4-manifolds and Gauge Theory Lecture Notes

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Lecture notes for the Fall 2022 graduate section of 4-manifolds and Gauge theory (Math 392C) at UT Austin, taught by Dr. Perutz. These notes were taken live in class (and so they may contain many errors). Source files: https://git.simonxiang.xyz/math_notes/files.html

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1 Classification problems in differential topology

The central problem of differential topology is the classification up to diffeomorphism of smooth manifolds. An ideal solution might have the following aspects:

- **Models.** A collection $\{X_i\}_{i\in I}$ of smooth connected manifolds of a particular dimension—maybe with some other traits like simple connectivity or compactness, representing all diffeomorphism types without redundancy.
- **Which manifold do I have in my hand?** When given some manifold M, we can decide which X_i it's diffeomorphic to, perhaps by computing invariants. If M is described by a finite set of data we can ask for an algorithm for this determination.
- **Are these two manifolds diffeomorphic?** We can compute invariants to decide this, or use an algorithm if the manifolds are presented in a finite fashion.
- **Families.** We want to understand smooth families, say $\{M_b\}_{b\in B}$ where B is a family. In other words we want to think about smooth fiber bundles $M_b \hookrightarrow \mathcal{M} \to B$, including understanding the homotopy type of $\mathrm{Diff}(M)$.

1.1 Dimension 1

We can say a smooth compact connected 1-manifold M is diffeomorphic to S^1. There is something interesting to ask about smooth families of 1-dimensional manifolds. There is a deformation retraction from $\mathrm{Diff}(S^1)$ to its subgroup S^1, which you can prove. So we have a homotopy equivalence $\mathrm{Diff}\, S^1 \simeq S^1$, and we can use this to show that families of copies of the circle are really interchangeable with unit circle bundles of rank 2 vector bundles. Using this (module an orientability issue), we are looking at $H^2(\text{base})$.

1.2 Dimensions 2 and 3

In dimension 2, we have our conditions being compact, oriented, and connected surfaces for simplicity.

(a) We have standard surfaces Σ_g, the surfaces of genus g.

(b) We can use the Euler characteristic $\chi(\Sigma) = 2-2g$ for Σ_g.

(c) The algorithmic aspect is also satisfied by the Euler characteristic.

(d) $\mathrm{Diff}^+(\Sigma)$ is the group of orientation preserving diffeomorphism. The mapping class group $\pi_0\,\mathrm{Diff}^+\,\Sigma$ (which is complicated). However, $\mathrm{Diff}_0(\Sigma)$ has diffeomorphisms isotopic to id.

  - The inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{Diff}^+(S^2)$ is a homotopy equivalence.
  - Writing $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, the inclusion $T^2 \to \mathrm{Diff}_0(T^2)$ (where T^2 acts on itself by translations) is a homotopy equivalence, while $\pi_0\mathrm{Diff}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$ (via the action of the mapping class group on $H_1(T^2;\mathbb{Z}) = \mathbb{Z}^2$).
  - For $g(\Sigma) > 1$, $\mathrm{Diff}_0(\Sigma)$ is contractible.

All of this "generalizes" to dimension 3 through Thurston's geometrization. The basic point is that for M^3 a closed 3-manifold, $\pi_1 M$ is (nearly) a complete invariant.[1]

[1] Minus the lens spaces, but those are truly exceptions.

1.3 Higher dimensions

The desired requirements are ambitious in higher dimensions. One issue is that arbitrary finitely presented groups can be represented by a manifold, and there are algorithmic problems with such groups (the word problem). Restricting to $\pi_1 = 1$, there are uncountably many 4-manifolds diffeomorphic to $\mathbb{R}^4$. But there are countably many (connected) compact manifolds. *Surgery theory* tells us there are excellent conceptual answers to classify of simply connected compact ($n \geq 5$)-manifolds. We need Poincaré duality, a tangent bundle, and "surgery obstruction". A candidate tangent bundle T must obey a constraint given by the *Hirzebruch index theorem*. Surgery theory says this is sufficient to produce a manifold, and how many.

All of this breaks down in dimension 4.

2 Classifying 4-manifolds

This is arguably the biggest mystery in geometric topology.

Question. What is the classification of closed, simply connected, smooth 4-manifolds up to diffeomorphism?

Why do we say simply connected? We want to avoid algorithmic decidability issues involving π_1 of multiply connected manifolds. The question of whether $\langle g_1, \cdots, g_\varphi \mid r_1, \cdots, r_q \rangle$ is the trivial group is an algorithmically unsolvable problem. There are ways to build a 4-manifold with fundamental group isomorphic group, so a classification scheme runs into a lot of danger of being a solution to this algorithmically unsolvable problem.

It turns out that the oriented homotopy type X of such a manifold (smooth or not) can be encoded in a *unimodular* symmetric bilinear form σ_X over $\mathbb{Z}$. What is a symmetric bilinear form over $\mathbb{Z}$? We need

- an abelian group H,
- a function $\sigma \colon H \times H \to \mathbb{Z}$ that is linear in each variable, and $\sigma(x, y) = \sigma(y, x)$.

Unimodular means that the map $H \to \mathrm{Hom}(H, \mathbb{Z}), x \mapsto \sigma(x, -)$ to its dual is an isomorphic. This implies that H is free abelian of finite rank. The claim is that attaching such a form knows the homotopy type of X. Namely, $H_X = H^2(X, \mathbb{Z})$ and $\sigma_X(x, y) = \langle (x \smile y) \in H^4(X, \mathbb{Z}), [X] \in H_4(X, \mathbb{Z}) \rangle$ where $[X]$ is the fundamental class for the orientation and $\langle\rangle$ is the evaluation pairing. This actually makes sense for a simply connected 4-dimensional Poincaré complex, which is a 4-dimensional CW complex that obeys Poincaré duality. Part of the story of surgery theory is that we start off with a blob that becomes a 4-manifold, and it turns out that blob needs to be a Poincaré complex. It follows from Poincaré duality and the universal coefficients theorem that this pairing is unimodular.

One way of getting at it is to say that $H^2(X) \cong_{\mathrm{PD}} H_2(X)$, so σ_X is a pairing on second homology. In these terms, σ_X is the *intersection form*. If Σ_1 and Σ_2 are closed oriented surfaces embedded in X (smooth), then $\sigma_X(\mathrm{PD}[\Sigma_1], \mathrm{PD}[\Sigma_2]) = \Sigma_1 \cdot \Sigma_2$ (where $\cdot$ is the oriented intersection number). To make things concrete, we can express a unimodular symmetric bilinear form (H, σ) in terms of a symmetric matrix Q over $\mathbb{Z}$. We pick an integer basis $(e_1, \cdots, e_j)$ for H, then $Q_{ij} = \sigma(e_i, e_j)$. Unimodularity implise that $\det Q = \pm 1$, and this pairing is a reversible process. Since we can change the basis, a unimodular form gives us a matrix Q as above modulo $\mathbb{Z}$-equivalence, where $Q \sim M^T Q M$ where $M \in \mathrm{GL}(\mathbb{Z}^b)$.

A more precise version of the question is this:

Question. (i) Which unimodular forms (H, σ) arise as intersection forms of simply connected smooth closed oriented 4-manifolds?

(ii) What are the possible diffeomorphism types of 4-manifolds representing a given form?

Working with topological simply connected, oriented, closed 4-manifolds up to homeomorphism, Mike Freedman from the early 80s showed that all unimodular forms arise. For example, the E_8 lattice and the Leech lattice all arise as topological 4-manifolds. When smoothable, 4-manifolds are homeomorphic iff they have isomorphic intersection forms. Essentially, Freedman showed that the theory of surgery which breaks down in 4 dimensions, can be amended when working with topological manifolds up to *homeo*morphism rather than smooth manifolds up to *diffeo*morphism.

We have formed a question and solved it in the topological category so to speak. Let's see what's different in the smooth category. Not all unimodular forms arise from *smooth* manifolds.

Theorem 2.1 (Rokhlin, 1952). *Suppose that X^4 is a smooth, closed, oriented, simply connected 4-manifold with* even *intersection form ($\sigma_X(x,x) \in 2\mathbb{Z}$). Then the* signature *(number of positive eigenvalues of Q minus the number of eeigenvalues) of σ_X is divisible by 16.*

If we take the signature of an even unimodular form, this is always divisible by 8 (by algebra). For example the E_8 form is positive definite, so the signature and rank are both 8. Rokhlin says that the for the form coming from a 4-manifold, looking at its signature and dividing by 8 is not merely an integer but an *even* integer. So E_8 cannot arise, but $E_8 \oplus E_8$ possibly could. There are many proofs but none are easy; some use the Pontryagin-Thom construction to make a connection with the stable homotopy group $\pi_{3+n}(S^n) \cong \mathbb{Z}/24$. We will prove this theorem but not by that route.

Matters stood here for more or less three decades. In 1982, this graduate student Simon Donaldson came up with an unbelievable beautiful proof of an unbelievable beautiful theorem.

Donaldson's Diagonalizability Theorem. *If a positive definite ($\sigma(x,x) > 0$ for all $x \neq 0$) unimodular form σ arises as the intersection form of a smooth closed oriented simply connected 4-manifold, then σ can be represented by the identity matrix.*

The number of isomorphism classes of unimodular positive definite forms of rank 32 is greater than 10^{16}. Donaldson says that precisely one of those come up as a 4-manifold. The proof uses **gauge theory**, the geometric analysis of a PDE with gauge symmetry, in particular the Yangs-Mills instanton. We will prove this but not using the Yangs-Mills instanton, but an alternative method brought in by Witten in 1994, the **Seiberg-Witten equations**, which arose by the work of Seiberg and Witten in string theory. These new equations bypassed many of the difficulties of instantons.

