Differential Geometry Notes

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February 3, 2021

Notes for the Spring 2021 section of Differential Geometry (Math 365G) at UT Austin, taught by Dr. Bowen. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

Curves and Surfaces

1.1 Curves

A curve $\mathscr{C} := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$. Curves in $\mathbb{R}^3$ are defined similarly. These are called **level curves**.

Definition 1.1. A **parametrized curve** in $\mathbb{R}^n$ is a map $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ for some α, β with $-\infty \le \alpha \le \beta \le \infty$. A parametrized curve whose image is contained in a level curve $\mathscr{C}$ is called a **parametrization** of $\mathscr{C}$.

Example 1.1. We parametrize the parabola. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the components γ_1 and γ_2 of γ must satisfy $\gamma_2(t) = \gamma_1(t)^2$ for all $t \in (\alpha,\beta)$. The parametrization $\gamma\colon (-\infty,\infty) \to \mathbb{R}^2$, $\gamma(t) = (t,t^2)$ works, as well as $\gamma(t) = (t^3,t^6)$, $\gamma(t) = (2t,4t^2)$, and so on.

For the circle $x^2 + y^2 = 1$, we could try $x = t$, but that only hits half of S^1. What satisfies $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$? $\gamma_1(t) = \cos t$ and $\gamma_2(t) = \sin t$ do. The interval $(-\infty,\infty)$ is overkill since the map has infinite degree.

Example 1.2. Consider the *astroid* $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in \mathbb{R}$. Since $\cos^2 t + \sin^2 t = 1$ for all t, then $x = \cos^3 t,\ y = \sin^3 t$ satisfy $x^{2/3} + y^{2/3} = 1$.

A function $f : (\alpha,\beta) \to \mathbb{R}$ is **smooth** if $\frac{d^n f}{dt^n}$ exists for all $n \ge 1$ and $t \in (\alpha,\beta)$. Smoothness is preserved under addition, multiplication, composition, etc. You differentiate vector valued functions componentwise, and we denote $d\gamma/dt$ by $\dot\gamma(t)$, $d^2\gamma/dt^2$ by $\ddot\gamma(t)$, etc.

Definition 1.2. If γ is a parametrized curve, then $\dot\gamma(t)$ is the **tangent vector** of γ at the point $\gamma(t)$.

Proposition 1.1. *If the tangent vector of a parametrized curve is constant, then the image of the curve is a straight line.*

Proof. If $\dot\gamma(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is constant, then

$$\gamma(t) = \int \frac{d\gamma}{dt}\,dt = \int \mathbf{a}\,dt = t\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is another constant vector. ⊠

Example 1.3. The **limaçon** is the parametrized curve $\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$, $t \in \mathbb{R}$. There's a self intersection at the origin, the tangent vector is $\dot\gamma(t) = (-\sin t - 2\sin 2t, \cos t + 2\cos 2t)$. This is well defined, but takes two different values at $t = 2\pi/3$ and $t = 4\pi/3$.

1.2 Arc Length

The length of a straight line segment between two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is $||\mathbf{u} - \mathbf{v}||$, given the standard norm/inner product/metric/blah on $\mathbb{R}^n$.

Definition 1.3. The **arc-length** of a curve γ starting at $\gamma(t_0)$ is the function $s(t)$ given by

$$s(t) = \int_{t_0}^{t} ||\dot\gamma(u)||\,du.$$

Example 1.4. For a **logarithmic spiral** $\gamma(t) = (e^{kt}\cos t, e^{kt}\sin t)$, we have $\dot\gamma = (e^{kt}(k\cos t - \sin t), e^{kt}(k\sin t + \cos t))$, so $||\dot\gamma||^2 = e^{2kt}(k\cos t - \sin t)^2 + e^{2kt}(k\sin t + \cos t)^2 = (k^2+1)e^{2kt}$. Then the arc length of γ starting at $\gamma(0) = (1,0)$ is

$$s = \int_0^t \sqrt{k^2+1}\,e^{ku}\,du = \frac{\sqrt{k^2+1}}{k}(e^{kt} - 1).$$

Note that the arc-length is differentiable, that is,

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du = \|\dot{\gamma}(t)\|.$$

Definition 1.4. If $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ is a parametrized curve, its **speed** at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be a **unit-speed** curve if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha,\beta)$.

