Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1 September 26, 2020: Homework 5

Hatcher Section 1.3 (p. 79): 5, 9, 14, 16, 20, 31
Hatcher Section 1.A (p. 86): 6, 7,

§1.1 Problem 5 Section 1.3

Problem. *Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $\widetilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\widetilde{X}$. Deduce that X has no simply-connected covering space.*

Solution. Let $\mathcal{G}$ be an open cover of $L := [0,1] \times \{0\}$, the left edge of X. Then since X is compact (and so is L), we have a finite subcover $\bigcup_{i=1}^{n} G_i$ of L. Our strategy: construct evenly covered neighborhoods from the finite subcovers that evenly cover the space. Choose a point, say $x = (0,0) \in G_1$. Then we have an evenly covered neighborhood U_x around x such that $p^{-1}(U_x)$ is a disjoint union of open sets that map homeomorphically to U_x. Ideally, our job would be really easy if the finite subcover was in the form $\amalg_{i=1}^{n} G_i$, so we could just take the union of every evenly covered open set around every x, finishing the construction. However, L is connected so there must be points at which the open sets in the cover intersect: say $G_i \cap G_{i+1}$ is nonempty for some $1 \leq i \leq n$, that is, there exists some $x_0 \in G_i \cap G_{i+1}$. We can find an evenly covered neighborhood $U_{x_0} \subseteq G_i \cap G_{i+1}$ containing x_0 by simply taking the intersection of the evenly covered neighborhoods of x_0 in G_i and G_{i+1}, respectively. Then this set is evenly covered (each disjoint copy of the neighborhood in G_i intersect the disjoint copy of the neighborhood in G_{i+1} is just a copy of $G_i \cap G_{i+1}$, and will map homeomorphically onto it, still disjoint). Now that we've taken care of every point, we formally construct the neighborhood by taking unions of everything: let $U_{(0,0)}$ be the evenly covered neighborhood of $(0,0)$, and U_2 be the evenly covered neighborhood of some other point $x_2 \in G_1$. Then their union evenly covers $\{(0,0), x_2\}$: repeat until we have an evenly covered neighborhood of G_1. Now repeat for every G_i: in the case of intersection, take the union with the evenly covered neighborhood(s) of $G_i \cap G_{i+1}$ for all $x \in G_i \cap G_{i+1}$. This concludes our construction of an evenly covered neighborhood of L.

Now it's not too difficult to see that this evenly covered neighborhood lifts homeomorphically to $\widetilde{X}$: since each sheet must contain a rectangle with all but a finite amount of lines in the box, it maps homeomorphically onto X. To see that X has no simply connected covering space, every single covering space $\widetilde{X}$ has sheets that have nontrivial loops (take any loop starting at $(0,0)$, then going to $(0,1)$, then $(1/n, 1)$ and finally $(1/n, 0)$ for some $n \in \mathbb{N}$: then this is a nontrivial loop). Therefore $\pi_1(\widetilde{X})$ cannot possibly be trivial, and so X has no simply connected covering space. ■

§1.2 Problem 9

Problem. *Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]*

Solution. See the commutative diagram below:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dotted, "f'"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

For this diagram to commute, we need the lift f' to exist: the lifting criterion tells us this happens when X is path-connected and locally path-connected (which we have by assumption), and when $f_*(\pi_1(X)) \subseteq p_*(\pi_1(S^1))$. But we have $\pi_1(X)$ finite by assumption, so $\pi_1(X) \subseteq \pi_1(S^1) = \mathbb{Z}$ and the lift f' exists by the lifting criterion. So the diagram commutes, and $p \circ f' = f$. Since $\mathbb{R}$ is contractible, take a homotopy from $\mathrm{id}_{\mathbb{R}}$ to the constant map. Then composing the lift f' with this homotopy gives a new homotopy from f to the constant map, and we are done. ■

§1.3 Problem 14

Problem. *Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.*

Solution. First, recall that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, since $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. So by van Kampens, we have $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which we can denote with the presentation $\langle a, b \mid a^2, b^2 \rangle$. By the fundamental theorem of Galois theory (not really), the task of classifying all covering spaces has been reduced to classifying subgroups of $\langle a, b \mid a^2, b^2 \rangle$ and seeing what covering they correspond to. Since $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$, if we define the map $\phi \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ by $0 \mapsto \mathrm{id}_{\mathbb{Z}}$, $1 \mapsto (x \mapsto -x)$, we have the subgroups of $\mathbb{Z} \rtimes \mathbb{Z}_2$ being either trivial or of the form

1. $\mathbb{Z}_2$,
2. $n\mathbb{Z} \rtimes \mathbb{Z}_2$,
3. $n\mathbb{Z}$,

for $n \in \mathbb{N}$. How do we know this is true? Just look at what happens to elements of $\mathbb{Z} \rtimes \mathbb{Z}_2$. Let $n \in \mathbb{N}$, and H be some subgroup of $Z_2 \rtimes \mathbb{Z}$. First, $\mathbb{Z}_2 \simeq \langle (0,1) \rangle$. If $(0,1) \notin H$ (the other element of $\mathbb{Z}_2$), H is generated by elements of the form $n\mathbb{Z}$. Similarly, if $(0,1) \in H$, H must be generated by elements of the form $(n, 0)$ and $(0,1)$, which corresponds to the semidirect product $n\mathbb{Z} \rtimes \mathbb{Z}_2$.

