Complex Analysis Lecture Notes

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These are my lecture notes for the Fall 2020 section of Complex Analysis (Math 361) at UT Austin with Dr. Radin. These were taken live in class, usually only formatting or typo related things were corrected after class. You can view the source code here: `https://git.simonxiang.xyz/math_notes/file/freshman_year/complex_analysis/master_notes.tex.html`.

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§1 August 27, 2020

§1.1 Basic Properties of Complex Numbers

We talk about functions $f : \mathbb{C} \to \mathbb{C}$ that map variables $z \mapsto f(z)$. This course is "not a very hard course" (it's a fun course!). Holomorphic functions have very nice properties automatically that real valued differentiable functions simply don't have.

Definition 1.1 (Complex Addition)**.** We define complex numbers as ordered pairs $z = (x, y)$ where $x, y \in \mathbb{R}$, with the binary operation of complex addition being defined as

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$

where $+$ denotes addition on the reals.

Once we define multiplication and additive/multiplicative inverses, we will have (almost) formed the field $\mathbb{C}$.

Definition 1.2 (Complex Multiplication)**.** For $x, y \in \mathbb{C}$, we have

$$(x_1, y_1)(x_2)(y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Note: for $a \in \mathbb{R}$, we define

$$a(x, y) = (ax, ay).$$

Recall $(a, 0)(x, y) = (ax, ay)$. So one can understand that $a \in \mathbb{R}$ is simply the real analog of $(a, 0)$ (or simply, $\mathrm{Re}(a, 0) = a \in \mathbb{R}$).

How do we define multiplication of a complex number by a real number? We can think of the reals acting (in a group sense) on the complex numbers, with the operation being the standard multiplication.

Example 1.1. Take $(1, 0)(x, y) = (x, y)$. So $1(x, y) = (x, y)$ (where $1 \in \mathbb{R}$).

Example 1.2 (Complex Addition is Commmutative)**.** We have already defined the sum of two complex numbers $z_1 + z_2$ as $z_3 = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$. Since addition is commutative on the real numbers, we have

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1,$$

so complex addition is commutative.

Claim: multiplication of complex numbers is commutative. You can verify this at home.

Theorem 1.1 (Distributive Law)**.** *We have*

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$

for $z_1, z_2, z_3 \in \mathbb{C}$.

Proof. This follows from the fact that $\mathbb{C}$ has a ring structure. ☒

§1.2 Real and Imaginary Parts

Definition 1.3. If $z = (x, y)$, then $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Furthermore, we can associate a complex number with a point in the plane in many ways:

(insert figure 1 later)

§1.3 Complex Numbers in the Plane

Point: the plane is just a plane. The plane doesn't have to have a coordinate system (coordinate axes don't have to be perpendicular). Any coordinate system is "useful" for adding complex numbers. For example, you can interpret complex addition as simply vector addition in the plane (no need for orthogonal axes!).

Definition 1.4 (Additive Inverse)**.** We have

$$-(x, y) = (-1)(x, y) = (-x, -y).$$

So $(x, y) + [-(x, y)] = (0, 0)$.

Note: $(x, y)(0, 1) = (-y, x)$, a *rotation* of (x, y) by 90°. Another note: We have $(x, y) \in \mathbb{C} \cong x + iy$ and $i = (0, 1)$. So

$$(x, y) \cong x + iy \cong (x, 0) + (0, 1)(y, 0).$$

§2 September 1, 2020

§2.1 Units and Zero Divisors in the Complex Numbers

Recall from last time: A complex number can be defined as $(x, y) = x + iy$, where $x, y \in \mathbb{R}$. Addition is easy: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + y_1) + i(y_1 + y_2)$. In particular, $(0, 0) = 0 + i \cdot 0 = 0$. For multiplication, assume $i^2 = -1$. Then

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 + iy_1x_2 + iy_2x_1 + i^2y_1y_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1).\end{aligned}$$

On division: what does it mean to divide complex numbers? We say the multiplicative unit of a complex number (wrt the ring $\mathbb{C}$) as the unique $\frac{1}{z} = z^{-1}$ s.t. $z \cdot z^{-1} = z^{-1} \cdot z = (1, 0) \in \mathbb{C}$ (the unity of $\mathbb{C}$). Assume $(x, y)(x, y)^{-1} = (1, 0)$. Then do u and v exist such that the system of equations

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases}$$

holds? Yes, iff the determinant $\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$ is non zero.

