4-manifolds and Gauge Theory Lecture Notes

Simon Xiang

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Contents

1 Complex structures and self duality, Hodge theory

Last time we talked about n-dimensional complex vector spaces V with $I = i \cdot -$ acts on V. Then $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^{1,0} \oplus V^{0,1}$ where $I^* = i$ acts on $V^{1,0}$, and $I^* = -i$ acts on $V^{0,1}$. We then took exterior powers $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \bigwedge_{\mathbb{C}}^k \left(V^{1,0} \oplus V^{0,1}\right) = \bigoplus_{p+q=k} \Lambda^{p,q}$, where $\Lambda^{p,q} = \mathrm{span}\{a_1 \wedge \cdots \wedge a_p \wedge b_1 \wedge \cdots \wedge b_q \mid a_j \in V^{1,0}, b_j \in V^{0,1}\} \cong \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$. Observe that $\Lambda^k I^*$ acts on $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ which acts on i^{p-q} on $\Lambda^{p,q}$.

These are complex forms in some sense, let us say something about real forms. On one hand, one could look at $\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$, or the complexification $\bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$. Canonically we have

$$\left(\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$$

which says that passing to exterior powers canonically commutes with extending to scalars. So $\left(\bigwedge_{\mathbb{R}}^k \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}$, e.g. $V = T_x M$, $\left(\Lambda^k T_x^* M\right) \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}$. We understand that $\Lambda^{q,p} = \overline{\Lambda^{p,q}}$ by taking the complex conjugate of $\mathbb{C}$. Then we have real forms $\Lambda_{\mathbb{R}}^k \mathrm{Hom}(V,\mathbb{R}) \subseteq \bigoplus \Lambda^{p,q}$, where $\omega = \sum_{p+q=k} \omega_{p,q}$, $\omega_{q,p} = \overline{\omega_{p,q}}$.

Let's look at this in complex dimension 2, real dimension 4. In the model case where $V = \mathbb{C}^2$, this has standard basis $\{e_1, e_2\}$, and $V^{1,0}$ has complex dual $\{e^1, e^2\}$ where $e^j(e_k) = \delta_{jk}$, e^j is $\mathbb{C}$-linear. $V^{0,1}$ then has basis given by the conjugates $\overline{e_1}, \overline{e_2}$ which are $\mathbb{C}$-antilinear. Then we are interested by the 2-forms

$$\Lambda^2 \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \left(\Lambda_{\mathbb{R}}^2 \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\right) \otimes \mathbb{C} = \underset{e^1 \wedge e^2}{\Lambda^{2,0}} \oplus \underset{e_1 \wedge \overline{e^1}}{\Lambda^{1,1}} \oplus \underset{\overline{e^1}, \overline{e^2}}{\Lambda^{0,2}}$$

($e_1 \wedge \overline{e^2}, e_2 \wedge \overline{e^1}, e^2 \wedge \overline{e^2}$ also lie in $\Lambda^{1,1}$). We can regard $\mathbb{C}^2$ as a real, oriented inner product space with basis $\left(e^1, \overline{e^1}, e^2, \overline{e^2}\right)$. Then $\Lambda_{\mathbb{R}}^2 \mathbb{C}^2 = \Lambda^+ \oplus \Lambda^-$, where

$$\Lambda^+ = \left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right)_{\mathbb{R}} \oplus \mathbb{R}(e^1 \wedge \overline{e^1} + e^2 \wedge \overline{e^2}),$$

and $\Lambda^- = \Lambda_-^{1,1} = \{\eta \in \Lambda^{1,1} \mid \eta \wedge \omega = 0\}$. This is a fairly trivial decomposition of 6-dimensional vector spaces; when we bring Hodge theory into the mix, we find that this trivial matter has a highly non-trivial aspect by the Hodge index theorem.

1.1 Hodge theory

Goal. Look at $(H^2(X^4;\mathbb{R}), Q_X) \cong (H_{\mathrm{DR}}^2(X))$ which comes with form $([\alpha],[\beta]) = \int_X \alpha \wedge \beta$. These two are canonically identified with a map called "integration". Say we have a conformal class of Riemannian metrics $[g]$, which leads to an orthogonal deomposition $H_{\mathrm{DR}}^2(X) = \mathcal{H}_{[g]}^+ \oplus \mathcal{H}_{[g]}^-$, so metrics give rise to a positive and negative definite decomposition of cohomology. Specifically, $\mathcal{H}_g^+$ is the space of 2-forms g which are self dual and harmonic, while $\mathcal{H}_g^-$ is the same but anti-self dual. **Harmonic forms** are the subject of Hodge theory.

The first part of Hodge theory is something called the **co-differential**. Here M^n is a manifold, then we have the exterior derivative $d\colon \Omega_M^k \to \Omega_M^{k+1}$. Assume an orientation exists and choose one, plus a Riemannian metric g (a symmetric pairing on each tangent space). We use these to construct the co-differential $d^*\colon \Omega^k \to \Omega^{k-1}$. There are two ways to define this:

- We have the Hodge star $*\colon \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ depending on the metric and the orientation. Then $d^* = (-1)^{k+1} *^{-1} \circ d \circ *$, where we conjugate the exterior derivative by the Hodge star. This indeed lowers the degree by one. $*$ is nearly an involution, where $* \circ * = \pm\,\mathrm{id}$. So this is just $(-1)^{k+1}(-1)^{k(n-k)} * \circ d \circ * = \boxed{(-1)^{kn+1} * \circ d \circ *}$. Since $d^2 = 0$, it follows that $(d^*)^2 = \pm * d * * d * = \pm * d d * = 0$.

- We have an L^2 inner product on k-forms: $\langle \alpha_1, \alpha_2 \rangle_{L^2} = \int_M g(\alpha_1, \alpha_2)\mathrm{vol}_g$, where $\alpha_1 \in \Lambda^k_C$ has compact support, $\alpha_2 \in \Omega^k$. We claim that
$$\langle d^*\alpha_1, \alpha_2 \rangle_{L^2} = \langle \alpha_1, d\alpha_2 \rangle_{L^2}.$$
  This is supposed to be a combination of the construction of the Hodge star with Stokes theorem and integration by parts. That is to say, if we look at $\int_M d(\alpha \wedge \beta)$ where $\deg(\alpha) = k, \deg(\beta) = n-k-1$, α has compact support, Stokes says that this integral is zero. On the other hand, $\int_M d\alpha \wedge \beta + (-1)^k \int_M \alpha \wedge d\beta$ which is integration by parts. This relation plus the definition of the Hodge star implies our claim.

1.2 Harmonic forms

Let's bring in harmonic forms now. The Hodge Laplacian $\Delta = (d + d^*)^2 = d \circ d^* + d^* \circ d$ is a degree two differential operator $\Lambda^k \to \Lambda^k$. The **harmonic k-forms** are defined as $\mathcal{H}^K = \ker \Delta$, or the kernel of the Laplacian. Clearly $\ker(d + d^*) \subseteq \mathcal{H}^k$, but the reverse inclusion holds if M is compact ($\ker(d + d^*) = \mathcal{H}^k$). Consider a form

$$\langle \alpha, \Delta\alpha \rangle_{L^2} = \langle \alpha, d^*d\alpha + dd^*\alpha \rangle_{L^2} = \langle d^*\alpha, d\alpha \rangle_{L^2} + \langle d^*\alpha, d^*\alpha \rangle_{L^2} = \|d\alpha\|^2_{L^2} + \|d^*\alpha\|_{L^2}.$$

This proves the claim, since if the LHS is zero then the two positive terms on the RHS must be zero. Next time we go on with Hodge theory and we will state the Hodge theorem, and combine it with self duality.

