

# Riemannian Geometry Notes

Simon Xiang

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# Part I

## Class Notes

Lecture 1

January 19, 2021

What is Riemannian geometry?? Consider  $\{(x, y) \mid x^2 + y^2 < 1\}$ , this is a coordinate chart. This doesn't tell us anything about the geometry of the surface, since  $z = \sqrt{1 - x^2 - y^2}$  and  $z = x^2 - y^2$  have the same coordinates. Geometry is extra data on a surface, besides the topological data. Our basic extra ingredient is the inner product, which allows us to talk about length, which tells us all sorts of things about the manifold.

### 1.1 Introduction to curvature

"In fact, with the inherited inner product on  $\mathbb{R}^2$ , this curved string is *not* curved!"

The map sending a curved string to a straight line is an isometry. So there are two different notions of curvature: extrinsic curvature talks about to what extent is something bent relative to something else, and intrinsic talks about what happens if you look locally. Locally, all 1-manifolds are isometric to  $\mathbb{R}$ .

Now to 2-manifolds. We have the circle, half-sphere, saddle, and half-cylinder discussed earlier. We can't really tell the difference between a cylinder and the plane, they're isometric. Extrinsic they're different, but intrinsically they're the same. The sphere is different—a circle has circumference  $2\pi R$ , that is, the circumference is the set of points of distance  $R$  away. What is the circumference of a sphere? Use spherical coordinates: a circle of radius  $R$  is a latitude line, and the length is  $2\pi$  times the distance of the great circle cut out at  $R$ , or  $2\pi \sin(R)$  if you draw out the angles. Approximately,  $2\pi \sin(R) \approx 2\pi(R - \frac{R^3}{6})$ . So circles are too small!

This means that spheres are in some sense, sphere shaped. Another thing is the area: the area of the cap is  $A = \int 2\pi \sin(R)dR = 2\pi(1 - \cos(R)) = 2\pi(\frac{R^2}{2} - \frac{R^4}{4!} + \dots) = \pi R^2 - \frac{\pi R^4}{12} + \dots$ . So circumferences and areas are a little bit too small. If we worked in  $z = x^2 + y^2$ , we would find that circumferences and areas would be a little bit too big. This is why you can't flatten an orange peel, or an accurate scaled map of the world preserving angles and area.

### 1.2 Dual space

Suppose  $V$  is an  $n$ -dimensional vector space, with basis  $\mathcal{E} = \{e_1, \dots, e_n\}$ . Then we can write any  $v \in V$  as the sum  $\sum v^i e_i$ . So we have a natural correspondence between vectors  $v$  and coordinates  $\{v^i\}$  where  $v^i \in \mathbb{R}^n$ . From now on, we use  $v^i e_i$  to denote  $\sum v^i e_i$ , this is called Einstein summation notation.

The dual space  $V^* = \text{Hom}(V, \mathbb{R})$  is the space of linear functionals from  $V$  into the base field, the reals. One element of  $V^*$  is the function that assigns each vector to its  $i$ th coordinate  $\phi^i(v) = v^i$ . So we have a nice set of transformations  $\{\phi^1, \phi^2, \dots, \phi^n\}$ .

**Claim.** The set  $\{\phi^1, \phi^2, \dots, \phi^n\}$  forms an  $n$ -dimensional basis for  $V^*$ , called the *dual basis* to  $\mathcal{E}$ .

*Proof.* Let  $\alpha = \alpha_i \phi^i$ . Suppose  $\alpha = 0$ , then for all  $v$ ,  $\alpha(v) = 0$ . So  $\alpha(e_j) = 0$ , and  $\alpha_i \phi^i(e_j) = 0$ . Now  $\phi^i(e_j) = \delta_j^i$ , so  $\alpha_i \delta_j^i = 0$  and therefore  $\alpha_j = 0$ . Now define  $\alpha_j = \alpha(e_j)$ . Applying this to a vector  $v$  gives  $(\alpha_j \phi^j)(v) = \alpha_j(v)(\phi^j(v)) = \alpha_j(v)v^j$ . Then  $\alpha(v) = \alpha(v^j e_j) = v^j \alpha(e_j) = v^j \alpha_j(v)$ , and these are equal. We conclude that  $\alpha = \alpha_j(v)v^j$ .  $\square$

Summary: write arbitrary elements of  $V$  as  $v^i e_i$ , and the dual space as  $\alpha = \alpha_j \phi^j$ . We have  $\phi^j(v) = v^j$ ,  $\alpha(e_i) = \alpha_i$ , and  $\alpha(v) = \alpha_i v^i$  (short for  $\alpha_i(v)v^i$ ). This is why we call  $V^*$  the dual space: pairing elements together in either order gives a number. On this vein,  $V^{**} = V$ , where the basis  $\{e_1, \dots, e_n\}$  is dual to  $\{\phi^1, \dots, \phi^n\}$ .  $V$  is the space of contravariant vectors, while  $V^*$  is the space of covectors, which are covariant in a categorical sense.

You can visualize this by thinking of covectors as rows and vectors as columns. Then  $e_i$  is a column with a 1 in the  $i$ th slot, while  $\phi^j$  is a row with a 1 in the  $j$ th slot. Then  $\alpha(e_i) = \alpha_i$ ,  $\phi^j(v) = v^j$ , as can be seen below.

$$\alpha = \begin{pmatrix} & \alpha & \end{pmatrix}, \quad v = \begin{pmatrix} v \\ & \end{pmatrix}, \quad \phi^j = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} & \alpha & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \alpha_i, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ & \end{pmatrix} = v^j.$$

Suppose we have a new basis  $\tilde{\mathcal{E}} = \{\tilde{e}_1, \dots, \tilde{e}_n\}$ . There must be some change of basis matrix, that is,  $\tilde{e}_i = A_i^j e_j$ , similarly we have a new dual basis with  $\tilde{\phi}^i = \phi^j B_j^{i1}$ . If we have a vector  $v = v^i e_i = \tilde{v}^i \tilde{e}_i$ , we have  $\tilde{v}^i = \tilde{\phi}^i(v) = B_j^i \phi^j(v) = B_j^i v^j$ . When your basis vectors get big, your coordinates get small, so they transform oppositely. This is the origin of the term contravariant. We claim that  $A_i^j B_j^k = \delta_i^k = B_i^j A_j^k$ . To see this, note that  $\delta_i^k = \tilde{\phi}^k(\tilde{e}_i) = \tilde{\phi}^k(A_i^j e_j) = \phi^l B_\ell^k (A_i^j e_j) = B_\ell^k A_i^j \delta_j^\ell = B_j^k A_i^j = A_i^j B_j^k$ .

We can kinda visualize vectors as columns and covectors as rows. Then  $e_i$  is a column with 1 in the  $i$ th while  $\phi^j$  is a row in the  $j$ th slot. Applying  $\alpha$  to  $e_i$  gives  $\alpha_i$ , while applying  $\phi^j$  to  $v$  gives  $v^i$ .

### 1.3 Tensors

A  $(k, \ell)$  tensor eats  $k$ -covectors and  $\ell$ -vectors, and pops out a number. These should be multilinear in each slot. In terms of coordinates,  $T(v, w) = T(v^i e_i, w^j e_j) = v^i w^j T(e_i, e_j)$ . We define  $T_{ij} = T(e_i, e_j)$ , so  $T(v, w)$  becomes  $T_{ij} v^i w^j$ . This is a  $(2, 0)$ -tensor. What is a  $(1, 1)$ -tensor? This is essentially a matrix, where  $S(v, \alpha) = S_i^j v^i \alpha_j$ . So  $S$  is the matrix,  $\alpha$  is a row, and  $v$  is a column. If you don't give a tensor enough information, it turns into a tensor of lower rank, which works the same in matrix multiplication. If you have a doubly covariant tensor, it turns vectors into covectors. If you have a doubly contravariant tensor, it turns covectors into vectors. Next time, we'll talk about the most important doubly covariant tensor, the inner product.

**Note.** The easiest way to remember why a  $(k, \ell)$ -tensor eats  $k$  covectors and  $\ell$  tensors is by first thinking of a tensor as an element of  $\bigotimes_k V \otimes \bigotimes_\ell V^*$ , then taking the dual to realize it as a multilinear map on  $(V^*)^{\times k} \times V^{\times \ell}$ . Also, a covariant tensor eats vectors because a covector does.

Lecture 2

January 21, 2021

### 2.1 A basis for tensors

Let's continue the algebra from yesterday. Recall a tensor takes two two vectors as input, denoted  $T(v, w) = T(v^i e_i, w^j e_j) = v^i w^j T(e_i, e_j) = \left(\sum_{i,j}\right) T_{ij} v^i w^j$ . If  $V = \mathbb{R}^2$ , what's an example of a covariant 2-tensor? The standard inner product  $\langle v | w \rangle = v^1 w^1 + v^2 w^2$  works. Using the notation  $g(v, w)$ , we have  $g_{11} = g(e_1, e_1) = 1, g_{12} = 0, g_{21} = 0, g_{22} = 1$ . In general, for an inner product  $g_{ij} = g_{ji}$ , and if you think of it as a matrix, this will be a positive definite matrix, or  $g_{ij} v^i v^j \geq 0$  if  $v \neq 0$  (a symmetric matrix with all eigenvectors are positive). In general, you can build

<sup>1</sup>In this convention,  $A_i^j$  is the entry  $i$ th row and  $j$ th column of a matrix  $A$  (sometimes you see  $a_i^j$ ) instead. Usually  $i$  denotes rows and  $j$  denotes columns, hence why the change of basis matrix for the dual basis is written as  $B_j^i$ .

an inner product out of a symmetric positive definite matrix. For example, the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with positive eigenvectors} \quad \begin{bmatrix} \frac{1 \pm \sqrt{5}}{2} \\ 1 \end{bmatrix}$$

works. Another interesting tensor is the determinant  $A(v, w) = v^1 w^2 - v^2 w^1 = -A(w, v)$ <sup>2</sup>. In this case,  $A_{12} = -A_{21} = 1$  and  $A_{11} = A_{22} = 0$ .

The natural next question is: what is the space of covariant 2-tensors on  $\mathbb{R}^2$ ? This is a vector space, what would the basis be? A natural basis would be the tensors  $\phi^i \otimes \phi^j$ . The tensor symbol in the middle is the **tensor product**: if  $S(v_1, \dots, v_n)$  and  $T(w_1, \dots, w_n)$  are covariant tensors (makes life simpler), then

$$S \otimes T(v_1, \dots, v_n, w_1, \dots, w_n) := S(v_1, \dots, v_n)T(w_1, \dots, w_n).$$

With this in mind, applying our basis tensor to  $(e_k, e_\ell)$  gives us  $\phi^i \otimes \phi^j(e_k, e_\ell) = \phi^i(e_k) \cdot \phi^j(e_\ell) = \delta_k^i \delta_\ell^j$ . This gives us 1 if you feed it  $(e_i, e_j)$ , and 0 if you feed it anything else. Since these form a basis for the space of covariant 2-tensors, our claim is that you can write any tensor  $T$  in the form  $T = T_{ij} \phi^i \otimes \phi^j$ . Then  $T(e_k, e_\ell) = T_{k\ell}$  by definition.

Let us return to the inner product. If we only feed it one vector, then  $g(v, )$  is waiting for a vector to spit out a number. This is just a covector, which gives an alternate use for the inner product besides taking two vectors and spitting out a number. This map is injective, since if it had a kernel (that is,  $g(v, ) = 0$ ), then  $g(v, )(v) \neq 0$ , since it's positive. Then  $g$  induces a map  $g : V \rightarrow V^*$  an isomorphism: if  $\alpha = g(v, )$ , then  $\alpha_j = g_{ij} v^i$ . We call this *lowering an index*: we take something with an up index, multiply it through  $g_{ij}$ , resulting in a down index. The inverse that *raises* indices is the inverse matrix  $g^{ij}$ , and  $g^{ij} g_{jk} = \delta_k^i$ .

Now that we have a basis for 2-tensors, what about a basis for  $\mathcal{T}^{k,\ell}$  (space of tensors)? It's going to be

$$\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes \phi^{j_1} \otimes \phi^{j_2} \otimes \cdots \otimes \phi^{j_\ell}\}$$

So a tensor  $T$  can be written as  $T = T_{j_1 \cdots j_\ell}^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes \phi^{j_\ell}$ . Furthermore, to calculate the coefficients, apply your tensor to the basis vectors to get  $T_{j_1 \cdots j_\ell}^{i_1 \cdots i_k} = T(\phi^{i_1}, \dots, \phi^{i_k}, e_{j_1}, \dots, e_{j_\ell})$ .

## 2.2 Trace of a matrix

Now let's talk about the trace of a matrix, defined as  $\text{Tr}(M) = M^i_i$ . This is actually independent of which basis we pick. If we change our basis by  $\tilde{e}_i = A_i^j e_j$ , recall that the dual basis transforms accordingly by  $\tilde{\phi}^i = B_j^i \phi^j$ , where  $B_j^i$  is an inverse for  $A_i^j$ . In our new basis,  $\tilde{M}_j^i = B_k^i A_j^\ell M_\ell^k = B_k^i M_\ell^k A_j^\ell$ , this is kind of like  $PMP^{-1}$  that we did in linear algebra. But this doesn't really work with tensors, what about 2, 3, 4, 5-tensors? That's why we're doing it this way.

