Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1.1 Problem 1 Section 1.2

Problem. *Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.*

Solution. Assume the center of the free product $G * H$ of nontrivial groups is nontrivial, that is, there exists a $z \in Z(G * H)$ such that $zw = wz$ for all $w \in G * H$. WLOG, take a nontrivial reduced word $w \in G * H$ (we can do this because G and H are nontrivial) that ends in $h \in H$. If z ends in $g \in G$, we are done, since zw will end in h while wz will end in g, contradicting the fact that z lies in $Z(G * H)$. If z ends in h, then if the ending letter of w is of the form h^n while the ending letter of z is of the form h^m, zw will end in h^n while wz will end in h^{n+m}, implying that $zw \neq wz$, a contradiction.

Next, we'll show the only elements of $G * H$ of finite-order are the conjugates of finite-order elements of G and H. We'll do this proof by cases.

Case 1: WLOG, $w \in G * H$ starts with $g \in G$ and ends with $h \in H$. Clearly this doesn't terminate as $(g_i \cdots h_i) \cdot (g_i \cdots h_i)$ can't reduce down at $h_i \cdot g_i$, so this product will just keep growing longer with each multiplication.

Case 2: WLOG, $w \in G * H$ starts and ends with g, but the ending term isn't the inverse of the beginning term. That is, $w = g^n \cdots g^m$, but $n + m \neq 0$. So $w^2 = (g^n \cdots g^m) \cdot (g^n \cdots g^m) = g^n \cdots g^{n+m} \cdots g^m$. Since g^{n+m} can't simplify, none of the other terms can, so this product will just keep growing longer to infinity like the previous one.

Case 3: WLOG, $w \in G * H$ starts with g and ends with g^{-1}, but is not the conjugate of some element. In the previous cases, the possibility of w being a conjugate wasn't even there, but now we can consider it (in the next case). For now, assume it isn't: then $(g \cdots g^{-1}) \cdot (g \cdots g^{-1})$ will reduce to $g \cdots 1 \cdots g$, but the middle dots won't reduce because w isn't the conjugate of some element. (Note that w starting with g and ending with g^{-1} is just a special case of w being a conjugate of some element with the $\omega \in G * H$ set to g. This time, the WLOG also includes if w starts with g^{-1} and ends with g).

Case 4: $w \in G * H$ is the conjugate of some element of $G * H$. We'll show that the only element that allows w to terminate are elements of finite-order in G or H. We have $w = \omega a \omega^{-1}$ for $\omega, a \in G * H$. Each multiplication of w gives $w^2 = (\omega a \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^2 \omega^{-1})$, $w^2 = (\omega a^2 \omega^{-1}) \cdot (\omega a \omega^{-1}) = (\omega a^3 \omega^{-1})$, and so on. Therefore $w^n = \omega a^n \omega^{-1}$. If a is a word (as in a is not just an element of G or H), it must be the conjugate of another word, which must be the conjugate of another word, and so on. We can't repeat this forever, so a can't be a word. Now assume a is an element of G or H. Then if a has infinite order in either of these groups, we have no n satisfying $a^n = 1$, which subsequently means $w = \omega a^n \omega^{-1}$ will never terminate. If a has finite order in G or H, then there exists an n such that $a^n = 1$. Then at that n, $w^n = \omega a^n \omega^{-1} = \omega \cdot 1 \cdot \omega^{-1} = 1$. Therefore w has order n if and only if w is the conjugate of some element of finite order in G or H, and we are done. ■

§1.2 Problem 10

Problem. *Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.*

Hint (from Dr. Allcock): this can be done directly with Van Kampen's, but it becomes easier if you manipulate $(D^2 \times I) \setminus (\alpha \cup \beta)$ first, being careful not to change the homotopy type, and carrying along the loop γ.

