Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source code: `https://git.simonxiang.xyz/math_notes/file/freshman_year/algebraic_topology/master_homework.tex.html`

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§1 August 26, 2020: Homework 0

§1.1 Question 1

Problem. Prove that the finite product of manifolds is a manifold.

Proof. We prove $M \times N$ is a manifold, where M is an m-manifold and N is an n-manifold, which is a sufficient condition for the finite product

$$\prod_{i=1}^{n} M_i$$

to be a manifold for M_i a m_i-manifold, $m_i \in \mathbb{N}$. First, note that the product of two T_2 spaces is T_2. Take τ_1, τ_2 to be topological spaces, and let X be their product. We have two distinct points $(a, b), (c, d)$ in X, which we can separate by open sets $X_1 \times U_2$, $X_1 \times V_2 \in X$ for $X_1 \in \tau_1$, $U_2, V_2 \in \tau_2$ if $a = c$ (which implies $b \neq d$), and $U_1 \times X_2$, $V_1 \times X_2$ for $U_1, V_1 \in \tau_1$, $X_2 \in \tau_2$ if $a \neq c$.

Now let (a, b) be in X, where a is in τ_1 and b is in τ_2. Then there exist $U_1 \in \tau_1$, $U_2 \in \tau_2$ such that $a \in U_1$, $b \in U_2$, and U_1 homeomorphic to $\mathbb{R}^m$, U_2 homeomorphic to $\mathbb{R}^n$. Simply take the open set $U_1 \times U_2 \in X$ containing the point (a, b), and define the homeomorphism $f : M \times N \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) = (g(x), h(y))$, where g and h are the homeomorphisms of U_1 onto $\mathbb{R}^m$ and U_2 onto $\mathbb{R}^n$ respectively. Clearly f is continuous since g and h are continuous, and by the same logic f^{-1} exists and is continuous, and is given by $f^{-1}(x, y) = (g^{-1}(x), h^{-1}(y))$ (whose components are continuous since g and h are homeomorphisms).

Finally, we have $\mathbb{R}^m \times \mathbb{R}^n$ homeomorphic to $\mathbb{R}^{m+n} = \mathbb{R}^{n+m}$, so we conclude the product manifold $M \times N$ is indeed an $n + m$-manifold. ⊠

§1.2 Question 2

Problem. Prove that a manifold is connected if and only if it is path-connected.

Proof. First, note that every path-connected space is connected. By way of contradiction, assume that a path-connected topological space (X, τ) is not connected, that is, there exist $U, V \in \tau$ such that $U \cap V = \emptyset$ and $U \cup V = X$.

Recall that an *interval* is a set $I \subset \mathbb{R}$ such that for all $a, b \in I$, $a < x < b$ implies $x \in I$. Furthermore, all intervals are connected (we omit the proof for brevity). Since τ is path-connected, for all $x, y \in X$ there exists a path $f : [a, b] \to X$ such that f is continuous and $f(a) = x$ and $f(b) = y$. Now the image of the path denoted $f([a, b])$ is connected, since the image of a connected set under a continuous function is connected. Choose $x \in U$ and $y \in V$: then the path f cannot connect x and y since $U \cap V = \emptyset$, and $f([a, b])$ must either be fully contained in U or V. Therefore path-connected spaces (and manifolds) are connected, proving the reverse implication.

For the forward implication, let $a \in M$. Consider X, the set of points that are path-connected to a. Note that $a \in X$ so $X \neq \emptyset$ (this is important). We claim X and X^c are open: to see this, let $x \in X$. Then we have an open neighborhood of x homeomorphic to $\mathbb{R}^n$, let us denote its image under the homeomorphism f as $U \subset \mathbb{R}^n$. We can find a convex neighborhood of $f(x)$ denoted $B(f(x), \epsilon) \subset U$ that is path-connected by definition. Since path-connectedness is preserved under a continuous map, the inverse image of the convex neighborhood containing $f(x)$ under the homeomorphism f denoted $f^{-1}(B(f(x), \epsilon))$ is path-connected. Note that $x \in f^{-1}(B(f(x), \epsilon))$, so there exists a path between every point in $f^{-1}(B(f(x), \epsilon))$ and x, therefore $x \in f^{-1}(B(f(x), \epsilon)) \subset X$ and is open. Since for all $x \in X$ we have $x \in X^\circ$, we conclude X is open. A similar argument follows for the fact that X^c is open: simply examine $y \in X^c$ and $B(f(y), \delta)$ instead.