We have the trace  $\text{Tr} = \tilde{M}_i^i = B_k^i M_\ell^k A_i^\ell$ . But  $B_k^i A_i^\ell = \delta_k^\ell$ , since these two are inverse matrices. So  $\text{Tr} = \delta_\ell^k M_\ell^k = M_k^k$ , which is the old definition of trace. We can apply this to higher rank tensors: suppose we have a tensor that takes in a covector and two vectors, denoted  $T(\alpha, v, w)$ . Define a tensor  $S(w) = T(\phi^i, e_i, w)$ , so  $S_i = T_{ji}^j$ . What happens if you change basis, that is, is  $T(\phi^i, e_i, w) = T(\tilde{\phi}^i, \tilde{e}_i, w)$ ?

*(breakout rooms)*

We want to show that  $T_{ij}^i w^j = \tilde{T}_{ij}^i \tilde{w}^j$ . For fixed  $w$ , let's define a new tensor,  $U(\alpha, v) = T(\alpha, v, w)$ . Let's take the trace of this, we've already shown that the trace of a  $(1, 1)$ -tensor doesn't depend on basis. So the  $w$  comes along for the ride. This is a slick solution, since  $\text{Tr } U$  is well defined. Once you know that the trace is defined for  $(1, 1)$ -tensors, you know it's defined for any  $(n, n)$ -tensor, where you pair one up index with one down index.

## 2.3 Tangent vectors

Enough about tensors. Now let's talk about manifolds. Usually the vector space at a point  $p$  we care about is the tangent space  $T_p(M)$  at that point. The object of importance is not a tensor at a point, but rather vector fields,

<sup>2</sup>You can also call this the area form, or the symplectic form.

covector fields, tensor fields, etc. The coefficients of such tensors are going to be functions of where you are, or the coordinates you're using.

In  $\mathbb{R}^n$ , consider a point  $p$ : what is a tangent vector there? There are several definitions, and they're all equivalent:

1. An arrow, add them head to tail.
2. An element of  $\mathbb{R}^n$ , a list of  $n$  numbers. This is pretty much an arrow, just take the coefficients and impose them on the standard basis.
3. Velocity. Consider all possible parametrized curves through a point, and identify all curves with the same velocity at time zero. So we mean the equivalence class of curves  $\gamma(t)$  with  $\gamma(0) = p$ <sup>3</sup>. The beauty of this third definition is it makes sense on any manifold. So we can consider the curves going through this point, and take them up to equivalence.
4. Directional Derivative. This is pretty much the same thing as velocity, since for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we can take  $\frac{df \cdot \gamma'(t)}{dt} \Big|_{t=0} = \frac{dy^i}{dt} \Big|_{t=0} \frac{\partial f}{\partial x^i} \Big|_p$ . The partial derivatives  $\{\partial_1, \dots, \partial_n\}$  gives a basis for this vector space.
5. Derivations. A derivation at  $p$  is a map  $D : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  with the following properties:
  - (1)  $D(af + bg) = aD(f) + bD(g)$ ,
  - (2)  $D(fg) = f(p)D(g) + g(p)D(f)$ .

The partial derivatives  $\partial_i|_p$  are derivations.

This idea of thinking about tangent vectors as derivations carries over very nicely to abstract manifolds, which is why we care. Understanding the ring of smooth functions means we understand what a tangent vector is, coordinate-free (we aren't anchored to  $\mathbb{R}^n$ ).

The directional derivative is a small step forward from velocity, which is a small step forward from the naïve arrow. Our goal is to show that the scary abstract idea of a derivation is just a small step forward from the directional derivative, and to do this, we show that they both use the same basis.

**Claim.** We have  $\{\partial_i\}$  a basis for the set of derivations at  $p$ , denoted  $\mathcal{D}_p(\mathbb{R}^n)$ .

We want to show a couple of things:

1. To show  $D(\text{constant}) = 0$ , note that  $D(cf) = cD(f) = cD(f) + f(p)D(c)$ , so we need  $D(\text{constant}) = 0$ .
2. If  $f(p) = g(p) = 0$ , then  $D(fg) = 0$ , this follows from the product rule.
3. Taylor series, we have  $f(x) = f(p) + \partial_i f(p)(x^i - p^i) + \text{higher order terms}$ . Then  $D(f) = 0 + \partial_i f(p)D(x^i) + 0 = D^i \partial_i f(p)$ , where  $D = D^i \partial_i$ .

This works for analytic functions. There is some cheating going on, but we don't need the entire Taylor series for the most part<sup>4</sup>. So every derivation is a linear combination of partial derivatives. From now on, think of vectors as a combination of partial derivatives, or inducing curves along a vector field.

## 2.4 Vector fields

A vector field  $v^i(x) \frac{\partial}{\partial x^i}$  is a bunch of coefficients in combination with the basis vectors of partial derivatives. How do we change coordinates here? We do this by the chain rule, that is,

$$\text{for } \{x\} \longleftrightarrow \{y\} \quad \text{we have} \quad \frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}.$$

<sup>3</sup>Since you consider the curves up to equivalence, a straight line is usually the one you visualize, since it's the simplest and they're all equivalent. Of course, you can choose a wigglier curve if you want.

<sup>4</sup>See my notes in `independent_reading/differentiable_manifolds` for more info (or Tu's book).

You can think of  $\partial x^j / \partial y^i$  as the change of basis matrix  $A_i^j$ . The inverse matrix  $B_j^i$  is just  $\partial y^i / \partial x^j$ . Let's try this in  $\mathbb{R}^2$  with  $(r, \theta) \leftrightarrow (x, y)$ . For  $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$  and  $\tilde{e}_1 = \frac{\partial}{\partial r}, \tilde{e}_2 = \frac{\partial}{\partial \theta}$ , recall that  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \tan^{-1}(\frac{y}{x})$ , and  $x = r \cos \theta, y = r \sin \theta$ . So

$$\frac{\partial y}{\partial r} = \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial y}{\partial \theta} = \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta = -y, \quad \frac{\partial x}{\partial r} = r \cos \theta = x.$$

So now we can convert between the two bases, that is, we have the change of basis matrix

$$\begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix}.$$

We can write the inverse matrix in terms of  $(x, y)$  or  $(r, \theta)$ , taking the inverse of a two by two is pretty easy. Now we know how to convert from one basis to the other.

Lecture 3 —

## January 26, 2021

### 3.1 A basis for the tangent space

Consider some manifold, then how do we find a basis for the tangent space? If each point is covered by a chart, consider the coordinates for each chart and take the partial derivatives in that direction. For a topologically interesting manifold, usually just one set of coordinates won't work for the whole manifold. So it becomes imperative that we know how to change between coordinate charts. Suppose we have two coordinates  $(x^1, x^2, \dots, x^n)$  and  $(y^1, y^2, \dots, y^n)$ , these give rise to the bases  $\{\partial_{x^i}\}$  and  $\{\partial_{y^i}\}$ . The chain rule tells us that

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Then the  $\frac{\partial y^j}{\partial x^i}$  form a change of basis matrix  $A_i^j$ , and we can take everything we know about vector spaces and tensors at a *point*, and apply it to tensor fields. A tensor on a *fixed* vector space is a certain object given by some coefficients. If we want a tensor *field*, we need a tensor at each point, which requires understanding what a basis is for the tangent space at each point, and what the coefficients are. Now the numbers  $T_{ij}^k(x)$  are *functions* of where you are, if  $T_{ij}^k$  gives the coefficients for a  $(1, 2)$ -tensor (1-contravariant 2-covariant).

*“Think globally, act locally.”*

This is our slogan for working with manifolds. You always want to keep a global picture of what's going on in your head, but when doing computations and such, always work in a neighborhood with coordinate charts, where everything looks like  $\mathbb{R}^n$ .

### 3.2 A basis for the dual space

What's the basis for the dual space of tangent vectors? Suppose we have a function  $f : M \rightarrow \mathbb{R}$ . Then we can take the derivative  $df(V) := V(f)$ , which is a covector field. In  $\mathbb{R}^n$ , we have  $(df)(\frac{\partial}{\partial x^1}) = \frac{\partial f}{\partial x^1}$ , and similarly  $(df)\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^j} = \partial_j f$ . So  $(df) = \frac{\partial f}{\partial x^j} \phi^j$ , since for a covector  $\alpha$ , we have  $\alpha_i = \alpha(e_i)$ . What happens when we take the derivative of the coordinate function  $x^i$ ? Note that we have

$$(dx^i)\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^i}{\partial x^j} = \delta_j^i.$$

So on a manifold with coordinates  $x^1, \dots, x^n$ , the basis for  $T_p M$  is  $\{\frac{\partial}{\partial x^i}\}$ , and the dual basis for  $T_p^* M$  is  $\{dx^i\}$ .

### 3.3 Applications of the dual basis

Now that we've characterized the bases for the tangent and cotangent spaces, what does a tensor actually look like? Consider a  $(2, 3)$ -tensor  $T(x) = T_{ijk}^{\ell m}(x) \frac{\partial}{\partial x^\ell} \otimes \frac{\partial}{\partial x^m} \otimes dx^i \otimes dx^j \otimes dx^k$ , where  $T_{ijk}^{\ell m} = T(dx^\ell, dx^m, \partial_i, \partial_j, \partial_k)$ . Once more: covariant is how many down indices you have, contravariant is how many up indices you have<sup>5</sup>. Fortunately we don't work with 5-tensors very often, but we do deal with a particular 2-covariant tensor all the time, which is the metric.

We are interested in doing geometry, so we need some way to measure lengths on a vector space. If we have the notion of an inner product at every point, then we have a metric  $g = g_{ij} dx^i \otimes dx^j$ , where  $g_{ij}^{(x)} = g(\partial_i, \partial_j)_x = \langle \partial_i, \partial_j \rangle_x$ . We will spend a ridiculous amount of time talking about this tensor at a point  $x$ . In  $\mathbb{R}^n$  this inner product is just  $\delta_{ij}^i$ , but for a manifold it isn't! Let's play around with the manifold of the upper hemisphere, given by  $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z > 0\}$ . Some possibilities for coordinates:

1.  $x, y$ , where  $z = \sqrt{1 - x^2 - y^2}$ ,
2.  $\theta, \phi$ , where  $\theta$  measures the angle from the north pole, and  $\phi$  measures the longitude. So  $x = \sin \theta \cos \phi$ ,  $y = \sin \theta \sin \phi$ ,  $z = \cos \theta$ .

How do we find a metric? We have  $\partial_x = (1, 0, -x/z)$ ,  $\partial_y = (0, 1, -y/z)$ . So

$$\begin{aligned} g_{11} &= 1 + \frac{x^2}{z^2} = 1 + \frac{x^2}{1 - x^2 - y^2} = \frac{1 - y^2}{1 - x^2 - y^2}, \\ g_{12} &= \frac{xy}{z^2} = \frac{xy}{1 - x^2 - y^2}, \\ g_{22} &= 1 + \frac{y^2}{z^2} = \frac{1 - x^2}{1 - x^2 - y^2}. \end{aligned}$$

These are, of course, functions of  $x$  (as a point). A perfectly reasonable question to ask now would be "what is  $\partial g_{ij}/\partial x^k$ "? Since we now have the metric as a function of the coordinates, we can ask at what rate does it change. Now let's move onto spherical coordinates. We have

$$\begin{aligned} \frac{dx}{dt} &= \cos \theta \cos \phi \frac{d\theta}{dt} - \sin \theta \sin \phi \frac{d\phi}{dt}, \\ \frac{dy}{dt} &= \cos \theta \sin \phi \frac{d\theta}{dt} + \sin \theta \cos \phi \frac{d\phi}{dt}, \\ \frac{dz}{dt} &= -\sin \theta \frac{d\theta}{dt}. \\ \frac{\partial}{\partial \theta} &= (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad \frac{\partial}{\partial \phi} = (-\sin \theta \sin \phi, \sin \theta \cos \phi, 0). \end{aligned}$$

So

$$\begin{aligned} g_{11} &= \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \phi = \cos^2 \theta + \sin^2 \theta = 1, \\ g_{12} &= 0, \\ g_{22} &= \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi = \sin^2 \theta. \end{aligned}$$

This makes sense because as you vary  $\theta$ , the rate of change down the sphere is 1, or you're moving with speed 1. This basis is nice and orthogonal, but it's not orthonormal. If we consider the basis  $\{\partial_\theta, (1/\sin^2(\theta))\partial_\phi\}$ , then we get an orthonormal basis. The problem is, there is no set of coordinates that satisfies that the expression is exactly the derivative with respect to the coordinates. It's kind of magic that our basis in  $\mathbb{R}^n$  is both orthonormal, and consists of the derivatives of the coordinates.

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<sup>5</sup>Dr. Sadun has been doing things the other way around (he says he can never keep it straight), but I feel like the covectors and vectors should be in this order.

### 3.4 Vector fields and vector flows

When you see a new vector field, you should ask yourself “what does the flow look like?”

Every time we have a vector field, it induces a **flow**. Suppose we have a vector field on a manifold. Given a point, we can find a curve that follows the vector field: that is, we can find a curve  $\{\gamma(t) \mid \gamma(0) = p, d\gamma/dt = V(\gamma(t))\}$ . Moving along these flow lines gives us a **flow**, a way to turn a point at time  $t = 0$  into a point at a later time. We denote this  $\Phi_t^V(p)$ , which is what we get when we start at a point  $p$  and flow along a vector field  $V$  for time  $t$ . Let’s look at some simple vector fields and figure out what their flows are in  $\mathbb{R}^2$ .