In summary:

(i) Nearly complete answers arise via SW theory.

(ii) SW *invariants* distinguish many diffeomorphism types.

A complete answer is not known and will require completely new ideas.

3 Basics of 4-manifold topology

Example 3.1. Let's get going with some first examples of closed simply connected 4 manifolds.

- Our first example is S^4. It's oriented as a hypersurface in $\mathbb{R}^5$.
- Our next example is $\mathbb{C}\mathrm{P}^2 = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^\times = S^5/\mathrm{U}(1) = \{\text{lines in } \mathbb{C}^3\}$. It has a CW structure wherer $\mathbb{C}\mathrm{P}^2 = e_0 \cup e_2 \cup e_4$, where the 0-cell and 2-cell make up a copy of $\mathbb{C}\mathrm{P}^1$ sitting in $\mathbb{C}\mathrm{P}^2$, which makes clear that $\pi_1(\mathbb{C}\mathrm{P}^2)$ is trivial (there are no 1-cells). It's canonically oriented, since its tangent spaces are *complex* (2-dimensional) vector spaces.

- Recall that $\mathbb{C}\mathrm{P}^2$ comes with a canonical orientation, so we regard it as an oriented manifold. Then there is a manifold $\overline{\mathbb{C}\mathrm{P}^2}$ which is $\mathbb{C}\mathrm{P}^2$ with the "wrong" orientation. It turns out not to be oriented homotopy equivalent to $\mathbb{C}\mathrm{P}^2$.
- Products of spheres $S^2 \times S^2$ are 4-manifolds, which can be oriented as a compact surface. There also exists $\overline{S^2 \times S^2}$, but there is an orientation reversing diffeomorphism of the 2-sphere—the antipodal map. So these two manifolds are actually diffeomorphic through $a \times \mathrm{id}$.
- There is also $S^1 \times S^3$, which is not simply connected since $\pi_1(S^1 \times S^3) = \mathbb{Z}$.

If X_1, X_2 are connected *oriented* smooth n-manifolds, we can form a new smooth n-manifold called the **connected sum** $X_1 \# X_2$ defined up to diffeomorphism. In dimension 2, we remove a coordinate disk from each, attached to it a cylinder, and identify the cylinders. More precisely, take oriented charts $\chi_i : D^n \hookrightarrow X_i$ $(i = 1, 2)$. Pick $\rho \in \mathrm{O}(n)$ where $\rho : \mathbb{R}^n \to \mathbb{R}^n$, $\det(\rho) = -1$, e.g. $\rho(x_1, \cdots, x_n) = (-x_1, x_2, \cdots, x_n)$. Let $X_i^\circ = X_i \setminus \chi_i(\frac{1}{2}D^n)$. So we throw out a smaller disk, and set

$$X_1 \# X_2 = X_1^\circ \cup X_2^\circ / \sim, \quad \chi_1(x) \sim \chi_2(\rho x), \quad x \in D^n \setminus \frac{1}{2}D^n.$$

The tricky thing is that there is an orientation reversing map built into this gluing. Well-definedness up to oriented diffeomorphism is not obvious. The fact that the connected sum doesn't depend on charts is not a triviality by any means.

3.1 (Co)homology

Now that we are confident that simply connected 4-manifolds exist, let us proceed. Let's start talking about homology and cohomology. We begin by talking about singular homology $H_*(X) = H_*(\to \cdots \to S_2(X) \to S_1(X) \to S_0(X) \to 0)$, where singular chains are maps $\Delta^n \to X$, and cohomology $H^*(X)$, the homology of the dual complex $H^*(\mathrm{Hom}(S_* X, \mathbb{Z}))$.

Recall the notion of an orientation for a topological n-manifold X (usually we think of orientations on tangent spaces). For $K \subset K$, write $H_*(X \mid K) = H_*(X, X \setminus K)$ for the relative homology of X with respect to K. We have local homology defined as $H_n(X \mid x)$, where $x \in X$. Cycles for this homology group are n-simplices in X whose boundary lies in the complement of x. If $x \in U \subset X$ where $U \cong \mathbb{R}^n$ is open, then we have

$$H_n(X \mid x) \underset{\text{excision}}{\overset{\cong}{\longleftarrow}} H_n(U \mid x) \cong H_n(\mathbb{R}^n \mid 0) \cong \mathbb{Z},$$

where the generator is a cycle around the origin. Then X leads to a family of copies of $\mathbb{Z}$ given by $\{H_n(X \mid x) \mid x \in X\}$, and an **orientation** is given by a *coherent* choice of generators for these groups. todo:details Closely related, a **fundamental class** $[X]$ is an element $[X] \in H_n(X)$ whose images in $H_n(X \mid x) \cong \mathbb{Z}$ are generators. Then it is clear that a fundamental class leads to an orientation, and a fundamental class exists only if X is compact.

Theorem 3.1. *Let X be a connected topological n-manifold. Then*

- $H_i(X) = 0$ *for all* $i > n$.
- *If X is non-orientable or non-compact, then* $H_n(X) = 0$.
- *If X is orientable and compact, then $H_n(X) \cong \mathbb{Z}$ generated by a fundamental class.*

Poincaré Duality Theorem. *For X a compact n-manifold, an orientation determines an isomorphism $D_X : H^k(X) \xrightarrow{\cong} H_{n-k}(X)$, where $D_{\overline{X}} = -D_X$.*

We will not name the isomorphism now since we don't have the tools to set it up yet. For X^4 a closed, connected, oriented 4-manifold, where does it have interesting homology and cohomology potentially?

- $H^0(X) \cong H_4(X) \cong \mathbb{Z}[X]$
- $H^1(X) \cong H_3(X)$
- $H^2(X) \cong H_2(X)$—note that these two are isomorphic.
- $H^3(X) \cong H_1(X)$
- $H^4(X) \cong H_0(X) = \mathbb{Z}[\text{pt.}]$

If X is simply connected, then $H_1(X) \cong H^3(X)$ becomes trivial, as well as $H^1(X) \cong H_3(X)$. So all the juice is in the H_2^2 example.

4 (Co)homology of 4-manifolds

Office hours are Monday from 3-4 PM in PMA 10.136.

Recall from algebraic topology that for any path connected space X, the Hurewicz map $h\colon \pi_1(X,x)^{\text{ab}} \to H_1(X)$ induces an isomorphism. For $[S^1] \in H_1(S^1)$, we have $h([\gamma]) = \gamma_*[S^1] \in H_1 X$, so $\gamma\colon (S^1,*) \to (X,x)$. If one looks in Hatcher, the general idea is that as a map into an abelian group, the Hurewicz map necessarily kills commutators, and that in fact is all it does. A lesser known "dual" fact is that

$$H^1(X) \cong \mathrm{Hom}(\pi_1(X),\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X),\mathbb{Z}) \cong \mathrm{Hom}(H_1X,\mathbb{Z}).$$

For example, if $\pi_1 = \{1\}$, then $H_1 = H^1 = 0$.

4.1 Universal coefficients

Suppose we look at the second cohomology of a space $H^2(X)$, and specifically we look at the torsion subgroup $H^2(X)_{\text{tors}}$. Universal coefficients says that this is isomorphic to homology one degree down, or $H^2(X)_{\text{tors}} \cong H_1(X)_{\text{tors}}$. On the other hand, if we mod out by torsion we get

$$H^2(X)/\text{tors} \xrightarrow[\text{eval pairing}]{\cong} \mathrm{Hom}(H_2X,\mathbb{Z}).$$

Recall the list of possible homology and cohomology groups for a closed, oriented, connected, topological 4-manifold X.

- $H^0(X) \cong H_4(X) \cong \mathbb{Z}[X]$, which is canonical. There is a cocycle unit that sends a point to one. $H_4(X) \cong \mathbb{Z}[X]$ depends on the fundamental class, which depends on orientation.
- $H^1(X) \cong H_3(X)$, where $H_1 = \mathrm{Hom}(\pi_1,\mathbb{Z})$.
- $H^2(X) \cong H_2(X)$—note that these two are isomorphic.
- $H^3(X) \cong H_1(X)$, where $H_1 = \pi_1^{\text{ab}}$.
- $H^4(X) \cong H_0(X) = \mathbb{Z}[\text{pt}]$, this depends on the orientation.

So the key invariants are π_1 and $H_2 = H^2$. When X is simply connected, the table simplifies to the following:

- $H^0(X) \cong H_4(X) \cong \mathbb{Z}[X]$
- $H^1(X) \cong H_3(X) = 0$.
- $H^2(X) \cong H_2(X)$.
- $H^3(X) \cong H_1(X) = 0$.

- $H^4(X) \cong H_0(X) = \mathbb{Z}[\text{pt.}]$

Applying universal coefficients, we see that $H^2_{\text{tors}} = (H_1)_{\text{tors}} = 0$, and $H^2 = \text{Hom}(H_2, \mathbb{Z})$. So H^2 is a free abelian group identified with its dual via Poincaré duality.

Example 4.1. Let's run through our basic stock of examples (subscripts denote homology in that dimension).

- S^4 has the CW decomposition is $e_0 \cup e_4$, so $H_*(S^4) = \mathbb{Z}_0 \oplus \mathbb{Z}_4$.
- $\mathbb{C}\text{P}^2 = e_0 \cup e_2 \cup e_4$, and $H_*(\mathbb{C}\text{P}^2) = \mathbb{Z}_0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong H^*(\mathbb{C}\text{P}^2)$.
- $S^2 \times S^2 = (e_0 \times e_0) \cup (e_0 \times e_2) \cup (e_2 \times e_0) \cup (e_2 \times e_2)$ as a product of $S^2 = e_0 \times e_2$. Reading off the homology, we have $H_*(S^2 \times S^2) = \mathbb{Z}_0 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_4$.
- If X_1, X_2 are 4-manifolds as above (smooth, closed, oriented, connected), we discussed how to form their connected sum $X_1 \# X_2$ so they meet along a copy of the 3-sphere. Using the Mayer-Vietoris sequence we can check that $H_i(X_1 \# X_2) = H_i(X_1) \oplus H_i(X_2)$, where $0 < i < 4$. To do this exercise, think about the effect of passing $X_i \rightsquigarrow X_i^\circ$, which itself is a MV exercise.

