2021 Summer Minicourses, UT Austin

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Notes for some of the Summer 2021 minicourses, taught by various graduate students here at UT Austin. If you taught one of these courses and don't want the notes to be viewed publicly (or if you have any comments or suggestions in general), please contact me at simonxiang@utexas.edu! Homepage: https://web.ma.utexas.edu/SMC/Minicourses.html

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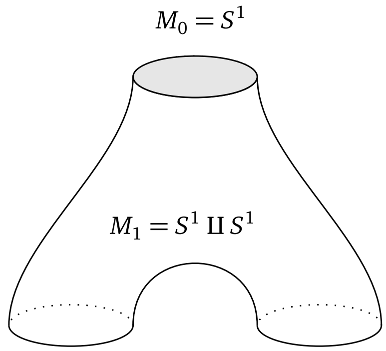


Figure 1: A bordism between M_0 and M_1.

1 Topological Field Theory (Arun Debray)

This is a minicourse on topological field theory, taught by Arun Debray. We will cover the generalities of TFTs, then a series of examples, namely invertible field theories, the finite path integral, state sums, and computing Hurewizc numbers. Existing introductions to TFTs tend to focus on dimension two and Frobenius algebras, or jump into higher category theory and the cobordism hypothesis, or cover lots of generalities and don't give many examples. This minicourse aims to construct many examples and give applications, since they're interesting and can be quite difficult to construct.

Regrettably time is finite, so we won't cover extended TFTs (involving higher category theory) and the cobordism hypothesis, Chern-Simons theory (look out for a minicourse on that later), and connections with physics. Many things that happen will be motivated by physics, but this minicourse is intended for mathematicians. For prerequisites, it would be good to have a solid handle on the language of smooth manifolds.

1.1 The general theory of TFTs (May 31)

Let us start off by clarifying some terminology. You may have heard "bordism" or "cobordism", and for a while cobordism was the more common word– we will use the word bordism. The etymology of cobordism comes from the Latin roots meaning "together (co) they bound (bord)". At some point Atiyah realized there is a generalized homology theory, and this is what we deal with today. So it makes sense to talk about bordisms and homology as opposed to cobordisms and cohomology. However if you read the literature and come across "cobordism groups", they probably mean the same thing as the bordism groups we discuss.

On the same vein, some people say "topological field theory" (TFT) and others say "topological quantum field theory" (TQFT). In mathematics, people tend to use these words interchangeably. Dan Freed pointed out that the Atiyah-Segal formalism we will discuss works for classical field theories as well, and mathematically saying something is quantum is kind of funky. So we will stick to the terminology of TFTs, but these two words are pretty much interchangeable.

Bordism

Bordism tells us when two manifolds together bound some manifold of one dimension higher.

Definition 1.1. Let M_0 and M_1 be closed n-manifolds. A **bordism** from M_0 to M_1 is a compact $(n+1)$-manifold X, a partition $\partial X = Y_0 \amalg Y_1$, and diffeomorphisms $\theta_i\colon Y_i \xrightarrow{\cong} M_i$. If there is a bordism from M_0 to M_1, we say M_0 and M_1 are **bordant**.

Bordism is an equivalence relation. It is reflexive since $M \times [0,1]$ is a bordism from M to M. If M and N are bordant, then flipping the manifold around is a bordism from N to M, and if we have bordisms M to N and N to P, just glue at N to get a bordism M to P. These ideas are illustrated in Figure 2, with $M = S^1$, $N = S^1 \amalg S^1$, and $P = S^1 \amalg S^1 \amalg S^1$.

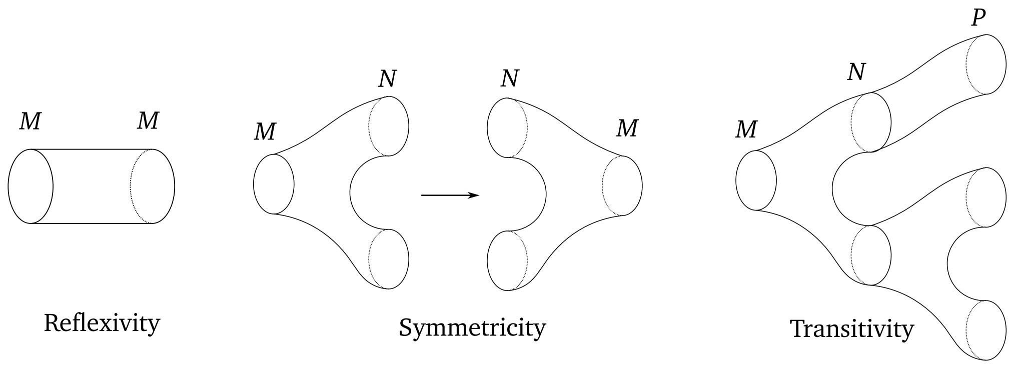


Figure 2: Illustration of the fact that bordism is an equivalence relation.

Denote the set of bordism classes of closed n-manifolds by Ω_n^O. The disjoint union gives Ω_n^O a commutative monoid structure, with $\emptyset$ the unit. It turns out this is actually an abelian group! Furthermore, the direct product turns $\Omega_*^O := \bigoplus_n \Omega_n^O$ into a $\mathbb{Z}$-graded ring. Additive inverses are easy to compute since the macaroni $M \times [0,1]$ gives a bordism from $2M$ to the point, so every element is its own inverse, and the bordism ring is actually an $\mathbb{F}_2$ algebra. These rings have been computed by René Thom in his thesis [Tho54], which are polynomial algebras on an infinite set of generators. It is known what manifolds correspond to these generators.

Now we introduce tangential structures, the goal of which is to generalize bordisms while taking into account topological information. For example, we have our diffeomorphisms θ_i that identify the boundary of a manifold. What if we want these to preserve orientation? Then we get a different notion of bordism, and non-isomorphic bordism groups. Tangential structures capture topological structure like spin structures and maps into spaces. They do *not* capture geometric information like a Riemannian metric or connections on principle bundles. Information must also be “local”– a single point inside a manifold is not a local notion, along with CW structures. This is deliberately vague, and we will see a precise definition soon.

Definition 1.2. Consider the **stable orthogonal group** $O := \mathrm{colim}_n \mathrm{O}_n$. The classifying space BO is the classifying space for stable virtual vector bundles, where “virtual” means we allow formal differences $E - F$ for bundles $E, F \to X$, and “stable” means we ignore the different between E and $E \oplus \mathbb{R}$.

Homotopy classes of maps to BO denoted $[M, BO]$ is identified with stable isomorphism classes of virtual vector bundles. The point is that a manifold has a canonical map (or homotopy class) $M \to BO$, which classifies its tangent bundle. Suppose we had an orientation on M, then we could repeat this whole process with *oriented* vector bundles, giving us the stable special orthogonal group $SO := \mathrm{colim}_n \mathrm{SO}_n$ with classifying space BSO.

It is true that a manifold admits an orientation iff the map $M \to BO$ factors through BSO. Different orientations of M are parametrized by choices of lift. Now we generalize this, and consider *any* space B and any map $B \to BO$ (it has to be a fibration, but up to homotopy we can make any map a fibration).

Definition 1.3. Let $\xi : B \to BO$ be a fibration. A ξ**-structure** on a manifold is a lift

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ M \arrow[ur, dotted] \arrow[r, "TM"'] & BO \end{tikzcd}$$

Two ξ-structures are equivalent if they are homotopic through lifts of the tangent bundle map.

Example 1.1. Given a family of maps $G_n \to O_n^t$, we can stabilize and take the colimit to obtain $\xi\colon BG \to BO$. For $BSO \to BO$ this is an orientation, and for $B\mathrm{Spin} \to BO$ this is a spin structure.

Example 1.2. We can also take $BO \times BG \to BO$ where G is some other group, so in this case a ξ-structure is equivalent to a map to BG, which is a principle G-bundle. More generally, $BO \times X \to BO$ encodes a map to X.

The main thing we want to do with this is induce a structure on the boundary. If M is a manifold with boundary, $T(\partial M) \oplus \nu \cong TM|_{\partial M}$, since the normal bundle is trivializable as a line bundle. There are two ways to do trivialize ν– by walking a distance ε into the manifold and walking ε outward (outward vs inward unit normal). As virtual stable vector bundles, $T(\partial M) \cong TM|_{\partial M} - \nu$, and a ξ-structure on TM induces a ξ-structure on $T(\partial M)$. Let ∂M refer to ∂M with its ξ-structure the *inner* unit normal, and $-\partial M$ refer to the ξ-structure via the *outer* unit normal.

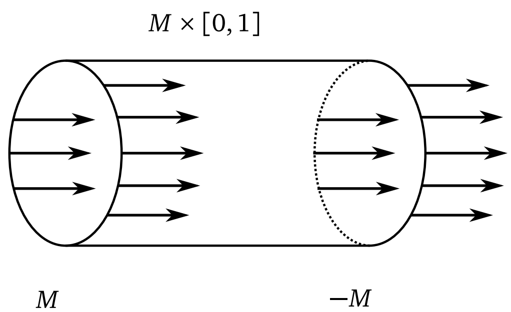


Figure 3: An induced structure on the boundary of $M \times [0,1]$.

Now we can define bordisms of ξ-manifolds, which are compact ξ-manifolds of one dimension higher, where everything has a ξ-structure and diffeomorphisms identify ξ-structures from ∂X to $M \amalg -N$. This also yields an equivalence relation compatible with the disjoint union, and so we get bordism groups Ω_n^ξ. However, it is not true that we always get a graded ring.

We want to upgrade or categorify this structure.

Definition 1.4. Define a **bordism category** Bord_n^ξ whose objects are closed $(n-1)$-dimensional ξ-manifolds, and whose morphisms are ξ-structured bordisms between them. Here, composition is just the gluing of bordisms. For composition to be associative, we need to take diffeomorphism classes rel boundary– so if two bordisms are diffeomorphic, they are the same bordism. The identity is a cylinder, a morphism from M to $-M$. So in order to have an identity morphism, we need to define bordisms with the minus sign.

It turns out that $(\amalg, \emptyset)$ turns the bordism category into a "categorical commutative monoid" structure, or in other words, $(\mathrm{Bord}_n^\xi, \amalg, \emptyset)$ is a **symmetric monoidal category**. Essentially, we have a unit and a "tensor prodouct" $\amalg$ which is associative and commutative up to natural isomorphism. For example, $(\mathsf{Vect}_\mathbb{C}, \otimes, \mathbb{C})$ is a symmetric monoidal category. The precise definition is in [Mac13], there you can also read about the corresponding notion of symmetric monoidal functors and symmetric monoidal natural transformations.

Topological field theories

Here comes the moment we have all been waiting for.

Definition 1.5. A **topological field theory** is a symmetric monoidal functor $Z\colon \mathrm{Bord}_n^\xi \to \mathsf{Vect}_\mathbb{C}$. We say n is the **spacetime dimension** of the theory, and $(n-1)$ is the **space dimension**.

In physics, the bordism category is interpreted as objects being space, and the morphism between them is like time evolution, hence spacetime. Let's unravel this definition.

- The objects in Bord_n^ξ are closed $(n-1)$-manifolds, and the bordisms/morphisms are n-manifolds.
- Every closed $(n-1)$-manifold M is associated to a vector space $Z(M)$ called the **state space**.[1]
- A bordism $X: M \to N$ defines a linear map $Z(X): Z(M) \to Z(N)$, where functoriality says that gluing goes to composition. You can think of this as time evolution from states on M to states on N.
- $Z(\emptyset) = \mathbb{C}$. Therefore a closed n-manifold X as a bordism $\emptyset \to \emptyset$ defines a linear map $\mathbb{C} \to \mathbb{C}$: the image of 1 is called the **partition function** of X.

Example 1.3 (The Euler TFT)**.** Assign to every closed $(n-1)$-manifold the state space $\mathbb{C}$, and to every bordism $X: M \to N$ the quantity $\lambda^{\chi(X,N)}$. Here, $\lambda \in \mathbb{C}^\times$ is fixed, and $\chi(X,N)$ is the relative Euler characteristic (an alternating sum of the relative homology groups). Then gluing (counting cells of a CW structure) and symmetric monoidality hold because of formulas for χ.

For $\lambda \neq 1$ it is not trivial, but it is almost trivial.

Theorem 1.1. *Let $Z: \mathrm{Bord}_n^\xi \to \mathsf{Vect}_\mathbb{C}$ be a TFT and M be a closed $(n-1)$-dimensional ξ-manifold. Then the vector space $Z(M)$ is finite-dimensional.*

This differs from QFT where state spaces tend to look like ℓ_2 of something. We prove this by defining a generalization of "finite-dimensional" in arbitrary symmetric monoidal categories, showing that this is preserved by symmetric monoidal functors, then showing that every object in Bord_n^ξ is "finite-dimensional".

Definition 1.6. Let $\mathcal{C}$ be a symmetric monoidal category and $x \in \mathcal{C}$. **Duality data** for x is an object $x^\vee \in \mathcal{C}$ and morphisms $e: x \otimes x^\vee \to 1$, $c: 1 \to x \otimes x^\vee$ such that the following maps compose to the identity:

$$x \xrightarrow{c \otimes \mathrm{id}_x} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{id}_x \otimes e} x, \tag{1}$$

$$x^\vee \xrightarrow{\mathrm{id}_{x^\vee} \otimes c} x^\vee \otimes x \otimes x^\vee \xrightarrow{e \otimes \mathrm{id}_{x^\vee}} x^\vee. \tag{2}$$

If duality data exists for x, we say X is **dualizable**, $x^\vee$ is the **dual** of x, e the **evaluation**, and c the **coevaluation**.

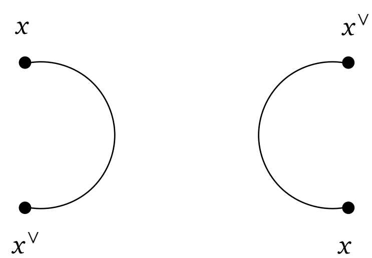


Figure 4: Evaluation (on the left) and coevaluation (on the right).

To visualize evaluation and coevaluation, people typically draw string diagrams, as in Figure 4. Evaluation can be thought of colliding things with time, and coevaluation can be thought of as making something new. Equations (1) and (2) tell us that the S-diagram (on the left) and Z-diagram (on the right) in Figure 5 both are just the identity, where the identity is a straight line.

[1] If you know about quantum mechanics, there is a Hilbert space of states. In QFT, we do this on arbitrary manifolds, and here we get a Hilbert space from a manifold of codimension 1. We will see soon why we only need to think about finite dimensional vector spaces.

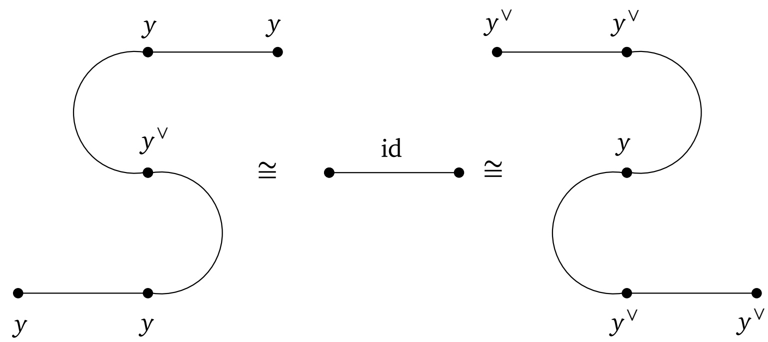


Figure 5: The S-diagram (left) encoding Equation (1) and the Z-diagram (right) encoding Equation (2).

Say V is dualizable with duality data $(V^\vee, c, e)$, then $c(1) = v^i \otimes v_i$, a *finite* sum.[2] Apply the Z-diagram to figure out that for any $x \in V$, we have $x = e(x, {}^i)v_i$, or the finite set $\{v_i\}$ spans V. Conversely, given a finite-dimensional vector space, let $V^\vee := \mathrm{Hom}(V, \mathbb{C})$, e be evaluation, and $c \colon 1 \mapsto e^i \otimes e_i$ ($\{e_i\}$ is a basis, $\{e^i\}$ is the dual basis). So V is finite-dimensional iff it is dualizable.

In Bord_n^ξ, *any* object is dualizable. Referring to Figure 6, evaluation is the outgoing macaroni and coevaluation is the incoming macaroni. We take diffeomorphism classes of bordisms as our morphisms, so clearly the diagrams are diffeomorphism to the identity cylinder. So they are the same. David Ben-Zvi likes to call this result "Zorro's lemma", modeled after the swordsman drawing a "Z" to complete his proof.

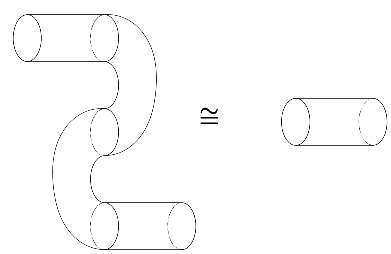


Figure 6: "Zorro's lemma", which shows all objects of Bord_n^ξ are dualizable.

Proof of Theorem 1.1. Symmetric monoidal functors preserve duality by definition, since they factor over the tensor product and must send the identity to the identity. Then if $f : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor and $x \in \mathcal{C}$ is dualizable, then $f(x)$ is dualizable. So if $Z \colon \mathrm{Bord}_n^\xi \to \mathsf{Vect}_{\mathbb{C}}$ is a TFT and M is a closed $(n-1)$-manifold, we know M is dualizable in Bord_n^ξ by Zorro's lemma. Therefore $Z(M)$ is dualizable in $\mathsf{Vect}_{\mathbb{C}}$, i.e., finite-dimensional. ⊠

We can also consider mapping class group actions of O and SO: general tangential structures are fine too, but require some extra care. The key idea is that $\mathrm{Diff}(M)$ acts on the state space $Z(M)$ by mapping cylinders: for $\varphi \in \mathrm{Diff}(M)$, this defines a bordism $M \to M$ by $[0,1] \times M$, where we attach 0 by id and 1 by φ. Because we defined diffeomorphisms rel boundary to make associativity work, this is not always the same morphism. This defines a group homomorphism from diffeomorphisms to automorphisms in the bordism category of M. By functoriality, we get an action of the diffeomorphisms on the state space, which is just the image of this mapping cylinder under Z.

If φ and φ' are isotopic, their mapping cylinders are diffeomorphic rel boundary, and they define the same morphism in Bord_n^ξ. So the $\mathrm{Diff}(M)$-action factors through the action of the **mapping class group** $\mathrm{MCG}(M) := \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$, where $\mathrm{Diff}_0(M)$ is the connected component of the identity. For example the diffeomorphism group of the torus is this infinite dimensional terrible thing, but the mapping class group is $\mathrm{SL}(2, \mathbb{Z})$.

[2]We use Einstein summation notation, so there is an implicit sum over i.

The **mapping torus** of $\varphi \in \mathrm{Diff}(M)$ is the mapping cylinder plus one extra identification, defined by $M_\varphi := [0,1]\times M/(0,x) \sim (1,\varphi(x))$. It shouldn't be difficult to show that for Z a TFT, the partition function of the mapping torus is the *trace* of the action of φ on the state space $Z(M)$. A special case is the mapping torus of the identity which is just $M \times S^1$, where $Z(M \times S^1) = \dim Z(M)$, since the trace in this case is just the dimension.

