An introduction to de Rham cohomology
How algebra and calculus relate to topology

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Motivation

Question

Does there exist a function that is the gradient of some other function? More precisely, when does there exists an $F\colon U \to \mathbb{R}^2$ for some open $U \subseteq \mathbb{R}^2$ that satisfies

$$\frac{\partial F}{\partial x} = f_1, \quad \frac{\partial F}{\partial y} = f_2 \quad \text{for a vector field } f = (f_1, f_2)?$$

(You could also think of this question as asking when vector fields have potential.)

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Answer

It depends on the topology of U!

Some vector calculus

Note that $\frac{\partial F}{\partial x} = f_1$, $\frac{\partial F}{\partial y} = f_2$ implies $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. Is this condition sufficient to show F is the gradient of some other function?

Example

The vector field $f(x, y) = (y, x)$ satisfies $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 1$, and the gradient of $F = xy$ is f.

Example

However, consider $f(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$. This vector field cannot be conservative, which you can show by integrating around a closed loop.
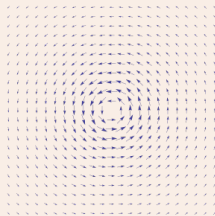
Div, grad, curl

Proposition

The condition $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$ is sufficient for the existence of a conservative vector field F if U looks like a ball (convex).

Definition

Define $C^\infty(U, \mathbb{R}) \overset{\text{grad}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{curl}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{div}}{\to} C^\infty(U, \mathbb{R})$,

$$\text{grad}\colon f \mapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\text{curl}\colon (f_1, f_2, f_3) \mapsto \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right),$$

$$\text{div}\colon (f_1, f_2, f_3) \mapsto \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

Sneak peek of de Rham cohomology

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Definition

The space $H^1(U)$ is the **1st de Rham cohomology group** of U.

Now the aforementioned proposition is equivalent to saying $H^1(U) = 0$ whenever U is convex. Here's why:

- All we need to show is that $\text{ker}(\text{curl}) \subseteq \text{im}(\text{grad})$.

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- In two dimensions, $f \in \ker(\operatorname{curl})$ means $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$.

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- All we need to show is that $\ker(\text{curl}) \subseteq \text{im}(\text{grad})$.
- In two dimensions, $f \in \ker(\text{curl})$ means $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$.
- So $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, and we know what to do from here!

Proposition

$\operatorname{div} \circ \operatorname{curl} = 0$.

Definition

Since the composition $\operatorname{div} \circ \operatorname{curl}$ is zero, we can also form the **2nd de Rham cohomology group** $H^2(U) := \ker(\operatorname{div})/\operatorname{im}(\operatorname{curl})$. To fit with the theme, define $H^0(U) = \ker(\operatorname{grad})$.

It turns out de Rham cohomology measures the amount of "holes" in a space. Since $\mathbb{R}^3$ is "completely solid", there should be no nontrivial de Rham cohomology. The following theorem demonstrates this.
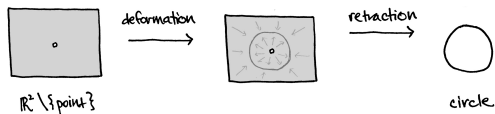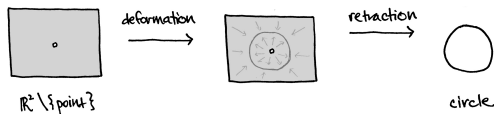
A quick summary

Theorem

For a convex open set $U \subseteq \mathbb{R}^3$, we have $H^0(U) = \mathbb{R}$, $H^1(U) = 0$, and $H^2(U) = 0$.

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$\mathbb{R}^2 \setminus \{\text{point}\}$ — deformation → — retraction → circle

Theorem

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Note: This is not an actual commutative diagram!

$$
\begin{array}{ccccccc}
\overset{\ker(\mathrm{grad})}{H^0(U)} & \longrightarrow & \overset{\ker(\mathrm{curl})/\operatorname{im}(\mathrm{grad})}{H^1(U) = 0} & \longrightarrow & \overset{\ker(\mathrm{div})/\operatorname{im}(\mathrm{curl})}{H^2(U) = 0} & \longrightarrow & H^3(U) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C^\infty(U, \mathbb{R}) & \underset{\mathrm{grad}}{\longrightarrow} & C^\infty(U, \mathbb{R}^3) & \underset{\mathrm{curl}}{\longrightarrow} & C^\infty(U, \mathbb{R}^3) & \underset{\mathrm{div}}{\longrightarrow} & C^\infty(U, \mathbb{R}) \\
& & \mathrm{exact} & & \mathrm{exact} & &
\end{array}
$$

The de Rham complex

Let's shift gears to a more abstract setting.