Definition 2.1 (Complex Conjugate)**.** We have $(x, -y)$ the complex conjugate of the complex number $z = (x, y)$, denoted $\bar{z}$.

We show that $\mathbb{C}$ has no zero divisors and is therefore an integral domain. WLOG, assume there exists z_1, z_2 such that $z_1 \neq 0$, $z_1z_2 = 0$: then we have z_1^{-1} exists. So $z_1^{-1}z_1z_2 = 1z_2 = 0$, therefore $z_2 = 0$. For example: the group $\mathrm{GL}_n(\mathbb{R})$ is not an integral domain, since we have zero divisors (two matrices that when multipled equal zero).

§2.2 Polar Coordinate Notation

Definition 2.2 (Polar Coordinates)**.** Think of (x, y) as rectangular coordinates in the xy-plane, and consider the *polar coordinate* notation $z = [r, \theta]$, where $r = \sqrt{x^2 + y^2} = |z|$ (modulus of z), and $\theta = \arctan(\frac{y}{x})$. So $[r, \theta] = (r \cos \theta, r \sin \theta)$.

Example 2.1 (Multiplication with Polar Coordinates)**.** We have

$$[r_1, \theta_1][r_2, \theta_2] = (r_1 \cos \theta_1, r_1 \sin \theta_1)(r_2 \cos \theta_2, r_2 \sin \theta_2).$$

Then

$$\begin{gathered}(r_1 \cos \theta_1 + ir_1 \sin \theta_1)(r_2 \cos \theta_2 + ir_2 \sin \theta_2) = \\ r_1r_2 \left[\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2\right] + ir_1r_2 \left[\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1\right] = \\ r_1r_2 \cos(\theta_1 + \theta_2) + r_1r_2 i \sin(\theta_1 + \theta_2) = \\ [r_1r_2, \theta_1 + \theta_2].\end{gathered}$$

Example 2.2. Assume that a complex number $z = (x, y)$ is nonzero. Then

$$\frac{1}{(x, y)} = \frac{1(x, -y)}{(x, y)(x, -y)} = \frac{(x, -y)}{x^2 + y^2}.$$

§2.3 On the Norm (Modulus) of a Complex Number

Example 2.3. Some properties of the modulus (norm) $|z|$:

1. $|z_1 z_2| = |z_1||z_2|$,
2. $\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = \left|z_1 \cdot \frac{\bar{z_2}}{|z_2|^2}\right| = |z_1|\frac{|z_2|}{|z_2|^2} = \frac{|z_1|}{|z_2|}$ (clearly $|\bar{z_2}| = |z_2|$),
3. $|z_1 + z_2| \leq |z_1| + |z_2|$ ($\mathbb{C}$ is a metric space, so the triangle inequality holds),
4. $|z_1 + z_2| \geq ||z_1| - |z_2||$ (reverse triangle inequality).

We prove the Reverse Triangle Inequality.

Proof. We have $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|$, so $|z_1 + z_2| \geq |z_1| - |z_2|$. A similar argument holds for z_2. ⊠

Think of the polar angle as only well defined for multiples of 2π. Define the argument (angle) as $\text{Arg} = -\pi < \theta \leq \pi$ (what??). So $\text{Arg}(1,1) = \frac{\pi}{4}$, $\text{Arg}(-1,0) = \pi$. OTOH, we would have $\arg(1,1) = \frac{\pi}{4} + 2\pi n$.

§2.4 Euler's Formula

Theorem 2.1 (Euler's Formula)**.** *We claim*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Try using Maclaurin series. ⊠

This suggests $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We proved this when we showed $[r_1, \theta_1][r_2, \theta_2] = [r_1 r_2, \theta_1 + \theta_2]$.

