Seminar on Reflection Positivity and Invertible Phases

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Notes for a seminar on Reflection Positivity and Invertible Phases, organized by Leon Liu and Cameron Krulewski. Source files: https://git.simonxiang.xyz/math_notes/files.html

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1 Quantum Mechanics

Today's speaker is Justin Kulp from the Perimeter Institute for Theoretical Physics. We will talk about "gapped phases of quantum matter". There are different camps interested in this: the condensed matter camp/quantum information (QI), the math camp, and the high energy physics (HEP) camp. They may say things like this:

- Cond-MAT/QI: Gapped system, microscopic Hamiltonians, phases, SPt, anyons
- MATH: Formal TFTs, π_0, homotopy, group cohomology, cobordism, MTC
- HEP: Gauge theory, TQFTs, field, Dijkgraaf-Witten.

We will not talk about quantum mechanics.[1] Regardless of choice of axioms, we have three objects everyone agrees on.

- $\mathcal{H}$ the **state space**, a complex separable Hilbert space
- $\mathrm{End}(\mathcal{H})$, some algebra of operators on $\mathcal{H}$, which will contain something called **observables**
- H the **Hamiltonian**, a non-negative self-adjoint operator.
- **Unitary evolution of states**, a one parameter group acting on $\mathcal{H}$ generated by H. In other words, a map $\mathbb{R} \mapsto U(\mathcal{H}), t \mapsto U_t = e^{-itH|\hbar}$?? called the time-evolution operator.

Since H is non-negative, $z \mapsto U_z = e^{i\tau H|\hbar}$, where τ is **Euclidian time**, $\tau \mapsto U_\tau = e^{-\tau H|\hbar}$, $\tau > 0$. Why? This turns oscillatory things into exponentially decaying things. This makes QFT analogous to Statistical Field Theory.

Example 1.1. A nice system is a particle on a ring. Consider the classical Lagrangian $L = \frac{1}{2}\dot{x}^2$, then after identifying $x \sim x+2\pi$ we can view x as a particle on a ring. After Hamiltonification we get $\hat{H} = -\frac{1}{2}\partial_x^2$, and $\mathcal{H} = L^2(S^1;\mathbb{C})$, $\tau \mapsto U_\tau = e^{-\tau\partial_x^2}$. Our eigenfunction is $\psi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, and evaluation is $E_n = \frac{n^2}{2}$. Then $L = \frac{1}{2}\dot{x}^2 + \frac{1}{2\pi}\theta\dot{x}$. Formally, $\hat{H} = \frac{1}{2}\left(-i\partial_x - \frac{1}{2\pi}\theta\right)^2, \mathcal{H} = L^2(S^1;\mathcal{L}_{e^{i\theta}})$. Okay this isn't worth it I will spend my time watching the previous lectures instead.

[1] Resources for QM: Ryan Hall- QM. Freed. Mackey's book on QM. Varadarajan. Kapustin 1303.6917?