Math Club Talks

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The UT Math Club meets weekly and invites speakers to give talks every Tuesday at 5:00 PM! Here are some notes I've TEX'd up from some of them (not all). Source: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

§1 The Borsuk-Ulam Theorem (9/15/20)

Today's speaker is Hannah Turner, a 6th year Ph.D student. We'll be talking about the Borsuk Ulam Theorem!

§1.1 Continuous Maps

We talk about maps from n-dimensional spheres to $\mathbb{R}^n$. Usually we talk about maps $f\colon \mathbb{R} \to \mathbb{R}$ that are continuous, "don't lift your pencil". In topology, preimage of open sets are open, AKA for $f\colon X \to Y$, points are close in Y imply sets are close in X. For the scope of this talk, assume topological spaces are metrizable.

Definition 1.1 (Sphere). We have $\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$ for $x_i \in \mathbb{R}$. We define the *sphere* notated S^{n-1} as the set

$$\{x_i \mid |x_i| = 1\},$$

or the set of points that are a distance 1 from the origin. For example, $S^1 \subseteq \mathbb{R}^2$, $S^2 \subseteq \mathbb{R}^3$.

Let talk about maps $S^1 \to \mathbb{R}$. Deform the circle into squiggly things then smash it. Or you can turn it into a square then squish it. Yay for deformation retractions! Also: S^1 is compact, so it maps onto a closed and bounded interval. Note this map isn't onto.

§1.2 The Borsuk-Ulam Theorem

Theorem 1.1 (Borsuk-Ulam). *Any map $f\colon S^n \to \mathbb{R}^n$ sends two antipodal points $(v \sim -v)$ in S^n to the same point in $\mathbb{R}^n$.*

Example 1.1. Any map $S^1 \xrightarrow{f} \mathbb{R}$ sends two antipodal points in S^1 to the same point in $\mathbb{R}$. Look at $g(x) = f(x) - f(-x)$, where $g\colon S^1 \to \mathbb{R}$. Our new goal: show that $g(x)$ has a zero (this shows BU for $n = 1$). Pick our favorite point $x_0 \in S^1$, and assume $g(x_0) \neq 0$. So $g(x_0)$ is either positive or negative, that is $g(x_0) > 0$ or $g(x_0) < 0$.
Assume $g(x_0) > 0$: what happends to $-x_0$, the antipodal point?

$$g(-x_0) = f(-x_0) - f(-(-x_0)) = f(-x_0) - f(x_0) = -(f(x_0) - f(-x_0)) = -g(x_0).$$

The $g(-x_0) < 0$. Now we apply the IVT, but we have to be a little careful. For the usual $\mathbb{R} \xrightarrow{f} \mathbb{R}$, say $f(x) = 5$, $f(y) = 7$, we hit every value in between 5 and 7. What's important: S^1 is *path-connected* (so the IVT still applies, since f is a function from a path-connected space into $\mathbb{R}$). Then there exists some $x \in S^1$ such that $g(x) = 0$, finishing the example.

The proof in higher dimensions is more difficult. There are three flavors:

1. Algebraic Topology: Assign an algebraic invariant. Weird equation: $H_*(\mathbb{R}P^n_i \mathbb{F}_2)$
2. Combinatorics: Tucker's Lemma,
3. Set covering (Lusternik-Schnirelmann): For S^n, any $n+1$ open sets covering one of the sets must contain antipodal points (in at least one of the covering sets).

§1.3 Corollaries of BU

Definition 1.2 (Homeomorphisms). A *homeomorphism* is a continuous function $f\colon X \to Y$ which has a continuous inverse $f^{-1}\colon Y \to X$, $f \circ f^{-1} = \mathrm{id}_X$.

Example 1.2. A map which is not injective cannot have an inverse! Because then one point would map to two, breaking the rules and causing society to fall into a complete collapse.

Example 1.3. Take the map from the half open interval to the circle, that is, $f\colon [0,1) \to S^1$. f is continuous, has an inverse, but the inverse isn't continuous. Intuition: points at the place where the "endpoints" are identified are now very far away in the preimage of the inverse. So f is a bijection but its inverse is not continuous, so f is NOT a homeomomorphism.

Corollary 1.1. *There is no homeomorphism from $S^n \to \mathbb{R}^n$. Any continuous function $f\colon S^n \to \mathbb{R}^n$ has $f(x) = f(-x)$, not even one to one!*

§1.4 Pancakes!