The reason why Dr. Radin says to "forget about Euler" is because he's trying to make a semi-rigorous (or self-contained) construction of the complex numbers. I think it's fine to rely on intuition from other courses, this isn't Real Analysis (nowhere near as rigorous). If we truly were to construct the field $\mathbb{C}$, we would have to cover polynomial rings and the fields generated by PID's quotient irreducible polynomials, then show that $\mathbb{C} \simeq \mathbb{R}[x]/\langle x^2 + 1\rangle$ (and show that this new field is algebraically closed too!). Of course this isn't feasible. So let's just think of this as Euler's Formula, and not some weird definition!

Back to math: using our newfound formula, we can simply say $\arg z = \theta$ such that $z = re^{i\theta}$ for any $z \in \mathbb{C}$. Similarly, $\text{Arg}\, z$ is just θ restricted to the interval $(-\pi, \pi]$.

Example 2.4. If $z = re^{i\theta}$ nonzero, then what is the polar form of $\frac{1}{z}$? It must be

$$\frac{1}{r}e^{-i\theta}.$$

Example 2.5. We've seen that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Then

$$e^{i\theta_1}\left(e^{i\theta_2}e^{i\theta_3}\right) = e^{i\theta_1}e^{i(\theta_2+\theta_3)} = e^{i(\theta_1+\theta_2+\theta_3)}.$$

So $(\cos\theta + i\sin\theta)^m = \cos(m\theta) + i\sin(m\theta)$. This is known as *de Moivre's formula* .

§3 September 3, 2020

§3.1 Fractional Powers

Let $z_0 \in \mathbb{C}$, and define the fractional power $(z_0)^{\frac{1}{m}}$ for $m \geq 2$. This is a complex number such that

$$\left[(z_0)^{\frac{1}{m}}\right]^m = z_0.$$

This many not be unique. To determine the value of the fractional power $(z_0)^{\frac{1}{m}}$, let $z_0 = r_0 e^{i\theta_0}$, $r_0 = |z_0|$, $\theta_0 \in \operatorname{Arg} z_0$. Then

$$(z_0)^{\frac{1}{m}} = (r_0)^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}.$$

Example 3.1. In polar form, $z_0 = i = e^{i\frac{\pi}{2}}$. We want $i^{\frac{1}{6}}$, one value is $e^{i\frac{\pi}{12}}$. Also,

$$e^{i\frac{\left[\frac{\pi}{2}+2\pi\right]}{6}} = e^{i\left[\frac{\pi}{12}+\frac{\pi}{3}\right]} = e^{i\frac{5\pi}{12}}.$$

In general, $i = e^{i\left[\frac{\pi}{2}+2\pi m\right]}$, so $e^{i\left[\frac{\pi}{12}+\frac{m\pi}{3}\right]}$ is a value of $i^{\frac{1}{6}}$ for any m. In particular, consider the choices $m = 0, 1, .., 5$. Then

(insert figure later- it has to do with roots of unity on the circle group tho)

This method gives all possible n-th roots. In particular, in the circle group U_1, each "walk" is equal to a multiplication of ζ.

We will eventually generalize the fractional power $z_0^{p/q}$ to z_0^w. Yada yada no exponentials allowed reeee. If you're going to formalize do it right or don't do it at all. Half baked rigor is about as useful as a potato (at least a potato can feed your family).

§3.2 Point Set Topology

Why are we studying abstract nonsense? We need topology to define limits of complex numbers. We will eventually define a derivative as a quotient of deltas, eg

$$\frac{\Delta f}{\Delta z} \to \frac{df}{dz} \quad \text{as } \Delta z \to 0.$$

We'll talk about open and closed sets and accumulation points and such (basic things needed for limits). Consider

$$\widetilde{S} = \{z \mid |z| \leq 1 \text{ and } |z| \neq 1 \text{ if } \operatorname{Re} z < 0\}.$$

Definition 3.1 (Open Ball)**.** We define an open ball

$$B(z_0, \epsilon) = \{z \mid |z - z_0| < \epsilon\}.$$

§3.3 Interior, Closure, Boundary

Definition 3.2 (Interior Point)**.** We have an *interior point* a point in a set such that there exists an open ball centered at the point entirely contained in the set. We define the set of all interior points of a set X as $\text{Int}(X)$.

Note that $\text{Int}(\widetilde{S}) = \{z \mid |z| < 1.\}$

Definition 3.3 (Exterior Point)**.** A point z_0 is an exterior point of S if there exists a ball

$$B(z_0, \epsilon) \subseteq S^c,$$

ie, $z_0 \in \text{Int}(S^c)$.