1.2 Invertible field theories (June 1)

These are the simplest possible non-trivial examples of TFTs, which is why we start here. People classify these by the tools of homotopy theory, which you may or may not like. So we try to extract the big idea of the proof to present, then go over the details at the end.

Throughout this talk, we will have the notion that symmetric monoidal categories are categorified versions of commutative rings. Given a commutative ring A, a unit x is an element invertible under multiplication (there exists an x^{-1} such that $x \cdot x^{-1} = 1$). When we categorify, the bad news is that x^{-1} and the isomorphism $x \otimes x^{-1} \xrightarrow{\cong} 1$ are now data. The good news is that this is contractible choice, like duality data, so we think of it as a condition. This is like choosing a Riemannian metric– there is a contractible space, so as far as homotopy theorists are concerned, a manifold and a manifold with some Riemannian metric are equivalent.

Example 1.4. In $(\mathsf{Vect}_\mathbb{C}, \otimes)$, a vector space is invertible iff it is one-dimensional. The dual is a good choice for the inverse, and evaluation is an isomorphism. Invertible vector spaces are trivializable, that is, they are all isomorphic to the trivial vector space $\mathbb{C}$. Maybe you have to make a choice– if we consider vector bundles or representations, we get non-trivial invertible objects (one dimensional representations, line bundles).

Picard groupoids

Let $\mathcal{C}^\times \subseteq \mathcal{C}$ denote the subcategory of $\otimes$-invertible objects and composition-invertible morphisms. This is a **Picard groupoid**, the key notion of today's talk. It has a symmetric monoidal structure, such that all objects are invertible.

Example 1.5. $\mathsf{Vect}_\mathbb{C}^\times$ consists of lines (or one-dimensional vector spaces) with nonzero linear maps between them.

Recall a TFT is a symmetric monoidal functor $\mathrm{Bord}_n^\xi \to \mathsf{Vect}_\mathbb{C}$, and they form a functor category where morphisms are symmetric monoidal natural transformations. This category of TFTs has a symmetric monoidal structure given by "pointwise tensor product":

$$(Z_1 \otimes Z_2)(M) := Z_1(M) \otimes Z_2(M).$$

The unit is the trivial theory, or the constant functor valued in $\mathbb{C}$ and $\mathrm{id}_\mathbb{C}$. An **invertible TFT** is an invertible object in this symmetric monoidal category. This is a concise definition that might make geometric topologists upset.

In a commutative ring of $\mathbb{C}$-valued functions, an element f is invertible iff $f(x) \in \mathbb{C}^\times$ for all x. The same idea for TFTs is analogous: a TFT Z is invertible iff for all closed $(n-1)$-manifolds M, $Z(M)$ is one-dimensional, and for all bordisms X, $Z(X) \neq 0$. You can think of these as "nearly trivial" TFTs.

A physics digression

How did these appear in physics? If you're a condensed matter theorist studying topological phases of matter, it is believed that there should be a way to classify these by taking a low energy approximation and obtaining a TFT. So the system behaves sort of like a quantum mechanical system, and you want to cut off all the eigenvalues of the Hamiltonian except for the bottom. Classifying these things is complicated, so you may want to start with a special case, known as "invertible phases" or "symmetry protected topological phases" or maybe even "short range entangled phases", which should be classified at low energy by invertible TFTs. A lot of open problems in physics and making the physics into math are related to this. We can make progress on this by comparing what the physicists have done to mathematical classifications of invertible TFTs. This is what Freed-Hopkins do in [FH19].

Another application is that sometimes quantum field theories or topological field theories are not quite precisely defined. There is a little extra data you have to give to define them, and the famous example is Chern-Simons theory. One way to define this in physics language is to say the theory has an anomaly. There are many ways to define this "anomaly", and Freed-Teleman do this in [FT14] by saying your theory is actually the boundary to some bulk theory which is invertible. The idea is you can understand anomalies by understand invertible field theories, and there are plenty of cases where people care about anomalies. Classifying invertible field theories is a helpful way to get your hands on what anomalies are. Plenty of papers do this, for example, Freed and Hopkins do this for M-theory in [FH21].

The Freed-Hopkins-Teleman theorem

The thing about invertible field theories is that we know what all of them are. This idea originates from [FHT07], but is really first explicitly stated in [FH19]. However they add additional assumptions, and Freed-Hopkins-Teleman is where this first really happened.

Theorem 1.2 (Freed-Hopkins-Teleman). *The abelian group of (isomorphism classes of) invertible n-dimensional TFTs of ξ-manifolds valued in* $\mathsf{sVect}_{\mathbb{C}}$ *is naturally isomorphic to the group of SKK ξ-bordism invariants* $\mathrm{Hom}(\mathrm{SKK}_n^{\xi}, \mathbb{C}^\times)$.

Here, n is the dimension of the manifold in the bordism, and $\mathsf{sVect}_{\mathbb{C}}$ is the category of **super vector spaces**. Super vector spaces are complex vector spaces with a $\mathbb{Z}_2$-grading (you can say when two elements are even are odd) which has the sign rule from cohomology. So if a and b are odd, then $a \otimes b = -b \otimes a$. We won't say what SKK is just yet, but it's a modified version of bordism invariants that are a bit harder to compute. The idea is that $\mathrm{Hom}(\mathrm{SKK}_n^{\xi}, \mathbb{C}^\times)$ is the partition function to an invertible TFT.

This is the proof sketch. We argue that classfying invertible TFTs is purely a question about Picard groupoids. It turns out we can express Picard groupoids and maps between them with algebraic data, which is known as the one-dimensional stable homotopy hypothesis combined with a little bit of Postnikov theory. Finally, we know this data for Bord_n^{ξ} (hard work) and $\mathsf{sVect}_{\mathbb{C}}^\times$ (fairly straightforward).

Here we review group completion. This may be new only because we don't talk about monoids that much. Say $f: M \to N$ is a homomorphisms of commutative monoids and $\mathrm{im}(f) \subseteq N^\times$. We can **group complete** M to an abelian group $\overline{M}$ by declaring new elements that satisfy $x \cdot x^{-1} = 1$, much like how $\mathbb{Z}$ is built from $\mathbb{N}$. $\overline{M}$ also goes by the Grothendieck group, K_0, or K. Then f extends to a map $\overline{f}: \overline{M} \to N^\times$ of abelian groups by setting $f(x^{-1}) := f(x)^{-1}$. Moreover, you can restrict $\overline{f}$ to M and obtain f, so f and $\overline{f}$ determine each other, or the abelian group of invertible maps $M \to N$ is naturally isomorphic to the abelian group of all maps $\overline{M} \to N^\times$.

This is the example we categorify. If a map has image in the invertible set, it factors through group completion and we lose no data this way. Given a symmetric monoidal category $\mathcal{C}$ and a symmetric monoidal functor f to a Picard groupoid $\mathcal{D}^\times$, the factors through the **Picard groupoid completion** $\overline{f}: \overline{\mathcal{C}} \to \mathcal{D}^\times$. $\overline{\mathcal{C}}$ is constructed by adding inverses to objects under tensor product and morphisms under composition, and similarly $\overline{f}$ is constructed by the formula $\overline{f}(x^{-1}) := f(x)^{-1}$. Again f and $\overline{f}$ determine each other.

That means that if $\mathcal{C} = \mathrm{Bord}_n^\xi$, classifying invertible TFTs (maps that factor through $\overline{f}$) is equivalent to computing the abelian group of isomorphism classes of Picard groupoid maps $\overline{\mathrm{Bord}_n^\xi} \to \mathsf{Vect}_{\mathbb{C}}^\times$. The key thing about Picard groupoids is that we can express them solely in terms of algebraic data.

- $\pi_0(\mathcal{C})$ is the abelian group of isomorphism classes of objects under tensor product.
- $\pi_1(\mathcal{C}) := \mathrm{Aut}_{\mathcal{C}}(1)$ is the automorphisms of the unit, and Eckmann-Hilton implies this group is abelian. This corresponds to the fact that $\pi_{\geq 2}$ or π_1 of a topological group is always abelian.
- Using $(-)\otimes \mathrm{id}_x$, we bring $\mathrm{Aut}_{\mathcal{C}}(1) \to \mathrm{Aut}_{\mathcal{C}}(1\otimes x) = \mathrm{Aut}_{\mathcal{C}}(x)$. So $\pi_1(\mathcal{C})$ is canonically identified wth $\mathrm{Aut}_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$.
- The **k-invariant** $k\colon \pi_0(\mathcal{C}) \otimes \mathbb{Z}/2 \to \pi_1(\mathcal{C})$ is defined as follows: given $x \in \pi_0(\mathcal{C})$, take the class of the symmetry map $\sigma \in \mathrm{Aut}_{\mathcal{C}}(x\otimes x) = \pi_1(\mathcal{C})$. We can think of $x \otimes x$ as the "swap" map, in $\mathsf{Vect}_{\mathbb{C}}$ this is the identity and in $\mathsf{sVect}_{\mathbb{C}}$ it is the identity on even elements and -1 on odd elements.

A theorem of Hoàng says that (π_0, π_1, k) determine a Picard groupoid up to equivalence. Moreover, homotopy classes of morphisms between Picard groupoid $f : \mathcal{C} \to \mathcal{D}$ are naturally identified with the abelian group of pairs of maps $f_0\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ which commute with the k-invariant. So to prove the classification of invertible TFTs, we need to determine (π_0, π_1, k) for $\overline{\mathrm{Bord}_n^\xi}$ and $\mathsf{sVect}_{\mathbb{C}}^\times$.

- For $\mathsf{Vect}_{\mathbb{C}}$, we do this directly: $\pi_0 = 0$, since invertible vector spaces up to isomorphism are all isomorphic to $\mathbb{C}$. What are the linear maps $\mathbb{C} \to \mathbb{C}$ that are invertible? This is $\mathbb{C}^\times$, so $\pi_1 = \mathbb{C}^\times$. For the k-invariant, there's only one map from $\mathbb{Z}/2 \otimes 0 \to \mathbb{C}^\times$, so $k = 0$.
- For sVect, $\pi_0 = \mathbb{Z}/2$ (the even and odd lines), $\pi_1 = \mathbb{C}^\times$, and the k-invariant is the unique injective nontrivial map $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \to \mathbb{C}^\times$.
- For Bord_n^ξ, this is a major theorem! In the celebrated paper [Gal+09][3] by Galatius-Madsen-Tillmann-Weiss, they determine the homotopy type of the cobordism category by obtaining a space from the topological category of bordisms, and they compute its homotopy. Nguyen goes further in [Ngu17] and extracts from that the Picard groupoid structure.

  π_0 is the ordinary bordism group Ω_{n-1}^ξ, and $\pi_1 = \mathrm{SKK}_n^\xi$, the **SKK bordism group**. [Gal+09] says it differently, but it is equivalent. S^1 has two framings up to equivalence: one is induced as the boundary of the trivial framing on the disk, and the other is the Lie group framing. This sends manifolds one dimension higher, and is well defined from bordism class to SKK bordism classes, which gives a k-invariant. A framing is a trivialization of the tangent bundle (a nullhomotopic map to BO), so it induces a ξ-structure.

Proof of Theorem 1.2. We have seen that invertible TFTs valued in $\mathsf{sVect}_{\mathbb{C}}$ are identified with pairs of maps $f_0\colon \Omega_{n-1}^\xi \to \mathbb{Z}/2$ and $f_1\colon \mathrm{SKK}_n^\xi \to \mathbb{C}^\times$. Crucially, the k-invariant for $\mathsf{sVect}_{\mathbb{C}}^\times$ is injective, so f_1 uniquely determines f_0 (if f_0 can exist). But the k-invariant tensors with $\mathbb{Z}/2$, so the image of $f_1 \circ k_{\mathrm{Bord}_n^\xi}$ is contained in $\mathbb{Z}/2 = \{\pm 1\} \subseteq \mathbb{C}^\times$, and therefore f_0 must exist. This means we can lose the data f_0, and so invertible TFTs are identified with maps $\mathrm{SKK}_n^\xi \to \mathbb{C}^\times$. ⊠

What is the SKK group exactly? Consider the notion of bordism where "bounding" means M bounds W, *and* the outward normal vector field on M extends to a nonvanishing vector field on W. This defines a commutative monoid under disjoint union, and group completion results in the **SKK group** SKK_n^ξ. Karras-Krek-Neumann-Ossa first studied this in [Kar+73], where they called it "Schneiden, Kleben, Kontrolle" (German for "cutting and pasting, controlled"), hence "SKK". This goes by many other names, including vector field bordism, Reinhardt bordism, Madsen-Tillmann bordism, and Lorentz bordism.

[3] If anybody knows why this is cited as `[Gal+09]` instead of `[GMTW09]`, please let me know.

In particular, the Euler characteristic is an SKK invariant, since $\chi(S^2) \neq 0$. Since we have a nonvanishing vector field on the whole thing, $\chi(W)$ vanishes by Poincaré-Hopf. Now pick a CW structure and use the gluing formula. Ordinary bordism invariants are also SKK invariants, since SKK bordism is stricter than bordism. Another interesting invariant is the **Kervaire semicharacteristic** in dimension $4k+1$ (where $\xi = \mathrm{SO}$), defined by

$$\kappa(M) := \sum_{i=o}^{2k} b_i(M) \pmod 2,$$

where $b_i(M)$ refers to the Betti numbers of M. SKK bordism invariants tend to be one of these three.

Here are many concrete examples of ordinary bordism invariants. The general idea is that integrating canonical cohomology classes by ξ-structures (for example, characteristic classes of the tangent bundle, or pulling back cohomology classes) gives bordism invariants of ξ-manifolds. This is super general, we can use generalized cohomology, twisted cohomology, etc. For example, say we want to study oriented 6-manifolds with principal U_1-bundles. If $p_1 \in H_1^4(M;\mathbb{Z})$ denotes the first Pontrjagin class, $c_1 \in H^2(M;\mathbb{Z})$ is the first Chern class, multiply them together to get something in H^6 and integrate. That is, there is an invariant $\Omega_6^{\mathrm{SO}}(B\mathrm{U}_1) \to \mathbb{Z}$ given by

$$M, P \mapsto \int p_1(M) c_1(P).$$

Why is this true? Both p_1 and c_1 admit de Rham models as closed forms, then this is true by Stokes' theorem. In general, Milnor-Stasheff show this for Whitney classes in [MS74] but their argument generalizes.

The homotopy theory in the background

The proof of Theorem 1.2 about the classification of invertible TFTs actually uses stable homotopy theory. Given a Picard groupoid $\mathcal{C}$, take the geometric realization of the *nerve* (a simplicial abelian group), resulting in a pointed CW complex with $\pi_i = 0$ for $i \geq 2$. This is a grouplike $\mathbb{E}_\infty$-space, so it defines a spectrum with $\pi_i = 0$ for $i \neq 0, 1$, called the "classifying spectrum" of $\mathcal{C}$.

The **1-dimensional stable homotopy hypothesis** conjectures that taking the classifying spectrum defines an equivalence of homotopy theories from the category of Picard groupoids to the category of 1-truncated connective spectra (only π_0 and π_1 nontrivial). This was a folk theorem proven by many (see [JO12]). Postnikov theory tells us how to determine homotopy classes of maps between such spectra using the k-invariant (see [JO12] again). Then [Gal+09] identified the classifying space of the groupoid of Bord_n^ξ, and [Ngu17] determined the grouplike $\mathbb{E}_\infty$-structure, hence the classifying spectra.

Why is there so much homotopy theory? We are often interested in classfiying **extended TFTs**, formulated in terms of higher bordism categories. Here, the homotopical approach generalizes *much* better. Invertible extended TFTs are classified by maps between Picard n-groupoids. Taking the nerve of a Picard n-groupoid then geometrically realizing results in a space, which has a grouplike $\mathbb{E}_\infty$-structure, so we get a spectra. The "categorified" n-dimensional stable homotopy hypothesis says that we get an equivalence between the homotopy theory of Picard n-groupoids and that of spectra with homotopy groups concentrated in degrees $[0, n]$. It was a conjecture up to last year, proven in [Mos+20]. Schommer-Pries computes the homotopy type of the bordism n-category in [Sch17][4] and gets out a classifying spectrum. Depending on the target $\mathcal{C}$, invertible extended TFTs are classified in terms of homotopy or cohomology groups of **Madsen-Tillman spectra**, again giving SKK groups.

If we only care about non-extended TFTs we can break it down into simpler components, but this is a sneak peek of what's going on behind the curtain in the proof of Theorem 1.2.

[4] Not sure if I got the right paper of Schommer-Pries...

1.3 The finite path integral (June 2)

You may have heard about the path integral spoken about in hush tones, as something the physicists do that is mathematically illegal. We're going to talk about a setting where you can actually do it, and product a topological theory. This puts strong constraints on what can happen, for example for gauge theory the gauge group is finite. The point is, this gives examples of TFTs that are not just invertible TFTs we talk about last time, and we can say some interesting properties about them (what are their state spaces, what are the values of their partition functions on various manifolds). Some of these go by names like Dijkgraaf-Witten theory, Yetters model, Quinn's finite homotopy TFT, etc.

The goal of today/the rest of the week is to construct interesting and nontrivial examples whose partition functions and state spaces are not too hard to calculate, and which (unlike yesterday) don't require too much homotopy theory to digest.

A quick review of yesterday's lecture. A **bordism invariant** is an abelian group homomorphism $\varphi\colon \Omega_n^\xi \to \mathbb{C}^\times$. Bordism invariants lift to an invertible TFT $Z_\varphi\colon \mathsf{Bord}_n^\xi \to \mathsf{sVect}_\mathbb{C}$ with partition function φ. Good examples of bordism invariants are integrating characteristic classes or natural cohomology classes for manifolds with a ξ-structure.

2 Spectra are your friends (Rok Gregoric)

This is a minicourse on stable homotopy theory, specifically spectra, taught by Rok Gregoric. The word "spectra" (or spectrum) shows up so much in math, for example the spectrum of an operator in functional analysis, the spectrum of a commutative ring in algebraic geometry, and spectra as a basic object of study in stable homotopy theory. The first two ideas are closely related while the third idea less so.

This minicourse will use ∞-categories as a warning to the faint of heart readers. As for prerequisites, it would be nice to know some algebraic topology (homotopy and homology) and a little bit of category theory. However, no background in stable homotopy theory will be assumed. The course will not prove much, the standard reference is [Lur17].

2.1 ∞-categories, loops, and suspension (May 24)

As an introduction, a story says that mathematics began with shepherds counting sheep by enumerating elements of $\mathbb{Z}_{\geq 0} = \mathbb{N}$. Adding two flocks of sheep induced a monoid (group without inverses) structure $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, and negative numbers came about by the group completion of the monoid $(\mathbb{Z}_{\geq 0})^{\mathrm{gp}} = \mathbb{Z}$. After the integers, we figured out how to multiply and got a ring structure. From here we obtain the abelian groups as the category of modules over the integers ($\mathsf{Ab} \simeq \mathsf{Mod}_\mathbb{Z}$), and everything in algebra follows. This is how algebra came about, and this is where we supposedly went wrong: we should have been working with the sphere spectrum all along.

What if instead of considering equivalent classes of sheep, the shepherds remembered they could "change" the sheep, for example saying "this is sheep 1 and sheep 2, but I can rename sheep 1 to sheep 2 and vice versa"? Instead of using equivalence classes of finite sets up to bijection, what if the shepherds remembered bijections as well? So if we worked with finite sets up to bijection (the category $\mathsf{Fin}^\simeq$), we can pass to $\mathbb{N}$ by the cardinality function $|\cdot|$, which is not a drastic change. The claim is we arrive at the sphere spectrum by imitating the historical development of the integers.

