}$ for any m. In particular, consider the choices $m = 0, 1, .., 5$. Then

(insert figure later- it has to do with roots of unity on the circle group tho)

This method gives all possible n-th roots. In particular, in the circle group U_1, each "walk" is equal to a multiplication of ζ.

We will eventually generalize the fractional power $z_0^{p/q}$ to z_0^w. Yada yada no exponentials allowed reeee. If you're going to formalize do it right or don't do it at all. Half baked rigor is about as useful as a potato (at least a potato can feed your family).

§3.2 Point Set Topology

Why are we studying abstract nonsense? We need topology to define limits of complex numbers. We will eventually define a derivative as a quotient of deltas, eg

$$\frac{\Delta f}{\Delta z} \to \frac{df}{dz} \quad \text{as } \Delta z \to 0.$$

We'll talk about open and closed sets and accumulation points and such (basic things needed for limits). Consider

$$\widetilde{S} = \{z \mid |z| \leq 1 \text{ and } |z| \neq 1 \text{ if } \operatorname{Re} z < 0\}.$$

Definition 3.1 (Open Ball)**.** We define an open ball

$$B(z_0, \epsilon) = \{z \mid |z - z_0| < \epsilon\}.$$

§3.3 Interior, Closure, Boundary

Definition 3.2 (Interior Point)**.** We have an *interior point* a point in a set such that there exists an open ball centered at the point entirely contained in the set. We define the set of all interior points of a set X as $\text{Int}(X)$.

Note that $\text{Int}(\widetilde{S}) = \{z \mid |z| < 1.\}$

Definition 3.3 (Exterior Point)**.** A point z_0 is an exterior point of S if there exists a ball

$$B(z_0, \epsilon) \subseteq S^c,$$

ie, $z_0 \in \text{Int}(S^c)$.

Definition 3.4 (Boundary Point)**.** A point z_0 is a boundary point of S if for ball $B(z_0, \epsilon)$ centered at z_0, $B(z_0, \epsilon) \cap S \neq \emptyset$ and $B(z_0, \epsilon) \cap S^c \neq \emptyset$. We define the *boundary* of a set S as the set of all boundary points, denoted ∂S.

Basic things: points can't be both in the interior and exterior, boundary and interior, etc etc.

Theorem 3.1. *For any set S, $\text{Int}(S)$, $\text{Ext}(S)$, and ∂S form a partition of S.*

We will use S° to denote the interior and $(S^c)^\circ$ to denote the exterior of a set from now on.

Example 3.2. $\partial \widetilde{S} = \{z \mid |z| = 1\}$.

Example 3.3. We have the unit circle $S = \{z \mid |z| = 1\} \cup zi$ (where zi is a point). $S^\circ = \emptyset$, $zi \in \partial S$, any point on the rim $\in \partial S$, so $\partial S = S$. By our previous theorem, $(S^c)^\circ = S^c$. (Who even studies the exterior of a set??)

§3.4 Open and Closed Sets

From now on a set refers to a subset of $\mathbb{C}$.

Definition 3.5 (Open Sets)**.** A set is open if it contains none of its boundary. Alternatively, a set is open iff $S = S^\circ$.

Example 3.4. $\mathbb{C}$ is open (and closed)! Furthermore, $\partial\mathbb{C} = \emptyset$ (which is an alternate condition for a set to be clopen). Note that $\emptyset$ is also both open and closed, since $\partial\emptyset = \emptyset$. This also makes sense if we look at it from the interior perspective (no interior points in $\emptyset$, every point has an open ball in $\mathbb{C}$).

Definition 3.6 (Closed Sets)**.** A set is closed if it contains all of its boundary. (What do you mean not the complement of open???)

Theorem 3.2. *S is closed $\iff$ S^c is open.*

Proof. Immediate. In general topology, we define open sets this way. $\boxtimes$

Example 3.5. Like I said earlier, both $\mathbb{C}$ and $\emptyset$ are closed. In general topology, we define both $S, \emptyset \in \tau$, since they're complements of course they're both open and closed. Exercise: prove that no other sets are both open and closed.

Definition 3.7 (Closure)**.** The closure $\bar{S}$ of S is the union

$$S \cup \partial S.$$

Clearly $\bar{S}$ is always closed (by our definition).

Theorem 3.3. *S° is open for any S.*

Doesn't this follow from the definition too??

§3.5 Jank Connectedness

Definition 3.8 (Path-connectedness)**.** A set S is path-connected if every pair of points $z_1, z_2 \in S$ is connected by a continuous path in S.

Every path-connected set is connected (can be written as the union of two disjoint sets). Something about polygonal paths?? Dr. Radin is right, this is most definitely not standard. Is this what physicists do to topology?

Now he's talking about the Topologist's sine curve (the classic counterexample). This is a counterexample to the (false) idea that connected implies path-connected by exhibiting a set that is connected but not path-connected (but we haven't even talked about the standard definition of connectedness yet!).