Now if H is trivial, this corresponds to the (path-connected) universal cover, given by an infinite chain of S^2's[1]. If H is $\mathbb{Z}_2$, this corresponds to the space itself, $\mathbb{R}P^2 \vee \mathbb{R}P^2$. $n\mathbb{Z}$ represents a chain of $2n$ spheres, which intuitively is 2 spheres covering $\mathbb{R}P^2 \vee \mathbb{R}P^2$ n times for a total of $2n$ spheres. This family of covering spaces corresponds to the subgroup of the free group generated by $(ab)^n$. Similarly, $n\mathbb{Z} \rtimes \mathbb{Z}_2$ corresponds to the subgroups $\langle a, b \rangle$, $\langle a, bab \rangle$, $\langle a, babab \rangle$ and so on. The family of covering spaces this corresponds to is $\mathbb{R}P^n \vee \overbrace{S^2 \vee \cdots \vee S^2}^{n \text{ times}} \vee \mathbb{R}P^n$ (note that $\mathbb{Z}_2$ corresponding to $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is just this at $n = 0$), and we are done. ■

§1.4 Problem 16

Problem. *Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.*

Solution. See the diagram below:

$$\begin{tikzcd}
X \arrow[d, dotted, "p"] \arrow[dd, bend right, "q \circ p"'] \\
Y \arrow[d, bend left, "q"] \\
Z
\end{tikzcd}$$

The solid lines indicate that the map is a covering map. We want to show that if Z is locally path-connected, then this diagram commutes. Also, we WTS that the covering space of p is normal if the covering space by $q \circ p \colon X \to Z$ is normal. Now since q is a covering map, let $y \in Y$: then there exists some neighborhood U_z of some $z \in Z$ such that a sheet of $q^{-1}(U_z)$ maps homeomorphically onto an open set in Y containing y, let's call it U_y. Since Z is locally path-connected, we can take a smaller neighborhood (say V_z) contained inside U_z such that any two points in V_z can be connected by a path in V_z. Then this property is preserved with $V_y :=$ the sheet of $q^{-1}(V_z)$ containing y, also note that V_z remains evenly covered by V_y. Furthermore, since $q \circ p$ is a covering map, we have $(q \circ p)^{-1}(V_z) = \amalg_\alpha X_\alpha$, where X_α are open sheets[2] that map homeomorphically onto V_z, and so the X_α are all path-connected. It also follows that the X_α map homeomorphically onto V_y as well. We want to show that we can evenly cover V_y (some neighborhood of any $y \in Y$) by p: now $p(X_\alpha) \subset q^{-1}V_z$. By the path-connectedness of the X_α's, either $p(X_\alpha)$ is homeomorphic to V_y or disjoint from V_y. Then $p^{-1}(V_y)$ is a disjoint union of open sets that map homeomorphically onto V_y, and we are done.

For the second part of the proof, assume that the covering $q \circ p$ is normal, then the induced subgroup $(q \circ p)_*(\pi_1(X))$ is normal in $\pi_1(Z)$. Furthermore, since

$$(q \circ p)_*(\pi_1(X)) \leq q_*(\pi_1(Y)) \leq \pi_1(Z),$$

$(q \circ p)_*(\pi_1(X))$ must be normal in $q_*(\pi_1(Y))$ as well, since it doesn't change under conjugation by any element of $\pi_1(Z)$, and $q_*(\pi_1(Y)) \leq \pi(Z)$. Now q_* is 1-1 (look at the kernel), so we can "pull back" and get the fact that $p_*(\pi_1(X))$ is normal in $\pi_1(Y)$, finishing the proof. ■

§1.5 Problem 20

Problem. *Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.*

Solution. It would be nice to do this geometrically, but working with covering spaces of Klein bottles is really hard to visualize. Instead, take a look at the fundamental polygon of the covering space of the Klein bottle by three Klein bottles. The reason why we choose three of them is to make ensure that the presentation of the subgroup correponding to this covering is non-normal in $\pi_1(K)$, where K is the Klein bottle.

[1] From Hatcher §1.3 Example 1.48.

[2] Sorry for denoting open sets with X, I just need a way to keep track of where I am.

