Differential Topology Notes

Simon Xiang

March 25, 2021

Notes for the Spring 2021 graduate section of Differential Topology (Math 382D) at UT Austin, taught by Dr. Freed. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

Part I
Class Notes

Lecture 1

January 19, 2021

> "Differential topology is a subset of geometry, which is a subset of math. Broadly, math is about space and numbers, and this is more on the space side. This isn't a partition, however."

1.1 Smooth Manifolds

Our main object of study is the smooth manifold, which is broadly a space on which you can do calculus. All these spaces look the same locally, the difference is in the global structure. We want to know how to do calculus on flat space first, which means doing calculus on open sets $U \subseteq \mathbb{A}^n = \{(x^1, \cdots, x^n) \mid x^i \in \mathbb{R}\}$, where $\mathbb{A}^n$ is affine n-space. Broadly, this means functions $f : U' \to U$, n functions of m variables, which are smooth (C^∞).

Smooth manifolds patch together these open sets, or a collection $\{U_\alpha\}\alpha \in A$. By patching together, we mean X is a smooth manifold, and a surjective map from this collection onto X. We go from one piece (chart) to the other by transition maps. Atlases will include some border towns when you're crossing over. The information must correspond, which is the idea of a diffeomorphism.

Example 1.1. Our first example will be two copies of the affine line $\mathbb{A}^1_x, \mathbb{A}^1_y$ projecting onto the circle $S^1 = \{\lambda \in \mathbb{C} \,\big|\, |\lambda| = 1\}$. If we define

$$\lambda = \begin{cases} e^{2i\tan^{-1} x} \\ e^{2i\left(\frac{\pi}{2} - \tan^{-1} y\right)} \end{cases}$$

we see that $0_x \mapsto 1$ and $0_y \mapsto -1$. So we patch $\mathbb{A}^1_x \setminus \{0\} \xrightarrow{f} \mathbb{A}^1_y \setminus \{0\}$ by the map $x \mapsto 1/x$.

Example 1.2. Another example is glueing two affine planes together by stereographic projection on a sphere. Work out what the transition function is in your free time.

Example 1.3. Take an affine plane $\mathbb{A}^2$ and a line ℓ in it, or the manifold X of affine lines through the plane.

Let's talk about the correspondence of manifolds and functions. For $f : X \to Y \ni c$, we can make shapes from functions like so:

(1) The image $f(X) \subseteq Y$,

(2) The fiber of f at c, $f^{-1}(c) \subseteq X$, and the inverse image $f^{-1}(z) \subseteq X$,

(3) The graph $\Gamma_f \subseteq X \times Y$.

1.2 Local-to-global and Classification theorems

Another idea is local vs global structure. An example of local structure is the inverse function theorem. Classification is also a big issue, it's good to know that manifolds are topologizable metric spaces. So we can talk about things like compactness and connectedness.

(1) Our only example in dimension 1 is the circle S^1.

(2) In dimension 2, we have the genus n-surfaces, the Klein bottle, projective space, etc.

(3) In dimension 3, if we add a simply-connected hypothesis this becomes the classic Poincaré (no longer!) conjecture.

(4) In dimension 4, it's a zoo

(5) In dimensions greater than 5, we have more wiggle room with the extra dimensions, so we can apply techniques from algebraic topology which are more effective with this wiggle room.

How do we classify functions? For smooth manifolds we consider manifolds up to diffeomorphism. For functions $f : S^1 \to S^1$, we can give it a nice topology (say the compact-open topology) and look at the path components, or classify the maps up to homotopy. In this case, homotopies are maps $F : [0,1] \times S^1 \to S^1$, classifying these are a kind of global property. A type of map from the circle to the circle is $f_n(\lambda) = \lambda^n$ for $n \in \mathbb{Z}$, these maps have winding numbers. In a homotopy, a path in the interval can wind around and intersect itself several times, but always evens out (points being born and dying). An important concept is the orientations, knowing which way things are facing. This is the first example of what's called *intersection theory*, which is what we use to make invariants.

Back to smooth manifolds. They arise in many places, including:

(1) Moduli spaces of geometric objects

(2) Solutions to (nonlinear) differential equations

This finishes the survey of the course. Now let's begin the actual content.

1.3 Topological Manifolds

Definition 1.1. Let X be a topological space.

(i) X is **locally Euclidian** if for all $x \in X$ there exists an open subset $U_x \subseteq X$ and a homeomorphism $U \to U'$ where $U' \subseteq \mathbb{A}^n$ for some $n \in \mathbb{Z}^{\geq 0}$.

(ii) X is a **topological manifold** if X is locally Euclidian, Hausdorff, and second countable.

Remark 1.1. At each $x \in X$, the dimension n is well-defined. So we have a function $\dim\colon X \to \mathbb{Z}^{\geq 0}$. If the dimension is constant, then we say such a manifold is an n-manifold. But this doesn't always have to be the case.

Remark 1.2. A topological manifold has a metrizable topology.

Example 1.4. Here we give some examples and nonexamples of topological manifolds.

(1) Consider $\mathbb{A}^1$ and S^2, then $X = \mathbb{A}^1 \amalg S^2$ is a topological manifold. It has two components with dimension 1 and 2, respectively.

(2) A nonexample is a circle with a line through it, since it's not locally Euclidian at the intersection point.

(3) Another nonexample is $\mathbb{A}^1 \cup \mathbb{A}^1 / \sim$ under the identification that glues every point together that isn't zero. So it's a line with a double point, each of which has an interval as an open point. Therefore we can't separate these points, and so this space is not Hausdorff.

(4) $\mathbb{A}^1_{\text{discrete}}$ is an uncountable set, so this is not second countable.

Remark 1.3. We do not study topological manifolds in this class. But smooth manifolds are topological manifolds with extra structure. In dimensions 1,2,3, they are the same, that is, every topological manifold admits a smooth structure.

In dimension four, TOP $\neq$ DIFF. For example, $\mathbb{A}^4$ has infinitely many unique smooth structures. In dimension seven, S^7 has 28 smooth structures. Milnor went on to classify smooth structures of spheres in all dimensions.

Lecture 2

January 21, 2021

2.1 Charts

Definition 2.1. Let X be a topological manifold.

(i) An n-dimensional **chart** on X is a pair (U,ϕ) where $U \subseteq X$ is open and $\phi : U \to \mathbb{A}^n$ is continuous such that ϕ is a homeomorphism onto $\phi(U)$.

(ii) Charts $(U,\phi),(V,\psi)$ are C^∞**-related** if $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$ is a C^∞ map as its inverse. This map is sometimes called the overlap between the charts or the transition function. We already know $\psi \circ \phi^{-1}$ is a bijection/homeomorphism, and we just need smoothness.

Example 2.1. Not all charts are C^∞ related. Let $X = \mathbb{R}$ and $U = V = \mathbb{R}$, $\phi(x) = x, \psi(x) = x^3$. Composing one direction sends $x \mapsto x^3$, while $y \mapsto y^{1/3}$ is not C^∞. These are perfectly valid charts, but not C^∞-related, they are in one direction but not in the other.

Example 2.2. Take $S^2 \subseteq \mathbb{A}^3$, and consider $U = \{x > 0\}$, $\phi(x,y,z) = (y,z)$, projecting onto the yz-plane. Given any point in this disc, we can solve for x^+ given by the equation $x^2 + y^2 + z^2 = 1$. Similarly, let $V = \{y > 0\}$ and $\psi(x,y,z) = (x,z)$. If we use α,β for xz coordinates and u,v for yz-coordinates, the transition map can be expressed on the domain of intersection as $\alpha = \sqrt{1-u^2-v^2}$ and $\beta = v$, where the inverse is also smooth.

2.2 Calculus on Affine Space

There are two arenas where we do calculus, $\mathbb{R}^n = \{(\xi^1,\cdots,\xi^n) \mid \xi^i \in \mathbb{R}\}$ as a vector space, and $\mathbb{A}^n = \{(x^1,\cdots,x^n) \mid x^i \in \mathbb{R}\}$ the affine space of points. As sets these are the same. We have some extra data on $\mathbb{R}^n$: first the zero vector $0 \in \mathbb{R}^n$, addition $+\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and multiplication $\cdot\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

The affine space $\mathbb{A}^n$ has one additional operation, $+\colon \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n$. This takes in a point and a vector, and displaces the point by the vector. So affine space has a vector space of translations, and for V a vector space we have A *affine over* V. Another way to say this is that V acts on A by translations, where the action is *simply transitive*. This means that given two points, we have the existence of a unique vector that takes one point to the other.

Let V,W be vector spaces, A,B be affine over V,W. For $U \subseteq A$ open, let $f : U \to B$. Then for $p \in U, \xi \in V$, we have the **directional derivative** as the map

$$\xi f(p) = \lim_{t\to 0} \frac{f(p+t\xi)-f(p)}{t}.$$

Of course, this may or may not exist.

Theorem 2.1. *If $\xi f(p)$ exists for all $\xi \in \mathbb{R}^n$, $p \in U$, and if each ξf is a continuous function of p, then for each $p \in U$, $\xi \mapsto \xi f(p)$ is a* linear *function of $\xi \in V$. This is called the **differential**, denoted $df_p \colon V \to W$. So $p+\xi \mapsto f(p)+df_p(\xi)$ is the best affine approximation of f at p.*

A conceptual approach to the differential is that $df_p \colon V \to W$ is the unique linear map such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\xi \in V$, $\|\xi\| < \delta$ implies $p + \xi \in U$ and $\|f(p+\xi) - f(p) - df_p(\xi)\| \le \varepsilon\|\xi\|$.

Example 2.3. Let $V = \mathbb{R}^n, A = \mathbb{A}^n_{x^1,\cdots,x^n}$, $W = \mathbb{R}^m, B = \mathbb{A}^m_{y^1,\cdots,y^m}$. Let

$$f = \begin{cases} y^1 = y^1(x^1,\cdots,x^n) \\ y^2 = y^2(x^1,\cdots,x^n) \\ \quad\vdots \\ y^m = y^m(x^1,\cdots,y^n) \end{cases}$$

, $x^i : \mathbb{A}^n \to \mathbb{R}, (x^1, \cdots, x^n) \mapsto x^i$. Then dx^i_p is independent of $p \in \mathbb{A}^n$, $dx^i : \mathbb{R}^n \to \mathbb{R}$ linear, $dx^i \in R^{n*} : (\xi^1, \cdots, \xi^n) \mapsto \xi^i$. Then $dx^1, \cdots, dx^n$ is a *basis* of $\mathbb{R}^{n*}$, and the dual of the dual is $\mathbb{R}^n$, so we also have a basis for $\mathbb{R}^n$ by $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$. We usually think of these as matrices, for example

$$\frac{\partial}{\partial x_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad dx^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j.$$

Since the differential $df_p : \mathbb{R}^n \to \mathbb{R}^n$ is linear, $df_p \left(\frac{\partial}{\partial x^j} \right) = A^i_j \frac{\partial}{\partial y^i}$. Here we use up-down indices and sum over i. We have $A^i_j = \frac{\partial y^i}{\partial x^j}$, the partial derivative. So

$$df_p \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}, \quad df_p = \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \otimes dx^j.$$

Definition 2.2. Given our standard setup, $f : U \to B$ is C^∞ if the iterated directional derivatives

$$\xi_1 \xi_2 \cdots \xi_k f : U \to W$$

exist and are continuous for all $k \in \mathbb{Z}^{>0}$, $\xi_1, \cdots, \xi_k \in V$.

Example 2.4. If f is C^∞, then for all $\xi_1, \xi_2 \in V$, $\xi_1 \xi_2 f = \xi_2 \xi_1 f$.

Example 2.5. For example,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

This idea of second derivates being symmetric functions will come in handy later.

2.3 Smooth Manifolds (for real this time)

Definition 2.3. Let X be a topological manifold.