2 Variational Characterization

todo:beginning of this lecture

Lemma 2.1. *A harmonic form strictly minimizes L^2 norm within its de Rham cohomology class.*

Proof. For $\alpha \in H^k_g, d\alpha = 0, d^*\alpha = 0, \|\alpha\|^2_{L^2} = \int_M g(\alpha, \alpha)\mathrm{vol}_g$. Then

$$\begin{aligned} \|\alpha + d\gamma\|^2 &= \langle \alpha + d\gamma, \alpha + d\gamma \rangle_{L^2} \\ &= \|\alpha\|^2_{L^2} + \|d\gamma\|^2_{L^2} + 2\langle \gamma, d^*\alpha \rangle_{L^2} \\ &> \|\alpha\|^2_{L^2} \end{aligned}$$

for $d\gamma \neq 0$. Conversely, it's easy to check that a minimizer for an L^2 norm in a fixed cohomology class is harmonic. Take some minimizer η, and look at $\frac{d}{dt}\Big|_{t=0} \left(\|\eta + t d\gamma\|^2_{L^2}\right)$. ⊠

We are in the world of calculus of variations. Here's the Hodge theorem.

2.1 The Hodge Theorem

Note that in Ω^k, we have $(\operatorname{im} d^*)^\perp = \ker d$. Then $\Omega^k = \ker d \oplus \operatorname{im} d^*$—this would follow formally in a *Hilbert space*, but the L^2 norm on Ω^k is *incomplete*. Nothing is for free.

Hodge Theorem. *We have L^2-orthogonal decompositions; $\Omega^k = \ker d \oplus \operatorname{im} d^*$, $\ker d = \mathcal{H}^k_g \oplus \operatorname{im} d$ (where $\mathcal{H}^k_g = \ker d \cap \ker d^*$. Hence the map*

$$\mathcal{H}^k_g \to H^k_{\mathrm{DR}}(M) = \frac{\ker d}{\operatorname{im} d}, \quad \eta \mapsto [\eta]$$

*is an **isomorphism**.*

At this point in the course we will not go into the proof. Later in the course we will discuss the types of techniques needed to prove this. The proof involves Hilbert space completions of Ω^k in which the existence of L^2 minimizers is a formality. There is something called an "elliptic regularity" step to prove that these minimizers lie in Ω^k and not its completions.

Hodge, being an algebraic geometer, did not think to recruit a collaborator who was an expert in the type of analysis that Hilbert developed. Of course, some mathematicians came in and fixed all the issues later (von Weyl, Kodaira).

2.2 Hodge theory and self duality

Now we come to the 4-dimensional case, where we relate Hodge theory to self duality. Let X^4_g be a compact Riemannian (we only need the conformal class of g, or $g \sim \lambda g$ for $\lambda \colon x \to (0,\infty) \subseteq \mathbb{R}$) oriented 4-manifold. The codifferential for 2-forms is $d^* = - * d * \colon \Omega^2 \to \Omega$. For $\eta \in \Omega^2$, $\eta \in \ker(d + d^*)$ or harmonic iff $*\eta \in \ker(d+d^*)$. If $\eta \in \mathcal{H}^2_g$, then $\eta^\pm = \frac{1}{2}(\eta \pm *\eta) \in \mathcal{H}^2_g$. Also, a self-dual 2-form is harmonic iff it's closed. The upshot is that $\mathcal{H}^2_g = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm = \mathcal{H}^2 \cap \Omega^\pm$. In otherwords, $\mathcal{H}^2_g$ is the sum of the self dual harmonic forms and the anti-self dual harmonic forms. OTOH, we have $\mathcal{H}^2 \xrightarrow{\cong} H^2_{\mathrm{DR}}(X)$ by the Hodge theorem. If $\omega \in \mathcal{H}^+$, then

$$\int_X \omega \frown \omega = \int (\omega * \omega) \smile 0) = \int |\omega|^2 \mathrm{vol} > 0.$$

If $\omega \in \mathcal{H}^-$, then

$$\int \omega \wedge \omega = - \int |\omega|^2 \mathrm{vol} < 0.$$

We have the second Betti number $b^2(X) = b^+ + b^-$, and $\tau(X) = b^+ - B^-$, where $b^\pm = \dim \mathcal{H}^\pm$.

2.3 The self-duality complex

Perhaps this is the title of one of the lesser known paper of Sigmund Freud. We are still on (X^4, g) our compact, oriented 4-dimensional Riemannian manifold. We examine the cochain complexes

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^+ \to 0$$

as follows. Consider $d^+\alpha = (d\alpha)^+ = \frac{1}{2}(1+*)d\alpha$ as a cochain complex $(\mathcal{E}^*, \delta)$. The result we want to explain is the following.

Theorem 2.1. *The cohomology of $\mathcal{E}^*$ is*

$$\begin{aligned} H^0(\mathcal{E}) &= H^0_{\mathrm{DR}}(X) \\ H^1(\mathcal{E}) &\cong H^1_{\mathrm{DR}}(X) \\ H^2(\mathcal{E}) &\cong \mathcal{H}^+_g. \end{aligned}$$

This is the prototype of things that come up in gauge theory a lot, specifically one often looks at 1-forms simultaneously in the kernel of d^+ and the codifferential adjoint of d.

Proof. It is easy to check the H^0 case. For $\alpha \in \Omega^1$, we have $d\alpha + d^+\alpha + d^-\alpha$, so

$$\int_X d\alpha \wedge d\alpha = \|d^+\alpha\|^2_{L^2} - \|d^-\alpha\|^2 = \int_X d(\alpha \frown d\alpha) \underset{\text{Stokes}}{=} 0.$$

So $\|d^+\alpha\|_{L^2} = \|d^-\alpha\|_{L^2}$, and $\ker d^+ = \ker d^- = \ker d$. Hence $H^1(\mathcal{E}) \to H^1_{\mathrm{DR}}(X)$, $[\alpha] \mapsto [\alpha]$ is an isomorphism.

We want $\mathcal{H}^+_g$ identified with $\omega \in \frac{\Omega^+}{\operatorname{im} d}$. Then $\omega = \omega_{\mathrm{harm}} + d\alpha + *d\eta$, OTOH $*\omega = \omega$ since we assumed it to be self dual. Then $d\alpha + *d\eta = 2d^+\alpha$. ⊠

3 The period map and the integer lattice

Last time we talked about how a choice of metric gives a splitting of the second de Rham cohomology. This week we look more closely at this splitting and how it interacts with the integer lattice within that second de Rham cohomology. There will be some differential geometry and a bit of analysis this week.

For any smooth manifold M^n and any $k \in \mathbb{Z}_{\geq 0}$, there is an additive subgroup $H^k_{\mathbb{Z}} \subseteq H^k_{\mathrm{DR}}(M)$ of "integer classes", i.e., classes $[\alpha]$ of closed k-forms α with **integer periods**; $\int_P \alpha \in \mathbb{Z}$, for all smooth singular k-cycles P (smooth compact oriented k-dimensional manifolds). This subgroup is a **lattice**, a 4-dimensional discrete subgroup, or the inclusion extends to an $\mathbb{R}$-linear isomorphism, and extending the coefficients to $\mathbb{R}$ is an isomorphism; $H^k_{\mathbb{Z}} \otimes \mathbb{R} \xrightarrow{\cong} H^k_{\mathrm{DR}}(M)$. Why is this a lattice?

$$H^k_{\mathrm{DR}}(M) \cong H^k(M;\mathbb{R}) \underset{\text{universal coefficients}}{\cong} H^k(M;\mathbb{Z}) \otimes \mathbb{R} = H^k(M;\mathbb{Z})' = \frac{H^k(M;\mathbb{Z})}{\text{tors}} \otimes \mathbb{R}.$$

Then $H^k(M;\mathbb{Z})' \hookrightarrow H^k_{\mathrm{DR}}(M)$ maps isomorphically onto $H^k_{\mathbb{Z}}$. Call the subgroup $H^k_{\mathbb{Z}}$ the **integer lattice**.