Definition

Define Ω^* to be the algebra generated by $dx_1, \cdots, dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0, \\ dx_i dx_j = -dx_j dx_i, \ i \neq j. \end{cases}$$

Then a **differential form** is an element of $\Omega^*(U)$, formally defined as $C^\infty(U, \mathbb{R}) \otimes_\mathbb{R} \Omega^*$, where $U \subseteq \mathbb{R}^n$ is open. This algebra has a natural grading $\Omega^*(U) = \bigoplus_{q=0}^n \Omega^q(U)$.

Example

Concretely, a form $\omega \in \Omega^q(U)$ can be written uniquely as $\sum f_I dx_I$, where I denotes a strictly increasing sequence of length $q < n$.

The exterior derivative

Definition

Define a *differential operator* $d \colon \Omega^q(U) \to \Omega^{q+1}(U)$ by the following properties:

1. if $f \in \Omega^0(U)$, then $df = \sum \frac{\partial f}{\partial x^i} dx^i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

This operator is called **exterior differentiation.**

Definition

The algebra $\Omega^*(U)$ paired with the differential operator d is called the **de Rham complex** on U.

Example

Let $\omega = xy\, dx \in \Omega^1(\mathbb{R})$ be a 1-form. Then
$d\omega = \left(\frac{\partial(xy)}{\partial x} dx + \frac{\partial(xy)}{\partial y} dy \right) dx = y\, dx\, dx + x\, dy\, dx = -x\, dx\, dy$.
We will see more exterior derivative calculations very soon.

In $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are one dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are both three dimensional. Then identify

$$
\begin{array}{ccccc}
\{\text{functions}\} & \simeq & \{0\text{-forms}\} & \simeq & \{3\text{-forms }\}, \\
f & & f & & f\,dx\,dy\,dz \\
\{\text{vector fields}\} & \simeq & \{1\text{-forms}\} & \simeq & \{2\text{-forms }\} \\
X=(f_1,f_2,f_3) & & f_1\,dx+f_2\,dy+f_3\,dz & & f_1\,dy\,dz-f_2\,dx\,dz+f_3\,dx\,dy
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So,

- Taking the exterior derivative of a 0-form (function) gives **gradient**,

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So,

- Taking the exterior derivative of a 0-form (function) gives $\boxed{\textbf{gradient}},$

- The exterior derivative of a 1-form (vector field) is $\boxed{\textbf{curl}},$

- And the exterior derivative of a 2-form is $\boxed{\textbf{divergence}}.$

Example

Consider a 2-form defined by $f_1\, dy\, dz - f_2\, dx\, dz + f_3\, dx\, dy$. Then

$$d(\text{2-form}) = \left(\frac{\partial f_1}{\partial x}\, dx + \frac{\partial f_1}{\partial y}\, dy + \frac{\partial f_1}{\partial z}\, dz\right) dy\, dz + \cdots$$

$$= \frac{\partial f_1}{\partial x}\, dx\, dy\, dz + 0 + 0 + \cdots$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz.$$

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$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \, dy \, dz.$$

This is precisely $\boxed{\text{divergence!}}$ Similarly, we also have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \boxed{\text{grad,}}$$

$$d(\text{1-form}) = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \, dy + \cdots = \boxed{\text{curl.}}$$

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So the exterior derivative generalizes all the previous notions of derivatives from calculus!

The derivative of the derivative is zero

Proposition

$d^2 = 0$, where d denotes the exterior derivative.

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Proof.

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0$$

since mixed partials commute, while the $dx_j dx_i$ anti-commute (the property that $dx_j dx_i = -dx_i dx_j$). $\boxtimes$

This generalizes the previous ideas $\mathrm{curl} \circ \mathrm{grad} = 0$, $\mathrm{div} \circ \mathrm{curl} = 0$, which allowed us to define $H^1(U)$ and $H^2(U)$. Since d is defined in all dimensions, we can define a more general de Rham cohomology group!

Closed and exact forms

Definition

If $d\omega = 0$, then ω is a **closed** form, while if $\omega = d\tau$ for some form τ, we say ω is an **exact** form. Precisely, $\ker d$ consists of all the closed forms, while $\operatorname{im} d$ are the exact forms.