We have somewhat of a monoid structure by taking the disjoint union of finite sets $\mathsf{Fin}^\simeq \times \mathsf{Fin}^\simeq \to \mathsf{Fin}^\simeq$, where $(I, J) \mapsto I \amalg J$. With the sufficient theory, we can consider the group completion $S := (\mathsf{Fin}^\simeq)^{\mathrm{gp}}$, which is an

Classical math	Homotopical math
equality ($=$)	equivalence ($\simeq$)
sets (Set)	spaces ($\mathcal{S}$) (homotopy types, anima...)
categories	∞-categories

Figure 7: A comparison of ideas in classical math versus homotopical math.

incarnation of the **sphere spectrum**. There is a natural multiplication $(I, J) \mapsto I \times J$ which, passing by cardinalities to the integers, is just the natural product. So S is a ring, and we can form $\mathsf{Sp} := \mathsf{Mod}_S$, the category of modules over S. Voilà, this is what a spectra is. This is an intuitive perspective and we glossed over many things, we will come back and clarify much of this later on.

Another story: an Indian emporer asked many blind men to touch and elephant and report back on what it is. Each blind man touched a different part of the elephant like the snout or leg, and thus reported back thinking they had found a tree or a sword or whatever. None of them had the complete picture, but each of them has said the truth. An elephant is quite complicated— you could describe it as a collection containing the ears, the trunk, the tusks, etc. Each of these is an *aspect* of the elephant, and in the same way spectra have many different incarnations:

- (co)homology theories,
- spaces stable under suspension,
- infinite loop spaces,
- homotopy-coherent analogue of abelian groups,
- modules over the sphere spectrum S,

and so on.

Definition 2.1. An **∞-category** $\mathcal{C}$ consists of:

(i) A set of objects $C \in \mathcal{C}$,

(ii) Spaces of maps $\mathrm{Map}_{\mathcal{C}}(C, C') \in \mathcal{S}$ for all $C, C' \in \mathcal{C}$,

(iii) A composition map

$$\mathrm{Map}_{\mathcal{C}}(C', C'') \times \mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{C}}(C, C''), \quad (f, g) \mapsto f \circ g.$$

which is associative and unital *up to coherent homotopy*.

What do we mean by "homotopy coherence"? Say we have two maps $f \circ (g \circ h)$ and $(f \circ g) \circ h$, these maps aren't actually equal, but equivalent. This equivalence $f \circ (g \circ h) \simeq (f \circ g) \circ h$ is a path in the space of maps $\mathrm{Map}_{\mathcal{C}}(C, C'')$. Given a triple composition $((f \circ g) \circ h) \circ k$, there are several different ways to express this, and chaining the homotopies that encode their equivalence gives a "loop" in a mapping space of sorts. This mapping space may give different (non-unique) choices of expression, which is bad, so homotopy coherence specifies that we can express composition up to contractible mapping spaces (so a unique choice). In essence, we can write expressions like $f \circ g \circ h$ without brackets, since it's unique up to contractible ambiguity. For example, when we say the following diagram commutes,

$$\begin{tikzcd} \bullet \arrow[d, "f"'] \arrow[dr, "h"] & \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

this implies that $h \simeq g \circ f$. This homotopy is implictly constructing a 1-simplex with the vertices of the triangle, and "filling" in the simplex to make the space contractible. This means that we have many choices of composite morphisms (the paths going through the middle), and they're all related by contractible ambiguity.

Example 2.1. What are some key examples of ∞-categories?

- An ordinary (locally small) category is an ∞-category where the mapping spaces are just discrete topologies (sets).
- The spaces $\mathcal{S}$ form an ∞-category.
- Cat_∞ is an ∞-category of ∞-categories.

The cool thing is that we can pass much of category theory through the language of ∞-categories. This is where we gloss over things a little bit— proving all of category theory for ∞-categories is hard, and some other people (Jacob Lurie and Emily Riehl, see [Lur17]) have already done all of it.

Example 2.2. Here are some familiar notions from category theory.

- The **initial object** $C \in \mathcal{C}$ is an object such that $\mathrm{Map}_{\mathcal{C}}(C, C')$ is contractible for all $C' \in \mathcal{C}$. Terminal objects are similarly defined. For example, in $\mathcal{S}$ (the ∞-category of spaces), the initial object is the empty space and terminal objects are contractible spaces.
- For $C \in \mathcal{C}$, we define the **overcategory** $\mathcal{C}/C$ with objects as morphisms $C' \to C$. A map in $\mathrm{Map}_{\mathcal{C}/C}(C' \to C, C'' \to C)$ makes the following diagram commute:

$$\begin{tikzcd} C' \arrow[rr] \arrow[dr] & & C'' \arrow[dl] \\ & C & \end{tikzcd}$$

  Undercategories $C/\mathcal{C}$ are defined analogously.
- An example of these constructions are **pointed spaces** $\mathcal{S}_*$, defined as the undercategory $*/\mathcal{S}$ of a contractible space. So morphisms between objects of pointed spaces $(x : * \to X) \in \mathcal{S}_*$ have to preserve the basepoint $*$.
- **Functors** also make sense, where the maps between morphisms are compatible with all the higher homotopies between them. This also gives rise to the ∞-category of functors.
- **Limits** and **colimits** give rise to **homotopy limits** (and **homotopy colimits**), and they have universal properties! For intuition on limits and colimits, consider a functor from the integers by comparison to a category $\mathcal{C}$.

$$\underbrace{\overset{\varprojlim F(n)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{F(-3)}{\bullet} \longrightarrow \overset{F(-2)}{\bullet} \longrightarrow \overset{F(-1)}{\bullet} \longrightarrow \overset{F(0)}{\bullet} \longrightarrow \overset{F(1)}{\bullet} \longrightarrow \overset{F(2)}{\bullet} \longrightarrow \overset{F(3)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\varinjlim F(n)}{\bullet}}_{\text{functor } F:(\mathbb{Z},\geq)\to\mathcal{C}}$$

  A *diagram* is just this functor F. Then a colimit (denoted $\varinjlim$) is the farthest thing you can put on the right, and a limit (denoted $\varprojlim$) is the farthest thing you can put on the left. Unlike analysis, we don't need a linear diagram, we just need to find the farthest things on the left/right of some diagram.

Here are two examples of limits and colimits. Suppose we have a terminal object $* \in \mathcal{C}$. Then the **suspension** of $C \in \mathcal{C}$ (denoted ΣC) is the pushout $* \amalg_C *$ of $C \to *$ with itself, or the colimit that completes the diagram on the left.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \Omega C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$$

Similarly, given an initial object $* \in \mathcal{C}$, the **(based) loops** on $C \in \mathcal{C}$ are defined as the pullback $\Omega C := * \times_C *$, the limit that completes the diagram on the right.

- **Adjunctions** between categories $\mathcal{C}, \mathcal{D}$ consist of two functors $F : \mathcal{C} \to \mathcal{D},\ G : \mathcal{D} \to \mathcal{C}$ such that

$$\mathrm{Map}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Map}_{\mathcal{C}(C,G(D))}$$

  for all $C \in \mathcal{C},\ D \in \mathcal{D}$. This relation is denoted $F \dashv G$. We say F is a **right adjoint functor** while G is a **left adjoint functor**. A basic yet profound fact in category theory is that left adjoints preserve colimits while right adjoints preserve limits.

Claim. *Let $\mathcal{C}$ be an ∞-category with a zero object*[5] $*$. *Then $\Sigma \dashv \Omega$.*

Proof. The claim is that $\mathrm{Map}_{\mathcal{C}}(\Sigma C, C') \simeq \mathrm{Map}_{\mathcal{C}}(C, \Omega C')$. How do we show this? Since colimits factor out of Hom to become limits, or more precisely $\mathrm{Hom}(\varinjlim F, N) = \varprojlim \mathrm{Hom}(F-, N)$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(\Sigma C, C') &\simeq \mathrm{Map}_{\mathcal{C}}(* \amalg_C *, C') \\
&\simeq \mathrm{Map}_{\mathcal{C}}(*, C') \times_{\mathrm{Map}_{\mathcal{C}}(C,C')} \mathrm{Map}_{\mathcal{C}}(*, C') \quad \text{by factoring out colimits} \\
&\simeq \mathrm{Map}_{\mathcal{C}}(C, *) \times_{\mathrm{Map}_{\mathcal{C}}(C,C')} \mathrm{Map}_{\mathcal{C}}(C, *) \quad \text{by contractibility} \\
&\simeq \mathrm{Map}_{\mathcal{C}}(C, * \times_C *) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(C, \Omega C').
\end{aligned}$$

⊠

Definition 2.2. A **homotopy category** of an ∞-category $\mathcal{C}$ is an ordinary 1-category $\mathrm{h}\mathcal{C} = \mathsf{Ho}(\mathcal{C})$, where

$$\mathrm{ob}(\mathsf{Ho}\mathcal{C}) := \mathrm{ob}(\mathcal{C}),$$
$$\mathrm{Hom}_{\mathsf{Ho}\mathcal{C}}(C, C') := \pi_0(\mathrm{Map}_{\mathcal{C}}(C, C')).$$

In other words, the objects are just the objects of $\mathcal{C}$, and the path components of the mapping space $\mathrm{Map}_{\mathcal{C}}(C, C')$ forms the set of morphisms of $\mathsf{Ho}\mathcal{C}$. For topological spaces, this is your familiar category of spaces up to homotopy hTop from algebraic topology.

This ends the "∞-category theory in a nutshell" section of this minicourse. Now let's talk about stability.

Definition 2.3. A (1-)category $\mathcal{C}$ is **abelian** if

(i) it has a zero object $0 \in \mathcal{C}$,

(ii) it has a biproduct[6] $\oplus$,

(iii) every morphism $f : C \to C'$ has a kernel $\ker(f) \to C$ and a cokernel $C' \to \mathrm{coker}(f)$,

(iv) for every $f : C \to C'$ in $\mathcal{C}$, $\mathrm{coker}(\ker(f) \to C) \simeq \ker(C' \to \mathrm{coker}(f))$. This is a generalization of the first isomorphism theorem, where if you think of $\mathrm{coker}(f)$ as $C'/\operatorname{im} f$, the RHS equals $\ker(C'/\operatorname{im} f) = \operatorname{im} f$, while the LHS becomes $\mathrm{coker}(\ker f) = C/\ker f$. So $C/\ker f \simeq \operatorname{im} f$. Diagramatically,

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
C \arrow[r, two heads] & \mathrm{coker}(f)
\end{tikzcd}$$

is both a pullback and a pushout. You can also think of this as saying that limits and colimits coincide in certain cases.

[5] A zero object is just an object that is both initial and terminal.

[6] A biproduct is both a product and a coproduct.

So how do we transport this notion to ∞-land?

Definition 2.4. An ∞-category $\mathcal{C}$ is **stable** if

(i) it has a zero object $0 \in \mathcal{C}$,

(ii) it has a biproduct $\oplus$,

(iii) every morphism $f : C \to C'$ admits a **fiber**[7] (or *homotopy kernel*) and **cofiber** (or *homotopy cokernel*), where the fiber is the pullback of f along the zero map, and the cofiber is the pushout of f along the zero map.

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ C \arrow[r, "f"] & C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d] & C' \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

(iv) any $C' \to C \to C''$ in $\mathcal{C}$ is a **fiber sequence** (fits into the left diagram) iff it is a **cofiber sequence** (fits into the right diagram). In other words, the square

$$\begin{tikzcd} C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\ 0 \arrow[r] & C'' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pullback iff it is a pushout.

Example 2.3. Any sequence $C \to 0 \to C'$ is a fiber if $C \simeq 0 \times_{C'} 0 \simeq \Omega C'$, and a cofiber if $C' \simeq 0 \amalg_C 0 \simeq \Sigma C$. This tells us that in stable ∞-categories, we can "undo" the suspension to get loops, and vice versa. This gives an equivalent characterization of stability— $\mathcal{C}$ with a zero object is stable iff

- it has all finite limits,
- $\Omega \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$,

or iff

- it has all finite colimits,
- $\Sigma \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$,

or iff

- it has all finite limits and colimits,
- any square

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & C'' \end{tikzcd}$$

in $\mathcal{C}$ is a pullback iff it is a pushout.

Since these two functors are adjoint, if either one exists the other characterization holds, or $\Omega \simeq \Sigma^{-1}$. So stability tells us that loops and suspension are not just in adjunction, but in *equivalence*. This means we study things *stable* under loops or suspension, hence the nomenclature.

[7] A way of motiviating this terminology is by realizing the kernel is the fiber of 0.

Example 2.4. The ∞-category $\mathcal{S}_*$ is not stable. There is no biproduct, and the initial object ($\emptyset$) and the terminal object ($\{*\}$) are not the same. To fix this, consider pointed spaces to get a zero object, and you can think about biproducts in your free time. We have (finite) limits and colimits in $\mathcal{S}_*$, however, $\Omega, \Sigma\colon S_* \to S_*$ are not equivalences. To see this, consider X any non-based space and $X_+ = X \amalg \{*\} \in \mathcal{S}_*$. Then loops $\Omega X_+ \simeq *$ because loops are pointed at $\{*\}$, and "undoing" loops means we can get back information about X, which should not be the case.

For example, we only have one map $S^2 \to S^1$, where $\pi_2(S^1) = 0$. However, suspending this results in a map $S^3 \to S^2$, including the Hopf fibration ($\pi_3(S^2) \simeq \mathbb{Z}$). Then desuspending kills the Hopf fibration.

The big idea is this: spaces are generally not stable, and to fix this lack of stability, we introduce spectra. How do we fix this? With spectra. The idea is to "make Ω invertible by force". Define the **∞-category of spectra** as the inverse limit

$$\mathsf{Sp} := \varprojlim\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right),$$

then the ∞-category of spectra has finite limits, and will respect $\Omega\colon \mathcal{S}_* \xrightarrow{\sim} \mathcal{S}_*$.

2.2 Spectra and stabilization (May 25)

Some observations on the definition of spectra:

- The category of spectra is a *limit* formed in the ∞-category of ∞-categories (Cat_∞), and hence a category. Each Ω is a functor between categories, where objects in Cat_∞ are the ∞-categories $\mathcal{S}_*$.
- Sp is a stable ∞-category by design, or $\Omega \simeq \Sigma^{-1}$.
- A **spectrum**, i.e. an object $X \in \mathsf{Sp}$, consists of spaces X_i associated to each $\mathcal{S}_*$ with equivalences $X_i \simeq \Omega X_{i+1}$ for all $i \in \mathbb{Z}$. If you've read *War and Peace* (the literature) on spectra, the equivalence $X_i \simeq \Omega X_{i+1}$ holds for **Ω-spectra**. There is an analogous notion of a **Σ-spectra** (or "pre-spectra"), defined as a collection of pointed spaces $X_i \in \mathcal{S}_*$ with pointed maps $\Sigma X_i \to X_{i+1}$. The difference between these two notions is we do not require the structure maps $\Sigma X_i \to X_{i+1}$ to be homotopy equivalences.

Example 2.5. Let $X \in \mathcal{S}_*$. Then we can form the **suspension spectrum** $\Sigma^\infty X \in \mathsf{Sp}$ as a Σ-spectrum, where $(\Sigma^\infty X)_i := \Sigma^i X$ for all $i \geq 0$, where $\Sigma^i X$ is the ith suspension of X. The structure maps are then given by

$$\Sigma\,(\Sigma^\infty X)_i \simeq \Sigma\left(\Sigma^i X\right) \simeq \Sigma^{i+1} X \simeq (\Sigma^\infty X)_{i+1}\,.$$

This is organically a Σ-spectrum and *not* an Ω-spectrum. Two example of suspension spectrum include the **zero spectrum** $0 \simeq \Sigma^\infty *$, and the **sphere spectrum** $S := \Sigma^\infty S^0$, so that $(\Sigma^\infty S)_i \simeq S^i$.

	Ω-spectra	Σ-spectra
PROS	∞-categorically meaningful	easier to find component spaces
CONS	more esoteric (complicated)	morphisms hard to define correctly

Figure 8: The pros and cons of Ω-spectra and Σ-spectra.

Ω-spectra arose naturally as a way to make ∞-categories stable, so in a sense they're "meaningful" in an ∞-categorical sense. However, this means explicit examples may be more esoteric, or not as nice as the examples we have for Σ-spectra.

On the other hand, morphisms are hard to define for Σ-spectra: consider a Σ-spectrum with component spaces X_i and structure maps. Naïvely, we could say a map between Σ-spectra sends spaces $X_i \to Y_i$ such that the corresponding squares commute. This holds for Ω-spectra, but the structure maps may not exist until we reach a certain i. For example, we have already seen the Hopf fibration that only exists after one iteration of suspension.

$$\begin{tikzcd}
\cdots \arrow[r] & X_{i-1} \arrow[r, red, "\exists ?"] \arrow[d] & X_i \arrow[r, red, "\exists ?"] \arrow[d] & X_{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y_{i-1} \arrow[r] & Y_i \arrow[r] & Y_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

So given these problems with Σ-spectra, how do we create Ω-spectra that contain the same information as Σ-spectra? To do this, start with a Σ-spectrum $(X_i, \Sigma X_i \to X_{i+1})_{i\geq 0}$, then use the adjunction $\Sigma \dashv \Omega$ to get structure maps $X_i \to \Omega_{i+1}$. To make these maps equivalences, we can iterate loops by applying the structure maps on the result of $X_i \to \Omega_{i+1}$, then taking the colimit, like so:

$$Y_i := \varinjlim \big(X_i \longrightarrow \Omega X_{i+1} \longrightarrow \Omega^2 X_{i+2} \longrightarrow \Omega^3 X_{i+3} \longrightarrow \cdots\big).$$

This gives rise to an Ω-spectrum $(Y_i, Y_i \simeq \Omega Y_{i+1})_{i\geq 0}$ by construction. The spaces Y_i and X_i will be different, *but they model the same spectrum!* In general the Y_i will be quite complicated, even if the X_i are simple— for example in the suspension spectrum, when $X_i = \Sigma^i X$, we have $Y_i = \varinjlim_j \Omega^j \Sigma^{j+i} X$.

Notation. For a spectrum $X \in \mathsf{Sp}$ and an Ω-spectrum model $(X_i, X_i \simeq \Omega X_{i+1})$ for X, we say $\Omega^\infty X := X_0$ is the **underlying (infinite loop) space of X**, and denote $\Omega^{\infty - n} X := X_n$. This is sensible notation because $\Omega^n(\Omega^{\infty-n}X) \simeq \Omega^n X_n \simeq X_0$, or $\Omega^\infty X$.

Note that $\Omega^\infty : \mathsf{Sp} \to \mathcal{S}_*$ is a functor, and is in adjunction with $\Sigma^\infty : \mathcal{S}_* \to \mathsf{Sp}$, where Σ^∞ preserves colimits and Ω^∞ preserves limits. Since Σ^∞ preserves colimits, we have an easy ∞-categorical way to describe Σ^∞. That is, $\Sigma^\infty : \mathcal{S}_* \to \mathsf{Sp}$ is purely determined by the fact that

(i) $\Sigma^\infty(S^0) \simeq S$, where S denotes the sphere spectrum,

(ii) Σ^∞ preserves colimits.