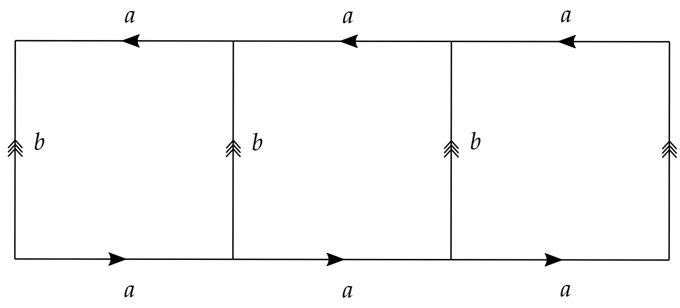

Figure 1: The covering space of the Klein bottle by Klein bottle(s), pre-identification.

Since we have the presentation for π_1 of the Klein bottle given by $\pi_1(K) = \langle a, b \mid abab^{-1} \rangle$, the presentation for the subgroup corresponding to this covering is given by the generators $\langle a^3, b \rangle$. If $\langle a^3, b \rangle$ were normal in $\pi_1(K)$, we would have any element of the subgroup conjugated by any element of $\pi_1(K)$, say a, in the subgroup. But $aba^{-1} = a(ab) = a^2 b \notin \langle a^3, b \rangle$ (if so, then $a \in \langle a^3, b \rangle$, so this would just be $\pi_1(K)$), and we conclude this covering is non-normal.

Now the 3-sheeted covering worked out very nicely for the Klein bottle. Let's look at coverings by the torus, where $\pi_1(\mathbb{T}) = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid [a, b] \rangle$. We can just directly consider the subgroups of $\pi_1(\mathbb{T})$ without drawing the diagram: the natural first choice, the fundamental group of the 2-sheeted covering by two tori is given by $\pi_1 = \langle a, b^2 \rangle$, which is normal in $\pi_1(\mathbb{T})$. So that doesn't work. Trying the same strategy as the previous part, if we put three of the 2-sheeted coverse together to get the 6-sheeted cover with fundamental group $\langle a^3, b^2 \rangle$, this is still normal in $\pi_1(\mathbb{T})$. Now the subgroup given by the generators $\langle a^3, a^2b^2 \rangle$ is non-normal, $a(a^2b^2)a^{-1} = ab(ba^{-1}) = (ab)a^{-1}b = b^2 \notin \langle a^3, a^2b^2 \rangle$, and we are done. ■

§1.6 Problem 31

Problem. *Show that the normal covering spaces of $S^1 \vee S^1$ are precisely the graphs that are Cayley graphs of groups with two generators. More generally, the normal covering spaces of the wedge sum of n circles are the Cayley graphs of groups with n generators.*

Solution. For $S^1 \vee S^1$, as seen in Hatcher, the covering spaces are just 2-oriented graphs, which can be shown is just any graph with four ends at each vertex. Similarly, we can generalize this to the n-oriented graph, which are the covering spaces of the n-wedge of circles (let's denote this with $\bigvee S^1$ to stand for $\bigvee_{i=1}^{n} S^1$). Now let N be the normal subgroup of $\pi_1(\bigvee S^1) = F_n$, corresponding to a normal cover $\widetilde{X}$: we show that this cover is the Cayley graph of $G := F_n/N$, a group with n generators.

We can do this by looking at the vertices and edges of the covering space and Cayley graph respectively (we'll now denote the Cayley graph of G by $\Gamma(G)$). Since vertices in $\Gamma(G)$ are just points in G, we have a 1-1 correspondence between the vertices of $\Gamma(G)$ and the vertices of $\widetilde{X}$, given by $\phi(g) = g \cdot \tilde{x}$ for some $g \in G$, $\tilde{x}$ a basepoint of $\widetilde{X}$. Take an edge in $\Gamma(G)$, say (x, y). Then $x = gy$ for some $g \in G$, so $\phi(x) = x \cdot \tilde{x} = gy \cdot \tilde{x} = g\phi(y)$, which implies $(\phi(x), \phi(y))$ is an edge in $\widetilde{X}$. Therefore $\phi \colon \Gamma(G) \to \widetilde{X}$ is a 1-1 correspondence between edges and points of $\Gamma(G)$ and $\widetilde{X}$. ■

§1.7 Problem 6 Section 1.A

Problem. *Let F be the free group on two generators and let F' be its commutator subgroup. Find a set of free generators for F' by considering the covering space of the graph $S^1 \vee S^1$ corresponding to F'.*