- (i) If  $V(x) = \partial_{x^1}$ , then  $\Phi_t^V(x, y) = (x + t, y)$ . Similarly, for  $W = \partial_{x^2}$ ,  $\Phi_t^W(x, y) = (x, y + t)$ .
- (ii) For  $X(x) = x^1 \partial_1 + x^2 \partial_2$ , our flow is given by  $\Phi_t^X(x, y) = e^t(x, y)$ . Note that the definition of flow sets up a differential equation, and the one we are solving right now is  $dx^1/dt = x^1, dx^2/dt = x^2$ .
- (iii) Say  $Y(x) = x^1 \partial_2 - x^2 \partial_1$ . This is rotation at a constant speed, since we’re solving the differential equations  $dx^1/dt = -x^2, dx^2/dt = x^1$ .

Recall that a vector field is a derivation. Suppose  $V$  and  $W$  are vector fields. If we have a function  $f : M \rightarrow \mathbb{R}$ , what kind of object is  $W(f)$ ? It’s also a function  $M \rightarrow \mathbb{R}$ , since the derivative of a function is a function. So is  $V(W(f))$ , and so on. What is  $V \circ W$ ? It’s not a vector field, but a second order differential operator, since we’re taking second derivatives.

We claim that  $V(W(f)) - W(V(f))$  is a *first order* differential operation. Note that  $W(f) = w^i \partial_i f$ , so

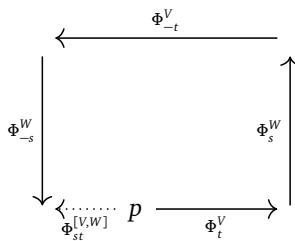
$$\begin{aligned} V(W(f)) &= v^j \partial_j(w^i \partial_i f) \\ &= v^j (\partial_j w^i) \partial_i f + v^j w^i \partial_j \partial_i f. \\ W(V(f)) &= w^i \partial_i(v^j \partial_j f) \\ &= w^i (\partial_i v^j) \partial_j f + w^i v^j \partial_i \partial_j f. \end{aligned}$$

Taking the difference of the two eliminate the second order term, since mixed partial derivatives commute. So

$$V(W(f)) - W(V(f)) = v^i \partial_j w^i (\partial_i f) - w^i (\partial_i v^j) \partial_j f = (v^j \partial_j w^i - w^i \partial_j v^i) \partial_i f.$$

This is called the **Lie bracket**, and is denoted  $[V, W]$ . It also goes by the *commutator*, or the *Lie derivative*  $\mathcal{L}_V(W)$ . What do these brackets mean? If we have *any* coordinates,  $[\partial_i, \partial_j] = 0$ . This tells us that if we have two vector fields with a nonzero Lie bracket, then they cannot be derivatives with respect to coordinates. It’s not that you haven’t thought up of the right coordinates, but that there *are no coordinates* that they’re derivatives of. If you recall the flows  $\Phi_t^V, \Phi_t^W$  above, they actually commute!

A basic question when you have two vector fields is what happens if you start at a point  $p$  and flow for time  $v$  according to  $V$ , then you flow for time  $s$  along time  $s$ ? Or better yet, you flow backwards along  $V$  with time  $-t$ , and backwards along  $W$  with time  $-s$ .



If the bracket is zero, you come back to where you started, and if it’s nonzero, you don’t. This bracket measures how far off you are! Showing that this is true is the homework.

### 3.5 The gradient

Suppose we have some  $f : M \rightarrow \mathbb{R}$ , then  $df = \partial_1 f dx^1 + \partial_2 f dx^2 + \dots$ . This isn't what a gradient is, it's the vector that has the property that  $\nabla f \cdot v = D_v f = v(f)$ , while  $df(v) = v(f)$ . The difference is the *inner product*. We have  $g(\nabla f, v) = df(v)$ , or  $g(\nabla f, \cdot) = df$ . The differential is what you get when you take the gradient and lower it an index. If you're working in  $\mathbb{R}^n$  this doesn't do anything, the metric is just the identity. However in more interesting coordinates, it does do something. While  $df$  is a covector, the **gradient**  $\nabla f^i$  is a vector. So

$$df_j = g_{ij}(\nabla f)^i, \quad \text{or} \quad (\nabla f)^i = g^{ij}(df)_j \quad (\text{note the inverse metric.})$$

The differential  $df$  is defined without reference to any inner product. It's a perfectly good covector field (or 1-form) on its own. The gradient is what you get when you raise the index of  $df$ .

Lecture 4

January 28, 2021

### 4.1 Inheriting metrics

A **Riemannian manifold** is a manifold with a nice, smoothly varying inner product at every point. Given a topological space, how do you come up with an interesting inner product on such space? One way to do this is to sit inside a space that already has one.

**Example 4.1.** The inner product of two vectors on the surface of a torus is their inner product in  $\mathbb{R}^3$ : but how do we actually write this down? We use coordinate charts. Say we have  $u, v$  coordinates and a map  $(u, v) \rightarrow \Sigma(u, v)$ . A basis for the domain (in  $\mathbb{R}^2$ ) is just  $\partial_u, \partial_v$ , so this gives rise to a basis  $\partial_u \Sigma = d\Sigma(\partial_u), \partial_v \Sigma = d\Sigma(\partial_v)$ . These are vectors in  $\mathbb{R}^3$ . Then

$$g_{11} = \partial_u \Sigma \cdot \partial_u \Sigma, \quad g_{12} = g_{21} = \partial_u \Sigma \cdot \partial_v \Sigma, \quad g_{22} = \partial_v \Sigma \cdot \partial_v \Sigma.$$

In general, for a submanifold, take the metric of the big space and restrict it to tangent vectors of the submanifold. When you parametrize surfaces, we need  $d\Sigma$  injective (or rank two), or else something else will map to 0, violating the rules of a metric.

**Example 4.2.** Consider the graph  $z = f(x, y)$ . This gives rise to a simple parametrization, where  $\Sigma(u, v) = (u, v, f(u, v))$ . Then  $\partial_u$  corresponds to  $(1, 0, \partial_u f)$  and  $\partial_v$  corresponds to  $(0, 1, \partial_v f)$ . So

$$g_{11} = 1 + (\partial_u f)^2, \quad g_{12} = g_{21} = (\partial_u f)(\partial_v f), \quad g_{22} = 1 + (\partial_v f)^2.$$

For the upper hemisphere where  $z = \sqrt{1-x^2-y^2} = f(x, y)$ , recall that  $\partial_u f = -u/\sqrt{1-x^2-y^2}, \partial_v f = -v/\sqrt{1-x^2-y^2}$ . So

$$g_{11} = \frac{1-v^2}{1-x^2-y^2}, \quad g_{12} = g_{21} = \frac{uv}{\sqrt{1-x^2-y^2}}, \quad g_{22} = \frac{1-u^2}{1-x^2-y^2}.$$

Work out the metric for the sphere on your own, where  $(u, v) \simeq (\theta, \phi) \rightarrow (\sin u \cos v, \sin u \sin v, \cos u)$ . For a parametrized curve  $\gamma(t)$  we only have one component, so  $g_{11} = \dot{\gamma} \cdot \dot{\gamma} = \|\dot{\gamma}\|^2$ . So  $g = \|\dot{\gamma}\|^2 dt^2$ .

### 4.2 Product spaces

Given two manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  with coordinates  $(x^1, \dots, x^n), (y^1, \dots, y^m)$ , we have the metric  $g_1 = (g_1)_{ij} dx^i \otimes dx^j$ , and the metric  $g_2 = (g_2)_{kl} dy^k \otimes dy^l$ . Then we can define the *product manifold*  $M_1 \times M_2$ , with coordinates  $(x^1, \dots, x^n, y^1, \dots, y^m)$ . Recall from a G&P question that  $T_{(p,q)}(M_1 \times M_2) = T_p M_1 \times T_q M_2$ . So we write a vector  $V$  in  $T_{(p,q)}(M_1 \times M_2)$  in the form  $V = (V_1, V_2)$ , where  $V_1$  is tangent to  $M_1$  and  $V_2$  is tangent to  $M_2$ . Then

$$\begin{aligned} g &= ((v_1, v_2), (w_1, w_2)) = g(v_1, w_1) + g_2(v_2, w_2) \quad \text{or} \\ g &= g(g_1)_{ij} dx^i \otimes dx^j + (g_2)_{kl} dy^k \otimes dy^l. \end{aligned}$$

The *simplest* example is  $\mathbb{R}^2$ , where each component is a vector in  $\mathbb{R}^1$ . Then  $g((v_1, v_2), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2) = v_1 w_1 + v_2 w_2$ , which is exactly how the inner product works in  $\mathbb{R}^2$ . This generalizes to  $\mathbb{R}^{n+m}$ , of course.

**Example 4.3.** The torus is the product of two circles  $\mathbb{T}^2 = S_{R_1}^1 \times S_{R_2}^1$ . If the metric on  $S_{R_i}^1$  is given by  $R_i^2 d\theta^2$ , the metric for the torus just  $g_1 = R_1^2 d\theta^2 + R_2^2 d\theta^2$ .

### 4.3 The warped product

Suppose we have two manifolds  $M_1, M_2$  and a function  $f : M_1 \rightarrow \mathbb{R}^+$  (excluding 0!!). Then define  $M_1 \times_{f_2} M_2 = (M_1 \times M_2, g)$ , where  $g$  is *not* the product metric. The metric  $g$  at a point  $(p, q)$  is given by  $g_{(p,q)} = (g_1)_{ij} dx^i \otimes dx^j + (f(p))^2 (g_2)_{kl} dy^k \otimes dy^l$ . This is a **warped product**. The warped product is indeed a metric, since it's positive definite, given  $f(p)$  never maps to zero.

Warped products are familiar: consider surfaces of revolution. If we parametrize  $\gamma(t)$  by arc length (so it's unit-speed), then  $\gamma(t) = (x(t), y(t))$ , a curve in the upper half plane. Then

$$\Sigma(t, \theta) = (x(t), y(t) \cos(\theta), y(t) \sin(\theta)).$$

We claim that this surface  $\Sigma$  is equal to a product  $\gamma \times_f S^1$ , where  $f(t) = y$ . To see this, note that  $\partial_t \Sigma(\dot{x}, \dot{y} \cos \theta, \dot{y} \sin \theta) = (0, -y \sin \theta, y \cos \theta)$ . So  $g_{11} = \dot{x}^2 + \dot{y}^2 = 1$ ,  $g_{12} = g_{21} = 0$ ,  $g_{22} = y^2$ . Then our metric is given by  $g = dt^2 + (y(t))^2 d\theta^2$ . The reason why  $\dot{x}^2 + \dot{y}^2 = 1$  is because we used a unit speed parametrization.

**Example 4.4.** Concrete example: consider the half circle  $\gamma(t) = (\cos t, \sin t)$  for  $0 < t < \pi$ . Then rotating around the  $x$ -axis gives a sphere minus antipodal poles<sup>6</sup>, and  $g = dt^2 + \sin^2(t) d\theta^2$ . This looks very familiar, namely, the metric on this warped product is the exact same as the standard one for the upper half sphere.

This demonstrates the idea of the warped product giving an interesting structure. Here,  $M_1 = (0, \pi)$ ,  $M_2 = S^1$  are both flat, but the warped product gives something round.

### 4.4 Quotients by discrete actions

Consider the quotient space  $\mathbb{R}/L\mathbb{Z}$ , or the set  $\{x \in \mathbb{R}\} / \sim$  where  $x \sim (x + nL)$ . Does the quotient space inherit a metric? Since  $\mathbb{R}/L\mathbb{Z}$  is a circle, a tangent vector  $v_i$  comes from a representative  $w_i$  in  $\mathbb{R}$ , that is  $v_i = \pi_*(w_i)$ , where  $\pi_*$  is projection onto the quotient space. Then define  $g(v_1, v_2) = g(w_1, w_2)$ . This fails when two vectors upstairs (say  $w_i$  and  $\tilde{w}_i$ ) have different values under the inner product. For  $w_i$  and  $\tilde{w}_i$  to have the same length, the identification needs to come from a group action by isometries. Here, the group action is “translate by  $L$ ”, it has no fixed points, and it acts by isometries.

**Proposition 4.1.** *If a group  $G$  acts freely on a space  $(X, g)$  by isometries, then  $X/G$  inherits a metric from  $X$ .*

In our example above,  $G = \mathbb{Z}$ , and the action is  $(n, x) \rightarrow x + nL$ . We are inclined to use the coordinates of  $X$  for  $X/G$ , since locally (pre-identification)  $X/G$  looks like  $X$ . This isn't a new concept, we have done this for years on the circle: we have always used  $\theta$  for coordinates by identifying the integers  $2\pi n$ . When would this fail?

**Example 4.5** (Non-examples). Suppose  $X = (\mathbb{R}^+, dx^2)$ , and  $G = \mathbb{Z}$ . Let the action be given by  $(n, x) \rightarrow 2^n x$ . This is a perfectly nice free group action (by way of our domain). However,  $G$  doesn't act by isometries. If we ask what the metric is at 1, this is the same as the metric at 2, 4, 1/2, and so on since they're all identified. This group action stretches things, so we don't have a well defined notion of length. The quotient is a perfectly good topological space, a circle: it just doesn't have a metric.