These conditions give rise to a unique functor Σ^∞ up to contractible space. Why does this equivalence work? There is also an adjuction between pointed spaces and spaces, where $\mathrm{oblv}: \mathcal{S}_* \to \mathcal{S}$ forgets the basepoint ($X \mapsto X$), and $+: \mathcal{S}_* \to \mathcal{S}$ adds a disjoint basepoint ($X \mapsto X \amalg \{*\}$). Then compose this adjuction with the adjunction $\Sigma^\infty \dashv \Omega^\infty$ to get an adjunction $\Sigma^\infty_+ \dashv \Omega^\infty_{\mathrm{oblv}}$ between spaces and spectra, like in the diagram below.

$$\begin{tikzcd}
\mathsf{Sp} \arrow[r, bend right, "\Omega^\infty"'] \arrow[rr, bend right=40, "\Omega^\infty_{\mathrm{oblv}}"'] \arrow[r, phantom, "\perp"] & \mathcal{S}_* \arrow[l, bend right, "\Sigma^\infty"'] \arrow[r, bend right, "\mathrm{oblv}"'] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, bend right, "+"'] \arrow[ll, bend right=40, "\Sigma^\infty_+"']
\end{tikzcd}$$

Then $\Sigma^\infty_+ : \mathcal{S} \to \mathsf{Sp}$ is characterized by preserving colimits and the fact that $\Sigma^\infty_+(*) \simeq S$.[8] Why do colimit preservation claims characterize functors uniquely? The reason is that $\mathcal{S}$ (resp $\mathcal{S}_*$) is freely generated by $*$ (resp S^0) under colimits, which is a universal property. One reason for why this is so simple is that we can present spaces

[8] An alternative notation is to denote $S[X] \simeq \Sigma^\infty_+(X)$, viewed as a sort of "spherical group ring". This notation is mostly used when X has additional structure.

as CW complexes— when you build a complex by attaching cells, you are actually giving a colimit formula for the space, where every object that your diagram is mapping to is contractible. How does this work? We start off with $S^0 \simeq * \amalg *$, a colimit (coproduct) of contractible things. Then $S^n \simeq \Sigma^n S^0$, where suspensions are also colimits. Finally, we create CW complexes by gluing cells together. Consider the following diagram:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ * \simeq D^n \arrow[r] & X' \end{tikzcd}$$

When we build CW complexes, we have a boundary map $S^n \to X$, where we have already built some complex X. Then we include the boundary $S^n \hookrightarrow D^n$, which is contractible, and form a *homotopy pushout* to glue in the cell. Alas, homotopy pushouts are also colimits, and we get that any CW complex is a colimit of contractible spaces.

That whole discussion was about Sp as the stabilization of $\mathcal{S}$. But we can stabilize any ∞-category $\mathcal{C}$ (as long as it has all finite limits)! The process is the same as for spectra:

(1) Take the **pointification** $\mathcal{C}_* := */\mathcal{C}$, the undercategory of $\mathcal{C}$ over the zero object.
(2) Perform **stabilization** by taking the inverse limit of loops, or by setting $\mathsf{Sp}(\mathcal{C}) := \varprojlim\big(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\big)$.

As stable categories are analogues of abelian categories, stabilization is an analogue of abelianization. The functor $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ preserves finite limits. The **universal property of stabilization** says this: $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ is terminal among finite limit preserving functors $F : \mathcal{D} \to \mathcal{C}$ with $\mathcal{D}$ stable. So stabilization is the "closest" you can get to $\mathcal{C}$ from a stable category $\mathcal{D}$. In particular, $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ iff $\mathcal{C}$ is stable.

The pressing question is, "what about Σ^∞?" The answer is that it sadly doesn't exist in general. If we assume $\mathcal{C}$ is nice enough, we can have an adjoint: what kind of structure do we need to add to $\mathcal{C}$? Suppose $\mathcal{C}$ is **presentable**, or has all limits and colimits and some additional set theoretic details[9]. If $\mathsf{Pr}^L \subseteq \mathsf{Cat}_\infty$ is the category of presentable categories with functors $F : \mathcal{C} \to \mathcal{D}$ preserving all colimits, which is equivalent to saying they admit a right adjoint.[10] The "L" in the name Pr^L is because the category consists of *left* adjoints if functors admit right adjoints. Similarly, define Pr^R to have the same objects, and preserve all limits or admit left adjoints. If we have adjoint functors $F \dashv G$, switching F and G gives an equivalence of ∞-categories $\mathsf{Pr}^L \simeq (\mathsf{Pr}^R)^{\mathrm{op}}$. Let $\mathsf{Pr}^{L,R}_{\mathrm{st}} \subseteq \mathsf{Pr}^{L,R}$ be the full subcategory spanned by stable presentable ∞-categories. Then

$$\begin{tikzcd} \mathsf{Pr}^L_{\mathrm{st}} \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathsf{Pr}^L \arrow[l, bend left, "\mathsf{Sp}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Pr}^R_{\mathrm{st}} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Pr}^R \arrow[l, bend right, "\mathsf{Sp}"'] \end{tikzcd}$$

where the unit[11] of the adjunction on the left is $\mathcal{C} \xrightarrow{\Sigma^\infty_+} \mathsf{Sp}(\mathcal{C})$, and the counit of the adjunction on the right is $\mathsf{Sp}(\mathcal{C}) \xrightarrow{\Omega^\infty} \mathcal{C}$. The unit and counit are in adjuction with each other by the equivalence $\mathsf{Pr}^L \simeq (\mathsf{Pr}^R)^{\mathrm{op}}$. The category of spaces $\mathcal{S}$ is a very presentable ∞-category— if we plug in $\mathcal{C} = \mathcal{S}$, we recover the adjuction $\Sigma \dashv \Omega$ that we discussed earlier.

[9] To avoid running into Russell's Paradox and similar things, mainly bounds on the cardinality, small category type stuff.
[10] The equivalence between preserving all colimits and admitting a right adjoint is the adjoint functor theorem.
[11] The **unit** of an adjuction $(L \dashv R) : X \underset{R}{\overset{L}{\rightleftarrows}} Y$ is the natural transformation $\eta : \mathrm{id}_X \to R \circ L$. Similarly, the **counit** of an adjuction is given by $\varepsilon : L \circ R \to \mathrm{id}_Y$.

Here's another pressing question: why are we inverting loops and not suspensions? Let's talk about history. In 1937, Freudenthal considered the identity $\Sigma X \xrightarrow{\mathrm{id}} \Sigma X$, and applied the adjunction $\Sigma \dashv \Omega$ to get a map $X \to \Omega\Sigma X$. We can iterate this operation like so,

$$X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \Omega^3\Sigma^3 X \longrightarrow \Omega^4\Sigma^4 X \longrightarrow \cdots$$

in a similar fashion to how we converted Σ-spectra to Ω-spectra. Then after applying the functor π_n, we get a sequence

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X) \longrightarrow \pi_{n+2}(\Sigma^2 X) \longrightarrow \pi_{n+3}(\Sigma^3 X) \longrightarrow \pi_{n+4}(\Sigma^4 X) \longrightarrow \cdots$$

from the isomorphism $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.

Freudenthal Suspension Theorem. *The sequence*

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X) \longrightarrow \pi_{n+2}(\Sigma^2 X) \longrightarrow \pi_{n+3}(\Sigma^3 X) \longrightarrow \pi_{n+4}(\Sigma^4 X) \longrightarrow \cdots$$

stabilizes, that is, $\pi_{n+i}(\Sigma^i X) \xrightarrow{\cong} \pi_{n+i+1}(\Sigma^{i+1} X)$ *for all* $i \gg 0$.

From the Freudenthal suspension theorem, we can talk about the stable homotopy groups $\pi_n^s(X) := \varinjlim_i \pi_{n+i}(\Sigma^i X)$, which gave rise to the field of stable homotopy theory. Spanier and Whitehead became interested this in 1953, and considered the colimit $\varinjlim\left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\right)$ in Cat_∞ (not with the modern language), and ran into the issue of not having enough limits, along with other categorical issues. However, Sp has all limits. Whitehead resolved this issue in 1962, with a two step process:

(1) Define $\mathcal{SW} := \varinjlim\left(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right)$, where $\mathcal{S}_*^{\mathrm{fin}}$ is a *finite* pointed category (e.g. a CW complex),
(2) Let $\mathsf{Sp} \simeq \mathrm{Ind}(\mathcal{SW})$, where we formally add in filtered colimits.

Frank Adams called the ind-completion procedure a "very esoteric concept" and "not very understandable", and so he constructed this notion for Σ-spectra in his book [Ada64]. The point is, the Spanier-Whitehead procedure works for any presentable ∞-category. Any presentable ∞-category $\mathcal{C}$ can be written as $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^\omega)$, where $\mathcal{C}^\omega$ is the category of compact objects (deformation retracts onto finite CW complexes) of $\mathcal{C}$. This motivates an analogous two step construction:

(1) Define $\mathcal{SW}(\mathcal{C}) := \varinjlim\left(\mathcal{C}_*^\omega \xrightarrow{\Sigma} \mathcal{C}_*^\omega \xrightarrow{\Sigma} \mathcal{C}_*^\omega \xrightarrow{\Sigma} \cdots\right)$,
(2) Let $\mathsf{Sp}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{SW}(\mathcal{C}))$.

Why are we doing this? Say $\mathcal{C}$ is a presentable ∞-category. Recall that

$$\mathsf{Sp}(\mathcal{C}) \simeq \underbrace{\varprojlim}_{\text{limit in } \mathsf{Cat}_\infty} \underbrace{\left(\cdots \xleftarrow{\overset{\text{right adj.}}{\Omega}} \underset{\text{pres.}}{\mathcal{C}_*} \xleftarrow{\overset{\text{right adj.}}{\Omega}} \underset{\text{pres.}}{\mathcal{C}_*} \xleftarrow{\overset{\text{right adj.}}{\Omega}} \underset{\text{pres.}}{\mathcal{C}_*}\right)}_{\text{diagram in } \mathsf{Pr}^R}.$$

There is an inclusion $\mathsf{Pr}^R \hookrightarrow \mathsf{Cat}_\infty$ that happens to preserve limits. So the limit in Cat_∞ could have also been computed in Pr^R. Applying the equivalence $\mathsf{Pr}^R \simeq (\mathsf{Pr}^L)^{\mathrm{op}}$, we get that

$$\mathsf{Sp}(\mathcal{C}) \simeq \underbrace{\varinjlim}_{\text{colimit in } \mathsf{Pr}^L} \underbrace{\left(\mathcal{C}_* \xrightarrow{\overset{\text{left adj}}{\Sigma}} \mathcal{C}_* \xrightarrow{\overset{\text{left adj}}{\Sigma}} \mathcal{C}_* \xrightarrow{\overset{\text{left adj}}{\Sigma}} \cdots\right)}_{\text{diagram in } \mathsf{Pr}^R}.$$

In this category, you can invert suspension by taking the colimit of the sequence of suspensions. But the inclusion $\mathsf{Pr}^L \hookrightarrow \mathsf{Cat}_\infty$ *does not preserve colimits!* This is the issue. To compute colimits in Pr^L, given a diagram $\phi \xrightarrow{F} \mathsf{Pr}^L$, we pass to compact objects by $\phi \xrightarrow{F} \mathsf{Pr}^L \xrightarrow{(-)^\omega} \mathsf{Cat}_\infty$, take the colimit in the ∞-category Cat_∞, and apply the functor $\mathrm{Ind}\colon \mathsf{Cat}_\infty \to \mathsf{Pr}^L$. For example,

$$\mathsf{Sp}(\mathcal{C}) \simeq \varinjlim_{\text{in } \mathsf{Pr}^L} \left(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \cdots\right) \simeq \mathrm{Ind}\Big(\varinjlim_{\text{in } \mathsf{Cat}_\infty} \left(\mathcal{C}_*^\omega \xrightarrow{\Sigma} \mathcal{C}_*^\omega \xrightarrow{\Sigma} \cdots\right)\Big) \simeq \mathrm{Ind}(\mathcal{SW}(\mathcal{C})).$$

Homotopy groups make sense in the land of spectra, but first let's review how they work in the land of spaces. Recall that for $X \in \mathcal{S}_*$, $\pi_n(X) := \pi_0(\Omega^n X)$ for all $n \geq 0$. In Sp, unlike $\mathcal{S}_*$, we also have Ω^{-1}, so Ω^n exists for $n < 0$. Then for $X \in \mathsf{Sp}$,

$$\pi_n(X) := \begin{cases} \pi_n(\Omega^\infty X), & \text{for } n \geq 0, \\ \pi_0(\Omega^{\infty - n} X), & \text{for } n < 0. \end{cases}$$

A unified formula is given by letting $\pi_n(X) \simeq \pi_0(\Omega^{\infty+n} X)$ for all $n \in \mathbb{Z}$.

Example 2.6. How do we compute homotopy groups of the suspension spectrum? We have

$$\pi_n(\Sigma^\infty X) \simeq \pi_n(\Omega^\infty \Sigma^\infty X) \simeq \pi_n\Big(\varinjlim_i \Omega^i \Sigma^i X\Big) \simeq \varinjlim_i \pi_n(\Omega^i \Sigma^i X) \simeq \varinjlim_i \pi_{n+i}(\Sigma^i X) \simeq \pi_n^s(X).$$

So stable homotopy groups of X are the same as the nth homotopy groups of suspension spectra. In particular, stable homotopy groups of spheres $\pi_n^s(S^0) \simeq \pi_n(S)$ are the same as homotopy groups of the sphere spectrum.

Claim. *For $X \in \mathsf{Sp}$, $\pi_n(X) \simeq \pi_0(\mathrm{Map}_{\mathsf{Sp}}(\Sigma^n(S), X))$ for all $n \in \mathbb{Z}$.*

Proof. Left as an exercise. The general idea is to note that $\Sigma^n(S) \simeq \Sigma^\infty(S^n)$, and use the adjunction $\Sigma \dashv \Omega$. ⊠

2.3 (Co)homology theories and the smash product (May 26)

Today we talk about some cohomology. Hopefully the notation will follow [May99].

Eilenberg-Steenrod Axioms. *A **cohomology theory** consists of functors $E^i : (\mathcal{CW}_*^{\text{fin}})^{\text{op}} \to \mathsf{Ab}$, satisfying*

*(i) **Homotopy invariance:** $f : X \xrightarrow{\sim} Y$ in $\mathcal{CW}_*^{\text{fin}}$ implies that $E^i(f)\colon E^i(Y) \xrightarrow{\cong} E^i(X)$ is an isomorphism.*

*(ii) **Additivity:** $E^i(X \vee Y) \simeq E^i(X) \oplus E^i(Y)$.*

*(iii) **Suspension:** $E^{i+1}(\Sigma X) \simeq E^i(X)$.*

*(iv) **Exactness:** For any $f : X \to Y$ in $\mathcal{CW}_*^{\text{fin}}$, the sequence*

$$E^i(\mathrm{cofib}(f)) \to E^i(Y) \to E^i(X)$$

is exact in Ab. If f is a CW-complex inclusion, then $\mathrm{cofib}(f) \simeq Y/X$.

There are some variants of cohomology, including

- Non-reduced cohomology, where $E^i_{\text{unred}}(X) := E^i(X \amalg \{*\})$,
- Cohomology of a pair $A \subseteq X$, where $E^i(X, A) := E^i(X/A)$,
- Homology $E_i : \mathcal{CW}_*^{\text{fin}} \to \mathsf{Ab}$, with similar axioms.

Another variant you may see in the axioms is the **long exact sequence**, which we claim we can get from the axioms (iii) and (iv). For a map $f : X \to Y$, we can iteratively take cofibers (or mapping cones) by adjoining the square on the right and taking the pushout.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{cofib}(f) \arrow[r] \arrow[d] \arrow[ur, phantom, "\color{red}\ulcorner"] & \color{red}\bullet \arrow[d] \\
& * \arrow[r] \arrow[ur, phantom, "\color{blue}\ulcorner"] & \color{blue}\bullet
\end{tikzcd}
$$

Then the red pushout (starting from X) has component maps factoring through Y and $\mathrm{cofib}(f)$, but the compositions are contractible, so this is precisely the suspension ${\color{red}\bullet} = \Sigma X$. Similarly, to get a map from ΣX to the blue pushout, the blue pushout factors through $\mathrm{cofib}(f)$ and $\Sigma(X)$, and is precisely ${\color{blue}\bullet} = \Sigma Y$. Then the map $\Sigma X \to \Sigma Y$ is just Σf, and by repeating we get our long exact sequence

$$
X \xrightarrow{f} Y \longrightarrow \mathrm{cofib}(f) \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma\mathrm{cofib}(f) \longrightarrow \Sigma^2 X \longrightarrow \cdots \tag{3}
$$

Applying suspension and exactness, we get our long exact sequence of abelian groups

$$
\begin{tikzcd}
\cdots \leftarrow E^i(X) & E^i(Y) \arrow[l, "E^i(f)"'] & E^i(\mathrm{cofib}(f)) \arrow[l] \\
E^{i-1}(X) \arrow[urr, rounded corners, to path={ -- ([xshift=-2ex]\tikztostart.west) |- ([yshift=-3ex]\tikztotarget.south) -| ([xshift=3ex]\tikztotarget.east) -- (\tikztotarget.east)}] & E^{i-1}(Y) \arrow[l, "E^{i-1}(f)"'] & E^{i-1}(\mathrm{cofib}(f)) \arrow[l] \\
E^{i-2}(X) \arrow[urr, rounded corners, to path={ -- ([xshift=-2ex]\tikztostart.west) |- ([yshift=-3ex]\tikztotarget.south) -| ([xshift=3ex]\tikztotarget.east) -- (\tikztotarget.east)}] & \cdots \arrow[l]
\end{tikzcd}
$$

The notation is a little unconventional since the arrows are pointing from right to left, but this emphasizes the idea that we get this long exact sequence by applying the contravariant functor E^i to Equation (3).

Example 2.7. Here are some standard examples of homology theories.

- For $A \in \mathsf{Ab}$, we can consider $H^i(X;A)$, the **ordinary cohomology with coefficients in A**. We can form ordinary cohomology in many ways, whether via singular or cellular or any approach really.
- **Topological complex K-theory** is the first cohomology theory that isn't "ordinary".[12] Define $KU^0(X)$ as the group completion of the isomorphism classes of complex vector bundles on X. **Bott periodicity** is a cool result that tells us $KU^0(\Sigma^2 X) \simeq KU^0(X)$, so we can extend KU^i by the suspension axiom to get

$$
KU^i(X) := \begin{cases} KU^0(X), & \text{for even } i, \\ KU^0(\Sigma X), & \text{for odd } i. \end{cases}
$$

- Another example of **topological real K-theory** (or *orthogonal K-theory*), where $KO^0(X)$ is obtained by replacing complex vector bundles with real vector bundles. Bott periodicity still holds, but in a more complicated fashion: $KO^0(\Sigma^8 X) \simeq KO^0(X)$, leading to the definition $KO^i(X) := KO^0(\Sigma^{-i \pmod 4} X)$.