Definition 3.4 (Boundary Point)**.** A point z_0 is a boundary point of S if for ball $B(z_0, \epsilon)$ centered at z_0, $B(z_0, \epsilon) \cap S \neq \emptyset$ and $B(z_0, \epsilon) \cap S^c \neq \emptyset$. We define the *boundary* of a set S as the set of all boundary points, denoted ∂S.

Basic things: points can't be both in the interior and exterior, boundary and interior, etc etc.

Theorem 3.1. *For any set S, $\text{Int}(S)$, $\text{Ext}(S)$, and ∂S form a partition of S.*

We will use S° to denote the interior and $(S^c)^\circ$ to denote the exterior of a set from now on.

Example 3.2. $\partial\widetilde{S} = \{z \mid |z| = 1\}$.

Example 3.3. We have the unit circle $S = \{z \mid |z| = 1\} \cup zi$ (where zi is a point). $S^\circ = \emptyset$, $zi \in \partial S$, any point on the rim $\in \partial S$, so $\partial S = S$. By our previous theorem, $(S^c)^\circ = S^c$. (Who even studies the exterior of a set??)

§3.4 Open and Closed Sets

From now on a set refers to a subset of $\mathbb{C}$.

Definition 3.5 (Open Sets)**.** A set is open if it contains none of its boundary. Alternatively, a set is open iff $S = S^\circ$.

Example 3.4. $\mathbb{C}$ is open (and closed)! Furthermore, $\partial\mathbb{C} = \emptyset$ (which is an alternate condition for a set to be clopen). Note that $\emptyset$ is also both open and closed, since $\partial\emptyset = \emptyset$. This also makes sense if we look at it from the interior perspective (no interior points in $\emptyset$, every point has an open ball in $\mathbb{C}$).

Definition 3.6 (Closed Sets)**.** A set is closed if it contains all of its boundary. (What do you mean not the complement of open???)

Theorem 3.2. *S is closed $\iff$ S^c is open.*

Proof. Immediate. In general topology, we define open sets this way. $\boxtimes$

Example 3.5. Like I said earlier, both $\mathbb{C}$ and $\emptyset$ are closed. In general topology, we define both $S, \emptyset \in \tau$, since they're complements of course they're both open and closed. Exercise: prove that no other sets are both open and closed.

Definition 3.7 (Closure)**.** The closure $\bar{S}$ of S is the union

$$S \cup \partial S.$$

Clearly $\bar{S}$ is always closed (by our definition).

Theorem 3.3. *S° is open for any S.*

Doesn't this follow from the definition too??

§3.5 Jank Connectedness

Definition 3.8 (Path-connectedness)**.** A set S is path-connected if every pair of points $z_1, z_2 \in S$ is connected by a continuous path in S.

Every path-connected set is connected (can be written as the union of two disjoint sets). Something about polygonal paths?? Dr. Radin is right, this is most definitely not standard. Is this what physicists do to topology?

Now he's talking about the Topologist's sine curve (the classic counterexample). This is a counterexample to the (false) idea that connected implies path-connected by exhibiting a set that is connected but not path-connected (but we haven't even talked about the standard definition of connectedness yet!).

§4 September 8, 2020

§4.1 Accumulation Points

Definition 4.1. A connected open set is a *domain*.

Definition 4.2. A *region* is a domain that contains none, some, or all of its boundary.

Definition 4.3 (Bounded Set)**.** A set S is bounded if

$$S \subseteq B(x_0, \epsilon).$$

for some $x_0 \in \mathbb{C}, \epsilon > 0$.

Definition 4.4 (Accumulation Points)**.** z_0 is an accumulation point of S if for all balls $B(z_0, \frac{1}{m})$ centered at z_0, we have

$$B(z_0, \frac{1}{m}) \setminus \{z_0\} \cap S \neq \varnothing.$$

Example 4.1. Let $S = \mathbb{Q}$. Then $\frac{1}{2}, \sqrt{2}$ etc are accumulation points of S (this relies on the fact that $\mathbb{Q}$ is *dense* in $\mathbb{R}$). This example shows that accumulation points don't have to be in the set themselves.

