Some Problems

Simon Xiang

I'm gonna try to type up my solutions to some problems here. They may or may not be correct.

Contents

Lecture 1

Euclidian Spaces

1.1 Smooth Functions on Euclidian Space

Problem. *Find a function $h\colon \mathbb{R} \to \mathbb{R}$ that is C^2 but not C^3 at $x = 0$.*

Solution. Take $h(x) = x^{5/2}$. Then $h''(x) = \frac{15}{4}\sqrt{x}$, but $h'''(x) = \frac{15}{8}x^{-1/2}$ for $x \neq 0$ and undefined at zero. Therefore h is C^2 but not C^3. ■

Problem. *Define $f(x)$ on $\mathbb{R}$ by*

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0; \\ 0 & \text{for } x \leq 0. \end{cases}$$

(a) *Show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.*

(b) *Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.*

Solution.

(a) We use induction on k. For $k = 1$, we have $f'(x) = \frac{1}{x^2}e^{-1/x}$. In this case, $p_2(y) = y^2$, and so $p_2(1/x) = \frac{1}{x^2}$. Now assume $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y. Then by the product rule,

$$f^{(k+1)}(x) = p'_{2k}(1/x)e^{-1/x} + \frac{1}{x^2}e^{-1/x}p_{2k}(1/x) = e^{-1/x}\left(p'_{2k}(1/x) + \frac{1}{x^2}p_{2k}(1/x)\right).$$

For the sum in the right expression, $p'_{2k}(1/x) + \frac{1}{x^2}p_{2k}(1/x)$ has degree $2(k+1)$: to see this, note that p'_{2k} has degree $2k-1$, so we can forget about it. If $p_{2k}(1/x) = a_{2k}\left(\frac{1}{x}\right)^{2k} + b_{2k-1}\left(\frac{1}{x}\right)^{2k-1} + \cdots$ for constants $a_{2k}, b_{2k}, \cdots$, we have $\frac{1}{x^2}p_{2k} = a_{2k}\frac{1}{x^{2k+2}} + b_{2k}\frac{1}{x^{2k+1}} + \cdots = a_{2k}\frac{1}{x^{2(k+1)}} + \cdots$. So this polynomial has degree $2(k+1)$. Therefore $f^{(k)}(x)$ is of the form $p_{2(k+1)}(1/x)e^{-1/x}$ for some polynomial $p_{2(k+1)}$ of degree $2(k+1)$, and we are done.

(b) Our strategy is to show that $f^{(k)}(x) = 0$ for $x < 0$, $f^{(k)} = 0$, and $\lim_{x\to 0} f^{(k)}(x) = 0$ for $x > 0$. These conditions ensure that f is smooth and the kth derivative vanishes at zero. First, note that $f^{(k)}(x) = 0$ for $x < 0$ by definition. To show $\lim_{x\to 0} f^{(k)}(x) = 0$ for $x > 0$, recall that $f^{(k)}(x) = p_{2k}(1/x)e^{-\frac{1}{x}}$. Using the genius substitution $u = \frac{1}{x}$, we can rewrite this limit as $\lim_{u\to\infty} \frac{p_{2k}}{e^u}$. From here, apply L'Hôpital's rule $2k$ times to get our desired result.

Finally, we show $f^{(k)}(0) = 0$. We do this by induction on k. The base case is true by definition. Assume $f^{(k)}(0) = 0$. Then $f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h-0} = \lim_{h\to 0} \frac{f^{(k)}(h)}{h} = \lim_{h\to 0} \frac{p_{2k}(1/h)e^{-1/h}}{h}$. Once again, make the substitution $u = 1/h$ to get $f^{(k+1)}(0) = \lim_{u\to\infty} \frac{up_{2k}(u)}{e^u} = 0$ by $2k+1$ applications of L'Hôpital's rule. ■

Problem. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a* ***diffeomorphism*** *if it is bijective and has a C^∞ inverse $F^{-1}: V \to U$.*

(a) Show that the function $f : (-\pi/2, \pi/2) \to \mathbb{R},\ f(x) = \tan x$ is a diffeomorphism.

(b) Find a linear function $h: (a, b) \to (-1, 1)$, thus proving that any two finite open intervals are diffeomorphic.

Then the composition $f \circ h: (a, b) \to \mathbb{R}$ is then a diffeomorphism of an open interval to $\mathbb{R}$.

Solution.

(a) We want to show that $\tan x$ is a smooth bijection and has a smooth inverse. Let $\tan(a) = \tan(b)$, then these numbers are associated to the same angle in $(-\pi/2, \pi/2)$, similarly, every real number is mapped onto by an angle in $(-\pi/2, \pi/2)$. For smoothness, note that $\tan'(x) = \sec^2(x)$, $\tan''(x) = 2\sec^2(x)\tan(x)$. From here you can see that the remaining derivatives are all products of sec and tan, which are both defined on $(-\pi/2, \pi/2)$ (since cos never hits zero on this interval). So $\tan x$ is smooth.

The C^∞ inverse has to be $\arctan: (-\pi/2, \pi/2) \to \mathbb{R}$, there are no better candidates. We have $\arctan \circ \tan(x) = \mathrm{id}_{\mathbb{R}}$ by definition, so arctan is an inverse: to see smoothness, note that $\arctan'(x) = \frac{1}{1+x^2}$, $\arctan''(x) = -\frac{2x}{(1+x^2)^2}$, and so on. These functions are all continuous on $(-\pi/2, \pi/2)$, and so arctan is a smooth inverse for tan. Therefore $\tan: (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism.

