# Complex Analysis Lecture Notes

Simon Xiang

These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. I was also unhappy with the textbook, so some supplementary notes from different texts are found at the bottom of the document. Since I took these live in class, there are many mistakes and gaps: if you have questions, comments, corrections, etc, feel free to email them to me at simonxiang@utexas.edu.

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# Part I
# Lecture Notes

Lecture 1

## August 27, 2020

### 1.1 Basic Properties of Complex Numbers

We talk about functions $f : \mathbb{C} \to \mathbb{C}$ that map variables $z \mapsto f(z)$. This course is "not a very hard course" (it's a fun course!). Holomorphic functions have very nice properties automatically that real valued differentiable functions simply don't have.

**Definition 1.1** (Complex Addition). We define complex numbers as ordered pairs $z = (x, y)$ where $x, y \in \mathbb{R}$, with the binary operation of complex addition being defined as

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$

where $+$ denotes addition on the reals.

Once we define multiplication and additive/multiplicative inverses, we will have (almost) formed the field $\mathbb{C}$.

**Definition 1.2** (Complex Multiplication). For $x, y \in \mathbb{C}$, we have

$$(x_1, y_1)(x_2)(y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Note: for $a \in \mathbb{R}$, we define

$$a(x, y) = (ax, ay).$$

Recall $(a, 0)(x, y) = (ax, ay)$. So one can understand that $a \in \mathbb{R}$ is simply the real analog of $(a, 0)$ (or simply, $\operatorname{Re}(a, 0) = a \in \mathbb{R}$).

How do we define multiplication of a complex number by a real number? We can think of the reals acting (in a group sense) on the complex numbers, with the operation being the standard multiplication.

**Example 1.1.** Take $(1, 0)(x, y) = (x, y)$. So $1(x, y) = (x, y)$ (where $1 \in \mathbb{R}$).

**Example 1.2** (Complex Addition is Commmutative). We have already defined the sum of two complex numbers $z_1 + z_2$ as $z_3 = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$. Since addition is commutative on the real numbers, we have

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1,$$

so complex addition is commutative.

Claim: multiplication of complex numbers is commutative. You can verify this at home.

**Theorem 1.1** (Distributive Law). *We have*

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$

*for* $z_1, z_2, z_3 \in \mathbb{C}$.

*Proof.* This follows from the fact that $\mathbb{C}$ has a ring structure. ⊠

### 1.2 Real and Imaginary Parts

**Definition 1.3.** If $z = (x, y)$, then $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Furthermore, we can associate a complex number with a point in the plane in many ways:

(insert figure 1 later)

### 1.3 Complex Numbers in the Plane

Point: the plane is just a plane. The plane doesn't have to have a coordinate system (coordinate axes don't have to be perpendicular). Any coordinate system is "useful" for adding complex numbers. For example, you can interpret complex addition as simply vector addition in the plane (no need for orthogonal axes!).

**Definition 1.4** (Additive Inverse). We have

$$-(x,y) = (-1)(x,y) = (-x,-y).$$

So $(x,y)+[-(x,y)] = (0,0)$.

Note: $(x,y)(0,1) = (-y,x)$, a *rotation* of $(x,y)$ by 90°. Another note: We have $(x,y) \in \mathbb{C} \cong x+iy$ and $i = (0,1)$. So

$$(x,y) \cong x+iy \cong (x,0)+(0,1)(y,0).$$

Lecture 2

## September 1, 2020

### 2.1 Units and Zero Divisors in the Complex Numbers

Recall from last time: A complex number can be defined as $(x,y) = x+iy$, where $x,y \in \mathbb{R}$. Addition is easy: $(x_1+iy_1)+(x_2+iy_2) = (x_1+y_1)+i(y_1+y_2)$. In particular, $(0,0) = 0+i\cdot 0 = 0$. For multiplication, assume $i^2 = -1$. Then

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= (x_1x_2+iy_1x_2+iy_2x_1+i^2y_1y_2)\\ &= x_1x_2-y_1y_2+i(y_1x_2+y_2x_1).\end{aligned}$$

On division: what does it mean to divide complex numbers? We say the multiplicative unit of a complex number (wrt the ring $\mathbb{C}$ ) as the unique $\frac{1}{z} = z^{-1}$ s.t. $z\cdot z^{-1} = z^{-1}\cdot z = (1,0) \in \mathbb{C}$ (the unity of $\mathbb{C}$). Assume $(x,y)(x,y)^{-1} = (1,0)$. Then do $u$ and $v$ exist such that the system of equations

$$\begin{cases} xu-yv = 1\\ xv+yu = 0\end{cases}$$

holds? Yes, iff the determinant $\begin{vmatrix} x & -y\\ y & x\end{vmatrix} = x^2+y^2$ is non zero.

**Definition 2.1** (Complex Conjugate). We have $(x,-y)$ the complex conjugate of the complex number $z = (x,y)$, denoted $\bar{z}$.

We show that $\mathbb{C}$ has no zero divisors and is therefore an integral domain. WLOG, assume there exists $z_1, z_2$ such that $z_1 \neq 0$, $z_1z_2 = 0$: then we have $z_1^{-1}$ exists. So $z_1^{-1}z_1z_2 = 1z_2 = 0$, therefore $z_2 = 0$. For example: the group $\mathrm{GL}_n(\mathbb{R})$ is not an integral domain, since we have zero divisors (two matrices that when multipled equal zero).

### 2.2 Polar Coordinate Notation

**Definition 2.2** (Polar Coordinates). Think of $(x,y)$ as rectangular coordinates in the $xy$-plane, and consider the *polar coordinate* notation $z = [r,\theta]$, where $r = \sqrt{x^2+y^2} = |z|$ (modulus of $z$), and $\theta = \arctan(\frac{y}{x})$. So $[r,\theta] = (r\cos\theta, r\sin\theta)$.

**Example 2.1** (Multiplication with Polar Coordinates). We have

$$[r_1,\theta_1][r_2,\theta_2] = (r_1\cos\theta_1, r_1\sin\theta_1)(r_2\cos\theta_2, r_2\sin\theta_2).$$

Then

$$\begin{gathered}(r_1\cos\theta_1+ir_1\sin\theta_1)(r_2\cos\theta_2+ir_2\sin\theta_2) =\\ r_1r_2[\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2]+ir_1r_2[\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_1] =\\ r_1r_2\cos(\theta_1+\theta_2)+r_1r_2i\sin(\theta_1+\theta_2) =\\ [r_1r_2,\theta_1+\theta_2].\end{gathered}$$

**Example 2.2.** Assume that a complex number $z = (x, y)$ is nonzero. Then

$$\frac{1}{(x,y)} = \frac{1(x,-y)}{(x,y)(x,-y)} = \frac{(x,-y)}{x^2+y^2}.$$

## 2.3 On the Norm (Modulus) of a Complex Number

**Example 2.3.** Some properties of the modulus (norm) $|z|$:

1. $|z_1 z_2| = |z_1||z_2|$,
2. $\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = \left|z_1 \cdot \frac{\bar{z_2}}{|z_2|^2}\right| = |z_1|\frac{|z_2|}{|z_2|^2} = \frac{|z_1|}{|z_2|}$ (clearly $|\bar{z_2}| = |z_2|$),
3. $|z_1 + z_2| \leq |z_1| + |z_2|$ ($\mathbb{C}$ is a metric space, so the triangle inequality holds),
4. $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$ (reverse triangle inequality).

We prove the Reverse Triangle Inequality.

*Proof.* We have $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|$, so $|z_1 + z_2| \geq |z_1| - |z_2|$. A similar argument holds for $z_2$. ⊠

Think of the polar angle as only well defined for multiples of $2\pi$. Define the argument (angle) as $\text{Arg} = -\pi < \theta \leq \pi$ (what??). So $\text{Arg}(1,1) = \frac{\pi}{4}$, $\text{Arg}(-1,0) = \pi$. OTOH, we would have $\arg(1,1) = \frac{\pi}{4} + 2\pi n$.

## 2.4 Euler's Formula

**Theorem 2.1** (Euler's Formula)**.** *We claim*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

*Proof.* Try using Maclaurin series. ⊠

This suggests $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We proved this when we showed $[r_1, \theta_1][r_2, \theta_2] = [r_1 r_2, \theta_1 + \theta_2]$.

The reason why Dr. Radin says to "forget about Euler" is because he's trying to make a semi-rigorous (or self-contained) construction of the complex numbers. I think it's fine to rely on intuition from other courses, this isn't Real Analysis (nowhere near as rigorous). If we truly were to construct the field $\mathbb{C}$, we would have to cover polynomial rings and the fields generated by PID's quotient irreducible polynomials, then show that $\mathbb{C} \simeq \mathbb{R}[x]/\langle x^2+1\rangle$ (and show that this new field is algebraically closed too!). Of course this isn't feasible. So let's just think of this as Euler's Formula, and not some weird definition!

Back to math: using our newfound formula, we can simply say $\arg z = \theta$ such that $z = re^{i\theta}$ for any $z \in \mathbb{C}$. Similarly, $\text{Arg}\, z$ is just $\theta$ restricted to the interval $(-\pi, \pi]$.

**Example 2.4.** If $z = re^{i\theta}$ nonzero, then what is the polar form of $\frac{1}{z}$? It must be

$$\frac{1}{r}e^{-i\theta}.$$

**Example 2.5.** We've seen that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Then

$$e^{i\theta_1}\left(e^{i\theta_2}e^{i\theta_3}\right) = e^{i\theta_1}e^{i(\theta_2+\theta_3)} = e^{i(\theta_1+\theta_2+\theta_3)}.$$

So $(\cos\theta + i\sin\theta)^m = \cos(m\theta) + i\sin(m\theta)$. This is known as *de Moivre's formula* .

Lecture 3

# September 3, 2020

## 3.1 Fractional Powers

Let $z_0 \in \mathbb{C}$, and define the fractional power $(z_0)^{\frac{1}{m}}$ for $m \geq 2$. This is a complex number such that

$$\left[(z_0)^{\frac{1}{m}}\right]^m = z_0.$$

This many not be unique. To determine the value of the fractional power $(z_0)^{\frac{1}{m}}$, let $z_0 = r_0 e^{i\theta_0}$, $r_0 = |z_0|$, $\theta_0 \in \operatorname{Arg} z_0$. Then

$$(z_0)^{\frac{1}{m}} = (r_0)^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}.$$

**Example 3.1.** In polar form, $z_0 = i = e^{i\frac{\pi}{2}}$. We want $i^{\frac{1}{6}}$, one value is $e^{i\frac{\pi}{12}}$. Also,

$$e^{i\frac{\left[\frac{\pi}{2}+2\pi\right]}{6}} = e^{i\left[\frac{\pi}{12}+\frac{\pi}{3}\right]} = e^{i\frac{5\pi}{12}}.$$

In general, $i = e^{i\left[\frac{\pi}{2}+2\pi m\right]}$, so $e^{i\left[\frac{\pi}{12}+\frac{m\pi}{3}\right]}$ is a value of $i^{\frac{1}{6}}$ for any $m$. In particular, consider the choices $m = 0, 1, .., 5$. Then

(insert figure later- it has to do with roots of unity on the circle group tho)

This method gives all possible $n$-th roots. In particular, in the circle group $U_1$, each “walk” is equal to a multiplication of $\zeta$.

We will eventually generalize the fractional power $z_0^{p/q}$ to $z_0^w$. Yada yada no exponentials allowed reeee. If you’re going to formalize do it right or don’t do it at all. Half baked rigor is about as useful as a potato (at least a potato can feed your family).

## 3.2 Point Set Topology

Why are we studying abstract nonsense? We need topology to define limits of complex numbers. We will eventually define a derivative as a quotient of deltas, eg

$$\frac{\Delta f}{\Delta z} \to \frac{df}{dz} \quad \text{as } \Delta z \to 0.$$

We’ll talk about open and closed sets and accumulation points and such (basic things needed for limits). Consider

$$\widetilde{S} = \{z \mid |z| \leq 1 \text{ and } |z| \neq 1 \text{ if } \operatorname{Re} z < 0\}.$$

**Definition 3.1** (Open Ball)**.** We define an open ball

$$B(z_0, \epsilon) = \{z \mid |z - z_0| < \epsilon\}.$$

## 3.3 Interior, Closure, Boundary

**Definition 3.2** (Interior Point)**.** We have an *interior point* a point in a set such that there exists an open ball centered at the point entirely contained in the set. We define the set of all interior points of a set $X$ as $\operatorname{Int}(X)$.

Note that $\operatorname{Int}(\widetilde{S}) = \{z \mid |z| < 1.\}$

**Definition 3.3** (Exterior Point)**.** A point $z_0$ is an exterior point of $S$ if there exists a ball

$$B(z_0, \epsilon) \subseteq S^c,$$

ie, $z_0 \in \operatorname{Int}(S^c)$.

**Definition 3.4** (Boundary Point)**.** A point $z_0$ is a boundary point of $S$ if for ball $B(z_0,\epsilon)$ centered at $z_0$, $B(z_0,\epsilon)\cap S \neq \emptyset$ and $B(z_0,\epsilon)\cap S^c \neq \emptyset$. We define the *boundary* of a set $S$ as the set of all boundary points, denoted $\partial S$.

Basic things: points can't be both in the interior and exterior, boundary and interior, etc etc.

**Theorem 3.1.** *For any set $S$,* $\operatorname{Int}(S)$ *,* $\operatorname{Ext}(S)$ *, and $\partial S$ form a partition of $S$.*

We will use $S^\circ$ to denote the interior and $(S^c)^\circ$ to denote the exterior of a set from now on.

**Example 3.2.** $\partial\widetilde{S} = \{z \mid |z| = 1\}$.

**Example 3.3.** We have the unit circle $S = \{z \mid |z| = 1\} \cup zi$ (where $zi$ is a point). $S^\circ = \emptyset$, $zi \in \partial S$, any point on the rim $\in \partial S$, so $\partial S = S$. By our previous theorem, $(S^c)^\circ = S^c$. (Who even studies the exterior of a set??)

## 3.4 Open and Closed Sets

From now on a set refers to a subset of $\mathbb{C}$.