Theorem 4.1. *We have S is closed if and only if S contains all of its accumulation points, the set of which is denoted S'. Furthermore, the closure of S denoted $\bar{S}$ is equal to $S \cup S'$.*

Proof. $\implies$ Accumulation points are either in the boundary of S or in S itself. Since S is closed, we have $S' \subseteq S$.
$\impliedby$ If $z_0 \in \partial S \cap S^c$ it would be an accumulation point of S, a contradiction. So $\partial S \subseteq S \implies S$ is closed. (I'll try to write a better proof later). ⊠

A quick summary of basic p-set topology:

1. S is open $\iff S = S^\circ$,
2. S is closed $\iff S^c$ is open,
3. S is open $\iff S$ contains none of ∂S,
4. S is closed $\iff S$ contains all of ∂S,
5. S is closed $\iff S$ contains all of S'.

§4.2 Limits

Consider a map $f\colon \operatorname{Dom}(f) \to \mathbb{C}$, $\operatorname{Ran}(f) \subseteq \mathbb{C}$ (I prefer the notation $f\colon X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$, and $\operatorname{Ran}(f) = f[X]$). The fact that f is well defined on X holds because define X to be a set on which f is well defined, duh).

We want to talk about whether a function is continuous or not. Intuitively, a function is continuous if points in the image being "close" together imply that points in the preimage are also "close" together (the preimage of an open set is open).

Definition 4.5 (Epsilon Delta Limits)**.** For z_0 an accumulation point of some subset X of $\mathbb{C}$ (a region), $\lim_{z\to z_0} f(z)$ exists and has a value of $L \iff$ for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon,$$

where $z \in X$. The modulus is just a distance metric: so the epsilon delta definition is the same as what I said earlier, if points are close to each other in the codomain ($|f(z) - L| < \epsilon$), then such points are close to each other in the domain ($0 < |z - z_0| < \delta$).

Some notes: the limit is only defined when z_0 is an accumulation point. This why accumulation points are also sometimes referred to as *limit points*.

§4.3 Continuity

Definition 4.6 (Continuity)**.** f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. f is said to be continuous on a set X if for all $x \in X$, f is continuous at x.

We want to *analyze* a function $f(z)$, let $z = (x, y)$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, $u(x, y) = \operatorname{Re} f$ and $v(x, y) = \operatorname{Im} f$.

Theorem 4.2. *We have*

$$\lim_{z\to z_0} f(z) = L \iff \begin{cases} \lim_{z\to z_0} \operatorname{Re} f(z) \to \operatorname{Re} L \\ \lim_{z\to zo} \operatorname{Im} f(z) \to \operatorname{Im} L. \end{cases}$$

Proof. Homework. ⊠

Theorem 4.3. *Let $f\colon X \to \mathbb{C}$, $g\colon Y \to \mathbb{C}$. For an accumulation point z_0 of $X \cap Y$, if $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then (excuse the abuse of notation)*

1. $\lim(f + g) = L + M$,
2. $\lim fg = LM$,
3. $\lim \frac{f}{g} = \frac{L}{M}$ *if* $M \neq 0$.

Proof. Same as the ones you'd find in any analysis course. ⊠

Continuity of sums, products, and quotients of functions follow from the above theorem. Now we turn our attention to the composition of functions.

Theorem 4.4. *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$. Let z_0 be an accumulation point of X. Then if f is continuous at z_0 and g is continous at $f(z_0)$, we have $f \circ g$ continuous at z_0.*

Example 4.2. $f(z) = |z^m|$ for a fixed m is equal to $(g \circ h)(z)$ where $h(z) = z^m$ and $g(w) = |w|$. Both h and g are continuous on $\mathbb{C}$, so $|z^m|$ is also continuous everywhere.

Example 4.3. The identity map is continuous. This is trivial (let $\delta = \epsilon$). It follows that maps of the form z^n is continuous for some positive integer n.

Corollary 4.1. *Functions of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $g(z)$ are polynomials are continuous given $g(z) \neq 0$.

Example 4.4. Let $f(z) = \frac{z}{\bar{z}}$, $z \neq 0$. Consider $z = x + iy$ near 0 with $x \neq 0, y = 0$, then $f(z) = 1$. If $x = 0, y \neq 0$ then $f(z) = -1$. Therefore $\lim_{z \to z_0} \frac{z}{\bar{z}}$ does not exist (standard technique for proving multivariate limits don't exist).