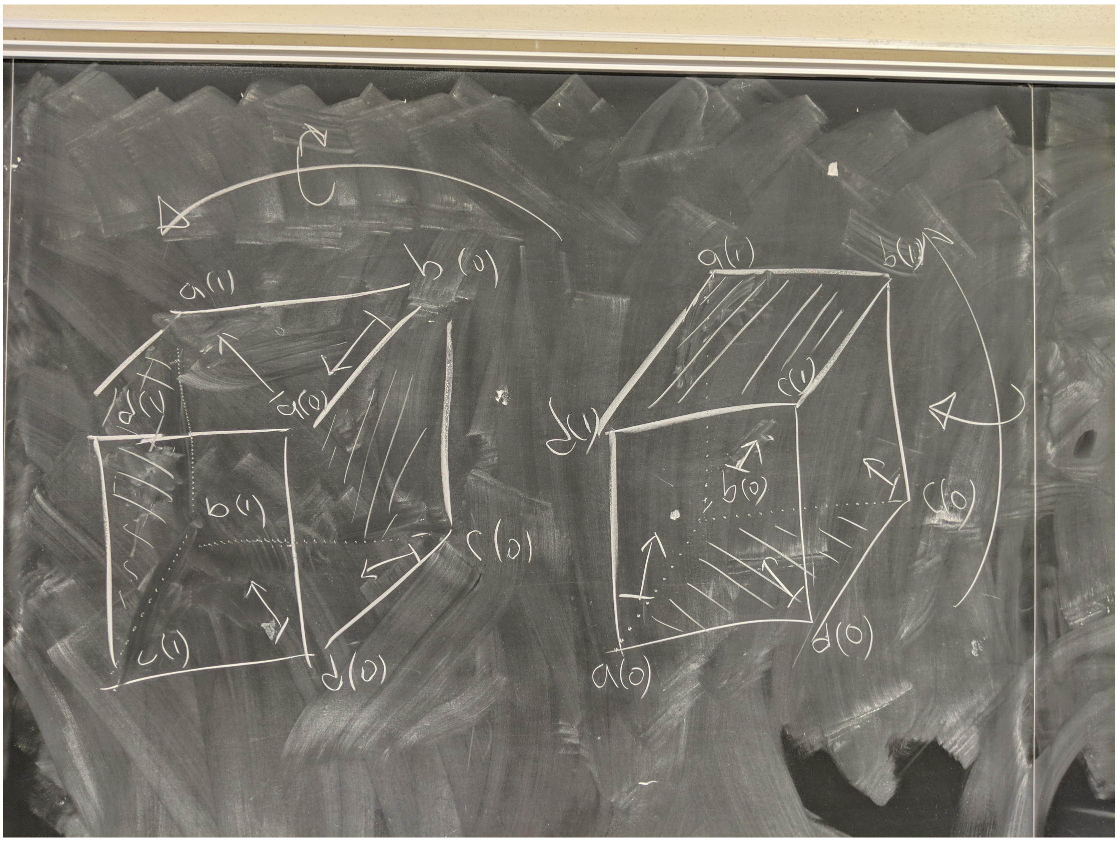
§1.3 Problem 14

Problem. *Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.*

Solution. Recall a group presentation for the quaternion group is given by

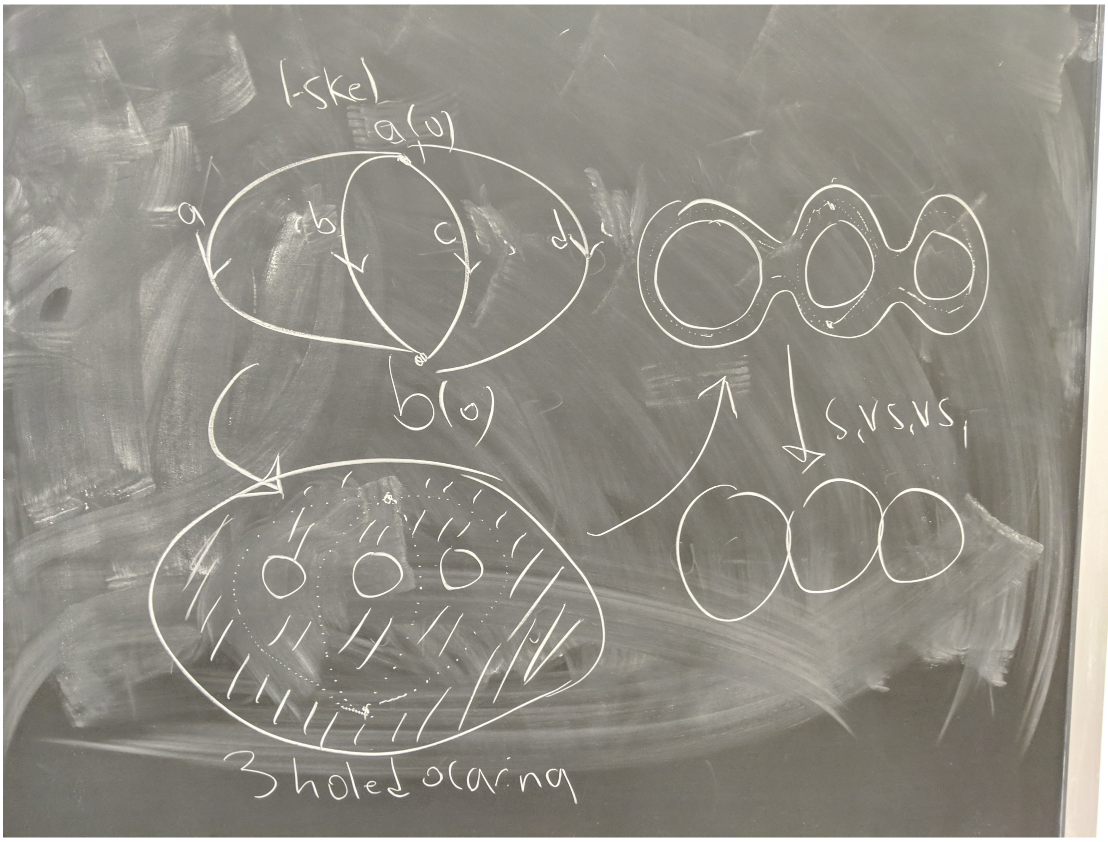
$$\langle -1, i, j, k \mid i^2 = j^2 = k^2 = -1, (-1)^2 = 1 \rangle.$$