Claim. *Suppose a cohomology theory $(E^i)_{i\in\mathbb{Z}}$ is **representable**, or $E^i(X) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(X, Y^i)$ for all $X \in \mathcal{S}_*^{\mathrm{fin}}$.*[13] *Then the collection $(Y_i \in \mathcal{S}_*)_{i\in\mathbb{Z}}$ is an Ω-spectrum.*

[12] Previously there was a dimension axiom uniquely characterizing ordinary cohomology theories, but that was dropped.
[13] This is actually always the case due to **Brown's representability theorem**.

Proof. Let X be some arbitrary space. Then by the definitions and the suspension axiom, we have

$$\mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(\Sigma X, Y_{i+1}) \simeq E^{i+1}(\Sigma X) \simeq E^i(X) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(X, Y_i).$$

Applying the adjunction between suspension and loops tells us that $\mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(\Sigma X, Y_{i+1}) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(X, \Omega Y_{i+1})$, which implies $Y_i \simeq \Omega Y_{i+1}$. ⊠

This tells us that we can always expect a spectrum out of a cohomology theory. We can go the other way by defining a cohomology theory $E_i \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(-, Y_i)$. Since each Y^i can be written as loops of something else, concatenation of loops induces a group structure on the E^i. This gives a bijection on objects of cohomology theories and hSp, but *not* an equivalence of categories, because of the existence of **phantom maps**. These are maps $f : Y \to Y'$ in Sp that induce 0 on the associated cohomology theories.

For $E \in \mathrm{Sp}$, we usually denote the associated cohomology theory by E^i. Since the homotopy category $\mathrm{h}\mathcal{S}_*$ is just π_0 of the mapping space, we can write

$$\begin{aligned} E^i(X) &\simeq \pi_0 \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^{\infty - i} E) \\ &\simeq \pi_0 \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, \Omega^{-i} E) \\ &\simeq \pi_i(\mathrm{Map}_{\mathcal{S}}(\Sigma^\infty X, E)) \end{aligned}$$

for all $i \in \mathbb{Z}$. Compare this to the fact that ordinary cohomology can be written as maps into the ith classifying space (a form of delooping) $H^i(X;A) \cong \pi_0 \mathrm{Map}_{\mathcal{S}_*}(X, B^i A)$. If HA denotes the spectrum associated to $H^*(-;A)$, then $\Omega^{\infty - i}(HA) = B^i A$. The last line says that we can write $E^i(X)$ as the ith homotopy group of the **mapping spectrum**, which is something we don't have the tools to talk about yet.

Let us discuss an ∞-categorical approach to homology. Suppose $\mathcal{C}$ is an ∞-category with all finite limits.

Definition 2.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is **excisive** if it takes pushouts to pullbacks, ie

$$\begin{tikzcd} C_{01} \arrow[r] \arrow[d] & C_1 \arrow[d] \\ C_0 \arrow[r] & C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \text{ in } \mathcal{C} \quad \rightsquigarrow \quad \begin{tikzcd} F(C_{01}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(C_1) \arrow[d] \\ F(C_0) \arrow[r] & F(C) \end{tikzcd} \text{ in } \mathcal{D}.$$

This is just like excision (or the Mayer-Vietoris property) in algebraic topology! Let $\mathcal{C} \approx \mathcal{S}$, $\mathcal{D} = \mathcal{S}_*$, and think of $E : \mathcal{C} \to \mathcal{S}_*$ as encoding a homology theory by $E_i(X) := \pi_i(E(X))$. Given a pushout square in $\mathcal{C}$ on the left, we get a pullback square in $\mathcal{S}_*$ on the right as seen below.

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad \overset{E \text{ is excisive}}{\rightsquigarrow} \quad \begin{tikzcd} E(U \cap V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(V) \arrow[d] \\ E(U) \arrow[r] & E(X) \end{tikzcd}$$

Another way to write this is that $E(U \cap V) \to E(U) \times E(V) \overset{f}{\underset{g}{\rightrightarrows}} E(X)$ is a homotopy equalizer upon passing to homotopy groups. An equalizer in abelian groups is the kernel of a difference, so we get a long exact sequence

$$\cdots \to \pi_{i+1} E(X) \longrightarrow \pi_i E(U \cap V) \longrightarrow \pi_i E(U) \oplus \pi_i E(V) \xrightarrow{f-g} \pi_i E(X) \longrightarrow \pi_{i-1} E(U \cap V) \to \cdots$$

which is precisely the Mayer-Vietoris sequence when we think of $\pi_i E(X) = E_i(X)$!

Define $\mathsf{Exc}_*(\mathcal{C},\mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{C},\mathcal{D})$ to be the full subcategory of excisive functors $F : \mathcal{C} \to \mathcal{D}$ such that $F(*) \simeq *$ (the functors are reduced). Observe that for $F \in \mathsf{Exc}_*(\mathcal{C},\mathcal{D})$ and any $C \in \mathcal{C}$,

$$\begin{tikzcd} C \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma C \end{tikzcd} \text{ in } \mathcal{C} \quad \rightsquigarrow \quad \begin{tikzcd} F(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & F(\Sigma C) \end{tikzcd} \text{ in } \mathcal{D},$$

i.e., for a pointed excisive functor F, we have $F(C) \xrightarrow{\sim} \Omega F(\Sigma C)$.

Proposition 2.1. *For any ∞-category $\mathcal{C}$ with finite limits and colimits,* $\mathsf{Sp}(\mathcal{C}) \simeq \mathsf{Exc}_*(\mathcal{S}_*^{\mathrm{fin}},\mathcal{C})$.

Proof. Define functors $\mathsf{Exc}_*(\mathcal{S}_*^{\mathrm{fin}},\mathcal{C}) \xrightarrow{f_n} \mathcal{C}_*$, where $F \mapsto F(S^n)$. Since $\Omega F(S^{n+1}) \simeq F(S^n)$, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Exc}_*(\mathcal{S}_*^{\mathrm{fin}},\mathcal{C}) \arrow[d] \arrow[drrr, bend left, "f_2"] \arrow[drrrr, bend left, "f_1"] \arrow[drrrrr, bend left, "f_0"] & & & & & \\ \mathsf{Sp}(\mathcal{C}) \arrow[r] & \cdots \arrow[r, "\Omega"] & & \mathcal{C}_* \arrow[r, "\Omega"] & \mathcal{C}_* \arrow[r, "\Omega"] & \mathcal{C}_* \end{tikzcd}$$

We can get a functor by passing to the limit of the sequence, which is precisely how we get $\mathsf{Sp}(\mathcal{C})$. The blue functor is an equivalence, since $\mathcal{S}_*^{\mathrm{fin}}$ is built by finite colimits from S^n. In particular, the excisive functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ corresponding to $X \in \mathsf{Sp}$ is given by $F(S^n) \simeq \Omega^{\infty - n} X$. ⊠

So stabilization of an ∞-category $\mathcal{C}$ has as objects ∞-categorical homology theories, valued in $\mathcal{C}$.

Now we talk about the smash product. We want an operation $\mathsf{Sp} \times \mathsf{Sp} \xrightarrow{\otimes} \mathsf{Sp}$ called the **smash product of spectra**, satisfying

(i) $\Sigma^\infty : \mathcal{S}_* \to \mathsf{Sp}$ where $\wedge \mapsto \otimes$, i.e., $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty(X) \otimes \Sigma^\infty(Y)$. Recall that $\wedge$ denotes the standard smash product of pointed spaces, where $X \wedge Y = (X \times Y)/(X \vee Y)$.

(ii) The sphere spectrum $S \in \mathsf{Sp}$ is the unit for $\otimes$, i.e., $S \otimes X \simeq X \otimes S \simeq X$ for all $X \in \mathsf{Sp}$. This is analogous to the idea that S^0 is the unit for the standard smash product, since $S^0 \wedge X \simeq X$.

(iii) The smash product $\otimes$ commutes with colimits in each variable, i.e.,

$$X \otimes \left(\varinjlim_i Y_i \right) \simeq \varinjlim_i (X \otimes Y_i) \quad \text{and} \quad \left(\varinjlim_i X_i \right) \otimes Y \simeq \varinjlim (X_i \otimes Y).$$

This tells us the smash product behaves more like the derived tensor product of abelian groups or modules, rather than the original one.

(iv) The smash product $\otimes$ makes Sp into a **symmetric monoidal ∞-category**, i.e.,

$$X \otimes Y \simeq Y \otimes X \quad \text{and} \quad (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$$

up to coherent homotopy.

Another note on the smash product of pointed spaces. You can also write $X \wedge Y$ as $\mathrm{cofib}(X \vee Y \hookrightarrow X \times Y)$. The "moral reason" for why smash products are useful and important is that $\mathrm{Map}_{\mathcal{S}_*}(X \wedge Y, Z) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{Map}_{\mathcal{S}_*}(Y,Z))$. Furthermore, $S^0 \wedge X \simeq X$, $S^1 \wedge X \simeq \Sigma X$, and $S^n \wedge X \simeq \Sigma^n X$.

So how do we create the smash product? The idea is to use the category Pr^L. For $\mathcal{C}, \mathcal{D}$ presentable, we can form the subcategories $\mathsf{Fun}^L(\mathcal{C}, \mathcal{D}), \mathsf{Fun}^R(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}(\mathsf{C}, \mathsf{D})$ of left and right adjoint functors respectively. Define the **Lurie tensor product** on Pr^L by $\mathcal{C} \otimes \mathcal{D} := \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. There is a very nice characterization of the Lurie tensor product by $\mathsf{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}^L(\mathcal{C}, \mathsf{Fun}^L(\mathcal{D}, \mathcal{E}))$. We can identify this with the full subcategory $\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$, where functors $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserve colimits in each variable. This can be viewed as an analogy with the tensor product of modules, where $\mathrm{Hom}_R(M \otimes_R^{\heartsuit} N, L) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$[14] can be identified with $\mathrm{Hom}_{\mathsf{Set}}(M \times N, L)$, with functions $f : M \times N \to L$ that are R-linear in each variable.

Example 2.8. Let us discuss examples of the Lurie tensor product. When we Lurie tensor a space with the category of spaces, we get

$$\mathcal{S} \otimes \mathcal{C} \simeq \mathsf{Fun}^R(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \simeq \mathsf{Fun}^L(\mathcal{S}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C}.$$

where the third equivalence follows due to the universal property of $\mathcal{S}$. What about pointed spaces? The Lurie tensor is symmetric monoidal, so

$$\mathcal{S}_* \otimes \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S}_* \simeq \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*) \simeq \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_* \simeq (\mathcal{C} \otimes \mathcal{S})_* \simeq \mathcal{C}_*.$$

The most exciting example is what happens when you tensor with spectra. We have

$$\begin{aligned}
\mathsf{Sp} \otimes \mathcal{C} &\simeq \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathsf{Sp}) \\
&\simeq \mathsf{Fun}^R\left(\mathcal{C}^{\mathrm{op}}, \varprojlim\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right)\right) \\
&\simeq \varprojlim\left(\cdots \xrightarrow{\Omega} \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*) \xrightarrow{\Omega} \mathsf{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*)\right) \\
&\simeq \varprojlim\left(\cdots \xrightarrow{\Omega} \mathcal{C} \otimes \mathcal{S}_* \xrightarrow{\Omega} \mathcal{C} \otimes \mathcal{S}_*\right) \\
&\simeq \varprojlim\left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\right) \\
&\simeq \mathsf{Sp}(\mathcal{C}).
\end{aligned}$$

The takeaway is that for $\mathcal{C}$ presentable, we can write $\mathcal{C} \simeq \mathcal{S} \otimes \mathcal{C}$, $\mathcal{C}_* \simeq \mathcal{S}_* \otimes \mathcal{C}$, and $\mathsf{Sp}(\mathcal{C}) \simeq \mathsf{Sp} \otimes \mathcal{C}$. Consider the pointed and stable full subcategories $\mathsf{Pr}^L_*, \mathsf{Pr}^L_{\mathrm{st}} \subseteq \mathsf{Pr}^L$ respectively, spanned by the pointed (resp stable) presentable ∞-categories. A way of rephrasing the relations above is then

$$\begin{tikzcd}
\mathsf{Pr}^L_* \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathsf{Pr}^L \arrow[l, bend left, "(-)^* \simeq \mathcal{S}_* \otimes -"]
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathsf{Pr}^L_{\mathrm{st}} \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathsf{Pr}^L \arrow[l, bend left, "\mathsf{Sp}(-) \simeq \mathsf{Sp} \otimes -"]
\end{tikzcd}$$

Both subcategories are closed under the Lurie tensor product! Consequently, $\mathcal{S} \in \mathsf{Pr}^L$, $\mathcal{S}_* \in \mathsf{Pr}^L_*$, and $\mathsf{Sp} \in \mathsf{Pr}^L_{\mathrm{st}}$ are the symmetric monoidal units.

Lemma 2.1. *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal ∞-category. Then the unit object $\mathbb{1} \in \mathcal{C}$ admits a canonical commutative algebra structure, i.e., $\mathbb{1} \in \mathsf{CAlg}(\mathcal{C})$.*

Proof. The structure of $A \in \mathsf{CAlg}(\mathcal{C})$ roughly consists of:

(i) the underlying object $A \in \mathcal{C}$,

(ii) "multiplication" $\mu : A \otimes A \to A$ in $\mathcal{C}$,

(iii) coherence data so that μ is associative, unital, and commutative.

[14] Here, $\otimes^{\heartsuit}$ is the standard tensor product that we know and love, nothing derived or anything like that.

This coherence data is also called $\mathbb{E}_\infty$. For $A \simeq \mathbb{1}$, we have $\mu\colon \mathbb{1}\otimes\mathbb{1} \xrightarrow{\sim} \mathbb{1}$, and coherence data is also determined contractibly from the fact that $\mathbb{1}$ is a unit, and the coherence data for $\otimes$ being symmetric monoidal. ⊠

Example 2.9. $\mathsf{CAlg}(\mathsf{Cat}_\infty)$ is precisely the symmetric monoidal ∞-categories. Similarly, $\mathsf{CAlg}(\mathsf{Cat})$ is just the symmetric monoidal categories, and $\mathsf{CAlg}(\mathsf{Vect}_K)$ consists of just the K-algebras. $\mathsf{CAlg}(\mathsf{Pr}^L)$ with respect to the Lurie tensor product gives the symmetric monoidal ∞-categories such that $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ commutes with colimits in each variable.

From Lemma 2.1, we get that for $\mathcal{S}\in\mathsf{CAlg}(\mathsf{Pr}^L)$ the operation is the Cartesian product, on pointed spaces $\mathcal{S}_*\in\mathsf{CAlg}(\mathsf{Pr}^L_*)$ we get the smash product for pointed spaces, and for spectra $\mathsf{Sp}\in\mathsf{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ we get the **smash product** $X\otimes Y$. This is how we define the smash product, and satisfies (iii) and (iv) of the "conditions we want" for smash product by construction! By design, the maps $\mathcal{S}\xrightarrow{(-)_*}\mathcal{S}_*$, $\mathcal{S}_*\xrightarrow{\Sigma^\infty}\mathsf{Sp}$ are symmetric monoidal functors. For $(-)_*$, this means $(X\times Y)_+\simeq X_+\wedge Y_+$, where $+$ is the operation of adding a distinguished basepoint. To check this, by definition we have

$$\begin{aligned} X_+\wedge Y_+ &= \frac{X_+\times Y_+}{X_+\vee Y_+} = \frac{(X\times Y)\cup\cancel{(X\times\{y\})}\cup\cancel{(\{x\}\cup Y)}\cup(\{x\}\times\{y\})}{\cancel{(X\times\{y\})}\cup\cancel{(\{x\}\times Y)}} \\ &= X\times Y\amalg\{\mathrm{pt}\} \\ &= (X\times Y)_+. \end{aligned}$$

The functor Σ^∞ is symmetric monoidal with respect to $\wedge$ and $\otimes$, and indeed, $\Sigma^\infty(X\wedge Y)\simeq\Sigma^\infty(X)\otimes\Sigma^\infty(Y)$. Furthermore, the unit for $\otimes$ on Sp is $\Sigma^\infty(S^0)\simeq S$, as desired.

2.4 $\mathbb{E}_\infty$ structures, loop spaces, and group completion (May 27)

Last time, we described the smash product of spectra $\otimes$, making Sp into a symmetric monoidal ∞-category, where

(i) $X\otimes Y\simeq Y\otimes X$, $X\otimes(Y\otimes Z)\simeq(X\otimes Y)\otimes Z$ up to coherent homotopy,

(ii) $S\otimes X\simeq X$, or the sphere spectrum S is the unit for $\otimes$.

Lemma 2.1 tells us that the unit object of a symmetric monoidal ∞-category admits a commutative algebra structure, or has a multiplication operation. Then $S\in\mathsf{CAlg}(\mathsf{Sp})$, i.e., S is an $\mathbb{E}_\infty$**-ring spectrum**. Here, $\mathbb{E}_\infty$ is a synonym for a commutative algebra. Let us discuss what $\mathbb{E}_n$-algebra objects. The setting to talk about $\mathbb{E}_n$-algebra objects are symmetric monoidal ∞-categories.

$(\mathsf{Cat}_\infty,\times,*)$	$(\mathsf{Cat},\times,*)$	$(\mathsf{Fin},\amalg,\emptyset)$	$(\mathsf{Fin}^{\simeq},\amalg,\emptyset)$
$(\mathsf{Pr}^L,\overset{\mathrm{Lurie}}{\otimes},\mathcal{S})$	$(\mathcal{S},\times,*)$	$(\mathsf{Set},\times,*)$	
$(\mathsf{Pr}^L_*,\overset{\mathrm{Lurie}}{\otimes},\mathcal{S}_*)$	$(\mathcal{S}_*,\wedge,S^0)$		$(\mathcal{D}(R),\otimes^L_R,R)$
$(\mathsf{Pr}^L_{\mathrm{st}},\overset{\mathrm{Lurie}}{\otimes},\mathsf{Sp})$	$(\mathsf{Sp},\otimes,S)$	$(\mathsf{Ab},\otimes^{\heartsuit},\mathbb{Z})$	$(\mathsf{Mod}^{\heartsuit}_R,\otimes^{\heartsuit}_R,R)$

Figure 9: Examples of symmetric monoidal ∞-categories.

The first column has our standard examples, and the second column consists of their units in CAlg. The third column has their analogues in 1-categories, and the fourth column consists of their generalizations (entry 34 is the derived category of left tensor products).

Definition 2.6. An $\mathbb{E}_\infty$**-algebra**, or **commutative algebra object** in $\mathcal{C}$ is a symmetric monoidal functor $A^\otimes\colon\mathsf{Fin}\to\mathcal{C}$, where $\amalg\mapsto\otimes$. They form an ∞-category $\mathsf{CAlg}(\mathcal{C}):=\mathsf{Fun}^\otimes(\mathsf{Fin},\mathcal{C})$.

Let's unpack this definition. We have $A := A^{\otimes}(*) \in \mathcal{C}$, the *underlying object* of the $\mathbb{E}_\infty$-algebra. For a set with n elements, we have $A^{\otimes}(\{1, \cdots, n\}) \simeq A^{\otimes}(* \amalg \cdots \amalg *) \simeq A^{\otimes}(*) \otimes \cdots \otimes A^{\otimes}(*) \simeq A^{\otimes n}$ by monoidality. We must have $A^{\otimes}(\emptyset) \simeq \mathbb{1} \in \mathcal{C}$, since symmetric monoidal functors take units to units. So now we know what the functor $A^{\otimes}$ does to every object. Now let's consider functoriality.