§5 September 10, 2020

§5.1 More on Continuity

Last time we talked about the function $\frac{z}{\overline{z}}$. What if we define the domain as $\mathbb{C} \setminus \{0\}$? Does $\lim_{z \to z_0} \frac{z}{\overline{z}}$ exist? (AKA: is $\frac{z}{\overline{z}}$ continuous on its domain?)

Theorem 5.1. *Let $f \colon \mathbb{C} \to \mathbb{C}$ be defined as $f = u + iv$. If f is continuous at z_0, then*

1. $\overline{f} = u - iv$ *is continuous at z_0. We can also write $\overline{f}$ as $g \circ f$ where $g(w) = \overline{w}$.*
2. $\frac{f+\overline{f}}{2} = \mathrm{Re}(f)$ *is continuous at z_0.*
3. $\frac{f-\overline{f}}{2i} = \mathrm{Im}(f)$ *is continuous at z_0.*

Proof. We prove that $f(z) = \overline{z}$ is continuous at any z_0. Given $\varepsilon > 0$, consider

$$|f(z) - f(z_0)| = |\overline{z} - \overline{z_0}|.$$

We need a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |\overline{z} - \overline{z_0}| < \varepsilon.$$

Claim: If $S = \varepsilon$, $|\overline{z} - \overline{z_0}| = |\overline{(z - z_0)}| = |z - z_0| = \delta = \varepsilon$. This is easy to see, so we are done. ⊠

Note. To show that

$$\lim_{z \to z_0} f(z) = L,$$

we consider neighborhoods (open sets around L), or the set of z such that $|f(z) - L| < \varepsilon$ (equivalently, the z such that $f(z) \in B(L, \varepsilon)$). Also, $\lim_{z \to z_0} f(z) - L \iff \lim_{z \to z_0}(f(z) - L) = 0 \iff \lim_{z - z_0 \to 0}(f(z) - L) = 0$.

§5.2 Limits near Infinity

Infinity is not a complex number!! Consider the limits

$$\lim_{z \to \infty} f(z)$$

and

$$\lim_{z \to z_0} f(z) = \infty.$$

To define these, we use neighborhoods of "∞ ". There is no notion of "$\pm\infty$" in the complex numbers. The definition is similar to the one you encountered in Real Analysis: z is "large" if $|z| > R$ for all $R \in \mathbb{R}$.

Definition 5.1 (Limits at Infinity)**.** For $z_0 \in \mathbb{C}$ we say

$$\lim_{z \to z_0} f(z) = \infty$$

if given some $R > 0$, $R \in \mathbb{R}$, there exists some $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z)| > R.$$

Example 5.1. We have $\lim_{z\to 0}(\frac{1}{z}) = \infty$ since given $R > 0$, there exists a $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|\frac{1}{z}| > R$, namely, $\delta = \frac{1}{R}$, because

$$|z| < \frac{1}{R} \implies \frac{1}{|z|} > R \iff \left|\frac{1}{z}\right| > R.$$

Definition 5.2 (Limits to Infinity)**.** We say $\lim_{z\to\infty} f(z) = L$, $L \in \mathbb{C}$ if and only if for all $\varepsilon > 0$, there exists some $R > 0$ such that

$$|z| > R \implies |f(z) - L| < \varepsilon.$$

Example 5.2. We have $\lim_{z\to\infty} \frac{1}{z} = 0$, let $\varepsilon > 0$, $R = \frac{1}{\varepsilon}$. Then $|f(z) - L| = \left|\frac{1}{z}\right|$, so

$$|z| > R \implies |z| > \frac{1}{\varepsilon} \implies \varepsilon > \frac{1}{|z|} = \left|\frac{1}{z}\right|,$$

and we are done.

Definition 5.3. Finally, we say

$$\lim_{z\to\infty} f(z) = \infty$$

if (for $R_1, R_2 \in \mathbb{C}$) given some $R_1 > 0$, there exists an $R_2 > 0$ such that

$$|z| > R_2 \implies |f(z)| > R_1.$$

Example 5.3. We have $\lim_{z\to\infty} z^2 = \infty$ since $|z^2| > R$ whenever $|z| > \sqrt{R}$.

