Algebraic Topology Homework

Simon Xiang

This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

Contents

§1 August 29, 2020: Homework 1

Hatcher Chapter 0 (p. 18): 1, 3ab, 17,
Hatcher Section 1.1 (p. 38): 3, 6, 7, 16.

§1.1 Question 1

Problem 1. Suppose X, Y are compact Hausdorff spaces and $f\colon X \to Y$ is continuous and onto. Define $\sim$ as the equivalence relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

(a) Prove the quotient space $X/\sim$ is Hausdorff.

(b) Use this to show that the induced map $X/\sim \to Y$ is a homeomorphism.

(c) Show that identifying the ends of the interval gives S^1.

(d) Give a cooler example.

Solution. We examine the structure of topologies generated by identifying points together who lie in the same "class" after a map.

(a) Let $[a], [b]$ be elements (equivalence classes) of the quotient space $X/\sim$. We want to separate $[a]$ and $[b]$ by open sets: since singletons are closed, we can separate $q^{-1}[a]$ and $q^{-1}[b]$ by open sets in X (where $q\colon X \to X/\sim$ is the canonical quotient map), due to the fact that the quotient map is continuous (and so the inverse image of closed sets are closed) and Hausdorff plus compact implies normal. Then their images are disjoint and open in $X/\sim$ as well, since the quotient map is open.

(b) We claim that f induces a map $g\colon X/\sim \to Y$ such that $g \circ q = f$. Then g is bijective (the equivalence classes all identify to the same point in Y), and continuous (by a theorem in Munkres, Corollary 22.3).We claim g is open: let a be open in $X/\sim$ ($q^{-1}(a)$ is open in X), then $g(a) = f(q^{-1}(a))$ is open in the topology induced by f on Y (since $q^{-1}(a)$ is open in X), so g is open. Then open, continuous, and bijective implies homeomorphism, and we are done.

(c) Identify the endpoints $\{0, 1\}$ in the interval $[0, 1]$: then the quotient space $[0, 1]/\{0, 1\}$ is homeomorphic to S^1 by defining the map $f\colon [0, 1] \to S^1$ as

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Then this identifies the endpoints $\{0, 1\}$ together, and points on the interval to points on the unit circle.

(d) A cooler example (from Munkres): Identify the corners of the edges of the box $X = [0, 1] \times [0, 1]$ by partitioning X into the singletons $\{(x, y)\}$ where $0 < x < 1$, $0 < y < 1$, the two point sets $\{(x, 0), (x, 1)\}$ where $0 < x < 1$ and $\{(0, y), (1, y)\}$ where $0 < y < 1$, and the four point set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Then $X/\sim$ is homeomorphic to the torus.

■

§1.2 Problem 1 Chapter 0

Problem. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. Informal idea: take the hole in the torus and stretch it all the way to the boundaries. So the torus becomes two cirles in parallel (in three dimensional space) connected by a point, "flatten" the two circles to obtian two circles intersecting at a point.

Formal idea: As seen earlier, we can glue the borders of a unit square to obtain a torus. Take $I = [-1, 1]$, then if we can show that $I^2 \setminus \{0\}$ retracts to ∂I^2 we are done (since ∂I^2 glued together is two circles). Since $I^2 \setminus \{0\}$ is convex, we can define $f\colon I^2 \setminus \{0\} \to S^1$ as the unit length $\frac{x}{|x|}$ of any ray from the origin to x, which is a retraction onto S^1. Restricting f to the boundary ∂I^2 then taking its inverse yields a map $g\colon S^1 \to \partial I^2$ from the circle to the boundary of the square: then the composition $g^{-1} \circ f$ is a retraction from the entire square to the circle then to the boundary. Define the homotopy $F\colon I^2 \setminus \{0\} \times [0, 1] \to \partial I^2$ as

$$F(x, t) = x(1 - t) + t(g^{-1} \circ f),$$

the desired homotopy from the torus (glued square) onto the two circles connected by a point (∂I^2). ∎