(i) An **atlas** on X is a collection $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ such that

  (a) The charts cover X, that is, $\bigcup_{\alpha \in A} U\alpha = X$,

  (b) For all $\alpha_1, \alpha_2 \in A$, $(U_{\alpha_1}, \phi_{\alpha_1})$ and $(U_{\alpha_1}, \phi_{\alpha_2})$ are C^∞ related.

(ii) An atlas is a **differentiable structure** on X if in addition

  (c) $\mathscr{A}$ is maximal: if (U, ϕ) is a chart which is C^∞-related to all $(U_\alpha, \phi_\alpha) \in \mathscr{A}$, then $(U, \phi) \in \mathscr{A}$.

(iii) A **smooth manifold** is a pair $(X, \mathscr{A})$ of a topological manifold and a differentiable structure.

Remark 2.1. Any atlas $\mathscr{A}$ is contained in a unique differentiable structure, given by

$$\overline{\mathscr{A}} = \{(U, \phi) \text{ charts} \mid (U, \phi) \text{is } C^\infty\text{-related to all } (U_\alpha, \phi_\alpha) \in \mathscr{A}\}.$$

Remark 2.2. We have an atlas $\mathscr{A}$ on S^2 with $|\mathscr{A}| = 6$ at the beginning of lecture, and there exists an $\mathscr{A}^1$ on S^2 with $|\mathscr{A}^1| = 2$. But there exists no $\mathscr{A}^n$ in S^2 with $|\mathscr{A}^n| = 1$.

Lecture 3

January 26, 2021

3.1 Examples of Smooth Manifolds

Let $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha\in A}$ be an atlas with $x_\alpha \colon U_\alpha \to A_\alpha$. Then we have a surjection

$$\coprod_{\alpha\in A} x(U_\alpha) \xrightarrow{\coprod_{\alpha\in A} x_\alpha^{-1}} M.$$

Is this disjoint union a manifold? It's clearly locally Euclidian and Hausdorff, but whether or not it's second countable depends on the indexing set A.

Example 3.1. Here are some examples of manifolds.

(1) We have $M = \emptyset$ a smooth manifold. This qualifies as a smooth manifold of any dimension, even negative one. This can be useful.

(2) An affine space on V a vector space is a smooth manifold (it has an atlas with a single chart and the identity map).

(3) If $n \geq 0$, $S^n \subseteq \mathbb{A}^{n+1}$ is a smooth manifold.

(4) We can also construct new manifolds from old.

   (a) If $\{M_\alpha\}_{\alpha\in A}$, where M_α are smooth manifolds and A is countable, then $\coprod_{x\in A} M_{\alpha A}$ is a smooth manifold.

   (b) Given $\{M_\alpha\}_{\alpha\in A}$, the Cartesian product product of manifold is also a manifold. For example, the torus $S^1 \times S^1$ is also a smooth manifold.

   (c) Let M be a smooth manifold. Then $N \subseteq M$ open means that N is also a smooth manifold, with the subspace topology. For example, $\mathrm{GL}_n(\mathbb{R}) \subseteq M_n\mathbb{R}$ as an n^2-dimensional vector space, so this forms a smooth manifold. This forms an open subset, which can be realized as the inverse image of an open set $(\mathbb{R} \setminus \{0\})$ under a continuous map, the determinant.

(5) Let V be a real vector space with positive dimension n, and $k \in \{0, 1, \cdots, n-1\}$. Then we define the **Grassmannian** $\mathrm{Gr}_k(V)$ as the set of $W \subseteq V$ subspaces of dimension k. For example, if $V = \mathbb{A}^2$ and $k = 1$, then this is $\mathbb{R}\mathrm{P}^1$. In general, $\mathrm{Gr}_1(V) = \mathbb{P}V$ which is projective space.

   To think about how to construct an atlas, let $w' \in \mathrm{Gr}_k(V)$. Then $w' \oplus w'' = V$ (dimension k and $n-k$). Consider $\psi \colon \mathrm{Hom}(w', w'') \to \mathrm{Gr}_k(V)$, $L \mapsto \Gamma_L$, the graph of L. This is an injective map, and $U_{w''} := \operatorname{im}\psi = \{W \in \mathrm{Gr}_k(V) \mid w \cap w'' = 0\}$ (can't be vertical). Then ψ^{-1} is a chart with values in the vector space $\mathrm{Hom}(w', w'')$, and image $\psi = U_{w''}$ only depends on w''. It can be given an *affine* space structure.

   Now we construct a topology and atlas on $\mathrm{Gr}_k(V)$. For $X \in \mathrm{Gr}_{n-k}(V)$, define $V_X = \mathrm{Hom}(V/X, X)$, $A_X = \{W \in \mathrm{Gr}_k(V) \mid W \cap X = 0\}$. Define on A_X the structure of an affine space over V_X. Namely, every $W \in A_X$ is a linear complement to X, or $V = W \oplus X$. todo: finish constructing the grassmannian

3.2 Functions on Smooth Manifolds

Say we have spaces A, B, C with $U \subseteq A, V \subseteq B$ open, and $f : U \to B$, $g \colon V \to C$. Since $f(U) \subseteq V$, $g \circ f : U \to C$.

Theorem 3.1. *If f, g are C^∞, then $g \circ f$ is C^∞. Furthermore, we have $d(g \circ f)_p = dg_{f(p)} \circ df_p$, where $df_p \colon V \to W$, $dg_{f(p)} \colon W \to X$, $d(g \circ f)_p \colon V \to X$.*

What does it mean for a map $f : M \to N$ between topological spaces to be smooth? If p is a point, pick a chart (U_α, x_α) containing p and another chart (V_β, y_β) containing $f(p)$.

Definition 3.1. A function f is C^∞ at $p \in M$ if for some charts (U_α, x_α) about p and (V_β, y_β) about $f(p)$, the function

$$y_\beta \circ f \circ x_\alpha^{-1} : x_\alpha(U_\alpha) \to y_\beta(V_\beta)$$

is C^∞.

Lemma 3.1. *If the condition above is true for one choice of chart, then it is true for all choices of charts.*

Proof. This relies on the fact that a composition of smooth maps is smooth and the chain rule. Explicitly, say f is smooth. Then $(y_\beta \circ f \circ x_\alpha^{-1})$ is C^∞, and we compose with the transition function $(x_\alpha \circ x_{\alpha'}^{-1})$ to change charts. But this is a composition of C^∞ maps, which is also C^∞. Similarly, changing charts in the codomain gives the composition $(y_{\beta'} \circ y_\beta^{-1}) \circ (y_\beta \circ f \circ x_\alpha^{-1})$, which is C^∞. ⊠

Example 3.2. Let $f : S^2 \to S^2$ be the antipodal map. This is just the restriction of an affine map $f : \mathbb{A}^3 \to \mathbb{A}^3, (x, y, z) \mapsto (-x, -y, -z)$. If $p = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $U_\alpha = \{x > 0\}$, $f(p) = (-1/\sqrt{2}, -1/\sqrt{2}, 0)$, $V_\beta = \{y < 0\}$. Then

$$y_\beta \circ f \circ x_\alpha^{-1}(u, v) = \left(-\sqrt{1 - u^2 - v^2}, -v\right).$$

3.3 The Tangent Space

This is how Freed defines the tangent space, aaaaa. Let $\coprod_{\alpha \in A} V_\alpha$ be the direct product of vector spaces V_α. An element ξ of the direct product looks like $\{\xi_\alpha\}$. The sum is defined by $(\xi + \eta)_\alpha = \xi_\alpha + \eta_\alpha$. Let X be a smooth manifold with atlas $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$, where $x_\alpha : U_\alpha \to \mathbb{A}_\alpha$. Note that $\mathbb{A}_\alpha$ is affine space with a vector space V_α of translations. For $p \in X$, let $A_p \subseteq A$ be the set of indices such $p \in U_\alpha$, and set $\mathscr{A}_p = \{(U_\alpha, x_\alpha)\}_{\alpha \in A_p}$.

Definition 3.2. The tangent space T_pX is the subspace of $\coprod_{\alpha \in A_p} V_\alpha$ consisting of the vectors $\xi = \{\xi_\alpha\}$ such that

$$\xi_\beta = d(x_\beta \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha)$$

for all $\alpha, \beta \in A_p$.

come back and finish notes on tangetn space+watch lecture

Lecture 4

January 28, 2021

4.1 The Tangent Space

come back for ntoes come back and take notes on good algebra stuff, like tangent, cotangent, germs ,etc

Lecture 5

February 2, 2021

"Now I'm out of blackboard, I'm out of time, so I'm out of both space *and* time."

todo: fill in these notes with multivariable analysis notes, pg 16-17, 24-26, 30, 58-63, 65

5.1 Preliminaries for the IFT

Definition 5.1. Let V be a real vector space. A **norm** on V is a function $\|-\|\colon V \to \mathbb{R}^{\geq 0}$ such that

(i) $\|\xi\| = 0$ iff $\xi = 0$,

(ii) $\|\lambda\xi\| = |\lambda|\|\xi\|$,

(iii) $\|\xi_1 + \xi_2\| \leq \|\xi_1\| + \|\xi_2\|$

for all $\xi, \xi_1, \xi_2 \in V_1$, $\lambda \in \mathbb{R}$. Then $(V, \|-\|)$ is a **normed linear space**.

Given $(V, \|-\|)$, define $d\colon V \times V \to \mathbb{R}^{\geq 0}$, $\xi_1, \xi_2 \mapsto \|\xi_1 - \xi_2\|$. You can check that (V, d) is a metric space. We Say $(V, \|-\|)$ is a **Banach space** if (V, d) is complete. This is always true if $\dim V < \infty$.

Example 5.1. Let $V = \mathbb{R}^n$. Then $\|(\xi^1, \cdots, \xi^n)\| = \sqrt{(\xi^1)^2 + \cdots + (\xi^n)^2}$.

Definition 5.2. Let V, W be normed linear spaces, and $T\colon V \to W$ be linear. Then T is **bounded** if there exists a $C > 0$ such that

$$\|T\xi\|_W \leq C\|\xi\|_V \quad \text{for all } \xi \in V.$$

Proposition 5.1. *T is bounded if and only if T is continuous if and only if T is uniformly continuous.*

Proof. Define $\|T\|_{\mathrm{Hom}(V,W)}$ on the least constant $C > 0$ such that $\|T\xi\| \leq C\|\xi\|$ for all $\xi \in V$. We can check that $\mathrm{Hom}(V, W)$ is a normed linear space and for $V \xrightarrow{T} W \xrightarrow{S} X$, we have $\|S \circ T\| \leq \|S\|\|T\|$. ⊠

Example 5.2 (An unbounded linear map)**.** Let $W = \ell^2 = \{(a_1, a_2, a_3, \cdots \mid a_i \in \mathbb{R})\}$ with the obvious norm. Consider $T\colon W \to W$, $e_n \mapsto ne_n$, where $e_n(0, \cdots, \hat{n}, \cdots)$. This function is unbounded. In general, you can't do this with complete spaces, but you can for incomplete ones.

