Differential Equations Notes

Simon Xiang

Notes for Differential Equations (M 427J) with Dr. Tsishchanka at UT Austin. May follow lectures or textbooks depending on the mood. Source code: `https://git.simonxiang.xyz/math_notes/file/freshman_year/differential_equations/master_notes.tex.html`.

Contents

§1 First Order Linear Differential Equations (8/27/20)

§1.1 Definitions

Definition 1.1 (Order)**.** We have the *order* of a differential equation the highest derivative of a function y that appears in the equation. For example, the order of the differential equation

$$\frac{dy}{dt} = 3y^2 \sin(t + y)$$

is 1, while the order of

$$\frac{d^3y}{dt^3} = e^{-y} + t + \frac{d^2y}{dt^2}$$

is 3. We would call the first example a *first-order differential equation* and the second a *third order differential equation.*

Definition 1.2 (Solution)**.** The *solution* of a differential equation is a continuous function $y(t)$ that together with its derivatives satisfies the given relationship.

Example 1.1. The function

$$y(t) = 2\sin t - \frac{1}{3}\cos 2t$$

is the solution of the second-order differential equation

$$\frac{d^2y}{dt^2} + y = \cos 2t$$

since

$$\begin{aligned} &\frac{d^2}{dt^2}\left(2\sin t - \frac{1}{3}\cos 2t\right) + \left(2\sin t - \frac{1}{3}\cos 2t\right) \\ = &\left(-2\sin t + \frac{4}{3}\cos 2t\right) + 2\sin t - \frac{1}{3}\cos 2t = \cos 2t. \end{aligned}$$

Goal: Given a differential equation of the form

$$\frac{dy}{dt} = f(t, y)$$

and the function $f(t, y)$, find all functions $y(t)$ that satisfy the equation above.

What we have: As of now, all we can solve is a differential equation of the form

$$\frac{dy}{dt} = g(t)$$

given $g(t)$ is integrable. Very sad!

Definition 1.3 (Linear ODE)**.** The general first-order linear differential equation is of the form

$$\frac{dy}{dt} + a(t)y = b(t),$$

where $a(t)$ and $b(t)$ are continuous (assumed to be functions of time).

§1.2 Homogeneous Linear Ordinary Differential Equations

Definition 1.4 (Homogeneous Linear ODE)**.** The equation

$$\frac{dy}{dt} + a(t)y = 0$$

is called the *homogeneous* first-order linear differential equation, and the previous definition is called the *nonhomogeneous* first-order linear differential equation for $b(t)$ not necessarily zero.

Example 1.2. Let us solve the homogeneous first-order linear differential equation. Rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t).$$

Second, note that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt}\ln|y(t)|.$$

Then we can write the differential equation in the form

$$\frac{d}{dt}\ln|y(t)| = -a(t),$$

so we have

$$\ln|y(t)| = -\int a(t)\,dt + c_1.$$

Continuing on,

$$|y(t)| = \exp\left(-\int a(t)\,dt + c_1\right) = c\exp\left(-\int a(t)\,dt\right)$$

or

$$\left|y(t)\exp\left(\int a(t)\,dt\right)\right| = c.$$

Now $y(t)\exp\left(\int a(t)\,dt\right)$ is continuous and we know its absolute value is constant which implies that the function itself is constant (which follows from the IVT, assuming $g(t_1) = c$ and $g(t_2) = -c$ for g a function, c a constant. So we have $y(t)\exp\left(\int a(t)\,dt\right) = c$, or

$$y(t) = c\exp\left(-\int a(t)\,dt\right). \tag{1}$$

Equation (1) is the *general solution* of the homogeneous equation. Note that there exist infinitely many solutions since for all c we have a distinct $y(t)$.

Example 1.3. To solve the Linear ODE

$$\frac{dy}{dt} + 2ty = 0,$$

simply apply Equation (1) to yield

$$y(t) = c\exp\left(-\int 2t\,dt\right) = c\exp\left(-t^2\right).$$

(This is taking too long! I'll type notes with less rigor next time).

§1.3 Initial Value Problems

Usually scientists are not interested in the general solution given by Equation (1), rather we look for solutions to a specific $y(t)$ which at some time t_0 has the value y_0, or we want to determine a $y(t)$ such that

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0.$$

Please accept the derivation that a general solution to this type of problem is

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)\, ds\right) \tag{2}$$

without proof (there is nothing of interest about the derivation process).