- For $\emptyset \to *$ in Fin, this gets sent to $\mathbb{1} \simeq A^{\otimes}(\emptyset) \xrightarrow{1} A^{\otimes}(*) \simeq A$ in $\mathcal{C}$.
- $* \amalg * \to *$ in Fin gets sent to $A \otimes A \simeq A^{\otimes}(* \amalg *) \xrightarrow{\mu} A^{\otimes}(*) \simeq A$.
- Permutations $\{1, \cdots, n\} \underset{\simeq}{\xrightarrow{\sigma}}$ get sent to a self-equivalence $A^{\otimes n} \xrightarrow{\sim} A^{\otimes n}$, where we can interpret this as an equivalence $a_1 \otimes \cdots \otimes a_n \simeq a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$ up to coherent homotopy.

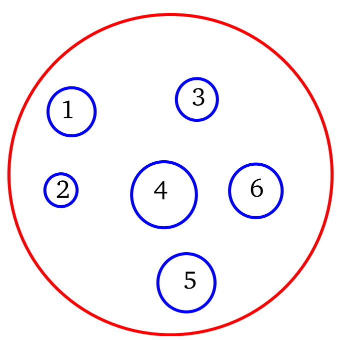
Together these classify all maps in Fin, since injections are built by chaining maps like $\emptyset \to *$, surjections by composing projections $* \amalg * \to *$, and bijections are just permutations.

Example 2.10. For $\mathbb{1} \in \mathsf{CAlg}(\mathcal{C})$ from Lemma 2.1, we can simply consider $\mathsf{Fin} \xrightarrow{\mathbb{1}^{\otimes}} \mathcal{C}$ that sends any finite set $I \in \mathsf{Fin}$ to $\mathbb{1} \in \mathcal{C}$. This is the only constant symmetric monoidal functor, since symmetric monoidality requires sending units to units.

Example 2.11. Commutative algebra structures are present everywhere— referring to Figure 9, Cat_∞ has commutative algebra objects as symmetric monoidal ∞-categories, in Pr^L they are symmetric monoidal ∞-categories such that the monoidal operation preserves colimits in each variable (similarly for Pr^L_* and $\mathsf{Pr}^L_{\mathrm{st}}$), in Cat they are the usual symmetric monoidal categories, in $\mathcal{S}$ we get commutative monoid spaces ($\mathbb{E}_\infty$-spaces), in Set we get commutative monoids (in Fin they must be finite), in Ab we get rings, in $\mathsf{Mod}_R^{\heartsuit}$ we get R-algebras, and in $\mathcal{D}(R)$ we get R-signature rings.

This idea of commutative algebra structures is the strongest version of coherent commutativity. From the perspective of operads (n-fold structures you can put on an object), $\mathbb{E}_\infty$-algebras have a canonical algebra structure over any operad. For a weaker notion, we replace Fin with a different ∞-category.

Example 2.12. Consider the **(n-dimensional framed) little disk ∞-category** denoted by $\mathsf{Disk}_n^{\mathrm{fr}}$, where objects are disjoint unions of n-dimensional disks indexed over a finite set I. Maps between disjoint unions of disks (indexed over J) consist of frame embeddings of the disjoint union $(D^n)^{\amalg I}$ into a bigger disk. Precisely, $D^n \amalg \cdots \amalg D^n = (D^n)^{\amalg I} \in \mathrm{ob}(\mathsf{Disk}_n^{\mathrm{fr}})$, and $\mathrm{Map}_{\mathsf{Disk}_n^{\mathrm{fr}}}((D^n)^{\amalg I}, (D^n)^{\amalg J}) := \mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n)^{\times J}$.



$\in \mathrm{Map}_{\mathsf{Disk}_2^{\mathrm{fr}}}((D^2)^{\amalg 6}, D^2)$

Figure 10: Six small disks embedded in a large 2-disk, an element of $\mathrm{Map}_{\mathsf{Disk}_2^{\mathrm{fr}}}((D^2)^{\amalg 6}, D^2)$.

The space $\mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n)^{\times J}$ can be topologized in the usual way, and that homotopy type is what the spaces of maps is. The frames just mean you can't rotate a disk inside a bigger one. The symmetric monoidal structure by $\amalg$ then tells us that $(D^n)^{\amalg I} \amalg (D^n)^{\amalg J} = (D^n)^{\amalg (I \cup J)}$.

A variant of this is obtained by shrinking the radii of the disks to zero, and expanding the larger disk to ∞, resulting in an embedding of points in $\mathbb{R}^n$. This homotopy equivalent space is the *configuration space* of however many points in $\mathbb{R}^n$, denoted by $\mathrm{Conf}_I(\mathbb{R}^n)^{\times J} \simeq \mathrm{Map}_{\mathsf{Disk}_n^{\mathrm{fr}}}((D^n)^{\amalg I}, (D^n)^{\amalg J})$. A good example to keep in mind is $n = 1$, where $\mathrm{Map}_{\mathsf{Disk}_1^{\mathrm{fr}}}((D_1)^{\amalg k}, D_1) \simeq \mathrm{Conf}_k(\mathbb{R})$ is the set of configurations of k points in $\mathbb{R}$, which is equivalent to the permutation group S_k.

Definition 2.7. An $\mathbb{E}_n$-algebra object in $\mathcal{C}$ is a symmetric monoidal functor $A^{\otimes} \colon \mathsf{Disk}_n \to \mathcal{C}$. They form an ∞-category $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) := \mathsf{Fun}^{\otimes}(\mathsf{Disk}_n, \mathcal{C})$.

Let's unpack this definition. We have an underlying object $A := A^{\otimes}(D_n) \in \mathcal{C}$, where $A^{\otimes}((D^n)^{\amalg I}) \simeq A^{\otimes I}$. Any frame embedding $D_n \amalg D_n \xhookrightarrow{i} D_n$ gives rise to a multiplication map $A \otimes A \xrightarrow{\mu_i} A$, which are homotopy commutative "in $\mathbb{R}^n$ directions". For example, when $n = 1$ we get associativity since you cannot "shift" two points past each other. Explicitly, for $\sigma \in S_k$ we get a map $A^{\otimes k} \xrightarrow{\mu_\sigma} A$, where $a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$. For $n = 2$, intuitively we have homotopy classes π_0 of spaces, and for higher dimension there are higher degrees of freedom.

Example 2.13. In the 2-category $(\mathsf{Cat}, \times, *)$, there are three notions of algebras:

(1) Monoidal categories, which are $\mathbb{E}_1$-structures,

(2) Braided monoidal categories, which are $\mathbb{E}_2$-structures,

(3) Symmetric monoidal categories, which are $\mathbb{E}_3 = \mathbb{E}_4 = \cdots = \mathbb{E}_\infty$-structures.

In general, if we have an n-category, we can see the difference from $\mathbb{E}_1$ all the way up to $\mathbb{E}_n$, but $\mathbb{E}_{n+1} = \mathbb{E}_{n+2} = \cdots$ and so on.

Something you might know is that π_n is abelian for $n \geq 2$. This is because for 1-categories, the $\mathbb{E}_1$ structure is different while $\mathbb{E}_2 = \mathbb{E}_3 = \cdots$. The usual proof of swapping around blocks from algebraic topology is an incarnation of the $\mathbb{E}_2$ structure, a functor from the 2-disks.

Example 2.14. An example of $\mathbb{E}_n$-structures for all n are the loopspaces from topology. The claim is that for $X \in \mathcal{S}_*$, we have $\Omega^n X \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S})$. To specify functoriality, we want to send $\mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n) \to \mathrm{Map}_{\mathcal{S}_*}((\Omega^n X)^I, \Omega^n X)$. First we go from frame embeddings of disks to a pointed map of spaces: we do this by starting with an embedding like in Figure 10, then quotienting out the region bound by the red disk, excluding the blue disks. This results in a wedge $D^n/\partial D^n \vee \cdots \vee D^n/\partial D^n \simeq S^n \vee \cdots \vee S^n$, so this gives a map $\mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n) \to \mathrm{Map}_{\mathcal{S}_*}(S^n, (S^n)^{\vee I})$. Then we apply the functor $\mathrm{Map}_{\mathcal{S}_*}(-; X)$, and since maps are contravariant in the first factor, the claim is that we get $\mathrm{Map}_{\mathcal{S}_*}((\Omega^n X)^I, \Omega^n X)$.

$$\begin{tikzcd}
\mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{S}_*}((\Omega^n X)^I, \Omega^n X) \\
\mathrm{Map}_{\mathcal{S}_*}(S^n, (S^n)^{\vee I}) \arrow[ur, bend right, "\mathrm{Map}_{\mathcal{S}_*}(-;X)"'] &
\end{tikzcd}$$

In the first factor, applying $\mathrm{Map}_{\mathcal{S}_*}(-; X)$ to S^n gives the maps from $S^n \to X$, which is precisely Ω^n. For the second factor, we have $(\Omega^n X)^I \simeq \mathrm{Map}_{\mathcal{S}_*}(S^n, X)^I \simeq \mathrm{Map}_{\mathcal{S}_*}((S^n)^{\vee I}, X)$. Applying I is a limit, and a limit on the outside becomes a colimit on the left factor, where the coproduct of pointed spaces is the wedge sum.

This actually turns out to be a generic example of $\mathbb{E}_n$ spaces.

Definition 2.8. An $\mathbb{E}_n$-space $X \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S})$ (for $1 \leq n \leq \infty$) is **group-like** if the monoid $\pi_0(X)$ in Set is a group. The group-like $\mathbb{E}_n$-spaces form a full subcategory $\mathsf{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S}) \subseteq \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S})$.

Theorem 2.1 (Boardman-Voigt and May)**.** *The map $\Omega^n \colon \mathcal{S}_*^{\geq n} \xrightarrow{\sim} \mathsf{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S})$ is an equivalence.*

Here, $\mathcal{S}_*^{\geq n}$ just means homotopy groups start existing from index n. This implies any $\mathbb{E}_n$-structure (at least group-like) comes from loop concatenation. Indeed, $\Omega^n X$ is group-like, since $\pi_0(\Omega^n X) \simeq \pi_n(X)$. Furthermore, for the n-connected cover $\tau_{\geq n} X \to X$ in $\mathcal{S}_*$, we have $\Omega^n(\tau_{\geq n} X) \xrightarrow{\sim} \Omega^n X$. This also implies the existence of an inverse functor $B^n \colon \mathsf{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S}) \to \mathcal{S}_*^{\geq n}$, which is the n-fold delooping functor (alternatively the n-fold classifying space). We have already seen this before when talking about cohomology, where $\Omega^n B^n G \simeq G$. Define $B^n \Omega^n X \simeq \tau_{\geq n} X$.

$$\begin{tikzcd}
\mathsf{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S}) \arrow[rr, bend right, hook] \arrow[rr, phantom, "\perp"] && \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \arrow[ll, bend right, "(-)^{\mathrm{gp}}"']
\end{tikzcd}$$

Finally, we have **group completion**. It is defined as the left adjoint to the inclusion of group like $\mathbb{E}_n$-spaces into all $\mathbb{E}_{n-1}$-spaces[15], which can be represented as ΩBX, or loops on the classifying space. How do we relate $\mathbb{E}_n$-algebras with $\mathbb{E}_{n-1}$-algebras? We can embed $D^n \hookrightarrow D^{n+1}$, inducing a symmetric monoidal functor $\mathsf{Disk}^{\mathrm{fr}}_n \to \mathsf{Disk}^{\mathrm{fr}}_{n+1}$. Given $A^{\otimes} \in \mathsf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C})$, compose it with our previous map on the left to get a symmetric monoidal functor $\mathsf{Disk}^{\mathrm{fr}}_n \to \mathsf{Disk}^{\mathrm{fr}}_{n+1} \xrightarrow{A^{\otimes}} \mathcal{C} \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$. This is how we get an underlying $\mathbb{E}_n$-algebra from an $\mathbb{E}_{n+1}$-algebra.

Warning. The functor $\mathsf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) \to \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is *not* fully faithful!

Proposition 2.2. *There is an equivalence of ∞-categories*

$$\mathsf{CAlg}(\mathcal{C}) \simeq \varprojlim\big(\cdots \to \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathsf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) \to \cdots \to \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})\big).$$

Proof. As usual this is just a sketch, but we want to compare mapping spaces

$$\mathrm{Map}_{\mathsf{Disk}^{\mathrm{fr}}_n}((D^n)^{\amalg I}, (D^n)^{\amalg J}) \simeq \mathrm{Emb}^{\mathrm{fr}}((D^n)^{\amalg I}, D^n)^{\times J}$$

and

$$\mathrm{Hom}_{\mathsf{Fin}}(I, J) \simeq \{f : I \to J\} \simeq \{I_j := f^{-1}(j) \subseteq I \text{ for all } j \in J\} \simeq \{I' \subseteq I\}^{\times J}.$$

To relate these two notions, we can write $(D^n)^{\amalg I} = (D^n)^{\amalg I'} \amalg (D^n)^{\amalg (I - I')}$. As $n \to \infty$, we "push out" the last complement into higher and higher dimensions. ⊠

The big idea is that an $\mathbb{E}_\infty$-algebra structure is isomorphic to a compatible family of $\mathbb{E}_n$-algebra structures for all $1 \leq n < \infty$. There is a similar result, **Dunn additivity**, which says that $\mathsf{Alg}_{\mathbb{E}_{n+m}}(\mathcal{C}) \simeq \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Alg}_{\mathbb{E}_m}(\mathcal{C}))$. So an $\mathbb{E}_n$ algebra is the same as $\mathbb{E}_1$ algebras nested n times inside each other.

Theorem 2.2 (May Recognition Principle)**.** *The functor $\Omega^\infty : \mathsf{Sp}^{\mathrm{cn}} \xrightarrow{\sim} \mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S})$ from connective spectra into $\mathbb{E}_\infty$ spaces is an equivalence.*

Let's unpack this. A spectrum $X \in \mathsf{Sp}$ is **connective** if $\pi_i(X) = 0$ for all $i < 0$. Spectra can have negative homotopy groups so this is a reasonable condition, but spaces can't, so this is saying that connective spectra are somewhat like spaces. The $\mathbb{E}_\infty$-structure on $\Omega^\infty X$ is supposed to be viewed as the "addition" on X, via the analogy of abelian groups and spectra in general.

Proof of Theorem 2.2. The idea is to use our equivalence $\Omega^n \colon \mathcal{S}_*^{\geq n} \xrightarrow{\sim} \mathsf{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S})$. As $n \to \infty$, via Proposition 2.2 $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S})$ will go to $\mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S})$. What does the left hand side to go? We can view this as the limit

$$\varprojlim\big(\cdots \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\big) \simeq \mathsf{Sp}^{\mathrm{cn}},$$

and so the Boardman-Voigt May equivalence gives rise to the May recognition principle. ⊠

[15] Not sure if it's $\mathbb{E}_n$ or $\mathbb{E}_{n-1}$...

The takeaway is that infinite loop spaces (that is, connected spaces) are equivalent to group-like $\mathbb{E}_\infty$-spaces. So as group-like $\mathbb{E}_n$-algebra structures arise from n-fold loops, a group-like $\mathbb{E}_\infty$-algebra structure comes from a connective spectrum.

The inclusion $\mathsf{Sp}^{\mathrm{cn}} \hookrightarrow \mathsf{Sp}$ has an adjoint: for any $X \in \mathsf{Sp}$, we can form the connected cover $\tau^{\geq 0}X \to X$. So the inclusion is the left adjoint, and the cover is the right adjoint. A variant of this is to consider $X \in \mathsf{Sp}_{\geq n}$ where $\pi_i(X) = 0$ for all $i < n$ (a t-structure on Sp), and taking the colimit we have $\varinjlim_{n\to\infty} \tau_{\geq -n}X \simeq X$.

Theorem 2.3 (Barrat-Priddy-Quillen)**.** *There is an equivalence* $\Omega^\infty S \simeq (\mathsf{Fin}^{\simeq})^{\mathrm{gp}}$.

This is the idea of counting sheep that we were talking about at the beginning of the minicourse. The claim is that both sides satisfy the universal property of the free group-like $\mathbb{E}_\infty$-space on a single generator. Finite sets up to bijection (thought of as $\mathsf{Fin}^{\simeq} \simeq \coprod_{n\geq 0} B\Sigma_n$) is the free $\mathbb{E}_\infty$-space with a single generator.

Proof. Use that for any symmetric monoidal ∞-category $\mathcal{C}$ with colimits, the forgetful functor $\mathsf{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{oblv}} \mathcal{C}$ admits a left adjoint Sym^*, the free $\mathbb{E}_\infty$-algebra functor.

$$\begin{tikzcd} \mathsf{CAlg}(\mathcal{C}) \arrow[rr, bend right, "\mathrm{oblv}"'] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend right, "\mathrm{Sym}^*"'] \end{tikzcd}, \qquad \mathrm{Sym}^*(\mathcal{C}) \simeq \coprod_{n\geq 0} (C^{\otimes n})/\Sigma_n$$

The claim is that $\mathrm{Sym}^*(\mathcal{C})$ is always in the form above, a disjoint union of tensor products modded out by the symmetric action by permuting tensor powers. This is a version of the symmetric algebra in classical algebra. So for $\mathcal{C} = \mathcal{S}$, we have objects being points or $C = *$, and

$$\mathrm{Sym}^*(*) \simeq \coprod_{n\geq 0} */\Sigma_n \simeq \coprod_{n\geq 0} B\Sigma_n \simeq \mathsf{Fin}^{\simeq}.$$

The meat of the proof uses the May Recognition Principle.

$$\begin{tikzcd}[column sep=large] \mathsf{Sp}^{\mathrm{cn}} \arrow[r, bend right, "\Omega^\infty"', blue] \arrow[r, phantom, "\simeq"] \arrow[rrr, bend right=40, "\Omega^\infty"'] & \mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S}) \arrow[l, bend right] \arrow[r, bend right, "\mathrm{oblv}"'] \arrow[r, phantom, "\perp"] & \mathsf{CAlg}(\mathcal{S}) \arrow[l, bend right, "(-)^{\mathrm{gp}}"', red] \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, bend right, "\mathrm{Sym}^*"', red] \arrow[lll, bend right=40, "S[-]"', blue] \end{tikzcd}$$

In the complicated diagram above, the left portion is just what the May recognition principle tells us. The middle portion tells us we can forget group-like spaces to just $\mathbb{E}_\infty$-spaces, and composing with the forgetful functor we get Ω^∞, not carrying any additional structure. Its left adjoint is the unbased suspension $S[-]$ (or Σ^∞_+).