**Definition 3.5** (Open Sets)**.** A set is open if it contains none of its boundary. Alternatively, a set is open iff $S = S^\circ$.

**Example 3.4.** $\mathbb{C}$ is open (and closed)! Furthermore, $\partial\mathbb{C} = \emptyset$ (which is an alternate condition for a set to be clopen). Note that $\emptyset$ is also both open and closed, since $\partial\emptyset = \emptyset$. This also makes sense if we look at it from the interior perspective (no interior points in $\emptyset$, every point has an open ball in $\mathbb{C}$).

**Definition 3.6** (Closed Sets)**.** A set is closed if it contains all of its boundary. (What do you mean not the complement of open???)

**Theorem 3.2.** *$S$ is closed $\iff$ $S^c$ is open.*

*Proof.* Immediate. In general topology, we define open sets this way. ⊠

**Example 3.5.** Like I said earlier, both $\mathbb{C}$ and $\emptyset$ are closed. In general topology, we define both $S, \emptyset \in \tau$, since they're complements of course they're both open and closed. Exercise: prove that no other sets are both open and closed.

**Definition 3.7** (Closure)**.** The closure $\bar{S}$ of $S$ is the union

$$S \cup \partial S.$$

Clearly $\bar{S}$ is always closed (by our definition).

**Theorem 3.3.** *$S^\circ$ is open for any $S$.*

Doesn't this follow from the definition too??

## 3.5 Jank Connectedness

**Definition 3.8** (Path-connectedness)**.** A set $S$ is path-connected if every pair of points $z_1, z_2 \in S$ is connected by a continuous path in $S$.

Every path-connected set is connected (can be written as the union of two disjoint sets). Something about polygonal paths?? Dr. Radin is right, this is most definitely not standard. Is this what physicists do to topology?

Now he's talking about the Topologist's sine curve (the classic counterexample). This is a counterexample to the (false) idea that connected implies path-connected by exhibiting a set that is connected but not path-connected (but we haven't even talked about the standard definition of connectedness yet!).

Lecture 4

# September 8, 2020

## 4.1 Accumulation Points

**Definition 4.1.** A connected open set is a *domain*.

**Definition 4.2.** A *region* is a domain that contains none, some, or all of its boundary.

**Definition 4.3** (Bounded Set)**.** A set $S$ is bounded if

$$S \subseteq B(x_0, \epsilon).$$

for some $x_0 \in \mathbb{C}$, $\epsilon > 0$.

**Definition 4.4** (Accumulation Points)**.** $z_0$ is an accumulation point of $S$ if for all balls $B(z_0, \frac{1}{m})$ centered at $z_0$, we have

$$B(z_0, \frac{1}{m}) \setminus \{z_0\} \cap S \neq \emptyset.$$

**Example 4.1.** Let $S = \mathbb{Q}$. Then $\frac{1}{2}, \sqrt{2}$ etc are accumulation points of $S$ (this relies on the fact that $\mathbb{Q}$ is *dense* in $\mathbb{R}$). This example shows that accumulation points don't have to be in the set themselves.

**Theorem 4.1.** *We have $S$ is closed if and only if $S$ contains all of its accumulation points, the set of which is denoted $S'$. Furthermore, the closure of $S$ denoted $\bar{S}$ is equal to $S \cup S'$.*

*Proof.* $\implies$ Accumulation points are either in the boundary of $S$ or in $S$ itself. Since $S$ is closed, we have $S' \subseteq S$. $\impliedby$ If $z_0 \in \partial S \cap S^c$ it would be an accumulation point of $S$, a contradiction. So $\partial S \subseteq S \implies S$ is closed. (I'll try to write a better proof later). ⊠

A quick summary of basic p-set topology:

1. $S$ is open $\iff S = S^\circ$,
2. $S$ is closed $\iff S^c$ is open,
3. $S$ is open $\iff S$ contains none of $\partial S$,
4. $S$ is closed $\iff S$ contains all of $\partial S$,
5. $S$ is closed $\iff S$ contains all of $S'$.

## 4.2 Limits

Consider a map $f : \operatorname{Dom}(f) \to \mathbb{C}$, $\operatorname{Ran}(f) \subseteq \mathbb{C}$ (I prefer the notation $f : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$, and $\operatorname{Ran}(f) = f[X]$). The fact that $f$ is well defined on $X$ holds because define $X$ to be a set on which $f$ is well defined, duh).

We want to talk about whether a function is continuous or not. Intuitively, a function is continuous if points in the image being "close" together imply that points in the preimage are also "close" together (the preimage of an open set is open).

**Definition 4.5** (Epsilon Delta Limits)**.** For $z_0$ an accumulation point of some subset $X$ of $\mathbb{C}$ (a region), $\lim_{z \to z_0} f(z)$ exists and has a value of $L \iff$ for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon,$$

where $z \in X$. The modulus is just a distance metric: so the epsilon delta definition is the same as what I said earlier, if points are close to each other in the codomain ($|f(z) - L| < \epsilon$), then such points are close to each other in the domain ($0 < |z - z_0| < \delta$).

Some notes: the limit is only defined when $z_0$ is an accumulation point. This why accumulation points are also sometimes referred to as *limit points*.

## 4.3 Continuity

**Definition 4.6** (Continuity)**.** $f$ is continuous at $z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$. $f$ is said to be continuous on a set $X$ if for all $x \in X$, $f$ is continuous at $x$.

We want to *analyze* a function $f(z)$, let $z = (x, y)$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, $u(x, y) = \operatorname{Re} f$ and $v(x, y) = \operatorname{Im} f$.

**Theorem 4.2.** *We have*

$$\lim_{z\to z_0} f(z) = L \iff \begin{cases} \lim_{z\to z_0} \operatorname{Re} f(z) \to \operatorname{Re} L \\ \lim_{z\to zo} \operatorname{Im} f(z) \to \operatorname{Im} L. \end{cases}$$

*Proof.* Homework. ⊠

**Theorem 4.3.** *Let $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. For an accumulation point $z_0$ of $X \cap Y$, if $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then (excuse the abuse of notation)*

1. $\lim(f + g) = L + M$,
2. $\lim f g = LM$,
3. $\lim \frac{f}{g} = \frac{L}{M}$ *if* $M \neq 0$.

*Proof.* Same as the ones you'd find in any analysis course. ⊠

Continuity of sums, products, and quotients of functions follow from the above theorem. Now we turn our attention to the composition of functions.

**Theorem 4.4.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ and $g : X \to \mathbb{C}$. Let $z_0$ be an accumulation point of $X$. Then if $f$ is continuous at $z_0$ and $g$ is continous at $f(z_0)$, we have $f \circ g$ continuous at $z_0$.*

**Example 4.2.** $f(z) = |z^m|$ for a fixed $m$ is equal to $(g \circ h)(z)$ where $h(z) = z^m$ and $g(w) = |w|$. Both $h$ and $g$ are continuous on $\mathbb{C}$, so $|z^m|$ is also continuous everywhere.

**Example 4.3.** The identity map is continuous. This is trivial (let $\delta = \epsilon$). It follows that maps of the form $z^n$ is continuous for some positive integer $n$.

**Corollary 4.1.** *Functions of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

*where $p(z)$ and $g(z)$ are polynomials are continuous given $g(z) \neq 0$.*

**Example 4.4.** Let $f(z) = \frac{z}{\bar{z}}$, $z \neq 0$. Consider $z = x + iy$ near 0 with $x \neq 0, y = 0$, then $f(z) = 1$. If $x = 0, y \neq 0$ then $f(z) = -1$. Therefore $\lim_{z\to z_0} \frac{z}{\bar{z}}$ does not exist (standard technique for proving multivariate limits don't exist).

Lecture 5

# September 10, 2020

## 5.1 More on Continuity

Last time we talked about the function $\frac{z}{\bar{z}}$. What if we define the domain as $\mathbb{C} \setminus \{0\}$? Does $\lim_{z\to z_0} \frac{z}{\bar{z}}$ exist? (AKA: is $\frac{z}{\bar{z}}$ continuous on its domain?)

**Theorem 5.1.** *Let $f : \mathbb{C} \to \mathbb{C}$ be defined as $f = u + iv$. If $f$ is continuous at $z_0$, then*

1. $\overline{f} = u - iv$ *is continuous at $z_0$. We can also write $\overline{f}$ as $g \circ f$ where $g(w) = \overline{w}$.*
2. $\frac{f+\overline{f}}{2} = \operatorname{Re}(f)$ *is continuous at $z_0$.*

3. $\frac{f-\overline{f}}{2i} = \text{Im}(f)$ *is continuous at* $z_0$.

*Proof.* We prove that $f(z) = \overline{z}$ is continuous at any $z_0$. Given $\varepsilon > 0$, consider

$$|f(z) - f(z_0)| = |\overline{z} - \overline{z_0}|.$$

We need a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |\overline{z} - \overline{z_0}| < \varepsilon.$$

Claim: If $S = \varepsilon$, $|\overline{z} - \overline{z_0}| = |\overline{(z - z_0)}| = |z - z_0| = \delta = \varepsilon$. This is easy to see, so we are done. ⊠

**Note.** To show that

$$\lim_{z \to z_0} f(z) = L,$$

we consider neighborhoods (open sets around $L$), or the set of $z$ such that $|f(z) - L| < \varepsilon$ (equivalently, the $z$ such that $f(z) \in B(L, \varepsilon)$). Also, $\lim_{z \to z_0} f(z) - L \iff \lim_{z \to z_0}(f(z) - L) = 0 \iff \lim_{z - z_0 \to 0}(f(z) - L) = 0$.

## 5.2 Limits near Infinity

Infinity is not a complex number!! Consider the limits

$$\lim_{z \to \infty} f(z)$$

and

$$\lim_{z \to z_0} f(z) = \infty.$$

To define these, we use neighborhoods of "$\infty$ ". There is no notion of "$\pm\infty$" in the complex numbers. The definition is similar to the one you encountered in Real Analysis: $z$ is "large" if $|z| > R$ for all $R \in \mathbb{R}$.

**Definition 5.1** (Limits at Infinity)**.** For $z_0 \in \mathbb{C}$ we say

$$\lim_{z \to z_0} f(z) = \infty$$

if given some $R > 0$, $R \in \mathbb{R}$, there exists some $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z)| > R.$$

**Example 5.1.** We have $\lim_{z \to 0}(\frac{1}{z}) = \infty$ since given $R > 0$, there exists a $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|\frac{1}{z}| > R$, namely, $\delta = \frac{1}{R}$, because

$$|z| < \frac{1}{R} \implies \frac{1}{|z|} > R \iff \left|\frac{1}{z}\right| > R.$$

**Definition 5.2** (Limits to Infinity)**.** We say $\lim_{z \to \infty} f(z) = L$, $L \in \mathbb{C}$ if and only if for all $\varepsilon > 0$, there exists some $R > 0$ such that

$$|z| > R \implies |f(z) - L| < \varepsilon.$$

**Example 5.2.** We have $\lim_{z \to \infty} \frac{1}{z} = 0$, let $\varepsilon > 0$, $R = \frac{1}{\varepsilon}$. Then $|f(z) - L| = \left|\frac{1}{z}\right|$, so

$$|z| > R \implies |z| > \frac{1}{\varepsilon} \implies \varepsilon > \frac{1}{|z|} = \left|\frac{1}{z}\right|,$$

and we are done.

**Definition 5.3.** Finally, we say

$$\lim_{z \to \infty} f(z) = \infty$$

if (for $R_1, R_2 \in \mathbb{C}$) given some $R_1 > 0$, there exists an $R_2 > 0$ such that

$$|z| > R_2 \implies |f(z)| > R_1.$$

**Example 5.3.** We have $\lim_{z \to \infty} z^2 = \infty$ since $|z^2| > R$ whenever $|z| > \sqrt{R}$.

## 5.3 Derivates

We are finally ready to define the derivative of a function (the good stuff). Given a function $f : X \to \mathbb{C}$, we will only define the derivative of $f$ at a point $z \in X^\circ$. Recall that $X^\circ = \{z \in X \mid B(z,\gamma) \subseteq X\}$ for some $\gamma > 0$.

**Definition 5.4** (Complex Derivative). A function $f : X \to \mathbb{C}$ is said to be *differentiable* at $z_0 \in X^\circ$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$ (so limits to infinity are not allowed. We will examine these "poles" later in the course). If the limit exists, we denote this limit as $f'(z_0)$.

**Example 5.4.** Let $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto 7$. We claim that $f'(z) = 0$ for all $z$, since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{7 - 7}{z - z_0} = 0.$$

We only look at the points $z$ "near" (accumulation points) $z_0$, so we don't have to worry about the case where $z = z_0$. So given $\varepsilon > 0$,

$$|z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \varepsilon$$

for any $\delta > 0$.

**Example 5.5.** Let $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z$. We claim $f'(z) = 1$ since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z - z_0}{z - z_0} = 1$$

for any $z \neq 0$. This limit is one since

$$\left| \frac{f(z) - f(z_0}{z - z_0} - 1 \right| = \left| \frac{z - z_0}{z - z_0} \right| = 0.$$

**Example 5.6.** Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. We will show $f'(z_0) = 2z_0$. We want to find a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - 2z_0 \right| < \varepsilon.$$

So

$$\left| \frac{z^2 - z_0^2}{z - z_0} - 2z_0 \right| = |(z + z_0) - 2z_0| = |z - z_0| < \varepsilon$$

if $|z - z_0| < \delta$ with $\delta = \varepsilon$. There aren't any limit signs because we directly invoked the epsilon-delta definition.

**Example 5.7.** Consider $f(z) = |z|$ (maps will map $\mathbb{C} \to \mathbb{C}$ unless otherwise stated from now on). We have showed $f$ is continuous for all $z$, but $f$ isn't differentiable at 0. Use the technique at the end of the last example (write out the piecewise definition of the absolute value and show that the limits don't agree).

