Math Club Talks

Simon Xiang

The UT Math Club meets weekly and invites speakers to give talks every Tuesday at 5:00 PM! Here are some notes I've TeX'd up from some of them (not all). Source: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

Lecture 1

4-Manifolds (10/13/20)

Today's speaker is Kai Nakamura, a 3rd(?) year Ph.D student.

Definition 1.1 (Manifolds). A **topological manifold** is a space that locally looks like $\mathbb{R}^n$ (locally Euclidian), and some other stuff too (Hausdorff, charts, etc). For example, S^n. A **smooth structure** on a topological manifold is essentially a way to do calculus on such space. For example, we do multivariable calculus on $\mathbb{R}^n$ with the standard smooth structure. So a **smooth manifold** is a topological manifold with a smooth structure.

We want to study these manifolds up to some notion of equivalence, so we study the manifolds up to **homeomorphism** (continuous map with continuous inverse) and **diffeomorphism** (differentiable map with differentiable inverse).

Example 1.1. Topologists say that S^1 and the ellipse are the same thing, because we can find a homeomorphisms. Similarly, $\mathbb{R}$ with the smooth structure and the open interval $(0, 1)$.

1.1 Classifying manifolds

In the 2-dimensional case, this is the classification of surfaces. The one dimensional case is pretty trivial, for example a line, open, half open, closed interval etc. Some examples of surfaces include $S^2, \mathbb{T}$, and the genus n-surface. These are also all smooth.

Classification of Surfaces. *Every closed orientable surface is homeomorphic or diffeomorphic to one of the examples we mentioned above.*

Something fun to think about: torus with a hole in it. In three dimensions, the most famous result is probably Poincare's conjecture (no longer a conjecture), which asks whether a simply-connected closed 3-manifold homeomorphic to a sphere. This is true! And Grigory Perelman ran off into the woods and was never to be seen again. Simply connected means π_1 is trivial.

In dimension 7, Milnor (1956) constructed smooth manifolds M that are homeomorphic to S^7 but not diffeomorphic. This shocked a lot of people, because in dimensions two or three we automatically get a smooth structure.

Smale (1961) proved the topological Poincare conjecture for all dimensions greater than or equal to 5, that is, a smooth n-manifold homotopy equivalent to S^n is homeomorphic to S^n. Although the theorem gives a homeomorphism, it actually uses smooth techniques. If you interset two spheres you can try to find a boundary circle (then rotate it?). This works for dimensions 5 and higher, and is known as **Whitney's trick**. However, this doesn't work for 4-manifolds.

Freedman (1981) showed that this worked topologically, which resulted in a classification of topologically simply connected closed 4-manifolds, also showing the topological 4-dimensional Poincare conjecture. This worked topologically, so maybe this works smoothly too?

Donaldson (1982) analyzed the anti-self dual Yang-Mills equations on a smooth 4-manifold to prove his famous Diagonalization theorem. The takeaway is that it showed the Whitney trick can't work smoothly in dimension four, unfortunately.

1.2 Exotic $\mathbb{R}^4$

These are examples of manifolds homeomorphic to $\mathbb{R}^4$, but not diffeomorphic to $\mathbb{R}^4$. In other words, there are different ways to do calculus on $\mathbb{R}^4$. This is a huge shock, because it doesn't happen in other dimensions, since smooth manifolds homeomorphic to $\mathbb{R}^n \implies$ diffeomorphic to $\mathbb{R}^n$ if $n \neq 4$. In fact, there are uncountably many distinct exotic $\mathbb{R}^4$. When you get to universal exotic $\mathbb{R}^4$ it gets very strange, since you have uncountably many submanifolds that aren't homeomorphic to each other. Furthermore we have small exotic $\mathbb{R}^4$, which are smooth manifolds of $\mathbb{R}^4$-std that are exotic $\mathbb{R}^4$.