Now suppose  $X = \mathbb{R}^2$  and  $G = \mathbb{Z}_2$ , so the action is rotation by  $\pi$ . This identifies the lower half of the plane with the upper half, and the positive real axis with the negative real axis. Besides the origin, this is a perfectly good group action acting isometrically and freely. However, the origin is a fixed point. A disk around the origin post quotient becomes a disk  $\mathbb{Z}/2$ , so this is not a manifold. This is an *orbifold*, a manifold with a singular point. In general, for nonfree group actions, big problems happen in neighborhoods of fixed points.

**Example 4.6** (Non-non-examples). Here we give some actual examples.

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<sup>6</sup>This also omits a line (the IDL?), but we can take care of that by identifying the lines and making the quotient space.

- We have already seen  $\mathbb{T}^2 = \mathbb{R}^2 / \text{lattice}$ , where a lattice is  $\mathbb{Z}^2$  with two basis vectors. In general, we can form  $\mathbb{T}^n = \mathbb{R}^n / \text{lattice}$ .
- Another important example is  $\mathbb{RP}^n = S^n / \pm 1$ .
- The **lens spaces**  $L(p, q)$  are formed by taking  $S^3 \subseteq \mathbb{C}^2 / \sim$ , where  $(z_0, z_1) \sim (\lambda z_0, \lambda^q z_1)$  and  $\lambda = e^{2\pi i/p}$ . The group here is  $\mathbb{Z}_p$ , and the identification is the group action, acting by isometries with no fixed points. Note that we want  $p, q$  to be relatively prime. Also note that  $L(2, 1)$ , since  $e^{2\pi i/2} = -1$ . These lens spaces look locally like  $S^3$ , but globally have some interesting topological properties: namely  $\pi_1(L(p, q)) = \mathbb{Z}_p$ .
- We will also eventually see some Riemann surfaces, given by  $\mathbb{H}^2 / G$ . Hyperbolic space has constant negative curvature, so you can realize a structure of Riemann surfaces with genus greater than 1, and give them all metrics of constant negative curvature. This is the same thing as talking about the *canonical* space of constant curvature  $\mathbb{H}$ , and modding it by a certain group action.

## 4.5 The complex projective plane

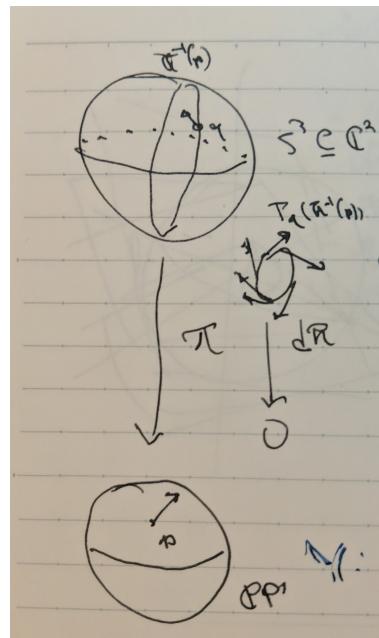
We can also quotient by continuous groups. Consider the complex projective plane  $\mathbb{CP}^1$ , defined by

$$\mathbb{CP}^1 := \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}\} / \sim, \quad (z_1, z_2) \sim (\lambda z_1, \lambda z_2), \lambda \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$$

Multiplication by a scalar  $\lambda$  (say  $3+i$ ) is *not* an isometry, since it scales lengths by  $\sqrt{10}$ . So the way we've written it as of now doesn't fit into our framework. However, we can break up  $\mathbb{C}^*$  into  $\mathbb{R}^+ \times U(1)^7$ , an element of the unit circle times a length. So we can rewrite the definition as

$$\mathbb{CP}^1 := \{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1)^2 + (z_2)^2 = 1\} / \sim, \quad (z_1, z_2) \sim (e^{i\theta} z_1, e^{i\theta} z_2).$$

Behold, *this* is an isometry. For now we restrict our discussion to lengths of vectors, since if you know what the length of a vector is, you know what the inner product is. This is because  $\langle v, w \rangle = (\langle v+w, v-w \rangle - \langle v-w, v-w \rangle)/4$  which is the sum of lengths squared over four. (This is called the *polarization identity*.) Consider  $S^3 \subseteq \mathbb{C}^2$  projecting onto  $\mathbb{CP}^1$  (topologically a sphere) by  $\pi$ , then a point  $p \in \mathbb{CP}^1$  corresponds to a circle in  $S^3$ . Consider a point  $q$  in this circle in  $S^3$  corresponding to  $p$ .



<sup>7</sup>Recall that  $U(1)$  denotes the unit complex numbers.

However,  $d\pi$  is *not* injective, since it maps any vector tangent to the sphere to zero. We can write  $T_q S^3$  as  $T_q(\pi^{-1}(p)) + T_q(\pi^{-1}(p))^\perp$ , so  $d\pi: T_q(\pi^{-1}(p)) \rightarrow 0$ . But  $d\pi: T_q(\pi^{-1}(p))^\perp \rightarrow T_p \mathbb{CP}^1$  is an isomorphism by the rank nullity theorem, since  $T_q(\pi^{-1}(p)) = \ker d\pi$ . Let us name this map  $\alpha$ . Then  $\langle v, w \rangle = g(\alpha^{-1}(v), \alpha^{-1}(w))$ . We say the vectors  $T_q(\pi^{-1}(p))$  are **vertical**, or they live in the fiber above a point. A vector that lives in  $T_q(\pi^{-1}(p))^\perp$  is **horizontal**, since it's orthogonal to a vertical vector. So the idea is to lift vectors in the base space uniquely to horizontal vectors, since they don't generally have a unique vertical left (since  $d\pi$  fails to be injective). What ingredients did we need to make this work?

- (1) The group acts by isometries. (We need two distances in the total space to be the same.)
- (2) The group needs to act freely.

This map  $\pi$  from  $S^3 \rightarrow S^2$  is called the **Hopf fibration**. This map has some nice properties, that is, the preimage of two points will always be linked circles (if you think of  $S^3$  as the one-point compactification of  $\mathbb{R}^3$ ). For  $\mathbb{CP}^n$ , we consider it as the quotient  $S^{2n+1}/S^1$ , or  $S^{2n+1} = \{(z_0, \dots, z_n) \mid |z_0|^2 + \dots + |z_n|^2 = 1\}/S^1$ , where  $(z_0, \dots, z_n) \sim (e^{i\theta}z_1, \dots, e^{i\theta}z_n)$ . What does the metric on  $\mathbb{CP}^1$  look like? This is the **Fubini-Study metric** todo:read about this in the book

Lecture 5

## February 2, 2021

Here we talk about model spaces, and when we get around to discussing curvature and such, we will compare them to these spaces, as in “how much does a space differ from  $S^n$ ?” for example.

### 5.1 Different notions of symmetry

**Definition 5.1.** A space  $X$  is **homogeneous** if all points look the same, that is, there exists an isometry  $X \rightarrow X$  that takes  $p$  to  $q$  for all  $p, q \in X$ . For spaces  $(X, g)$  and  $(Y, \tilde{g})$ , we say  $X, Y$  are **isometric** if we have a map  $f: X \rightarrow Y$  such that  $f$  is a diffeomorphism, and  $g$  is the pullback of  $\tilde{g}$  (that is,  $g = f^*\tilde{g}$ ).

**Note.** Some differential topology. Recall that the derivative of a map  $F: M \rightarrow N$  is also called the **push-forward**, since it induces a map  $F_*: T_p M \rightarrow T_{F(p)}N$ . Then this gives rise to a dual map on the cotangent spaces  $(F_*)^*: T_{F(p)}^*N \rightarrow T_p^*M$ , which *pulls back* tangent covectors in  $T_{F(p)}^*N$  to tangent covectors in  $T_p^*M$ . Rewrite  $(F_*)^*$  as  $F^*$  and call this the **pullback** of  $F$ .

**Example 5.1.** Some examples of homogeneous spaces:

1. Sphere
2. Plane
3. Torus

If you think of the torus as  $\mathbb{R}^2/\mathbb{Z}^2$ , then translations form a nice symmetry of the torus. However, even though every point the same, this doesn't hold for directions. This motivates the following definition:

**Definition 5.2.** A space  $X$  is **isotropic** if for all  $p, q \in X$ , where  $v, w$  are unit tangent vectors to  $p, q$  respectively, there exists an isometry taking  $(p, v) \rightarrow (q, w)$ .

Here, the sphere and plane are isotropic, but the torus is not. However, there is an even stronger condition than being isotropic. Rather than taking a single vector, you take an **orthonormal frame** at  $p$ . An orthonormal frame is just an ordered orthonormal (with respect to  $g$ ) basis for the tangent space at  $p$ .

**Definition 5.3.** A space  $X$  is **frame isotropic** if for all  $p, q \in X$ ,  $\{v_i\}, \{w_i\}$  orthonormal frames of  $X$  at  $p, q$  respectively, for all  $i$  there exists an isometry taking  $(p, v_i) \rightarrow (q, w_i)$ .

An example of a space that's isotropic but not frame isotropic is  $\mathbb{C}\mathbb{P}^n$  for  $n > 1$ . In  $\mathbb{C}^n$ , all directions are the same: given a vector, you can send it to  $(1, 0, 0, \dots)$  through a change of basis. Given an orthogonal pair of vectors, not all pairs of the same. For example, the pair  $(1, 0, \dots), (0, 1, \dots)$  is related in a different way than  $(1, 0, \dots)$  and  $(i, 0, \dots)$ , this is a complex geometric notion. What this means is that if we do something that really uses the complex structure, you can get something where the geometry tells the difference. This is what we do in  $\mathbb{C}\mathbb{P}^n$ : we start off with  $\mathbb{C}^{n+1} \setminus \{0\}$ , then we mod out by multiplication by  $\lambda \in \mathbb{C} \setminus \{0\}$ . Then pairs of vectors with certain relations will be preserved or not preserved by this action. That is, all points in  $\mathbb{C}\mathbb{P}^n$  look the same, but given a vector and  $i$  times that vector, it behaves differently than a vector and some other vector. This is a handwavy idea, but it illustrates the point. A recap:

$$\text{frame isotropic} \implies \text{isotropic} \implies \text{homogeneous}$$

We study three different frame isotropic spaces. One is  $S^n$ , the other will be  $\mathbb{R}^n$ , and the third will be  $\mathbb{H}^n$ .

## 5.2 The action of the group of isometries on a space

Let a group  $G$  act transitively on a space  $X$ , that is, for all  $x, y \in X$  there exists a  $g$  such that  $gx = y$ . Define  $H_x = \{h \in G \mid h \cdot x = x\}$ . This is the subgroup of  $G$  that fixes  $x$ , or the **stabilizer** of  $x$ . Let us show that

(1)  $H_x$  and  $H_y$  are conjugate.

(2)  $X \simeq G/H$ .

*Proof.* For (1), we want to show there exists a  $g \in G$  such that  $gH_xg^{-1} = H_y$ . Since the action is transitive, we have  $gx = y$  for some  $g \in G$ . Then for  $h_x \in H_x$  (where  $h_x x = x$ ),

$$gh_xg^{-1}(y) = gh_xx = gx = y.$$

So this fixes  $y$ , and is therefore an element of  $H_y$ . Similarly, consider  $g^{-1}h_yg \in g^{-1}H_yg$ , then

$$g^{-1}h_yg(x) = g^{-1}h_yy = g^{-1}y = x,$$

and therefore is an element of  $H_x$ . For (2), choose  $x \in X$ . Suppose we have a coset  $aH_x$  corresponding to  $ax$ , since  $a(H_x x) = ax$ . Since the action is transitive, we get all the points in the space.  $\square$

In general, for a homogeneous space, we say  $G = \{\text{isometries of } X\}$ . These isometries act transitively, and we can apply our result above. For nonhomogeneous spaces this set may be small and doing this may be useless, but within the category of homogeneous spaces, we always want to look at the group of isometries. We also have  $H_x = \{\text{isometries that sent } x \text{ to itself}\}$ , the **isotropy group at  $x$**  (which we proved is the same for all  $x$ ). This means we can just speak of the **isotropy group** of homogeneous spaces<sup>8</sup>, rather than the isotropy group of the north pole or Austin, TX on  $S^2$ . So when we study homogeneous spaces, we always try to write it as  $G/H$ .

When can we tell when groups are isotropic? An isotropy means that  $H$  acts transitively on unit tangent vectors at a point. This means that it's big enough to send every vector to another vector, we simply look at  $H$  applied to an arbitrary vector and ask whether it hits any other vector. Frame isotropy in this context means that it takes every frame to every other frame. The group that does this is  $O(n)$ , so if  $H = O(n)$  then we are frame isotropic.

**Example 5.2.** What are the symmetries of  $\mathbb{C}\mathbb{P}^n$ ? In this case,  $H = \tilde{U}(n)$ , and since  $U(n)$  sends every vector to every other vector it is indeed isotropic. But it doesn't send every frame to every other frame, so  $\tilde{U}(n) \neq O(2n)$ . However,  $U(1) = SO(2)$ , so  $\mathbb{C}\mathbb{P}^1$  is frame isotropic.