For the forgetful functor $\mathsf{CAlg}(\mathcal{S}) \to \mathcal{S}$, its left adjoint is the free functor Sym^*, and group completion is the left adjoint ot the middle forgetful functor, and equivalences have inverses which are adjoints. Note that we don't really have a specific description of the functor $\mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S}) \to \mathsf{Sp}^{\mathrm{cn}}$. This implies that $S[-] \dashv \Omega^\infty$. So following the blue path is the same as following the red path, and this gives us our equivalence. For $X \in \mathcal{S}$, the blue path gives us $\Omega^\infty S[X]$ and the red path is just $\mathrm{Sym}^*(X)^{\mathrm{gp}}$. So $\Omega^\infty S[X] \simeq \mathrm{Sym}^*(X)^{\mathrm{gp}}$, and setting $X \simeq *$ turns the left hand side into the sphere spectrum and the right hand side into $\mathsf{Fin}^{\simeq}$ by the first part of the proof. Then $\Omega^\infty S \simeq (\mathsf{Fin}^{\simeq})^{\mathrm{gp}}$, and we are done. $\boxtimes$

2.5 Spectra as modules, a Brave New Algebra, and some examples (May 28)

Recall that connective spectra are defined as

$$\mathsf{Sp}^{\mathrm{cn}} = \mathsf{Sp}_{\geq 0} = \{X \in \mathsf{Sp} \mid \pi_i(X) = 0 \quad \text{for all } i < 0\},$$

or the set of spectra with no negative homotopy groups. A variant of this is the **n-connective spectra**, where $\mathsf{Sp}_{\geq n} = \{X \in \mathsf{Sp} \mid \pi_i(X) = 0 \text{ for all } i < n\}$, or where the ith homotopy groups don't exist for $i < n$. This admits a left adjoint $\mathsf{Sp} \xrightarrow{\tau_{\geq n}} \mathsf{Sp}_{\geq n}$ in the n-connective cover, which comes about by killing all the lower homotopy groups. These connective covers for different n form a tower, and decreasing n allows homotopy groups in decreasing degrees, which ends up recovering the entirety of the original spectrum.

$$\varinjlim \underbrace{\left(\cdots \longrightarrow \tau_{\geq n+1}X \longrightarrow \tau_{\geq n}X \longrightarrow \tau_{\geq n-1}X \longrightarrow \cdots \right)}_{\text{Whitehead tower}} \simeq X$$

This tower is actually less useful than a variant called the **Postnikov tower**, which is constructed by just flipping everything around. Consider $\mathsf{Sp}_{\leq n} = \{X \in \mathsf{Sp} \mid \pi_i(X) = 0 \text{ for all } i > n\} \hookrightarrow X$ with the right adjoint $\tau_{\leq n}$ by **n-truncation**, which cuts away higher homotopy groups. If we recover X by the *colimit* of the n-connective covers in the Whitehead tower, this time we take a *limit* of the Postnikov tower.

$$X \simeq \varprojlim \underbrace{\left(\cdots \longrightarrow \tau_{\leq n+1}X \longrightarrow \tau_{\leq n}X \longrightarrow \tau_{\leq n-1}X \longrightarrow \cdots \right)}_{\text{Postnikov tower}}$$

The reason why the Postnikov tower is slightly more useful than the Whitehead tower is that when we encounter spectra they are usually connective. Then we have homotopy groups in only a bounded number of degrees, whereas with a non-connective spectrum, Whitehead towers allow us to reduce the problem to connective spectra (or shifts of connective spectra). Together, $\mathsf{Sp}_{\leq n}, \mathsf{Sp}_{\geq m} \subseteq \mathsf{Sp}$ form a **t-structure** with **heart**, where we define

$$\mathsf{Sp}^{\heartsuit} = \mathsf{Sp}_{\geq 0} \cap \mathsf{Sp}_{\leq 0} = \{X \in \mathsf{Sp} \mid \pi_i(X) = 0 \text{ for all } i \neq 0\}.$$

Then we have an equivalence between discrete spectra and abelian groups $\mathsf{Sp}^{\heartsuit} \simeq \mathsf{Ab}$, where $X \mapsto \pi_0(X)$ and $A \mapsto HA$ by the Eilenberg-Maclane construction of classifying spaces. This is an equivalence of categories, so the "heart" of spectra are just abelian groups! We often view $\mathsf{Ab} \subseteq \mathsf{Sp}$ as a fully faithful subcategory, where $A = HA$.

Example 2.15. We have the functor π_0 between symmetric monoidal ∞-categories $(\mathsf{Sp}, \otimes, S) \xrightarrow{\pi_0} (\mathsf{Ab}, \otimes^{\heartsuit}, \mathbb{Z})$. Unfortunately, it is *not true* that $A \otimes B \simeq A \overset{\heartsuit}{\otimes} B$ for $A, B \in \mathsf{Ab}$. Take $\mathbb{F}_2$, the integers mod 2, and smash it with itself as a spectrum. We find that

$$\pi_*(\mathbb{F}_2 \otimes \mathbb{F}_2) \simeq \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \cdots] \quad \text{such that } |\xi_i| = 2^i - 1,$$

or a polynomial ring with coefficients in $\mathbb{F}_2$ with variables in degrees $2^i - 1$ for all i. We call this the **dual Steenrod algebra** $\mathscr{A}_*$. This very much has non-negative homotopy groups in non-negative degrees (since i is independent of π_*), so $\mathbb{F}_2 \otimes \mathbb{F}_2 \notin \mathsf{Sp}^{\heartsuit}$.

However, it is true that $\pi_0(A \otimes B) \simeq A \overset{\heartsuit}{\otimes} B$. More generally, for $X, Y \in \mathsf{Sp}^{\mathrm{cn}}$, we have $X \otimes Y \in \mathsf{Sp}^{\mathrm{cn}}$ and $\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes^{\heartsuit} \pi_0(Y)$. The inclusion $(\mathsf{Ab}, \otimes^{\heartsuit}, \mathbb{Z}) \subseteq (\mathsf{Sp}, \otimes, S)$ is not quite symmetric monoidal, but is instead **lax symmetric monoidal**, which means that there is a canonical map $A \otimes B \to \pi_0(A \otimes B)$. This induces a functor on commutative algebra objects

$$\mathsf{CAlg}^{\heartsuit} \simeq \mathsf{CAlg}(\mathsf{Ab}) \hookrightarrow \mathsf{CAlg}(\mathsf{Sp}) \simeq \mathsf{CAlg}$$

into $\mathbb{E}_\infty$-rings. The claim is that $\mathbb{E}_\infty$-rings are a powerful ∞-categorical analogue of commutative rings.

Example 2.16. Some examples of $\mathbb{E}_\infty$-rings:

- The sphere spectrum $S \in \mathsf{CAlg}$ works because $S \otimes S \xrightarrow{\mu,\,\simeq} S$.
- Any ordinary commutative ring $A \in \mathsf{CAlg}^\heartsuit \subseteq \mathsf{CAlg}$ can be viewed as an $\mathbb{E}_\infty$-ring.
- K-theory spectra form another family of examples, where $KU, KO \in \mathsf{CAlg}$. The $\mathbb{E}_\infty$-multiplication comes from the tensor product of vector bundles $V \otimes V'$.
- For any $\mathbb{E}_\infty$-ring $A \in \mathsf{CAlg}$, its connective cover $\tau_{\geq 0}(A) \in \mathsf{CAlg}$ is also going to be an $\mathbb{E}_\infty$-ring. For example, consider $ku := \tau_{\geq 0}(KU)$, $ko := \tau_{\geq 0}(KO)$ the connected covers of KU and KO.
- For $M \in \mathsf{Sp}$, consider the **free $\mathbb{E}_\infty$-ring generated by M**, defined by $\mathrm{Sym}^*(M) \simeq \bigoplus_{n\geq 0}(M^{\otimes n})/\Sigma_n$. Then it satisifes the universal property as an adjunction to the functor $\mathsf{CAlg} \xrightarrow{\text{oblv}} \mathsf{Sp}$, and

  $$\mathrm{Map}_{\mathsf{CAlg}}(\mathrm{Sym}^*(M), A) \simeq \mathrm{Map}_{\mathsf{Sp}}(M, A).$$

  In particular, for a single generator, $S\{t\} := \mathrm{Sym}^*(S)$ is the free $\mathbb{E}_\infty$-ring on one generator (in other words, a polynomial ring).
- Speaking of that, for $X \in \mathsf{CAlg}(\mathcal{S})$ an $\mathbb{E}_\infty$-space, consider its suspension spectrum $S[X] \in \mathsf{CAlg}$ which inherits an $\mathbb{E}_\infty$-ring structure from the $\mathbb{E}_\infty$-structure on X. This is because the unbased suspension spectrum $S[-]\colon (\mathcal{S}, \times, *) \to (\mathsf{Sp}, \otimes, S)$ is symmetric monoidal. In particular, we can plug in the non-negative integers (or the free commutative monoid on one generator) to get $S[t] := S[\mathbb{Z}_{\geq 0}]$, which we call the **polynomial $\mathbb{E}_\infty$-ring on one generator**.

One might think that the free $\mathbb{E}_\infty$-ring and polynomial $\mathbb{E}_\infty$-ring on one generator should be the same. Sadly they are not. For the polynomial ring, note that $\mathbb{Z}_{\geq 0}$ is a disjoint union of points, so $S[t] \simeq \bigoplus_{n\geq 0} S$. On the other hand, for the free $\mathbb{E}_\infty$-ring we factor out the oroginal spectrum and mod out by the symmetric group action, or $S\{t\} \simeq \bigoplus_{n\geq 0} S/\Sigma_n$. However,

$$S[t] \simeq \bigoplus_{n\geq 0} S \not\simeq S\{t\} \simeq \bigoplus_{n\geq 0} S/\Sigma_n.$$

The action of Σ_n may be trivial, but recall that modding out S by the trivial action of points gets us the classifying space. The homotopy groups of the spectra have to do with *group homology* of the symmetric group, which makes things complicated. The point is that the S/Σ_n (called the **homotopy quotient spectra**) are *not* just the same as the sphere spectra. The consequence is that there are two sorts of affine lines in SAG, or *spectral algebraic geometry*, where $\mathbb{A}'_{\text{smooth}} = \mathrm{Spec}\, S\{t\}$ and $\mathbb{A}'_\flat = \mathrm{Spec}\, S[t]$.

Example 2.17. Another example is **Thom's bordism spectra**. Here is a contemporary take on a classical idea. Consider a functor $\mathrm{Vect}_{\mathbb{R},\mathbb{C}} \to \mathsf{Sp}$ by sending $V \mapsto \Sigma^\infty S^V$, where S^V is the one-point compactification of V. Then we get maps $BO, BU \xrightarrow{\mathcal{J}} \mathrm{BGL}_1(S)$, where $\mathrm{BGL}_1(S) \simeq \mathrm{Aut}_{\mathsf{Sp}}(\Sigma^\infty S^V) \simeq \mathrm{Aut}_{\mathsf{Sp}}(S)$. These are called **J-homomorphisms**, maps of $\mathbb{E}_\infty$-spaces. We can consider $\mathrm{BGL}_1(S) \subseteq \mathsf{Sp}$ as the "full"[16] subcategory of spectra spanned by S. Then we can define

$$MO \overset{\text{def}}{=} \varinjlim\big(BO \xrightarrow{\mathcal{J}} \mathrm{BGL}_1(S) \subseteq \mathsf{Sp}\big), \quad MU \overset{\text{def}}{=} \varinjlim\big(BU \xrightarrow{\mathcal{J}} \mathrm{BGL}_1(S) \subseteq \mathsf{Sp}\big).$$

Since $\mathcal{J}$ is a functor of $\mathbb{E}_\infty$-spaces and is symmetric monoidal, the colimits MO and MU will naturally acquire an $\mathbb{E}_\infty$-ring structure. These are the real and complex bordism spectra respectively. Some remarks:

(1) The connection to bordisms is that "$\Omega^\infty MO \simeq \mathrm{Bord}_O^\simeq$ and $\Omega^\infty MU \simeq \mathrm{Bord}_U^\simeq$"[17]. This was initially defined by the cohomology theory description, which explicitly counts some kind of bordisms.

[16] This would be a full subcategory of $\mathsf{Sp}^\simeq$.

[17] This is the intuition, but the technicalities are more subtle. I didn't catch the author names of the paper referenced.

(2) There exists a generalization of the Thom spectrum construction for an arbitrary vector bundle. For $E \to X$ a rank n vector bundle, it is classified by a map $X \xrightarrow{\zeta_E} BO(n)$. Then the **Thom spectrum** of E is given by

$$X^E = \mathrm{Th}(E) \overset{\text{def}}{=} \varinjlim\left(X \xrightarrow{\zeta_E} BO(n) \longrightarrow BO \xrightarrow{\mathcal{J}} \mathsf{Sp}\right).$$

Example 2.18. Here are some more things we can do with $\mathbb{E}_\infty$-rings.

- The key example is that we can form *modules*. For $A \in \mathsf{CAlg}$, we can form the ∞-category $\mathsf{Mod}_A = \mathsf{Mod}_A(\mathsf{Sp})$. The data of an A-module consists of some $M \in \mathsf{Sp}$ and a structure map $A \otimes M \to M$, plus the usual module axioms (interpreted homotopy-coherently). For example, consider $X \in \mathsf{Sp}$. We have the structure map $S \otimes X \xrightarrow{\sim} X$ being the identity, so $\mathsf{Sp} \simeq \mathsf{Mod}_S$.
- For $M, N \in \mathsf{Mod}_A$, we can form the **relative smash product** $M \otimes_A N \in \mathsf{Mod}_A$ analogous to the usual tensor product over a ring. How we do this is by taking the colimit of the diagram

$$M \otimes_A N := \varinjlim\left(\cdots \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} M \otimes A \otimes A \otimes N \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} M \otimes A \otimes N \rightrightarrows M \otimes N \right),$$

  which equips Mod_A with a symmetric monoidal structure. This is an analogue of the usual relative tensor product, where for $R \in \mathsf{CAlg}^\heartsuit$ and $M, N \in \mathsf{Mod}_R^\heartsuit$, we define

$$M \otimes_R^\heartsuit N \cong \mathrm{Coeq}\left(M \overset{\heartsuit}{\otimes} R \overset{\heartsuit}{\otimes} N \rightrightarrows M \overset{\heartsuit}{\otimes} N\right),$$

  the coequalizer of the map above. In an ∞-categorical perspective, we can't just stop at one map, so we need to take the colimit of an ever increasing diagram.

  This is the idea of a Brave New Algebra: here we are just doing algebra, but in the setting of spectra.

  Remark 2.1. For A an $\mathbb{E}_n$-ring this whole process works too, but $\otimes_A$ only makes Mod_A into an $\mathbb{E}_{n-1}$-monoidal ∞-category.

- For an $\mathbb{E}_\infty$-ring R, the category of R-algebras CAlg_R can be defined either as the overcategory CAlg/R (maps from $\mathbb{E}_\infty$-rings $R \to A$), or as $\mathbb{E}_\infty$-algebra objects in Mod_R (the category $\mathsf{CAlg}(\mathsf{Mod}_R)$). So saying you have an $\mathbb{E}_\infty$-structure where everything is compatible with an R-module structure is the same as saying you have an $\mathbb{E}_\infty$-map $R \to A$.

  For two $\mathbb{E}_\infty$-rings $A, B \in \mathsf{CAlg}_R$, taking the relative smash product $A \otimes_R B$ is the pushout in the ∞-category CAlg. This is analogous to the idea that the tensor product is the pushout in the category of ordinary commutative rings CRing. However, this is *false* for $\mathbb{E}_n$-algebras where $n < \infty$. So $\mathbb{E}_\infty$-rings are in a sense the "best" analogue of commutative rings.
- Let A be a connective $\mathbb{E}_\infty$-ring. In that case, Mod_A has a t-structure with $\mathsf{Mod}_A^\heartsuit \simeq \mathsf{Mod}_{\pi_0(A)}^\heartsuit$, the ordinary 1-category of $\pi_0(A)$-modules.
- For an *ordinary* ring $R \in \mathsf{CAlg}^\heartsuit \subseteq \mathsf{CAlg}$, we have an equivalence $(\mathsf{Mod}_R, \otimes_R, R) \simeq (\mathcal{D}(R), \otimes_R^L, R)$ with the **derived (∞)-category** of modules with the **left derived tensor product** and unit R. Note that the usual tensor product doesn't make sense in derived categories. What do we mean by a derived ∞-category? We take the *chain complex* of ordinary R-modules and inverting **quasi-isomorphisms** (maps between chain complexes that induce isomorphisms on all cohomology groups). Concisely, we have $\mathsf{D}(R) \simeq (\mathsf{ChCplx}(\mathsf{Mod}_R^\heartsuit))[\simeq_{\text{qiso}}^{-1}]$.[18]

[18]There is a subtlety about being monoidally equivalent but not symmetric monoidally equivalent, but we don't have to worry about it.

$$\begin{array}{rcl} \Sigma & \longleftrightarrow & [1] \text{ shift up} \\ \Omega & \longleftrightarrow & [-1] \text{ shift down} \\ \pi_i & \longleftrightarrow & H^{-i} \text{ chain-complex cohomology} \\ \text{fib} & \longleftrightarrow & \text{hKer} = \text{Cocone} \\ \text{cofib} & \longleftrightarrow & \text{hCoker} = \text{Cone} \end{array}$$

Figure 11: Analogues of homotopical operations in the land of homological algebra.

In Figure 11, suspension corresponds to the shift up functor (shifting chain complexes up one degree), and loops correspond to shifting down. Homotopy groups correspond to the chain complex cohomology, and fibers/cofibers correspond to cones and cocones. Some people call these homotopy kernels and homotopy cokernels. This should be a refinement or justification for the analogy between $\mathsf{Sp} \simeq \mathsf{Mod}_{S}$ and $\mathsf{Ab} \simeq \mathsf{Mod}_{\mathbb{Z}}^{\heartsuit}$.

Example 2.19. Our next example is **algebraic K-theory**. Let A be a connective $\mathbb{E}_\infty$-ring.

(1) Consider the category $\mathsf{Mod}_A^{\mathrm{proj}} \subseteq \mathsf{Mod}_A$, the full subcategory spanned by **projective A-modules**. A projective A-module M is a direct summand of free modules, or $M \oplus N \simeq A^{\otimes n}$ for some $n \geq 0$ and $N \in \mathsf{Mod}_A$.

(2) Discard all non-invertible morphisms to get $(\mathsf{Mod}_A^{\mathrm{proj}})^{\simeq} \in \mathsf{Grpd}_\infty \simeq \mathcal{S}$.

(3) Note that the direct sum operation and zero makes $(\mathsf{Mod}_A^{\mathrm{proj}}, \oplus, 0)$ into a symmetric-monoidal ∞-category, therefore $(\mathsf{Mod}_A^{\mathrm{proj}})^{\simeq}$ is an $\mathbb{E}_\infty$-space.

(4) Group complete, then note that $\left((\mathsf{Mod}_A^{\mathrm{proj}})^{\simeq}\right)^{\mathrm{gp}} \in \mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S}) \xleftarrow{\Omega^\infty} \mathsf{Sp}^{\mathrm{cn}} \ni K(A)$ is a correspondence by the May recognition principle. We say that $K(A)$ is an **algebraic K-theory**. This is a post-modern interpretation of a classical idea called "Quillen Q-construction".

Some remarks:

- This is really only correct for connective $\mathbb{E}_\infty$-rings. The "correct" idea in the other case is Waldhousen K-theory, there is also non-connective K-theory, and so on.
- The symmetric monoidal structure on projective modules from $\otimes_A$ makes $K(A)$ into an $\mathbb{E}_\infty$-ring. So $K(A)$ is not only a spectrum, but also an $\mathbb{E}_\infty$-ring.
- The construction of $K(A)$ works for $\mathbb{E}_n$-rings as well, but we only get $K(A) \in \mathsf{Alg}_{\mathbb{E}_{n-1}}(\mathsf{Sp})$ for $n \geq 2$.
- One might know of the **higher K-theory groups**, defined by $K_i(A) := \pi_i K(A)$.