What about $z_0 \neq 0$? Is $f : \mathbb{C} \setminus \{0\} \to C$ differentiable? Let $z_0 \in C \setminus \{0\}$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z| - |z_0|}{z - z_0} = \frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}}.$$

We let $z$ get close to $z_0$ in two different ways. First, assume $r = r_0$ but $\theta \neq \theta_0$ (vary the angle, but all having length $r$). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{0}{r(e^{i\theta} - e^{i\theta_0})} = 0.$$

Next, assume $r \neq r_0$ but $\theta = \theta_0$ (points on a line with angle $\theta$, vary the length). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{r - r_0}{e^{i\theta}(r - r_0)} = e^{-i\theta} \neq 0.$$

So $f$ is nowhere differentiable.

## 5.4 Product, Quotient, and Chain Rules

To get $f'(z)$ for $f(z) = z^m$, we want a formula. Time for induction!

**Theorem 5.2.** *If $f'(z_0)$ and $g'(z_0)$ exist for two functions $f$ and $g$, then so do the derivatives*

1. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$,
2. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,
3. $(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$ *provided* $g(z_0) \neq 0$.

**Theorem 5.3.** *If $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$ then $f \circ g$ is differentiable at $z_0$ and*

$$(f \circ g)'(z_0) = f'[g(z_0)]g'(z_0).$$

**Note** (Leibniz Rule)**.** Suppose we have $f_1, f_2, ..., fn$ functions all differentiable at $z_0$. Then

$$(f_1 f_2 f_3 \cdots f_n)'(z_0) = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + f_1 f_2 f_3' f_4 \cdots f_n + \cdots.$$

In particular, $(z^n)' = n(z'z^{n-1}) = nz^{n-1}$ (just take $f_i = f$ and it becomes clear that this is true).

Lecture 6

# September 15, 2020

I could be studying fundamental groups right now, but instead I'm sitting here verifying limits and derivatives by hand. Why?? OK so Gradescope is a meme. Anything new?

Everything so far has been awfully boring. But now it gets interesting. Finally, I've been waiting for this.

## 6.1 Cauchy-Riemann Equations

Suppose we have a function $f : \mathbb{C} \to \mathbb{C}$, write it as $f(z) = u(x,y) = iv(x,y)$. Assume $f'(z_0)$ exists, and is equal to

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(u(x,y) + iv(x,y)) - (u(x_0,y_0) + iv(x_0,y_0))}{(x+iy) - (x_0 + iy_0)}.$$

Then we rewrite this to get

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y) - u(x_0,y_0) + i[v(x,y) - v(x_0,y_0)]}{(x-x_0) + i(y-y_0)}.$$

Consider two special ways $(x,y)$ can be "near" $(x_0,y_0)$. First, let $x = x_0$ but $y \neq y_0$. Then the quotient becomes

$$\frac{u(x_0,y) - u(x_0,y_0) + i[v(x_0,y) - v(x_0,y_0)]}{i(y-y_0)} =$$
$$\frac{u(x_0,y) - u(x_0,y_0)}{i(y-y_0)} + \frac{v(x_0,y) - v(x_0,y_0)}{y-y_0}.$$

Then the limit is equal to

$$\frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0).$$

Now let $y = y_0$ but $x \neq x_0$. Then the quotient becomes

$$\frac{u(x,y_0) - u(x_0,y_0) + i[v(x,y_0) - v(x_0,y_0)]}{x-x_0} = \frac{u(x,y_0) - u(x_0,y_0)}{x-x_0} + \frac{i[v(x,y_0) - v(x_0,y_0)]}{x-x_0},$$

so the limit $f'(z_0)$ is equal to

$$\frac{\partial u(x_0,y_0)}{\partial x} + i\frac{\partial v}{\partial x}(x_0,y_0).$$

Why are we doing this? It's because if the limit exists, it should be the same whichever direction you approach it from, so you can derive some cool equalities.

The two equations must agree, so

$$\frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0).$$

Examine the real and imaginary parts, so we have

$$\frac{\partial v}{\partial y}(x_0,y_0)=\frac{\partial u}{\partial x}(x_0,y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0,y_0)=-\frac{\partial u}{\partial y}(x_0,y_0). \tag{1}$$

These are known as the *Cauchy-Riemann Equations*. Furthermore,

$$f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0).$$

What does this tell us? If your function is differentiable at a point, then we have a way to compute the derivative at that point. The converse does not hold, that is, if the Cauchy-Riemann equations hold this doesn't necessarily guarantee the existence of a derivative at that point.

**Example 6.1.** Recall that the function $f(z)=|z|$ is nowhere differentiable. However, consider $g(z)=|z|^2=x^2+y^2=u(x,y)+iv(x,y)$, where $v(x,y)=0$ and $u(x,y)=x^2+y^2$. Let's check to see if this function satisfies the Cauchy-Riemann equations. $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=2y$, $\frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=0$. Does $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$? Only if $2x=0 \implies x=0$. Does $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$? Only if $2y=0 \implies y=0$. So this function only satisfies the Cauchy-Riemann equations at the origin, which implies that the function is nowhere differentiable at any other point ($g'(z_0)$ does not exists if $z_0\neq 0$). Now it does satisfy the CR equations at 0, but we don't have the existence of the derivative guaranteed.

Let's check the case if $z_0=0$.

$$\frac{g(z)-g(0)}{z-0}=\frac{|z|^2-0}{z-0}=\frac{\overline{z}z}{z}=\overline{z}.$$

Does $\lim_{z\to 0}\overline{z}$ exist? Yes, and it's equal to 0. So $g'(0)$ exists and is equal to 0.

## 6.2 Weak Converse of CR Equations

Let's talk about the opposite of the CR equations.

**Theorem 6.1** (Weak Converse of Cauchy-Riemann Equations)**.** *Let $f=u+iv$ be defined on a neighborhood of $z_0=x_0+iy_0$. Suppose the partial derivatives of $u$ and $v$ exist in that neighborhood, and are continuous at $z_0$. Furthermore, suppose the functions $u$ and $v$ satisfy the CR-equations at $z_0$. Then $f'(z_0)$ exists.*

**Note.** We claim the hypotheses hold for $|z|^2: u(x,y)=u^2+y^2$, $v(x,y)=0$. Oops, I went to the restroom here. I don't think I missed anything interesting though.

Now the next topic is very important.

**Example 6.2.** Let $f(x,y)=e^x(\cos(y)+i\sin(y))=e^x\cos(y)+ie^x\sin(v)$. Note: $u$ and $v$ are *nice*. Let's compute the CR equations: $\frac{\partial u}{\partial x}=e^x\cos(y)=\frac{\partial v}{\partial y}=e^x\cos(y)$. We also have $\frac{\partial u}{\partial y}=-e^x\sin(y)=-\frac{\partial v}{\partial x}=-e^x\sin(y)$. Then $f$ is differentiable everywhere, furthermore,

$$f'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=e^x\cos(y)+ie^x\sin(y)=e^x(\cos(y)+i\sin(y))=f.$$

So $f$ is equal to its derivative everywhere. This is probably the single most important function in the entire course.

We will eventually denote this function as $\exp(z)$. Notee: if we use Euler's formula $e^{i\theta}=\cos\theta+i\sin\theta$, then

$$\exp(z)=e^xe^{iy}=e^{x+iy}=e^z.$$

But we have to make sure we can add the exponents first.

## 6.3 CR Equations in Polar Coordinates

Before discussing this further, consider polar coordinates for $z$. For any function $g$, write $f(z) = u(r,\theta) + iv(r,\theta)$. Then after the change of coordinates we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

where $x = r\cos\theta$, $y = r\sin\theta$. We have $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$. Use these to get

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

and

$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta.$$

If $f$ is differentiable, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Plugging these into the CR equations, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, ??stopmoving the page$$

Next time: no. We'll do it next time. We have a test in 2 weeks BTW.

Lecture 7

# September 17, 2020

## 7.1 CR Equations (cont)

Last time: Cauchy Riemann equations for $f = u + iv = u(x,y) + iv(x,y)$. They are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Also discussed this for polar coordinates, yada yada.

**Example 7.1.** Let $f(z) = \frac{1}{z^4}$.

$$\frac{1}{z^4} = \frac{1}{r^4 e^{i4\theta}} = \frac{1}{r^4}e^{-i4\theta} = \frac{1}{r^4}[\cos(4\theta) - i\sin(4\theta)].$$

So

$$\begin{aligned} f' &= e^{-i\theta}\left(-\frac{4}{r^5}\cos(4\theta) + i\frac{4}{r^5}\sin(4\theta)\right) \\ &= -\frac{4}{r^5}e^{-i\theta}(e^{-i4\theta}) \\ &= -\frac{4}{r^5}e^{-5i\theta} = -\frac{4}{r^5 e^{5i\theta}} \\ &= -\frac{4}{z^5}, \end{aligned}$$

which is a known formula for the derivative.

## 7.2 Analytic Functions

**Definition 7.1** (Analytic)**.** A function $f$ is said to be analytic at $z_0$ if $f$ is differentiable at all $z$ in some ball centered at $z_0$. $f$ is said to be analytic on a set $S$ if for all $z_0 \in S$, $f$ is analytic at $z_0$.

**Example 7.2.** Let $f(z) = z^m$, $m \in \mathbb{N}$. Then $f'(z_0) = mz_0^{m-1}$ for all $z_0$. So such $f$ are analytic in $\mathbb{C}$. If $m \in \mathbb{Z} \setminus \mathbb{N}$, this formula still holds for $z_0$ nonzero. So $f$ is analytic on the punctured plane $\mathbb{C} \setminus \{0\}$.

**Definition 7.2** (Entire function)**.** We say a function $f$ is *entire* if $f$ is analytic on $\mathbb{C}$. For example, $f(z) = z^3$ is entire. More generally, all polynomials are entire.

**Theorem 7.1.** *If $f'(z) = 0$ for all $z \in D$ a domain, then $f$ is constant in $D$. (Is this weak Liouville's Theorem?)*

*Proof.* We will show that for any pair $z_1, z_2 \in D$, $f(z_1) = f(z_2)$. Let $z_1, z_2 \in D$, then there is some finite set of straight lines connecting $z_1$ and $z_2$ (what is this definition reeee). Consider $f$ on a segment $z = z(t)$, $0 \le t \le 1$. Then $F(t) = f[z(t)]$, $0 \le t \le 1$ which is equal to $u[x(t), y(t)] + iv[x(t), y(t)]$ So

$$\frac{dF}{dt} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\right].$$

By assumption, $\frac{df}{dz} = 0$ in $D$. We can write this in two ways: $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0$. So $\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial x}{\partial y}$ in $D$, and so $\frac{dF}{dt} = 0$, $0 \le t \le 1$.

Consider $\mathrm{Re}[F(t)] = u[x(t), y(t)]$. It follows that $\frac{d}{dt}u(x(t), y(t)) = 0$, $0 \le t \le 1$. It follows that $u(x(t), y(t))$ is constant, since

$$\int_0^1 \frac{d}{dt}u(x(t), y(t))\,dt = 0.$$

Similarly, $V(x(1), y(1)) - V(x(0), y(0)) = 0$. So $F(t) = f(z(t))$ has the same values at $z_a$ and $z_b$. (What?? Why??) Once we get Louiville's theorem we will get a less bad proof. This proof was big bad. ⊠

We will show later that if $f = u + iv$ is analytic at $z_0$, then $u(x, y)$ and $v(x, y)$ have partial derivatives of all orders in a neighborhood of $z_0$. If a function is analytic, then it is infinitely differentiable: what?? Complex analysis is crazy.

## 7.3 Harmonic Equations

**Definition 7.3** (Harmonic)**.** If $u_{xx} + u_{yy} = 0$, $u$ is *harmonic* in that nbd of $z_0$, similarly for $v$. Laplace equation.

**Definition 7.4.** We say $v$ is a *harmonic conjugate* of $u$ in some region $D$ if $u$ and $v$ are harmonic in $D$, and $u, v$ satisfy the CR equations.

**Note.** For some analytic function $f$ its true that $\overline{f(z)} = f(z)$.

**Theorem 7.2** (The Reflection Principle)**.** *"Apparently this is famous, but I've never used it" -Dr. Radin*

*Suppose $f$ is analytic in a domain $D$ which is symmetric WRT the $x$-axis. Then for all $z \in D$,*

$$\overline{f(z)} = f(\overline{z}) \iff f \text{ is real on the segment of the } x\text{-axis in } D.$$

**Note.** I've been taking less notes because I'm simultaneously doing my weekly Differential Equations quiz while TEXing notes. Just wanted to say that

Lecture 8

# September 22, 2020

Last time: for any $z = (x, y) \in \mathbb{C}$, $\exp(z) = e^x[\cos y + i \sin y] = e^x e^{iy}$. $e^z = \mathrm{Exp}(z) = \exp(z)$. We showed this function is differentiable on $\mathbb{C}$ and that its derivative is itself.

The product of complex numbers $e^{z_1}e^{z_2} = (e^{x_1}e^{iy_1})(e^{x_2}e^{iy_2})$. Since multiplication is commutative, we have $(e^{x_1}e^{x_2})(e^{iy_1}e^{iy_2}) = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{z_1+z_2}$. This follows from our defintions, its not an assumption.

**Corollary 8.1.** *$e^z e^{-z} = e^0 = 1$. So $e^{-z} = \frac{1}{e^z}$. Also, for $m = 1, 2, \cdots$ $(e^z)^m = e^{mz}$. This also holds for negative integers. Finally, by our differentiation rules,*

$$\frac{d}{dz}e^{az^n} = naz^{n-1}e^{az^n}.$$

So far we've covered how to differentiate polynomials (or more generally, rational functions), and now we've added $e^z$ to our arsenal. Let's introduce some more basic functions to our list. Why do we differentiate? This is a course in functions of a complex variable, differentiating them, integrating the, etc. (I wish we covered analytic continuity). The next set of functions are trig functions.