3.1 The 4-dimensional case

For X^4 closed and oriented, we have our quadratic form on $H^2_{\mathrm{DR}}(X)$, where $\eta \mapsto \int_X \eta \wedge \eta$. From last time, we saw using Hodge theory that $H^2_{\mathrm{DR}}(X) = \mathcal{H}^+_{[g]} \oplus \mathcal{H}^-_{[g]}$, where these subspaces depend on a choice of conformal structure. Then $\mathcal{H}^{\pm}_{[g]}(\mathbb{Z}) := H^2_{\mathbb{Z}} \cap \mathcal{H}^{\pm}_{[g]}$. Recall that $b^+ = \dim \mathcal{H}^+$, $b^- = \dim \mathcal{H}^-$.

Theorem 3.1. *Assume $b^+(X) > 0$. Then for generic conformal structures $[g]$, we have $\mathcal{H}^-_{\mathbb{Z}} = 0$.*

Precisely, we work with conformal classes of C^r Riemannian metrics, $r \in \mathbb{N}, r \geq 3$. "Generic" means it holds on a countable intersection of open dense subsets. It turns out spaces $\mathcal{C}_r(X)$ of conformal structures identified with an open ball in a Banach space.

Corollary 3.1. $\mathcal{H}^-_{\mathbb{Z}} = 0$ *for a **dense** set of C' conformal structures.*

Proof. Apply the Baire category theorem in the closure of the open ball. ⊠

3.2 The period map

This is a map $P : \mathcal{G}_r(X) \to \mathrm{Gr}^-$, $[g] \mapsto \mathcal{H}^-_{[g]} \subseteq H^2 s_{\mathrm{DR}}(X)$, where Gr^- is short for $\mathrm{Gr}^-_{b_-}\left(H^2_{\mathrm{DR}}(X)\right)$, the Grassmannian of $b_-(X)$-dimensional subspaces of $H^2_{\mathrm{DR}}(X)$. The minus means we want the subspace to be negative definite for the quadratic form. The proof will involve P and its derivative. We've taken calculus and know how to differentiate things. How do we differentiate the period map?

The simplest part of the story, yet the most instructive and important, has to do with the Grassmannian itself. The Grassmannian Gr^- has submanifolds S_c for any $0 \neq c \in H^2_{\mathrm{DR}}(X)$, where $S_c := \{J \in \mathrm{Gr}^- \mid c \in J\}$.

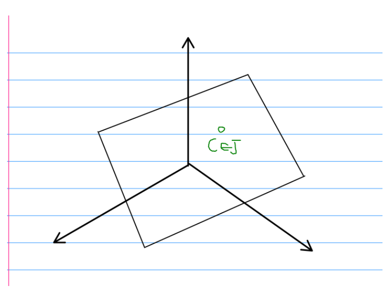

Figure 1: Visualizing some $J \in S_c$.

Why is S_c a submanifold? Consider charts on Gr (or Gr^-). For $J \in \mathrm{Gr}$, choose a complement K, so $H = J \oplus K = H^2_{\mathrm{DR}}(X)$. Then $\mathrm{Hom}_{\mathbb{R}}(J,K) \xrightarrow{\phi_{J,K}} \mathrm{Gr}, \theta \mapsto \mathrm{graph}(\theta)$, where $\phi_{J,K}$ is an atlas for Gr. Suppose now that $c \in J$, or $J \subseteq S_c$. $\phi_{J,K}$ maps $\{\theta \in \mathrm{Hom}(J,K) \mid \theta(c) = 0\}$ to a neighborhood of J in S_c, i.e. we get a *submanifold* chart. Since $b^+ > 0$, $\mathrm{Gr}^- \setminus S_c$ is open and dense in Gr^-. We can look at $\bigcup_{0 \neq c \in H^2_{\mathbb{Z}}} S_c \subseteq \mathrm{Gr}^-$, a countable union of closed submanifolds of positive codimension, and is *generic* (no idemp sets). What this tells us is that our intution that if we look at a Grassmannian and a random subspace to see how it intersects the integer lattice is that it only intersects at the origin.

Our next step is to show that P is transverse to each of the submanifolds S_c where $0 \neq c \in H^2_{\mathbb{Z}}$. This implies that $P^{-1}(S_c) \subseteq \mathcal{C}_r(X)$ is a closed codimensional b^+ submanifold (where $\mathcal{C}_r(X)$ is an ∞-dimensional manifold). This implies our theorem. Transversality means that if $P[g] \in S_c$, then $\boxed{T^{\mathrm{Gr}^-}_{P[g]} = T_{P[g]}S_c + \mathrm{im}\, D_{[g]}P}$ (the derivative of the period map). This is the technical statement that is going to be proved. Some of the proof will be next time, primarily using a Hodge theoretic calculation. First you differentiate the Hodge star with respect to the conformal structure, then you differentiate the period map. It also uses a principle from PDEs, called unique continuation for harmonic 2-forms.

4 The period map and the integer lattice, continued

Let X^4 be a closed, oriented manifold, and $P\colon \mathcal{C}(X) \to \mathrm{Gr}^- \subseteq \mathrm{Gr}_{b^-(X)}(H^2_{\mathrm{DR}}(X))$, where $\mathcal{C}(X)$ is the set of conformal structures. Then $P[g] = \mathcal{H}^-_{[g]} = \{g\text{-ASD harmonic functions}\}$. So $\mathcal{H}^-_{[g]}(\mathbb{Z}) = \mathcal{H}^-_{[g]} \cap H^2(X,\mathbb{Z}) \subseteq H^2_{\mathrm{DR}}(X) = P[g] \cap H^2(X,\mathbb{Z})$. The generic non-existence theorem says that for generic $[g], b^+ > 0$, we have $\mathcal{H}^-_{[g]}(\mathbb{Z}) = 0$. For $S_c \subseteq \mathrm{Gr}^-$ (submanifolds of codimension b^+), the claim is that $P \pitchfork c$ for each c. Hence $P^{-1}(S_c)$ is a codimension b^+ submanifold of $\mathcal{C}(X)$. The theorem holds since generic conformal structures don't lie in the countable union $\bigcup_c P^{-1}(S_c)$.

Remark 4.1. In families of conformal structures, $\mathcal{H}^-_{[g]}(\mathbb{Z}) \neq 0$ occurs with codimension b^+ in the parameter space.

We need to understand the manifold structure of $\mathcal{C}(X)$.

4.1 Conformal structures as maps $\Lambda^- \to \Lambda^+$

Let V be a 4-dimensional oriented vector space. Choose a reference inner product g_0, which gives rise to a splitting $\Lambda^2 V = \Lambda^+_0 \oplus \Lambda^-_0$. If Λ^- is another 3-dimensional subspace of $\Lambda^2 V$ that is negative definite with respect to the squaring form $\eta \mapsto \eta \wedge \eta$, then the projection $\Lambda^- \xrightarrow{\cong} \Lambda^-_0$ is an isomorphism. So $\Lambda^- = \Gamma_m = \mathrm{graph}\big(m\colon \Lambda^-_0 \to \Lambda^+_0\big)$, where $m \subseteq \mathrm{Hom}(\Lambda^-_0, \Lambda^+_0)$.