Example 2.20. Let's give an analogous construction of the **topological complex K-theory spectrum**:

(1) Consider the topological category $\mathsf{Vect}_{\mathbb{C}}$ of finite dimensional vector spaces, and pass to homotopy types of mapping spaces to get an ∞-category $\mathsf{Vect}_{\mathbb{C}}$.

(2) Discard the non-invertible morphisms to get an ∞-groupoid $\mathsf{Vect}_{\mathbb{C}}^{\simeq}$, which is just a space.

(3) The direct sum of vector spaces $(\mathsf{Vect}_{\mathbb{C}}, \oplus, 0)$ makes $\mathsf{Vect}_{\mathbb{C}}^{\simeq}$ a symmetric monoidal ∞-category, and so an $\mathbb{E}_\infty$-space.

(4) Group complete and apply the May recognition principle to get $\Omega^\infty(ku) \simeq \left(\mathsf{Vect}_{\mathbb{C}}^{\simeq}\right)^{\mathrm{gp}} \in \mathsf{CAlg}^{\mathrm{gp}}(\mathcal{S})$ for $ku \in \mathsf{Sp}^{\mathrm{cn}}$. The fact that ku is connective is strange– one of the defining properties of complex K-theory is that it is 2-periodic, or its homotopy groups are $\mathbb{Z}$ in all even degrees and 0 in all odd degrees. So we only get the connective cover of KU out of this construction, however, we can recover KU out of ku in a canonical way.

(5) Note that via the tensor product of complex vector spaces, $(\mathsf{Vect}_{\mathbb{C}}^{\simeq}, \otimes_{\mathbb{C}}, \mathbb{C})$ is a symmetric monoidal ∞-category, and ku is an $\mathbb{E}_\infty$-ring.

(6) There is a canonical element $\beta \in \pi_2(ku)$, the *Bott generator*, which comes from the inclusion $\mathsf{Vect}_{\mathbb{C}}^{\dim=1} \subseteq \mathsf{Vect}_{\mathbb{C}}$. Then define $KU := ku[\beta^{-1}]$, and so we obtain the topological K-theory spectrum.

In Conclusion

What have we done this week? We started off with the metaphor of blind men groping an elephant, and realized spectra as

$$\mathsf{Sp} \simeq \mathsf{Sp}(\mathcal{S}) \simeq \varprojlim\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right) \simeq \mathsf{Ind}(\mathcal{SW}(\mathcal{S}_*^{\mathrm{fin}})) \simeq \mathsf{Exc}_*(\mathcal{S}_*, \mathcal{S}) \simeq \mathbb{1}_{\mathsf{Pr}^L_{\mathrm{st}}, \overset{\mathrm{Lurie}}{\otimes}} \simeq \mathsf{Mod}_{\mathbb{S}}.$$

Hopefully spectra are our friends now.

3 Introduction to Varieties and Schemes (Desmond Coles and Saad Slaoui)

Even more so than what we call "key theorems" in mathematics, it is the fertile viewpoints wihch, in our art, constitute the most powerful tools of discovery — or rather, they are not tools, but they are the very eyes of the researcher who passionately strives to understand the nature of mathematical things.

— *Alexander Grothendieck, Récoltes et Semailles*

3.1 The basic framework, and problems the algebro-geometrically minded think about (June 28)

Welcome to "Introduction to varieties and schemes", or "Grothendieck's paradise" (based off Hilbert's quote "No one shall be able to expel us from the paradise that Cantor created for us.") Today we cover the basic framework, some problems algebraic geometers think about, and varieties vs schemes. The standard references include the introduction section of [Gro60][19], Springer edition, and some lecture notes by another French mathematician.

The goal is to study spaces of solutions to systems of polynomial equations with coefficients in a commutative ring k via geometric methods. Classically, k was a field, say $\mathbb{R}$. Descartes, Fermat, and Euler worked with polynomial equations in two or three variables, which leads to things like curves or surfaces in $\mathbb{R}^2$ or $\mathbb{R}^3$. The next big step was letting $k = \mathbb{C}$, and work from Abel and Riemann laid the foundation for complex algebraic geometry. This gives an interplay between analysis, complex geometry, and algebra.

One of the successes of scheme theory is that it allows for k to be a general commutative ring, like the integers. When we involve integers, we are actually doing number theory, in this case Diophantine geometry. At first glance, looking for rational solutions seems to be a discrete problem that doesn't involve geometry at all (there is no real sense in where the integers form a space). And yet, we will see by the end of this lecture how scheme theory gives a place for these sorts of questions in a geometric setting.

[19] I want this to be EGA but it seems difficult to do with with BibTeX...

Recall the goal is to study spaces of solutions to systems of polynomial equations with coefficients in a commutative ring k via geometric methods. Let $S = \{f_j\}_j$ be a family of polynomial equations over some indexing set, so we can view $\{f_j\}_j \subseteq k[(t_i)_{i\in I}]$. Most often $I = \{1, \cdots, n\}$ and we are studying $k[t_1, \cdots, t_n]$. We are interested in the subset $V(S) := \{a \in k^I \mid f_j(a) = 0\} \subseteq k^I$ for all j in our family. We often denote k^I by $\mathbb{A}_k^{|I|}$, called **affine k-space**. Then $V(S)$ is an **affine algebraic set** associated to the system S.

Note. As long as I is finite and k is a Noetherian ring (thought of as a finiteness condition on the ring), we get that (S), the ideal generated by S, is equivalent to the ideal generated by finitely many polynomials, denoted $(g_1, \cdots, g_r)$. In other words, (S) is finitely generated. The point is that for I finite and k Noetherian, affine algebraic sets are always cut out by finitely many equations. Then $V(-)$ assigns to any subset S of $k[(t_i)_{i\in I}]$ a subset of affine space $\mathbb{A}_k^{|I|}$ by $S \mapsto V(S)$.

Exercise 3.1. You can check that the choice of S doesn't sense the passing to ideals. In other words, $V(S) = V(I)$ where $I = (S) \triangleleft k[(t_i)_{i\in I}]$.[20]

The equation $f(a) = 0$ for f a polynomial (in one variable, $f = \sum a_i t^i$, $a_i \in k$) in $k[(t_i)_{i\in I}]$ makes sense not only for a a tuple in k^I, but also for *any* field extension L of k (if k is a field). There is more to this: consider $f = \sum a_i t^i$ in one variable, for $a_i \in k$. In fact, a can live in A^I for any k-algebra $A \in \mathsf{CAlg}_k$. For a given system S, we don't just get one vanishing set associated to k, we get infinitely many vanishing sets associated to any possible k-algebra A. So we have an assignment $V_s(-)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ by $A \mapsto V_s(A) := \{a \in A^I \mid f_j(a) = 0 \text{ for all } j\}$.

Exercise 3.2. The suggestive notation from earlier was chosen for a purpose. Show that any k-algebra homomorphism $A \to B$ induces a map between the solution sets $V_s(A) \to V_s(B)$. This implies that S gives rise to a set valued functor out of the category of k-algebras.

Exercise 3.3. Let $R := k[(t_i)_{i\in I}]/(S) \in \mathsf{CAlg}_k$. Check that we have isomorphisms $\mathrm{Hom}_{\mathsf{CAlg}_k}(R, A) \xrightarrow{\simeq} V_s(A)$ for all $A \in \mathsf{CAlg}_k$.

The principle is that a given system $S = \{f_1, \cdots, f_r\}$ should be thought of as a recipe for producing infinitely many solution sets as the k-algebra of admissible solutions varies. The goal is to take the categorically defined object (the functor $\mathsf{CAlg}_k \to \mathsf{Set}$) and build a "geometric avatar" for this large amount of data. In essence, what we're after is a certain geometric space that captures the geometric data that can be extracted from all of these solution spaces. That is the point of scheme theory.

Goal. Given a system $S = \{f_j\}_j \subseteq k[(t_i)_{i\in I}]$, we want to associate with it a **scheme** X, encoding the geometric data contained in $V_s(A)$ for $A \in \mathsf{CAlg}_k$.

Definition 3.1. A **scheme** (in the sense of Grothendieck) is a figure offering a simplified and functional (faithful) representation of an object (of algebraic origin) or equivalently, of an (instantiation) procedure.

This is not really a precise definition. We'll work toward it. The idea is that $S \rightsquigarrow X$ (a geometric object) $\rightsquigarrow V_s(k) = X(k)$, the "classical" solutions or $V_s(A) = X(A)$, the "A-valued" solutions of the system.

In the Diophantine case where $k = \mathbb{Z}$, we start with a system of polynomial equations with integral coefficients, and to it we asssociate a scheme X. We are really after $X(\mathbb{Z})$, the integral solutions to the system, but we can also look at reducing the solutions mod p and looking for solutions in the field $\mathbb{Z}/p\mathbb{Z}$. So to the scheme we associate $X(\mathbb{F}_p)$ the solutions mod p, or we could go the other road and look for complex solutions $X(\mathbb{C})$. Because the complex plane has an analytic topology, we are hoping this is a more geometric set. Both of these come from the same scheme. The **Weil conjectures** say we can draw a bridge between $X(\mathbb{F}_p)$ and $X(\mathbb{C})$.

[20]Each f_j is written in terms of one t_i right? Are coefficients polynomials (elements of $k[(t_i)_{i\in I}]$) or elements of k?

Note. So far, we've only described the "affine case", where S is associated to some geometric object Spec(R), where $R = k[(t_i)_i]/(S)$, the **affine scheme** associated to R. In parallel to smooth/complex manifold theory, we would like to use these as building blocks for general schemes, in the same way a smooth manifold is made up of an atlas of charts locally resembling Euclidian spaces glued together in a sensible way.

Why do we care? The central object of study in complex algebraic geometry ($k = \mathbb{C}$) are called **smooth projective varieties** X, which come with an embedding $X \hookrightarrow \mathbb{C}\mathrm{P}^r$. To formulate this we need to make sense of complex projective space, which is a non-affine scheme.

Let's talk about some problems in algebraic geometry.

- **Counting problems**: "Is there a solution?" to a system S. For example with $k = \mathbb{F}_q$, this is a matter of checking finitely many values. The question is, what is $\#V_s(\mathbb{F}_{q=p^n})$? This is closely related to the Weil conjectures. This is an extremely hard question for $k = \mathbb{Z}$, and a lot of number theory studies this question studies this question for integral or rational solutions. One of the simplest and hardest examples is $x^n + y^n - z^n = 0$. For $n \geq 3$, $V_s(\mathbb{Z}) = \{(0,0,0)\}$, and this is Fermat's last theorem.

  The other counting problem is intersecting various algebro-geometric objects and asking how many points are in the intersection, aptly named *intersection theory*.
- **Relationship to "other geometries"**: If k is a topological field (eg $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$)[21], we can ask whether we can exploit the geometry of $X(k)$ equipped with the corresponding "analytic" topology as a subspace of a topological field.

  - **Serre's GAGA principle**: Here $k = \mathbb{C}$, $S = f_1, \cdots, f_r \in \mathbb{C}[t_0, \cdots, t_n]$ where the f_i are homogeneous polynomials. Then we can associate a scheme X which is naturally embedded in $\mathbb{P}^n$, since homogeneous polynomials do not care if you scale the input. Similarly, $X(\mathbb{C}) \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, and if we denote $X(\mathbb{C})$ by X^{an} this is the **analytification of X**, and has the structure of a compact complex Kähler manifold. Serre's GAGA principle establishes a correspondence between structures on this complex manifold and the algebraic variety X, which Desmond will be talking about on day 8.
  - If $k \hookrightarrow \mathbb{C}$ is a subfield, eg $\mathbb{Q}$, and X is the scheme associated to the system S, then $X(\mathbb{C}) = X^{\mathrm{an}}$ has a topological fundamental group $\pi_1(X^{\mathrm{an}})$. On the other hand, there exists a theory of **étale fundamental groups** in algebraic geometry denoted $\pi_1^{\text{ét}}(X)$ which make sense for schemes. A theorem says that there is an "inclusion" $\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}})} \to \pi_1^{\text{ét}}(X)$ where the big hat denotes the profinite completion of the standard fundamental group. Then we have a sequence

    $$1 \to \underbrace{\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}})}}_{\text{geometric}} \to \pi_1^{\text{ét}}(X) \to \underbrace{\mathrm{Gal}(\bar{k}/k)}_{\text{arithmetic}} \to 1$$

    where the absolute Galois group of the seperable closure of k over the ground field k measures in a sense the failure of the map to be an isomorphism. So this gives a link between the geometric and arithmetic.

    The principle is that doing algebraic geometry over a field k is approximately doing geometry over the algebraic closure $\bar{k}$, and amalgamate this with the arithmetic k, often in the form $\mathrm{Gal}(\bar{k}/k)$. For example in $\mathbb{Q}_p$ the topology is really nasty, it's totally disconnected and each ball has every point as the center. There isn't an obvious way to do analysis, but there is a theory called (todo:brokenage?) geometry, which has a GAGA theorem and other good stuff. Tropical geometry falls out of this.

[21] Here $\mathbb{Q}_p$ is the ring of p-adic numbers.

  - We can consider $H^*_{\text{sing}}(X^{\text{an}};\mathbb{Q})$, our classic singular cohomology with $\mathbb{Q}$ coefficients from algebraic topology. We can ask "which cohomology classes can be obtained algebraically?" By algebraically, we consider our scheme X as a geometric space and analyze the cycles $Z \hookrightarrow X$, subobjects of a certain codimension. If we have an algebraic object, then Poincaré duality can extend this to the Poincaré dual class α_Z. The **Hodge conjecture** pretty much asks if given an arbitrary cohomology class of $H^*_{\text{sing}}(X^{\text{an}};\mathbb{Q})$, can we express it as some linear combination of Poincaré dual classes of algebraic cycles on X?

- **Classification problems**: When are two varieties or schemes non isomorphic? We start by building invariants, including:

  - *Numerical*: What's the dimension? Genus? Degree of the polynomial?
  - *Analogues of algebro-topological invariants*: π_1 becomes $\pi_1^{\text{ét}}$, which detects a lot more than the classical fundamental group. We can also look for analogues of the singular cohomology for X defined over (with coefficients in) finite field $\mathbb{F}_q$, which exists and is called **étale cohomology** $H^*_{\text{ét}}(X;\mathbb{Q}_t)$.

    K-theory is the study of vector bundles over a space, and step one is to make sense of algebraic bundles. Then the question is "can we look at equivalence classes of vector bundles?", then we do some group completion. This is called the **algebraic K-theory of schemes**, which makes sense.
  - *Birational geometry*: The basic idea is that open sets in the Zariski topology is huge. The question is then "do two varieties or schemes have dense open subsets that are isomorphic?" The minimal model program asks about the geometry of these dense open sets, and the resolution of singularities uses birational transformations to make varieties smooth.

- **Classification of structures**: Can we classify certain types of structures in algebraic geometry over a fixed scheme X? For example, vector bundles over a fixed spaces or elliptic curves over k.

  - *Moduli spaces*: Can we geometrically study families of objects of a certain type? This results again in a functor, and we again ask if we can study that geometrically. The answer is "yes", and this is the theory of moduli spaces. The functor $\mathcal{M}_{\mathcal{B},X} : \mathsf{CAlg}_k \to \mathsf{Set}$ is defined by

    $$\mathcal{M}_{\mathcal{B},X}(A) := \{\text{families of } \mathcal{B}\text{-objects over } X \text{ parametrized by } \operatorname{Spec} A\}.$$

    The best case scenario is if this functor is **representable**, i.e. there exists some **classifying space** Z such that for any $A \in \mathsf{CAlg}_k$, we have an isomorphism

    $$\mathcal{M}_{\mathcal{B},X}(A) \xrightarrow{\simeq} \operatorname{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} A, Z)$$

    where Sch_k denotes schemes over k. Note that Z has a canonical family of $\mathcal{B}$-objects, say $\mathcal{U}$. Then a map $\operatorname{Spec} A \to Z$ formally pulls back to $f^*\mathcal{U}$, and this is how the isomorphism is established.

    Example 3.1. Projective space $\mathbb{P}^n_k$ for k a field can be built this way. The intuition is that projective space "parametrizes lines in k^{n+1}", and we can turn this intuition into a functor of this type. This is an example where we have a representing object.

    Other times we don't, like $\mathcal{M}_g$, the "absolute" moduli functor of curves (schemes of dimension 1) of genus g, which is *not* representable. Other examples include principal G bundles on a scheme X for G a group, where $\mathrm{Bun}_G(X)$ is also not representable.

    The problem with these objects is that the family of $\mathcal{B}$-objects is too sophisticated for them to fit into Set-valued functors. Automorphisms of elliptic curves make Set-valued functors too coarse to accomodate this kind of data. The solution is to change the category of sets to the category of groupoids, and this is where **stacks** come into play.

Comparing varities and schemes

Fix $S = \{f_j\} \subseteq k[(t_i)_{i \in I}]$ throughout this section.

References

[Ada64] J.F. Adams. *Stable Homotopy Theory*. Lectures Notes in Mathematics. Springer, 1964. ISBN: 978-3-662-15942-2. URL: https://link.springer.com/chapter/10.1007/978-3-662-15942-2_3.

[FH19] Daniel S. Freed and Michael J. Hopkins. *Reflection positivity and invertible topological phases*. 2019. arXiv: 1604.06527 [hep-th].

[FH21] Daniel S. Freed and Michael J. Hopkins. *Consistency of M-Theory on nonorientable manifolds*. 2021. arXiv: 1908.09916 [hep-th].

[FHT07] Daniel S Freed, Michael J Hopkins, and Constantin Teleman. "Consistent orientation of moduli spaces". In: *arXiv preprint arXiv:0711.1909* (2007).

[FT14] Daniel S. Freed and Constantin Teleman. *Relative quantum field theory*. 2014. arXiv: 1212.1692 [hep-th].

[Gal+09] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. "The homotopy type of the cobordism category". In: *Acta mathematica* 202.2 (2009), pp. 195–239.

[Gro60] Alexander Grothendieck. "Éléments de géométrie algébrique: I. Le langage des schémas". In: *Publications Mathématiques de l'IHÉS* 4 (1960), pp. 5–228.

[JO12] Niles Johnson and Angélica M Osorno. "Modeling stable one-types". In: *arXiv preprint arXiv:1201.2686* (2012).

[Kar+73] U Karras, M Kreck, W.D. Neumann, and E Ossa. "Cutting and Pasting of Manifolds". In: *Mathematics Lecture Series* 1 (1973).

[Lur17] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Mac13] Saunders Mac Lane. *Categories for the working mathematician*. Vol. 5. Springer Science & Business Media, 2013.

[May99] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832. URL: http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

[Mos+20] Lyne Moser, Viktoriya Ozornova, Simona Paoli, Maru Sarazola, and Paula Verdugo. "Stable homotopy hypothesis in the Tamsamani model". In: *arXiv preprint arXiv:2001.05577* (2020).

[MS74] John Willard Milnor and James Stasheff. *Characteristic Classes*. Princeton University Press, 1974.

[Ngu17] Hoang Kim Nguyen. "On the infinite loop space structure of the cobordism category". In: *Algebraic & Geometric Topology* 17.2 (2017), pp. 1021–1040.

[Sch17] Christopher Schommer-Pries. *Invertible Topological Field Theories*. 2017. arXiv: 1712.08029 [math.AT].

[Tho54] René Thom. "Quelques propriétés globales des variétés différentiables". In: *Commentarii Mathematici Helvetici* 28.1 (1954), pp. 17–86.