Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html`.

Contents

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§1.1 Group Actions

Definition 1.1 (Group Action)**.** An *action* of a group G on a set X is a map

$$a\colon G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

such that

1. $(1_G, x) \mapsto x$,
2. $g_1(g_2 \cdot x) = (g_1 g_2) \cdot x$

for all $x \in X$, $g_1, g_2 \in G$. Notation: $G \hookrightarrow X$, G acts on X.

Proposition 1.1. Let G be a group and X a set. Actions of G on X ($a\colon G \times X \to X$) are in bijection with homomorphisms $\phi\colon G \to S_X$.

Proof. Given an action $a\colon G \times X \to X$, define $\phi_a\colon G \to S_X$, $g \mapsto (x \mapsto a(g, x))$, $a(g, x) \in X$. Verify that

1. $x \mapsto a(g, x)$ is a bijection on X ($\iff$ $[x \mapsto a(g, x)] \in S_x$),
2. ϕ_a is a homomorphism.

$\boxtimes$

Given $\phi\colon G \to S_X$ a homomorphism, define $a_\phi\colon G \times X \to X$, $(g, x) \mapsto \phi(g)(x) \in X$. We have to verify that

1. a_ϕ is a group action, i.e., a_ϕ is a well-defined map.
2. $a_\phi(1_G, x) = x$. $\phi(1_G)(x) = 1_{S_X}(x) = \mathrm{id}_X(x) = x$.
3. $a_\phi(g_1, a_\phi(??)$

Finally, we must verify that

$$a \mapsto \phi_a \mapsto a_{\phi_a} = a$$

and

$$\phi \mapsto a_\phi \mapsto \phi_{a_\phi} = \phi.$$

§1.2 Orbits and Stabilizers

Given an action $a\colon G \times X \to X$ and an element $x \in X$, we can talk about the *orbit* of this action under x.

Definition 1.2 (Orbits)**.** We define an *orbit* of x as

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

The *stabilizer* of x stabilizes x.

Definition 1.3 (Stabilizer)**.** We define the *stabilizer* of x as

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

Remark 1.1. We have $1_G \in G_x$ for all $x \in X$.

Claim. *G_x is a subgroup of G.*

1. $1_G \in G_x \impliedby (1_G, x) = x$,
2. $g \in G_x, \quad g^{-1} \in G_x$.

Definition 1.4 (Transitive Action)**.** An action is *transitive* if

$$Gx = X$$

for some $x \in X$. Prove that if you have this property *for some* $x \in X$, then this is the same as *every* $x \in X$ having this property.

Lemma 1.1. *If $x, y \in X$ lie in the same orbit (there exists a $g \in G$ such that $gx = y$), then $G_x = g^{-1}G_y g$.*

Proof. Let $h \in G_y \iff hy = y \implies hgx = gx \implies g^{-1}hgx = g^{-1}(gx) = (g^{-1}g)x = x$. So $ghg^{-1} \in G_x$, and $g^{-1}G_y g \in G_x$. ??? $\boxtimes$

Lemma 1.2. *Let $G \hookrightarrow X$. Then two orbits are either equal or disjoint.*

Proof. $G_x \cap G_y \neq \emptyset \implies G_x = G_y$. Let $z \in G_x \cap G_y \implies G_x = G_z = G_y$. $\boxtimes$

General idea of group actions: for every element of the set, you have its stabilizer, and you can look at its orbits (are the same or are they disjoint?).

§1.3 Quotient Group of Orbits

Let $G \hookrightarrow X$, $x \in X$. Consider the map

$$G/G_x \to G_x, \quad gG_x \mapsto g \cdot x.$$

Notice this is well defined because $gh \mapsto gh \cdot x = g(hx) = gx$ since $h \in G_x$.

Claim. *The map $G/G_x \mapsto G_x$ is a bijection.*

Surjectivity follows from the definition of an orbit, and injectivity ... is up to you to prove. (Not hard, think about the definitions). But what does this mean?

Proposition 1.2. If G is finite, then the size of each orbit divides the size of G.

Proof. $x \in X$, $G_x \leftrightarrow G/G_x \implies |G_x| = |G/G_x| \mid |G|$. $\boxtimes$

Example 1.1. Every group acts on itself in three different ways, that is, $G \hookrightarrow X$, $X = G$.

1. Left multiplication: $g \cdot x = gx$,
2. Conjugation: $g \cdot x = gxg^{-1}$,
3. Right multiplication: $g \cdot x = xg^{-1}$ (if we define it as xg some properties of group actions will not hold). Why? $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.

Orbits and Stabilizers WRT the above actions:

1. $Gx = X = G$ for all $x \in X$, $G_x = 1_G$,
2. $Gx =$ conjugacy class of x, $G_x =$centralizer of $x = \{g \in G \mid gx = xg\}$,
3. $Gx = X = G$ for all $x \in X$, $G_x = ??$

Proposition 1.3. Let G be a group of order n, then $G \simeq$ subgroup of S_n.