Differential Geometry Notes

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Lecture 1

January 22, 2021

Brief review on dot products. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$, $\lambda\colon \mathbb{R} \to \mathbb{R}^n$. Then define a new function $f(t) = \gamma(t) \cdot \lambda(t)$. Precisely, if $\gamma(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t))$, and $\lambda(t)$ is similarly defined, then

$$f(t) = \gamma_1(t)\lambda_1(t) + \gamma_2(t)\lambda_2(t) + \cdots + \gamma_n(t)\lambda_n(t),\; f'(t) = \sum_{i=1}^{n} \left(\gamma_i'(t) + \lambda_i(t) + \gamma_i(t)\lambda_i'(t)\right) = \lambda'(t)\cdot\lambda(t) + \gamma(t)\cdot\lambda'(t).$$

So $\frac{d(\gamma\cdot\lambda)}{dt} = \frac{d\gamma}{dt} \cdot \lambda + \gamma \cdot \frac{d\lambda}{dt}$.

Proposition 1.1. *Suppose $\|\gamma(t)\|$ is a constant, then $\gamma(t) \perp \gamma'(t)$.*

Proof. We want to show that $\gamma(t) \cdot \gamma'(t) = 0$. We have FINISH THIS ⊠

Example 1.1. Let $\gamma(t) = (t, \sqrt{1-t^2})$. We have $\|\gamma\| = 1$ at all times, since $\gamma'(t) = \left(1, -\frac{t}{\sqrt{1-t^2}}\right)$, which is orthogonal. Neat visualization!

1.1 Reparametrization

This is in the book notes. In the proof that curves are regular iff they have a unit speed parametrization, one direction is easy.

Lecture 2

January 25, 2021

2.1 Closed curves

Definition 2.1. We say $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is **T-periodic** (where $T > 0$) if $\gamma(T + t) = \gamma(t)$. We say γ is **closed** if it is T-period for some T.

A natural question to ask is whether or not we can parametrize level curves? You know what a gradient is.

Theorem 2.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x,y) \neq \vec{0}$ for all (x,y) with $f(x,y) = 0$. Then for all (x_0, y_0) with $f(x_0, y_0) = 0$, there exists a regular $\gamma\colon (\alpha,\beta) \to \mathbb{R}^2$ such that $\alpha < 0 < \beta$, $\gamma(0) = (x_0, y_0)$ and $f(\gamma(t)) = 0$ for all t.*

Note. The proof uses the inverse function theorem. Note that we can parametrize the entire curve under fairly broad conditions, that is, if $f^{-1}(0)$ is *connected* then we can choose γ to parametrize all of $f^{-1}(0)$.

Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. A **global inverse** is a map $G\colon \mathbb{R}^n \to \mathbb{R}^n$ with $F \circ G(\vec{x}) = \vec{x}$. A **local inverse** at $\vec{x}$ is a map $G\colon U_{\vec{x}} \to \mathbb{R}^n$ with $F \circ G(\vec{y}) = \vec{y}$ for all $\vec{y}$, where $U_{\vec{x}}$ is a neighborhood of $\vec{x}$. An **infinitesmal inverse** at $\vec{x}$ is a linear map A such that $(D_{\vec{x}}F) \circ A$ is the identity, where $D_{\vec{x}}F$ is the Jacobian matrix.

The Inverse Function Theorem. *If F is smooth and has an infinitesmal inverse at $\vec{x}$, then it has a smooth local inverse at $\vec{x}$.*

Theorem 2.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $\nabla f(x,y)$ is not horizontal for all (x,y) with $f(x,y) = 0$, then there exists a regular $\gamma\colon (\alpha,\beta) \to \mathbb{R}^2$ with $\gamma(t) = (t, g(t))$ and $f(\gamma(t)) = 0$ (and $\gamma(0) = (x_0, y_0)$ like in the previous theorem).*

Proof. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x,y) = (x, f(x,y))$. Then

$$DF = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}, \quad \det(DF) = \frac{\partial f}{\partial y} \neq 0.$$

By the inverse function theorem, since DF is invertible, there exists a local smooth inverse G, where $F \circ G(x,y) = (x,y) = (G_1(x,y), f(G_1,(x,y), G_2(x,y)))$. This implies that $G_1(x,y) = x$, $f(x, G_2(x,y)) = y$. Define $\gamma(t) = (t, G_2(t,0))$. Since F and G are smooth, γ is regular, so

$$f(\gamma(t)) = f(t, G_2(t,0)) = 0.$$

Something happened here. ⊠

Lecture 3

January 27, 2021

3.1 Curvature

Definition 3.1. Assume γ is a unit-speed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$. Define the curvature by $\kappa(s) = \|\ddot{(}s)\| = \left\|\left(\frac{d^2}{ds^2}\gamma\right)(s)\right\|$.