Proposition 1.2. *Let $\mathbf{n}(t)$ be a unit vector that is a smooth function of parameter t. Then the dot product $\dot{\mathbf{n}}(t)\cdot\mathbf{n}(t) = 0$ for all t, i.e., $\dot{\mathbf{n}}(t)$ is zero or orthogonal to $\mathbf{n}(t)$ for all t. If γ is a unit-speed curve, then $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.*

Proof. We differentiate $\mathbf{n}\cdot\mathbf{n} = 1$ to get $\dot{\mathbf{n}}\cdot\mathbf{n} + \mathbf{n}\cdot\dot{\mathbf{n}} = 0$, so $\dot{\mathbf{n}}\cdot\mathbf{n} = 0$. ⊠

1.3 Reparametrization

A parametrized curve $\tilde{\gamma}\colon (\tilde{\alpha},\tilde{\beta}) \to \mathbb{R}^n$ is a **reparametrization** of a parametrized curve $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta)$ (the *reparametrization map*) such that the inverse map $\phi^{-1}\colon (\alpha,\beta) \to (\tilde{\alpha},\tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $\tilde{t} \in (\tilde{\alpha},\tilde{\beta})$.

Note that since ϕ has a smooth inverse, γ is a reparametrization of $\tilde{\gamma}$, since $\tilde{\gamma}(\phi^{-1}(t)) = \gamma(\phi(\phi^{-1}(t))) = \gamma(t)$ for all $t \in (\alpha,\beta)$.

Example 1.5. We can reparametrize the circle as $\tilde{\gamma}(t) = (\sin t, \cos t)$. To show this, we want to find a reparametrization map ϕ such that $(\cos\phi(t), \sin\phi(t)) = (\sin t, \cos t)$. $\phi(t) = \pi/2 - t$ works.

Definition 1.5. A point $\gamma(t)$ of a parametrized curve γ is called a **regular point** if $\dot{\gamma}(t) \neq \mathbf{0}$; otherwise $\gamma(t)$ is a **singular point** of γ. A curve is **regular** if all of its points are regular.

Proposition 1.3. *Any reparametrization of a regular curve is regular.*

Proof. Suppose $\tilde{\gamma}$ is a reparametrization of γ, let $t = \phi(\tilde{t})$ and $\psi = \phi^{-1}$ such that $\tilde{t} = \psi(t)$. Differentiating both sides of $\phi(\psi(t)) = t$ WRT t gives $\frac{d\phi}{d\tilde{t}}\frac{d\psi}{dt} = 1$. So $d\phi/d\tilde{t}$ is never zero. Since $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$, differentiating again gives $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt}\frac{d\phi}{d\tilde{t}}$, so $d\tilde{\gamma}/d\tilde{t}$ is never zero, if $d\gamma/dt$ is never zero. ⊠

Proposition 1.4. *If $\gamma(t)$ is regular, then s is a smooth function of t.*

Proof. Recall that $\frac{ds}{dt} = \|\dot{\gamma}(t)\| = \sqrt{\dot{u}^2 + \dot{v}^2}$. Since $f(x) = \sqrt{x}$ is *smooth* on $(0,\infty)$, along with u and v, and $\dot{u}^2 + \dot{v}^2 > 0$ for all t (since γ is regular), s itself is also smooth. ⊠

Proposition 1.5. *A parametrized curve has a unit-speed reparametrization iff it is regular.*

Proof. Suppose a parametrized curve $\gamma\colon (\alpha,\beta) \to \mathbb{R}^n$ has a unit-speed reparametrization $\tilde{\gamma}$, with a reparametrization map ϕ. Letting $t = \phi(\tilde{t})$, we have $\tilde{\gamma}(\tilde{t}) = \gamma(t)$ and so

$$\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt}\frac{dt}{d\tilde{t}} \implies \left\|\frac{d\tilde{\gamma}}{d\tilde{t}}\right\| = \left\|\frac{d\gamma}{dt}\right\|\left|\frac{dt}{d\tilde{t}}\right|.$$

Since $\tilde{\gamma}$ is unit speed, $\|d\tilde{\gamma}/d\tilde{t}\| = 1$, so $d\gamma/dt$ cannot be zero. ⊠

Lecture 2

January 20, 2021

We start by talking about curves in space. Differential geometry is about infinitesmal stuff, tangent lines, things like that. Curvature is about approximating things by the radius of a circle, it's pretty intuitive. After curves, we get into surfaces. Geodesics are like the shortest way to connect two points, a locally length-minimizing curve. We have extrinsic and intrisic curvature, which depend and don't depend on embeddings. The natural next step after curves and surfaces is Riemannian geometry (woohoo).