## 5.3 Nice spaces (Part I): The plane $\mathbb{R}^n$

What's the nicest space there is? Consider  $\mathbb{R}^n$  with the standard metric. The group of isometries  $G$  is generated by translations and rotations, where translations correspond to  $\mathbb{R}^n$  and rotations correspond to  $O(n)$ . There is a

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<sup>8</sup>This group is sometimes called the **little group**.

natural action of  $O(n)$  on  $\mathbb{R}^n$ , but the actual thing we get is  $\mathbb{R}^n \rtimes O(n)$ . The simplest way to describe this is by matrices; consider

$$H = \left( \begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array} \right), \quad A \in O(n), v \in \mathbb{R}^n.$$

Our composition rule is that

$$\left( \begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array} \right) \left( \begin{array}{c|c} B & w \\ \hline 0 & 1 \end{array} \right) = \left( \begin{array}{c|c} AB & Aw + v \\ \hline 0 & 1 \end{array} \right).$$

You compose the rotations, and the subgroup where  $A$  is the identity turns the operation  $Aw + v$  into simply adding vectors. If we take the quotient by  $SO(n)$ , we get the translations. There is a subgroup where  $A$  is the identity, if we take the quotient by that subgroup we wind up with  $O(n)$ . It isn't just the product with  $O(n)$  and  $\mathbb{R}^n$  because the law mixes the two. So we don't call it the product, we call it the *semidirect* product. On the  $O(n)$  factor it's just usual multiplication, while on the  $\mathbb{R}^n$  factor we have this funny business where  $O(n)$  interferes.

We want to figure out the isotropy group at a point, say the origin. This acts on a vector  $x$  by

$$\left( \begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array} \right) \left( \begin{array}{c} x \\ 1 \end{array} \right) = Ax + v.$$

The way to think about this is first rotating by  $A$ , then doing a translation by  $v$ . We want to see what happens when you act on the origin, and we want the isotropy group acting on the origin to give us back the origin. But it actually gives us back  $(v | 1)$ , so  $H$  is the set of rotations with no translational piece:

$$\left( \begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \stackrel{\text{(expected)}}{=} \left( \begin{array}{c} v \\ 1 \end{array} \right) \stackrel{\text{(actual)}}{=} \left( \begin{array}{c} v \\ 1 \end{array} \right) \implies H_0 = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right).$$

So  $H_0 = O(n)$ , the rotation group. Hooray! This shows that the little group of  $\mathbb{R}^n$  is the rotation group, which means  $\mathbb{R}^n$  is frame isotropic. Recall  $g_{ij} = \delta_{ij}$ , and  $g(v, w) = \sum v^i w^i$ .

## 5.4 Nice spaces (Part II): The sphere $S^n$

The book works with spheres of radius  $R$ , but let's work with the standard unit sphere. This is defined as a subspace of  $\mathbb{R}^{n+1}$ , where  $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ . The set of isometries of the sphere is just the rotations in  $(n+1)$ -space  $G = O(n+1)$ . Out of these, the rotations that fix the north pole just consist of the lower dimensional rotations, that is,  $H_{(0,0,\dots,1)} = O(n)$ .

If we wanted to restrict to things that preserve orientation, we would talk about *oriented* frames and whether a space is *oriented* frame isotropic, and ask whether  $H = SO(n)$ , taking the quotient  $SO(n+1)/SO(n)$ . We have shown above that

$$\begin{aligned} S^n &= O(n+1)/O(n) \\ &= SO(n+1)/SO(n). \end{aligned}$$

This isn't how we usually think of a sphere, but if we really want to get into the symmetry, the quotient by (orientation preserving) rotations is the right way to think about a sphere. We can give an explicit map  $O(n+1) \rightarrow S^n$ : an element of  $O(n+1)$  is an orthogonal matrix of the form  $(\vec{b}_1, \dots, \vec{b}_{n+1})$ , where the columns are orthonormal. So  $O(n+1)$  is the set of all orthonormal frames of  $n+1$  dimensions. The map simply sends  $(\vec{b}_1, \dots, \vec{b}_{n+1}) \rightarrow \vec{b}_{n+1}$ .

Right multiplying by an element of  $H_0 = O(n)$  (as in the equation in the last section) does exactly this; it pulls out the last element.

Now we want to get coordinates on  $S^n$  and see what they look like. We do this by stereographic projection, measuring the length of a ray starting at the north pole going through a point, up to where it intersects the equatorial plane. Working out the steps is the homework, but it turns out that

$$g_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{4}{(1+\|u\|^2)^2} & \text{if } i = j. \end{cases}$$

Notice at each point,  $g_{ij} = 4/(1+\|u\|^2)^2$  times the usual metric on  $\mathbb{R}^n$ . In other words, up to an overall scale, what happens at every point is just like what you would have on  $\mathbb{R}^n$ . If you take the metric on  $S^n$  and do a **conformal transformation** (rescale the metric at each point by a certain amount), we can turn  $S^n \setminus \{\text{point}\}$  into  $\mathbb{R}^n$ . So the sphere and  $\mathbb{R}^n$  are **conformal** to each other. This map doesn't preserve length, but preserves angles, which is the definition of conformal.

## 5.5 Nice spaces (Part III): Hyperbolic space $\mathbb{H}^n$

How about a space that curves the other way? We work in  $\mathbb{R}^{n+1}$  with a **pseudometric**, since it's not positive definite. Define a pseudo inner product by

$$\langle v, w \rangle = v^1 w^1 + v^2 w^2 + \cdots + v^n w^n - v^{n+1} w^{n+1}.$$

This is **Minkowski spacetime**, with  $n$  space coordinates and one time coordinate. We usually write a vector in  $\mathbb{R}^{n+1}$  as  $(x, t)$  for  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . So  $\langle (v, t), (w, s) \rangle = v \cdot w - ts$ . What would the unit sphere be? We actually have two interesting unit spheres now, since one is where you set  $\langle x, x \rangle = 1$ , and the other where you set  $\langle x, x \rangle = -1$ . Let's use  $\langle x, x \rangle = -1$ . This is no longer the familiar sphere you know, since it's defined by the equation

$$(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 - t^2 = -1, \quad \text{or}$$

$$t^2 = 1 + (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2.$$

If  $n = 2$ , then  $t^2 = 1 + (x^1)^2 + (x^2)^2$  defines the two sheeted hyperboloid. If we let  $\mathbb{H}^n$  just be the upper half of the plane, this is *sort of* like talking about a sphere, since we *sort of* have a metric (it's indefinite), and we only consider the upper connected component. The next question to ask is "what is the metric?" Our metric is as follows: given a point  $p$  on the upper half plane and two tangent vectors, take the indefinite inner product of  $\mathbb{R}^{n+1}$  and restrict it to the tangent space at a particular point. Our claim is that this doesn't give us a *pseudometric*, it gives us an *actual metric*.

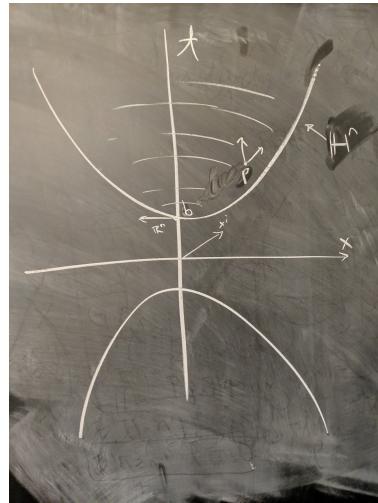


Figure 1: A model of hyperbolic space in  $\mathbb{R}^{n+1}$ .

For example, at the base  $b$  the tangent vectors have no  $t$  component, so the inner product is just the one on  $\mathbb{R}^n$ . The set of isometries are going to be the set of linear isometries of  $\mathbb{R}^{n+1}$  with this funny metric  $O(n, 1)$ , and it's not hard to cook up one that sends  $b \rightarrow p$ . So the metric on  $T_p \mathbb{H}^n$  behaves like the metric on  $T_b \mathbb{H}^n$ , because the map that sends  $b \rightarrow q$  preserves the  $(n, 1)$  inner product, and once you end up at the base  $b$  the time coordinate doesn't affect you at all.

So this defines a space where the points all look alike, since  $O(n, 1)$  sends every point to every other point. We need to explicitly construct an element of  $O(n, 1)$  that sends  $b \rightarrow q$ . Let us do this in two dimensions. Say  $p$  is of the form  $(p, t) = (p^1, 0, 0, \dots, t)$ . Consider

$$\left( \begin{array}{cccc|c} \cosh(\alpha) & & & & \sinh(\alpha) \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ \hline \sinh(\alpha) & & & & \cosh(\alpha) \end{array} \right).$$

The claim is that this is an element of  $O(n, 1)$ , sending  $(0, 0, \dots, 1)$  to  $(\sinh \alpha, 0, 0, \dots, \cosh \alpha)$ . Since  $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$ , there exists an  $\alpha$  such that for  $(p^1, 0, 0, \dots, t)$ ,  $p^1 = \sinh(\alpha)$  and  $t = \cosh(\alpha)$  (since  $t^2 - (x^1)^2 = 1$  by the definition of our pseudometric). More explicitly, this is the map

$$\left( \begin{array}{cccc|c} t & & & & x \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ \hline x & & & & t \end{array} \right), \text{ since } t^2 - x^2 = 1.$$

This is an explicit map in  $O(n, 1)$  sending  $(0, 0, \dots, 1) \rightarrow (x, 0, \dots, t)$ . In  $n$  dimensions, replace the component  $(x, 0, 0, \dots)$  with any vector of length  $n$ . This gives us our isometries, since if we have an isometry of the underlying vector space  $\mathbb{R}^{n+1}$ , this induces an isometry on the unit sphere. This unit sphere is actually a **Riemannian** metric since we have an isometry taking a point with a positive definite metric to another point. So for hyperbolic space,

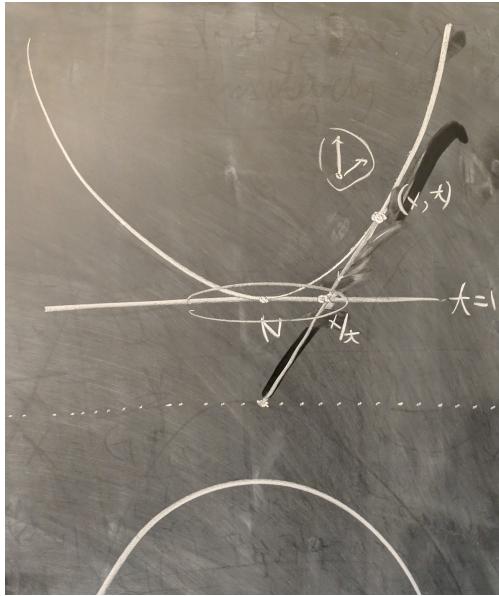
$$G = O(n, 1) = \left\{ A \in M(n+1) \mid A^T \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \right\}$$

If we replaced the  $-1$  factor with  $1$ , this would become the set of matrices  $A$  such that  $A^T A = I$ , which is just  $O(n+1)$ . Similarly,

$$H = O(n) = \left\{ \left( \begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array} \right) \mid B^T B = I \right\} < O(n+1, 1).$$

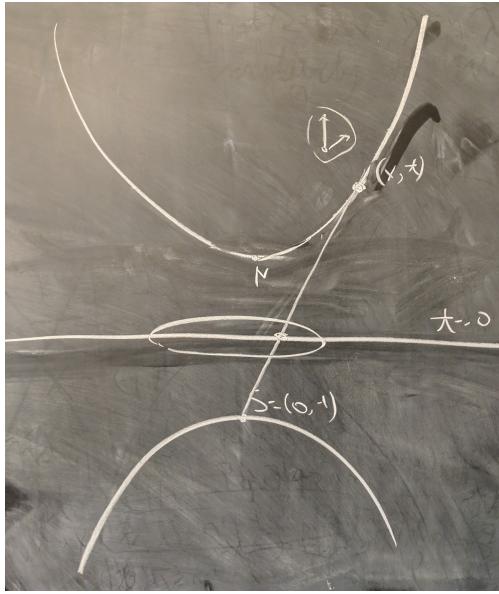
So hyperbolic space is frame isotropic! This is the abstract definition of hyperbolic space, but what does the metric look like? We need to get coordinates on  $\mathbb{H}^n$ , for which there will be several different models. There are three ways:

- (1) Consider the plane  $t = 1$ . For each point  $(x, t)$ , project down to the origin through that plane. This intersects  $t = 1$  at a point  $x/t$  contained in the unit disk, since  $t^2 - |x^2| = 1$ . So  $x/t < 1$  since  $t$  is bigger than  $x$ .



Parametrizing by points in the unit disk by projection gives us the **Beltrami-Klein model**.

- (2) The second way is by the analogue of stereographic projection. Rather than consider a line through the north pole, we consider a line through  $(x, t)$  and the *south pole*  $S = (0, -1)$ .



The point ends up in a unit disk here too, but a different one: in the previous model, we had the Beltrami-Klein disk, while this is called the **Poincaré disk model**. Both will give you metrics (but it turns out this metric will be conformal).

- (3) The final model will give us a metric on the **upper half space**, where we take  $\mathbb{R}^{n-1} \times [0, \infty)$  and map the disk onto it.

For all of these, first we work it out in two dimensions, then  $n$  dimensions. However, we actually understand  $n$  dimensional hyperbolic space abstractly: we have seen that  $\mathbb{H}^n = O(n, 1)/O(n)$  and is frame isotropic. What is this mysterious frame isotropic space that is not a sphere nor a plane? We will see next time.