Example 3.1. The circle curve $\gamma(t) = (R\cos t, R\sin t)$ is not unit speed. So $\gamma'(t) = (-R\sin t, R\cos t)$, and $\|\gamma'(t)\| = R$. The arclength $s(t) = \int_0^t R\,du = tR$, so $s^{-1}(t) = \frac{t}{R}$. A reparametrization is $\widetilde{\gamma}(t) = \left(R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)\right)$.

Say $\gamma(s) = \left(R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)\right)$ for simplicity. Then $\dot{\gamma} = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right)\right)$, and $\ddot{\gamma} = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)$. So $\|\ddot{\gamma}\| = \kappa(s) = \frac{1}{R}$.

Parametrizing by arc length is painful. So we can define (if γ is regular) $\kappa = \frac{\|\dot{\kappa} \times \ddot{\kappa}\|}{\|\kappa\|^3}$. This makes life easier, since in this definition, $\kappa(t) = \kappa(s(t))$. What is a cross product?? Let $\vec{v}, \vec{w} \in \mathbb{R}^3$, then $\vec{v} \times \vec{w} \in \mathbb{R}^3$ as well. One way to find the cross product is by computing

$$\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = (v_2w_3 - v_3w_2)\mathbf{i} + (-v_1w_3 + v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}.$$

The cross product is **bilinear**, that is, $(\mathbf{v} + \mathbf{u}) \times \mathbf{w} = \mathbf{v} \times \mathbf{w} + \mathbf{u} \times \mathbf{w}$, and satisfies homogeneity, and antisymmetric like the determinant. Also, $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$, and finally we have the right hand rule.

We can simplify our old formula to $\frac{\|\ddot{\gamma}\|\sin\theta}{\|\dot{\gamma}\|^2}$.

Proof of the formula for curvature. Let $s(t) = \int_{t_0}^t \|\gamma'(u)\|\,du$ be the arc length of a curve, and $\widetilde{\gamma}(t) = \gamma(s^{-1}(t))$. So $\widetilde{\gamma}(s(t)) = \gamma(t)$. Then

$$\widetilde{\gamma}'(s(t))s'(t) = \gamma'(t) \implies \widetilde{\gamma}'(s(t)) = \frac{\gamma'(t)}{s'(t)}.$$

Then $\widetilde{\gamma}''(s(t))s'(t)^2 + \widetilde{\gamma}'(s(t))s''(t) = \gamma''(t)$ by the chain rule. So

$$\kappa(t) = \widetilde{\gamma}''(s(t)) = \frac{\gamma''(t) - \widetilde{\gamma}'(s(t))s''(t)}{s'(t)^2} = \frac{\gamma''(t) - \frac{\gamma'(t)}{s'(t)} \cdot s''(t)}{s'(t)^2}.$$

Recall that $s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\|\,du$, so $s'(t) = \|\dot{\gamma}(t)\|$. We use inner products, now $s'(t)^2 = \|\dot{\gamma}(t)\|^2 = \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle$. So differentiating gives $2s'(t)s''(t) = 2\langle \dot{\gamma}(t), \ddot{\gamma}(t)\rangle$. Then $s''(t) = \frac{\langle \dot{\gamma}(t), \ddot{\gamma}(t)\rangle}{s'(t)}$. Plugging everything gives

$$\kappa(t) = \left\|\frac{\ddot{\gamma} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\frac{\dot{\gamma}\cdot\ddot{\gamma}}{\|\dot{\gamma}\|}}{\|\dot{\gamma}\|^2}\right\| = \left\|\left(\frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\frac{\dot{\gamma}\cdot\ddot{\gamma}}{\|\dot{\gamma}\|\|\ddot{\gamma}\|}\right)\right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \left\|\frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} - \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\cos\theta\right\| \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2} = \sin\theta \cdot \frac{\|\ddot{\gamma}\|}{\|\dot{\gamma}\|^2}.$$

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