Example 1.4. To solve

$$\frac{dy}{dt} + (\sin t)y = 0, \quad y(0) = \frac{3}{2},$$

let $a(t) = \sin t$, $t_0 = 0$, $y_0 = \frac{3}{2}$. Then

$$y(t) = \frac{3}{2}\exp\left(-\int_0^t \sin s\, ds\right) = \frac{3}{2}\exp\left(\cos t - 1\right).$$

Example 1.5. To solve the initial value problem

$$\frac{dy}{dt} + \exp(t^2)y = 0, \quad y(1) = 2,$$

simply PLUG IT IN (reee) to get

$$y(t) = 2\exp\left(-\int_1^t e^{s^2}\, ds\right).$$

Recall that this is the *Gaussian Integral* and can be solved by a change to double integration by polar coordinates (yielding $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$), but in general has no closed form solution.

§1.4 Non-homogeneous Linear Differential Equations

Recall the non-homogeneous linear differential equation of the form $\frac{dy}{dt} + a(t)y = b(t)$. Let $\mu(t)$ be a continuous function. Then multiply both sides by a continuous function $\mu(t)$ to get

$$\mu(t)\frac{dy}{dt} + a(t)\mu(t)y = \mu(t)b(t),$$

which is equivalent to the above form of a non-homogeneous linear differential equation. What we want: $\mu(t)\frac{dy}{dt} + a(t)\mu(t)y$ equal to the derivative of some simple expression. To get this, notice that

$$\frac{d}{dt}\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

by the product rule, so if $\frac{d\mu(t)}{dt} = a(t)\mu(t)$, our expression above will simply be equal to the derivative of $\mu(t)y$. Since our new expression is just a linear homogeneous differential equation, we have

$$\mu(t) = \exp\left(\int a(t)\,dt\right).$$

Now the expressions

$$\frac{d}{dt}\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

and

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t)$$

are equivalent, so we can integrate both sides to obtain

$$\mu(t)y = \int \mu(t)b(t)\,dt + c$$

or

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right) = \exp\left(-\int a(t)\,dt\right)\left(\int \mu(t)b(t)\,dt + c\right).$$

A similar integration between t_0 and t yields

$$\mu(t)y - \mu(t_0)y_0 = \int_{t_0}^{t} \mu(s)b(s)\,ds$$

or

$$y = \frac{1}{\mu(t)}\left(\mu(t_0)y_0 + \int_{t_0}^{t} \mu(s)b(s)\,ds\right),$$

solving initial-value problems.

§1.5 Integrating Factor

Remark: $\mu(t)$ is called an *integrating factor* for the nonhomogeneous equation since after multiplying both sides by $\mu(t)$ we can immediately integrate to find all solutions.

Example 1.6. We find the general solution of the differential equation

$$\frac{dy}{dt} - 2ty = t.$$

We know the integrating factor $\mu(t)$ is equal to $\exp\left(\int -2t\,dt\right) = e^{-t^2}$, so multiplying both sides by $\mu(t)$ yields

$$e^{-t^2}\left(\frac{dy}{dt} - 2ty\right) = e^{-t^2}t$$

which is equivalent to

$$\frac{d}{dt}\left(e^{-t^2}y\right) = e^{-t^2}t$$

by our choice of $\mu(t)$. So

$$\int \frac{d}{dt}\left(e^{-t^2}y\right)\,dt = \int te^{-t^2}\,dt,$$

and by the Fundamental Theorem we have

$$e^{-t^2}y = -\frac{1}{2}e^{-t^2} + c.$$

Finally, we conclude that

$$y = -\frac{1}{2} + ce^{t^2}.$$

Example 1.7. Here we solve an initial value problem. Let

$$\frac{dy}{dx} + xy = xe^{x^2/2}, \quad y(0) = 1.$$

The integrating factor $\mu(x)$ is equal to $\exp\left(\int x\,dx\right) = e^{x^2/2}$. Multiply both sides by $\mu(x)$ to obtain

$$e^{x^2/2}\left(\frac{dy}{dx} + xy\right) = e^{x^2}\left(xe^{x^2/2}\right),$$

which is equivalent to

$$\frac{d}{dx}\left(e^{x^2/2}y\right) = xe^{x^2}$$

by our choice of $\mu(x)$. By the Fundamental Theorem, we have

$$e^{x^2/2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c.$$

Let $x = 0$ and $y = 1$, then $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$. We conclude that

$$y(x) = \frac{1}{2}e^{x^2/2} + \frac{1}{2}e^{-x^2/2}.$$

We can simplify this to

$$y(x) = \frac{1}{2}e^{x^2/2}\left(1 + e^{-x^2}\right).$$