Lemma 5.1. *If W is complete, then* $\mathrm{Hom}(V, W)$ *is complete.*

Proposition 5.2. $\mathrm{Iso}(V, W) \subseteq \mathrm{Hom}(V, W)$ *is open, where*

$$\mathrm{Iso}(V, W) = \{T\colon V \to W \mid T \textit{is a continuous isomorphism, } T^{-1} \textit{ is continuous.}\}$$

Sketch of Proof. Let $T \in \mathrm{Iso}(V, W)$, $a \in \mathrm{Hom}(V, W)$, $\|a\| < \frac{1}{\|T\|}$. We claim that $T + a$ is invertible. Set

$$S_N = \sum_{n=0}^{N} (-1)^n (T^{-1}a)^n T^{-1} \in \mathrm{Hom}(W, V).$$

We claim that $\{S_N\}_N$ is a Cauchy sequence. Note that $\mathrm{id}_V - S_N(T + a) = (-1)^{n+1}(T^{-1}a)^{N+1}$ and $\mathrm{id}_W - (T + a)S_N = (-1)^{N+1}(aT^{-1})^{N+1}$. So

$$\begin{aligned}
\left\| \sum_{n=M+1}^{N} (-1)^n (T^{-1}a)^n T^{-1} \right\| &\leq \sum_{n=M+1}^{N} \|T^{-1}\|^{n+1}\|a\|^n \\
&= \|T^{-1}\| \cdot \sum_{n=M+1}^{N} \left(\frac{\|a\|}{\|T\|}\right)^n \\
&\leq \|T^{-1}\| \frac{\delta^{M+1}}{1 - \delta} \text{ and } M \to \infty.
\end{aligned}$$

Then use use completeness to produce $\lim_{N\to\infty} S_N = S$, and claim that $S = (T + a)^{-1}$. ⊠

5.2 The Contraction Mapping Fixed Point Theorem

Theorem 5.1. *Let (X,d) be a complete metric space, and $\phi: X \to X$. Suppose there exists some $0 < C < 1$ such that*

$$d(\phi(x_1),\phi(x_2)) \leq Cd(x_1,x_2) \quad \textit{for all } x_1,x_2 \in X.$$

[1]*Then there exists a unique $x \in X$ such that $\phi(x) = x$*[2].

Sketch of Proof. Uniqueness is immediate. Choose any $x_0 \in X$. Inductively set $x_{n+1} = \phi(x_n)$. Then $\{x_n\}$ is Cauchy, and $\lim_{n\to\infty} x_n = x$ exists. Essentially we use the NIP and find a fixed point by nesting subsets infinitely. ⊠

Notation. Let V,W be complete normed linear spaces, A,B be affine sets with associated vector spaces V,W, $U \subseteq A$ be open, $f: U \to B$, and $df: U \to \mathrm{Hom}(V,W)$.

Definition 5.3. Let $p \in V$. Then f is **differentiable at** p if there exists a $T \in \mathrm{Hom}(V,W)$ such that

$$\forall \varepsilon > 0\ \exists \delta > 0 \ni \|\xi\|_V < \delta \implies p+\xi \in U \quad \text{and} \quad \|f(p+\xi)-f(p)-T(\xi)\|_W \leq \varepsilon\|\xi\|_V$$

for all $\xi \in V$.

5.3 The Inverse Function Theorem

Proposition 5.3. *Let $p_0,p_1 \in U$ and $(1-t)p_0+tp_1 \in V$ for all $t \in [0,1]$. Suppose f is differentiable, and $\|df_p\| \leq C$. Then $\|f(p_1)-f(p_0)\|_W \leq C\|\xi\|_V$, where $p_1 = p_0+\xi$.*

Proof. Note that $f(p_1)-f(p_0) = \int_0^1 dt\ df_{p_t}(\xi)$, where $p_t = (1-t)p_0+tp_1$. Then

$$\begin{aligned}\|f(p_1)-f(p_0)\| &\leq \int_0^1 dt\ \|df_{p_t}\|\|\xi\| \\ &\leq \int_0^1 dt C\|\xi\| \\ &= C\|\xi\|.\end{aligned}$$

⊠

Theorem 5.2. *With our standard setup, and assume $f \in C^1$, $p \in U$, $df_p: V \to W$ is invertible. Then there exists an $U' \subseteq U$ open, $g: V' \to U' \subseteq A$, $V' \subseteq B$ open such that g and f are inverses. Also, $g \in C'$ and $dg_{f(p)} = df_p^{-1}$ for all $p \in U'$.*

Proof. Define $\widetilde{f}(\xi) = df_p^{-1}(f(p+\xi)-f(p)): U-p \to V$. Set $\phi(\xi) = \xi - \widetilde{f}(\xi)$, $\phi(0) = 0$, $d\phi_0 = 0: V \to V$. Choose $r > 0$ such that $\|d\phi_\xi\| < 1/2$ if $\xi \in \overline{B_r}$. Then by our previous theorem, $\phi(\overline{B_r}) \subseteq \overline{B}_{r/2}$. Say $\eta \in \overline{B_{r/2}}$. Define

$$\phi^\eta(\xi) = \eta + \xi - \widetilde{f}(\xi) = \eta + \phi(\xi): \overline{B_r} \to \overline{B_r}.$$

Then $\phi^\eta(\xi) = \xi$ if and only if $\widetilde{f}(\xi) = \eta$. Estimate

$$\|\phi^\eta(\xi_2)-\phi^\eta(\xi_1)\| = \|\phi(\xi_1)-\phi(\xi_1)\| \leq \frac{1}{2}\|\xi_2-\xi_1\|.$$

Apply the fixed point theorem to produce fixed points $\phi^\eta: \overline{B_r} \to \overline{B_r}$. We have a unique solution given by $\widetilde{g}: \overline{B}_{r/2} \to \overline{B_r}$. Set $g(q) = p + \widetilde{g}(df_p^{-1}(q-f(p)))$. Then

- $\widetilde{g}$ is Lipschitz continuous with constant 2,
- $\widetilde{g}$ is differentiable,

[1]This is called **Lipschitz continuity**, and is stronger than uniform continuity, which is stronger than continuity.
[2]We say that ϕ is a **contraction**.

- $d\widetilde{g}$ is continuous.

So the following diagram commutes:

$$\begin{tikzcd} B_{r/2} \arrow[r, "\widetilde{g}"] & U'' \arrow[r, "d\widetilde{f}"] & \mathrm{Iso}(V) \arrow[d, "\text{invert}"] \\ & & \mathrm{Iso}(V) \end{tikzcd}$$

We conclude that $d\widetilde{g}_\eta = \left(d\widetilde{f}_{\widetilde{g}(\eta)}\right)^{-1}$. ⊠

Lecture 6

February 4, 2021

6.1 The Implicit Function Theorem

When can we use an equation like $f(x,y) = x^2 + xy + y^3 = 0$ to define y as a function of x near a point (x_0, y_0)? We want the tangent line not to be vertical, or $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \neq 0$.

Implicit Function Theorem. *Let A_1, A_2, B be affine spaces, $U_1 \subseteq A_1$, $U_2 \subseteq A_2$ be open sets, $f : U_1 \times U_2 \to B$ be a smooth function, and $(\hat{p_1}, \hat{p_2} \in U_1 \times U_2$ be a point. Assume $(df)^2_{(\hat{p_1},\hat{p_2})} : V_2 \to W$ is invertible. Then we can locally find a function ϕ which solves*

$$f(p_1, \phi(p_1)) = \hat{q}, \quad p_1 \in U_1',$$

where $\phi : U_1' \to U_2$ is smooth.

6.2 Maximal rank, immersions, submersions

Let V, W be finite dimensional real vector spaces.

Definition 6.1. Let $T : V \to W$ be linear. Then

(i) $\operatorname{rank} T = \dim T(V) \leq \min(\dim V, \dim W)$

(ii) T has **maximal rank** if there is equality above.

A maximal rank map is injective if $\dim V \leq \dim W$, bijective if $\dim V = \dim W$, and surjective if $\dim V \geq \dim W$.

Lemma 6.1. *Let V, W be finite dimensional vector spaces.*

(i) The space of maximal rank linear maps $\mathrm{MaxRank}(V, W) \subseteq \mathrm{Hom}(V, W)$ is open.

(ii) If $T \in \mathrm{Max\,Rank}(V, W)$, then there exist $\{e_1, \cdots, e_m\}$ a basis for V, $\{f_1, \cdots, f_n\}$ a basis for W such that

$$T(e_j) = f_j, \quad j = 1, \cdots, m \qquad \text{if } \dim V \leq \dim W,$$

$$T(e_j) = \begin{cases} f_j, & j = 1, \cdots, m \\ 0, & j = m+1, \cdots, n \end{cases} \qquad \text{if } \dim V \geq \dim W.$$

Proof. For (1), if $\dim V = \dim W$, we have already given the argument that the subset of isomorphisms $\mathrm{Iso}(V, W) \subseteq \mathrm{Hom}(V, W)$ is open, by showing that this is the preimage of an open set (specifically $\mathbb{R} \setminus \{0\}$) under the smooth function $\det\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$.

If $\dim V < \dim W$ and $T_0 : V \to W$ has maximal rank, choose $W_0 \subseteq W$ complementary to $T_0(V)$. In other words, $W = T_0(V) \oplus W$. If $\pi\colon W \to W/W_0$ is the projection map, define

$$p\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W/W_0), \quad T \mapsto \pi \circ T.$$

todo: finish this ⊠

6.3 The maximal rank condition for smooth maps of manifolds

Definition 6.2. Let $f: M \to N$ be smooth, $p \in M$, $q = f(p)$. The differential f at p is a linear map $df_p: T_pM \to T_qN$.

(i) If df_p is injective, then f is an **immersion** at p.

(ii) If df_p is surjective, then f is a **submersion** at p (or **regular**, or p is a regular point).

(iii) If df_p is not surjective, then p is a **critical point**.

(iv) If all $p \in f^{-1}(q)$ are regular points, then q is a **regular value** of f.

(v) If there exists a $p \in f^{-1}(q)$ a critical point, then q is a **critical value** of f.

Remark 6.1. If $q \notin f(M)$, then q is trivially a regular value.

Sard's Theorem. *Let* $\mathrm{Crit}(f) \subseteq N$ *be the subset of critical values. Then* $\mathrm{Crit}(f)$ *has measure zero for* f *smooth.*

As a corollary, we have the set of regular values dense, and in particular nonempty. The set of critical values can be empty however, an example being the identity map $\mathrm{id}_M: M \to M$ on any M.

Definition 6.3. $f: M \to N$ is a **diffeomorphism** if f is bijective and f^{-1} is smooth.

Remark 6.2.

- Differentiate $f^{-1} \circ f = \mathrm{id}_M$. Then $(df^{-1})_{f(p)} \circ df_p = \mathrm{id}_{T_pM}$ for all $p \in M$, i.e., $(df^{-1})_{f(p)} = (df_p)^{-1}$.
- A composition of diffeomorphisms is a diffeomorphism. In categorical language, build a category of smooth manifolds, then isomorphism are diffeomorphisms (since we can find inverses).
- $U \subseteq M$ open, $x: U \to A$ affine is a chart iff $x|_U: U \to x(U)$ is a diffeomorphism.

Corollary of IVT. *Let* $f: M \to N$, $p \in M$, $df_p: T_pM \to T_{f(p)}N$ *is invertible. Then there exists* $p \in U \subseteq M$ *open,* $f(p) \in V \subseteq N$ *such that* $f|_U: U \to V$ *is a diffeomorphism, or* f *is a* ***local diffeomorphism*** *at* p.

Sketch of Proof. This is just a manifold version of the inverse function theorem: how do we convert? Choose charts around $p, f(p)$ given by $(\widetilde{U}, x)$, and $(\widetilde{V}, y)$ mapping into affine spaces A, B. Then apply the IVT to $y \circ f \circ x^{-1}: x(\widetilde{U}) \cap x(f^{-1}(\widetilde{U})) \to B$, $d(y \circ f \circ x^{-1}) = dy_{f(p)} \circ df_p \circ (dx^{-1})_{x(p)}$ bijective. ⊠

Proposition 6.1. *Say* $p \in M$ *a smooth manifold,* $n = \dim_p M$, *and* $U \subseteq M$ *open containing* p.