We can write down more examples—if we wanted to look at n copies of connected sums of $\mathbb{C}\text{P}^2$'s with n copies of the other orientation of $\mathbb{C}\text{P}^2$, we have

$$H_*\left(\overset{m}{\#}\mathbb{C}\text{P}^2 \overset{n}{\#}\overline{\mathbb{C}\text{P}^2}\right) = \mathbb{Z}_0 \oplus \mathbb{Z}_2^{m+n} \oplus \mathbb{Z}.$$

Furthermore, $S^2 \times S^2, \mathbb{C}\text{P}^2 \# \mathbb{C}\text{P}^2, \mathbb{C}\text{P}^2 \# \overline{\mathbb{C}\text{P}^2}$, and $\overline{\mathbb{C}\text{P}^2} \# \overline{\mathbb{C}\text{P}^2}$ have the same H_* (additively).

4.2 Multiplicative structure

The main thing is the **cup product**. For all spaces Y we have the singular chain complex $(S_*(Y), \partial)$. We have the singular *co*chain complex (S^*Y, δ), where $S^n(Y) = \text{Hom}(S_n Y, \mathbb{Z})$, and $\delta = \partial^\vee$. Then there is the cup product map, which is the cochain map $S^*Y \otimes S^*Y \to S^*Y$. There is a result called the **Eilenberg-Zilber Theorem**, which concerns natural chain maps $S_*(Y \times Z) \xrightarrow{\lambda} S_*(Y) \otimes S_*(Z)$ wrt spaces Y, Z. The condition we want to impose is that there results a map on H_0 such that $\lambda[\text{pt}, \text{pt}] \to [\text{pt}] \otimes [\text{pt}]$. The theorem says that:

(i) λ exists (and is standard, the so called Alexander-Whitney map)
(ii) λ is unique up to natural chain homotopies
(iii) λ is a natural chain homotopy equivalence

The point is that we get maps $S^*Y \otimes S^*Y \xrightarrow{\lambda^\vee} S^*(Y \times Y)$ sending tensor products of *co*chains to *co*chains of a product. Then apply the *diagonal* map diag: $Y \to Y \times Y, y \mapsto (y, y)$. In summary, this leads to a map

$$\smile : S^*Y \otimes S^*Y \xrightarrow{\lambda^\vee} S^*(Y \times Y) \xrightarrow{\text{diag}^*} S^*(Y)$$

inducing inducing the cup product

$$\smile : H^pY \otimes H^qY \to H^{p+q}Y, \quad [\alpha] \otimes [\beta] \mapsto [\alpha \smile \beta]$$

by passing to representatives. The cup product has the properties of being

- Associative, makes $H^*Y = \bigoplus_{p \geq 0} H^pY$ a graded ring,
- Unital ($1 \in H^0Y$),
- Graded commutative, $b \smile a = (-1)^{|a||b|} a \smile b$.

We still need to discuss the **cap product**, which is a graded module over $(H^*Y, \smile)$, as follows:

$$\frown : H^p(Y) \otimes H_q(Y) \to H_{q-p}(Y),$$

where $(a \smile b) \frown h = a \frown (b \frown h)$. There is a similar construction using the Eilenberg-Zilber theorem which we will not write out now. However we can better state what the Poincaré duality theorem says now.

Poincaré Duality. *For X^n a closed oriented topological n-manifold, the map*

$$D_X : H^k(X) \to H_{n-k}(X), \quad D_X(a) = a \frown \underset{\in H_n(X)}{[X]}$$

is an isomorphism.

If now X is smooth (as well as the other conditions), say $Y^{n-j} \hookrightarrow X^n, Z^{n-k} \hookrightarrow X^n$ are compact oriented embedded submanifolds. We have fundamental classes $[Y] \in H_{n-j}(X), [Z] \in H_{n-k}(X)$. Suppose these have Poincaré duals $D_X^{-1}[Y] \in H^j(X), D_X^{-1}[Z] \in H^k(X)$. We can now cup together their homology classes to get $D_X^{-1}[Y] \smile D_X^{-1}[Z] \in H^{j+k}(X)$. The assertion is that this class is Poincaré dual to $[Y \cap Z]$ (dimension $n-(j+k)$) assuming $Y \pitchfork Z$. So Poincaré duality is just intersecting submanifolds. Using this we can compute cup products in the cohomology rings of our basic 4-manifolds, which we will do next time.

5 Intersection forms

Tim forgot to show up to class today. Take M to be a closed oriented $2n$-manifold. Then we have the cup product pairing on $H^n(M)$, where $x, y \in H^nM$, $\mathbb{Z} \ni x \cdot y = \langle (x \smile y) \in H^{2n}(M), [M] \in H_{2n}(M) \rangle$. This pairing is skew-symmetric when n is odd, and symmetric when n is even, i.e. when $\dim M$ is a multiple of 4. It is closely related to Poincaré duality, namely:

$$\underbrace{(x \cdot y)}_{\in \mathbb{Z}} \underset{\in H_0 M}{[\mathrm{pt}]} = \langle x \smile y, [M] \rangle [\mathrm{pt}] = \underbrace{(x \smile y) \frown [M]}_{H_0 M} = x \frown (y \frown [M]) = x \frown D_M y.$$

So the cup product makes H_{-*} a graded module over H^*, where $(x \cdot y) = \langle x \in H^n, D_M y \in H_n \rangle \in \mathbb{Z}$. We discussed this last time, the cup product in H^* leads to Poincaré duality for intersections in H_n. In dimension 4, we have $x \in H^2(M), y \in H^2(M)$ Poincaré dual to $[\Sigma], [\Sigma'] \in H_2(M)$. Here $\Sigma, \Sigma' \subset M$ are oriented compact embedded surfaces. The claim is that $x \cdot y$ is given by the intersection number $\#\Sigma \cap \Sigma'$, provided that $\Sigma \pitchfork \Sigma'$.

At $x \in \Sigma \cap \Sigma'$, we have $T_x M = T_x \Sigma \oplus T_x \Sigma'$. Compare orientations $o_{M,x}$ and $o_{\Sigma,x} \oplus o_{\Sigma',x}$. Attach a sign $\varepsilon_x = \pm 1$ according to whether orientations match or not, so $\#\Sigma \cap \Sigma' = \Sigma_{x \in \Sigma \cap \Sigma'} \varepsilon_x$. Can we always find such represented surfaces? The answer is yes, but we will not discuss this today.

Example 5.1. Let us examine $\mathbb{C}\mathrm{P}^2$ as a complex surface. We have $T_z \mathbb{C}\mathrm{P}^2$ a complex vector space. Any finite dimensional complex vector space V is *oriented* as a real vector space. If $(e_1, \cdots, e_d)$ is a complex basis, we have an oriented real basis $(e_1, ie_1, e_2, ie_2, \cdots, e_d, ie_d)$. We also have $H_2(\mathbb{C}\mathrm{P}^2) \cong H^2(\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z}$, generated by $[\mathbb{C}\mathrm{P}^1] = [L]$ for any (projective) line $L \subset \mathbb{C}\mathrm{P}^2$ (since $\mathbb{C}\mathrm{P}^2 = \mathbb{P}(\mathbb{C}^3), L = \mathbb{P}(V)$ for V a dimension 2 complex subspace).

When trying to apply the intersection number method, the immediate problem is that $\mathbb{C}\mathrm{P}^1 = \ell$ is not transverse to itself. This is easy to fix, just take a second representative $\ell \cdot \ell = [L] \cdot [L'] = \pm 1$ for lines $L \pitchfork L'$.

Notation. Let Q_M denote the intersection (cup product) pairing on $H_2(M) = H^2(M)$ for M^4.

So $(H_2(\mathbb{C}\mathrm{P}^2), Q_{\mathbb{C}\mathrm{P}^2}) \cong (\mathbb{Z}, \times)$, the matrix in basis $1 \in \mathbb{Z}$, or $[1]$.

Example 5.2. For $\overline{\mathbb{C}\mathrm{P}^2}$, we have $Q_{\overline{\mathbb{C}\mathrm{P}^2}} = (\mathbb{Z}, [-1])$.

Example 5.3. For $S^2 \times S^2$, a basis for H_2 is $h = [S^2 \times \mathrm{pt}], v = [\mathrm{pt} \times S^2]$. They intersect transversely at one place, so $h \cdot v = \pm 1$. These are complex submanifolds, so their tangent spaces are invariant by multiplication by i. So $h \cdot v = +1$. What about $h \cdot h$? Resolving the lack of transversality is done by perturbing it to be $h \cdot h = \#(S^2 \times x) \cap (S^2 \times y)$ for $x \neq y$. This intersection is empty and tranverse, and is 0. Similarly $v \cdot v = 0$. So the matrix for $Q_{S^2 \times S^2}$ in basis (h, v) is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which is a basic example of a hyperbolic quadratic form. This is an *even* form, where $Q_{S^2 \times S^2}(c, c) \in 2\mathbb{Z}$. To see this, note that $Q(ab + bv, ab + bv) = 2ab \in 2\mathbb{Z}$. In contrast, $Q_{\mathbb{CP}^2}$ is *odd* (i.e. not even), since the self intersection of the line is 1.

Note. $H_2(M_1 \# M_2) = H_2(M_1) \oplus H_2(M_2)$.

