Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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Hatcher Section 1.2 (p. 52): 1, 10, 14, 16, 21,
Hatcher Section 1.3 (p. 79): 30,
Hatcher Section 1.A (p. 86): 5.

§1.1 Problem 1 Section 1.2

Problem. *Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.*

§1.2 Problem 10

Problem. *Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.*

Hint (from Dr. Allcock): this can be done directly with Van Kampen's, but it becomes easier if you manipulate $(D^2 \times I) \setminus (\alpha \cup \beta)$ first, being careful not to change the homotopy type, and carrying along the loop γ.

§1.3 Problem 14

Problem. *Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.*

§1.4 Problem 16

Problem. *Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.*

§1.5 Problem 21

Problem. *Show that the join $X * Y$ of two nonempty spaces X and Y is simply-connected if X is path-connected.*

Hint (from Dr. Allcock): If you are not comfortable with the join of spaces then wrap your mind around the following examples in order:

1. *Join of two points*
2. *Join of a point and an interval*

3. *Join of a point and a circle*
4. *Join of 2 copies of the interval*
5. *Join of a circle and an interval*
6. *Join of two circles (doesn't embed in $\mathbb{R}^3$, but still understandable).*

That might be enough: if not, work out examples using the figure 8 *or S^2, or use your imagination.*

§1.6 Problem 30 Section 1.3

Problem. *Dray the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.*

§1.7 Problem 5 Section 1.A

Problem. *Construct a connected graph X and maps $f, g \colon X \to X$ such that $fg = \mathbb{1}$ but f and g do not induce isomorphisms on π_1. [Note that $f_* g_* = \mathbb{1}$ implies that f_* is surjective and g_* is injective.]*