(i) Suppose $x^1, \cdots, x^n: U \to \mathbb{R}$, *and* $dx^1_p, \cdots, dx^n_p$ *form a basis of* $T^*_pM = (T_pM)^*$. *Then there exists a* $U' \subseteq U$ *such that* $(U'; x^1, \cdots, x^n)$ *is a chart.*

(ii) For $x^1, \cdots, x^k: U \to \mathbb{R}$, $k < n$, *and the* $dx^1_p, \cdots, dx^k_p$ *are linearly independent, then there exists a* $U' \subseteq U$ *andd* $x^{k+1}, \cdots, x^n: U' \to \mathbb{R}$ *such that* $(U'; x^1, \cdots, x^k)$ *is a chart.*

(iii) For $x^1, \cdots, x^\ell: U \to \mathbb{R}$, $\ell > n$, $dx^1_p, \cdots, dx^\ell_p$ *spanning* T^*_pM, *then there exists a* $U' \subseteq U$, $\{i_1, \cdots, i_n\} \subseteq \{1, \cdots, \ell\}$ *such that* $(U'; x^{i_1}, \cdots, x^{i_\ell}$ *is a chart.*

todo:prove this stuff and the stuff below it

Theorem 6.1. *Let* $f: M \to N$, $p \in M$, $\dim_p M = m$, $\dim_{f(p)} N = n$, df_p *have maximal rank. Then there exist charts* $p \in (U, x), f(p) \in (V, y)$ *such that*

$$y^i = x^i, \qquad i = 1, \cdots, n \quad m \geq n;$$
$$y^i = \begin{cases} x^i, & i = 1, \cdots, m \\ 0, & i < m+1, \cdots, n \end{cases}, \quad m \leq n.$$

Definition 6.4. We say $f: M \to N$ is an **embedding** if f is a 1-1 (global) immersion (local) which is a homeomorphism onto its image.

Definition 6.5. Let M be a smooth manifold and $Q \subseteq N$ a subset. Then Q is a **submanifold** of N if for all $q \in Q$ there exists $\ell \in \{0, \cdots, n\}$ and a chart (V, y) of N about q such that

$$y(Q \cap V) = \{(y^1, \cdots, y^n) \in \mathbb{A}^n \mid y^{\ell+1} = \cdots = y^n = 0\} \cap y(V).$$

The integer ℓ is the **codimension** of Q in N at the point q.

Remark 6.3. Note that submanifolds are manifolds, with the chart $\mathscr{A} = \{(V_\alpha \cap Q; y^1_\alpha, \cdots, y^\ell_\alpha)\}$, where $\mathscr{A} = \{(U_\alpha, y_\alpha)\}$ is a covering of Q by submanifold charts.

Example 6.1. Consider $f : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$, $t \mapsto (tx_0, ty_0) \pmod{\mathbb{Z}^2}$. If y_0/x_0 is irrational, it doesn't hit any points on the $\mathbb{Z}^2$ lattice besides zero, so it winds densely around the torus. So this is an injective immersion, but *not* an embedding, and the image $Q \subseteq \mathbb{R}^2/\mathbb{Z}^2$ is *not* a submanifold.

Lecture 7

February 9, 2021

We have three basic ways to associate a "shape" to a function $f : M \to N$: the *image* $f(M) \subseteq N$, the *preimage* $f^{-1}(q) \subseteq M$, and the *graph* $\Gamma(f) \subseteq M \times N$ of f.

- For M, N smooth manifolds and f a smooth function, the graph $\Gamma(f)$ is always a submanifold of $M \times N$, and is diffeomorphic to the domain M.
- If f is an embedding, then $f(M)$ is diffeomorphic to M.
- If q is a regular value, then $f^{-1}(q)$ is a submanifold of the domain M.
- We will soon study *transversality*, the condition for the inverse image $f^{-1}(Q) \subseteq M$ of a submanifold $Q \subseteq N$ to be a submanifold.

7.1 Embeddings and submanifolds

Theorem 7.1. *Let $f : M \to N$ be an embedding. Then $f(M) \subseteq N$ is a submanifold.*

Proof. Let Q denote $f(M)$. Fix $q \in Q$: we must construct a submanifold chart about q. Let $p \in M$ be the unique point so that $f(p) = q$. Since f is immersive, we have charts (U, x) about p and (V, y) about q such that $y \circ f \circ x^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^m, 0, \cdots, 0)$. We claim there exists an open subset $V' \subseteq V$ such that the restricted chart (V', y) is a submanifold chart.

If the condition for being a submanifold chart fails, then we have a sequence $\{p_k\}_{k=1}^\infty \subseteq M \setminus U$ such that $\lim_{k\to\infty} y^j(f(q_k)) = 0$ for $j = m+1, \cdots, n$. So the sequence $\{f(q_k)\} \subseteq V$ converges to a point of $f(U)$, and since f is a homeomorphism onto its image we conclude that $\{p_k\} \subseteq M \setminus U$ converges to a point of U, which is a contraction[3] since $M \setminus U$ is closed in M. ⊠

7.2 Regular values and submanifolds

Let us talk about the algebra.

Definition 7.1. A sequence

$$V \xrightarrow{T} W \xrightarrow{S} X$$

of linear maps of vector spsaces is **exact** if $S \circ T = 0$ and $\ker S = \operatorname{im} V$ as subspaces of W. A **long exact sequence**

$$\cdots \longrightarrow V^i \longrightarrow V^{i+1} \longrightarrow V^{i+2} \longrightarrow \cdots$$

[3] This is supposed to be "contradiction", but I find my typo funnier.

is a sequence of linear maps in which every two consecutive maps forms an exact sequence. A **short exact sequence** is a LES of the form

$$0 \longrightarrow V' \xrightarrow{T} V \xrightarrow{S} V'' \longrightarrow 0.$$

In the above equation, the linear map $T\colon V' \to V$ is injective with cokernel $(V/\operatorname{im} T)$ isomorphic to V'', and the linear map $S\colon V \to V''$ is surjective with kernel isomorphic to V'. Furthermore, if V', V, V'' are finite dimensional, then $\dim V = \dim V' + \dim V''$.

Definition 7.2. Let $P \subseteq M$ be a submanifold and $p \in P$.

(1) The **codimension** of P in M at p is defined by $\operatorname{codim}_p(P \subseteq M) = \dim_p M - \dim_p Q = \dim(T_pM/T_pP)$.

(2) The quotient space T_pM/T_pP is the **normal (space)** to P at p.

Sometimes we use the notation $\nu_p = T_pM/T_pP$ to denote the normal space at p. Observe that there is a short exact sequence

$$0 \longrightarrow T_pP \longrightarrow T_pM \longrightarrow \nu_p \longrightarrow 0.$$

Theorem 7.2. *Let $f: M \to N$ be a smooth map of smooth manifolds and $q \in N$ a regular value. Then $P := f^{-1}(q) \subseteq M$ is a submanifold of codimension equal to $\dim_q N$. Furthermore, if $p \in P$,*

$$T_pP = \ker(df_p\colon T_pM \to T_pN).$$

We can express this with the short exact sequence

$$0 \longrightarrow T_pP \longrightarrow T_pM \xrightarrow{df_p} T_qN \longrightarrow 0,$$

illustrated in todo:figure. In general, the codimension is a locally constant function $\operatorname{codim}\colon P \to \mathbb{Z}^{\geq 0}$. Our theorem asserts that if P is cut out by a single function, then codim is constant.

Proof. todo: ⊠

Example 7.1. The 2-sphere $S^2 \subseteq \mathbb{A}^3_{x,y,z}$ is cut out by the single function $f: \mathbb{A}^3 \to \mathbb{R}$, $(x,y,z) \mapsto x^2+y^2+z^2$. Namely, $S^2 = f^{-1}(1)$. The differential $df = 2x\,dx + 2y\,dy + 2z\,dz$ does not vanish at any point of $f^{-1}(-1)$. Note that 0 is a critical point, but $f^{-1}(0) \subseteq \mathbb{A}^3$ is a submanifold (not of the expected codimension of $\dim \mathbb{R}$, though).

Example 7.2. todo:O^n orthogonal group, lie groups, a proposition,

7.3 A counting invariant; the fundamental theorem of algebra

Theorem 7.3. *Let M be a compact smooth manifold, N a smooth manifold with $\dim M = \dim N$, and $f: M \to N$ a smooth function. Set $N_{reg} \subseteq N$ the subset of regular values. Then the function*

$$\#\colon N_{reg} \to \mathbb{Z}^{\geq 0}, \quad q \mapsto \# f^{-1}(q)$$

The conclusion is that for any regular value $q \in N$ the subset $f^{-1}(q) \subseteq M$ is finite and its cardinality is a locally constant function of the regular value.

Proof. todo:this. also todo: fundamental thm of algebra ⊠

Lecture 8

February 23, 2021, Review

Cancelled due to snow day, so here are notes on stuff we should have learned plus review.

8.1 Sard's Theorem

This theorem has a long history involving Morse, Sard, Brown, Dubrovockii, and Thom. But we just call it Sard's theorem.

Sard's Theorem. *Let X, Y be C^∞ manifolds and $f : X \to Y$ a C^∞ map. Denote $C \subseteq X$ the subset of critical points of f. Then $f(C) \subseteq Y$ has measure zero.*

We will talk about measure zero later. Also, recall that $f(C)$ is the set of critical values of f, and its complement in Y is the set of regular values of f. This implies that the set of regular values is dense (since sets of measure zero have nonempty interior), or nonempty, a fact which we often use. Since a finite or countable union of measure zero sets has measure zero, we have the following result:

Corollary 8.1. *Let $\{X_i\}_{i\in I}$ be a collection of smooth manifolds, where I is finite or countable. Let Y be a smooth manifold and $f_i : X_i \to Y, i \in I$ a smooth map. Then the set of simultaneous regular values of f_i is a dense subset of Y.*

Corollary 8.2. *Suppose X, Y are smooth manifolds with $\dim X < \dim Y$ and $f : X \to Y$ a smooth map. Then $f(X) \subseteq Y$ has measure zero.*

In this case, every point of the domain is critical, since the differential cannot be surjective. We can actually prove Corollary 8.2 without Sard's theorem in a more elementary fashion.

Corollary 8.3. *Any smooth map $f : S^n \to S^m$ is nullhomotopic if $n < m$.*

Proof. By the previous corollary there exists a point $q \in S^m$ not in the image of f, so f factors through a map $f' : S^n \to S^m \setminus \{q\}$. Stereographic projection is a diffeomorphism $\varphi : S^m \setminus \{q\} \xrightarrow{\cong} \mathbb{R}^m$. Define the family of homotheties

$$h_t : \mathbb{R}^m \to \mathbb{R}^m, \quad \xi \mapsto (1-t)\xi.$$

Let $\iota : \mathbb{R}^m \hookrightarrow S^m$ denote the inclusion. Then $\iota \circ h_t \circ \varphi \circ f' : S^n \to \mathbb{R}^m$ is a nullhomotopy of f'. ⊠

8.2 Measure zero in affine space

We define the measure (or volume) of subsets of $\mathbb{A}^n$ and use them to define when some $E \subseteq \mathbb{A}^n$ has measure zero.

Definition 8.1.

(i) A **standard box** defined by real numbers $a^1, \cdots, a^n,\ b^1, \cdots, b^n$ with $a^i < b^i$, $i = 1, \cdots, n$ is the set

$$S = S(a^1, b^1; \cdots; a^n, b^n) = \{(x^1, \cdots, x^n) \in \mathbb{A}^n \mid a^i < x^i < b^i \text{ for all } i = 1, \cdots, n\}.$$

If $b^i - a^i$ is the same independent of i, then S is a **standard cube** of length $b - a$.

(ii) The **volume** of the standard box is

$$\mu(S) = \prod_{i=1}^{n} (b^i - a^i)$$

(iii) A set $E \subseteq \mathbb{A}^n$ has **(n-dimensional) measure zero** if for all $\varepsilon > 0$ there exists a covering $\{S_i\}_{i\in I}$ of E with I finite or countable such that $\sum_{i\in I} \mu(S_i) < \varepsilon$.

Note that this depends on the dimension: an open interval in $\mathbb{A}^1$ does not have 1-dimensional measure zero, but it does have n-dimensional measure zero for $n > 1$.

Proposition 8.1.

(1) Let $E \subseteq \mathbb{A}^n$ be a set of measure zero and $E' \subseteq E$ be a subset. Then E' has measure zero.
(2) Let $\{E_i\}_{i\in I}$ be a finite or countable collection of measure zero subsets of $\mathbb{A}^n$. Then $\bigcup_{i\in I} E_i$ has measure zero.
(3) The affine subspace $\mathbb{A}^m \subset A^n$ has n-dimensional measure zero for $m < n$.
(4) Let $U \subseteq \mathbb{A}^n$ be open, $E \subseteq U$ be measure zero, and $f : U \to \mathbb{A}^n$ a C^1 map. Then $f(E) \subseteq \mathbb{A}^n$ has measure zero.
(5) A standard box does not have measure zero.