Solution. This is very reminiscent of Problem 30, §1.3. We know the universal covering of $S^1 \vee S^1$ (which we'll now denote by $\widetilde{X}$) is $\Gamma(F)$, which looks really cool and is the first image on the Wikipedia page for Cayley graphs. Now the commutator subgroup F' is isomorphic to the fundamental group of $\widetilde{X}/F'$. Now $\widetilde{X}/F'$ is just the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$, so $\pi_1(\widetilde{X}/F')$ is the free group on the number of edges in $E \setminus T$, where T is a spanning tree and E is the set of edges in $\widetilde{X}/F'$. So we know F' is free on an infinite set of generators, since any spanning tree will leave an infinite set of edges behind. To find an explicit set of such generators, look at a specific spanning tree. For example, we could have a tree that is just all the horizontal lines in $\mathbb{Z} \times \mathbb{Z}$, with a vertical line down the middle to make sure the tree is connected. This can be explicitly described by letting $T = (0, r) \cup \bigcup_i (m, i)$ for $r, m \in \mathbb{R}$, $i \in \mathbb{Z}$. See Figure 2 for how this looks.

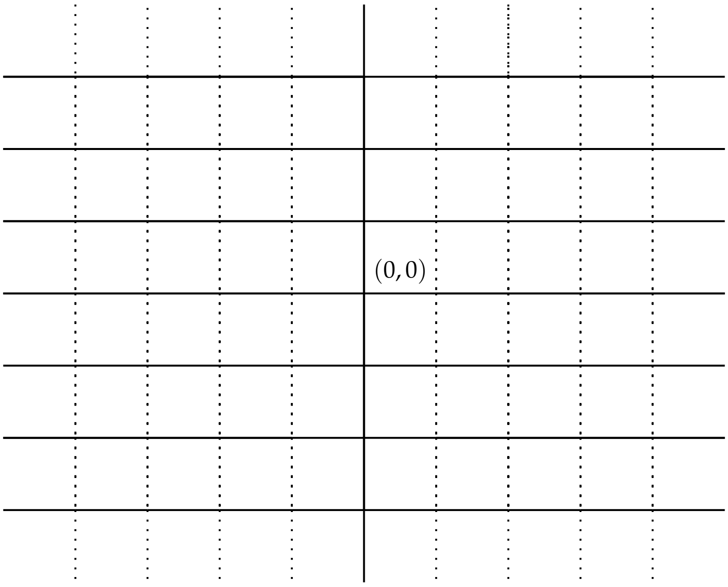

Figure 2: A spanning tree T of $\mathbb{Z} \times \mathbb{Z}$, indicated by the bold lines.

The set $(0, r)$ just corresponds to the line down the origin, and each i line in the big union is just a horizontal line at height i (that cycles through all $m \in \mathbb{R}$). Then an explicit set of generators can be given by $\langle a^i b^j a^{-1} a^{1-i}, a^{-i} b^j a b^{-j} a^{i-1} \rangle$ for $i \in \mathbb{N}$, $j \in \mathbb{Z} \setminus \{0\}$. There are more possibilities by considering more spanning trees, of course, but this is just one of them. ■

§1.8 Problem 7

Problem. *If F is a finitely generated free group and N is a nontrivial normal subgroup of infinite index, show, using covering spaces, that N is not finitely presented.*

Solution. We know F is the fundamental group of a wedge of n circles (let's call this X), and so we can give a universal covering of X by $\Gamma(F)$. Now let's look at the covering corresponding to the normal subgroup of infinite index N, we'll call it $\widetilde{X}$. Let T be a spanning tree of $\widetilde{X}$: we have $\widetilde{X}/T$ having the same homotopy type as $\widetilde{X}$, so they have the same fundamental group N. Since $\pi_1(\widetilde{X}/T)$ is free on E/T generators (where E is the set of edges of $\widetilde{X}$) like the last problem, assume that N is finitely presented. Then T would have to cover all but a finite set of edges.

Trees are required to be cycle-free: our strategy will involve using our hypotheses to show that T has a self-loop, a contradiction. Let v be a vertex in $\widetilde{X}$: then if $p\colon \widetilde{X} \to X$ is the covering map, the induced subgroup $p_*(\pi_1(\widetilde{X}, x_0)) = N$ for x_0 the basepoint of the wedge of circles. Furthermore, since N is nontrivial by assumption, there exists some nontrivial loop $\gamma \in \pi_1(X)$ that correponds to an element of N, say γ has length $n > 0$. Our strategy: lift this loop to $\widetilde{\gamma}$ based at some $v \in T$, which is a nontrivial loop in T, deriving our contradiction. To do this, note that since N has infinite index, we have an infinite set of vertices of $\widetilde{X}$, connected to a finite amount of edges (since we only have a finite wedge of circles). If we consider the set of vertices that have an edge attached to it not in T, the set of vertices at least n lengths away (as in can only be reached by a path with at least n edges) must be infinite, because T covers all but a finite set of edges. So take any v in this set, $\widetilde{\gamma}$ based at v is a word of length n and therefore won't hit any vertices that are connected to edges not in T. But then $\widetilde{\gamma}$ is a loop contained entirely in T, a contradiction, and we are done. ■