Refer to the figure to see what's being identified to what: in the left side, I identified the right face with the left (0 and 1 just denoting what goes where, probably should have used subscripts) and in the right side of the figure I identified the bottom face with the top: together they describe the identification space of I^3.



The cube I^3 has the cell complex structure of eight 0-cells, twelve 1-cells, six 2-cells, and one 3-cell. Applying the first identification will identity eight 0-cells with four, and the second will identify four with two. Similarly, for 1-cells we identify twelve 1-cells onto eight (by mashing four together) and eight onto four (by mashing another four together). Finally, the identification will identify two 2-cells with each other, bringing us from four 2-cells to 3 2-cells. So $I^3/\sim$ has the desired cell complex structure.

We examine the 1-skeleton: See the figure to see why the resultant structure and $S^1 \vee S^1 \vee S^1$ are homotopy equivalent.



Therefore $\pi_1(X^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. We have the loops $a \cdot \overline{b}, a \cdot \overline{c}, a \cdot \overline{d}$. Let i, j, and k denote the homotopy classes of these loops (to make it feel more like the quaternions).

We have three 2-cells attached, being f_1, f_2, and f_3. It can be seen that $f_1 = [a\overline{b}c\overline{d}]$, $f_2 = [c\overline{b}d\overline{a}]$, $f_3 = [d\overline{b}\overline{a}c]$ by following the lines. We can perform some calculations to rewrite these in a form we like. We have

- $f_1 = [a\overline{b}c\overline{d}] = [a\overline{b}c\overline{a}a\overline{d}] = [ij^{-1}k]$.
- $f_2 = [c\overline{b}d\overline{a}] = [c\overline{a}a\overline{b}d\overline{a}] = [j^{-1}ik^{-1}]$.
- $f_3 = [d\overline{b}\overline{a}c] = [d\overline{a}a\overline{b}\overline{a}c] = k^{-1}ij$.

