Pre-homework for M382C (Algebraic Topology I)
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NOT DUE. If you can solve them all then you are very well-prepared. If you can only solve a few, even after looking up unknown words, then please come talk to me about your preparation for the course.

Following Hatcher, an n-dimensional *manifold* means a Hausdorff topological space, each of whose points has an open neighborhood that is homeomorphic to $\mathbb{R}^n$. (Many other sources include the technical hypothesis of *paracompactness*, which is difficult to motivate at this beginning of the course.)

Problem 1. The product of finitely many manifolds is a manifold.

Problem 2. A manifold is connected if and only if it is path-connected.

Problem 3. Suppose a finite group G acts on a manifold M. Suppose the action is *free*, meaning that only the identity element has any fixed points. Then the orbit space M/G is also a manifold. ("Lying in the same G-orbit" is an equivalence relation on M. M/G means the set of equivalence classes. The topology on M induces one on M/G, which is the one you must work with.)

Problem 4. If freeness is dropped in the previous problem, then M/G may or may not be manifold. (Hint: examples exist with $M = \mathbb{R}^1$ or $\mathbb{R}^2$.)

The rest of the exercises concern cell complexes; see the appendix in Hatcher.

Problem 5. Suppose X is a cell complex. Then the following are equivalent.

(1) X is connected.
(2) X is path-connected.
(3) The 1-skeleton of X is connected.

Problem 6. Suppose X is a cell complex. Then X is locally compact if and only if each point of X has a neighborhood which meets only finitely many cells.

Problem 7. There exists a cell complex X which is not first countable. In particular, the topology on X cannot be induced by any metric on X. (Hint: you can take X to be 1-dimensional.)

Problem 8. Suppose X is a cell complex and $S \subseteq X$ is compact. Then S lies in the union of finitely many cells. (This is proven in the appendix, but don't look.)

Problem 9. Suppose f is a function from a cell complex X to a topological space Y. Prove that f is continuous if and only if its restriction to each cell is continuous. (A little care is required to make this precise. The cell complex structure includes, for each n-cell α, a continuous function from the n-ball into X. This is called the characteristic map of α. The "restriction" of f to α means the composition $B^n \to X \to Y$.)

Problem 10 (Important: composing infinitely many homotopies). Suppose X is a cell complex, and for each $n > 0$ the n-skeleton $X^{(n)}$ is contractible. Show that X is also contractible! (Hint: chain together the homotopies to get a "homotopy" $H : X \times [0, \infty) \to X$. This is not an actual homotopy because $[0, \infty) \ncong [0, 1]$. But extend H to a continuous function $X \times [0, \infty] \to X$.)