We could ask which linear maps m give rise to a negative definite subspace. For $m \in \mathrm{Hom}(\Lambda^-_0, \Lambda^+_0)$, Γ_m is negative definite iff $|m|_{\mathrm{op}} < 1$. Then $g(\eta) = \eta^2$, and

$$g(\alpha + m\alpha) = g(\alpha) + g(m\alpha) = \underbrace{\big(-|\alpha|^2 + |m\alpha|^2\big)}_{<0}\mathrm{vol}.$$

Proposition 4.1. *The map $m \mapsto \Gamma_m$ identifies linear maps $\Lambda^-_0 \to \Lambda^+_0$ of operation norm < 1, with negative definite 3-dimensional subspaces of $\Lambda^2 V$. Then conformal structures are in bijection with the set $\big\{m \,\big|\, |m| < 1\big\}$.*

This tells us that conformal structures are not identified by some complicated manifold, but just some ball in a vector space.

Remark 4.2. If $\Lambda^- = \Gamma_m$, then $\Lambda^+ = (\Lambda^-)^+ = \gamma_{m^*}$, where m^* is adjoint to m.

This is the linear algebra picture, the globalization is essentially immediate. The conformal structures on our 4-manifold are identified with vector bundle maps $\Lambda^-_{g_0} \xrightarrow{m} \Lambda^+_{g_0}$ with reference metric g_0, with the property that $|m_x|_{\mathrm{op}} < 1$ for all $x \in X$. Here X is compact, and $c \mapsto C^r$ is a bundle map, an open subset of a Banach space.

We compute that

$$D_{[g_0]}P\colon T_{[g_0]}\mathcal{C}(X) \to T_{P[g_0]}\mathrm{Gr}^- = C^r(X, \mathrm{Hom}(\Lambda^-_{g_0}, \Lambda^+_{g_0})) \to \mathrm{Hom}(\mathcal{H}^-, \mathcal{H}^+).$$

Proposition 4.2. *For m a bundle map, $\alpha^- \in \mathcal{H}^-$,*

$$(D_{[g_0]}P)(m)(\alpha^-) = m(\alpha^-)_{\mathrm{harm}} \in \mathcal{H}^+.$$

So this is a harmonic orthogonal projection $\Omega^+_{g_0} = \mathcal{H}^+ \oplus \mathrm{im}\, d^+ \to \mathcal{H}^+$ (conformal self-duality complex).

This is the claimed answer, and it is as clean as can be. In differential topology, often you are faced with scary tasks like differentiating a map between complex structures. If you can unravel to the point where you can really formulate what you need, it is more feasible to guess what the answer should be and go from there.

We use this to show that $P \mathrel{\overline{\pitchfork}} S_c$, i.e., $T_{P[g]}\mathrm{Gr}^- = T_{P[g]}S_c + \mathrm{im}\, D_{[g]}P$. We need that $\mathrm{im}\, DP$ spans the normal space $T_{P_{g_0}}\mathrm{Gr}^-/T_{P_g}S_c = N_{P_g}(S_c)$. We have

$$N_J S_c = \frac{T_J \mathrm{Gr}^-}{T_J S_c} = \frac{\mathrm{Hom}(J, J^\perp)}{\{\theta \in \mathrm{Hom}(J, J^\perp) \mid J(c) = 0\} \underset{\theta \mapsto \theta(c)}{\xrightarrow{\cong}} J^\perp}.$$

Unraveling this lot, we find out that what we need to prove is the following:

- If $\alpha^- \in \mathcal{H}^-_g$ represents $0 \neq c \in H^2(X, \mathbb{Z})$, then for all $\alpha^+ \in \mathcal{H}^+_g$, there exists a bundle map $m\colon \Lambda^-_g \to \Lambda^+_g$ with the property that $m(\alpha^-)_{\mathrm{harm}} = \alpha^+$.

If not, then there exist forms $\alpha^\pm \in \mathcal{H}^\pm_g$ such that $\alpha^+ \perp_{L^2} m(\alpha^-)_{\mathrm{harm}}$ for all m. So $0 = \langle \alpha^+, m(\alpha^-)_{\mathrm{harm}}\rangle_{L^2} = \langle \alpha^+, m(\alpha^-)\rangle_{L^2}$ for all m. For such a vanishing to hold for all bundle maps m seems very strong. Say there exists an $x \in X$ where $\alpha^+_x \neq 0, \alpha^-_x \neq 0$. Use bundle maps m supported by x to get a contradiction. We conclude that either α^+ or α^- must be identically zero on some open set. There is a unique continuation principle that says a harmonic form vanishing to infinite order of a point vanishes everywhere. Hence $\alpha^- = 0$ or $\alpha^+ = 0$ implies they both are zero, or $\alpha^+ = \alpha^- = 0$.

Next time, complex geometry.

5 Hodge Theory on complex surfaces

Today we will discuss the Hodge index theorem and KS surfaces.

5.1 Kähler manifolds

Let X be a complex manifold (holomorphic atlas), and J be a complex structure, where $J\colon TX \to TX$, $J = i \cdot -$. Then we have the **hermitian metric** on X :

$$h_x\colon T_xX \times T_xX \to \mathbb{C}$$

for all $x \in X$.[1] It satisfies the following properties:

- It is $\mathbb{R}$-bilinear,

[1] Something about this is different from the quantum mechanical definition.

- $h_x(Ju,v) = ih_x(u,v) = -h_x(u,Jv)$,
- $h_x(v,n) = \overline{h_x(v,v)}$,
- $h_x(u,u \in \mathbb{R}) > 0$ for every $u \neq 0$.

Then $h = g + i\omega$, where g is a Riemannian metric for $J \in O(g)$, and ω is a 2-form, where $\omega(J_n, J_v) = \omega(u,v)$. Think of J as given. Then $\omega \longleftrightarrow g$, with $g(u,v) = \omega(u,Jv)$, etc. So $\Lambda^2(X) \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda_{1,1} \otimes \alpha^{0,2}$, with eigenvalues i^2, i^0, i^{-2}, since this is i^{p-q} on Λ^{p-q}. This distinguishes ω as a $(1,1)$-form since it lives in the right eigenvalue of J. We can think of h as determining and being determined by this $(1,1)$-form ω, such that $\omega(v,Jv) > 0$ for all $v \neq 0$.

Definition 5.1. A **Kähler manifold** $(X,J,h = g + i\omega)$ is a complex manifold plus a hermitian metric h such that $d\omega = 0$. In other words, its imaginary part ω is a closed 2-form.

This is a mysterious yet natural condition, since 2-forms being closed is something to think about. Instead of h, you can specify a closed positive $(1,1)$-form ω (positive means $\omega(v,Jv) > 0$). Here g is called a **Kähler metric** and ω is called a **Kähler form**. So Kähler forms are the harmonious intersection between complex, symplectic, and Riemannian geometry.