§5.3 Derivates

We are finally ready to define the derivative of a function (the good stuff). Given a function $f\colon X \to \mathbb{C}$, we will only define the derivative of f at a point $z \in X^\circ$. Recall that $X^\circ = \{z \in X \mid B(z, \gamma) \subseteq X\}$ for some $\gamma > 0$.

Definition 5.4 (Complex Derivative)**.** A function $f\colon X \to \mathbb{C}$ is said to be *differentiable* at $z_0 \in X^\circ$ if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$ (so limits to infinity are not allowed. We will examine these "poles" later in the course). If the limit exists, we denote this limit as $f'(z_0)$.

Example 5.4. Let $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto 7$. We claim that $f'(z) = 0$ for all z, since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{7 - 7}{z - z_0} = 0.$$

We only look at the points z "near" (accumulation points) z_0, so we don't have to worry about the case where $z = z_0$. So given $\varepsilon > 0$,

$$|z - z_0| < \delta \implies \left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \varepsilon$$

for any $\delta > 0$.

Example 5.5. Let $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z$. We claim $f'(z) = 1$ since

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z - z_0}{z - z_0} = 1$$

for any $z \neq 0$. This limit is one since

$$\left|\frac{f(z) - f(z_0}{z - z_0} - 1\right| = \left|\frac{z - z_0}{z - z_0}\right| = 0.$$

Example 5.6. Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. We will show $f'(z_0) = 2z_0$. We want to find a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left|\frac{f(z) - f(z_0)}{z - z_0} - 2z_0\right| < \varepsilon.$$

So

$$\left|\frac{z^2 - z_0^2}{z - z_0} - 2z_0\right| = |(z + z_0) - 2z_0| = |z - z_0| < \varepsilon$$

if $|z - z_0| < \delta$ with $\delta = \varepsilon$. There aren't any limit signs because we directly invoked the epsilon-delta definition.

Example 5.7. Consider $f(z) = |z|$ (maps will map $\mathbb{C} \to \mathbb{C}$ unless otherwise stated from now on). We have showed f is continuous for all z, but f isn't differentiable at 0. Use the technique at the end of the last example (write out the piecewise definition of the absolute value and show that the limits don't agree).

What about $z_0 \neq 0$? Is $f\colon \mathbb{C} \setminus \{0\} \to C$ differentiable? Let $z_0 \in C \setminus \{0\}$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z| - |z_0|}{z - z_0} = \frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}}.$$

We let z get close to z_0 in two different ways. First, assume $r = r_0$ but $\theta \neq \theta_0$ (vary the angle, but all having length r). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{0}{r(e^{i\theta} - e^{i\theta_0})} = 0.$$

Next, assume $r \neq r_0$ but $\theta = \theta_0$ (points on a line with angle θ, vary the length). Then

$$\frac{r - r_0}{re^{i\theta} - r_0 e^{i\theta_0}} = \frac{r - r_0}{e^{i\theta}(r - r_0)} = e^{-i\theta} \neq 0.$$

So f is nowhere differentiable.

§5.4 Product, Quotient, and Chain Rules

To get $f'(z)$ for $f(z) = z^m$, we want a formula. Time for induction!

Theorem 5.2. *If $f'(z_0)$ and $g'(z_0)$ exist for two functions f and g, then so do the derivatives*

1. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$,
2. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,
3. $(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$ *provided* $g(z_0) \neq 0$.

Theorem 5.3. *If g is differentiable at z_0 and f is differentiable at $g(z_0)$ then $f \circ g$ is differentiable at z_0 and*

$$(f \circ g)'(z_0) = f'[g(z_0)]g'(z_0).$$

Note (Leibniz Rule)**.** Suppose we have $f_1, f_2, ..., fn$ functions all differentiable at z_0. Then

$$(f_1 f_2 f_3 \cdots f_n)'(z_0) = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + f_1 f_2 f_3' f_4 \cdots f_n + \cdots .$$

In particular, $(z^n)' = n(z' z^{n-1}) = nz^{n-1}$ (just take $f_i = f$ and it becomes clear that this is true).