(6) *If $F \subseteq \mathbb{A}^n$ has nonempty interior, then F does not have measure zero.*
(7) *Let $E \subseteq \mathbb{A}^n$ be closed. Suppose that for all $c \in \mathbb{R}$ the set $E \cap (\{c\} \times \mathbb{A}^{n-1}) \subseteq \mathbb{A}^{n-1}$ has $(n-1)$-dimensional measure zero. Then E has n-dimensional measure zero.*

Since C^∞ maps are C^1, a special case of (4) is that the image of a measure zero set under a C^∞ diffeomorphism has measure zero.

Proof. todo:proof Assertion (1) is immediate since the same cover for E will cover E'. ⊠

8.3 Measure zero on smooth manifolds

Definition 8.2. Let Y be a smooth manifold. A subset $E \subseteq Y$ has **measure zero** if for all $\mathbb{A}^n$-valued charts $(V, y) \subseteq Y$, the set $y(E \cap Y) \subseteq \mathbb{A}^n$ has measure zero.

The dimension n may change based off which connected component you choose of Y. The above definition would be impractical if we had to verify every chart in a maximal atlas, but Proposition 8.1(4) guarantees the following.

Proposition 8.2. *A subset $E \subseteq Y$ has measure zero if the condition of Definition 8.2 holds for a set of charts of Y which cover E.*

Proposition 8.3. *Let $E \subseteq Y$ have measure zero. Then $Y \setminus E$ is dense.*

Proof. $(\overline{Y \setminus E})^c \subseteq E$ is open (complement of a closed set) and has measure zero (since E has measure zero), so it must be empty since it has no interior by Proposition 8.1(6). So $\overline{Y \setminus E} = Y$. ⊠

Proof of Corollary 8.2. todo: ⊠

8.4 Introduction to fiber bundles

A quick recap. We have introduced conditions on smooth maps, like the rank of the differential being maximal, leading to a local normal form for f. An injective immersion which is a homeomorphism onto its image is an embedding, and the image of an embedding is a submanifold. The differential of a submersion is surjective everywhere, so the fibers are submanifolds.

Definition 8.3. Let $\pi \colon E \to M$ be a smooth map of smooth manifolds. We say π is a **fiber bundle** if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ about p and a diffeomorphism $\varphi \colon U \times \pi^{-1}(p) \to \pi^{-1}(U)$ such that the following diagram commutes:

$$\begin{tikzcd} U \times \pi^{-1}(p) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

The domain E is the **total space** and the codomain M is the **base** of the fiber bundle π. Here pr_1 is projection onto the first factor, then inclusion $U \hookrightarrow M$. Note that $\pi^{-1}(p)$ can be empty if π is not onto.

Remark 8.1. Local triviality provides a diffeomorphism $\pi^{-1}(p') \xrightarrow{\cong} \pi^{-1}(p)$ of the fiber over any $p' \in U$ with the fixed fiber $\pi^{-1}(p)$. If we rewrite π as a map from M to sets, then local triviality expresses the local constancy of this map.

Example 8.1. Let M, F be smooth manifolds. Then projection $\mathrm{pr}_1 \colon M \times F \to M$ is a fiber bundle. For any $p \in M$, choose $U = M$ and $\varphi = \mathrm{id}$. This is the trivial fiber bundle over M with fiber F. Any fiber bundle is locally isomorphic (← will make this precise next lecture) to a trivial fiber bundle.

Example 8.2. The map $\pi \colon O_n \to S^{n-1}$, $A \mapsto A\xi_0$ is a fiber bundle, where $\xi_0 = (1, 0, \cdots, 0)$. It is nontrivial if $n \geq 3$.

Any map (of sets) $\pi \colon E \to M$ induces a partition of the domain into its fibers. Fiber bundles induce "regular" partitions in that the fibers are locally diffeomorphic to each other. In this way fiber bundles provide useful decompositions of smooth manifolds. Fiber bundles can encode the geometry of the base manifold, for example the tangent bundle. Since the total space is a smooth manifold, we can apply our tools to learn about the base.

8.5 The tangent bundle

What is the tangent space? Suppose we have an abstract smooth manifold, which doesn't come embedded in affine space. (We don't know that everything can be embedded yet in affine space yet.) The G&P approach is that M comes embedded in affine space, where $M \subseteq \mathbb{A}^n$. Let $\xi \in T_pM \subseteq \mathbb{R}^n$, so ξ is a column vector in $\mathbb{R}^n$, where T_pM is a linear subspace. So when you define the tangent bundle, we have $TM \subseteq M \times \mathbb{R}^n$, $TM = \{(p, \xi) \mid p \in M, \xi \in \mathbb{R}^n, \xi \in T_pM \subseteq \mathbb{R}^n\}$.

For example, we have the 4-manifold $TS^2 \subseteq$ a subspace of the 5-manifold $S^2 \times \mathbb{R}^3$. So $S^2 \to \mathbb{R}^3$, $p \mapsto (p - 0)$, with the displacement vector from the origin denoted by η. So

$$\begin{aligned} F &: S^2 \times \mathbb{R}^3 \to \mathbb{R}, \\ p, \xi &\mapsto \langle \eta(p), \xi \rangle, \end{aligned}$$

where $\langle a, b \rangle$ is the standard inner product in $\mathbb{R}^3$. So $TS^2 = F^{-1}(0)$.

Claim. The origin 0 is a regular value.

Let $(p, \xi) \in TS^2$, $\langle \eta(p), \xi \rangle = 0$. We have $dF_{(p,\xi)} \colon T_pS^2 \times \mathbb{R}^3 \to \mathbb{R}$ is linear, surjective, and nonzero. Since SO_3 acts on $S^2 \times \mathbb{R}^3$, we can assume

$$\begin{aligned} p &= (1, 0, 0) \in \mathbb{A}^3 \\ \xi &\in (0, a, 0) \in \mathbb{R}^3. \end{aligned}$$

We need to check that $dF_{(p,\xi)}(0; (1, 0, 0))$ is nonzero. By the Liebniz rule,

$$\begin{aligned} d\langle \eta(p), \xi \rangle &= \langle d\eta(p), \xi \rangle + \langle \eta(p), d\xi \rangle \\ &= \langle (1, 0, 0), (1, 0, 0) \rangle \\ &= 1 \neq 0. \end{aligned}$$

Now let M be an abstract manifold, p be a point, and (U, x) be a chart at p. Then we have isomorphisms

$$\begin{tikzcd} T_pM \arrow[r, "(U_1,x_1)", "\cong"'] \arrow[dr, "\cong", "(U_2,x_2)"'] & \mathbb{R}^n \\ & \mathbb{R}^n \end{tikzcd}$$

So the tangent bundle is constructed by a map

$$\coprod_{p \in U_1} T_pM \xrightarrow[\cong]{} U_1 \times \mathbb{R}^n.$$

We will elaborate on this next time. Small things: the topology on $\mathrm{Hom}(V, W)$ is the one that comes from a vector space. You can form a topology on SO_2 by considering it as a subspace of $\mathbb{R}^4$.

8.6 Transversality

Say we have manifolds $X, Z \subseteq Y$ and a map $f : X \to Z$, $p \mapsto f(p)$. We consider the linearizations T_pX mapping onto $T_{f(p)}Z$ a subspace of $T_{f(p)}Y$ by $T = df_p$. Something?? then $W + T(X) = Y$. In general transversality measures the failure of a map to be submersive, and surjective implies T transverse. Lol we talked about tennis

8.7 Relating the Grassmannian with the Stiefel manifold

Let V be a vector space with an inner product $a < k \leq \dim V$. Then define

$$\mathrm{St} = \{T : \mathbb{R}^k \to V \mid T \text{ is an isometry}\}$$

Define $\mathrm{Hom}(\mathbb{R}, V)$ as the set of linear maps $T : \mathbb{R} \to V$. Then we have a map $\mathrm{Hom}(\mathbb{R}, V) \to V$, $T \mapsto T(1)$. Consider $f:\ \mathrm{St}_k(V) \to \mathrm{Gr}_k(V)$, where $T \mapsto \operatorname{im} T = T(\mathbb{R}^k) \subseteq V$. Then T is surjective, smooth, and a submersion, so $W \in \mathrm{Gr}_k(V)$, $f^{-1}(W) =$? For $k = 1$, $\mathrm{St}_1(V) = S(V) \to \mathrm{Gr}_1(V)$ is a double cover, or covering map. Say $k = 2$, for $V = \mathbb{R}^3$ what does $\mathrm{St}_2(V) \to \mathrm{Gr}_2(V)$ look like? We have

$$\begin{aligned} f^{-1}(w) &= \{e_1, e_2 \mid e_1, e_2 \text{ form an orthonormal basis of } W\} \\ &= \{\mathbb{R}^2 \xrightarrow[b]{\cong} W \mid \text{isometries}\}. \\ O_2 &= \{\mathbb{R}^2 \xrightarrow[\cong]{g} \mathbb{R}^2 \text{ isometries}\} \text{ a group.} \\ &f^{-1}(W) \times O_2 \to f^{-1}(W) \\ &b, g \mapsto b \circ g. \end{aligned}$$

If $g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, O_2 acts on $f^{-1}(W)$ on the right by $b \circ (g_1 \circ g_2) = (b \circ g_1) \circ g_2$. Now this action is transitive and free, so the action being simply transitive is true iff $f^{-1}(W)$ is a (right) O_2-torsor.

Lecture 9

February 25, 2021

9.1 Fiber Bundles

Definition 9.1. Let $\pi : E \to M$ be a smooth map of smooth manifolds. We say π is a **fiber bundle** if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ around p and a diffeomorphism $\varphi : U \times \pi^{-1}(p) \to \pi^{-1}(U)$ such that the following diagram commutes:

$$\begin{tikzcd} U \times \pi^{-1}(p) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

The domain E is the **total space** and the codomain M is the **base** of the fiber bundle π. We denote $E_p = \pi^{-1}(p)$ to be the fiber over the point p in the base.

The parametrized version of a smooth map of manifolds is a map of fiber bundles. Let $\pi' : E' \to M$ and $\pi : E \to M$ be fiber bundles over the same base. Then a map of fiber bundles is a smooth map $\varphi : E' \to E$ which fits into the commutative diagram

$$\begin{tikzcd} E' \arrow[rr, "\varphi"] \arrow[dr, "\pi'"'] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

It is a smooth family of smooth maps $\varphi_p : E'_p \to E_p$ parametrized by $p \in M$. When the fiber bundles π', π have extra structure—like bundles of affine spaces, vector spaces, Lie groups, etc.—then we may require that φ_p preserve that structure.

Definition 9.2. A **section** of a fiber bundle $\pi\colon E \to M$ is a map $\sigma\colon M \to E$ such that σ is a right inverse for π, that is, $\pi(\sigma(x)) = x$ for all $x \in M$. A **local section** is a section on an open $U \subseteq M$.

Definition 9.3 (Fiber product)**.** The fiber product is the parametrized version of the Cartesian product of manifolds. Let M be a smooth manifold and $\pi_i\colon E_i \to M$, $i = 1,2$ fiber bundles over M. Define

$$E_1 \times_M E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$$

Then $E_1 \times_M E_2 \subseteq E_1 \times E_2$ is a submanifold. (We prove this once we have the tool of transversality.) The maps π_1, π_2 agree and determine a map $\pi\colon E_1 \times_M E_2 \to M$, which is a fiber bundle. Namely, local trivializations φ_1, φ_2 of E_1, E_2 over open neighborhoods U_1, U_2 of a point $p \in M$ combine to a local trivialization of π over $U_1 \cap U_2$. This generalizes to a fiber product of a finite set of fiber bundles over a common base.