Example 5.1. Some examples:

- For the complex tori $T = \mathbb{C}^n/\text{lattice}$, it has a complex structure and ω an intersection form on $\mathbb{C}^n$, which is translation invariant.
- A crucial example is $\mathbb{C}\mathrm{P}^{\mathrm{N}}$. The assertion is that this carries a unique Kähler form $\omega_{\mathbb{C}\mathrm{P}^{\mathrm{N}}}$ such that it is as symmetric as can be, that is, $\omega_{\mathbb{C}\mathrm{P}^{\mathrm{N}}}$ is invariant under the action of $PU(N+1)$ on $\mathbb{C}\mathrm{P}^{\mathrm{N}}$. So $\int_{\mathbb{C}\mathrm{P}^1} \omega_{\mathbb{C}\mathrm{P}^{\mathrm{N}}} = 1$. We will see this form often as the curvature of a line bundle. In general we have a Kähler class $[\omega] \in H^2_{\mathrm{DR}}(X)$, but in this case $[\omega_{\mathbb{C}\mathrm{P}^{\mathrm{N}}}]$ actually lies in the integer lattice $H^2_{\mathbb{Z}}$.
- If $X \subseteq \mathbb{C}\mathrm{P}^{\mathrm{N}}$ is a complex submanifold, then the restriction $\omega_{\mathbb{C}\mathrm{P}^{\mathrm{N}}}\Big|_X$ is a Kähler form. Then smooth algebraic varieties (things cut out of $\mathbb{C}\mathrm{P}^{\mathrm{N}}$ with homogeneous equations) admit Kähler forms, moreover *integral* Kähler forms ($[\omega] \in H^2_{\mathbb{Z}}$).
- The **Kodaira embedding theorem** says that if X a complex complex manifold admits an integral Kähler form, then X embeds in $\mathbb{C}\mathrm{P}^{\mathrm{N}}$ for sufficiently large N. For example, integral Kähler forms are extremely cheap for Riemann surfaces; take any volume form with integral one. Then by Kodaira's theorem this embeds in complex projective space.
- **Chow's theorem** says that X is furthermore algebraic. So the Riemann surface is actually cut out by homogeneous polynomial equations.

5.2 Hodge theory

We will only make assertions here, no proofs. For $(X,J,h = g + i\omega)$ a compact Kähler manifold, the (p,q)-components of a g-harmonic k-form are still harmonic. The differential forms split up as $\Omega^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}$. It turns out that the *harmonic* k-forms split up the same way; $\mathcal{H}^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$, where $\mathcal{H}^{p,q}$ is $\Lambda^{p,q} \cap (\mathcal{H}^k \otimes \mathbb{C})$. This turns out to be *extremely* useful.

Recall that $\overline{\Lambda^{p,q}} = \Lambda^{q,p}$. So $\mathcal{H}^{p-q} = \mathcal{H}^{q,p}$. We have $h^{p,q} = \dim \mathcal{H}^{p,q}$, so the Betti number is given by $b^k = \sum_{p+q=k} h^{p,q}$, where $h^{qp} = h^{pq}$.

Definition 5.2. A **Hodge structure** of weight k on an abelian group $H_{\mathbb{Z}}$ is a vector space decomposition

$$H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}, \quad H^{q,p} = \overline{H^{p,q}}.$$

For X compact Kähler, we get a weight k Hodge structure on $H^k(X;\mathbb{Z})$, because

$$H^k(X;\mathbb{Z}) \otimes \mathbb{Z} \cong \mathcal{H}^k \otimes \mathbb{C} = \bigoplus \mathcal{H}^{p,q}.$$

Note that this is is a similar flavor to the things we've been talking about our Riemannian 4-manifolds, with the decomposition into the self dual and anti-self dual parts. The word "period map" is really borrowed from Hodge theory. Even more is true. The space $\mathcal{H}^{p,q}$ is canonically identified with a gadget form complex analytic geometry, the q-th *sheaf cohomology* $H^q(X;\mathcal{A}^p)$, where $\mathcal{A}^p = \Lambda^p_{\mathcal{O}_X}(J^*X)$ is the p-th exterior power of the holomorphic cotangent sheaf, and $\mathcal{O}_X$ is the sheaf of holomorphic functions. For example, if X is a compact Riemann surface,

$$\underbrace{H^1(X,\mathbb{C})}_{\dim 2g} = \underbrace{H^0(J^*X)}_{g,\text{ holomorphic 1-forms}} \oplus \underbrace{H^1(\mathcal{O}_X)}_{g}$$

We cannot say much about H^1 and H^2, but $H^0(\mathcal{A}^p)$ is the set of holomorphic p-forms, which is locally (on $(z_1,\cdots,z_n)$) the sum $\sum_{|I|=p} f_I dz_I$ for f_I holomorphic.

5.3 Complex Kähler surfaces

We have $H^2(X;\mathbb{C}) = \mathcal{H}^{2,0} \cong H^0(\mathcal{A}^2) \oplus \mathcal{H}^{1,1} \cong H^1(\mathcal{A}^1) \oplus \mathcal{H}^{0,2} \cong H^2(\mathcal{O})$, where $\mathcal{H}^{2,0} \longleftrightarrow \mathcal{H}^{0,2}$ and $\mathcal{H}^{1,1} \longleftrightarrow \mathcal{H}^{1,1}$ by complex conjugation. We saw (pointwise) that $\Lambda^2 \otimes \mathbb{C} = \Lambda^+ \otimes \mathbb{C} \oplus \Lambda^- \otimes \mathbb{C}$. We observed that $\Lambda^+ \otimes \mathbb{C} = \Lambda^{2,0} \otimes \mathbb{C} \cdot \omega \oplus \Lambda^{0,2}$, and $\Lambda^- \otimes \mathbb{C} = \Lambda^{1,1}_0$ for $\omega^\perp$. What we can do is apply this *globally* to harmonic 2-forms; what we get is that self-dual harmonic forms complexified consist of the the harmonic $(2,0)$ forms, harmonic $(0,2)$ forms, and complex copies of the Kähler form ω.

$$\mathcal{H}^+ \otimes \mathbb{C} = \mathcal{H}^{2,0} \otimes \mathbb{C} \cdot \omega \oplus \mathcal{H}^{0,2},$$

where $\mathcal{H}^- = \mathcal{H}^{1,1}_0$ for $\omega^+ \subseteq \mathcal{H}^{1,1}$. This just follows from the pointwise calculation we just did. This immediately implies what we call the Hodge index theorem.

Hodge index theorem. *For X a compact Kähler surface,*

- $b^+ = 1 + 2h^{2,0}$.
- *Moreover, the wedge product form on $\mathcal{H}^{1,1}$, $\eta \mapsto \int_X \eta \wedge \eta$ has "signature" $(1, b^- - 1)$.*

For example, if you look at the Néron-Severi group $\mathrm{NS}(X) = \mathcal{H}^{1,1} \cap H^2_{\mathbb{Z}}$, it has a complex curve $C \subseteq X$, and $PD[C] \in \mathrm{NS}(X)$. So the intersection form on $\mathrm{NS}(X)$ is negative definite on $\perp$ to $[\omega]$.

6 *K*3 surfaces and periods

New topic: covariant derivatives, curvature, and gauge transformations.

6.1 *K*3 surfaces

Definition 6.1. A $K3$ **surface** (X,J) is a compact complex surface with $b^1(X) = 0$ with admits:

- a Kähler structure $g + i\omega$
- a nowhere vanishing holomorphic 2-form σ

i.e., has local holomorphic coordinates (z_1,z_2), $\sigma = h(z_1,z_2)dz_1 \wedge dz_2$ where h is holomorphic and non-zero.

