Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

§1 September 14, 2020: Homework 3

Hatcher Section 1.2 (p. 52): 8, 11, 13,
Hatcher Section 1.3 (p. 79): 1, 4,
Assigned problem parts (a) and (b).

§1.1 Problem 8 Section 1.2

Problem. *Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.*

Solution. To see what this group looks like, if we take an "outer" and "inner" circle of the respective tori and glue them together, we get a two hulahoops stacked and glued inside each other. This is homeomorphic to the figure 8 cross the circle (it looks like a figure 8 if you cut it open), so this space is homeomorphic to $(S^1 \vee S^1) \times S^1$.

Formally, we can look at this in terms of presentations and relations: say π_1 of a torus is given by the presentation $\langle a, b \mid a^{-1}b^{-1}ab = [a,b] \rangle$ ($[a,b]$ denotes the commutator of a and b). Then we can similarly represent the second torus' fundamental group by $\langle c, d \mid [c,d] \rangle$. Note that π_1 of the intersection is equal to $\pi_1(S^1) = \langle a \rangle$. What this identification does is glue the second circle of this first torus to the second circle of the second torus, such that the homotopy class $[b] = [d]$. Then by van Kampens, we have π_1 of the two tori joined at a circle as

$$\langle a, b, c \mid [a,b], [c,b] \rangle.$$

Since b commutes with a and c but a and c don't commute, this is isomorphic to $(\mathbb{Z} * \mathbb{Z}) \times Z$, which is our expected result if we think of this space as $(S^1 \vee S^1) \times S^1$. ■

§1.2 Problem 11

Problem. *The* mapping torus T_f *of a map $f\colon X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the undiced map $f_*\colon \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [One way to do this is to regard T_f as built from $X \vee S^1$ by attaching cells.]*

Remark: In the most important case, f_* is an isomorphism. In this case, observe that $\pi_1(T_f)$ is a semidirect of $\pi_1(X)$ by $\mathbb{Z}$.

§1.3 Problem 13

Problem. *The space Y in the preceding exercise (Klein bottle embedded in $\mathbb{R}^3$ with a deleted open disk at the circle of self-intersection) can be obtained from a disk with two holes by identifying its three boundary circles. There are only two essentially different ways of identifying the three boundary circles. Show that the other way yields a space Z with $\pi_1(Z)$ not isomorphic to $\pi_1(Y)$. [Abelianize the fundamental groups to show they are not isomorphic.]*

§1.4 Problem 1 Section 1.3

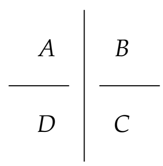
Problem. *For a covering space $p\colon \widetilde{X} \to X$ and a subspace $A \subseteq X$, let $\widetilde{A} = p^{-1}(A)$. Show that the restriction $p\colon \widetilde{A} \to A$ is a covering space.*

§1.5 Problem 4

Problem. *Construct a simply-connected covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff iff X is compact Hausdorff.*

§1.6 Problem (Dehn presentation)

Problem. *The Dehn presentation of a $\pi_1(\mathbb{R}^3 - L)$, for L a link, is different from the Wirtinger presentation from class (see also Hatcher' §1.2#22). Write p for the projection $\mathbb{R}^3 \to \mathbb{R}^2$. Suppose the projection of L has the usual form for a link diagram. Then $p(L)$ divides $\mathbb{R}^2$ into components. The Dehn presentation has one generator for each bounded region. (We take the identity element as the "generator" associated to the unbounded region.) There is one relation for each crossing. Suppose the regions involved at that crossing are A, B, C, D as in the picture. The relation is $AB^{-1}CD^{-1} = 1$.*



Prove that this presentation gives $\pi_1(\mathbb{R}^3 - L)$.

Here is the meaning of the generators. (You should be drawing pictures like mad throughout this problem.) We think of the link as lying close to $\mathbb{R}^2$, and the basepoint far above it. A generator goes down from the basepoint, through the corresponding region, and then comes back up through the unbounded region. (This is why the "generator" associated to the outside region is the identity.)

Here is an approach to proving that the presentation is legit. We start by building a 2-complex X containing L. Its 1-skeleton is the union of L and one vertical segment for each crossing, joining the upper strand to the lower strand at that crossing. Then you add one 2-cell for each region. For a particular (closed) region, consider its preimage in L. You get a subset of L. Take the union of this with the vertical segments corresponding to the corners of the region. This gives a circle in the 1-skeleton, along which you attach a disk. You can see the disk inside $\mathbb{R}^3$; it projects bijectively to its image in $\mathbb{R}^2$, except that each vertical segment projects to a corner of the region. (You should be drawing pictures like mad throughout this problem.)

Work out $\pi_1(\mathbb{R}^3 - L)$ by starting with $\mathbb{R}^3 - X$ and adding stuff until we have $\mathbb{R}^3 - L$, using VK at each stage. To start, show $\mathbb{R}^3 - X$ is simply connected. Then take the union of $\mathbb{R}^2 - X$ with the interior of one of the 2-cells of X. Use Van Kampen's theorem to show that that this adjoins a generator. (You should be drawing pictures like mad throughout this problem.) Repeating this for the other regions yields the generators in the presentation. This gives $\mathbb{R}^3 - X^{(1)}$. Next add in the interior of one of the vertical segments. Use VK's theorem to show that this imposes one relation, namely the one associated to that crossing in the link diagram. (You should be drawing pictures like mad throughout this problem.) Repeating this gives $\mathbb{R}^3 - K$, with known generators and relations for its fundamental group.

This approach is "dual" to the usual approach of "1-cells are generators, 2-cells are relations". To use the usual approach, one can find a sort of "dual" complex to X in $\mathbb{R}^3$, to which $\mathbb{R}^3 - L$ deformation-retracts. But I have an easier time seeing the relation by the approach above, because they are represented by very small loops around the vertical segments.

§1.7 Problem (Dehn presentation example)

Problem. *Example of the Dehn presentation:*

(a) *Use the Dehn presentation to present $\pi_1(\mathbb{R}^3 - K)$ where K is the $(2, n)$ torus knot (in particular, n is odd).*

(b) *You'll get a presentation with $n + 1$ generators. Simplify it down to 2 generators. How similar is the result to the elegant presentation $\langle x, y | x^2 = y^n \rangle$ from the book?*