We reach the final stage of this proof. By assumption, our manifold M is connected. This is equivalent to the fact that the only subsets of M that are both open and closed are M and $\emptyset$: if there existed an $A \subset M$ that were both open and closed, then $A \cap A^c = \emptyset$ and $A \cup A^c = M$, contradicting the fact that M is connected. Now we have constructed a path-connected set X that is both open and closed— both X and X^c are open, and $X \neq \emptyset$ as stated earlier in the proof. We conclude that $X = M$, and so the manifold M is path-connected. $\boxtimes$

§1.3 Question 3

Problem. Suppose a finite group G acts on a manifold M. Suppose the action is *free*, meaning that only the identity element has any fixed points. Then the orbit space M/G is also a manifold. ("Lying in the same G-orbit" is an equivalence relation on M. M/G means the set of equivalence classes. The topology on M induces one on M/G, which is the one you must work with.)

Proof. Why helpppppp

G acts on M: a map $* : G \times M \to M \ni ex = x \forall x \in M$, $*(g1, g2)x = *(g1 * (g2, x)) \forall g_1, g_2 \in G$, $x \in M$ or alternatively $(g_1 g_2)x = g_1(g_2 x) \forall x \in M$, $g_1, g_2 \in G$. M is a G-set.

Free action: $g \in G \wedge \exists x \in X \ni gx = x \implies g = e$.

Orbit of $x \in X$: $Gx = \{gx \mid g \in G\}$ for some $x \in X$. x $\sim$ y iff $\exists g \in G \ni gx = y \implies$ orbits are equivalence classes under this relation.

Orbit space: set of all equivalence classes (under the same orbit relation), denoted X/G (also called quotient, space of coinvariants).

Induces a topology: Recall the quotient map induces the quotient topology, e.g. $U \in G/M$ open iff $f^{-1}(U) \in G$ open.

Things to consider: G is finite, M is a manifold (locally Euclidian), the action is *free*, WTS: M/G is locally euclidian, T2, etc.

Let G be a finite group acting on a manifold M. Consider the quotient map $p : M \to M/G$, where p maps elements of M to their respective G-orbit in G/M. Then p induces the desired topology on the quotient space M/G (that is, sets of orbits are open if their union is open in M).

First, we show G/M is T_2. Since the action of G is free, we have $\boxtimes$

§2 August 29, 2020: Homework 1

Hatcher Chapter 0 (p. 18): 1, 3ab, 17,
Hatcher Section 1.1 (p. 38): 3, 6, 7, 16.

§2.1 Question 1

Problem 1. Suppose X, Y are compact Hausdorff spaces and $f\colon X \to Y$ is continuous and onto. Define $\sim$ as the equivalence relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

(a) Prove the quotient space $X/\sim$ is Hausdorff.

(b) Use this to show that the induced map $X/\sim \to Y$ is a homeomorphism.

(c) Show that identifying the ends of the interval gives S^1.

(d) Give a cooler example.

Solution. We do this by

(a) Let $[a], [b]$ be elements (equivalence classes) of the quotient space $X/\sim$. We want to separate $[a]$ and $[b]$ by open sets: since t

(note X is T_4 and you can also separate compact sets). Canonical projection map is a quotient map: therefore it maps closed sents onto closed (and preimage of closed is also closed)

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§2.2 Problem 1 Chapter 0

Problem. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

§2.3 Problem 3a

Problem. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

§2.4 Problem 3b

Problem. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

§2.5 Problem 17a

Problem. Show that the mapping cylinder of every map $f\colon S^1 \to S^1$ is a CW complex.

§2.6 Problem 17b

Problem. Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

§2.7 Problem 3 Section 1.1

Problem. For a path-connected space X, show that $\pi(X)$ is abelian if and only if all basepoint-change homeomorphisms β_h depend only on the endpoints of the path h.

§2.8 Problem 6

Problem. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $\left(S^1, s_0\right) \to (X, x_0)$. Let $\left[S^1, X\right]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to \left[S^1, X\right]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $\left[S^1, X\right]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

§2.9 Problem 7

Problem. Define $f\colon S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on both boundary circles. [Consider what f does to the map $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.

§2.10 Problem 17

Problem. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.