Bordism and TQFTs

A brief introduction to topological quantum field theories

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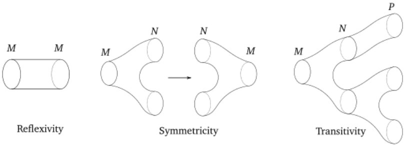
Bordism

Definition

Let Y_0, Y_1 be closed n-manifolds. A **bordism** X from Y_0 to Y_1 is a compact $(n+1)$-manifold X with boundary, a decomposition $\partial X = M_0 \amalg M_1$, and diffeomorphisms $\theta_i \colon Y_i \xrightarrow{\cong} M_i$.



Definition

Let Ω_n denote the set of equivalence classes of n-manifolds under the equivalence relation of bordism. An element of Ω_n is called a **bordism class**.

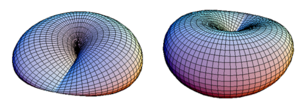
Bordism groups

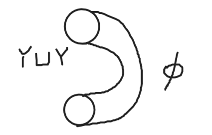
Disjoint union gives Ω_n a *commutative monoid* structure, and for $Y \in \Omega_n$, the manifold $[0,1] \times Y$ is a bordism between $Y \amalg Y$ and the unit $\emptyset^n$. So Ω_n is an *abelian group*.

Example

Some calculations:

- $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ with generator pt. Even points are bordant by intervals, and the single point cannot bound by classification.
- $\Omega_1 \cong 0$. Closed 1-manifolds are copies of circles which bound.
- $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ with generator $\mathbb{RP}^2$.





Symmetric monoidal categories

Definition (Symmetric monoidal categories)

Let C be a category. A **symmetric monoidal structure** on C consists of an object $1_C \in C$, a functor $\otimes\colon C \otimes C \to C$, and natural isomorphisms

$$\begin{tikzcd}[column sep=huge] C \times C \times C \arrow[r, bend left=40, "-\otimes(-\otimes-)"] \arrow[r, bend right=40, "(-\otimes-)\otimes-"'] \arrow[r, phantom, "\Uparrow\alpha"] & C \end{tikzcd},$$

$$\begin{tikzcd}[column sep=huge] C \times C \arrow[r, bend left=40, "(-\otimes-)\circ\tau"] \arrow[r, bend right=40, "-\otimes-"'] \arrow[r, phantom, "\Uparrow\sigma"] & C \end{tikzcd}, \quad \text{and} \quad \begin{tikzcd}[column sep=huge] C \arrow[r, bend left=40, "\mathrm{id}_C"] \arrow[r, bend right=40, "1_C\otimes-"'] \arrow[r, phantom, "\Uparrow\iota"] & C \end{tikzcd}.$$

Some other compatibility axioms are required, like $\sigma^2 = \mathrm{id}$, they essentially say that α, ι, and σ behave well with each other.

Symmetric monoidal categories

Example

Consider $(\mathsf{Vect}_k, \otimes, k)$, then this is a symmetric monoidal category. Here $1_{\mathsf{Vect}_k} \in \mathsf{Vect}_k$ is k, the functor $\otimes \colon \mathsf{Vect}_k \otimes \mathsf{Vect}_k \to \mathsf{Vect}_k$ is the standard tensor product, and the natural isomorphisms exist.

Definition (Symmetric monoidal functors)

Let C, D be symmetric monoidal categories. A **symmetric monoidal functor** $F \colon C \to D$ is a functor with additional data, namely an isomorphism $1_D \to F(1_C)$ and a natural isomorphism

$$\begin{tikzcd} C \times C \arrow[rr, bend left, "F(- \otimes -)"] \arrow[rr, bend right, "F(-) \otimes F(-)"'] \arrow[rr, phantom, "\Uparrow \psi"] & & C \end{tikzcd} .$$

and many compatibility conditions.

Bordism categories

Definition

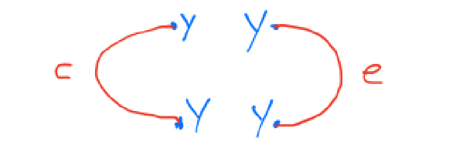
Fix $n \in \mathbb{Z}^{\geq 0}$. The **bordism category** $\text{Bord}_{\langle n-1,n \rangle}$ is the symmetric monoidal category defined as follows.