2.1 Curves

We have two kinds of curves: level curves and parametrized curves. A **parametrized curve** is a map $\gamma\colon(\alpha,\beta)\to\mathbb{R}^n$, for example, $\gamma(t)=(t^2,t^3)$. My take on open vs closed intervals: paths take one point to another, while curves describe a, well, curve in $\mathbb{R}^2$. They don't necessarily have to start somewhere or end somewhere, and aren't necessarily compact of course.

A **level curve** is (informally) something of the form $f^{-1}(x_0)$ where $f:\mathbb{R}^n\to\mathbb{R}^{n-1}$, $x_0\in\mathbb{R}^{n-1}$. We usually study the special case $n=2$.

Example 2.1. Precisely, $f^{-1}(x_0)=\{y\in\mathbb{R}^n \mid f(y)=x_0\}$. Take $f:\mathbb{R}^2\to\mathbb{R}, f(x,y)=x^3+y^3-3xy$, this is called the *Foliom of Descartes*.

Usually in this course we study parametrized curves, since they're easy to compute arc length ($\int_{t_0}^t \|\dot\gamma(t)\|\,dt$ and curvature. Meanwhile, level curves are good for applications, as they arise naturally as graphs of functions. If $\gamma\colon\mathbb{R}\to\mathbb{R}^n$, we have $\gamma(t)=(\gamma_1(t),\gamma_2(t),\cdots,\gamma_n(t))$, where $\gamma_1\colon\mathbb{R}\to\mathbb{R}$, $\gamma_2\colon\mathbb{R}\to\mathbb{R}$, and so on. Then the derivative is given by the n-tuple

$$\dot\gamma=\gamma'=\frac{d\gamma}{dt}=\left(\frac{d\gamma_1}{dt},\cdots,\frac{d\gamma_n}{dt}\right).$$

We say γ is **smooth** if $\frac{d^n\gamma}{dt^n}$ exists for all $n\ge 0$. We don't really care about curves that aren't smooth.

2.2 Tangent Vectors

We have $\gamma'(t)$ the **tangent vector** at time t. The **tangent line** at time t is $\{\gamma(t)+u\gamma'(t)\mid u\in\mathbb{R}\}$, the direction is much more important than the magnitude (speed). The **speed** of γ at time t is $\|\gamma'(t)\|$. The **arc length** of γ from time t_0 is

$$s(t)=\int_{t_0}^t \|\dot\gamma(u)\|\,du.$$

Integrating over speed gives distance traveled, which is arc length.

Example 2.2. If $\gamma(t)=(t^2,t^3)$, the length from zero to one is $s(1)=\int_0^1\sqrt{4u^2+9u^4}\,du=\int_0^1 u\sqrt{4+9u^2}\,du=\frac{(4+9u^2)^{3/2}}{27}\Big|_0^1=\text{blah}$.

Note that we can differentiate dot products. Let $\gamma\colon\mathbb{R}\to\mathbb{R}^n$, $\lambda\colon\mathbb{R}\to\mathbb{R}^n$. Then $\phi:\mathbb{R}\to\mathbb{R},\phi(t)=\langle\gamma(t),\lambda(t)\rangle$. How do you compute $\frac{d\phi}{dt}$? It's the product rule, $\frac{d\phi}{dt}=\frac{d\gamma}{dt}\cdot\lambda+\gamma\cdot\frac{d\lambda}{dt}$.