Any *other* holomorphic 2-form σ' is $\sigma' = f\sigma$, where f is holomorphic on X. The maximum modulus theorem then implies that f is constant, so $H^0(\mathcal{A}^2)$ (global holomorphic 2-forms) is $\mathbb{C}\sigma$ which implies $h^{20} = \dim H^0(\mathcal{A}^2) = \dim \mathcal{H}^{20} = 1$, since $h^{02} = h^{20} = 1$ and $\mathcal{H}^{02} = \overline{\mathcal{H}^{20}}$. Last time we ended with the Hodge index theorem, which says that for a Hähler surface (compact), $b^+ = 1 + h^{02} + h^{20} = 3$. So $b^+(K3) = 3$. Moreover we get that the self dual harmonic forms $\mathcal{H}^+_g$ are spanned by the real part of $\mathbb{C}\sigma \oplus \mathbb{C}\overline{\sigma} \oplus \mathbb{R}\omega$ (where elements of $\mathbb{C}\overline{\sigma}$ look like $a\alpha + \overline{a\sigma}$ for $a \in \mathbb{C}$).

There is a magic formula, a version of the Riemann-Roch theorem which says that

$$\sum (-1)^q h^{0,q} = \frac{1}{12}\left(c_1^2(TX) + c_2(TX)\right)[X]$$

leading to a beautiful characterisation of the Euler characteristic, where $\chi(X) = c_2(TX)[X] = 24$. Here $b^1 = 0$ by assumption, so $\chi(X) = 2 + b^2$. This implies that $b^2 = 22$, $b^+ = 3$, and so $b^- = 19$. Then you have a Hodge structure $H^2(X) \otimes \mathbb{C} = \mathcal{H}^{20} \oplus \mathcal{H}^{11} \oplus \mathcal{H}^{02}$. Knowing $\mathbb{C}\sigma$, we get $\mathcal{H}^{20} = \mathbb{C}\sigma$, $\mathcal{H}^{02} = \overline{\mathbb{C}\sigma}$, and $\mathcal{H}^{11} = (\mathbb{C}\sigma \oplus \mathbb{C}\overline{\sigma})^\perp$, where $[\eta] \in H^2_{\mathrm{DR}}(X, \mathbb{C})$, and $\int_X \eta \wedge \sigma = \int_X \eta \wedge \overline{\sigma} = 0$. Here $\mathbb{C}\sigma$ is by the Hodge theorem; note that $\int \sigma \wedge \sigma = 0$, while $\int \sigma \wedge \overline{\sigma} > 0$. We can record the Hodge structure of a $K3$ surface (X, J) as its **period point** $\mathbb{C}\sigma$ in the **period domain**

$$P = \left\{ \mathbb{C}\sigma \in \mathbb{P}(H^2(X, \mathbb{C})) \,\middle|\, \int \sigma \wedge \sigma = 0, \int \sigma \wedge \overline{\sigma} > 0 \right\}$$

This is a 20-dimensional complex manifold; $H^2(X, \mathbb{C})$ is 22 dimensional, projection takes this down to 21 dimensions, the quadratic equation $\sigma \wedge \sigma$ takes this down to 20 dimensions, and $\sigma \wedge \overline{\sigma}$ is an open condition. This is equivalent to the Hodge structure. It is a fact that all points of P occur as period points. It is not quite true that the period point determines the $K3$ structure up to isomorphism, but "nearly". Some $K3$ surfaces are "algebraic", i.e. there exists a Kähler class $[\omega] \in H^2\mathbb{Z}$. For example:

- quartic surfaces in $\mathbb{C}\mathrm{P}^3$
- quadric and cubic in $\mathbb{C}\mathrm{P}^4$
- double cover of $\mathbb{C}\mathrm{P}^2$
- branched along a smooth sextic curve

This takes the dimension down to 19. We briefly mentioned the Néron-Severi group $\mathrm{NS}(X, J) = H^{11} \cap H^2_{\mathbb{Z}}$, the classes of complex curves on X. Then $\rho(X) = rk\mathrm{NS}$, the "Picard point", and $\rho \geq 1$ for X algebraic. Having ρ *big* is *special*, e.g. for a *generic* quartic surface, $\rho = 1$. This ends our sketchy overview of $K3$ surfaces.

6.2 Connections and vector bundles

We move toward Gauge theory proper. To set a convention, let $E \to X$ be a smooth complex vector bundle, with a Hermitian inner product h in E.

Definition 6.2. A **covariant derivative** (also known as a **connection**) $\nabla \in E$ is a $\mathbb{C}$-linear map $\nabla : C^\infty(X, E) \to C^\infty(X, T^*X \otimes_{\mathbb{R}} E)$ from the space of sections to the sections of the cotangent bundle tensored with E. It has the property that for a C^∞ function f and section s, we want to the Liebniz rule to hold in a sense, or

$$\nabla(fs) = df \otimes s + f\nabla s$$

where df is the exterior derivative of f (a 1-form). It is called **unitary** (with respect to a Hermitian inner product h) if

$$d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2).$$

In other words, we want the connection to obey some form of the product rule.

For any vector field $V \in C^\infty(X, TX)$, we get $\nabla_V : C^\infty(X, E) \to C^\infty(X, E)$; for $\nabla_V s$, contract X into ∇s. It is a quick and easy check that ∇ is a *local operator*, that is to say, $(\nabla s)(x)$ depends only on the germ of s near x (in fact this is a weaker statement, it only depends on the first order of the germ of s).

Example 6.1. In the trivial line bundle $\mathbb{C}$, the projection $X \times \mathbb{C} \to \mathbb{C}$, a section of the covariant derivative amounts to a $\mathbb{C}$-linear map $\nabla\colon C^\infty(X, \mathbb{C}) \to C^\infty(T^*X \otimes \mathbb{C})$ obeying the Liebniz rule. For example, the exterior derivative d does the job, called the trivial covariant derivative. If we use the obvious Hermitian metric on $\mathbb{C}$ (given by the standard Hermitian metric of $\mathbb{C}$ independent of where you are on X), then d is unital. This is essentially the product rule.

If V is a complex vector space, we get a trivial vector bundle $V = V \times X \to X$, which carries a trivial connection as well; $d_V = d \otimes \mathrm{id}_V$.

The difference $\nabla - \nabla'$ between two covariant derivatives $E \to X$ has the property of being linear over functions($C^\infty(X)$-linear), or

$$\nabla(fs) = df \otimes s + f\nabla s$$
$$\nabla'(fs) = df \otimes s + f\nabla' s$$

Taking the difference will result in something linear over functions. So $(\nabla - \nabla')s = \alpha s$, where $\alpha_x \in \mathrm{End}_{\mathbb{C}}(E_x)$. What we find is that given a covariant derivative ∇ in E, the space of covariant derivatives is given by ∇ plus the vector bundle consisting of the cotangent bundle tensored with endormorphisms of E , or

$$\{\text{covariant derivatives in } E\} = \nabla + C^\infty\left(X, T^*X \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}} E\right)$$

The space of covariant derivatives is an affine vector space modelled on the vector space of 1-forms valued in $\mathrm{End}(E)$. In some sense they are pretty straightforward objects. We will continue with this on Wednesday.

7 Gauge transformations, curvature, and characteristic classes

7.1 Gauge transformations

Let $E \to M$ be a complex vector bundle. A **gauge transformation** is an automorphism $u\colon E \xrightarrow{\cong} E$, which form a group G_E. There is a bundle (*not* a principle bundle) of Lie groups

$$\begin{tikzcd}
\mathrm{GL}(E) \arrow[dr] & \subset & \mathrm{End}_{\mathbb{C}} E \arrow[dl] \\
& M &
\end{tikzcd}$$

with G_E the group of sections of $\mathrm{GL}(E) \to M$. When a hermitian metric in E is given, consider the **unitary** gauge transformations (sections of $U(E) \to M$). If $u\colon E \to E'$ is a vector bundle isomorphism, a covariant derivative ∇ in E induces Δ' in E'. Then

$$\Delta'_v = u \circ \nabla_v \circ u^{-1}$$

for all vector fields v. So G_E acts on the left on $\mathcal{C}_E = \{\text{covariant derivatives in } E\}$, and

$$u \cdot \nabla = u \circ \nabla \circ u^{-1}.$$

Some cultural confusion; in physics (QFT), they will sometimes describe the unitary group as the gauge group. In math, it's the group of gauge transformations. What mathematicians call the structure group is what physicists call the gauge group.