## 8.1 Trig functions

Recall that $e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$, so $\frac{e^{ix}+e^{-x}}{2} = \cos x$, $\frac{e^{ix}-e^{-x}}{2} = \sin x$. (Not sure if I got the definitions right). We can extend these to the complex plane, we define $\cos z = (e^{iz}+e^{-iz})/2$, $\sin z = (e^{iz}-e^{-iz})/2i$ for all $z$[1]. So $\frac{d}{dz}\cos z = (ie^{iz} - ie^{-iz})/2 = i^2\sin z = -\sin z$. Similarly, $\frac{d}{dz}\sin z = (ie^{iz}+ie^{-iz})/2i = \cos z$. So these formulas agree with their real analog. We write the definitions again for clarity:

**Definition 8.1.** We define the trigonometric functions $\sin z$ and $\cos z$ on $\mathbb{C}$ as

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz}+e^{-iz}}{2}.$$

Now by our new definitions of trig functions,

$$\cos z + i\sin z = \frac{e^{iz}+e^{-iz}}{2} + i\frac{e^{iz}-e^{-iz}}{2i} = e^{iz}.$$

From our definition,

$$\sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{zi}.$$

We claim that this is equal to $\sin z_1 \cos z_2 + \cos z_2 \sin z_1$. This is just a bunch of tedious manual labor. I don't really want to type this out, but here I am. We have this equal to

$$\begin{aligned}
&\left(\frac{e^{iz_1}-e^{iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right)+\left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)\\
&= \frac{1}{4i}\left[e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)} + e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(z_1+z_2)} - e^{i(-z_1-z_2)}\right]\\
&= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{i(-z_1-z_2)}\right]\\
&= \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{i(-z_1-z_2)}\right]\\
&= \sin(z_1+z_2).
\end{aligned}$$

I think I may have typed the second (long) equation incorrectly, but I am not in the mood for going back and double checking this. Manual labor should be reserved for homework (and even then, I am still unwilling to do it).

We have a special case: $\sin(z+2\pi) = \sin(z)\cos(2\pi) + \cos(z)\sin(2\pi) = \sin(z)$. Clearly this generalizes to $\sin(z+2\pi n)$ for $n \in \mathbb{Z}$. So sin is periodic.

**Definition 8.2** (Tangent)**.** Let

$$\tan z = \frac{\sin z}{\cos z}.$$

Note that this isn't defined at $\cos z = 0$: when does this happen? I stopped taking notes here for a little bit.

[1]Note that $e^{iz}$ is differentiable, and $\frac{d}{dz}e^{iz} = ie^{iz}$.

## 8.2 Hyperbolic trig functions

Let's look at another class of functions: the cool dudes, hyperbolic trig functions (sinh is pronounced "sinch", cosh is pronounced "coush", etc). This gives me good memories of my first Calculus class with Dr. Neal Brand at UNT.

**Definition 8.3.** We define the hyperbolic trig functions $\cosh z$ and $\sinh z$ as

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Similarly, $\tanh z$ is defined as

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

Lecture 9

# September 24, 2020

Last time: hyperbolic trig functions. We have some functions and theorems to deal with them, like rational functions $p/q$, trig functions $\cos z, \sin z, \tan z$, and hyperbolic trig functions $\cosh z, \sinh z$, the exponential function $\exp(z)$, etc. Where these functions are defined they are analytic.

## 9.1 Logarithmic functions

Let $f$ be a function. Then $f(z) = w \iff z = f^{-1}(w)$. This is only a function if $f$ is onto. Consider the case where $f(z) = e^z$ and $e^z = w$.

**Definition 9.1** (Logarithm)**.** We define the functional inverse of the exponentatial function as the *logarithm*, that is,

$$\log w = \{z \mid e^z = w\},$$

that is, $\log = \exp^{-1}$. Suppose $z = x + iy$, so $e^z = e^{x+iy} = e^x(\cos x + i \sin y) = w$. Write $w$ in polar form, so $w = |w|e^{i \arg w}$. What values of $z$ give rise to this? We want $e^x = |w|$, $e^{iy} = e^{i \arg w}$. $e^x = |w| \iff x = \ln|w|$, so $y \in \arg w$. Therefore we have

$$\log w = \ln|w| + i \arg w$$

for $w \neq 0$. So this function is multivariate.

We want to do calculus with this function. We are not terribly interested in multivalued things (actual functions lmao). One way to get an actual function is to define

$$\operatorname{Log} w = \ln|w| + i \operatorname{Arg}(w)$$

for $w \neq 0$. But that's not enough: we also want functions to be differentiable, etc. Note that this function Log wouldn't be continuous on the negative real axis. If $w \in S^1$, then ... I stopped paying attention here. Because of the discontinuity (which I was not paying attention for), we define $\operatorname{Log}(w) = \ln|w| + i \operatorname{Arg} w$ only for $w$ not a negative real number.

**Claim.** Log satisfies the CR equations. Verify this in your free time.

Note that $\frac{d}{dz} \operatorname{Log} z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{1}{r} + 0) = \frac{1}{z}$ on its domain. This is very cool, thank you logarithm. This is useful, but we need other ways to get "honest" functions from $\log w = \ln|w| + i \arg w$.

aa hayaku ouchi ni kaeritai

Recall the identity that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, with addition being componentwise for the infinite sets. So $\log(z_1 z_2) = \ln|z_1 z_2| + i \arg(z_1 z_2) = \ln|z_1| + \ln|z_2| + i \arg z_1 + i \arg z_2 = \log z_1 + \log z_2$.

Where are we headed? To the next class of functions $z^\alpha$ and beyond.

Lecture 10

# October 1, 2020

Last time: we had a test. Today we'll talk about branches and next time move onto integration, finally. So I had a test until 11 today, and I probably won't take many (any) notes because I'm doing my homework (due at 2) during the lecture. (If anybody wants to send me notes, please do, my email is simonxiang@utexas.edu).

## 10.1 Catching up: logarithms and branches

So I didn't pay attention for the last week and now I'm paying for it, because I have to take extra notes to stay on track (and finish the homework). A side note: if we think of branches as sheets and the logarithm only being defined on simply connected domains, this is suprisingly similar to the theory of covering spaces in algebraic topology. Maybe I can find a way to connected a fundamental group to the codomain of the logarithm... or am I just spouting nonsense?

Motivation: solve equations of the form $e^w = z$, $z$ is nonzer0 (or else the earth collapses). Write this as $e^u e^{iv} = re^{i\Theta}$ where $w = u + iv$, $\Theta = \operatorname{Arg}\theta$. Since $r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff r_1 = r_2$, $\theta_1 = \theta_2 + 2\pi n$ for some $n \in \mathbb{Z}$, we have

$$e^u e^{iv} = re^{i\Theta} \iff e^u = r \text{ and } v = \Theta + 2\pi n$$

for some $n \in \mathbb{Z}$. Now $e^u = r \implies u = \ln r$ in the traditional real valued sense. So $e^w = z$ if and only if

$$w = \ln r + i(\Theta + 2\pi n)$$

for some $n \in \mathbb{Z}$. If we write $\log z = \ln r + i(\Theta + 2\pi n)$, then $e^{\log z} = z$ for $z \neq 0$.

**Definition 10.1** (Multivalued logarithm)**.** We have $\log z$ for $z \in \mathbb{C}$, $z \neq 0$ defined by

$$\log z = \ln r + i(\Theta + 2\pi n),$$

where $n \in \mathbb{Z}$, $z = re^{i\Theta}$, $\Theta = \operatorname{Arg}\theta$.

**Example 10.1.** Let $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -\frac{2\pi}{3}$. So

$$\log(z) = \ln 2 + i - \left(\frac{2\pi}{3} + 2\pi n\right) = \ln 2 + 2\pi i\left(n - \frac{1}{3}\right)$$

for $n \in \mathbb{Z}$.

Note that $\log(e^z) = z$ does not necessarily hold. We can write $\log z = \ln|z| + i\arg z$, which implies

$$\log(e^z) = \ln|e^z| + i\arg(e^z) = \ln(e^x) + i(y + 2\pi n) = (x + iy) + 2i\pi n,$$

since $|e^z| = e^x$ and $\arg(e^z) = y + 2\pi n$ for some $n \in \mathbb{Z}$ (this can be seen by writing $\exp z$ as $e^x e^{iy}$). So

$$\log(e^z) = z + 2\pi i n$$

for $n$ an integer. I'm sick of this. From now on, $n$ denotes an integer, that is, some $n \in \mathbb{Z}$. You can figure out when this abuse of quantifiers ends by context. We can define the principle value of $\log z$ at $n = 0$ by

$$\operatorname{Log} z = \ln r + i\Theta.$$

Note that $\operatorname{Log} z$ is well-defined and single-valued when $z \neq 0$, furthermore, $\log z = \operatorname{Log} z + 2\pi i n$. If $z \in \mathbb{R}$, this is just the standard logarithm from calculus.

**Example 10.2.** Here's a cool trick: we can define the logarithm of negative numbers now (something we couldn't do in calculus), since $\log(-1) = \ln 1 + i(\pi + 2\pi n) = i\pi(2n + 1)$, $\operatorname{Log}(-1) = i\pi$.

## 10.2 Branches of the logarithm

If we restrict $\theta$ such that for some $\alpha \in \mathbb{R}$, $\alpha < \theta < \alpha + 2\pi$, then $\log z = \ln r + i\theta$ with components $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ is single-valued and cointuous in the stated domain (?). It's defined from the $x$-axis to the angle it makes with $\alpha$— note that it isn't defined on the ray $\theta = \alpha$, because some neighborhood of $z$ (on such ray) will contain points near $\alpha$ and $\alpha + 2\pi$. Not only is this restricted logarithm continuous, but it's also analytic on its domain, because of CR. Then by the derivative of polar stuff, we have

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i\theta\right) = \frac{1}{re^{i\theta}}.$$

So $\frac{d}{dz}\log z = \frac{1}{z}$ when $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$. In particular, $\frac{d}{dz}\operatorname{Log} = \frac{1}{z}$ for $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$.

A *branch* of a multivalued function $f$ is any single-valued function $F$ that's analytic in some domain $\Omega$, where for any $z \in \Omega$ we have $F(z)$ one of the values of $f$. Whoever wrote this textbook needs to stop overusing references, please. Note that for $\alpha \in \mathbb{R}$, log restricted to $\alpha$ is a branch of the general logarithm. We say $\operatorname{Log} z = \ln r + i\Theta$ for $r > 0$, $-\pi < \Theta < \pi$ is the principal branch. A *branch cut* is a portion a line or curve that is introduced in order to define a branch $F$ of a multivalued function $f$. Point on the branch cut are singular (have no well-defined nbd) and any point common to every branch cut of $f$ is a branch point. For example, the branch cut for the principal branch is the origin plus the ray $\Theta = \pi$, and the origin is a branch point for log.

**Example 10.3.** Take the principle branch $\operatorname{Log} z = \ln r + i\Theta$. Then $\operatorname{Log}(i^3) = \operatorname{Log}(-i) = \ln 1 - i\frac{\pi}{2} = -i\frac{\pi}{2}$, but $3\operatorname{Log} i = 3\left(\ln 1 + i\frac{\pi}{2}\right) = i\frac{3\pi}{2}$. So $\operatorname{Log}(i^3) \neq 3\operatorname{Log} i$.

## 10.3 Logarithmic identities

These derivations aren't interesting IMO. If $z_1, z_2 \in \mathbb{C}$, we have

$$\log(z_1 z_2) = \log z_1 + \log z_2, \tag{2}$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2. \tag{3}$$

Let $z \in \mathbb{C}$. Then if you write $z = re^{i\theta}$, it can be seen that

$$z^n = e^{n\log z}. \tag{4}$$

Furthermore, for $z \neq 0$, $k \in \mathbb{N}$, we have

$$z^{1/k} = \exp\left(\frac{1}{k}\log z\right), \tag{5}$$

$$\exp\left(\frac{1}{k}\log z\right) = \sqrt[k]{r}\exp\left[i\left(\frac{\Theta}{k} + \frac{2\pi n}{k}\right)\right]. \tag{6}$$

## 10.4 Complex exponents

**Definition 10.2** (Complex exponential function)**.** For $z \neq 0$, $c \in \mathbb{C}$, we have the function $z^c$ defined as

$$z^c = e^{c\log z}. \tag{7}$$

We already know this holds for $c = n$ or $c = \frac{1}{n}$. Usually powers of $z$ are multivalued.

**Example 10.4.** We have $i^{-2i} = \exp(-2i\log i)$, where $\log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = i\pi\left(2n + \frac{1}{2}\right)$. Then $i^{-2i} = \exp[\pi(4n+1)]$. Note that every value of $i^{-2i}$ lies in $\mathbb{R}$.

Since $1/e^z = e^{-z}$, then $\frac{1}{z^c} = \frac{1}{\exp(c\log z)} = \exp(-c\log z) = z^{-c}$. So $1/i^{2i} = i^{-2i}$, and we have $\frac{1}{2i} = \exp[\pi(4n+1)]$. For $z = re^{i\theta}$ and $\alpha \in \mathbb{R}$, the branch $\log z$ of $\alpha$ is single-valued and analytic on its domain. Using that branch, it follows that $z^c = \exp(c\log z)$ is also single-valued and analytic on such domain. Then the derivtive of that branch of $z^c$ is given by

$$\frac{d}{dz}z^c = \frac{d}{dz}\exp(c\log z) = \frac{c}{z}\exp(c\log z) = c\frac{\exp(c\log z)}{\exp(\log z)} = c\exp[(c-1)\log z] = cz^{c-1},$$

for $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$. The principle value is what you think it is: P.V. $z^c = e^{c \operatorname{Log} z}$, and so is the principal branch of $z^c$, when $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$.

**Example 10.5.** The principal value of $(-i)^i$ is

$$\exp[i \operatorname{Log}(-i)] = \exp\left[i\left(\ln 1 - i\frac{\pi}{2}\right)\right] = \exp\frac{\pi}{2},$$

that is, P.V.$(-i)^i = \exp\frac{\pi}{2}$.

**Example 10.6.** The principle branch of $z^{2/3}$ can be written as

$$\exp\left(\frac{2}{3}\operatorname{Log} z\right) = \exp\left(\frac{2}{3}\ln r + \frac{2}{3}i\Theta\right) = \sqrt[3]{r^2}\exp\left(i\frac{2\Theta}{3}\right).$$

So P.V. $z^{2/3} = \sqrt[3]{r^2}\cos\frac{2\Theta}{3} + i\sqrt[3]{r^2}\sin\frac{2\Theta}{3}$.