§1.3 Problem 3a

Problem. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution. Recall that two spaces X and Y are homotopy equivalent if there exists a pair of continuous maps $f\colon X \to Y$, $g\colon Y \to X$ such that $gf = \iota_X$, $fg = \iota_Y$. Clearly the relation is reflexive (consider $f\colon X \to X$ and f^{-1}) and symmetric (use the same pair of maps f and g to show that Y and X are homotopy equivalence). We show the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$, fulfilling the transitive requirement. Let $f_1\colon X \to Y$, $g_1\colon Y \to X$ be the maps on X, Y, and $f_2\colon Y \to Z$, $g_2\colon Z \to Y$ the maps on Y, Z. Let $f_3\colon X \to Z$ (and $g_2\colon Z \to X$) be defined by $f_3 = f_2 \circ f_1$ ($g_3 = g_1 \circ g_2$). Then $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1 = g_1 \circ \iota_Y \circ f_1 = g_1 \circ f_1 = \iota_X$. Similarly, $f_3 \circ g_3 = f_2 \circ f_1 \circ g_1 \circ g_2 = f_2 \circ \iota_Y \circ g_2 = f_2 \circ g_2 = \iota_Z$, and we are done. ∎

§1.4 Problem 3b

Problem. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution. Clearly a map $f\colon X \to Y$ is homotopic to itself (take a constant homotopy $f_t = f$ for all t). If $f \simeq g$, then define a homotopy g_t from g to f as $g_t = f_{1-t}$ where f_t

denotes the original homotopy. Then $g_0 = g$ and $g_1 = f$ (connecting g and f) so g_t is a homotopy between g and f. Finally, let maps f and g be homotopic, along with maps g and h. We want to find a homotopy from f to h: If f_t is the homotopy connecting f and g and g_t is the homotopy connecting g and h, define the homotopy h_t on the interval as

$$h_t = \begin{cases} f_{2t} & \text{if } t \in [0, 0.5], \\ g_{2t-1} & \text{if } t \in [0.5, 1]. \end{cases}$$

The homotopy agrees with itself at $t = 0.5$ since $f_{2 \cdot 0.5} = f_1 = g$, and $g_{2 \cdot 0.5 - 1} = g_0 = g$. Furthermore, $h_0 = f_0 = f$, and $h_1 = g_1 = h$, so h is a homotopy between f and h, and we are done. ■

§1.5 Problem 17a

Problem. Show that the mapping cylinder of every map $f\colon S^1 \to S^1$ is a CW complex.

Solution. This is the generalization of the idea that if we have a cellular map between two CW complexes, then the mapping cylinder is also a CW complex. To do this, note the product of two CW complexes is also a CW complex: so $S^1 \times I$ is a CW complex. Then we define cells in the mapping cylinder as the cells of $S^1 \times I$ and S^1. ■

§1.6 Problem 17b

Problem. Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Solution. Both the annulus and the Möbius band retract to S^1. So construct a 2-dimensional CW complex by pasting points to S^1 to the 2-cells $S^1 \times I$ and the Möbius band, then identifying the annulus and the Möbius band by S^1. So this CW has both the annulus and the Möbius band as deformation retracts. ■

§1.7 Problem 3 Section 1.1

Problem. For a path-connected space X, show that $\pi(X)$ is abelian if and only if all basepoint-change homeomorphisms β_h depend only on the endpoints of the path h.

Solution. (Not attempted). ■

§1.8 Problem 6

Problem. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $\left(S^1, s_0\right) \to (X, x_0)$. Let $\left[S^1, X\right]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to \left[S^1, X\right]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $\left[S^1, X\right]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Solution. (Not attempted). ■

§1.9 Problem 7

Problem. Define $f\colon S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on both boundary circles. [Consider what f does to the map $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$].

Solution. (didn't finish). ■

§1.10 Problem 16

Problem. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

Solution. (didn't finish) Take the family of retractions that map the first circle to itself (identity) and wrap the second circle around the first n times, then if $n \in \mathbb{N}$ this is an infinite family of retractions. (Not sure how to formalize this or show they're nonhomotopic). ■