7.2 Curvature

What is the curvature of this induced covariant derivative (pullback)? It is exactly what you think it is;

$$F_{u\cdot\nabla} = u \circ F_\nabla \circ u^{-1}.$$

If you have a flat connection, e.g. if $F_\nabla = 0$ (a flat connection), then $F_{u\cdot\nabla} = 0$ for every $u \in G_E$. There are some formulas that one can work out relatively simply.

Lemma 7.1. *Suppose we want to compare the gauge transformation u acting on ∇ given by $u\cdot\nabla$, and ∇. It is given by*

$$u\cdot\nabla - \nabla = -(\nabla_u)_{u^{-1}},$$

where $(\nabla_u)_s = [\nabla, u]s = \nabla(us) - u\nabla s$.

Proof. This is just the product rule; apply the Liebniz rule for ∇ to $s = uu^{-1}s$ for s a section of E, or $\nabla s = \nabla(uu^{-1}s)$. ⊠

Example 7.1. Some noteworthy cases:

- For the trivialized bundle, $\nabla = d + A$, where A is an endomorphism valued 1-form. Then $u(d+A) - (d+A) = -[d+A, u]u^{-1}$.
- For a line bundle (rank $E = 1$), any automorphism of E_x is multiplication by a complex number. We can think of a gauge transformation $u \in G_E$ as a complex valued function on M, or $u \colon M \to \mathbb{C}$. The formula simplifies; we have $u\cdot\nabla - \nabla = -(du)u^{-1}$.

Bianchi identity. *Consider the exterior derivative d_∇ associated with ∇ and apply it to the curvature 2-form $F_\nabla \in \Omega^2_M(\mathrm{End}E)$. Taking the commutator, we have*

$$[d_\nabla, F_\nabla] = 0.$$

Proof. In local coordinates $(x_1, ijk\cdots, x_n)$, write $F_\nabla = \sum_{i,j} F_{ij}dx_{ij}$ (where $dx_{ij} = dx_i \wedge dx_j$). Write $\nabla_i = \nabla_{\partial/\partial x_i}$. Then

$$\begin{aligned}
[d_\nabla, F_\nabla] &= \sum [\nabla_i, F_{jk}]dx_{ijk} \\
&= \sum_{i,j,k} [\nabla_i, [\nabla_j, \nabla_k]]dx_{ijk} \\
&= 2\sum_{i<j<k} \big([\nabla_i, [\nabla_j, \nabla_k] + [\nabla_j, [\nabla_k, \nabla_i]] + [\nabla_k, [\nabla_i, \nabla_j]\big)dx_{ijk}
\end{aligned}$$

which are all zero by the Jacobian of the derivative. ⊠

If we consider trivialized bundles $\nabla = d + A$, the Bianchi identity is equivalent to saying that

$$d(F_{d_A}) = F_\nabla \wedge A - A \wedge F_\nabla.$$

7.3 A little bit of Chern-Weil theory

This has to deal with the topological significance of curvature.

Lemma 7.2. *The $\mathbb{C}$-valued 2-form given by $\operatorname{tr} F_\nabla \in \Omega^2_M(\mathbb{C})$ is closed ($d(\operatorname{tr} F_\nabla) = 0$) and its cohomology class in $H^2_{\mathrm{DR}}(M, \mathbb{C})$ is independent of the covariant derivative ∇.*

Proof. Bianchi says that in a local trivialization $dF_\nabla = F_\nabla \wedge A - A \wedge F_\nabla$, we have $\nabla = d + A$. Taking the trace $\operatorname{tr}(dF_\nabla)$, we have

$$\operatorname{tr}(dF_\nabla) = \operatorname{tr}(\text{commutator}) = 0.$$

On the other hand, $\operatorname{tr}(dF_\nabla) = d(\operatorname{tr} F_\nabla)$. This shows closedness. Now we want to show that $\operatorname{tr} F_{\nabla'} - \operatorname{tr} F_\nabla = d(\text{something})$. We have $\nabla' = \nabla + a$, and $F_{\nabla'} = F_\nabla + [d_{\nabla'} a] + a\nabla$, which implies that $\operatorname{tr} F_{\nabla'} = \operatorname{tr} F_\nabla + \operatorname{tr}[d_{\nabla'} a]$. It turns out that $\operatorname{tr}[d_{\nabla'} a] = \operatorname{tr} d a^{-1}$, i.e., $\operatorname{tr} F_{\nabla + a} - \operatorname{tr} F_\nabla = d(\operatorname{tr} a)$. ⊠

So from E we get a cohomology class

$$c_1(E) = \left[\frac{i}{2\pi} \operatorname{tr} F_\nabla\right] \in H^2_{\mathrm{DR}}(M),$$

called the **first Chern class**. If $f : N \to M$ is a smooth map, then $f^*E \to N$ carries a pullback connection $f^*\nabla$ with curvature $F_{f^*\nabla} = f^*F_\nabla$. From this we get right away that the first Chern class is natural, or $c_1(f^*E) = f^*c_1(E)$. This exactly says that c_1 is a **characteristic class** for complex vector bundles. It is essentially immediate that $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$, as $\oplus$ carries a direct sum covariant derivative $\nabla^1 \oplus \nabla^2$.

If we choose ∇ *unitary*, then $\operatorname{tr} F_\nabla$ is an *imaginary* 2-form. So $\frac{i}{2\pi} F_\nabla$ is **real**, or c_1 lives in a *real* (not complex) de Rham cohomology. Why the 2π? It turns out that this implies the Chern class is actually integral, or $c_1(E) \in H^2_{\mathbb{Z}} \subseteq H^2_{\mathrm{DR}}$. Chern-Weil theory goes on to consider more complicated expressions involving the curvature, e.g.,

$$\left[\frac{1}{8\pi^2} \operatorname{tr}(F_\nabla \wedge F_\nabla)\right] = c_2(E) - \frac{1}{2} c_1(E)^2.$$

We can check that this is closed an independent of the choice of covariant derivative, and we finish by writing down the identity that represents this.

$$\operatorname{tr} F^2_{\nabla + a} - \operatorname{tr} F^2_\nabla = d \operatorname{tr} \underbrace{\left\{[a \wedge d_\nabla, a] + \frac{2}{3} a \wedge a \wedge a\right\}}_{\mathrm{CS}(a)}.$$

This $\mathrm{CS}(a)$ is called the **Chern-Simons functional**, which is key in the gauge theory on 3-manifolds.

8 Two equations with gauge symmetry

todo:flat connections and gauge symmetry

8.1 Instantons

Definition 8.1. Let X be a 4-manifold equipped with a conformal structure. A **Yangs-Mills instanton**, or *anti-self-dual connection*, in the hermitian vector bundle $E \to X$, is a unitary connection $\nabla \in \mathcal{A}_E$ such that

$$(F_\nabla)^+ = 0.$$

Here $(\cdot)^+$ is the self-dual projection $\frac{1}{2}(1 + *)$, mapping $\Omega^2(\mathfrak{u}(E))$ to $\Omega^+_g(\mathfrak{u}(E))$.