We deduce that $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$. This becomes clear when you know all H_2-classes are representable by surfaces. For example, for $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ we have $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, which is odd. So there exists no (oriented)[2] homotopy equivalence $S^2 \times S^2 \longleftrightarrow \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. So the intersection form is an invariant of oriented manifolds modulo oriented homotopy equivalence.

5.1 Non-degeneracy

For A an abelian group, write $A' = A/A_{\mathrm{tors}}$. The assertion is that for M^4, the intersection pairing O_M is a non-degenerate pairing on the second homology modulo torsion $H_2(M)'$. That is to say, $Q_M : H_2(M)' \times H_2(M)' \to \mathbb{Z}$, and the map $H_2(M)' \to \mathrm{Hom}((H_2 M)', \mathbb{Z}), h \mapsto Q_M(h, -)$ is an isomorphism of abelian groups, which follows readily from Poincaré duality. In the prescence of a fundamental group we may have to kill torsion, but as we discussed, when $\pi_1 M = 0$, the torsion subgroup of H_2 is 0. So Q_M is non-degenerate on $H_2 M$ itself.

6 The intersection form over $\mathbb{R}$

Today we will talk about the intersection form of closed 4-manifolds, its signature, and its cobordism invariance.

6.1 Sylvester's law of inertia

In the 19th century, "inertia" was used in roughly the same way as "invariance" today. Let $(V, \cdot)$ be a symmetric bilinear form $[V \ni (x, y) \mapsto (x \cdot y) \in \mathbb{R}]$ on a finite dimensional vector space $V/\mathbb{R}$. Assume $(V, \cdot)$ is non-degenerate. Then there exists a basis $(v_1, \cdots, v_r)$ in which the matrix of the form is

$$\langle v_i \cdot v_j \rangle_{i,j} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

We have that $\langle V, \cdot \rangle$ determines (p, q), and any maximal positive definite spanning $\{v_1, \cdots, v_p\}$ (resp negative definite spanning $\{v_{p+1}, \cdots, v_q\}$) subspace of V has dimension p (resp q).

Sketch of proof. Over any field k with $\mathrm{char}(k) \neq 2$, we can find a basis $(e_1, \cdots, e_r)$ in which the matrix $(v_i \cdot v_j)_{i,j}$ is *diagonal* (orthonormal basis). Over $\mathbb{R}$, set $f_i = e_i / \sqrt{|e_i \cdot e_i|}$. Then $(f_i \cdot f_j)_{i,j} = \mathrm{diag}(\pm 1, \cdots, \pm 1)$. Rearrange the basis to make the matrix diagonal as so:

$$\mathrm{diag}(\underbrace{1, \cdots, 1}_{p}, \underbrace{-1, \cdots, -1}_{q})$$

[2] For M_1, M_2 closed oriented n-manifolds, an **oriented homotopy equivalence** is a homotopy equivalence $h \colon M_1 \to M_2$ with the property that the pushforward $h_*[M_1] = [M_2]$.

Call this basis (v_i). The next claim is that if $V = V^+ \oplus V^-$ where V^+ is positive definite, $\oplus$ is orthogonal, and V^- is negative definite, then $\dim V^+ = p, \dim V^- = q$. To prove this, note that $V^+ \cap \operatorname{span}\{v_{p+1}, \cdots, v_{p+q}\} = 0$. Use a dimension estimate to get $V = V^+ \oplus \operatorname{span}\{v_{p+1}, \cdots, v_{p+q}\}$ and $\dim V^+ = p$.

The final claim is that a maximal positive definite subspace has dimension p, and the reason is that its orthogonal complement has a splitting like above. ⊠

Define the signature $\tau(v, \cdot) = p - q = \dim V^+ - \dim V^-$. Complete invariants for a real non-degenerate symmetric bilinear forms are of rank $r(V, \cdot) = \dim = p + q$, where $\tau(V, \cdot) = p - q$. Note that the rank $r(V, -\sigma) = r(V, \sigma)$ doesn't care about the sign, while $\tau(V, -\sigma) = -\tau(V, \sigma)$.

So if M^4 is a closed oriented 4-manifold, it has an intersection form $\cdot$ on its second integer homology $H_2(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$, hence a real valued non-degenerate form on $H_2(M; \mathbb{R}) = H_2(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H^2(M; \mathbb{R}) = H^2(M, \mathbb{Z}) \otimes \mathbb{R}$ (extending integers). This has a rank $\dim H^2(M; \mathbb{R}) = b^2(M)$ (the second Betti number) which has signature $\tau(M)$.

Example 6.1. Some examples:

- For $\mathbb{C}\mathrm{P}^2$, $Q_{\mathbb{C}\mathrm{P}^2} = [I_1], \tau = 1$. In general, for an orientation reversed manifold the sign of the fundamental class is flipped, so $\tau(-M) = -\tau(M)$, eg $\tau(\overline{\mathbb{C}\mathrm{P}^2}) = -1$.
- For $\tau(S^2 \times S^2)$, the intersection form is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and has signature 0. Or, there exists an orientation reversing diffeomorphism $S^2 \times S^2 \to S^2 \times S^2$implies that $\tau = -\tau$, or $\tau = 0$.
- Last time we saw that $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$, so $\tau(M_1 \# M_2) = \tau(M_1) + \tau(M_2)$. For example, $\tau\left(p\mathbb{C}\mathrm{P}^2 \# q\overline{\mathbb{C}\mathrm{P}^2}\right) = p - q$, while $b^2\left(p\mathbb{C}\mathrm{P}^2 \# q\overline{\mathbb{C}\mathrm{P}^2}\right) = p + q$. So an oriented 4-manifold $p\mathbb{C}\mathrm{P}^2 \# \overline{q\mathbb{C}\mathrm{P}^2}$ determines p, q. If $p\mathbb{C}\mathrm{P}^2 \# q\overline{\mathbb{C}\mathrm{P}^2} \cong r\mathbb{C}\mathrm{P}^2 \# s\overline{\mathbb{C}\mathrm{P}^2}$, then $p = r$ and $q = s$.

6.2 Cobordism invariance of τ

We need to set up Poincaré-Lefschetz duality. For N a compact oriented $(d+1)$-manifold with boundary ∂N, there are canonical isomorphisms as follows: $H^k(N) \cong H_{d+1-k}(N, \partial N)$ and $H_k(N) \cong H^{d+1-k}(N, \partial N)$. The picture is as follows: for a closed manifold, the intersection pairing is on cycles with complementary dimension. With boundary, we can let one of the loops run into the boundary.

There is a commutative diagram with exact rows as follows; we have $\partial N \overset{i}{\hookrightarrow} N^{d+1}$. We write the exact sequence of the pair $(N, \partial N)$ in cohomology. We have

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(N, \partial N) \arrow[r] \arrow[d, "\cong"'] & H^k(N) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^k(\partial N) \arrow[r] \arrow[d, "\cong"', "D_{\partial N}"] & H^{k+1}(N, \partial N) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{d+1-k}(N) \arrow[r] & H_{d+1-k}(N, \partial N) \arrow[r, "\partial"] & H_{d-k}(\partial N) \arrow[r, "i^*"] & H_{d-k}(N) \arrow[r] & \cdots
\end{tikzcd}$$

So the Poincaré-Lefschetz duality isomorphisms fit nicely into a commutative diagram.

Theorem 6.1. *Say N^{2n+1} is a compact oriented manifold with boundary $i\colon \partial N \hookrightarrow N$ where ∂N has dimension $2n$ (here H^* uses real coefficients). Define*

$$L = \operatorname{im}(i^*\colon H^n(N) \to H^n(\partial N)) \subseteq H^n(\partial N).$$

Then the following holds:

(i) *L is isotropic for the cup product form, or $x, y \in L$ implies that $x \smile y \in H^{2n}(\partial N)$ is 0.*

(ii) $\dim L = \frac{1}{2} \dim H^n(\partial N)$.

Proof. The proof of (i) is easy; take $x, y \in H^n(N)$ and cup together their restrictions to get

$$i^*x \cdot i^*y = \langle i^*x \smile i^*y, [\partial N]\rangle = \langle i^*(x \smile y), [\partial N]\rangle = \langle x \smile y, i_*[\partial N] = 0\rangle = 0.$$

For the proof of (ii), it's a diagram chase. ⊠

The upshot is this; $\tau(\partial N^5) = 0$. We will get to this next time.

7 More about the intersection form

7.1 Cobordism and signature

Here $x \cdot y = \sigma(x, y)$. Through an algebraic lens, (V, σ) is a symmetric bilinear form that is non-degenerate over $\mathbb{R}$. Suppose $L \subseteq V$ is an isotropic subspace such that $x \cdot y$ for all $x, y \in L$, and $\dim L = \frac{1}{2} \dim V$. Then the signature $\tau(V, \sigma) = 0$.

Proof. Say $V = L \oplus K$, $L \to K^*$, $\ell \mapsto \sigma(\ell, \cdot \in K)$. It is injective by non-degeneracy, and $\dim K = \dim L$, so its an isomorphism. This implies $(V, \sigma) \cong \underset{x}{K} \oplus \underset{\alpha}{K^*}$, where $(x + \alpha) \cdot (x' + \alpha') = \alpha(x') + x'(x)$. Do the same for $(V, -\sigma)$, so $(V, -\sigma) \cong (V, \sigma)$ which implies $\tau = 0$. ⊠

Theorem 7.1. *If there exists an oriented cobordism from X_1 to X_2, then $\tau(X_1) = \tau(X_2)$. In particular, if X is an oriented boundary, then $\tau(X) = 0$.*

Proof. Last time, we saw that $H^2(-X_1 \amalg X_2)$ has a middle dimension isotropic subspace, namely the image of $H^2 Y$. The lemma implies that $\tau(-X_1 \amalg X_2) = 0$, i.e. $\tau(X_1) = \tau(X_2)$. ⊠

Example 7.1. Some examples:

- We have $S^2 \times S^2 = \partial(D^3 \times S^2)$ which implies that $\tau(S^2 \times S^2) = 0$ (we knew this already).
- We have $\tau(\mathbb{C}\mathrm{P}^2) = 1$, so $\mathbb{C}\mathrm{P}^2$ is *not* an oriented boundary. There exists no oriented cobordism $\mathbb{C}\mathrm{P}^2 \sim -\mathbb{C}\mathrm{P}^2$.
- For $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$, we have $\tau = 1 - 1 = 0$. Is this the boundary of a compact 5-manifold? It is (connect sum is cobordant to the disjoint union), and in general $M^4 \# -M^4$ is also a boundary.
- The cylinder $[0, 1] \times M$ gives a cobordism from $\emptyset$ to $-M \amalg M$.
- There exists a cobordism from $-M_1 \amalg M_2$ to $-M \# M_2$ which goes by the name of "attaching a 1-handle". More references will be in the notes.