1.3 The Conway knot is (k)not slice

Say we have the Conway knot C, then it has a sibling (mutation) the Kinoshita-Terasaka knot, by flipping a region around.

Definition 1.2 (Knot)**.** A **knot** K is an embedding $S^1 \hookrightarrow S^3$, where $S^3 = \partial B^4$, $D^2 \hookrightarrow B^4$, $\partial D^2 \hookrightarrow K \subseteq B^4$. If the disk above doesn't intersect itself as an embedding, we say the knot is **slice**.

The Kinoshita-Terasaka knot is slice. For a long time, it was an open problem whether the Conway knot was slice. Usually mutants provided a pretty good picture, but it didn't work, and neither did a bunch of invariants. For example $T(C) = 0, S(C) = 0$, nobody could figure it out. For a while, this was the last knot with 11 crossing that we hadn't classified it yet.

Lisa Piccirillo's Proof. Consider the disk embedded in S^4, with an equatorial S^3. If it's slice, it must bound a disk here. If the knot is slice, we can thicken up the disk, and take the union of B^4 (hemisphere) and union it with the thickened slice disk. This is called the **knot trace** $X(K)$. A knot K is slice iff $X(K) \subseteq S^4$. Now asking whether C is slice is the same thing as asking whether $X(C)$ embeds in S^4. Piccirillo's insight was to use another knot D such that $X(C) \cong X(D)$ a diffeomorphism. Now is D slice? $S(D) = 2 \implies D$ is not slice, so $X(D)$ doesn't embed in S^4 and so $X(C)$ doesn't embed in S^4, therefore C is not slice.

Lecture 2

Curvature and Gauss-Bonnet (1/26/21)

Today's speaker is Rok Gregoric, a 4$^{\text{th}}$ year PhD student. Today we'll be talking about some differential geometry!

Question. What is the sum of angles in a triangle? π! Or is it?

Imagine a globe, start at the north pole and walk to the equator. Then walk to the north pole, and this makes a triangle with entirely right angles. The correct statement is that the sum of a triangle with *straight edges* in *the plane* is π. Our goal is to remove these two hypotheses.

Corollary 2.1. *The sum of the angles of an n-gon with straight sides in the plane is* $(n-2)\pi$.

Proof. Assume our polygon is convex. Then pick a point, and connect it to each edge by straight lines, giving a triangulation of the n-gon. Do some stuff and it will work out. ⊠

Definition 2.1. The **geodesic curvature** of a curve at a point $p = u(s)$ is $\kappa_g(p) = \|\frac{d\vec{t}}{ds}\|$.

This makes sense by considering the radius of an osculating circle. Fun stuff! Let $\vec{v} \in T_pS$ be a *covariant (directional) derivative* of a function f on S, then

$$(\nabla_{\vec{v}} f)(p) := \text{projection onto } T_pS \text{ along } \vec{n} \text{ of } (D_{\vec{v}} f)(p).$$

This defines an endomorphism, and so $\kappa(p) = \det\big(T_pS \to T_pS, \vec{v} \mapsto \nabla_{\vec{v}}\vec{n}\big)$.

We have some approaches to geodesic curvature, like $\kappa_g(p) = \frac{1}{r} = |\nabla_{\vec{t}}\vec{t}|$. So we can talk about straight lines.

Definition 2.2. A **geodesic** g on a surface S is any curve so that $\kappa_g \equiv 0$. An equivalent definition is that geodesics are length minimizing curves.

Example 2.1. Geodesics in the plane are straight lines, while geodesics on the sphere are great circles.

Definition 2.3. A **polygon** in S is a bounded subsurface $P \subseteq S$, whose boundary curve ∂P is piecewise smooth.

Gauss-Bonnet Theorem. *Let $P \subseteq S$ be a simply-connected n-gon. The sum of its angles φi_i is*

$$\sum_{1 \le i \le n} \varphi_i = (n-2)\pi + \int_{\partial P} \kappa_g \, ds + \int_P \kappa \, dS.$$