**Example 10.7.** Let $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$. Then $(z_1z_2)^i = 2^i = e^{i\ln 2}$, and

$$\begin{aligned} z_1^i &= e^{i\operatorname{Log}(1+i)} = e^{i(\ln\sqrt{2}+i\pi/4} = e^{-\pi/4}e^{i(\ln 2)/2}, \\ z_2^i &= e^{i\operatorname{Log}(1-i)} = e^{i(\ln\sqrt{2}-i\pi/4)} = e^{\pi/4}e^{i(\ln 2)/2}. \end{aligned}$$

So $(z_1z_2)^i = z_1^iz_2^i$ as expected. But do some similar stuff with $(z_2z_3)^i = e^{-\pi}e^{i\ln 2}$, $z_3^i = e^{3\pi/4}e^{i(\ln 2)/2}$, and we find that $z_2^iz_3^i = e^{2\pi}(z_2z_3)^i$.

We have the exponential with base $c$ defined as

$$c^z = e^{z\log c}$$

for $c$ a nonzero constant in $\mathbb{C}$. When we specify a value for $\log c$, this function is entire, and

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = e^{z\log c}\log c = c^z\log c.$$

Lecture 11

# October 6, 2020

## 11.1 Parametrized curves

Let's talk about integration! Define $I = \int f(z)\,dz$ for *parametrized curves* $\Gamma$, given by $w\colon [a,b] \to \mathbb{C}$. We say the set of points

$$\{w(t) \mid a \le t \le b\}$$

is the "trace" of the curve $w$, denoted $\operatorname{tr} w$[2].

**Example 11.1.** Let $w(t) = e^{i2\pi t}$ for $t \in [0, \frac{1}{2}]$. Unfortunately, I'm not cool enough to live-TEX figures in class, so try to use your imagination to see what this curve would look like (a semicircle). Also consider $w(t) = (1+i)t$ for $t \in [1,3]$. This one looks like a straight line.

**Example 11.2.** Consider $w(t) = e^{it}$ for $t \in [0, 4\pi]$. What is $\operatorname{tr} w$? It's simply the unit circle.

Given a parametrized $\Gamma$ and a function $f(z)$ defined on at least $\operatorname{tr}\Gamma$, we define $\int_\Gamma f(z)\,dz$ by a limit of Riemann sums (inb4 not as powerful as integration by Lebesgue measure). I know we never defined what $t$ is, but in each interval $[t_j, t_{j+1}]$ pick some $\hat{t}_{j+i}$ and compute $f(\hat{t}_1)[w(t_1) - w(t_0)] + f(\hat{t}_2)[w(t_2) - w(t_1)] + \cdots$. Using summation notation, we have the sum written as

$$\sum_{j=1}^{m} f(w(\hat{t}_j))[w(t_j) - w(t_{j-1})].$$

[2]Wait, I'm not sure all of a sudden. If this notation is non-standard call me a fool, but this is how it works in linear algebra.

**Example 11.3.** Consider $w(t) = e^{i2\pi t}$ for $t \in [0,1]$, and $f(z) = z^3$. Choose $t_j = \frac{j}{m}$, so $t_0 = 0$ and $t_m = 1$. Let $\hat{t}_j = t_j$.Consider the approximations of $\int_\Gamma z^3\,dz$, given by

$$\sum_{j=1}^{m} f[w(t_j)][w(t_j) - w(t_{j-1})] =$$
$$\sum_{j=1}^{m} f[e^{i2\pi t_j}][e^{i2\pi t_j} - e^{i2\pi t_{j-1}}] =$$
$$\sum_{j=1}^{m} f[e^{i2\pi \frac{j}{m}}][e^{\frac{i2\pi j}{m}} - e^{\frac{i2\pi(j-1)}{m}}] =$$
$$\sum_{j=1}^{m} f[e^{i2\pi \frac{j}{m}}][e^{\frac{i2\pi j}{m}} - e^{\frac{i2\pi(j-1)}{m}}] = \cdots$$

Unfortunately I was too slow to finish the work. This simplifies to

$$\left(1 - e^{\frac{2\pi i}{m}}\right) \sum_{j=1}^{m} \left(e^{i\frac{8\pi}{m}}\right)^j,$$

which is a geometric series of the form $S = \sum_{j=K}^{L} a^j$ and are easy to compute. We have $aS = S - a^K + a^{L+1}$, so $(1-a)S = a^K - a^{L+1} \implies S = \frac{a^K - a^{L+1}}{1-a}$. This proof is pretty much the same as any one in a calculus course. For $S = \sum_{j=1}^{m} e^{\left(\frac{i8\pi}{m}\right)^j}$, we have

$$S = \frac{e^{\frac{i8\pi}{m}} - \left(e^{i\frac{8\pi}{m}}\right)^{m+1}}{1 - e^{i\frac{8\pi}{m}}} \tag{8}$$

I missed something else big, gotta get faster. So $\int_\Gamma z^3\,dz = 0$.

Lecture 12

# October 8, 2020

## 12.1 More on integration

Last time: we were defining definite integrals on functions $f(z)$ on parametrized curves $\Gamma$, denoted by $\int_\Gamma f(z)\,dz$. He's talking about Riemann integration by adding squares, we learned this back in high school. The difference is that the $\Delta z_i$'s are occurring on a curve rather than an axis or interval. Wait, I missed something important. We have

$$I = \int_a^b f[w(t)]w'(t)\,dt.$$

This is how we actually define $\int_\Gamma f(z)\,dz$. Since $f[w(t)]w'(t) = g_1(t) + ig_2(t)$, where the $g_i$ for $i \in \{1,2\}$ are real, we have $\int_a^b f[w(t)]w'(t)\,dt$ equal to

$$\int_a^b g_1(t)\,dt + i\int_a^b g_2(t)\,dt.$$

Now it's just first year calculus.

**Example 12.1.** For $\Gamma$, $z = w(t) = e^{i\pi t}$, $0 \le t \le 1$ a semicircle, we have $f(z) = z^2$. Then

$$I = \int_\Gamma f(z)\,dz = \int_a^b f[w(t)]w'(t)\,dt = \int_0^1 e^{2\pi i t} i\pi e^{i\pi t}\,dt = i\pi \int_0^1 e^{3\pi i t}\,dt.$$

Recall that $\frac{d}{dz}e^{cz} = ce^{cz}$. So this result is just a special case of something we already know. We claim this integral is equal to $i\pi \frac{e^{3\pi it}}{3\pi i}\Big|_0^1$. Refer to earlier, then assume $g_1(t) = \frac{dG_1(t)}{dt}$, $g_2(t) = \frac{dG_2(t)}{dt}$. Then

$$I = \big(G_1(b) - G_1(a)\big) + i\big(G_2(b) - G_2(a)\big) = [G_1(b) + iG_2(b)] - i[G_1(a) + iG_2(a)].$$

If $g_1 + ig_2 = \frac{d}{dt}[G_1 + iG_2]$, (not sure about the logical stuff I'm just writing words at this point), we have $I = (G_1 + iG_2)b - (G_1 + iG_2)a$. So the fundamental theorem "generalizes". Going back to the claim, the integral is equal to $i\pi \frac{e^{3\pi i} - 1}{3\pi i} = -\frac{2}{3}$.

**Example 12.2.** Let $\Gamma_1 : z = w(t) = e^{i\pi t}$, $0 \le t \le \frac{1}{4}$, a cheesecake slice of the unit circle up to $e^{i\frac{\pi}{4}}$, with the function $f(z) = z^3$. So

$$I_1 = \int_0^{\frac{1}{4}} e^{i3\pi t} i\pi e^{i\pi t}\, dt = i\pi \int_0^{\frac{1}{4}} e^{i\pi 4t}\, dt = i\pi \frac{e^{i\pi 4t}}{4\pi i}\Big|_0^{\frac{1}{4}} = \frac{1}{4}[e^{i\pi} - 1] = -\frac{1}{2}.$$

**Example 12.3.** More example spam. $\Gamma_2 \colon w(s) = e^{i\pi^2}$, $a \le s \le \frac{1}{2}$, where $f(z) = z^3$. Then $I_2 = \int_0^{\frac{1}{2}} e^{i3\pi s^2} 2i\pi s e^{i\pi s^2}\, ds$. Why are we changing variables back to the previous example? Something about the invariance principle, I'll read more on this later. I'm barely paying attention, but I think what's happening is that depedence on the orientation of parametrization or the actual parametrization itself isn't that important, as it should: why would changing the direction screw up the entire integral? Well, it might somewhere else, but not here.

## 12.2 Numerical methods for estimating integrals

Welcome to the section where those who only talk in abstract nonsense stop paying attention (aka, how is it even fathomable that the things you study might be **applied** to a *real life* scenario?? Unacceptable!). I didn't intend to actually stop paying attention, but I spent too much time figuring out how to get rainbow colors and emoji in LaTeX that I missed a big chunk of information. Conclusion: $|I| \le ML$. We get an upper bound on the integral. A slightly more complicated upper bound is

$$\begin{aligned} \left|\sum_j f(w(\hat{t}_j))w'(t)[t_j - t_{j-1}]\right| &\le \sum \left|f(w(\hat{t}_j))w'(t)[t_j - t_{j-1}]\right| \\ &\le \langle please\, give\, me\, some\, time\, to\, copy\, down\, the\, equations \rangle \end{aligned}$$

What about $\Gamma_1, \Gamma_2, \Gamma_1 \neq \Gamma_2$? Nah.

**Example 12.4** (important)**.** Let $\Gamma \colon e^{it}$, $0 \le t \le 2\pi$, where $f(z) = \frac{1}{z}$. We have

$$I = \int_0^{2\pi} e^{-it} ie^{it}\, dt = 2\pi i.$$

What we're doing is integrating over the full circle, where the function is bad on the origin but nice on the circle. What if we did it on the open circle $\widetilde{\Gamma}$ instead (homeomorphic to $0, 1)$)? Then $\int_{\widetilde{\Gamma}}$ is approximately $2\pi i$.

Lecture 13

# October 13, 2020

## 13.1 More on integration

Soon we'll get to Cauchy's theorem, the most important theorem in this course (integral on a closed curve is equal to zero). Thank goodness I read the other book, it covered in two pages what we cover in two weeks (sans calculations).

**Theorem 13.1.** *Suppose $f$ is continuous on a domain $D$. Then TFAE:*

1. *$f$ has a primitive $F$ on $D$.*
2. *$\int_\Gamma f(z)\,dz$ along paths $\Gamma \subseteq D$ only depend on the endpoints of $\Gamma$.*
3. *$\oint_\Gamma^0 f(z)\,dz = 0$ for all $\Gamma$ a closed path.*

*Proof.* ($1 \implies 2$) Assume $f = \frac{df}{dz}$ in $D$. Then

$$\begin{aligned}\int_\Gamma f\,dz &= \int_a^b f[(w(t)]w'(t)\,dt \\ &= \int_a^b \frac{dF}{dz}[w(t)]w'(t)\,dt \\ &= \int_a^b \frac{d}{dt}F[w(t)]\,dt \\ &= F[w(b)] - F[w(a)]\end{aligned}$$

by the FTC, finishing the first implication.

($2 \implies 3$) Assume $\Gamma$ is closed loop, choose a basepoint $\gamma$, then $\oint_\Gamma f(z)\,dz = F[w(\gamma)] - F[w(\gamma)] = 0$. Wait, is my proof wrong? Dr. Radin is splitting the curve in two, then noting that they have opposite orientation, implying that the left and right derivatives will cancel.

($3 \implies 1$) Assume that $\oint_\Gamma f(z)\,dz = 0$ for $\Gamma$ a closed path. Define $F(w) = \int_\Gamma f(z)\,dz$ where $w$ is an endpoint of $\Gamma$. From here, it's not hard to show that $\frac{dF}{dz} = f$, finising the proof. ⊠

RIP for Dr. Radin's internet, we lost a good one.

**Example 13.1.** Let $\Gamma\colon w(t) = e^{it}$, where $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Let $f(z) = z^4$. Then

$$I = \int_\Gamma f(z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{4it} i e^{it}\,dt = i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{5it}\,dt = i\frac{e^{5it}}{5i}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{5[2i]} = \frac{2i}{5}.$$

Alternatively, use the theorem above.

**Example 13.2.** Let $D$ be the annulus and $\Gamma$ be a path in the annulus, then $\oint_\Gamma f(z)dz = 0$.

**Example 13.3.** Let $w(t) = e^{it}$ for $0 \leq t \leq 2\pi$, set $f(z) = \frac{1}{z}$. Now $f$ has an antiderivative, say $F = \mathrm{Log}(z)$, but $F$ is not defined on any $D$ containing $\Gamma$, so we can't directly apply the theorem. Trick: define $I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{1}{z}\,dz$, where $\Gamma_\varepsilon$ is the open interval $I \setminus B(z_0, \varepsilon)$ for some basepoint $z_0$, $\varepsilon > 0$. Then this integral is equal to $\mathrm{Log}(-1+\varepsilon i) - \mathrm{Log}(-1-\varepsilon i) = \ln|-1+\varepsilon i| + i\,\mathrm{Arg}(-1+i\varepsilon) - \ln|-1-\varepsilon i| - i\,\mathrm{Arg}(-1-i\varepsilon)$. After some estimates, the ln's reduce to approximately zero, and we get $\sim \pi i - (\sim -\pi i) \cong 2\pi i$.

**Definition 13.1** (Simple curve)**.** A parametrized path is simple if $w(t) \neq w(t')$ for $t \neq t'$. If $\Gamma$ is closed, we can make an exception for the endpoints $w(a) = w(b)$.

## 13.2 Cauchy's Theorem

Here we are.

**Theorem 13.2** (Cauchy's Theorem)**.** *If $f$ is analytic on $\Gamma^\circ$ for $\Gamma$ a simple closed curve, then*

$$\oint_\Gamma f(z)\,dz = 0.$$

Basically, when we talk about interiors we mean the connected open bounded region made from a simple closed path, which exists by the Jordan curve theorem. This is really powerful, since we don't require the existence of an antiderivative, we just need $f$ to be analytic. The earlier theorem is quite elementary, but Cauchy's theorem is much more advanced. Once we prove this, it will follow that analytic functions $f$ have antiderivatives, and we can go on to say that $f$ has second, third, and so on derivatives. This is one of the first "cool" theorems of complex analysis in that it demonstrates how much nicer analytic functions are compared to real valued functions. No proof, unfortunately.

