380C PROBLEM SET 1

DUE WEDNESDAY, SEPTEMBER 8TH

For sets X and Y, we let $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)$ be the set of functions $f : X \to Y$. For groups G and H, we let $\mathrm{Hom}_{\mathrm{Groups}}(G, H) \subseteq \mathrm{Hom}_{\mathrm{Sets}}(G, H)$ be the set of group homomorphisms $\varphi : G \to H$.

Problem 1.

(a) Let G be a group acting on a set X. For $g \in G$, let $\varphi_g : X \to X$ be the map $x \mapsto gx$.

   (i) For $g \in G$, show that φ_g is a bijection with inverse $\varphi_{g^{-1}}$.

   (ii) By the above, for any $g \in G$, $\varphi_g \in \mathrm{Aut}(X)$.

   Show that the function $G \xrightarrow{g \mapsto \varphi_g} \mathrm{Aut}(X)$ is a homomorphism.

(b) Let G be a group and let X be a set. Let $\mathsf{A}_{G,X} \subseteq \mathrm{Hom}_{\mathrm{Sets}}(G \times X, X)$ be the subset of maps defining an action of G on X; we can think of $\mathsf{A}_{G,X}$ as the set of possible actions of G on X.

   We have a map:

$$\mathsf{A}_{G,X} \to \mathrm{Hom}_{\mathrm{Groups}}(G, \mathrm{Aut}(X))$$

   constructed in (a)(ii). Show that this map is a bijection.

Problem 2. Let G be a group.

In this problem, we consider $\mathbb{Z}$ and $\mathbb{Z}/n$ as groups under addition.

(a) Show that the map:

$$\mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z}, G) \to G$$
$$\varphi \mapsto \varphi(1)$$

   is a bijection.

   Deduce that an action of $\mathbb{Z}$ on a set X is equivalent to an automorphism T of X, as stated in class.

(b) Show that the map:

$$\mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z}/n, G) \to G$$
$$\varphi \mapsto \varphi(1)$$

   is injective with image $\{g \in G \mid g^n = 1\}$.

Deduce that an action of $\mathbb{Z}/n$ on a set X is equivalent to automorphism T of X with $\underbrace{T \circ \ldots \circ T}_{n \text{ times}} = \mathrm{id}_X$, as stated in class.

Problem 3.

(a) Let $\varphi : G \to H$ be a homomorphism. Suppose $\mathrm{Ker}(\varphi) = \{1\}$. Show that the induced map $G \to \mathrm{Image}(\varphi)$ is an isomorphism of groups.

(b) Suppose G acts on a set X. Suppose that for any $g \in G$ with $g \neq 1$, there exists an element $x \in X$ with $gx \neq x$.

Show that the map $G \xrightarrow{g \mapsto \varphi_g} \mathrm{Aut}(X)$ yields an isomorphism between G and a subgroup of $\mathrm{Aut}(X)$.

(c) Let G be a finite group. Show that G is isomorphic to a subgroup of $S_{|G|}$, the symmetric group on $|G|$ letters.

Problem 4.

(a) Show that $|S_n| = n!$.

(b) Suppose $n > 1$. Let S_n act on the set $X = \{1, \ldots, n\}$ in its tautological way. Show that for the induced[1] action on $X \times X$, there are exactly two orbits.

Problem 5. Let p be a prime number. Show that any group of order p is isomorphic to $\mathbb{Z}/p$.

[1]If a group G acts on sets X and Y, then G naturally acts on $X \times Y$ by the formula: $g \cdot (x, y) = (g \cdot x, g \cdot y)$.