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Definition

The q-th **de Rham cohomology** of U is the space

$$H_{DR}^q(U) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

Since $d^2 = 0$, $\operatorname{im} d \subseteq \ker d$ trivially. Now the question at the beginning of the talk reduces to "can we find a nontrivial closed form on U"? The generalized de Rham cohomology measures to what extent we can do this, by collapsing the trivial solutions to zero.

Example

Let us compute the cohomology of $\mathbb{R}^1$. $\ker d$ in $\Omega^0(\mathbb{R}^1)$ consists of constant functions, so $H^0(\mathbb{R}^1) = \mathbb{R}$. Every 1-form $\omega = g(x)dx$ is exact, since $d\int_0^x g(u)\,du = \omega$; this implies $H^1(\mathbb{R}^1) = 0$, since we mod out by the entire space. Succinctly, we have

$$H^q(\mathbb{R}^1) = \begin{cases} \mathbb{R}, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$$

More generally, it is true that

$$H^*(\mathbb{R}^n) \begin{cases} \mathbb{R} & \text{in dimension 0}, \\ 0 & \text{otherwise}. \end{cases}$$

This result is called the *Poincaré lemma*.

The Mayer-Vietoris sequence

A map between spaces induces a map on forms, formally stated below:

Remark

Note that a smooth map $f\colon X \to Y$ induces a **pullback**
$f^*\colon \Omega^0(Y) \to \Omega^0(X), g \mapsto g \circ f$, which naturally extends to a pullback
on forms $f^*\colon \Omega^*(X) \to \Omega^*(Y)$

$$f^* \left(\sum g_I \, dy_{i_1} \cdots dy_{i_q} \right) = \sum \left(g_I \circ f \right) d(y_{i_1} \circ f) \cdots d(y_{i_q} \circ f)$$

which commutes with d. So assigning the complexes Ω^* to a sequence of
maps is a **contravariant functor**.

Suppose our space $X = U \cup V$ for U, V open. Then we have a sequence
of inclusions

$$X \leftarrow U \amalg V \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} U \cap V$$

where $U \amalg V$ is the (set-theoretic) disjoint union, and ∂_0, ∂_1 denote
inclusions in V, U respectively.

The Mayer-Vietoris sequence

Applying the contravariant functor Ω^* to the sequence of inclusions gives

$$\Omega^*(X) \to \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

Take the difference of the maps to get the **Mayer-Vietoris sequence**

$$0 \longrightarrow \Omega^*(X) \longrightarrow \underset{(\omega,\tau)}{\Omega^*(U) \oplus \Omega^*(V)} \underset{\mapsto}{\longrightarrow} \underset{\tau-\omega}{\Omega^*(U \cap V)} \longrightarrow 0,$$

which turns out to be exact. This induces a long exact sequence on cohomology

$$\cdots \to H^q(X) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \overset{d^*}{\to} H^{q+1}(X) \to \cdots$$

So how do we actually use this weird algebraic construction?

The de Rham cohomology of the punctured plane

Example

For $X = \mathbb{R}^2 \setminus \{0\}$, we can cover it with two open sets U, V, whose intersection $U \cap V$ is just two solid chunks of $\mathbb{R}^2$. So

$$H^0(U) \oplus H^0(V) = H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R},$$
$$H^1(U) = H^1(V) = H^1(U \cap V) = 0.$$

Clearly H^2 and above of X, U, V, etc are all zero. Our long exact sequence from Mayer-Vietoris becomes

$$H^0(X) \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1(X) \to 0 \to \cdots$$

Since $\delta \colon (\omega, \tau) \mapsto (\tau - \omega, \tau - \omega)$, im δ is 1-dimensional, and so is ker δ. Then by the first isomorphism theorem, $H^0(X)/0 \cong \ker \delta = \mathbb{R}$, and $H^1(X) \cong \mathbb{R} \oplus \mathbb{R}/(\ker d^* = \operatorname{im} \delta) = \mathbb{R}$. This shows that $\mathbb{R}^2 \setminus \{0\}$ has a nontrivial first cohomology, like expected!

Thank you!

Thank you for listening to my talk, and a special thank you to Arun Debray for mentoring me and answering all my dumb questions! These slides and detailed notes can be found on my website: `https://simonxiang.xyz/math`