We want to get some consequences from this theorem.

**Definition 13.2** (Simply connected)**.** A domain $D$ is simply connected if every loop $\gamma$ is nullhomotopic, that is, the fundamental group $\pi_1(D)$ is trivial. How it's formulated in complex analysis: the interior of every loop is contained in the domain.

**Corollary 13.1.** *If $f$ is analytic in a simply-connected domain $D$, then*

$$\oint_\Gamma f(z)\,dz = 0$$

*for any closed $\Gamma \subseteq D$.*

*Proof.* Since the interio(unfinished) ⊠

**Corollary 13.2.** *If $f$ is analytic in a simply-connected domain, then $f$ has an antiderivative there.*

Lecture 14

## October 15, 2020

Yay Dr. Radin fixed his internet

### 14.1 Cauchy's integral formula

Last time: see Theorem 13.1 and Theorem 13.2. Say we're working with domains that aren't simply connected, or multiply connected[3] domain, with loops $\Gamma_1$, $\Gamma_2$ encircling the "holes". Say $f$ is analytic on the rest of the domain and continuous on $\Gamma, \Gamma_1, \Gamma_2$, where $\Gamma$ is the boundary of the domain.

**Claim.** We claim that the sum

$$\int_\Gamma f(z)\,dz + \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0.$$

To see this is true, just draw some loops to split the domain into two regions, and since $f$ is analytic on such regions we have the sum equal to zero (take care to note the orientations of the curves).

This is cool because our result tells us about $\int_\Gamma f(z)\,dz$ even though $f$ isn't differentiable everywhere in $\Gamma$.

**Theorem 14.1** (Cauchy's integral formula)**.** *Let $f$ be analytic on a simple closed curve $\Gamma$ with positive orientation. Suppose $z_0 \in \Gamma^\circ$. Then*

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}\,dz = f(z_0).$$

This is called **Cauchy's integral formula**. Note to self: finish reading the section on power series so I can prove this. A disturbing trend in this course is presenting big, important theorems without proof.

[3] A **multiply connected** set is a connected set that isn't simply connected.

**Example 14.1.** Let $f(z) = k$ be a constant. We claim that

$$\int \frac{f(z)}{z-z_0}\,dz = k2\pi i.$$

To see this path in $\Gamma_1$, $\int_\Gamma \frac{k}{z-z_0}\,dz = \int_{\Gamma_1} \frac{k}{z-z_0}\,dz = k2\pi i$, by our earlier claim.

Now let's use this to get a formula for $f^{(m)}(z_0)$. Start with Cauchy's formula for $f(z)$ and $f(z+h)$, then

$$\frac{f(z+h)-f(z)}{h} = \frac{\frac{1}{2\pi i}\oint\left(\frac{f(w)}{w-(z+h)} - \frac{f(w)}{w-z}\right)dz}{h} = \frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{(w-z-h)(w-z)}\,dz.$$

Consider

$$\begin{aligned}
&\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{(z-w)^2}\,dz\right| \\
=&\left|\frac{1}{2\pi i}\oint\left(\frac{f(w)}{(w-z-h)(w-z)} - \frac{f(w)}{(z-w)^2}\right)dz\right| \\
=&\left|\frac{1}{2\pi i}\int \frac{f(w)h\cdot dz}{(w-z-h)(w-z)^2}\,d\right| \\
=&\frac{|h|}{2\pi}\left|\oint_\Gamma \frac{f(w)}{(w-z-h)(w-z)^2}\,dz\right| \le \max\left|\frac{f(w)}{(w-z-h)(w-z)^2}\right|\cdot \text{length}(\Gamma).
\end{aligned}$$

This function doesn't blow up, so it goes to zero as $|h| \to 0$. So we've proven that

$$\frac{1}{2\pi i}\oint \frac{f(w)}{(w-z)^2}\,dz = f^1(z).$$

More generally, one can prove that

$$\frac{m!}{2\pi i}\int \frac{f(w)}{(w-z)^{m+1}}\,dz = f^{(m)}(z)$$

by induction. This prove that $f$ has derivatives of all orders (by explicitly stating them).

## 14.2 Consequences of Cauchy's integral formula

Some consequences:

**Corollary 14.1.** *If $f$ is analytic at $z$, then $f$ has derivatives of all orders at $z$.*

**Corollary 14.2.** *If $f$ is continuous on a domain $D$ and $\oint_\Gamma f(z)\,dz = 0$ for all closed $\Gamma \subseteq D$, then $f$ is analytic in $D$.*

Another consequence: suppose that $f$ is entire, and

$$f^{(m)}(z) = \frac{m!}{2\pi i}\oint \frac{f(z)}{(w-z)^{m+1}}\,dz$$

where $\Gamma$ is the curve $w(t) = z + re^{it}$ for $0 \le t \le 2\pi$. Then

$$\left|f^{(m)}(z)\right| \le \frac{m!}{2\pi}\cdot\frac{M_r}{r^{m+1}}\cdot 2\pi r \le m!\frac{M_r}{r^m},$$

where $M_r$ is a number such that $|f(z)| \le M_r$ on the circle. This is called **Cauchy's inequality**. Reminder that $\left|\int_\Gamma f(z)\,dz\right| \le \max|g| \cdot missed\,sometihng\,here$. We apply this to $g(w) = \frac{f(w)}{(w-z)}^{m+1}$.

**Theorem 14.2** (Liouville's theorem)**.** *The only bounded entire functions are constant.*

Wow, this is a lot of stuff.

Lecture 15

# October 20, 2020

Reminder: we have an exam next week. Covers everything up until now, including stuff on the homework due today (still doing it whoops!). We might do new stuff, or we might review. Also, the homework due next week is due on Thursday, since we don't want it to interfere with studying for the exam.

## 15.1 The Fundamental Theorem of Algebra

Last time: we proved Liouville's theorem (Theorem 14.2), although I don't see a proof in my notes whoops. Let's talk about some applications (we're going to prove the Fundamental Theorem of Algebra, I can feel it!).

Let $p(z)$ be a polynomial in $\mathbb{C}$ with real or complex coefficients, denoted by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for $m \geq 1$, $a_m \neq 0$. Then this function is entire.

**Claim.** This polynomial has at least one root[4].

*Proof.* Assume $p(z) \neq 0$ for all $z$. Consider $f(z) = \frac{1}{p(z)}$, which is also entire. We claim that there is some $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z$. It's easy to see that

$$|z_1 + \cdots + z_i + \cdots + z_n| \geq |z_1| - \cdots |z_i| \cdots - |z_n|$$

by a simple application of the reverse triangle inequality, given $z_i \in \mathbb{C}$. So

$$|p(z)| \geq |a_m z^m| - |a_0| - |a_1 z| - \cdots |a_{m-1} z^{m-1}| \implies \tag{9}$$

$$\frac{|p(z)|}{|z|^m} \geq |a_m| - \frac{|a_0|}{|z|^m} - \cdots - \frac{|a_{m-1}|}{|z|}. \tag{10}$$

We can make each one smaller than $\frac{|a_m|}{|z|^m}$ or something like that, so $\frac{p(z)}{|z|^m} \geq |a_m| - \frac{|a_m|}{2} = \frac{|a_m|}{2}$. On the other hand, $|f(z)| = \frac{1}{|p(z)|}$ is continuous for $|z| \leq K$. So it has a maximum somewhere, say $M$, such that $|f(z)| \leq M$, $|z| \leq K$. This implies that $\left|\frac{1}{f(z)}\right| \geq M$ for $|z| \leq K$, but since $\left|\frac{1}{f(z)}\right| = |p(z)|$, and $|p(z)| \geq \frac{|a_m|}{2}|z| \geq \frac{|a_m|}{2}K^m$ for $|z| \geq K$, this is a contradiction. Don't ask how. When did we apply Liouiville's theorem? Oh right, it's coming up soon. Basically, all this work was to show that $|f(z)|$ is bounded (and entire), and therefore constant, and that's basically the proof. ⊠

The fundamental theorem of algebra is an application of a result from algebra (w0w) that states that $\mathbb{C}$ is an **algebraically closed field**, that is, every polynomial in $\mathbb{C}$ has a zero in $\mathbb{C}$. Since $\mathbb{C}$ is a field extension of $\mathbb{R}$, then every polynomial in $\mathbb{R}$ has solutions in $\mathbb{C} \supset \mathbb{R}$, which is our fundamental theorem. This shows that $\mathbb{C}$ is the **algebraic closure** of $\mathbb{R}$, and can be denoted $\mathbb{R}/\langle x^2 + 1\rangle$.

I also don't see why we need a contradiction for the proof above: we've shown that every polynomial with no roots is constant. The contrapositive is that every non-constant polynomial has a root. What else is there to see?

Suppose $f$ is analytic at $z_0$ and $|f(z)| \leq |f(z_0)|$ in some neighborhood of $z_0$. Consider $w(t) = z_0 + \varepsilon e^{it}$ where $0 \leq t \leq 2\pi$. I missed something, and don't feel like covering it. JK, it was actually a local version of Liouville's theorem, which was a consequence of Cauchy's integral theorem.

OK, he said something about finding a better proof, I'm interested again.

**Theorem 15.1** (Maximum modulus theorem). *If $f$ is analytic and non-constant in some domain, then $|f(z)|$ has no local maximum in such domain.*

[4] So, apparently the proof of the fundamental theorem of algebra is left to the homework, RIP..

## 15.2 Introduction to power series

OK, now we're finally gonna talk about power series. I've been waiting for this. Basically, we've been saying analytic functions, but we never knew that all along, analytic functions really just mean they have a convergent power series expansion around a nbd of such point. AHahahA

We'll show that $f$ is analytic at $z_0$ if and only if

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Lecture 16

# October 22, 2020

Exam next tuesday, don't forget. Today, we'll review logarithms and branches, and after that, maybe cover some new material. Three problems, two on new material, one on old.

## 16.1 Review on logarithms and branches

Recall that

$$\log(z) = \ln|z| + i\arg(z).$$

Here's how we define branches: choose a branch cut (the simplest way is to make a straight line), and choose some value for $\arg(z)$ for some $z_0 \notin$ the branch cut: this defines a well-defined function

$$\widetilde{\log}(z) = \ln|z| + i\,\widetilde{\arg}(z)$$

It turns out this function is defined everywhere besides the branch, and the derivative of any branch is $1/z$. Now every logarithm with the same branch cut differs by a multiple of $2\pi$, so the constant difference will cancel. Question from me: why do two branches have the same derivative? It turns out because it extends uniquely by continuity, so the fixed $z_0$ will only vary by a constant.

Question in class: why do we need branch cuts? Good question, the straight answer is that log wouldn't be well defined otherwise, because things will "overlap". Refer to the Riemann surface of a complex valued logarithm below:

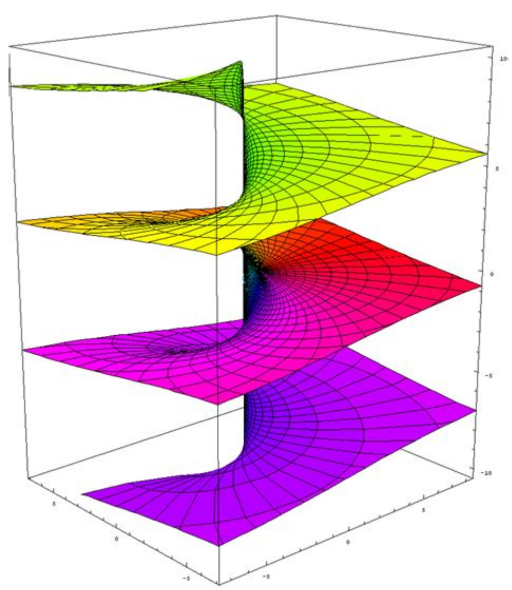

Figure 1: The Riemann surface of the complex logarithm.

Although I haven't really said what a Riemann surface is, you can see that it sort of represents the complex logarithm in 3-d space. Note that it spirals up like a helix, so if you take a vertical line in the $z$-axis it'll hit the surface a (countable) number of times. So to get rid of this, take a random line outward (a branch), and start following the surface upward until you reach the line again but up one level. Then if you restrict the values of the logarithm to everything you just followed, this restricts the spiral to just one coil, starting and ending (not inclusive) at such branch (since at the branch, it'll be defined twice). This gives a well defined branch of the logarithm.

That was my personal answer by the way, Dr. Radin said something different but to the same extent. Now we're talking about $\int_\Gamma \widetilde{\log z}\, dz$. Say it's not defined at some number of points, is this a problem? Ohh, so $f$ has to be continuous on the curve. Of course this is true then. Basically, we want $\operatorname{tr}\Gamma$ to be a closed and bounded (and therefore compact) set, so any function on it will attain a min and a max.

## 16.2 Basic notions of power series

Moving on: maybe at the end of this lecture, we'll have proven the equivalence of analycity and holomorphicity. Oh boy, let's have some fun with convergence of sequences and series (let's break out the epsilon's and delta's!).

**Definition 16.1** (Sequence). A **sequence** is an ordered set

$$\{z_1, z_2, \cdots\},$$

possibly with repeats. A sequence $\{z_j\}$ **converges** to $\widetilde{z}$ if for all $\varepsilon > 0$ there exists some integer $N > 0$ such that if $j \geq N$, we have

$$|z_j - \widetilde{z}| < \varepsilon.$$

In this case, we write $\{z_j\} \to \widetilde{z}$. A **Cauchy sequence** is a sequence such that for some positive integer $n$, for all $j, k \geq n$, we have

$$|z_j - z_k| < \varepsilon$$

for all $\varepsilon < 0$ (I did this from memory from real analysis, it might be wrong).

Some notes: Cauchy completeness has been discussed (equivalence of Cauchy sequences and convergent sequences in a complete metric space), limits of sequences are unique.

