Abstract Algebra Lecture Notes

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Lecture notes for the Fall 2020 graduate section of Abstract Algebra (Math 380C) at UT Austin, taught by Dr. Ciperiani. I'm currently auditing this course due to the fact that I'm not officially enrolled in it. These notes were taken live in class (and so they may contain many errors). You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/abstract_algebra/master_notes.tex.html`.

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§1.1 Applications of Group Actions

Group actions are connected to Representation Theory, a step forward from group actions (eg a group acting on a vector space). Then you can understand your "random group" through Linear Algebra.

Proposition 1.1. Let G be a group of order n, then G is isomorphic to a subgroup of S_n.

Proof. Consider $G \hookrightarrow G$, $g \mapsto [x \mapsto gx]$, with the corresponding homomorphism $\varphi\colon G \to S_G \simeq S_n$. $\operatorname{Ker}\varphi = ?$ $g \in \operatorname{Ker}\varphi \iff \varphi(g) = 1_G$, since $x \mapsto gx$, $x = gx \implies g = 1_G$. φ is an injective homomorphism implies that $\varphi\colon G \to \operatorname{im}\varphi \trianglelefteq S_n$ is an isomorphism. ⊠

Definition 1.1 (Faithful Group Actions)**.** Let $G \hookrightarrow X$. Then the group action is faithful if

$$\bigcap_{x \in X} G_x = \{1_G\}.$$

(Recall that the G_x are the stabilizing sets of x).

Example 1.1. Let G be a group, H some subgroup of G. Consider $X = G/H$ to be the set of left cosets. Then $G \hookrightarrow X$, $g \cdot (xH) = gxH$.

Orbits: $O_{xH} = G/H$, since $(yx^{-1})xH = yH$ for all $x, y \in G$. This is an example of a *transitive* group action.

Stabilizers: $G_{xH} = \{y \in G \mid yxH = xH\}$. $yxH = xH \iff x^{-1}yxH = H \iff x^{-1}yx \in H \iff y \in xHx^{-1}$. So $G_{xH} = xHx^{-1}$.

Example 1.2. Let $G \hookrightarrow X$, $X = \{xHx^{-1} \mid x \in G\}, H \leq G$. Then the action is given by

$$g \cdot xHx^{-1} = gxHx^{-1}g^{-1},$$

which works because $gxH(gx)^{-1} \in X$. Then $O_{xHx^{-1}} = X$ for all $x \in G$ (so the action is transitive). What is the stabilizer of an element? Let $x = 1_G$, then $G_H = \{g \in G \mid gHg^{-1} = H\} =: N_G(H)$ ($N_G(H)$ denotes the normalizer of H in G). Verify that $G_{xHx^{-1}} = xN_G(H)x^{-1}$.

Theorem 1.1. *Let $H \leq G$ be a subgroup of index n. Then there exists an $N \trianglelefteq G$ such that $N \leq H$, $|G/N| \mid n!$.*

Proof. Consider $G \hookrightarrow G/H$, $g \cdot xH = gxH$. Observe that $|G/H| = n$. Then

$$\varphi\colon G \to S_{G/H} \simeq S_n.$$

Let $N = \operatorname{Ker}\varphi = \bigcap_{x \in G} G_{xH}$. $x = 1_G \implies G_H = H, gH = H$. Since N is the kernel of a group homomorphism, it is automatically a normal subgroup of G. $\operatorname{Ker}\varphi = N \implies \varphi : G/N \hookrightarrow S_n$. $G/N \simeq \operatorname{im}\varphi \leq S_n$ which implies $|G/N| = |\operatorname{im}\varphi| \mid |S_n| = n! \implies |G/N| \mid n!$. ⊠

Corollary 1.1. *If G has a group of finite index, then G has a normal subgroup of finite index.*

Corollary 1.2. *Let G be a finite group and p be the smallest prime that divides $|G|$. Then every subgroup of index p is normal.*

Proof. We have $H \leq G$ such that $[G : H] = p$. Then by our theorem, there exists some normal subgroup $N \trianglelefteq H$ such that $N \leq H$, $|G/N \,\big|\, p!$. $p! = p \cdot (p-1)!$, which is only divisible by primes smaller than p. But $|G|$ is not divisible by any primes smaller than p, or any of the $(p-1)!$, so $\gcd(|G/N, (p-1)!) = 1$, which implies $|G/N| = p \implies N = H$, so H is normal. ⊠

§1.2 More on Group Actions

Let $G \hookrightarrow X$ ($Z(G)$ denotes the center of the group). Then

1. $G/G_x \longleftrightarrow G_x$ a bijection $\implies$ $[G : G_x] = |G_x|$. This is a bijection because $gG_x = \{gh \mid h \in G_x\} \mapsto ghx = gx$.

2. X is a disjoint union of the distinct orbits. $1_G x = x \to x \in G_x$ and two orbits are equal or disjoint. So $|x|$ = number of orbits of size $1 + \sum$ sizes of other larger distinct orbits. If $Gx = \{x\}, x$ is a fixed point of the action, so the number of orbits of size 1 are the fixed points of the action. $G \hookrightarrow G$ by conjugation, $g \cdot x = gxg^{-1} \implies |G| = |Z(G)| + \sum$ larger distinct conjugacy classes. This is known as *the class equation.* Formally,

$$|G| = |Z(G)| + \sum [G : C_G(g)]\,, \, C_G(g) = G_g.$$

   The conjugacy class of $x \in G = Gx = [G : G_x]$.

What can we tell from the class equation? If $|G| = p^n$ for p a prime, then $x \notin Z(G) \implies [G : C_G(x)]$ is divisible by p. $p^n = |Z(G)| + p \cdot m$ for $m \in \mathbb{Z}$. In addition, $Z(G) \ni 1_G \implies p \,\big|\, |Z(G)|$. Non trivial by the way.