Theorem 7.2 (Thom, 1952)**.** *If X^4 is a closed oriented 4-manifold with $\tau(X) = 0$, then there exists a compact oriented 5-manifold Y with $\partial Y = X$.*

The proof of this theorem uses the Pontryagin-Thom construction.

7.2 Characteristic vectors

Let (Λ, σ) be a unimodular form, where τ comes from $\Lambda \otimes \mathbb{R}$. Now look at $\Lambda \otimes \mathbb{Z}/2$. Recall (Λ, σ) is *even* (or *odd* else) if $x \cdot x \in 2\mathbb{Z}$ for all $x \in \Lambda$

Definition 7.1. A **characteristic vector** $c \in \Lambda$ is one such that $c \cdot x \equiv x \cdot x \pmod 2$ for all $x \in \Lambda$.

Example 7.2. If Λ is even, the zero vector 0 is characteristic. So is any $c = 2\lambda, \lambda \in \Lambda$. Let $I_+ = \langle \mathbb{Z}, \text{mult} \rangle$, $I_+ = [1]$m $I_- = -I_+$. Then $pI_+ \oplus qI_-$ is an orthogonal direct sum.

$$pI_+ \oplus qI_- = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

The vector $1 \in \mathbb{Z}$ is a characteristic vector for $I_\pm$. So $e_1 + \cdots + e_{p+q} \in \mathbb{Z}^{p+q}$ is characteristic for $pI_+ \oplus qI_-$

Proposition 7.1. *For any unimodular form, characteristic vectors exist, and form a coset of* $2\Lambda \subseteq \Lambda$.

Proof. Let $\overline{\Lambda} = \Lambda_{\mathbb{Z}} \otimes \mathbb{Z}/2$ be a mod 2 reduction. It inherits from $x \subseteq \Lambda$ has an image $\overline{x} = x \otimes 1 \in \overline{\Lambda}$. Our condition $c \cdot x \equiv x \cdot x \pmod 2$ for all x means that $\overline{c} \cdot \overline{x} = \overline{x} \cdot \overline{x}$ for all $\overline{x} \in \overline{\Lambda}$. In $\overline{\Lambda}$, $v \mapsto v \cdot v \in \mathbb{Z}/2$ is $\mathbb{Z}/2$-linear, i.e. it lies in $\overline{\Lambda}^* = \text{Hom}_{\mathbb{Z}/2}(\overline{\Lambda}, \mathbb{Z}/2)$. The form on $\overline{\Lambda}$ remains non-degenerate. So there exists a $\overline{c}$ such that $\overline{c} * 8v = v \cdot v$ for all v. Characteristic vectors are lifts of $\overline{c}$ to Λ, and as such they form a coset of 2Λ. ⊠

We will later talk about characteristic classes, which are a different use of characteristic.

Observation. If $c, c' \in \Lambda$ are characteristic, then $c \cdot c - c' \cdot c' \in 8\mathbb{Z}$.

Proof. Say $c' = c + 2x$. Then

$$c' \cdot c' = c \cdot c + \underbrace{4(c \cdot x + x \cdot x)}_{\text{even}},$$

which is a multiple of 8. ⊠

Theorem 7.3 (Hasse-Minkowski)**.** *Two indefinite (there exist v, w with $v \cdot v > 0, w \cdot w < 0$) unimodular forms are isomorphic if they have the same rank $\in \mathbb{N}$, signature $\in \mathbb{Z}$, and type (even or odd) $\in \mathbb{Z}/2$.*

From Serre's "A course in arithmetic", an indefinite *odd* unimodular form is isomorphic to $pI_+ \oplus qI_-$.

Example 7.3. For $U = (\mathbb{Z}^2, \cdot)$, it is odd with rank 3 and signature -1. By Hasse-Minkowski this is isomorphic to $I_+ \oplus 2I_-$, which is easily checked. In $U \oplus I$, take $v_1 = e_1 + e_2 + e_3, v_2 = e_1 + e_3, v_3 = e_2 + e_3$. We can check that $v_1^2 = 1, v_2^2 = -1, v_3^2 = -1$, which is a $\mathbb{Z}$-basis.

8 Classification of indefinite unimodular forms

Today we will discuss the proof of Hasse-Minkowski, even forms, and the E_8 lattice. Let us restate the Hasse-Minkowski theorem.

Hasse-Minkowski Theorem. *An indefinite unimodular form is determined up to isomorphism by its rank, signature, and type (even or odd).*

Example 8.1. An odd indefinite unimodular form is isomorphic to $pI_+ \oplus qI_-$

Some comments on the proof: the proof hingers on finding an *isotropic* vector $x \neq 0$, i.e. $x \cdot x = 0$. Over $\mathbb{R}$, indefiniteness is equivalent to the existence of an isotropic vector. Over $\mathbb{Z}$, it suffices to find x over $\mathbb{Q}$, from then we can scale it up to lie in $\mathbb{Z}$.

There is a big idea in number thoery which deals with thinking about *local* and *global* principles. We have (Λ, σ) an indefinite unimodular form. Then considering $\Lambda \otimes \mathbb{Q}$, over $\mathbb{Q}$ the quadratic form $x \mapsto x \cdot x$ is diagonalizable;

$$x = (x_1, \cdots, x_r) \mapsto a_1 x_1^2 + \cdots + a_r x_r^2 = q(x) \quad \text{for } 0 \neq a_i \in \mathbb{Q}.$$

We want to solve $q(x) = 0$; this is what we did in high school adding a few numbers and the condition that the solution must be rational. "Global" solutions over $\mathbb{Q}$ give rise to "local" solutions over the "places" (completion) of $\mathbb{Q}$. An isotropic vector over $\mathbb{Q}$ gives rise to an isotropic vector over $\mathbb{R}$, and isotropic vectors over the p-adics $\mathbb{Q}_p$ for p a prime. So solutions over $\mathbb{Q}_p$ are roughly equivalent to a sequence of vectors $x(n) \in \Lambda$ such that $q(x(n)) \equiv 0 \pmod{p^n}$, such that $x(n)$ refines $x(n-1)$.

8.1 The Hasse-Minkowski local-to-global principle

The Hasse-Minkowski local to global principle says that for quadratic forms q, existence of local isotropic vectors for all p over $\mathbb{R}$ implies existence of a global isotropic vector. This works best in this area of number theory; if we move to elliptic curves there are examples a plenty of local objects that don't translate globally. See Serre's *A Course in Arithmetic* for a reference for all of this.

It turns out that over $\mathbb{Q}_p$, quadratic forms in ≥ 5 variables *always* have an isotropic vector for all p over $\mathbb{R}$. In fewer variables (≤ 4), being indefinite and unimodular implies that a p-adic isotropic vector exists. Over the reals, indefiniteness gives rise to an isotropic vector. How does this help? Say Λ is odd, indefinite, and unimodular. Find $0 \neq x \in \Lambda$, $x \cdot x = 0$.

Check. It is not trivial but not too complicated either to check that $x = me + nf$, where $e \cdot e = 1, f \cdot f - 1, e \cdot f = 0$ for some $m, n \in \mathbb{Z}$. In other words, x lies in a copy of $I_+ \oplus I_-$.

Since we are working with quadratic forms, we must do what every proof in quadratic forms does, which is proceed by induction on the rank. Now $\Lambda = (I_+ \oplus I_-) \oplus_{\text{orthog}} \Lambda'$, so either $\Lambda' \oplus I_+$ or $\Lambda' \oplus I_-$ is indefinite. From here induct on the rank.

Corollary 8.1 (Odd case of H-M)**.** *For all unimodular forms Λ, if $c \in \Lambda$ is a characteristic vector ($c * x \equiv x \cdot x \pmod 2$ for all x), then $c \cdot c \equiv \tau(\Lambda) \pmod 8$.*

Example 8.2. If Λ is even, then $\tau(\Lambda) \in 8\mathbb{Z}$.

Proof. Stabilizing Λ by adding a copy of I_+ or I_- adds ± 1 to τ, ± 1 to $c \cdot c$. So we reduce to the case of being odd or indefinite, so this leads to $pI_+ \oplus qI_-, \tau = p - q$, $c = (1, \cdots, 1)$. Note that $c \cdot c = p - q$, and the fact that this particular characteristic vector has square *equal* to the signature implies that *any* characteristic vector is congruent to the signature $\pmod 8$. ⊠

8.2 The even case

The proof for Hasse-Minkowski for *even* unimodular forms is an intricate reduction to the *odd* case. The two basic even unimodular forms $U, \mathbb{Z}^2$, $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is an even pairing over $\mathbb{Z} \oplus \mathbb{Z}^k$. The other is $\boxed{E_8}$, which is positive definite of rank 8. Then $U \cong -U$, and $-E_8$ is negative definite. So every even unimodular form is isomorphic to $rU \oplus s(\pm E_8)$ for some non-negative integers r, s.