Lecture 6 —

**February 4, 2021**

todo:hyperbolic spce

Lecture 7 —

**February 9, 2021**

## 7.1 Recap of hyperbolic space

Last time we talked about how to view a hyperbolic space a couple different ways, listed below. (For now we limit our attention to  $\mathbb{H}^2$ , but in  $n$  dimensions arguments work pretty much the same way.)

- (1) The hyperboloid model. Here we consider the upper half plane  $z^2 = x^2 + y^2 + 1$  (for  $z \geq 1$ ) sitting in  $\mathbb{R}^3$ , with the metric  $ds^2 = dx^2 + dy^2 - dz^2$ .
- (2) The Klein-Beltrami model. Consider the disk  $u^2 + v^2 < 1$ , with the metric  $d^2 = \text{something messy}$ .
- (3) The Poincare disk model. Once again consider  $u^2 + v^2 < 1$ , with the metric  $ds^2 = \frac{4(du^2 + dv^2)}{(1-u^2-v^2)^2}$ .
- (4) The upper half plane. Consider all pairs  $(x, y)$  for  $y > 0$ , with the metric  $d^2 = \frac{dx^2 + dy^2}{y^2}$ .

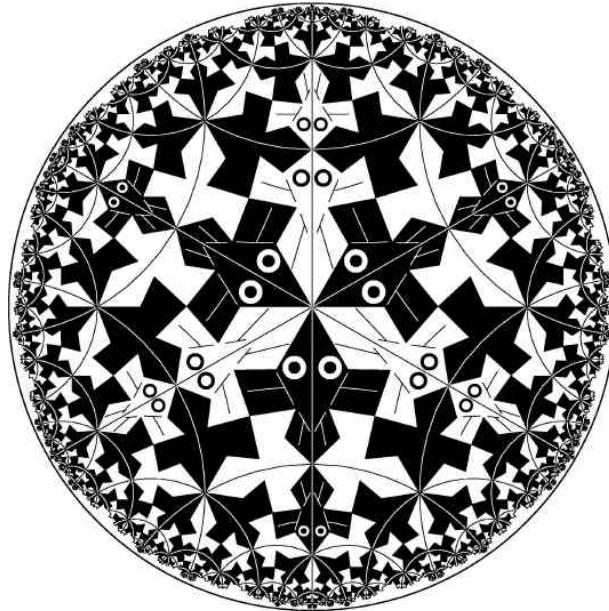


Figure 2: Obligatory Escher print.

This tells us a couple of things: all the little dudes are the same size. You can also track straight lines on the disk, the diameter lines definitely are but so are the arcs outlined in Section 7.1. It looks a little funny because this is a disk of infinite volume; the metric gets closer to zero as you approach the boundary, and if you try to integrate from the center out you'll see that it diverges.

## 7.2 The upper half plane

Let  $x, y$  for  $y > 0$ . It's easier to package them in a complex number  $z = x + iy$ . So our metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz d\bar{z}}{(\operatorname{Im} z)^2} = \frac{|dz|^2}{(\operatorname{Im} z)^2}.$$

We claim there is a natural group acting on this space, namely  $\operatorname{SL}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$ . This acts on  $\mathbb{C}$  by the rule  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$ . You can think of this action as  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$ , so when the matrix acts on  $z$  itself, it transforms the ratio of  $z : 1$  to  $az + b : cz + d$ . If you think about the linear action of matrices on  $\mathbb{C}^2$  and the induced action it has on quotients, this is really an action on  $\mathbb{CP}^1$ . Call this ratio  $\frac{az+b}{cz+d} = w$ , and this transformation is called a **Möbius transformation**. Does this really take the upper half plane to the upper half plane? To do this, let's see if  $\operatorname{Im} w = (w - \bar{w})^9$  is positive. We have

$$\begin{aligned} w - \bar{w} &= \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} \\ &= \frac{z - \bar{z}}{|cz+d|^2}. \end{aligned}$$

<sup>10</sup>So  $\operatorname{Im} w = \frac{\operatorname{Im} z}{|cz+d|^2} > 0$ , and this action is really a transformation of the upper half plane. Further more, this is a local isometry. Since  $w = \frac{az+b}{cz+d}$ ,  $dw = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} dz = \frac{dz}{(cz+d)^2}$ . Similarly,  $d\bar{w} = \frac{d\bar{z}}{(c\bar{z}+d)^2}$ , and  $dw d\bar{w} = \frac{dz d\bar{z}}{|cz+d|^4}$ . But the imaginary part transforms by  $\frac{1}{|cz+d|^2}$ , so

$$\frac{|dw|^2}{(\operatorname{Im} w)^2} = \frac{|dz|^2}{(\operatorname{Im} z)^2},$$

and our metric is preserved. We have just shown that we have a group acting on the upper half plane by isometries; the next question to ask is “is the action transitive?” Here are some particularly interesting elements of  $\operatorname{SL}(2, \mathbb{R})$ .

- $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ : this sends  $z \mapsto z + a$ , which translates  $z$  by  $a$  in the *real* direction.
- $\begin{pmatrix} \sqrt{b} & 0 \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix}$ : this sends  $z \mapsto bz$ , which scales  $z$  by  $b$ .

If you consider  $i$  as a “center” point for your model, we have  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{b} & 0 \\ 0 & 1/\sqrt{b} \end{pmatrix} \cdot i = a + bi$ , and composing the matrices gives  $\begin{pmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{pmatrix}$  as the map sending  $i \mapsto a + bi$ . Our action is transitive, since  $i$  goes to any point, and the inverse matrix sends any point to  $i$ . Then any point gets sent to any point by first sending it to  $i$ , then the other point.

The next question to ask is “what is the isotropy group of a point?” We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} i = i, \quad \frac{ai + b}{ci + d} = i \implies ai + b = di - c.$$

Then  $a = d$  and  $c = -b$ , so we are looking at matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ ; furthermore these matrices have determinant 1, so  $a^2 + b^2 = 1$ . This becomes all matrices of the form  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ , which is just  $\operatorname{SO}(2)$ . The point is applying one of these matrices to  $i$  rotates the plane by  $2\theta$ . Note that

$$i \mapsto \frac{\cos \theta i + \sin \theta}{-\sin \theta i + \cos \theta} = \frac{ie^{-i\theta}}{e^{-i\theta}}.$$

The derivative map, how much you rotate tangent vectors, is  $\frac{1}{(cz+d)^2} = \frac{1}{e^{-2i\theta}} = e^{2i\theta}$ . In particular, if  $\theta = \pi$ , then we rotate back to  $2\pi$ . This is not intrinsic to  $i$ , in general,  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = z$ . So  $\operatorname{Isom}(\mathbb{H}^2) = \operatorname{SL}(2, \mathbb{R}) / \pm I = \operatorname{PSL}(2, \mathbb{R})$ , and

$$\begin{aligned} \mathbb{H} &= \operatorname{SL}(2, \mathbb{R}) / \operatorname{SO}(2) \\ &= \operatorname{PSL}(2, \mathbb{R}) / \operatorname{PSO}(2). \end{aligned}$$

<sup>9</sup>It's actually  $\frac{w - \bar{w}}{2i}$ , but we're just trying to check whether it's positive or not.

<sup>10</sup>Some algebra was skipped.

### 7.3 Geodesics

Let  $(M, g)$  be a Riemannian manifold, and  $p, q \in M$ . We want to define the distance from  $p$  to  $q$ . The first thing we do is define distance along a path: think of a path  $\gamma(t)$  such that  $\gamma(t_0) = p$  and  $\gamma(t_f) = q$ . Then we define the **length** of  $\gamma$  by

$$L(\gamma) = \int_{t_0}^{t_f} \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt.$$

This doesn't depend on parametrization, if you reparametrize you get the same integral. Define the **distance** from  $p$  to  $q$  as

$$d(p, q) = \inf_{\gamma} L(\gamma),$$

where  $\gamma$  is a path from  $p$  to  $q$ . Unfortunately, a **geodesic** has two meanings, and we use both. One is that a geodesic is a locally length minimizing path. The second definition of the geodesic is something that minimizes a different functional called the **energy** of the path. Things that minimize energy are length minimizing paths parametrized at constant speed. So for every geodesic, there's a *best* parametrization where you move along at constant speed, and we say "geodesic" we speak of that parametrization. For now we're just talking about geometric paths, and want something that minimizes length.

The geodesic joining two points is not always unique. Consider  $\mathbb{R}^2 \setminus \{0\}$ , and the line connecting  $(-1, 0)$  and  $(1, 0)$ . There is no path of length 2 connecting them, since it would have to go through the origin (not in the space). You can find a path that takes a small detour, as small as you want, so the  $\inf(\text{all paths})$  still equals 2. But there is no geodesic from  $(-1, 0)$  to  $(1, 0)$ . So geodesics don't always exist globally<sup>11</sup>: they do always exist locally however, which we will soon see.

### 7.4 Geodesics in nice spaces

Our three nice spaces are the plane, the sphere, and hyperbolic space.

- (1) Consider  $p, q \in \mathbb{R}^2$  with  $\gamma(0) = p$  and  $\gamma(1) = q$ . Can you show the shortest path is a straight line? (This was homework for diff geo.) Assume  $p = (0, 0)$  and  $q = (1, 0)$  for simplicity (since length is invariant under isometries, and the argument works for arbitrary lengths). If our path is given by  $y = \gamma(x)$ , then

$$\begin{aligned} L(\gamma) &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &\geq \int_1^0 \sqrt{1} dx = 1. \end{aligned}$$

We get equality iff  $y$  is constant, since wiggling in the  $y$  direction only makes the  $(dy/dx)^2$  factor bigger, increasing the length. (Wow this was a lot simpler than the diff geo proof.)

- (2) Now we move to  $S^2$ . If  $(\varphi, \theta)$  parametrize the sphere, then  $ds^2 = d\varphi^2 + \sin^2 \varphi (d\theta)^2$ . (Figure out which coordinate corresponds to the azimuthal or whatever angle from that expression.) What's the shortest path from the north pole to a point  $(\varphi, \theta)$ ? We have

$$\begin{aligned} &\int \sqrt{\left(\frac{d\varphi}{dt}\right)^2 + \sin^2 \varphi \left(\frac{d\theta}{dt}\right)^2} dt \\ &\geq \int \left| \frac{d\varphi}{dt} \right| dt \geq \varphi_{\text{final}}. \end{aligned}$$

This is the same argument as before: any wiggles in the  $\theta$  direction only contribute to  $(d\theta/dt)^2$ , increasing the length. You also always want  $d\varphi/dt = |d\varphi/dt|$ , i.e. no doubling back. So the length minimizing path is when  $\theta$  is constant and  $\varphi$  is monotonic, i.e. you're going straight down the longitudinal line. This continues

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<sup>11</sup>They always exist for *compact* manifolds.

all the way down to the south pole. So we have just shown that a great circle is a geodesic. To parametrize this, you can send  $t = \cos \theta(0, 0, 1) + \sin \theta \vec{v}$ , where  $\vec{v}$  lives on a great circle.

Suppose we start at some other point  $p$  with tangent vector  $v$ , where  $p \in \mathbb{R}^3$  and  $p \perp v \in \mathbb{R}^3$ . Can you find a geodesic that goes through  $p$  along  $v$ ? Geodesics are invariant under isometry, so send  $p$  to the north pole. Now find a geodesic/great circle going through  $v$ , and rotate back. We have shown that  $\gamma(t) = \cos(t)p + \sin(t)v$  is a geodesic.

Spherical triangles have area of the sum of the angles minus  $\pi$ . In a plane this sum minus  $\pi$  is zero, but in the sphere it's not. (A neat proof is in the diff geo textbook/my diff geo notes.)

**(Digression!).** We never discussed what area means for a Riemannian manifold. The idea of area is that you chop things up into little pieces, and find the area of each piece. This is a parallelogram spanned by two vectors  $v, w$ . This is going to be some sort of determinant. We want

$$\begin{aligned} |v||w|\sin\theta &= \sqrt{|v|^2|w|^2 - |v|^2|w|^2\cos^2\theta} \\ &= \sqrt{(v \cdot v)(w \cdot w) - (v \cdot w)^2} \\ &= \sqrt{g_{11}g_{22} - g_{12}^2} \\ &= \sqrt{\det g}. \end{aligned}$$

We gave an argument for two dimensions, but the same thing works for arbitrary dimensions. In general

$$\int \sqrt{\det g} d^n x = \text{volume}.$$

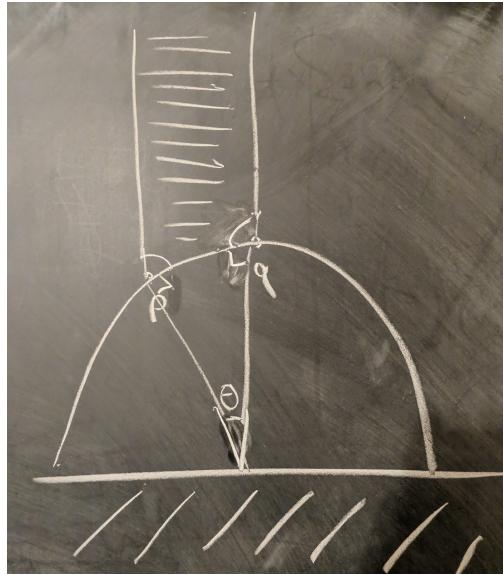
In the case of  $S^2$ ,  $g_{11} = 1, g_{12} = g_{21} = 0, g_{22} = \sin^2 \varphi$ . So  $dA = \sin \varphi d\varphi d\theta$ , and then simply integrate. For stereographic coordinates,  $g_{11} = g_{22} = \frac{4}{(1+u^2+v^2)^2}, g_{12} = 0$ , so  $\sqrt{\det} = \frac{4}{(1+u^2+v^2)}$ . So  $dA = (4du dv)/(1+u^2+v^2)^2$ . In the Poincare disk, just replace  $u^2, v^2$  with minus signs. In the upper half plane,  $g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = 0$ ,  $\det = \frac{1}{y^r}, \sqrt{\det} = \frac{1}{y^2}, dA = \frac{1}{y^2} dx dy$ .

