Complex Analysis Lecture Notes

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These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`.

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§1.1 Basic Properties of Complex Numbers

We talk about functions $f \colon \mathbb{C} \to \mathbb{C}$ that map variables $z \mapsto f(z)$. This course is "not a very hard course" (it's a fun course!). Holomorphic functions have very nice properties automatically that real valued differentiable functions simply don't have.

Definition 1.1 (Complex Addition)**.** We define complex numbers as ordered pairs $z = (x, y)$ where $x, y \in \mathbb{R}$, with the binary operation of complex addition being defined as

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$

where $+$ denotes addition on the reals.

Once we define multiplication and additive/multiplicative inverses, we will have (almost) formed the field $\mathbb{C}$.

Definition 1.2 (Complex Multiplication)**.** For $x, y \in \mathbb{C}$, we have

$$(x_1, y_1)(x_2)(y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2).$$

Note: for $a \in \mathbb{R}$, we define
$$a(x, y) = (ax, ay).$$
Recall $(a, 0)(x, y) = (ax, ay)$. So one can understand that $a \in \mathbb{R}$ is simply the real analog of $(a, 0)$ (or simply, $\mathrm{Re}(a, 0) = a \in \mathbb{R}$).

How do we define multiplication of a complex number by a real number? We can think of the reals acting (in a group sense) on the complex numbers, with the operation being the standard multiplication.

Example 1.1. Take $(1, 0)(x, y) = (x, y)$. So $1(x, y) = (x, y)$ (where $1 \in \mathbb{R}$).

Example 1.2 (Complex Addition is Commmutative)**.** We have already defined the sum of two complex numbers $z_1 + z_2$ as $z_3 = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$. Since addition is commutative on the real numbers, we have

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1,$$

so complex addition is commutative.

Claim: multiplication of complex numbers is commutative. You can verify this at home.

Theorem 1.1 (Distributive Law)**.** *We have*

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3,$$

for $z_1, z_2, z_3 \in \mathbb{C}$.

Proof. This follows from the fact that $\mathbb{C}$ has a ring structure. $\boxtimes$

§1.2 Real and Imaginary Parts

Definition 1.3. If $z = (x, y)$, then $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Furthermore, we can associate a complex number with a point in the plane in many ways:

(insert figure 1 later)

§1.3 Complex Numbers in the Plane

Point: the plane is just a plane. The plane doesn't have to have a coordinate system (coordinate axes don't have to be perpendicular). Any coordinate system is "useful" for adding complex numbers. For example, you can interpret complex addition as simply vector addition in the plane (no need for orthogonal axes!).

Definition 1.4 (Additive Inverse)**.** We have

$$-(x, y) = (-1)(x, y) = (-x, -y).$$

So $(x, y) + [-(x, y)] = (0, 0)$.

Note: $(x, y)(0, 1) = (-y, x)$, a *rotation* of (x, y) by $90°$. Another note: We have $(x, y) \in \mathbb{C} \cong x + iy$ and $i = (0, 1)$. So

$$(x, y) \cong x + iy \cong (x, 0) + (0, 1)(y, 0).$$

§2 hello?

Definition 2.1. ok