Lecture 3

January 22, 2021

Brief review on dot products. Let $\gamma\colon\mathbb{R}\to\mathbb{R}^n$, $\lambda\colon\mathbb{R}\to\mathbb{R}^n$. Then define a new function $f(t)=\gamma(t)\cdot\lambda(t)$. Precisely, if $\gamma(t)=(\gamma_1(t),\gamma_2(t),\cdots,\gamma_n(t))$, and $\lambda(t)$ is similarly defined, then

$$f(t)=\gamma_1(t)\lambda_1(t)+\gamma_2(t)\lambda_2(t)+\cdots+\gamma_n(t)\lambda_n(t),\ f'(t)=\sum_{i=1}^n\left(\gamma_i'(t)+\lambda_i(t)+\gamma_i(t)\lambda_i'(t)\right)=\lambda'(t)\cdot\lambda(t)+\gamma(t)\cdot\lambda'(t).$$

So $\frac{d(\gamma\cdot\lambda)}{dt}=\frac{d\gamma}{dt}\cdot\lambda+\gamma\cdot\frac{d\lambda}{dt}$.

Proposition 3.1. *Suppose $\|\gamma(t)\|$ is a constant, then $\gamma(t)\perp\gamma'(t)$.*

Proof. We want to show that $\gamma(t)\cdot\gamma'(t)=0$. We have FINISH THIS ⊠

Example 3.1. Let $\gamma(t)=(t,\sqrt{1-t^2})$. We have $\|\gamma\|=1$ at all times, since $\gamma'(t)=\left(1,-\frac{t}{\sqrt{1-t^2}}\right)$, which is orthogonal. Neat visualization!

3.1 Reparametrization

This is in the book notes. In the proof that curves are regular iff they have a unit speed parametrization, one direction is easy.

Lecture 4

January 25, 2021

4.1 Closed curves

Definition 4.1. We say $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is **T-periodic** (where $T > 0$) if $\gamma(T+t) = \gamma(t)$. We say γ is **closed** if it is T-period for some T.

A natural question to ask is whether or not we can parametrize level curves? You know what a gradient is.

Theorem 4.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x,y) \neq \vec{0}$ for all (x,y) with $f(x,y) = 0$. Then for all (x_0, y_0) with $f(x_0,y_0) = 0$, there exists a regular $\gamma\colon (\alpha,\beta) \to \mathbb{R}^2$ such that $\alpha < 0 < \beta$, $\gamma(0) = (x_0,y_0)$ and $f(\gamma(t)) = 0$ for all t.*

Note. The proof uses the inverse function theorem. Note that we can parametrize the entire curve under fairly broad conditions, that is, if $f^{-1}(0)$ is *connected* then we can choose γ to parametrize all of $f^{-1}(0)$.

Assume $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth. A **global inverse** is a map $G\colon \mathbb{R}^n \to \mathbb{R}^n$ with $F \circ G(\vec{x}) = \vec{x}$. A **local inverse** at $\vec{x}$ is a map $G\colon U_{\vec{x}} \to \mathbb{R}^n$ with $F \circ G(\vec{y}) = \vec{y}$ for all $\vec{y}$, where $U_{\vec{x}}$ is a neighborhood of $\vec{x}$. An **infinitesmal inverse** at $\vec{x}$ is a linear map A such that $(D_{\vec{x}}F) \circ A$ is the identity, where $D_{\vec{x}}F$ is the Jacobian matrix.

The Inverse Function Theorem. *If F is smooth and has an infinitesmal inverse at $\vec{x}$, then it has a smooth local inverse at $\vec{x}$.*

Theorem 4.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x,y)$ is not horizontal for all (x,y) with $f(x,y) = 0$, then there exists a regular $\gamma\colon (\alpha,\beta) \to \mathbb{R}^2$ with $\gamma(t) = (t, g(t))$ and $f(\gamma(t)) = 0$ (and $\gamma(0) = (x_0,y_0)$ like in the previous theorem).*

Proof. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x,y) = (x, f(x,y))$. Then

$$DF = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}, \quad \det(DF) = \frac{\partial f}{\partial y} \neq 0.$$

By the inverse function theorem, since DF is invertible, there exists a local smooth inverse G, where $F \circ G(x,y) = (x,y) = (G_1(x,y), f(G_1,(x,y), G_2(x,y)))$. This implies that $G_1(x,y) = x$, $f(x, G_2(x,y)) = y$. Define $\gamma(t) = (t, G_2(t,0))$. Since F and G are smooth, γ is regular, so

$$f(\gamma(t)) = f(t, G_2(t,0)) = 0.$$

Something happened here. ⊠

Lecture 5

January 27, 2021

5.1 Curvature

Definition 5.1. Assume γ is a unit-speed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$. Define the curvature by $\kappa(s) = \|\ddot{}(s)\| = \left\| \left(\frac{d^2}{ds^2}\gamma \right)(s) \right\|$.