Example 8.3. It turns out that if $\mathbb{Q} \subseteq \mathbb{C}\mathrm{P}^3$ is a smooth quartic surface, then $b_2(Q) = 22, \tau(Q) = -16$, and has even type. Then by H-M we get that this indefinite form is isomorphic to $3U \oplus 2(-E_8)$.

8.3 The E_8 lattice

Let us start by defining the D_8 lattice. Define

$$D_n := \left\{ x \in \mathbb{Z}^n \,\middle|\, \sum x_i \in 2\mathbb{Z} \right\} \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n.$$

In two dimensions, D_2 consists of squares around the origin. Let $\gamma = \frac{1}{2}(e_1+\cdots+e_8) \in \mathbb{R}^8$. Set $E_8 = D_8+(\gamma+D_8) \subseteq \mathbb{R}^8$ with $\cdot$ being the dot product. We can check that this takes integer values. We could also say that

$$E_8 = \left\{ x \in \mathbb{Z}^8 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^8 \,\middle|\, \sum x_i \in 2\mathbb{Z} \right\}.$$

This is an even lattice; if $x \in E_8$, then $x \cdot x = \sum x_i^2 \equiv \sum x_i \equiv 0 \pmod 2$. There is a basis $(v_1, \cdots, v_8)$, where $v_i = e_{i+1} - e_i$ for $i \le i \le 6$, $v_7 = \frac{1}{2}(e_1+e_8) - \frac{1}{2}(e_2+\cdots+e_7)$, and $v_8 = e_1 + e_2$. The v_i's are all *roots*, i.e. vectors of square two, or $v_i \cdot v_i = 2$. For $i \neq j$, $v_i \cdot v_j$ is usually zero, but there are some exceptions. We have $v_i \cdot v_{i+1} = -1$ for $1 \le i \le 5$, $v_7 \cdot v_1 = -1$, and $v_8 \cdot v_2 = -1$. A more memorable way of encoding this diagram is by the Dynken diagram, where nodes are basis vectors and edges are vectors v_i, v_j such that $v_i \cdot v_j = -1$. todo:figure

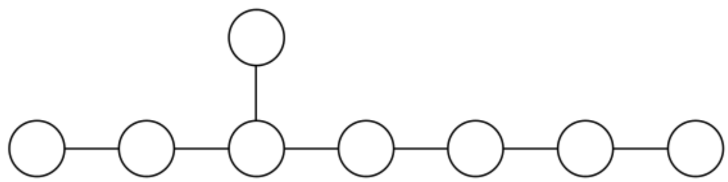

Figure 1: The Dynkin diagram of E_8.

To see that E_8 is unimodular, we have $\Lambda \supset \Lambda'$ a sublattice with $\det \Lambda' = [\Lambda : \Lambda'] \det \Lambda$. Both $\mathbb{Z}_8$ and E_8 contain D_2 as an index 2 sublattice, which implies that $\det E_8 = \frac{2}{2} \det \mathbb{Z}^8 = 1$.

9 J.H.C Whitehead's Theorem on 4-dimensional homotopy types

Theorem 9.1. *If X, X' are simply connected, oriented typical 4-manifolds, then any isometry $Q_X \xrightarrow{\cong} Q_X$ of intersection forms comes from an oriented (deg 1) homotopy equivalence $X \to X'$.*

This comes from a 1949 paper by Whitehead called "*On simply connected 4-dimensional polyhedra*", with complications due to H^3. For manifolds, $H^3 = H = 0$, so we get a simpler statement and simpler proof.

9.1 Overview of relevant homotopy theory

For a space X, call it 0-connected if it's path-connected and for $n \ge 0$, n-connected if all based maps $(S^k, *) \to (X, *)$ $(k \le n)$ are based homotopies to constants. In other words, $\pi_k(X, *) = [S^k, X] = 0$ for all $k \le n$.

Hurewicz Theorem. *Consider the Hurewicz map $h_n \colon \pi_n(X) \to H_n(X), [\gamma] \mapsto \gamma^*[S^n]$. If X is $(n-1)$-connected (where $n-1 \ge 1$), then h_n is an* ***isomorphism****.*

It is often useful to think of the homotopy groups as $\pi_n(X, *) = [(I^n, \partial I^n), (X, x)]$, since $S^n = I^n/\partial I^n$. In other words, we map the boundary of the n-cube to the basepoint. Using this picture, composition is easy to draw. There is a relative version of this setup; for $A \subseteq X$, we have a relative homology group $H_*(X, A)$ and a long exact sequence

$$H_k(A) \to H_k(X) \to H_k(X, A) \xrightarrow{\partial} H_{k-1}(A)$$

There is something similar for homotopy groups, where $* \subseteq A \subseteq X$. We get relative homotopy groups $\pi_k(X, A)$ sitting in an exact sequence of the same sort:

$$\cdots \to \pi_k(A) \to \pi_k(X) \to \pi_k(X, A) \to \pi_{k-1}(A) \to \cdots$$

The absolute homotopy gives a small clue of what to do; I^n should map to X while ∂I^n should map to A. We have $\pi_n(X,A,*) = \{\text{homotopy classes of maps } (I^n, \partial I^n, J^{n-1}) \to (X,A,*)\}$, where $J^{n-1} := (\partial I^n) \setminus \left(\{1\} \times I^{n-1}\right)$, which is the boundary minus the top face. One can think of J^{n-1} as the "open" or lidless cardboard box. Then we have the relative Hurewicz map $\pi_n(X,A,*) \to H_n(X,A)$, which is an isomorphism if (X,A) is $(n-1)$-connected $(n-1 \geq 1)$, or has trivial $\pi_k(X,A),\ k \leq n-1$.

9.2 Whitehead's Theorem

A map of path-connected spaces $f : X \to Y$ is called a **weak homotopy equivalence** if it induces bijections on $\pi_k(X) \to \pi_k(Y)$ for all $k \in \mathbb{N}$. Some facts:

- If X, Y have the same homotopy type of a CW complex, a weak homotopy equivalence is a homotopy equivalence.
- If X, Y are simply connected, and $f : X \to Y$ is an H_*-isomorphism, then it's a weak homotopy equivalence.
- If X is a compact smooth manifold, then X is homotopy equivalent to a finite CW complex.[3]
- The same is true for X a compact topological manifold (hard).

Back to 4 dimensions, let us dicsus the construction of 4-dimensional CW complexes. Start with a wedge sum of n copies of the 2-sphere, $\bigvee^n S^2$. Attach a 4-cell D^4 via an attaching map $f : \partial D^4 = S^3 \to \bigvee^n S^2$ which lead to a CW complex X_f. This is not yet a 4-manifold and will most likely not be. These spaces tend to be model homotopy types for 4-manifolds. The homotopy type of X_f depends only on the homotopy class of f. We can asssume that f respects given basepoints. The main lemma is as follows:

Lemma 9.1. *We have $\pi^3\left(\bigvee^n S^2\right) \cong \{n \times n \text{ symmetric matrices over } \mathbb{Z}\}$, where $[f] \mapsto Q_f$.*

Note that $H^*(X_f) = \mathbb{Z}_0 \oplus \mathbb{Z}_2^n \oplus \mathbb{Z}_4$ with basis $e_1, \cdots, e_n$ of fundamental classes of the 2-spheres. Then $H^2(X_f) = \mathrm{Hom}(H_2(X_f), \mathbb{Z}) \cong \mathbb{Z}^n$ with dual basis $(e^1, \cdots, e^n)$. Now we can write down the matrix Q_f. We have

$$(Q_f)_{ij} = \left\langle \underbrace{e^i \smile e^j}_{H^4(X_f)}, [X_f] \right\rangle \in \mathbb{Z}.$$

In the $n = 1$ case, we have $\pi_3(S^2) \xrightarrow[\cong]{\text{Hopf invariant}} \mathbb{Z}$.

Proof of the case where $n = 1$. Think of $S^2 = \mathbb{P}^1 \sim \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{P}^2$. We want to know $\pi_3(\mathbb{P}^1)$, which sits inside an exact sequence

$$\pi_4(\mathbb{P}^2) \to \pi_4(\mathbb{P}^2, \mathbb{P}^1) \to \pi_3(\mathbb{P}^1) \to \pi_3(\mathbb{P}^2)$$

We have a fibration sequence $S^1 \hookrightarrow S^5 \to \mathbb{P}^2$ which is a fiber bundle with fiber S^1. This implies that $\pi_3(\mathbb{P}^2) = 0, \pi_4(\mathbb{P}^2) = 0$. So $(\mathbb{P}^2, \mathbb{P}^1)$ is 4-connected, and $\pi_4(\mathbb{P}^2, \mathbb{P}^1) \cong H_4(\mathbb{P}^2, \mathbb{P}^1) = \mathbb{Z}$. ⊠

The proof of this key lemma is the generalization of the Hopf invariant.

[3] Some ways to show this in clude Morse theory which gives a handlebody decomposition, or that X is homotopy equivalent to the nerve of a good covering.

10 Complex structures and self duality, Hodge theory

Last time we talked about n-dimensional complex vector spaces V with $I = i \cdot -$ acts on V. Then $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^{1,0} \oplus V^{0,1}$ where $I^* = i$ acts on $V^{1,0}$, and $I^* = -i$ acts on $V^{0,1}$. We then took exterior powers $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \bigwedge_{\mathbb{C}}^k \left(V^{1,0} \oplus V^{0,1}\right) = \bigoplus_{p+q=k} \Lambda^{p,q}$, where $\Lambda^{p,q} = \mathrm{span}\{a_1 \wedge \cdots \wedge a_p \wedge b_1 \wedge \cdots \wedge b_q \mid a_j \in V^{1,0}, b_j \in V^{0,1}\} \cong \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$. Observe that $\Lambda^k I^*$ acts on $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ which acts on i^{p-q} on $\Lambda^{p,q}$.