9.2 Examples of fiber bundles

Example 9.1. Let V be a 2 dimensional vector space and A be an affine space over V. Let E be the space of affine lines in A. Given a line in A, we get a corresponding line in V by something, todo. We claim that π is a fiber bundle. So $\pi^{-1}(L)$ is the set of parallel lines with tangent direction L. If we choose K the complement to L, where $K \oplus L = V$, we identify K with $\pi^{-1}(L)$ by the isomorphism $\xi \mapsto \ell + \xi$.

Example 9.2. Here we give an example of a surjective submersion that is not a fiber bundle. Let

$$E = \{(x, y, z) \in \mathbb{A}^3 \mid y^2 + z^2 = 1\} \setminus \{(0, 0, 1), (0, 0, -1)\},$$

which looks like an infinite cylinder minus two points.

Example 9.3 (Covering spaces)**.** A (smooth) covering space $\pi\colon E \to M$ is a fiber bundle. To see this, recall that every $p \in M$ has an open neighborhood $U \subseteq M$ which is evenly covered, i.e., there is a discrete set S and a homeomorphism

$$\varphi\colon U \times S \to \pi^{-1}(U)$$

which commutes with projection to U. This is precisely the local trivialization condition. So a fiber bundle with discrete fibers is a covering space.

Example 9.4 (Affine lines in a plane)**.** Let V be a 2-dimensional $\mathbb{R}$-vector space, and let A be affine over V. Let E be the 2-manifold of affine lines in A. Each affine line determines a 1-subspace, its tangent line. Assigning the line is a smooth map $\pi\colon E \to \mathbb{P}V$. We claim that π is a fiber bundle. Fix $K \in \mathbb{P}V$ and $p \in A$. Let us produce a local trivialization of π on $U = \mathbb{P}V \setminus \{K\}$. First, observe that p determines a section $s_p\colon \mathbb{P}V \to E$ of π, which assigns to each $L \in \mathbb{P}V$ the unique affine line through p with tangent line L. Define

$$\begin{aligned} \varphi\colon U \times K &\to \pi^{-1}(U) \\ L, \xi &\mapsto s_p(L) + \xi. \end{aligned}$$

Then φ is a diffeomorphism which commutes with projection, and is therefore a local trivialization, and fiber bundle.

Remark 9.1. The section s_p is an exampe of a "smoothly varying" family of affine lines, and the fiber bundle makes this notion precise.

Remark 9.2. The fibers of Example 9.4 have more structure, they are affine spaces. More precisely, $\pi^{-1}(L)$ is affine over V/L. In fact, there is a vector bundle $Q \to \mathbb{P}V$ whose fiber at $L \in \mathbb{P}V$ is the vector space V/L, and π is a bundle of affine spaces over $Q \to \mathbb{P}V$, a parametrized version of a single affine space over a single vector space. This is an example of a nontrivial fiber bundle.

Example 9.5 (Nonexample no.1)**.** Here we give a (non)example of a surjective submersion that isn't a fiber bundle. Define

$$E = \{(x, y, z) \in \mathbb{A}^3 \mid y^2 + z^2 = 1\} \setminus \{(0, 0, 1), (0, 0, -1)\}.$$

This is a cylinder minus two points n, s. Let P denote the space of affine planes in $\mathbb{A}^3$ which contain the z-axis, then P is diffeomorphic to $\mathbb{RP}^1$ (consider the natural projection onto the the xy-plane).

Define $\pi\colon E \to P$ to be the map taking $p \in E$ to the plane containing the distint non-colinear points n, s, p, where n and s are the deleted north and south poles. Then π is surjective and a submersion: for the latter, a motion germ in P is represented by a curve Π_ℓ of planes through the z-axis. Intersect with the affine line $x = 1$, $z = 0$ to lift to a motion p_t in E such that $\pi(p_t) = \Pi_t$.

However, π is *not* a fiber bundle. The typical fiber bundle of π has total space an ellipse minus n, s, whereas the fiber over the xz-plane Π_{xz} is the union of two affine lines minus $n, s,$, which is not diffeomorphic to the other fibers. So π cannot be locally trivial over Π_{xz}.

9.3 Vector Bundles

Definition 9.4. A **vector space** consists of the data $(V, 0, +, \times)$ where V is a set, $0 \in V$ is a distinguished element (the zero vector), $+\colon V \times V \to V$ and $x\colon \mathbb{R} \times V \to V$ are addition and scalar multiplication. We have some axioms, like $(V, 0, +)$ is an abelian group, scalar multiplication distributes over vector addition, etc.

Definition 9.5 (Vector bundle)**.** A **vector bundle** $(\pi, 0, +, \times)$ consists of a fiber bundle $\pi\colon E \to M$; a section $0\colon M \to E$ of π, called the **zero section**; a smooth map $+\colon E \times_M E \to E$ such that

$$\begin{tikzcd} E \times_M E \arrow[rr, "+"] \arrow[dr] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes; and a smooth map $x\colon \mathbb{R} \times E \to E$ such that

$$\begin{tikzcd} \mathbb{R} \times E \arrow[rr, "\times"] \arrow[dr] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. We also require the vector space axioms and that local trivializations for π be linear maps on fibers. We know that each fiber E_p, $p \in M$ of $\pi\colon E \to M$ is a vector space. The last condition, that local trivializations be linear on fibers requires this be a locally trivial family of vector spaces. Explicity, it asserts that for each $p \in M$ there exists an open neighborhood $U \subseteq M$ and a diffeomorphism φ in the diagram

$$\begin{tikzcd} U \times E_p \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

such that $\varphi|_{p' \times E_p}\colon E_p \to E_{p'}$ is a linear isomorphism for all $p' \in U$.

9.4 Constructions of vector bundles

todo:this section, also todo: tangetn/cotangent bundle

Lecture 10

March (cold), 2021: Recorded lecture

todo:bump functions and partitions of unity

Lecture 11

March 2, 2021

11.1 Embedding manifolds into affine space

Example 11.1. Any 1-manifold is a circle, the affine line, or a union of the two. When embedding manifolds it suffices to consider connected manifolds, sincif we can embed one component we an embed all of them. The circle embeds in $\mathbb{A}^2$, while the line is just $\mathbb{A}^1$.

In dimension two, the manifolds $S^2, S^1 \times S^1$, the two holed torus, etc. embed in $\mathbb{A}^3$. However, $\mathbb{RP}^2$ does not embed in $\mathbb{A}^3$. It does embed in $\mathbb{A}^4$, where $xyz \neq 0$, $[x,y,z] = [\lambda x, \lambda y, \lambda z]$. This can be seen by the embedding $f : \mathbb{RP}^2 \to \mathbb{A}^4$, $[x,y,z] \mapsto \frac{1}{x^2+y^2+z^2}(x^2, xy, xz, yz)$. No we want to show that:

- f separates points (injective)
- f is an immersion
- f is an embedding.

todo:something happened about embedding $\mathbb{RP}^n$ in affine space

Theorem 11.1. *Let M be a smooth manifold. Then there exists an embedding $M \hookrightarrow \mathbb{A}^n$ for some N.*

Theorem 11.2 (easy Whitney)**.** *There exists an embedding $M^n \hookrightarrow \mathbb{A}^{2n+1}$.*

Theorem 11.3 (hard Whitney)**.** *There exists an embedding $M^n \hookrightarrow \mathbb{A}^{2n}$.*

Proof of Theorem 11.1. Assume M is compact. Cover M by finitely many $\mathbb{A}^n$-defined charts $\{((U_\alpha, x_\alpha)\}_{\alpha \in A}$ such that

- $C(2) \subseteq x_\alpha(U_\alpha)$ ($C(2)$ denotes the cube of radius 2)
- $\bigcup_\alpha x_\alpha^{-1}(C(1)) = M$.

Fix a cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \le \chi(x) \le 1$, $\chi(x) = 1$ if $|x| \le 1$, $\chi(x) = 0$ if $|x| \ge 2$. Define

$$\widetilde{x}_\alpha^i : M \to \mathbb{R}, \quad \alpha \in A, i \in \{1, \cdots, n\}$$
$$\widetilde{x}_\alpha^i = \begin{cases} \chi \circ x_\alpha^i, & \text{on } U_\alpha; \\ 0, & \text{on } M \setminus x_\alpha^{-1}\left(\overline{C(2)}\right) \end{cases}$$

$$\rho_\alpha : M \to \mathbb{R}, \quad \alpha \in A$$
$$\rho_\alpha = \begin{cases} \prod_{i=1}^n \chi \circ x_\alpha^i & \text{on } U_\alpha \\ 0, & \text{on } M \setminus x_\alpha^{-1}\left(\overline{C(2)}\right). \end{cases}$$

Set $B_\alpha = \rho_\alpha^{-1}(1)$. Then $x_\alpha^{-1}(C(1)) \subseteq B_\alpha$. So $\bigcup_{\alpha \in A} B_\alpha = M$. Set $f : M \to \mathbb{A}^{(n+1)\cdot \# A}$, $f = \{(\rho_\alpha, \widetilde{x}_\alpha^1, \cdots, \widetilde{x}_\alpha^n\}_{\alpha \in A}$.

Claim. f is an injective immersion.

To see that f is an immersion, let $p \in M$. Choose $\alpha \in A$ such that $p \in B_\alpha \subseteq U_\alpha$. Then $d\widetilde{x}_\alpha^1(p), \cdots, d\widetilde{x}_\alpha^n(p)$ are linearly independent. To see that f is injective, choose $p, q \in M$, $p \in B_\alpha$. If $q \in B_\alpha$, then $\widetilde{x}_\alpha^1, \cdots, \widetilde{x}_\alpha^n$ separates. If $q \notin B_\alpha$, then $\rho_\alpha(p) = 1$, $\rho_\alpha(q) \neq 1$. ⊠

Theorem 11.4. *If M^n is embedded in a finite dimensional affine space, then*

(1) M admits an immersion into $\mathbb{A}^{2n}$,

(2) M admits an injective immersion into $\mathbb{A}^{2n+1}$.

Corollary 11.1. *If M is compact, then M embeds into $\mathbb{A}^{2n+1}$.*

Proof. Suppose $f : M \hookrightarrow A$ is the embedding, where A is affine over V. If todo:help something happened come back to this proof ⊠

11.2 Open covers and partitions of unity

todo:watch the recorded lecture and read the section from warner Here's a quick recap.

Definition 11.1. Let M be a topological space. Let $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$ be sets of open subsets of M.

(i) $\{U_\alpha\}_{\alpha\in A}$ is an **open cover** of M if $\bigcup_{\alpha\in A} U_\alpha = M$.

(ii) $\{V_\beta\}_{\beta\in B}$ is a **subcover** of $\{U_\alpha\}_{\alpha\in A}$ if there exists an injection $r: B \to A$ such that $V_\beta = U_{r(\beta)}$ for all $\beta \in B$.

(iii) A **refinement** of $\{U_\alpha\}_{\alpha\in A}$ is an open cover $\{V_\beta\}_{\beta\in B}$ together with a function $r: B \to A$ such that $V_\beta \subseteq U_{r(\beta)}$ for all $\beta \in B$.

(iv) todo:to see the rest watch the lecture

Definition 11.2. Let M be a topological space and $\rho: M \to \mathbb{R}$ a continuous function. The **support** of ρ is the closed set

$$\operatorname{supp}\rho = \overline{\rho^{-1}(\mathbb{R}^{\neq 0})}.$$

Definition 11.3. Let M be a smooth manifold.

(i) A **partition of unity** $\{\rho_i\}_{i\in I}$ is a set of C^∞ functions $\rho_i: M \to \mathbb{R}$ such that

  (a) $\{\operatorname{supp}\rho_i\}_{i\in I}$ is locally finite

  (b) $\rho_i \geq 0$

  (c) $\sum_{i\in I}\rho_i(p) = 1$ for all $p \in M$

(ii) If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, then $\{\rho_i\}_{i\in I}$ is **subordinate** to $\{U_\alpha\}_{\alpha\in A}$ if there exists a function $r: I \to A$ such that $\operatorname{supp}\rho_i \subseteq U_{r(i)}$ for all $i \in I$.

