Differential Topology Notes

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Lecture 1

April 6, 2020

(last time: universal properties, motivating differential forms: watch!)

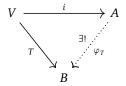
A lot of definitions, here's the new ones:

Definition 1.1. A **subalgebra** of an algebra A is a linear subspace $A' \subseteq A$ containing 1 such that $a'_1 a'_2 \in A'$ for all $a'_1, a'_2 \in A'$. A **2-sided ideal** $I \subseteq A$ is a linear subspace such that AI = I and IA = I. A \mathbb{Z} -**grading** of an algebra A is a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that $A^{k_1} A^{k_2} \subseteq A^{k_1 + k_2}$ for all $k_1, k_2 \in \mathbb{Z}$. If A is a \mathbb{Z} -graded algebra and $a \in A^k, k \in \mathbb{Z}^{>0}$, then a is **decomposable** if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \cdots, a_k \in A^1$. If not, a is **indecomposable**.

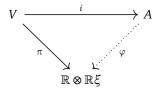
1.1 Tensor algebras

Let V be a vector space. We want to make the "free-est" algebra possible without relations, the tensor algebra $\bigotimes V$, thought of as the "free algebra generated by V".

Definition 1.2. Let V be a vector space. A **tensor algebra** (A, i) over V is an algebra A and a linear map $i: V \to A$ such that for all (B, T) of an algebra B and a linear map $T: V \to B$ such that φ_T is a homomorphism of algebras.



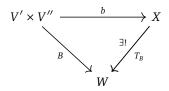
(A, i) is unique up to unique isomorphisms by a universal property argument (last time?). i is injective? If $(\xi \neq 0) \in V$ and $i(\xi) = 0$, set $B = \mathbb{R} \oplus \mathbb{R} \xi$ and define $\xi^2 = 0$.



Note that $\pi|_{\mathbb{R}\xi} = \text{id}$. But $\xi = \pi(\xi) = \varphi_1(\xi) = 0$, a contradiction. Furthermore, A has a canonical \mathbb{Z} -grading. $\lambda \in \mathbb{R}^{\neq 0, \neq 1}$, $T_\lambda : V \to V$ is scalar multiplication, $\varphi_\lambda : A \to A$ is a homomorphism. (look at notes)

Now let's define a new product of vector spaces, the tensor product, which is universal for bilinear forms.

Definition 1.3. Let V' and V'' be vector spaces. A **tensor product** (X, b) of V', V'' is a vector space X and a bilinear map $b: V \times V'' \to X$ such that for all (W, B),



We denote $X = V' \otimes V''$, and $b(\xi', \xi'') = \xi' \otimes \xi'', \xi' \in V', \xi'' \in V''$.

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If S' is a basis of V', S" a basis of V", then $S' \times S''$ is a basis of $V' \otimes V''$, where

$$S' \times S'' \cong \{ \xi' \otimes \xi'' \mid \xi' \in S', \xi'' \in S'' \}.$$

Note that \otimes is "commutative" and "associative" with unit \mathbb{R} , so

$$\mathbb{R} \otimes V \to V$$

$$V_1 \otimes V_2 \to V_2 \otimes V_1$$

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),$$

forming what we call a **symmetric monoidal category**. We write $\otimes^1 V = V$, $\otimes^2 V = V \otimes V$, $\otimes^3 V = V \otimes V \otimes V$ and so on. We also write $\otimes^0 V = \mathbb{R}$, and sometimes replace $\otimes^n V$ with $V^{\otimes n}$.

1.2 Existence of tensor algebras

Let V be a vector space, and $A = \bigoplus_{k=0}^{\infty} \otimes^k V$. Let $i: V \hookrightarrow A$ be the inclusion into $\otimes' V = V$.

Claim. (A, i) is a tensor algebra over V.

To see this, note that

$$\xi_1 \otimes \cdots \otimes \xi_k) \cdot_A \eta_1 \otimes \cdots \otimes \eta_\ell = \xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_1 \otimes \cdots \otimes \eta_\ell \in \otimes^{k+\ell} V.$$

Note that $A = \otimes' V$ is *not* commutative.

1.3 The Exterior Algebra

We want to impose todo:come back

- Lecture 2 -

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todo:see notes on chapter 21, multivariate analysis

2.1 Exterior algebra of a direct sum

Definition 2.1. Let V be a vector space. An **exterior algebra** (E, j) over V is an algebra E and a linear map $f: V \to E$ satisfying $f(\xi)^2 = 0$ for all $\xi \in V$ such that for all pairs (B, T) consisting of an algebra E and a linear map E: E and E satisfying E and E such that E and E such that E and E such that E and E are E and E such that E and E are E are E and E are E and E are E are E are E are E and E are E are E and E are E and E are E are E are E and E are E and E are E and E are E are E are E and E are E are

Let L_1, L_2 be linear, and $\bigwedge^* (L_1 \oplus L_2 = V)$.