**Definition 16.2** (Series). A **series** is a sum $\sum_j z_j$. A series $\sum_{j=j_0}^{\infty} z_j$ **converges** to some $s$ if the sequence of partial sums

$$\sum_{j_0}^{1} z_j, \sum_{j_0}^{2} z_j, \sum_{j_0}^{3} z_j, \cdots$$

converges to $s$.

**Theorem 16.1.** *We have*

$$\sum^{\infty} z_j = s \iff \begin{cases} \sum^{\infty} \operatorname{Re} z_j = \operatorname{Re} s \\ \sum^{\infty} \operatorname{Im} z_j = \operatorname{Im} s \end{cases}.$$

**Note.** In order that $\sum^{\infty} a_m = s$, it must be the case that $a_m \to 0$. We can prove this by the Cauchy criterion. Obviously the converse doesn't hold.

**Definition 16.3** (Absolute convergence). We say $\sum^{\infty} a_m$ is absolutely convergent if $\sum^{\infty} |a_m|$ is convergent.

There's a theorem that absolute convergence implies convergence, this should be clear. The converse doesn't hold! Take the alternating harmonic series $\sum \frac{(-1)^m}{m}$ is converges to $\ln 2$, but it doesn't converge conditionally (the standard harmonic series diverges). It's also not true that absolutely convergent series have to converge to real numbers, although I can see where this came from: something is absolutely convergent if the absolute value of the terms converge (clearly to a real number)— however, we're talking about what the original thing converges to, not the absolute value of it! It could be complex, for example, just plug $i$ behind everything for an easy counterexample.

**Theorem 16.2.** *If $f$ is analytic on some $B(z_0,R_0)$, then*

$$f(z_0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n=f(z)$$

*for such $z\in B(z_0,R_0)$.*

**Example 16.1.** We have

$$1+\sum_{n=1}^{\infty}z^n=\frac{1}{1-z}\quad\text{for } |z|<1,$$

since $\left(\frac{1}{1-z}\right)^{(n)}\Big|_0=n!$. This is a computation that every first year calculus student has (or should have) done. We could also do this with geometric series, but here we just want to show that this follows from Theorem 16.2.

Apparently this next theorem is the craziest in the course (I thought that Cauchy's theorem was already pretty crazy).

**Theorem 16.3** (Laurent's Theorem)**.** *Suppose $f$ is analytic in an annulus $R_1<|z-z_0|<R_2$, denoted by $A$. Then there exists a sequence of complex numbers $\{c_n\}$ such that for all $z\in A$,*

$$\sum_{-\infty}^{\infty}c_n(z-z_0)^n=f(z),$$

*where $\sum_{-\infty}^{\infty}c_n(z-z_0)^n=\sum_{-\infty}^{-1}+c_0+\sum_{1}^{\infty}$. Furthermore,*

$$C_n=\frac{1}{2\pi i}\oint_{\Gamma}f(z)(z-z_0)^{-n-1}\,dz$$

*over any closed curve $\Gamma$ with positive orientation such that $\operatorname{tr}\Gamma\subseteq A$, $z_0<R_1$, and the loop $\Gamma$ around $z_0$ is nontrivial.*

**Example 16.2.** Let $f(z)=\frac{1}{z}$, $z_0=0$, $R_1=1$, $R_2=3$. Take $c_{-1}=1$, $c_n=0$ for all other $n$. Then

$$\cdots+c_{-2}(z)^{-2}+c_{-1}(z)^{-1}+c_0+c_1(z)^{+1}+c_2(z)^{+2}+\cdots$$

goes to $z_0$ for $c_0$ and $\frac{1}{z}$ for everything else.

# Part II
# Miscellaneous Notes

Lecture 17

## Actual notes

I want to do some real math! These notes will follow Stein and Shakarchi §1.2.

### 17.1 Continuous functions

We've already seen the standard epsilon-delta definition of continuity. An equivalent definition is the sequential definition, that is, for every sequence $\{z_1,z_2,\cdots\}\subseteq\Omega\subseteq\mathbb{C}$ such that $\lim z_n=z_0$, then $f$ is continuous at $z_0$ if $\lim f(z_n)=f(z_0)$. Since the notions for convergence of complex numbers and $\mathbb{R}^2$ is the same, $f$ of $z=x+iy$ is continuous iff it's continuously viewed as a function of two real variables $x$ and $y$. If $f$ is continuous, then the real valued function defined by $z\mapsto|f(z)|$ is clearly continuous (by the triangle inequality).

We say $f$ attains a **maximum** at the point $z_0\in\Omega$ if

$$|f(z)|\le|f(z_0)|\text{ for all } z\in\Omega.$$

The definition of a minimum is what you think it is.

**Theorem 17.1.** *A continuous function on a compact set $\Omega$ is bounded and attains a maximum and minimum on $\Omega$.*

*Proof.* Same as the any one you'd find in a Real Analysis course. ⊠

## 17.2 Holomorphic functions

Let's talk about the good stuff. Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then $f$ is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, such that the quotient is well-defined. This limit is called the **derivative of** $f$ and $z_0$, and is denoted by

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Note that $h$ approaches 0 from any direction. The function $f$ is said to be **holomorphic on** $\Omega$ if $f$ is holomorphic at every point of $\Omega$. If $C \subseteq \mathbb{C}$ is closed, then $f$ is **holomorphic on** $C$ if $f$ is holomorphic on some open set containing $C$. Finally, if $f$ is holomorphic on $\mathbb{C}$ then $f$ is said to be **entire**. Sometimes the terms **regular** or **complex differentiable** are used in place of holomorphic, but holomorphic functions are much much nicer than real values differentiable functions. Furthermore, every holomorphic function is analytic (that is, it has a power series expansion near every point), and so we also use the term **analytic** to refer to holomorphic functions. Once again, things are not as nice in Real Analysis, with infinitely differentiable real valued functions not having power series expansions.

**Example 17.1.** We have any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ entire, and $f(z) = \frac{1}{z}$ holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$. However, $f(z) = \frac{\bar{z}}{z}$ is not entire, as $\frac{f(z_0+h)-f(z_0)}{h} = \frac{\bar{h}}{h}$, which has no limit as $h \to 0$.

If we write the definition of a holomorphic function as $f$ being holomorphic at $z_0 \in \mathbb{C}$ iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0+h) - f(z_0) - ah = h\psi(h),$$

where $\psi$ is a function defined for all "small" $h$, and $\lim_{h\to 0}\psi(h) = 0$, we can see that $f$ is holomorphic implies $f$ is continuous (clearly $a = f'(z_0)$). The basic stuff, distribution over addition, product rule, quotient rule, chain rule, yada yada all apply.

## 17.3 Cauchy-Riemann equations and the Jacobian

OK, here's the difference between real and complex valued functions again. In terms of real variables, $f(z) = \bar{z}$ corresponds to $F : (x, y) \to (x, -y)$, which is differentiable in the real sense, its derivative at a point being the map corresponding to its Jacobian. In fact, $F$ is linear and is equal to its derivative at any point, and is therefore infinitely differentiable. So a complex valued function having a real derivative need not imply the complex valued function is holomorphic. We can associate complex valued functions $f = u + iv$ to the mapping $F(x, y) = (u(x, y), v(x, y))$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Recall that $F(x, y)$ is differentiable at a $P_0 = (x_0, y_0) \in \mathbb{R}^2$ if there exists a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P_0+H) - F(P_0) - J(H)|}{|H|} \to 0 \quad \text{as } |H| \to 0,\ H \in \mathbb{R}^2.$$

We could also write

$$F(P_0 + H) - F(P_0) = J(H) + |H|\Psi(H),$$

where $|\Psi(H)| \to 0$ as $|H| \to 0$. Very similar to what just happened above. Then the transformation $J$ is unique and called the **derivative** of $F$ at $P_0$. Given that $F$ is differentiable and the partial derivatives exist, we have

$$J = J_F(x, y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

**Note.** In the case of complex differentiation, the derivative is a complex number $f'(z_0)$, but for real derivatives, it's a matrix.

However, there is a way to link these two notions. Let's consider the limit of $h \in \mathbb{R}$, that is, $h = h_1 + ih_2$ with $h_2 = 0$. Then if $z_0 = x_0 + iy_0$, we have

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0). \end{aligned}$$

Similarly, for $h \in \mathbb{C} \setminus \mathbb{R}$, say $h = ih_2$ ($h$ is purely imaginary), we have

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0). \end{aligned}$$

So if $f$ is holomorphic, then we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

Now $f = u + iv$, so $\partial f / \partial x = \partial u / \partial x + i \partial v / \partial x$, and similarly $\partial f / \partial y = \partial u / \partial y + i \partial v / \partial y$. Separate the real and imaginary parts and note that $1/i = -i$, then we get

$$\begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= \frac{1}{i} \left( \frac{\partial u}{\partial y} \right) + \frac{1}{i} \cdot i \left( \frac{\partial v}{\partial y} \right) \implies \\ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \end{aligned}$$

which implies the following nontrivial relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{11}$$

The relations described in Equation (11) are known as the **Cauchy-Riemann** equations, which link real and complex analysis. Here we state the converse of the Cauchy-Riemann theorems in an important theorem.

**Theorem 17.2.** *Suppose $f = u + iv$ is a complex-valued function on an open set $\Omega$. If $u$ and $v$ are continuously differentiable and satisfy the Cauchy-Riemann equations for all $\omega \in \Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \partial f / \partial z$.*

*Proof.* Recall that $u(x+h_1, y+h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \psi_1(h)$ and $v(x+h_1, y+h_2) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \psi_2(h)$, where $\psi_j(h) \to 0$ (for $j \in \{1, 2\}$) as $|h|$ tends to 0, and $h = h_1 + ih_2$. Then by the Cauchy-Riemann equations we have

$$f(z + h) - f(z) = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + |h| \psi(h),$$

where $\psi(h) = \psi_1(h) + \psi_2(h) \to 0$, as $|h| \to 0$. Therefore $f$ is holomorphic and $f'(z) = 2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$. ⊠

## 17.4 Power series (todo)

I was wondering when we were going to cover these?

## 17.5 Integration along curves

**Definition 17.1** (Parametrized curve)**.** A **parametrized curve** is a function $z(t) \colon [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. We say a parametrized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for $t \in [a, b]$.

At the endpoints $t = a$ and $t = b$, we interpret $z'(a)$ and $z'(b)$ as the one-sided limits

$$z'(a) = \lim_{h\to 0, h>0} \frac{z(a+h)-z(a)}{h} \quad \text{and} \quad \lim_{h\to 0, h<0} \frac{z(b+h)-z(b)}{h}.$$

We call these the right-hand derivative of $z(t)$ at $a$ and the left-hand derivative of $z(t)$ at $b$, respectively[5]. Similarly, we say the parametrized curve is **piecewise-smooth** if $z$ is continuous on $[a,b]$, and there exist points $\{a_0, a_1, \cdots, a_n\}$ such that

$$a = a_0 < a_1 < \cdots < a_n = b,$$

where $z(t)$ is smooth on $[a_k, a_{k+1}]$ for $1 \le k \le n$. This differs from the standard definition of a smooth curve in that the right and left hand derivatives at $a_k$ may differ for some $1 \le k \le n-1$. Why don't we just say parametrizations are just paths (like in the definition of path-connectedness)? That would save me a lot of typing, because the word "path" is much shorter than the word "parametrization"[6].

We say two parametrizations $z\colon [a,b] \to \mathbb{C}$ and $\widetilde{z}\colon [c,d] \to \mathbb{C}$ are **equivalent** if there exists a continouously differentiable bijection $s \mapsto t(s)$ from $[c,d]$ to $[a,b]$ such that $t'(s) > 0$ and $\widetilde{z}(x) = z(t(s))$. The fact that we require the derivative to be positive says that the orientation is preserved: as $s$ walks on the path from $c$ to $d$, $t(s)$ walks on the path from $a$ to $b$. The family of all parametrizations that are equivalent to $z(t)$ determine a **smooth curve** $\gamma \subseteq \mathbb{C}$, which is the image of $[a,b]$ under $z$ with the given orientation. We can define a curve $\gamma^-$ obtained from $\gamma$ by reversing the orientation, for example consider the parametrization $z^-\colon [a,b] \to \mathbb{R}^2$ defined by

$$z^-(t) = z(b+a-t).$$

We can also define a **piecewise-smooth curve** in the same way: let $z(a)$ and $z(b)$ be the end-points of the curve (independent of parametrization). Then $\gamma$ begins at $z(a)$ and ends at $z(b)$. A curve is **closed** if $z(a) = z(b)$ for any parameterization (it forms a loop), and is **simple** if it isn't self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$ (of course, we make an exception if it's closed). From now on, we'll call a piecewise-smooth curve a **curve** for brevity, since we don't really care if it's not piecewise-smooth.

**Example 17.2.** The standard example of a curve is a circle. Consider the circle

$$C_r(z_0) = \{z \in \mathbb{C} \bigm| |z - z_0| = r\}.$$

The **positive orientation** (counterclockwise) is given by $z(t) = z_0 + re^{it}$ while the **negative orientation** is given by $z(t) = z_0 + re^{-it}$, for $t \in [0, 2\pi]$. When we talk about circles $C$, we'll usually be talking about the positively oriented circle.

Now let's talk about integration along curves! A key theorem says that if a complex valued function is holomorphic in the interior of a closed circle $\gamma$, then

$$\int_\gamma f(z)\, dz = 0.$$

A version of this theorem is called *Cauchy's Theorem*, which we'll talk about later. Given a smooth curve $\gamma \subseteq \mathbb{C}$ parametrized by $z\colon [a,b] \to \mathbb{C}$ and $f$ a continuous function on $\gamma$, we define the **integral of $f$ along** $\gamma$ by

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt.$$

How do we know this doesn't depend on the parametrization of $\gamma$? Say $\widetilde{z}$ is an equivalent parametrization of $z$, then

$$\int_a^b f(z(t)) z'(t)\, dt = \int_c^d f(z(t(s))) z'(t(s)) t'(s)\, ds = \int_c^d f(\widetilde{z}(s)) \widetilde{z}'(s)\, ds,$$

[5] Intuitively, shouldn't it be the other way around? Like left-hand derivative corresponds to $a$, since it's on the "left-hand side" of the interval...