(iii) If $I = A$ and $r = \mathrm{id}_A$, then we say $\{\rho_i\}_{i\in I}$ is **subordinate with the same index set.**

Theorem 11.5. *Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ an open cover.* todo:watch the lecture

Back to the lecture.

Theorem 11.6. *If M^n is a submanifold of an affine space, then there exists an embedding $M \subseteq \mathbb{A}^{2n+1}$.*

Proof. todo:read in GP, uses the fact that embeding iff injective proper immersion. so we just have to exhibit a proper map ⊠

11.3 Transversality

Definition 11.4 (Transversality for linear maps)**.** Let $T: V \to W$ be a linear map between vector space and $U \subseteq W$ a subspace. Then we say **T is transverse to U**, written $T \pitchfork U$, if and only if the subspaces $T(V)$ and U span W:

$$W = T(V) + W.$$

This is equivalent to the condition that the composition

$$V \xrightarrow{T} W \longrightarrow W/U$$

be surjective, where the second map is projection onto the quotient. (This was a homework problem.)

Definition 11.5. Let X, Y be smooth manifolds, $Z \subseteq Y$ a submanifold, $f: X \to Y$ a smooth map, and $p \in X$ such that $f(p) \in Z$. Then **f is transverse to Z at p**, written $f \pitchfork_p Z$ if

$$T_{f(p)}Y = df_p(T_pX) + T_{f(p)}Z.$$

We say **f is transverse to Z**, written $f \pitchfork Z$, if $f \pitchfork_p Z$ for all $p \in X$ such that $f(p) \in Z$.

Remark 11.1.

(1) For $q \in Y$ we have $f \mathrel{\overline{\pitchfork}} \{q\}$ iff q is a regular value of f.
(2) Any map f satisfies $f \mathrel{\overline{\pitchfork}} Y$.
(3) If $\dim X + \dim Z < \dim Y$, then $f \mathrel{\overline{\pitchfork}} Z$ iff $f(X) \cap Z = \emptyset$.
(4) If $Z_1, Z_2 \subseteq Y$ are submanifolds, and $f_i : Z_i \to Y$ is the inclusion map, then $Z_1 \mathrel{\overline{\pitchfork}} Z_2$ iff $f_1 \mathrel{\overline{\pitchfork}} Z_2$. This relation is symmetric: $f_1 \mathrel{\overline{\pitchfork}} Z_2$ iff $f_2 \mathrel{\overline{\pitchfork}} Z_1$.

Theorem 11.7. *Let X, Y be smooth manifolds, $Z \subseteq Y$ a submanifold, and $f : X \to Y$ a smooth map. Assume $f \mathrel{\overline{\pitchfork}} Z$. Then $W := f^{-1}(Z)$ is a submanifold. Furthermore, if $p \in X$ satisfies $f(p) \in Z$, then*

(1) $T_pW = df_p^{-1}\left(T_{f(p)}Z\right)$.
(2) df_p *induces an isomorphism of normal spaces* $\mathscr{V}_p(W \subseteq X) \to \mathscr{V}_{f(p)}(Z \subseteq Y)$.
(3) $\operatorname{codim}_p(W \subseteq X) = \operatorname{codim}_{f(p)}(Z \subseteq Y)$.

Proof. Choose a submanifold chart (W, y) on Y with $f(p) \in W$, and suppose $y : W \to A$, where A is an affine space over a vector space V. Furthermore, let $A' \subseteq A$ be an affine subpsace so that $y^{-1}(A') = W \cap Z$. Suppose $V' \subseteq V$ is the subspace of translation that preserve A'. Let $\pi : A \to A/V'$ be projection onto the quotient affine space, and let $q \in A/V'$ be the image of A' under π. Since $f \mathrel{\overline{\pitchfork}}_p Z$ we have $d(\pi \circ y \circ f)_p$ surjective. Since surjectivity is an open condition, choose an open neighborhood $U \subseteq X$ of p such that $\pi \circ y \circ f|_U : U \to A/V'$ is a submersion; in particular, $q \in A/V'$ is a regular value and $(\pi \circ y \circ f|_U)^{-1}(q) = W \cap U$. Then apply the preimage theorem. ⊠

Lecture 12

March 4, 2021

12.1 Maximal rank and open conditions

What is a family of maps? We know what a family of smooth manifolds is, with a fiber bundle $\pi : X \to S$. Then we have a family of manifolds X_s, $s \in S$. For a smooth manifold, fix some manifolds X, Y. Then in our parameter space S, we have for $s \in S$ a map $F_s : X \to Y$, where $F : S \times X \to Y$, $F_s(x) = F(s, x)$, for $s \in S$.

Now say we have varying manifolds $F_s : X_s \to Y_s$. Then we have two fiber bundles

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & {} & \end{tikzcd}$$

where π_X, π_Y are fiber bundles, and F is smooth. Certain properties are *stable* todo:(? not sure), or they hold in *open* subsets of S. Consider the diagram

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

where π_E, π_F are vector bundles. Let $m \in M$, and T_m be of maximal rank. Then there exists an open set $U \subseteq M$ containing m such that $T_{m'}$ has maximal rank if $m' \in U$. We have $V \subseteq M$ an open neighborhood of m with local trivializations. A technique we will use in the proof is:

$$\begin{tikzcd} E|_V \arrow[r, "T"] & F|_V \\ V \times E_m \arrow[u, "\cong"] \arrow[r, "S_{m'}"] & V \times F_m \arrow[u, "\cong"'] \end{tikzcd} \quad \text{where } S_{m'}: E_m \to F_m,\ m' \in V.$$

Theorem 12.1. *Let X, Y, S be smooth manifolds, X be compact, $Z \subseteq Y$ be a submanifold, $F : S \times X \to Y$ be smooth. Then there exists an open neighborhood $W \subseteq S$ containing s_0 such that $F_s, s \in W$ is a*

(i) local diffeomorphism
(ii) immersion
(iii) submersion
(iv) $\pitchfork Z$
(v) injective immersion
(vi) embedding
(vii) diffeomorphism

if F_{s_0} *has that property.*

Remark 12.1. Given X, Y, we can look at different topologies on $C^\infty(X, Y) = \{f : X \to Y \mid f \text{ is } C^\infty\}$. Then the notion of **stability** is equivalent to saying maps of type () form an *open* subset. **Approximations** are saying that maps of type () form a *dense* subset. Even in point set topology you thought about function spaces, for example a homotopy is a path in the function space between the domain and the codomain.

Proof. (i) to (iv) are maximal rank properties (we have to convert (iv) to a submersion property by the last lecture). todo:draw a picture? something about uniformity and compactness. For $x \in X$ choose $U_x \subseteq X$, $W_x \subseteq W$ such that dF is maximal rank on $W_x \times U_x$ (base comes before the fiber). Then for $\{U_x\}_{x\in X}$ an open cover of X, we have a finite $F \subseteq X$ such that $\{U_x\}_{x\in F}$ also covers X.

(v) If there is no neighborhood of s_0 where F_s is injective, find sequences $s_n \to s_n$ in S and $\{x_n\}, \{x_n'\}$ in X such that $F(s_n, x_n) = F(s_n, x_n')$. Since X is compact, we can find subsequences $x_{n_k} \to x_0$, $x'_{n_k} \to x'_0$. Then as $k \to \infty$,

$$F_{s_0}(x_0) = F_{s_0}(x_0') \implies x_0 = x_0' \quad \text{since } F_{s_0} \text{ is injective.}$$

Define

$$G : S \times X \to S \times Y, \quad s, x \mapsto s, F(s, x)$$

$$dG_{(s_0,x_0)} = \begin{pmatrix} \mathrm{id}_{T_{s_0}S} & * \\ 0 & dF_{s_0} \end{pmatrix}$$

$$dF_{s_0} \text{ injective} \implies dG_{(s_0,x_0)} \text{ injective} \implies G \text{ is injective in an open nbd of } (s_0, x_0).$$

(vii) Choose a connected neighborhood W of s_0 in S such that F_s, $s \in W$ is an injective local diffeomorphism. We want to show that F_s is surjective. Let $Y_0 \subseteq Y$ be compact, choose $X_0 \subseteq X$ compact such that $F_{s_0}(X_0) = Y_0$. We know $F_s(X_0) \subseteq Y$ is open since F_{s_0} is a local diffeomorphism. $F_s(X_0) \subseteq Y$ is closed since X_0 is compact. Also, $F_s(X_0)$: if $x_0 \in X_0$, then $t \mapsto s_t$ is a path from s_0 to s, then $t \mapsto F_{s_t}(x_0)$ is a path from $F_{s_0}(x_0)$ to $F_s(x_0)$. ⊠

Remark 12.2. For a manifold, path components and components agree.

12.2 Manifolds with boundary

Why do we need manifolds with boundary?

- We have smooth homotopies $[0, 1] \times X \to Y$. If we want to talk about smooth manifolds, at $\partial[0, 1]$ it is not true that the manifold is $[0, 1] \times X$ is locally Euclidian. So we need to add a 'boundary'.
- We do calculus on closed intervals, like finding minima, maxima, etc. Naturally we want to do this on curvy abstract spaces as well.

- Bordism, or "smooth homology". We have the notion of homology, and two 1-cycles are homologous when they differ by a boundary. We can do this with smooth manifolds, where a map between manifolds gives rise to manifolds with boundary.

  We can "cone off" a cycle or circle to get a disk, etc. In singular homology you can cone things off and get a smooth space. But this won't work for something like the torus. A question is "for a given manifold, can I write it as the boundary of a compact manifold"? The answer is generally no (but you can for the torus).

Let us define manifolds with boundary. For the *local model*, let A be affine space, $H \subseteq A$ be an affine hyperplane (a subspace of one less dimension), and A^- the closure of a component of $A \setminus H$. In the *standard local model*, we have $A = \mathbb{A}^n$, $H = \{x^1 = 0\}$, $A^- = \{x^1 \le 0\}$. We have $A^- = \mathrm{Int}(A^-) \amalg \partial A^-$, where $\partial A^- = H$.

12.3 Calculus on manifolds with boundary

If $U \subseteq A^-$, we do calculus by a map $f : U \to B^-$. Note that this is a generalization of our previous notion of calculus, where U may have been contained in the interior of A^-.

- We say f is C^∞ at $p \in U$ if there exists $p \in \widetilde{U} \subseteq A$ open and $\widetilde{f} : \widetilde{U} \to B$ C^∞ such that $\widetilde{f}|_{U\cap\widetilde{U}}$ is the composition

$$U \cap \widetilde{U} \xrightarrow{f|_{U\cap\widetilde{U}}} B^- \hookrightarrow B.$$

Lemma 12.1. *$d\widetilde{f}_p : V \to W$ is independent of extension $\widetilde{f}$.*

Definition 12.1. $df_p : V \to W$ is $d\widetilde{f}_p$ for any $\widetilde{f}$.

Idea: $p \in \partial A^- = H$, then

$$d\widetilde{f}_p = \lim_{p' \to p} d\widetilde{f}_{p'} = \lim_{\substack{p' \to p \\ p' \in U \cap \mathrm{Int}(A^-)}} d\widetilde{f}_{p'} = \lim_{\substack{p' \to p \\ p' \in \widetilde{U} \cap \mathrm{Int}(A^-)}} df_p.$$

If f is a diffeomorphism onto its image, then $f(U \cap H \subseteq K$, and $f(U \cap \mathrm{Int}(A)) \subseteq \mathrm{Int}(B^-)$.

Definition 12.2.

(1) A **topological manifold with boundary** is a topological space X which is Hausdorff, paracompact, and locally homeomorphic to an open subset of a closed half affine space.

(2) An **atlas** is ...

If X is a manifold with boundary, then

- $p \in X$, T_pX is defined as before
- $TX \to X$ is the tangent bundle

$X = \mathrm{Int}(X) \amalg \partial X$.

Lecture 13

March 9, 2021

Definition 13.1.