- (3) Now we turn our attention to hyperbolic space. Suppose we have two points  $i, iy_1$ . Then the geodesic is also a straight line, since

$$d = \int \frac{\sqrt{dx^2 + dy^2}}{y} \geq \int_1^{y_1} \left| \frac{dy}{y} \right| = |\ln|y_1||.$$

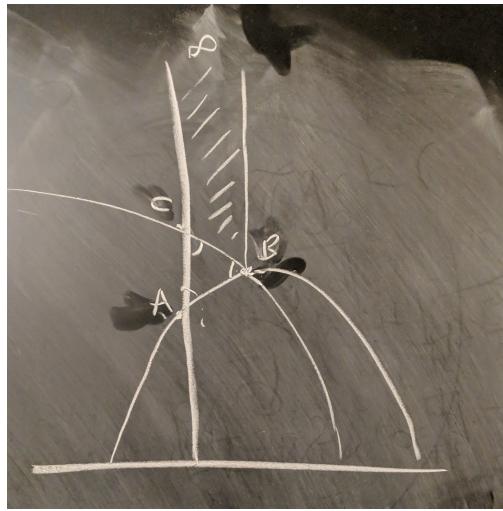
This diverges as you approach the outer edge, and also as you approach the center. Now for arbitrary  $p, q$ , send  $p \rightarrow i$  and rotate until  $q$  winds up above  $i$  by our aforementioned transformations. What happens when you send a line through an isometry? The translation  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  shifts a vertical line to the right by a factor of  $a$ , while dilation  $\begin{pmatrix} \sqrt{b} & 0 \\ 0 & 1/(\sqrt{b}) \end{pmatrix}$  does nothing to a vertical line. Now for the Möbius transformation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $z \mapsto \frac{az+b}{cz+d}$ , as  $z$  gets large, this is approximately  $a/c$ . As  $z$  gets small, this is approximately  $b/d$ . So the image of a vertical line is going to be something that starts on the real axis at  $b/d$  and goes to  $a/d$  (you know  $b/d$  and  $a/c$  are distinct since  $ad - bc = 1$ ). The remarkable thing is that this turns out to be a circle centered at the midpoint  $(b/d + a/c)/2$ , with radius  $(a/c - b/d)/2 = (ad - bc)/2cd = 1/2cd$ . So our geodesics are either vertical lines or semicircles.

Consider an **ideal triangle**, which is the limit of a triangle as one of the vertices goes off to infinity.



The sum of the angles at  $p$  and  $q$  will be less than  $\pi$ . How much less? The claim is that it will be less than  $\pi$  by exactly  $\theta$ , since summing up  $\angle p, \angle q$ , the right angles, and  $\theta$  gives exactly  $2\pi$ . The right angles plus  $\theta$  is  $\pi + \theta$ , so  $\angle p + \angle q = \pi - \theta$ .

How does this relate to areas? Recall that we want to integrate  $\int \frac{dx dy}{y^2} = \int \frac{dx}{y}$ , since  $\int_{y_0}^{\infty} \frac{dy}{y^2} = \frac{1}{y_0}$ . On a circle centered at the  $x$ -axis,  $\int \frac{dx}{y} = \int d\theta = \theta$ . So in hyperbolic space, the angle deficit is the area. One last thing: usually you don't care about ideal triangles, you care about real triangles. Send it through an isometry takes one vertex to  $i$ , the other above  $i$ , and the third somewhere else.



The claim is that the sum of the angles of  $\Delta ABC$  is the sum of the angles of  $\Delta AB\infty$  (ideal triangle) minus the sum of the angles of  $\Delta BC\infty$  plus  $\pi$ . So the deficit for the real triangle is the deficit of the big ideal triangle minus the deficit of the little ideal triangle. Skipping some steps, we have shown that the sum of the angles in hyperbolic space is  $\pi - \text{Area } \Delta$ .

**Corollary 7.1.** *The area of a triangle is less than  $\pi$ .*

Hyperbolic geometry is wack. See you next time.

Lecture 8

## February 11, 2021

### 8.1 More on geodesics in hyperbolic space

Last time we were talking about geodesics in hyperbolic space. We have seen that vertical lines are geodesics, and an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acting on the line sends it to an arc starting at  $a/c$  and ending at  $b/d$  WLOG. Then shifting by isometries, we can assume that  $a/c = -b/d$ , which implies  $c/d = -a/b$ . So

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : iy \rightarrow \frac{b + aiy}{d + ciy} = \frac{b(1 + \frac{a}{b}iy)}{d(1 + \frac{c}{d}iy)}.$$

Since the expressions in the numerator and denominator are conjugate, the image of  $iy$  is a point of constant magnitude  $b/d$ , which implies that geodesics are indeed parts of a circle centered at the  $x$ -axis. Now that we know what geodesics are, this tells us a lot of things. Consider a point  $(a, 0)$ , with our geodesics a circle of radius  $R$ . Parametrize by  $\theta$  where  $x = a + R \cos \theta$ ,  $y = R \sin \theta$ ,  $dx = -R \sin \theta d\theta$ ,  $dy = R \cos \theta d\theta$ , with metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{R^2 d\theta^2}{R^2 \sin^2 \theta} = \frac{d\theta^2}{\sin^2 \theta} \implies ds = \pm \frac{d\theta}{\sin \theta}.$$

The  $\pm$  sign stems from the fact that when calculating distance we can go around a circle either clockwise or counterclockwise. Integrating gives us  $S = \pm \ln |\csc \theta + \cot \theta| = \pm \ln |R/y + (x-a)/y| = \pm \ln |(R+x-a)/y|$ . This diverges logarithmically as  $x \rightarrow a+R$ ,  $y \rightarrow 0$ , which is expecting since this is an isometry of the vertical line. As  $x \rightarrow a-R$  and  $y \rightarrow 0$ , both factors approach zero so it's a little trickier. However you can rewrite this as  $\ln |y/(R-x-a)|$ , and so the denominator approaches zero as expected and this also diverges logarithmically. So because of our metric, this arc is infinitely long on both sides.

Given two points  $p, q \in \mathbb{H}^2$ , we want to find the semicircle going through  $p$  and  $q$ . To do this, draw a line through  $p$  and  $q$ , then the perpendicular bisector will hit the  $x$  axis at a point an equal distance  $R$  from both  $p$  and  $q$ . This forms a triangle, and now you can just compare values of  $\csc \theta$ ,  $\cot \theta$ , etc. So we have just figured out the *entire* metric in hyperbolic space, since given any two points we can calculate the distance between them.

### 8.2 Triangles in hyperbolic space

Recall from last time that the sum of angles  $\alpha, \beta$  of an ideal triangle is  $\pi - \theta$ , and  $\text{Area} = \theta$ . How do we know this? We have

$$\iint_A \frac{dx dy}{y^2} = \int dx \int_{y(x)}^{\infty} \frac{1}{\tilde{y}^2} d\tilde{y} = \int \frac{dx}{y} = \int \frac{dx}{y} = \int \frac{R \sin \theta d\theta}{R \sin \theta} = \int -d\theta = -\theta.$$

The negative value stems from the fact that we integrate from left to right, while  $\theta$  is measured from right to left. So  $\alpha + \beta = \pi - \text{Area}$ , a surprisingly nice result when compared to our weird expression for length. To find a similar expression for *actual* triangles, say  $\alpha, \beta, \gamma$  are the angles of a triangle  $T$  where  $\beta, \gamma$  lie on a semicircle. Then the complement of  $\alpha$  is  $\pi - \alpha$ , and denote the angle of the ideal triangle corresponding to  $\beta$  by  $\delta$ .<sup>12</sup> Let  $B$  denote the ideal triangle formed by  $\pi - \alpha$  and  $\delta$ . We have

$$\begin{aligned} \pi - \alpha + \delta &= \pi - (\text{Area of } B) \\ \delta + \beta + \gamma &= \pi - (\text{Area of } B) - (\text{Area of } T) \\ \alpha + \beta + \gamma - \pi &= -\text{Area}(T) \\ \pi - (\alpha + \beta + \gamma) &= \text{Area}(T). \end{aligned}$$

Wow! We figured out the area of an arbitrary triangle just by using symmetry, no messing with Christoffel symbols or curvature or anything.

<sup>12</sup>This is for triangles with two vertices  $\alpha, \gamma$  positioned directly on top of each other, but you can write any triangle this way up to isometry.

**Corollary 8.1.** *The area of a triangle is always less than  $\pi$ .*

To visualize this result, imagine a triangle with two base points infinitely close to the  $x$ -axis, and a third point all the way out in the  $y$  direction (say  $y = 3 \times 10^{45}$ ). Then by our metric  $ds^2 = (dx^2 + dy^2)/y^2$ , as  $y$  becomes very large the two “parallel” vertical geodesics stemming from our two basepoints get very close. Likewise, the parallel lines get infinitely close to the endpoints of the semicircle connecting the basepoints. So just imagine a triangle in the Poincare disk model with infinitely tight pinched edges.

This is in contrast with the sphere, where a triangle has bloated edges. There, a triangle can even have three right angles. This completes our tour of hyperbolic space, and chapter three of the book! Now we move on to talking about geodesics.<sup>13</sup>

### 8.3 Geodesics as energy minimizing curves

For a Riemannian manifold  $(M, g)$ , let  $\gamma$  be a path taking a point  $p \in M$  to a point  $q \in M$ , where  $\gamma(0) = p$  and  $\gamma(t) = q$ . Recall that  $L(\gamma) = \int \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt$ . The inner product of velocity with itself is velocity squared, taking the square root gives speed and integrating WRT time gives length. Our previous notion of geodesics was thinking about them as length minimizing curves. Now consider the **energy** of a path, where

$$E(\gamma) = \int g(\dot{\gamma}, \dot{\gamma}) dt.$$

We will study curves that minimize energy and see what they look like. In one dimension, consider  $M = (\mathbb{R}, dx^2)$ . We want a path from 0 to  $L$  in time  $T$ , that is,  $\gamma(0) = 0$ ,  $\gamma(T) = L$ , and for this path to minimize  $\int_0^T (d\gamma/dt)^2 dt = \int_0^T (dx/dt)^2 dt$ . We expect this to be the path with constant velocity, since any other path gets traversed faster, ergo more energy.

**The Cauchy-Schwarz inequality for  $L_2$  space.** Recall that Cauchy-Schwarz tells us that  $\|\langle u, v \rangle\| \leq \langle u, u \rangle \cdot \langle v, v \rangle$ . Applying this to the norm on the space of square-integrable functions (where  $\langle f, g \rangle = \int_A \overline{f(x)} g(x) dx$ ) gives

$$\left| \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx \right|^2 \leq \int_{\mathbb{R}^n} |f(x)|^2 dx \int_{\mathbb{R}^n} |g(x)|^2 dx.$$

Now that we know this very useful inequality, note that

$$\begin{aligned} TE(\gamma) &= \int_0^T 1^2 dt \int_0^T \left( \frac{dx}{dt} \right)^2 dt \\ &\geq \left( \int_0^T \left( 1 \cdot \frac{dx}{dt} dt \right)^2 \right) = L^2. \\ E(\gamma) &\geq \frac{L^2}{T}. \end{aligned}$$

We have equality precisely when  $(dx/dt)$  is a constant. A more elementary argument is that

$$\begin{aligned} &\int \left( \frac{dx}{dt} \right)^2 dt - \int \left( \frac{dx}{dt} - \frac{L}{T} + \frac{L}{T} \right)^2 dt \\ &= \int \left( \frac{dx}{dt} - \frac{L}{T} \right)^2 + \underbrace{\frac{2L}{T} \left( \frac{dx}{dt} - \frac{L}{T} \right)}_{=0} + \frac{L^2}{T^2} dt \\ &= \int \left( \frac{dx}{dt} - \frac{L^2}{T} \right) dt + \frac{L^2}{T}. \end{aligned}$$

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<sup>13</sup>The subtitles for the recorded lecture always say “genetics” instead of geodesics, and I find it hilarious that we so often abruptly switch from talking about math to talking about biology.

You minimize this expression by setting the left integral to zero, or setting  $(dx/dt) = L/T$  everywhere. This is essentially the same proof, since proving Cauchy-Schwartz uses a complete the square argument like this. Let us also solve this by using an overkill technique, called calculus of variations. This generalizes and is quite useful, which is why we're demonstrating how to prove this three times over.

## 8.4 Calculus of variations

Given a function  $f(x)$  on  $\mathbb{R}^n$ , how do you find the minima? We compute  $\nabla f$  and set it equal to zero. This may also find maxima and saddle points, but if we're at a minima, it's locally constant. Another way to say this is that  $f(\vec{x} + \delta\vec{x}) = f(\vec{x}) + L(\delta\vec{x}) + O(\delta\vec{x})$ . We want  $L(\delta\vec{x}) = 0$ , otherwise we would have a nontrivial change in  $f$  to the first order. This linear function  $L$  is  $\nabla \cdot f$ , or  $df$  applied to the change in  $x$ .