[6] Thank you for the observation, very cool.

proving that $\int_\gamma f(z)\,dz$ is well-defined. Now if $\gamma$ is piecewise-smooth, given a piecewise-smooth parametrization $z(t)$ we have

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,dt.$$

We can also define the **length** of the smooth curve $\gamma$ as $\text{length}(\gamma) = \int_a^b |z'(t)|\,dt$. Apply the same arguments as before to get that $\text{length}(\gamma)$ is parametrization independent and that if $\gamma$ is piecewise smooth, $\text{length}(\gamma)$ is the sum of the lengths of the smoooth components.

**Proposition 17.1.** *Let $\gamma$ be a curve, and $f, g$ be functions. Then*

1. *Integration is a linear operation, that is, for $\alpha, \beta \in \mathbb{C}$ we have*

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$

2. *For $\gamma^-$ the curve representing the reverse orientation of $\gamma$, we have*

$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$

3. *The following inequality holds:*

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$$

*Proof.* Basically, do it yourself.

1. Follows from the definition and linearity of the Riemann integral.
2. Exercise for the reader.
3. Note that

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\,dt \le \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$$

⊠

Suppose $f$ is define on some open set $\Omega \in \mathbb{C}$. A **primitive** for $f$ on $\Omega$ is a function $F$ that's holomorphic on $\Omega$ such that $F'(z) = f(z)$ for all $z \in \Omega$. Now let's look at the Fundamental Theorem of Calculus again.

**Theorem 17.3.** *If a continuous function $f$ has a primitive $F$ in $\Omega$, and $\gamma$ is a curve in $\Omega$ starting at $\omega_1$ and ending at $\omega_2$, then*

$$\int_\gamma f(z)\,dz = F(\omega_2) - F(\omega_1).$$

*Proof.* Good thing we have the FTC from Real Analysis. If $z(t)\colon [a,b] \to \mathbb{C}$ is a parametrization of $\gamma$ with $z(a) = \omega_1$ and $z(b) = \omega_2$, then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
&= \int_a^b F'(z(t))z'(t)\,dt \\
&= \frac{d}{dt} F(z(t))dt \\
&= F(z(b)) - F(z(a)).
\end{aligned}$$

Clearly we're done if $\gamma$ is smooth. If $\gamma$ is only piecewise-smooth, then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\ &= F(z(a_n)) - F(z(a_0)) \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

⊠

**Corollary 17.1.** *If $\gamma$ is a closed curve in some open $\gamma \in \mathbb{C}$ and $f$ is continuous and has a primitive in $\Omega$, then*

$$\int_\gamma f(z)\,dz = 0.$$

*Proof.* Note that if $\gamma$ is a closed curve, then $\omega_1 = \omega_2$. ⊠

**Example 17.3.** We can use this corollary to show that functions don't have primitives. For example, $f(z) = 1/z$ doesn't have a primitive in $\mathbb{C} \setminus \{0\}$, since if $C$ is the unit circle parametrized by $z(t) = e^{it}$ when $0 \le t \le 2\pi$, we have

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i \neq 0.$$

**Corollary 17.2.** *If $f$ is holomorphic on a region $\Omega$ and $f' = 0$, then $f$ is a constant function.*

*Proof.* Let $\omega_0 \in \Omega$. We WTS that $f(\omega) = f(\omega_0)$ for all $\omega \in \Omega$: since $\Omega$ is connected we can find a curve (path) joining $\omega_0$ and $\omega$. Clearly $f$ is a primitive for $f'$. Then

$$\int_\gamma f'(z)\,dz = f(\omega) - f(\omega_0) = 0$$

by assumption, which implies that $f(\omega) = f(\omega_0)$, finishing the proof. ⊠

Lecture 18

# Cauchy's Theorem and Its Applications

These will follow Stein and Shakarchi §2.

Last time we talked about some good stuff: open sets, holomorphic functions, integration along curves. Our first cool theorem relates the three, which is (you guessed it) Cauchy's theorem. It loosely states that if $f$ is holomorphic on an open set $\Omega$ and $\gamma \subseteq \Omega$ is a closed curve whose interior is also contained in $\Gamma$, then

$$\int_\gamma f(z)\,dz = 0.$$

Cool stuff happens, including the calculus of residues. Right now we restrict ourselves to easy curves (toy contours) for simplicity, but we'll deal with the general case soon. We'll also get to Cauchy's integral formula, which says that for $f$ holomorphic in an open set containing a circle $C$ and its interior, then for all $z \in C$,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

From here, by differentiating we'll see that as long as the first derivative exists, all of them do! From there, we'll get

- The property of "analytic continuation" (finally), namely that a holomorphic function is determined by its restriction to any open subset of its domain, which follows from the fact that holomorphic functions are analytic.
- Liouville's theorem, which can prove the fundamental theorem of algebra.
- Morera's theorem, which gives a simple integral characterization of holomorphic functions, preserved under uniform limits.

## 18.1 Goursat's theorem

We know that if $f$ has a primitive in $\Omega$ open, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve $\gamma \subseteq \Omega$. We also can show that if the relation above holds, then a primitive will exist.

**Theorem 18.1** (Goursat's theorem)**.** *If $\Omega$ is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in $\Omega$, then*

$$\int_T f(z)\,dz = 0,$$

*given $f$ is holomorphic in $\Omega$.*

*Proof.* Given a triangle $T^{(0)}$ with a fixed positive orientation, consider its barycentric subdivision yielding the similar triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$, defined to be the same orientation as $T^{(0)}$. So

$$\int_{T^{(0)}} f(z)\,dz = \int_{T_1^{(1)}} f(z)\,dz + \int_{T_2^{(1)}} f(z)\,dz + \int_{T_3^{(1)}} f(z)\,dz + \int_{T_4^{(1)}} f(z)\,dz. \tag{12}$$

Then there exists a $j$ such that

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4\left|\int_{T_j^{(1)}} f(z)\,dz\right|,$$

because if that weren't the case, it would contradict Equation (12). Denote such $T_j^{(1)}$ by $T^{(1)}$: note that $d^{(1)} = \frac{1}{2}d^{(0)}$ and $p^{(1)} = \frac{1}{2}p^{(0)}$, where $d^{(i)}$ and $p^{(i)}$ are the diameters and perimeters of the $T^{(i)}$'th triangle, respectively. Continuing on, we have a sequences of triangle $T^{(0)}, T^{(1)}, \cdots, T^{(n)}, \cdots$ such that

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,dz\right|, \quad d^{(n)} = 2^{-n}d^0,\ p^{(n)} = 2^{-n}p^{(0)}.$$

Denote the *solid* closed triangle with boundary by $\mathcal{T}^{(n)}$, and note that our construction yields a sequence of nested compact sets

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots \supset \mathcal{T}^{(n)} \supset \cdots$$

whose diameter approaches zero. By the NIP, there exists a unique $z_0 \in \bigcup_{i=1} \mathcal{T}^{(i)}$. Now $f$ is holomorphic at $z_0$, so we write $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$, where $\psi(z) \to 0$ as $z \to z_0$. Since $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, then we integrate and get

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} \psi(z)(z - z_0)\,dz.$$

Now $z_0 \in \overline{\mathcal{T}^{(n)}}$ and $z \in \partial\mathcal{T}^{(n)} \implies |z - z_0| \le d^{(n)}$. Recall that $\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma}|f(z)| \cdot \mathrm{length}(\gamma)$. So we have the estimate

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \le \varepsilon_n d^{(n)} p^{(n)},$$

where $\varepsilon_n = \sup_{z\in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$. Therefore we have

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \le \varepsilon_n 4^{-n} d^{(0)} p^{(0)} \implies \left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \le \varepsilon_n d^{(0)} p^{(0)}.$$

Letting $n \to \infty$ concludes the proof, since $\varepsilon_n \to 0$. ⊠

Attempt to sum up the proof in easy to read words: take a triangle and subdivide it naturally into four more triangles. Choose one of them such that it follows a nice inequality, and continue subdiving until infinity. Since $\mathbb{C}$ is complete, we can find a point in all their closures: rewriting the formula for a derivative, we can apply the inequalities and get our result.

**Corollary 18.1.** *If $f$ is holomorphic in an open set $\Omega$ that contains a rectangle $R$ and its interior, then*

$$\int_R f(z)\,dz.$$

*Proof.* Immediate by constructing the $n$-gon with $n-2$ simplices. ⊠

## 18.2 Local existence of primitives and Cauchy's theorem in a disk

Ah, so I was complaining about the jank-connectedness definition by polygonal paths earlier in my notes. I see why they do it that way now.

**Theorem 18.2.** *A holomorphic function in an open disc has a primitive in that disk.*

*Proof.* WLOG, let $D$ be a disk centered at the origin, $z \in D$ be a point. Consider a curve from 0 to $z$ made by joining straight line segments, starting by joining 0 to $\operatorname{Re} z$ then from $\operatorname{Re} z$ to $z$, with the orientation you'd expect. We'll denote this polygonal path by $\gamma_z$. Define the unique function $F(z) = \int_{\gamma_z} f(w)\,dw$: we claim $F$ is holomorphic in $D$ and $F'(z) = f(z)$. Let $z, h \in \mathbb{C}$ and choose $h > 0$ such that $z + h \in D$. Consider

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw.$$

Draw some lines and make a geometric argument that $F(z+h) - F(z) = \int_\eta f(w)\,dw$, where $\eta$ is the straight line segment from $z$ to $z+h$. Then since $f$ is continuous at $z$ we have $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. Therefore

$$F(z+h) - F(z) = \int_\eta f(z)\,dw + \int_\eta \psi(w)\,dw = f(z)\int_\eta dw + \int_\eta \psi(w)\,dw.$$

On one hand, 1 has $w$ as a primitive, so the first integral is just $h$. On the other hand, we have the estimate

$$\left|\int_\eta \psi(w)\,dw\right| \le \sup_{w\in\eta} |\psi(w)||h|.$$

Since the supremum goes to zero as $h \to 0$, we conclude that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

⊠

Basically, every holomorphic function has a local primitive. It's crucial that this holds not only for arbitrary disks, but other sets as well.

**Theorem 18.3** (Cauchy's theorem for a disc). *If $f$ is holomorphic in a disk, then*

$$\int_\gamma f(z)\,dz = 0$$

*for any closed curve $\gamma$ in that disc.*

*Proof.* Since $f$ has a primitive, apply Corollary 17.1. ⊠

**Corollary 18.2.** *Suppose $f$ is holomorphic in an open set containing the circle $C$ and its interior. Then*

$$\int_C f(z)\,dz = 0.$$

*Proof.* Let $D$ be the disk with boundary circle $C$, then there exists a slightly larger disc $D'$ containing $D$ such that $f$ is holomorphic on $D'$: apply Cauchy's theorem on $D'$ to conclude that $\int_C f(z)\,dz = 0$. ⊠

Basically, **toy contours** are curves for which the notion of interior is clear. For a toy countour $\gamma$ we easily have

$$\int_\gamma f(z)\,dz - 0,$$

if $f$ is holomorphic in an open set containing $\gamma$ and its interior. You might ask what about general curves? Maybe we'll talk about Jordan's theorem for pieceweise-smooth curves, which states that a simple closed piecewise-smooth curve has a well defined interior that is simply connected.

## 18.3 Evaluating some integrals

I was just thinking about the Fourier transform today! What a coincidence— also, this is the second book in a three part series, so I'm already expected to know what this is and what it means, yay.

**Example 18.1.** If $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx.$$

This shows that $e - \pi x^2$ is its own Fourier transform. If $\xi = 0$, then this formula just becomes the Gaussian integral, which we know to be

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \sqrt{\frac{\pi}{\pi}} = 1.$$

Assume $\xi > 0$, and consider $f(z) = e^{-\pi z^2}$ an entire function. Oh my gosh no I'm not finishing this proof.

**Example 18.2.** Another classical example is

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

Consider $f(z) = (1-e^{iz})/z^2$, and integrate over the indented semicircle in the upper half of the annulus in the positive plane, with $\gamma_\varepsilon^+$ and $\gamma_R^+$ denoting the semicircles of radii $\varepsilon$ and $R$ with negative and positive orientations, respectively (where $R > \varepsilon$). Then by Cauchy's theorem, we have

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz + \int_\varepsilon^R \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0.$$

Let $R \to \infty$, and note that

$$\left|\frac{1-e^{iz}}{z^2}\right| \le \frac{2}{|z|^2},$$

so the integral over $\gamma_R^+$ goes to zero. Therefore

$$\int_{|x|\ge\varepsilon} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz.$$

Next, note that $f(z) = \frac{-iz}{z^2} + E(z)$, where $E(z)$ is bounded at $z \to 0$, while on $\gamma_\varepsilon^+$ we have $z = \varepsilon e^{i\theta}$ and $dz = i\varepsilon e^{i\theta}\, d\theta$. Therefore

$$\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\, dz \to \int_\pi^0 (-ii)\, d\theta = -\pi \quad \text{as } \varepsilon \to 0.$$

Taking real parts yields

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\, dx = \pi.$$

Since the integrand is even, we are done.

## 18.4 Cauchy's integral formula (todo skipped some stuff)

This is nice because we can learn about the behavior of functions on big sets by looking at small sets.

**Theorem 18.4** (Cauchy's integral formula)**.** *Suppose $f$ is holomorphic in an open set that contains the closure of a disk $D$. If $C$ denotes the boundary circle of the disk with positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \textit{for any point } z \in D.$$

*Proof.* keyhole ⊠

**Corollary 18.3.** *If $f$ is holomorphic in an open set $\Omega$, then $f$ has infinitely many complex derivatives in $\Omega$. Moreover, if $C \subseteq \Omega$ is a circle whose interior is also contained in $\Omega$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

*for all $z \in C^\circ$.*

*Proof.* induction, base case is CIF ⊠

There are hereby known as the **Cauchy integral formulas**.

**Theorem 18.5** (Liouville's theorem)**.** *If $f$ is entire and bounded, then $f$ is constant.*

*Proof.* If suffices to prove that $f' = 0$, since $\mathbb{C}$ is connected, then we can apply Corollary 17.2. ⊠