1. Objects are closed $(n-1)$-manifolds.
2. The hom-set $\text{Bord}_{\langle n-1,n \rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $X \colon Y_0 \to Y_1$.
3. Composition of morphisms is by gluing.
4. For each Y the bordism $[0,1] \times Y$ is $\text{id}_Y \colon Y \to Y$.
5. The monoidal product is disjoint union.
6. The empty manifold $\emptyset^{n-1}$ is the tensor unit (for the symmetric monoidal structure).

Examples of bordism categories

Example

- $\mathrm{Bord}_{\langle -1,0\rangle}$ is a category with a single object $\emptyset^{n-1}$, hence a monoid, namely the set of morphisms $\mathrm{Bord}_{\langle -1,0\rangle}(\emptyset^{-1}, \emptyset^{-1})$. These are finite unions of points with diffeomorphism class $\mathbb{Z}^{\geq 0}$, and composition/disjoint union both induce addition.
- $\mathrm{Bord}_{\langle 0,1\rangle}$ has objects points, with four distinct connected bordisms up to diffeomorphism.



Duality

Definition (Duality data)

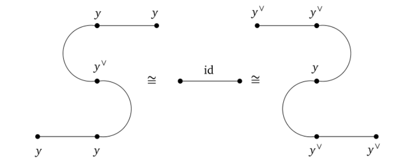
For a symmetric monoidal category C and $y \in C$, we say y is **dualizable** if there exists *duality data* $(y^\vee, c, e)$, where $y^\vee \in C$, $c\colon 1_C \to y \otimes y^\vee$, $e\colon y^\vee \otimes y \to 1_C$, such that

$$\left(y \xrightarrow{c \otimes \mathrm{id}_y} y \otimes y^\vee \otimes y \xrightarrow{\mathrm{id}_y \otimes e} y \right) = \mathrm{id}_y, \tag{1}$$

$$\left(y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes c} y^\vee \otimes y \otimes y^\vee \xrightarrow{e \otimes \mathrm{id}_{y^\vee}} y^\vee \right) = \mathrm{id}_{y^\vee}\,. \tag{2}$$

Example

Recall "evaluation" and "coevaluation" from $\mathrm{Bord}_{\langle 0,1 \rangle}$.



Duality in vector spaces

Example

Let $V \in \mathrm{FdVect}_k$. Then we have duality data consisting of the algebraic dual V^*, along with the following maps:

- $e\colon V^* \otimes V \to k, (f, v) \mapsto f(v)$ (evaluation)
- $c\colon k \to V \otimes V^*, \lambda \mapsto \sum_i \lambda v_i \otimes v_i^*$ (coevaluation)

Send a vector $v_j \in V$ and a covector $f \in V^*$ through the duality data:

$$v_j \to \left(\sum_i v_i \otimes v_i^*\right) \otimes v_j \to \sum_i v_i \otimes \delta_j^i = v_j,$$
$$f \to f \otimes \left(\sum_i v_i \otimes v_i^*\right) \to \sum_i f(v_i) \otimes v_i^* = f.$$

We need V to be *finite dimensional* because otherwise we cannot write down coevaluation, which requires a basis.

TQFTs

Definition (Topological quantum field theories)

Let C be a symmetric monoidal category. Then an **n-dimensional topological quantum field theory** with values in C is a symmetric monoidal functor

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to C$$

Usually we consider Vect_k for $k = \mathbb{C}$. Note that n-dimensional TQFTs form a symmetric monoidal category TQFT_n under the natural tensor product.

More on TQFTs

Example

A closed n-manifold M can be seen as a bordism $\emptyset^{n-1} \to \emptyset^{n-1}$, under a TQFT F this gets send to an endomorphism of k, or a number.

Proposition (Finiteness)

Let $F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ *be a TQFT. Then for all* $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$, $F(Y)$ *is finite dimensional.*

Proof.

Note that a point (manifold) in $\mathrm{Bord}_{\langle n-1,n\rangle}$ is dualizable. Symmetric monoidal functors preserve duality data, so the vector space $F(Y)$ is dualizable, which is true iff $F(Y)$ is finite dimensional. $\boxtimes$

Classification of 1-dimensional TQFTs

Definition (Categorical stuff)

Let C be a symmetric monoidal category. Define C^{fd} as the full subcategory of dualizable objects, and the *groupoid of units* $C^{\sim}$ containing only invertible morphisms.

Theorem (Cobordism hypothesis, 1-categorical version)

Let C be a symmetric monoidal category. Then the map

$$\Phi\colon \mathsf{TQFT}^{\mathrm{or}}_{\langle 0,1\rangle}(C) \to (C^{\mathrm{fd}})^{\sim}, \quad F \mapsto F(\mathrm{pt}_+)$$

is an equivalence of groupoids.