Back to our original problem. We have  $x(t)$ , where  $x(0) = 0$  and  $x(T) = L$ . We change  $x(t)$  to  $x(t) + \delta x(t)$ , and compute  $E(x + \delta x) - E(x) = L(\delta x) + O(\delta x^2)$ . We throw away the higher order terms, and now we want  $L(\delta x) = 0$ . So

$$\begin{aligned} E(x + \delta x) &= \int \frac{d(x + \delta x)^2}{dt} \\ &= \underbrace{\int \left( \frac{dx}{dt} \right)^2 dt}_{=E(x)} + 2 \frac{dx}{dt} \frac{d(\delta x)}{dt} + \underbrace{\left( \frac{d\delta x}{dt} \right)^2 dt}_{\text{second order, ignore}}. \end{aligned}$$

Then

$$\begin{aligned} \delta E &= \int 2 \frac{dx}{dt} \frac{d(\delta x)}{dt} dt \\ &= \underbrace{2 \frac{dx}{dt} \delta(x) \Big|_0^T}_{=0, \delta x(0)=0, \delta(T)=0} - \int 2 \frac{d^2 x}{dt^2} \delta x(t) dt, \\ \delta E &= - \int 2 \ddot{x} \delta x dt, \quad \frac{\delta E}{\delta x(t)} = -2 \ddot{x}(t). \end{aligned}$$

When will this linear function of  $\delta x$  be zero? This is true only if  $\ddot{x} = 0$ . So our solution must have  $\ddot{x}(t) = 0$ ,  $\dot{x}(t) = \text{constant}$ . Since we're going from position 0 to position  $L$  over time  $T$ , we must have  $\dot{x} = L/T$ .

For an arbitrary manifold, consider a path  $\gamma$  from  $p$  to  $q$ . How would you minimize  $E(\gamma)$ ? Geometrically we want to take the shortest path possible: once you go along a path with total length, we basically arrive at the one dimensional problem, where we travel a distance  $L$  along time  $T$ . We do this by going at constant speed  $L/T$ . So for an energy minimizing geodesic, we want to:

- (1) Minimize length.
- (2) Go at constant speed, given by  $\sqrt{g(\dot{\gamma}, \dot{\gamma})}$ .

By minimizing energy, we're killing two birds with one stone: we get minimal length for free, while also achieving constant speed. Functionally, minimizing energy is easier than minimizing length because there's no square root involved.

## 8.5 Motivating connections

In  $\mathbb{R}^n$ , you minimize length by requiring  $\ddot{x} = 0$  and  $\dot{x} = \text{constant}$ . In  $S^2 \subseteq \mathbb{R}^3$ , we have already determined geodesics to be great circles, so we go around the equator (by sending any point to a pole) at constant speed. Our acceleration points towards the center of the sphere, so  $\ddot{x} \perp T_p S^2$ . If you're driving on the sphere, you

never turn (acceleration points toward a different  $T_p S^2$ ) and never slow down or speed up (acceleration in the forwards/backwards direction).

In hyperbolic space, what is  $\ddot{x}$ ? This is a trick question. You know what it means to take a second derivative with respect to coordinates. When we say  $\ddot{x} = 0$  in  $\mathbb{R}^n$ , we really mean  $\ddot{x}_i = 0$ . Using different coordinates (such as polar), it would not be true that  $\dot{r}, \theta = 0$ . This statement that  $\ddot{x} = 0$  is coordinate dependent, and in Euclidian space we have these beautiful coordinates to play with. For  $S^2$ , the acceleration refers to taking the second derivative with respect to coordinates in  $\mathbb{R}^3$ .

We don't have these coordinates in hyperbolic space: we have models, but those are all different! To answer this, we'll need to develop a coordinate free way of taking coordinates, to answer the question "what does  $\frac{d}{dt}$  (vector) even mean?" We know  $\frac{d}{dt}$  (position) is just a tangent vector. But  $\ddot{x}$  is the derivative of a *tangent vector*, and we take this derivative at different points. Given a vector field on a manifold, what is its derivative? This is what **connections** are for, along with *covariant derivatives* and *parallel transport*. Given a nice connection, we can ask what the second derivative is, then set the acceleration to zero, and nice things happen. But we need to develop this machinery first.

Lecture 9

## February 23, 2021

### 9.1 Review

Let's see what we did this semester.

- (1) We talked about tensors and metrics.
- (2) Then we talked about spaces with symmetry in relation to transitive group actions ( $X = G/H$ ).
- (3) We talked about important examples like  $S^n$  and  $\mathbb{H}^n$ .
- (4) For geodesics, we talked about length minimizing and energy minimizing stuff (no acceleration).

### 9.2 Spaces with symmetry

A space is **homogeneous** if every point looks the same. Let  $X$  be a space, and  $G$  be the set of isometries of  $X$ . Then  $G$  acts transitively on  $X$ . Consider two points  $p, q$  on the torus. Then there exists an isometry that translates  $p$  to  $q$ , so the torus is homogeneous.

A space is **isotropic** if every unit tangent vector looks the same. That is, is there an isometry that takes one point and tangent vector to another? For the torus, the answer is no. An example:  $\mathbb{C}\mathbb{P}^n$  for  $n > 1$ . (This list is non exhaustive).

A space is **frame isotropic** if every (oriented) orthonormal basis looks the same. Some examples include  $S^n$ ,  $\mathbb{R}^n$ , and  $\mathbb{H}^n$ . (This list may be exhaustive).

### 9.3 Geodesics

The more naive notion of geodesics comes from minimizing length, while a more analytic version comes from minimizing energy. Say we have a curve  $\gamma$  from  $p$  to  $q$ , where  $\gamma(0) = p$ ,  $\gamma(T) = q$ . For  $L = \int ds$ ,  $\dot{\gamma} \equiv \frac{d\gamma(t)}{dt}$ ,  $ds^2 = g(\dot{\gamma}, \dot{\gamma}) dt^2$ ,  $ds = \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt$ . Then

$$L = \int_0^T \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt.$$

Say we have a curve from the origin to  $(t, 0)$ , where the length is given by

$$\int_0^T \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} dt \geq \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2} dt = \int_0^T \left| \frac{dx}{dt} \right| dt \geq L?$$

The way for this to minimize length is to have all the inequalities be equalities. For example, say we have two points on the sphere. Since the sphere is isotropic, rotate to set one of the points as the north pole, so a geodesic is a great circle that goes through these two points.

Using the upper half plane model, using the same idea we have a symmetry that takes one point to another. So vertical lines minimize length.

We can also talk about minimizing energy, given by  $\int_0^T g(\dot{\gamma}, \dot{\gamma}) dt$ , with  $\gamma(0) = p, \gamma(T) = q$ . The minimum energy is  $L^2/T$ , where  $L$  is the minimum length. If you go along a geodesic you get this value, while if you go along another curve you get a higher energy. To see this, note that

$$\begin{aligned} E &= \int \left( \frac{ds}{dt} \right)^2 dt \\ &= \int \left( \frac{ds}{dt} - \frac{L}{T} + \frac{L}{T} \right)^2 dt \\ &= \int \left( \frac{ds}{dt} - \frac{L}{T} \right)^2 + \left( \frac{L}{T} \right)^2 + 2 \frac{L}{T} \left( \frac{ds}{dt} - \frac{L}{T} \right) dt \\ &= \int_0^T \left( \frac{ds}{dt} - \frac{L}{T} \right)^2 + \frac{L^2}{T^2} dt \\ &\geq \frac{L^2}{T}. \end{aligned}$$

In Euclidian space, going along a straight line with constant speed means zero acceleration. On a sphere, we have  $\ddot{X} \perp S^n$ , using the fact that  $S^n \subseteq \mathbb{R}^{n+1}$ . In hyperbolic space, we don't even know what acceleration is. We know how to take the second derivative of a coordinate, but we don't know how to take the derivative of a velocity vector. We need connections to make sense of this, which we will talk about on Thursday since the university has decided we cannot talk about new things.

## 9.4 Tensors

A  $(k, \ell)$ -tensor is a multi-linear map that takes  $k$  vectors and  $\ell$  covectors and gives a number. A covector is a  $(1, 0)$ -tensor (circular definition!), or a linear map on a vector space  $V$  with basis  $\{e_1, \dots, e_n\}$ . Then  $V^* = \text{Hom}(V, \mathbb{R})$  with basis  $\{\phi^1, \dots, \phi^n\}$ . The most important example is the metric tensor

$$\begin{aligned} g(v, w) &= g(v^i e_i, w^j e_j) \\ &= v^i w^j g(e_i, e_j) \\ &= v^i w^j g_{ij}. \end{aligned}$$

In general, for a tensor  $T(v, w, \dots, \alpha, \beta, \dots)$ , we can always expand this to

$$\begin{aligned} T(v^i e_i, w^j e_j, \dots, \alpha_k \phi^k, \beta_\ell \phi^\ell, \dots) \\ &= v^i w^j \dots \alpha_k \beta_\ell T(e_i, e_j, \dots, \phi^k, \phi^\ell, \dots) \\ &= (T_{ij\dots}^{kl\dots} v^i w^j \dots \alpha_k \beta_\ell \dots). \end{aligned}$$

Under a change of basis, we have a new basis  $\{\tilde{e}_i\}$ , where  $\tilde{e}_i = A_i^j e_j$ , and  $B_i^j A_j^k = \delta_i^k$ ,  $\tilde{\phi}^i = B_i^j \phi^j$ . So

$$\tilde{T}_{ij\dots}^{kl\dots} = A_i^{i'} A_j^{j'} \dots B_k^k B_\ell^\ell \dots T_{i' j' \dots}^{k' l' \dots}.$$

Everything we've talked about is a tensor at a point, but the important things are really tensor fields. For a manifold, how are we going to get a basis for the tangent space to a point  $p$ ? We have two conventions:

- (1) Say we have coordinates  $x^1, \dots, x^n$ . Then we have a basis  $\partial_1, \dots, \partial_n$ .
- (2) We can also pick an orthonormal basis  $E_1, \dots, E_n$ .

Most of the time we use convention (1), and everything we've done so far. Say we have coordinates  $y^1, \dots, y^n$  with basis  $\partial/\partial y^1, \dots, \partial/\partial y^n$ . Then

$$\frac{\partial}{\partial y^i} = \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^j}$$

by the chain rule, which gives a change of basis matrix. So  $A_j^i = \frac{\partial x^i}{\partial y^j}$ , and  $B_j^i = \frac{\partial y^i}{\partial x^j}$ . Once we have these change of basis matrices, we know how to convert from one set of coordinates to another.

## 9.5 Examples

Say we have the plane  $\mathbb{R}^2$ . Some people like the coordinates  $(x, y)$ , and others like polar coordinates  $(r, \theta)$ . So a basis for the tangent space could be given by  $\partial_x, \partial_y$  or  $\partial_r, \partial_\theta$ . To compute  $A_j^i$ , we want to find  $\partial(x, y)/\partial(r, \theta)$ . So for  $x = r \cos \theta, y = r \sin \theta$ , we have

$$A_j^i = \begin{bmatrix} A_1^1 = \frac{\partial x}{\partial r} & A_2^1 = \frac{\partial x}{\partial \theta} \\ A_1^2 = \frac{\partial y}{\partial r} & A_2^2 = \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta = \frac{x}{r} & -r \sin \theta = -y \\ \sin \theta = \frac{y}{r} & r \cos \theta = x \end{bmatrix}.$$

To find  $B_i^j$ , we have  $r = \sqrt{x^2 + y^2}$ , so  $\partial_x r = \frac{x}{x^2 + y^2}, \partial_y r = \frac{y}{x^2 + y^2}$ . For  $\theta = \arctan^{-1}(y/x)$ , we have

$$d\theta = \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) =$$

## 9.6 Important Examples of Spaces with Symmetry

What are the isometries of  $S^n$ ? We want the orientation preserving rotations starting at the origin, given by  $\text{SO}(n+1)$ . Within these, which isometries preserve the north pole? This is given by

$$H = \left( \begin{array}{ccc|c} & & & \vdots \\ & \text{SO}(n) & & 0 \\ & & & \vdots \\ \hline \cdots & 0 & \cdots & 1 \end{array} \right) = \text{SO}(n).$$

Then  $S^n = \text{SO}(n+1)/\text{SO}(n)$ . At a point  $x$ , we have tangent vectors  $v^1, v^2, \dots, v^n$ , which are all isotropic (?). So  $S^n$  is frame isotropic. With Lie stuff, we have

$$T_{G/H} = \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{h}^\perp.$$

For hyperbolic space, this really is the same thing. If we think of hyperbolic space as a sphere with the funny metric  $ds^2 = dx^2 - dt^2$  (abuse of notation,  $dx$  is a vector while  $dt$  is a number) sitting in  $\mathbb{R}^{n,1}$ , what is the group of transformations preserving this hyperplane? This is  $\text{SO}(n, 1)^+$ , with our subgroup  $H = \text{SO}(n)$  (similar to  $S^n$ ).

Note that

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2), \quad \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \in \text{SO}(1, 1)^+, \quad \text{SO}(2), \text{SO}(1, 1)^+ \subseteq \text{SL}(1, 1?).$$

Neither of these elements are in the other group (didn't catch why?).