These are complex forms in some sense, let us say something about real forms. On one hand, one could look at $\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$, or the complexification $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$. Canonically we have

$$\left(\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$$

which says that passing to exterior powers canonically commutes with extending to scalars. So $\left(\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}$, e.g. $V = T_xM$, $\left(\Lambda^k T_x^*M\right) \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}$. We understand that $\Lambda^{q,p} = \overline{\Lambda^{p,q}}$ by taking the complex conjugate of $\mathbb{C}$. Then we have real forms $\Lambda_{\mathbb{R}}^k \mathrm{Hom}(V,\mathbb{R}) \subseteq \bigoplus \Lambda^{p,q}$, where $\omega = \sum_{p+q=k} \omega_{p,q}$, $\omega_{q,p} = \overline{\omega_{p,q}}$.

Let's look at this in complex dimension 2, real dimension 4. In the model case where $V = \mathbb{C}^2$, this has standard basis $\{e_1, e_2\}$, and $V^{1,0}$ has complex dual $\{e^1, e^2\}$ where $e^j(e_k) = \delta_{jk}$, e^j is $\mathbb{C}$-linear. $V^{0,1}$ then has basis given by the conjugates $\overline{e_1}, \overline{e_2}$ which are $\mathbb{C}$-antilinear. Then we are interested by the 2-forms

$$\Lambda^2 \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \left(\Lambda_{\mathbb{R}}^2 \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \underset{e^1 \wedge e^2}{\Lambda^{2,0}} \oplus \underset{e_1 \wedge \overline{e^1}}{\Lambda^{1,1}} \oplus \underset{\overline{e^1}, \overline{e^2}}{\Lambda^{0,2}}$$

($e_1 \wedge \overline{e^2}, e_2 \wedge \overline{e^1}, e^2 \wedge \overline{e^2}$ also lie in $\Lambda^{1,1}$). We can regard $\mathbb{C}^2$ as a real, oriented inner product space with basis $\left(e^1, \overline{e^1}, e^2, \overline{e^2}\right)$. Then $\Lambda_{\mathbb{R}}^2 \mathbb{C}^2 = \Lambda^+ \oplus \Lambda^-$, where

$$\Lambda^+ = \left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right)_{\mathbb{R}} \oplus \mathbb{R}(e^1 \wedge \overline{e^1} + e^2 \wedge \overline{e^2}),$$

and $\Lambda^- = \Lambda_-^{1,1} = \{\eta \in \Lambda^{1,1} \mid \eta \wedge \omega = 0\}$. This is a fairly trivial decomposition of 6-dimensional vector spaces; when we bring Hodge theory into the mix, we find that this trivial matter has a highly non-trivial aspect by the Hodge index theorem.

10.1 Hodge theory

Goal. Look at $(H^2(X^4;\mathbb{R}), Q_X) \cong (H^2_{\mathrm{DR}}(X))$ which comes with form $([\alpha],[\beta]) = \int_X \alpha \wedge \beta$. These two are canonically identified with a map called "integration". Say we have a conformal class of Riemannian metrics $[g]$, which leads to an orthogonal deomposition $H^2_{\mathrm{DR}}(X) = \mathcal{H}^+_{[g]} \oplus \mathcal{H}^-_{[g]}$, so metrics give rise to a positive and negative definite decomposition of cohomology. Specifically, $\mathcal{H}^+_g$ is the space of 2-forms g which are self dual and harmonic, while $\mathcal{H}^-_g$ is the same but anti-self dual. **Harmonic forms** are the subject of Hodge theory.

The first part of Hodge theory is something called the **co-differential**. Here M^n is a manifold, then we have the exterior derivative $d\colon \Omega^k_M \to \Omega^{k+1}_M$. Assume an orientation exists and choose one, plus a Riemannian metric g (a symmetric pairing on each tangent space). We use these to construct the co-differential $d^*\colon \Omega^k \to \Omega^{k-1}$. There are two ways to define this:

- We have the Hodge star $*\colon \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ depending on the metric and the orientation. Then $d^* = (-1)^{k+1} *^{-1} \circ d \circ *$, where we conjugate the exterior derivative by the Hodge star. This indeed lowers the degree by one. $*$ is nearly an involution, where $* \circ * = \pm \mathrm{id}$. So this is just $(-1)^{k+1}(-1)^{k(n-k)} * \circ d \circ * = \boxed{(-1)^{kn+1} * \circ d \circ *}$. Since $d^2 = 0$, it follows that $(d^*)^2 = \pm * d * * d * = \pm * dd * = 0$.

- We have an L^2 inner product on k-forms: $\langle\alpha_1,\alpha_2\rangle_{L^2} = \int_M g(\alpha_1,\alpha_2)\mathrm{vol}_g$, where $\alpha_1 \in \Lambda^k_C$ has compact support, $\alpha_2 \in \Omega^k$. We claim that
$$\langle d^*\alpha_1,\alpha_2\rangle_{L^2} = \langle\alpha_1, d\alpha_2\rangle_{L^2}.$$
This is supposed to be a combination of the construction of the Hodge star with Stokes theorem and integration by parts. That is to say, if we look at $\int_M d(\alpha\wedge\beta)$ where $\deg(\alpha)=k, \deg(\beta)=n-k-1$, α has compact support, Stokes says that this integral is zero. On the other hand, $\int_M d\alpha\wedge\beta + (-1)^k\int_M \alpha\wedge d\beta$ which is integration by parts. This relation plus the definition of the Hodge star implies our claim.

10.2 Harmonic forms

Let's bring in harmonic forms now. The Hodge Laplacian $\Delta = (d+d^*)^2 = d\circ d^* + d^*\circ d$ is a degree two differential operator $\Lambda^k \to \Lambda^k$. The **harmonic k-forms** are defined as $\mathcal{H}^K = \ker\Delta$, or the kernel of the Laplacian. Clearly $\ker(d+d^*) \subseteq \mathcal{H}^k$, but the reverse inclusion holds if M is compact ($\ker(d+d^*) = \mathcal{H}^k$). Consider a form

$$\langle\alpha,\Delta\alpha\rangle_{L^2} = \langle\alpha, d^*d\alpha + dd^*\alpha\rangle_{L^2} = \langle d^*\alpha, d\alpha\rangle_{L^2} + \langle d^*\alpha, d^*\alpha\rangle_{L^2} = \|d\alpha\|^2_{L^2} + \|d^*\alpha\|_{L^2}.$$

This proves the claim, since if the LHS is zero then the two positive terms on the RHS must be zero. Next time we go on with Hodge theory and we will state the Hodge theorem, and combine it with self duality.

11 Variational Characterization

todo:beginning of this lecture

Lemma 11.1. *A harmonic form strictly minimizes L^2 norm within its de Rham cohomology class.*

Proof. For $\alpha \in H^k_g, d\alpha = 0, d^*\alpha = 0, \|\alpha\|^2_{L^2} = \int_M g(\alpha,\alpha)\mathrm{vol}_g$. Then

$$\begin{aligned}\|\alpha + d\gamma\|^2 &= \langle\alpha + d\gamma, \alpha + d\gamma\rangle_{L^2}\\ &= \|\alpha\|^2_{L^2} + \|d\gamma\|^2_{L^2} + 2\langle\gamma, d^*\alpha\rangle_{L^2}\\ &> \|\alpha\|^2_{L^2}\end{aligned}$$

for $d\gamma \neq 0$. Conversely, it's easy to check that a minimizer for an L^2 norm in a fixed cohomology class is harmonic. Take some minimizer η, and look at $\frac{d}{dt}\Big|_{t=0}\left(\|\eta + td\gamma\|^2_{L^2}\right)$. ⊠

We are in the world of calculus of variations. Here's the Hodge theorem.

11.1 The Hodge Theorem

Note that in Ω^k, we have $(\operatorname{im} d^*)^\perp = \ker d$. Then $\Omega^k = \ker d \oplus \operatorname{im} d^*$—this would follow formally in a *Hilbert space*, but the L^2 norm on Ω^k is *incomplete*. Nothing is for free.

Hodge Theorem. *We have L^2-orthogonal decompositions; $\Omega^k = \ker d \oplus \operatorname{im} d^*$, $\ker d = \mathcal{H}^k_g \oplus \operatorname{im} d$ (where $\mathcal{H}^k_g = \ker d \cap \ker d^*$. Hence the map*

$$\mathcal{H}^k_g \to H^k_{\mathrm{DR}}(M) = \frac{\ker d}{\operatorname{im} d}, \quad \eta \mapsto [\eta]$$

is an ***isomorphism***.

At this point in the course we will not go into the proof. Later in the course we will discuss the types of techniques needed to prove this. The proof involves Hilbert space completions of Ω^k in which the existence of L^2 minimizers is a formality. There is something called an "elliptic regularity" step to prove that these minimizers lie in Ω^k and not its completions.

Hodge, being an algebraic geometer, did not think to recruit a collaborator who was an expert in the type of analysis that Hilbert developed. Of course, some mathematicians came in and fixed all the issues later (von Weyl, Kodaira).