Example 5.1. The circle curve $\gamma(t) = (R\cos t, R\sin t)$ is not unit speed. So $\gamma'(t) = (-R\sin t, R\cos t)$, and $\|\gamma'(t)\| = R$. The arclength $s(t) = \int_0^t R\,du = tR$, so $s^{-1}(t) = \frac{t}{R}$. A reparametrization is $\widetilde{\gamma}(t) = (R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)$.

Say $\gamma(s) = \left(R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)\right)$ for simplicity. Then $\dot{\gamma} = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right)\right)$, and $\ddot{\gamma} = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)$. So $\|\ddot{\gamma}\| = \kappa(s) = \frac{1}{R}$.

Parametrizing by arc length is painful. So we can define (if γ is regular) $\kappa = \frac{\|\dot{\kappa}\times\ddot{\kappa}\|}{\|\kappa\|^3}$. This makes life easier, since in this definition, $\kappa(t) = \kappa(s(t))$. What is a cross product?? Let $\vec{v}, \vec{w} \in \mathbb{R}^3$, then $\vec{v} \times \vec{w} \in \mathbb{R}^3$ as well. One way to find the cross product is by computing

$$\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = (v_2w_3 - v_3w_2)\mathbf{i} + (-v_1w_3 + v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}.$$

The cross product is **bilinear**, that is, $(\mathbf{v}+\mathbf{u}) \times \mathbf{w} = \mathbf{v} \times \mathbf{w} + \mathbf{u} \times \mathbf{w}$, and satisfies homogeneity, and antisymmetric like the determinant. Also, $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$, and finally we have the right hand rule.

We can simplify our old formula to $\frac{\|\ddot{\gamma}\|\sin\theta}{\|\dot{\gamma}\|^2}$.

Proof of the formula for curvature. Let $s(t) = \int_{t_0}^t \|\gamma'(u)\|\,du$ be the arc length of a curve, and $\widetilde{\gamma}(t) = \gamma(s^{-1}(t))$. So $\widetilde{\gamma}(s(t)) = \gamma(t)$. Then

$$\widetilde{\gamma}'(s(t))s'(t) = \gamma'(t) \implies \widetilde{\gamma}'(s(t)) = \frac{\gamma'(t)}{s'(t)}.$$

Then $\widetilde{\gamma}''(s(t))s'(t)^2 + \widetilde{\gamma}'(s(t))s''(t) = \gamma''(t)$ by the chain rule. So

$$\kappa(t) = \widetilde{\gamma}''(s(t)) = \frac{\gamma''(t) - \widetilde{\gamma}'(s(t))s''(t)}{s'(t)^2} = \frac{\gamma''(t) - \frac{\gamma'(t)}{s'(t)} \cdot s''(t)}{s'(t)^2}.$$

Recall that $s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\|\,du$, so $s'(t) = \|\dot{\gamma}(t)\|$. We use inner products, now $s'(t)^2 = \|\dot{\gamma}(t)\|^2 = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle$. So differentiating gives $2s'(t)s''(t) = 2\langle\dot{\gamma}(t), \ddot{\gamma}(t)\rangle$. Then $s''(t) = \frac{\langle\dot{\gamma}(t), \ddot{\gamma}(t)\rangle}{s'(t)}$. Plugging everything gives

$$\kappa(t) = \left\|\frac{\ddot{\gamma} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\frac{\dot{\gamma}\cdot\ddot{\gamma}}{\|\dot{\gamma}\|}}{\|\dot{\gamma}\|^2}\right\| = \left\|\left(\frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\frac{\dot{\gamma}\cdot\ddot{\gamma}}{\|\dot{\gamma}\|\|\ddot{\gamma}\|}\right)\right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \left\|\frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\cos\theta\right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \sin\theta \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2}.$$

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Lecture 6

January 29, 2021

Whoops, showed up 30 min late because I was eating breakfast. We're talking about signed curvature.

6.1 Signed Curvature

The idea is to find a unit normal vector for each point on the curve. Then the signed curvature is positive if the curve bends in the direction of the unit normal vector field, and negative if it isn't.