Corollary 1.2 (Pancake Theorem)**.** *Any two disks in the place can be cut exactly in half by one slice. This includes weirdly shaped disks! In general, if we have n amount of n-dimensional blobs, we would have an n-dimensional hyperplane (locally homeo to $\mathbb{R}^{n-1}$) in $\mathbb{R}^n$ that slices each n-dimensional blob exactly in half.*

Proof. Sketch of a proof: take our 3 objects A_1, A_2, A_3. Something about normal vectors and perpendicular planes. Measure the volume? (Measures??) Pick the plane that gives half of the sandwich. Repeat for every plane in the sphere, call each plane P_x (where half of the sandwich is on each side of any P_x). Define a map $f\colon S^2 \to \mathbb{R}^2$ by $x \mapsto (\mathrm{vol}(A_2)$ on the positive side of P_x, $\mathrm{vol}(A_3)$ on the positive side of $P_x)$. We know there are x_0 and $-x_0$ with $f(x_0) = f(-x_0)$ by BU. Man, I wish I could TeX figures in real time. So

$$\begin{aligned} x_0 &\mapsto (\mathrm{vol}(A_2)P^+_{x_0}, \mathrm{vol}(A_3)P^+_{x_0}), \\ -x_0 &\mapsto (\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}), \end{aligned}$$

which are equal. The point is, we get the same plane but we're looking at it from two different directions, because $(\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}) = \left(\mathrm{vol}(A_2)P^-_{x_0}, \mathrm{vol}(A_3)P^-_{x_0}\right)$. $\mathrm{vol}(A_2)$ is cut in half by P_{x_0}, $\mathrm{vol}(A_3)$ is cut in half by P_{x_0}. ⊠

§2 What's the Putnam? (9/22/20)

Announcements: the reading groups are ready! We're studying analytic number theory, graph theory, and complexity theory. Also: social this Friday, Tiffs treats! Next week: Quantum computing, stay tuned.

Today's speaker is Dr. Rusin, an assistant professor here in the math department. He likes working with students that make their lives difficult for themselves (by doing hard problems). Some alternatives:

- The Bennett competition (only for calculus students). Problems that are not allowed to go on a final exam because they're hard. (I've read these on the walls before!) We also have linear algebra and differential equations exams.
- "Spy people" are mathematicians working for the NSA. In other words, traitors. There is some competition for math modeling that runs in February, in teams of three. It's the "anti-Putnam".

§2.1 About the Putnam

Now let's talk about the Putnam: it's an annual math competition, open to undergraduates in the US and Canada (no more than 4, no bachelors). Mathematics only, once a year (historically, the first Saturday in December). It runs all day, from 9:00 to 3:00 in two groups of six questions. Reset your progress at the lunch break? `[Yes:No]`. College level topics: calculus, linear algebra, differential equations, number theory, topology, real analysis, abstract algebra, even statistics and mathematical physics. The questions are quite "accessible" on the surface: syke!

Some are about games: flashbacks to Fishman's Banach-Mazur game (Alice and Bob). It was actually me, linear algebra! Some practice strategies include working on old questions, learn the tricks. By tradition, the questions are arranged from easier to harder. So most people try the first question. The classic: the median score out of 120 is 1.

§2.2 Problem A1 2019

Problem (2019 Putnam Question A1)**.** *Determine all possible values of the expression*

$$A^3 + B^3 + C^3 - 3ABC,$$

where A, B, and C are nonnegative integers.

What do? Let's see... looks like number theory or linear algebra to me. Does this remind me of a group I know? Find a pattern, generalize the pattern, determine the relation, write a proof. If $A = B = C$, then $3A^3 - 3AAA = 0$, taking care of the trivial case. I hate how Zoom kills my battery.

By FLT?? Complete madlad. Unfortunately it's not relevant (very sad). I wish I could see more examples of proof by overkill. What if we have the numbers of the form $A - 1, A, A + 1$? Expanding the cubes, we get $3A^3 + 6A - 3ABC = 3A^3 + 6A - 3A(A^2 - 1) = 9A$. So we have all multiples of 9 at the least. Is the output all integers, and we just show it this way? Seems easier than classifying stuff in the domain (close integers, etc). Similarly, for $A, A, A + 1$, we have

$$\sum = A^3 + A^3 + A^3 + 3A^2 + 3A + 1 - 3A^2(A + 1) = 3A^3 - 3A^3 + 3A^2 - 3A^2 + 3A + 1 = 3A + 1.$$

Eventually, you keep plugging stuff in but you can't find a solution set with some and some, anything congruent to 3 or 6 $\pmod 9$. Then the fact that the domain is non-negative, that messes with the output formulas, eventually only non-negative outputs. This follows from the AM-GM inequality (oldest trick in the book).

We're not done yet: factor the polynomial, plug it into a matrix. There's a connection with something called a *circulant* matrix, do stuff with the eigen-whatever. Not all solutions have to be elegant, just solve them. Let's look at the winners: all from MIT, great. But we got honorable mention yay!

This year, the tentative date for the Putnam is February 20, 2021. If everyone's back and running on campus, they intend to hold the competition as usual. Backup plan: they're still going to run the competition, but no prizes and no winners. Maybe hybrid too. Look at the web pages at the math department (Dr. Rusin's website) for the Zoom link. Hook' em!

Digest this problem in your free time: you can prove it in two words. ???

Problem. *Given a lattice grid, you can make triangles with the vertices as points. Is there an equilateral triangle with integer coordinates?*