11.2 Hodge theory and self duality

Now we come to the 4-dimensional case, where we relate Hodge theory to self duality. Let X_g^4 be a compact Riemannian (we only need the conformal class of g, or $g \sim \lambda g$ for $\lambda \colon x \to (0,\infty) \subseteq \mathbb{R}$) oriented 4-manifold. The codifferential for 2-forms is $d^* = -*d* \colon \Omega^2 \to \Omega$. For $\eta \in \Omega^2$, $\eta \in \ker(d+d^*)$ or harmonic iff $*\eta \in \ker(d+d^*)$. If $\eta \in \mathcal{H}_g^2$, then $\eta^\pm = \frac{1}{2}(\eta \pm *\eta) \in \mathcal{H}_g^2$. Also, a self-dual 2-form is harmonic iff it's closed. The upshot is that $\mathcal{H}_g^2 = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm = \mathcal{H}^2 \cap \Omega^\pm$. In otherwords, $\mathcal{H}_g^2$ is the sum of the self dual harmonic forms and the anti-self dual harmonic forms. OTOH, we have $\mathcal{H}^2 \xrightarrow{\cong} H^2_{\mathrm{DR}}(X)$ by the Hodge theorem. If $\omega \in \mathcal{H}^+$, then

$$\int_X \omega \frown \omega = \int (\omega * \omega) \smile 0) = \int |\omega|^2 \mathrm{vol} > 0.$$

If $\omega \in \mathcal{H}^-$, then

$$\int \omega \wedge \omega = -\int |\omega|^2 \mathrm{vol} < 0.$$

We have the second Betti number $b^2(X) = b^+ + b^-$, and $\tau(X) = b^+ - B^-$, where $b^\pm = \dim \mathcal{H}^\pm$.

11.3 The self-duality complex

Perhaps this is the title of one of the lesser known paper of Sigmund Freud. We are still on (X^4, g) our compact, oriented 4-dimensional Riemannian manifold. We examine the cochain complexes

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^+ \to 0$$

as follows. Consider $d^+\alpha = (d\alpha)^+ = \frac{1}{2}(1+*)d\alpha$ as a cochain complex $(\mathcal{E}^*, \delta)$. The result we want to explain is the following.

Theorem 11.1. *The cohomology of $\mathcal{E}^*$ is*

$$\begin{aligned} H^0(\mathcal{E}) &= H^0_{\mathrm{DR}}(X) \\ H^1(\mathcal{E}) &\cong H^1_{\mathrm{DR}}(X) \\ H^2(\mathcal{E}) &\cong \mathcal{H}_g^+. \end{aligned}$$

This is the prototype of things that come up in gauge theory a lot, specifically one often looks at 1-forms simultaneously in the kernel of d^+ and the codifferential adjoint of d.

Proof. It is easy to check the H^0 case. For $\alpha \in \Omega^1$, we have $d\alpha + d^+\alpha + d^-\alpha$, so

$$\int_X d\alpha \wedge d\alpha = \|d^+\alpha\|^2_{L^2} - \|d^-\alpha\|^2 = \int_X d(\alpha \frown d\alpha) \underset{\text{Stokes}}{=} 0.$$

So $\|d^+\alpha\|_{L^2} = \|d^-\alpha\|_{L^2}$, and $\ker d^+ = \ker d^- = \ker d$. Hence $H^1(\mathcal{E}) \to H^1_{\mathrm{DR}}(X)$, $[\alpha] \mapsto [\alpha]$ is an isomorphism.

We want $\mathcal{H}_g^+$ identified with $\omega \in \frac{\Omega^+}{\operatorname{im} d}$. Then $\omega = \omega_{\mathrm{harm}} + d\alpha + *d\eta$, OTOH $*\omega = \omega$ since we assumed it to be self dual. Then $d\alpha + *d\eta = 2d^+\alpha$. ⊠

12 The period map and the integer lattice

Last time we talked about how a choice of metric gives a splitting of the second de Rham cohomology. This week we look more closely at this splitting and how it interacts with the integer lattice within that second de Rham cohomology. There will be some differential geometry and a bit of analysis this week.

For any smooth manifold M^n and any $k \in \mathbb{Z}_{\geq 0}$, there is an additive subgroup $H^k_{\mathbb{Z}} \subseteq H^k_{\mathrm{DR}}(M)$ of "integer classes", i.e., classes $[\alpha]$ of closed k-forms α with **integer periods**; $\int_P \alpha \in \mathbb{Z}$, for all smooth singular k-cycles P (smooth compact oriented k-dimensional manifolds). This subgroup is a **lattice**, a 4-dimensional discrete subgroup, or the inclusion extends to an $\mathbb{R}$-linear isomorphism, and extending the coefficients to $\mathbb{R}$ is an isomorphism; $H^k_{\mathbb{Z}} \otimes \mathbb{R} \xrightarrow{\cong} H^k_{\mathrm{DR}}(M)$. Why is this a lattice?

$$H^k_{\mathrm{DR}}(M) \cong H^k(M;\mathbb{R}) \underset{\text{universal coefficients}}{\cong} H^k(M;\mathbb{Z}) \otimes \mathbb{R} = H^k(M;\mathbb{Z})' = \frac{H^k(M;\mathbb{Z})}{\text{tors}} \otimes \mathbb{R}.$$

Then $H^k(M;\mathbb{Z})' \hookrightarrow H^k_{\mathrm{DR}}(M)$ maps isomorphically onto $H^k_{\mathbb{Z}}$. Call the subgroup $H^k_{\mathbb{Z}}$ the **integer lattice**.

12.1 The 4-dimensional case

For X^4 closed and oriented, we have our quadratic form on $H^2_{\mathrm{DR}}(X)$, where $\eta \mapsto \int_X \eta \wedge \eta$. From last time, we saw using Hodge theory that $H^2_{\mathrm{DR}}(X) = \mathcal{H}^+_{[g]} \oplus \mathcal{H}^-_{[g]}$, where these subspaces depend on a choice of conformal structure. Then $\mathcal{H}^{\pm}_{[g]}(\mathbb{Z}) := H^2_{\mathbb{Z}} \cap \mathcal{H}^{\pm}_{[g]}$. Recall that $b^+ = \dim \mathcal{H}^+$, $b^- = \dim \mathcal{H}^-$.

Theorem 12.1. *Assume $b^+(X) > 0$. Then for generic conformal structures $[g]$, we have $\mathcal{H}^-_{\mathbb{Z}} = 0$.*

Precisely, we work with conformal classes of C^r Riemannian metrics, $r \in \mathbb{N}, r \geq 3$. "Generic" means it holds on a countable intersection of open dense subsets. It turns out spaces $\mathcal{C}_r(X)$ of conformal structures identified with an open ball in a Banach space.

Corollary 12.1. $\mathcal{H}^-_{\mathbb{Z}} = 0$ *for a **dense** set of C' conformal structures.*

Proof. Apply the Baire category theorem in the closure of the open ball. ⊠

12.2 The period map

This is a map $P : \mathcal{G}_r(X) \to \mathrm{Gr}^-$, $[g] \mapsto \mathcal{H}^-_{[g]} \subseteq H^2 s_{\mathrm{DR}}(X)$, where Gr^- is short for $\mathrm{Gr}^-_{b_-}\left(H^2_{\mathrm{DR}}(X)\right)$, the Grassmannian of $b_-(X)$-dimensional subspaces of $H^2_{\mathrm{DR}}(X)$. The minus means we want the subspace to be negative definite for the quadratic form. The proof will involve P and its derivative. We've taken calculus and know how to differentiate things. How do we differentiate the period map?

The simplest part of the story, yet the most instructive and important, has to do with the Grassmannian itself. The Grassmannian Gr^- has submanifolds S_c for any $0 \neq c \in H^2_{\mathrm{DR}}(X)$, where $S_c := \{J \in \mathrm{Gr}^- \mid c \in J\}$.

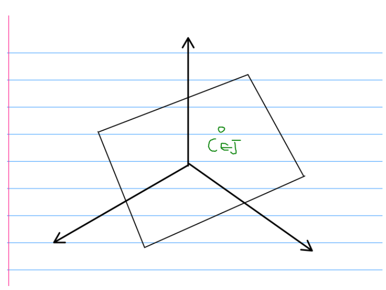

Figure 2: Visualizing some $J \in S_c$.

Why is S_c a submanifold? Consider charts on Gr (or Gr^-). For $J \in \mathrm{Gr}$, choose a complement K, so $H = J \oplus K = H^2_{\mathrm{DR}}(X)$. Then $\mathrm{Hom}_{\mathbb{R}}(J,K) \xrightarrow{\phi_{J,K}} \mathrm{Gr}$, $\theta \mapsto \mathrm{graph}(\theta)$, where $\phi_{J,K}$ is an atlas for Gr. Suppose now that $c \in J$, or $J \subseteq S_c$. $\phi_{J,K}$ maps $\{\theta \in \mathrm{Hom}(J,K) \mid \theta(c) = 0\}$ to a neighborhood of J in S_c, i.e. we get a *submanifold* chart. Since $b^+ > 0$, $\mathrm{Gr}^- \setminus S_c$ is open and dense in Gr^-. We can look at $\bigcup_{0 \neq c \in H^2_{\mathbb{Z}}} S_c \subseteq \mathrm{Gr}^-$, a countable union of closed submanifolds of positive codimension, and is *generic* (no idemp sets). What this tells us is that our intution that if we look at a Grassmannian and a random subspace to see how it intersects the integer lattice is that it only intersects at the origin.

Our next step is to show that P is transverse to each of the submanifolds S_c where $0 \neq c \in H^2_{\mathbb{Z}}$. This implies that $P^{-1}(S_c) \subseteq \mathcal{C}_r(X)$ is a closed codimensional b^+ submanifold (where $\mathcal{C}_r(X)$ is an ∞-dimensional manifold). This implies our theorem. Transversality means that if $P[g] \in S_c$, then $\boxed{T^{\mathrm{Gr}^-}_{P[g]} = T_{P[g]} S_c + \mathrm{im}\, D_{[g]} P}$ (the derivative of the period map). This is the technical statement that is going to be proved. Some of the proof will be next time, primarily using a Hodge theoretic calculation. First you differentiate the Hodge star with respect to the conformal structure, then you differentiate the period map. It also uses a principle from PDEs, called unique continuation for harmonic 2-forms.