Then by Proposition 1.26, we have

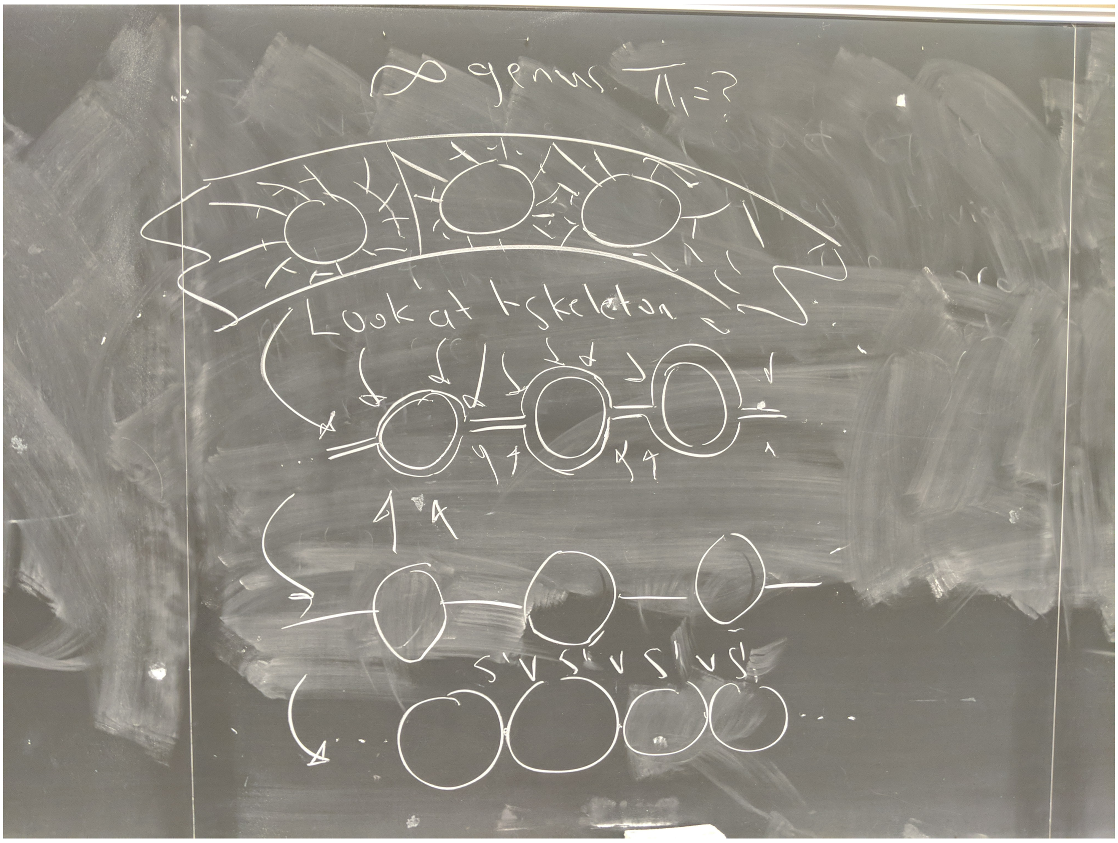
$$\pi_1(X) \simeq \pi_1(X_2) \simeq= \langle i, j, k \mid ij^{-1}k = j^{-1}ik^{-1} = k^{-1}ij = 1 \rangle.$$

This is just the quaternion group in disguise: we have $ki = j, i = kj, ij = k$. So $i^2 = (ik)j = j^2$, $j^2 = k(ij) = k^2$. Therefore $i^2 = k^2 = j^2$, denote this element as -1. Finally, all we have to do is show that $(-1)^2 = 1$ (by the group presentation given at the beginning). We have $(-1)^2 = i^2j^2 = i(ij)j = (k^{-1}k)ikj = k^{-1}(ki)kj = k^{-1}j(kj) = k^{-1}(ji) = k^{-1}k = 1$, and we are done. ■

§1.4 Problem 16

Problem. *Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.*

Solution. I don't have a way to explicitly describe the deformation retraction (since I don't have a way to describe the surface), but this is an image I drew showing that the surface has the same homotopy type as an infinite wedge of S^1's, denoted $S^1 \vee S^1 \vee \cdots$.



(This is why I TeX my homework, by the way). Note that even though the circles are side by side, we can slide them together and glue at a single point. So by Hatcher Example 1.21, the fundamental group of this surface of infinite genus is just

$$\pi_1(\text{surface}) = \pi_1(S^1 \vee S^1 \vee \cdots) = *_\alpha \pi_1(S^1) = \mathbb{Z} * \mathbb{Z} * \cdots .$$

Then π_1 is just the free group on an infinite number of generators (one for each $\mathbb{Z}$), and we are done (although a free product may not always be free, a free product of free groups is, and $\mathbb{Z} \simeq F_1$). ■

§1.5 Problem 21

Problem. *Show that the join $X * Y$ of two nonempty spaces X and Y is simply-connected if X is path-connected.*

Hint (from Dr. Allcock): If you are not comfortable with the join of spaces then wrap your mind around the following examples in order:

1. *Join of two points*
2. *Join of a point and an interval*
3. *Join of a point and a circle*
4. *Join of 2 copies of the interval*
5. *Join of a circle and an interval*
6. *Join of two circles (doesn't embed in $\mathbb{R}^3$, but still understandable).*

That might be enough: if not, work out examples using the figure 8 or S^2.

§1.6 Problem 30 Section 1.3

Problem. *Dray the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.*

§1.7 Problem 5 Section 1.A

Problem. *Construct a connected graph X and maps $f, g \colon X \to X$ such that $fg = \mathbb{1}$ but f and g do not induce isomorphisms on π_1. [Note that $f_* g_* = \mathbb{1}$ implies that f_* is surjective and g_* is injective.]*