(b) Consider the function with its graph being a line segment joining $(a, 1)$ to $(b, -1)$. ■

1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

Problem. *Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.*

Solution. Since $Xf = \sum a^i \left(\frac{\partial f}{\partial x^i}\right)$, we have $Xf = x\left(\frac{\partial f}{\partial x}\right) + y\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 2y^2$. ■

Problem. *Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.*

Solution. For reference: C_p^∞ is the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p. A germ is an equivalence class of a pair (f, U) where for U, V nbds of p, two pairs $(f, U), (g, V)$ are related if there exists an open set $W_p \subseteq U \cap V$ such that $f = g$ on W. Let $[f]_p, [g]_p$ be germs of two functions f, g.

We define addition and multiplication by $[f]_p + [g_p] = [f + g]_p$, $[f]_p \times [g]_p = [f \times g]_p$ and scalar multiplication by $a[f]_p = [af]_p$ for $a \in \mathbb{R}$. If $f : U_p \to \mathbb{R}, u \mapsto f(u)$ and $g : V_p \to \mathbb{R}, v \mapsto g(v)$, then we define $f + g : U \cap V \to \mathbb{R}$ by $(f + g)(w) = f(w) +_{\mathbb{R}} g(w)$ and $(f \times g)(w) = f(w) \times_{\mathbb{R}} g(w)$ for $w \in U \cap V$. Addition is commutative because addition in $\mathbb{R}$ is commutative. ■

Problem. *Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that*

(a) the sum $D + D'$ is a derivation at p,

(b) the scalar multiple cD is a derivation at p.

Solution. Recall that derivations are $\mathbb{R}$-linear maps $D\colon C_p^\infty \to \mathbb{R}$ satisfying the Liebniz rule $D(fg) = (Df)g(p) + f(p)Dg$. Let D and D' be derivations. Then $(D+D')(fg) = D(fg) + D'(fg) = (Df)g(p) + f(p)Dg + (D'f)g(p) + f(p)D'g = (Df + D'f)g(p) + f(p)(Dg + D'g) = ((D+D')f)g(p) + f(p)(D+D')g$. So $D+D'$ is a derivation.

On the same vein, consider cD for D a derivation. Then $(cD)(fg) = cD(fg) = c(Df)g(p) + cf(p)Dg = (cD)fg(p) + f(p)(cD)g$, so cD is also a point-derivation at p. ■

Problem. *Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.*

Solution. Let D_1 and D_2 be the standard derivative at p that sends functions to $\mathbb{R}$, and $[f]_1 \in C_p^\infty, f : x \mapsto x^2, [g]_1 \in C_p^\infty, g : x \mapsto x$. If $D_1 \circ D_2$ were a derivation, then $(D_1 \circ D_2)(fg) = (D_1 \circ D_2)(f)g(p) + f(p)(D_1 \circ D_2)(g) = f''g(1) + f(1)g'' = 2$. But in reality, $(fg)'' = 6$. So this is false. ■

1.3 Alternating k-Linear Functions

Problem. *If f is a trilinear function on a vector space V, what is $(Af)(v_1, v_2, v_3)$, where $v_1, v_2, v_3 \in V$?*

Solution. Recall the six elements of S_3 are $1, (12), (23), (13), (123), (132)$. So

$$\begin{aligned}(Af)(v_1, v_2, v_3) = &\sum_{\sigma \in S_3} (\operatorname{sgn}\sigma)\sigma f \\ =&f(v_1, v_2, v_3) - \\ &f(v_2, v_1, v_3) - \\ &f(v_1, v_3, v_2) - \\ &f(v_3, v_2, v_1) + \\ &f(v_2, v_3, v_1) + \\ &f(v_3, v_1, v_2).\end{aligned}$$

■

Problem. *Show that if f, g, h are multilinear functions on V, then $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.*

Solution. Let f, g, h be j, k, and ℓ-linear functions respectively. Then

$$\begin{aligned}(f \otimes g) \otimes h &= \big(f(v_1, \cdots, v_j)g(v_{j+1}, \cdots, v_{j+k})\big) \otimes h \\ &= f(v_1, \cdots, v_j)g(v_{j+1}, \cdots, v_{j+k})h(v_{j+k+1}, \cdots, v_{j+k+\ell}) \\ &= f(v_1, \cdots, v_j)(g \otimes h) \\ &= f \otimes (g \otimes h).\end{aligned}$$

■

Problem. *For $f, g \in A_2(V)$, write out the definition of $f \wedge g$ using $(2,2)$-shuffles.*

Solution. Note that the images of $\{1, 2, 3, 4\}$ under the $(2,2)$-shuffles of S_4 are $1234, 1324, 1423, 2314, 2413, 3412$. So the $(2,2)$-shuffles are $1, (23), (243), (123), (1243)$ and $(13)(24)$.

$$\begin{aligned}f \wedge g = &\sum_{(2,2)\text{-shuffles } \sigma} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, v_{\sigma(2)})g(v_{\sigma(3)}, v_{\sigma(4)}) \\ =&f(v_1, v_2)g(v_3, v_4) - \\ &f(v_1, v_3)g(v_2, v_4) + \\ &f(v_1, v_4)g(v_2, v_3) + \\ &f(v_2, v_3)g(v_1, v_4) - \\ &f(v_2, v_4)g(v_1, v_3) + \\ &f(v_3, v_4)g(v_1, v_2).\end{aligned}$$

■