(1) A **topological manifold-with-boundary** is a topological space X which is:

- Hausdorff,
- second countable,
- locally homeomorphic to closed affine half-space.

(2) An **atlas** is, as before, a covering by charts into closed affine half space with C^∞ overlaps.

Note that $X = \operatorname{Int} X \amalg \partial X$.

Proposition 13.1.

(1) $\operatorname{Int} X$ *is a manifold,*

(2) ∂X *is a manifold.*

Example 13.1. Consider $D^n \subseteq \mathbb{A}^n$ a manifold-with-boundary, where $D^n = \{(x^1, \cdots, x^n) \mid (x^1)^2 + \cdots + (x^n)^2 \le 1\}$. We have $\operatorname{Int} D^n = B^n$ an open ball, and $\partial D^n = S^{n-1}$.

Proof.

(1) Let $p \in \operatorname{Int} X$, and (U, x) be a chart around p where $x \colon U \to A^-$. Set $U' = U \cap \operatorname{Int} A^-$, $x' \colon U' \xrightarrow{X|_{U'}} \operatorname{Int} A^- \hookrightarrow A$. Then (U', x') is a chart on $\operatorname{Int} X$.

(2) Let $p \in \partial X$, and (U, x) a chart about p. Set $U'' = U \cap \partial X$ (this is an open subset of ∂X).

$$\begin{tikzcd} U'' \arrow[r, "X|_{U''}"] \arrow[dr, dotted, "X''"'] & A^- \\ & H \arrow[u, hook] \end{tikzcd}$$

Then (U'', x'') is a chart on ∂X. ⊠

13.1 The tangent space of manifolds with boundary

Let X be a manifold w/∂[4]. If $p \in X$, then $T_pX \xrightarrow{(U,x)\cong} V$ a vector space. (If $x \colon U \to (\mathbb{A}^n)^-$, then $T_pX \cong \mathbb{R}^n$. As before, the vector space T_pX patches to a vector bundle

$$TX = \coprod_{p \in X} T_pX \xrightarrow{\pi} X.$$

Let $p \in \partial X \subseteq X$, then we have a vector subspace

$$\begin{tikzcd} T_p(\partial X) \arrow[d, "(U,x'')"'] & \subset & T_pX \arrow[d, "(U,x')"] \\ V' & \subset & V \end{tikzcd}$$

todo:diagram may be incorrect, also todo: draw some pictures Then since we have $\frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}$ is a basis of $T_p(\partial X)$ and $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ is a basis of T_pX, we have a short exact sequence

$$0 \longrightarrow T_p(\partial X) \longrightarrow T_pX \longrightarrow \underbrace{T_pX/T_p(\partial X)}_{=\mathscr{V}_p(\partial X \subseteq X), \dim = 1} \longrightarrow 0.$$

Definition 13.2. An **orientation** of a real line L is a choice of component of $L \setminus \{0\}$. Conventions:

(1) We orient $\mathscr{V}(\partial X \subseteq X)$ by *outward* normals.

(2) "quotient before sub"

(3) "ONF", which stands for "outward normal first". You can remember this by noting ONF also stands for "one never forgets".

Over ∂X we have a short exact sequence of vector bundles

$$0 \longrightarrow T(\partial X) \longrightarrow TX|_{\partial X} \longrightarrow \mathscr{V}_{\partial X \subseteq X} \longrightarrow 0.$$

[4] I wonder if typing this or "manifold with boundary" is faster?

13.2 Submanifolds of manifolds with boundary

todo:figure: what kind of submanifolds are not allowed? Let A be affine, $H \subseteq A$ be an affine hyperplane, A^- be the closure of components of $A \setminus H$, $S \subseteq A$ be an affine subspace where $S \pitchfork H$ $(V' + V'' = V)$. Define $S^- = S \cap A^-$.

Definition 13.3. Let X be a manifold with boundary, $W \subseteq X$ be a subset. Then W is a **neat submanifold** if for all $p \in W$, we have a local chart of this form: todo:figure. It "straightens out" the submanifold.

13.3 Construction via regular values/tranverse pullback

Proposition 13.2. *Let X be a standard manifold, $f : X \to \mathbb{R}$ be smooth, and $c \in R$ a regular value. Then $f^{-1}(\mathbb{R}^{\leq c})$ is a manifold with boundary $f^{-1}(c)$.*

todo:insert drawn figure about torus. This picture is the poster child for *Morse theory,* where the set of critical values is a discrete set of points. Something about bordism and pinching circles at the critical value. Notation: let $f : X \to Y$, X is a manifold with boundary, $\partial f = f|_{\partial X}, \partial X \to Y$.

Theorem 13.1. *Let X be a manifold with boundary, Y be a manifold, $f : X \to Y$ smooth. Then the subset of Y consisting of simultaneous regular values of $f, \partial f$ is dense.*

This is a conseqeunce of Sard's theorem.

Proof. A regular point $p \in \partial X$ of ∂f is regular for f:

$$\begin{tikzcd} T_p(\partial X) \arrow[dr, "d(\partial f)_p"] \arrow[dd, hook] & \\ & T_{f(p)}Y \\ T_pX \arrow[ur, "df_p"'] & \end{tikzcd}$$

Look for simultaneous regular values of $\partial f : \partial X \to Y$, $f|_{\mathrm{Int} X} : \mathrm{Int} X \to Y$. ⊠

Theorem 13.2. *Let X be a manifold with boundary and Y a manifold, $f : X \to Y$. Let $q \in Y$ be a regular value under $f, \partial f$. Then $W = f^{-1}(q) \subseteq X$ is a neat submanifold.*

$$\mathscr{V}_p(W \subseteq X) \xrightarrow[\cong]{df_p} T_{f(p)}Y \implies \operatorname{codim}_p(W \subseteq X) = \dim_{f(p)} Y.$$

Remark 13.1. There is a generalization to $Z \subseteq Y$ a submanifold, where $f, \partial f \pitchfork Z \implies W := f^{-1}(Z)$ is a neat submanifold of X.

Proof. Let $p \in W \cap \mathrm{Int} X$ as before, $p \in W \cap \partial X$. Choose $(V; y^1, \cdots, y^n$ about q, $y^\alpha(q) = 0$, $U; x^1, \cdots, x^m)$ about p, $f(U) \subseteq V$.

Claim. $x^1, f^\alpha y', \cdots < f^\alpha y^n$ have linearly independent differentials at p. $f^* y^\alpha = y^\alpha \circ f$. $W \cap U = \{f^\alpha y' = \cdots = f^\alpha y^n = 0\}$. todo:?? see notes for this proof Complete to a chart at p:

$$x^1, \tilde{x}^2, \cdots, \tilde{x}^{m-n}, f^\alpha y', \cdots, f^\alpha y^n$$

⊠

Theorem 13.3. *Let X be a connected 1-manifold with boundary. Then X is diffeomorphic to one of the following:*

- S^1 *(compact, no boundary)*
- $[0, 1]$ *(compact, no boundary)*
- $\mathbb{R}$ *(noncompact, no boundary)*
- $[0, 1)$ *(nocompact, boundary)*

Corollary 13.1. *If X is a* compact *1-manifold with boudary, then $\#\partial X \in \partial \mathbb{Z}$.*

You can prove this with Morse functions or Riemannian metrics.

Lecture 14

March 25, 2021

Recap:

Theorem 14.1. *Let X be a manifold with boundary, Y a smooth manifold, $Z \subseteq Y$ a submanifold, and $f : X \to Y$ a smooth map. Then there exists a homotopy $H : [0,1] \times X \to Y$ such that $H_0 = f$ and $H_1, \partial H_1 \pitchfork Z$.*

Theorem 14.2. *Let X be a manifold with boundary, Y a smooth manifold, and $Z \subseteq Y$ a closed submanifold. Say $C \subseteq X$ is a closed subset of X. If we have an $f : X \to Y$ such $f|_C, \partial f|_{C\cap\partial X} \pitchfork Z$, then there exists an $H : [0,1] \times X \to Y$ such that*

$$H_0 = f, \quad H_1, \partial H_1 \pitchfork Z, \quad H_t|_C = f|_C \text{ for all } t \in [0,1].$$

Proof. todo:see the notes ⊠

14.1 Mod 2 degree

Let $f : X \to Y$ be a smooth map for X a compact n-manifold and Y a connected n-manifold. If $q \in Y$ is a regular value, then $f^{-1}(q)$ is a 0-dimensional submanifold of X, hence a finite set of points since X is compact. todo:see the notes for good stuff As we move between the regular values, the inverse images are created an annihilated in pairs.

Theorem 14.3. *Fix $n \in \mathbb{Z}^{>0}$.*

(1) The mod 2 cardinality $\#f^{-1}(q) \pmod 2$ of the inverse image of a regular value $q \in Y$ is independent of q.

(2) If $F : [0,1] \times X \to Y$ is a smooth homotopy of maps, and $q \in Y$ is a simultaneous regular value of F, F_0, F_1, then $\#F_0^{-1}(q) = \#F_1^{-1}(q) \pmod 2$.

Proof. For (2), todo:see the notes ⊠

Proposition 14.1. *Let X be a compact connected manifold of nonzero dimension. Then $\mathrm{id}_X \not\simeq$ the constant map, i.e., X is not contractible.*

Proof. We have $\deg_2 \mathrm{id}_X = \# \mathrm{id}_X^{-1}(q) = \#\{q\} = 1$ for all $q \in X$. Then $\underline{q_0} : X \to X$ is a constant map with value q_0. So $\mathrm{def}_2\, \underline{q_0} = \#\underline{q_0}^{-1}(q) = 0$ $(q \neq q_0)$. ⊠

Proposition 14.2. *Let X be a compact connected manifold of nonzero dimension. Then there exists an $f : X \to S^n$ with $\deg_2 f = 1$.*

todo:see the sketch, apparently it's important

14.2 Mod 2 intersection

The setup is similar to the degree mod 2. Let Y be a smooth manifold, X a compact manifold, $Z \subseteq Y$ a closed submanifold. Let $f : X \to Y$ be smooth, and $\dim X + \dim Z = \dim Y$.

Definition 14.1. If $f \pitchfork Z$, define $\#_2(f,Z) = \#f^{-1}(Z) \pmod 2$. This is a compact 0-submanifold of X, and is therefore a finite set of points in X. If $f \not\pitchfork Z$, use a homotopy of f to achieve transversality. If $f_0 \sim f_1$, $f_0, f_1 \pitchfork Z$ implies $\#f_0^{-1}(Z) = \#f_1^{-1}s(Z) \pmod 2$. If $f : X \hookrightarrow Y$ is an inclusion of a submanifold, write $\#_2(X,Z) = \#_2(Z,X)$.

If W is compact with boundary, $F : W \to Y \supset Z$ closed, and $\dim \partial W + \dim Z = \dim Y$, then $\#_2(\partial F, Z) = 0$.

Example 14.1.

(1) Let $Y = S^1 \times S^1, * \in S^1, X = S^1 \times *, Z = * \times S^1$. Then $\#_2(X,Z) = 1$.

(2) Let $Y = S^2$, $f : S^1 \to S^2$. Then f is nullhomotopic, so $f = \partial F$, where $F : D^2 \to S^2$. This implies that $\#_2(f,Z) = 0$ for all ?? This gives us the following:

Theorem 14.4. *The torus is not diffeomorphic to S^2.*

(3) Let $Y = \mathbb{R}\mathrm{P}^2$, and $L = \mathbb{R}\mathrm{P}^1$ be a line in $\mathbb{R}\mathrm{P}^2$. We can think of this as compactifying $\mathbb{A}^2$ with $\mathbb{R}\mathrm{P}^1$ (at the boundary, identify antipodal points). Since 1+1=2, perturb L to L' by a homotopy, so $\#_2(L, L) = 1$.

(4) The same story happens in $\mathbb{C}\mathrm{P}^2$.

(5) Let $Y = \mathbb{R}\mathrm{P}^2$, and $Z = L$ a line. Take $X = C$ a cubic curve (crazy stuff happens)