For a gauge transformation $u \in \mathcal{G}_E$, one has

$$(u^* F_\nabla)^+ = (u F_\nabla u^{-1})^+ = u F^+_\nabla u^{-1}$$

so $\mathcal{G}_E$ preserves the instantons. Thus the instanton equation, like the flatness equation, posseses gauge symmetry. The equation $(F_{\nabla + A})^+ = 0$ amounts to a first order differential equation for A. Donaldson theory is the study of this equation; the focus is largely on the case of rank 2 vector bundles. Our purpose here is to understand a much simpler case, that of instantons in line bundles $L \to X$.

Theorem 8.1. *Let (X, g) be a closed, oriented Riemannian 4-manifold, and $L \to X$ a hermitian line bundle. Then L admits an instanton iff $c_1(L)$ lies in the group $\mathcal{H}^-_{[g]} \cap H^2_{\mathbb{Z}}$. If $b^+(X) > 0$ then, for a generic conformal structure $[g]$, no non-trivial line bundle admits instantons.*

Proof. todo: ⊠

9 Connections, gauge transformations, instantons in line bundles

Last time we discussed instantons in the case of line bundles. We will come back to the theorem giving criteria for the existence of instantons in line bundles later. Today we discuss the space of connections in a line bundle, and also the quotient space (or orbit space) of covariant derivatives modulo the action of the gauge group.

Notation. Let X^4 be a closed oriented 4-manifold, $L \to X$ be a hermitian line bundle (complex vector bundle of rank 1). Denote the space of unitary covariant derivatives in L by $\mathcal{A}_X = \nabla = \Omega^1_X(\mathfrak{u}(L))$, where $\mathfrak{u}(L)$ denotes skew invariant endomorphisms of L. This is equal to $\nabla + i\Omega^1_X$, a reference covariant derivative plus imaginary 1-forms.

We use the C^∞ topology on Ω^1_X (hence on $\mathcal{A}_X$). For any compact M^n, for any vector bundle $E \to M$, there is a C^∞ topology on $C^\infty(M; E)$. We will not get into the functional analysis details of this; it is defined by a translation invariant metric, making $C^\infty(M; E)$ a topological vector space (more specifically a *Fréchet space*). The metric is defined by a sequence of C^∞ norms. We will not go into it anymore, but we will say the following. Say a compact set $K \subseteq U$ open in M^n, where $U \cong \mathbb{R}^n$, so we have a chart. Say $E|_U \to U$ is trivialized, or isomorphic to $\mathbb{C}^r$. So sections of E supported in K are functions $\mathbb{R}^n \to \mathbb{R}^r$ supported in k. For sections (s_n), s supported in k, to say that $s_n \to s$ in C^∞ means that $s_n \to s$ uniformly and also the sequence of partial derivatives

$$\frac{\partial^\alpha}{\partial x^\alpha} s_n \to \frac{\partial^\alpha}{\partial x^\alpha} s$$

converges uniformly for all multi-indices α.

We have our space $\mathcal{A}_L = \nabla \to i\Omega^1_M$ an affine Fréchet space. We also have the gauge group $\mathcal{G}_L$ of unitary gauge transformations which is just $C^\infty(X, U(L))$. Gauge transformations necessarily act by complex scalars, which on the complex line have norm 1. So this is the space of circle valued functions $C^\infty(X, U(1))$. $\mathcal{G}_L$ also has a C^∞ topology and acts on the space of covariant derivatives. In the case of line bundles, the action is simply given by $u \cdot \nabla = \nabla - (du)u^{-1}$. Then we have the orbit space $\mathcal{B}_L = \mathcal{A}_L / \mathcal{G}_L$.

Nothing about this quotient space is given for free. For example, it is not terribly obvious that this quotient space is Hausdorff. It turns out that it is Hausdorff, even if we replace L by a higher rank bundle. In the line bundle case there's a concrete picture. We find that $\mathcal{B}_L \cong (\text{Fréchet space}) \times (S^1)^{b_1(X)}$ which is Hausdorff, much more than that rather something nice. There is no such simple picture in higher rank.

Observations. Let X be connected.

- There is a subgroup $U(1) \subseteq \mathcal{G})L$ (continuous gauge transformations)
- This subgroup acts trivially on $\mathcal{A}_L$, or $du = 0$.
- The quotient $\mathcal{G}_L / U(1)$ acts *freely* on covariant derivatives.

What is the group of connected components $\pi_0(\mathcal{G}_L)$? This is the group of homotopy classes $X \to S^1$, which is the first cohomology group $H^1(X, \mathbb{Z})$. One way to see the bijection is that $H^1(X, \mathbb{Z}) \subseteq H^1_{\text{DR}}(X)$ as the integer lattice, then map $[u] \mapsto [du] \in H^1_{\text{DR}}(X)$. We can concretely write down the identity component $\mathcal{G}^0_L \subseteq \mathcal{G}_L$, which consists of gauge transformations u which have a *logarithm*:

$$u = e^{i\xi}, \quad \xi : X \to \mathbb{R}.$$

It acts on covariant derivatives as follows:

$$\left(e^{i\xi}\right)\cdot\nabla = \nabla - i\,d\xi.$$

So it acts on $\mathcal{A}_L$ by adding an exact 1-form. Fix a reference covariant derivative ∇. Then $\mathcal{A}_L/\mathcal{G}_L^0 \cong i\left(\frac{\Omega_X^1}{d(\Omega_X^0)}\right)$.

9.1 Gauge fixing

We talk about the gauge slice, or gauge fixing. The Hodge decomposition implies that $\Omega_X^1 = d\Omega_X^0 \oplus \ker d^*$. We take the "*Coulomb gauge slice*", defined by

$$\mathcal{S} = \nabla + i\ker d^* \subseteq \mathcal{A}_L.$$

Then projection $\mathcal{S}\to\mathcal{A}_L/\mathcal{G}_L^0$ is a homeomorphism $\mathcal{S}\xrightarrow[\text{homeo}]{\cong}\mathcal{A}_L/\mathcal{G}_L^0$. There is nothing quite so simple for higher dimensional vector bundles. The nomenclature comes the fact that line bundles is a good language for describing electromagnetism. The Coulomb condition measure some condition for measuring the potential of the divergence of some electric field.

We have that $\pi_0\mathcal{G}_L = H^1(X,\mathbb{Z})$ acts on $\mathcal{A}_L/\mathcal{G}_L^0 = \mathcal{S}_L$. We know it acts because the whole gauge group acts, but what is the action specifically? Take a gauge transformation u, then

$$u\cdot\nabla = \nabla - (du)u^{-1} = \nabla - d(\log u)$$

for $d(\log u)$ a closed 1-form. Even though $\log u$ is not well defined as a random branch of the complex logarithm, $d(\log u)$ is. Then the class $[d(\log u)]\in H^1(X;\mathbb{Z})$ lives in the first integer cohomology. We can find a function ξ such that the closed 1-form $d(\log u)+d\xi$ is harmonic, or co-closed (in $\ker d^*$). Then $\nabla + d(\log u) = d\xi$ lies in $\mathcal{S}_L$. This describes the action of $H^1(X;\mathbb{Z})$ on $\mathcal{S}_L$. Then the orbit space

$$\mathcal{B}_L = \mathcal{A}_L/\mathcal{G}_L \cong \mathcal{S}_L/\pi_0\mathcal{G}_L \cong \underbrace{\frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})}}_{\text{Picard torus}} \times \operatorname{im} d^*.$$

What we use here is that $\mathcal{S}_L = \nabla + i\ker d^*$, but $\ker d^* = \mathcal{H}^1\oplus\operatorname{im} d^*$ by the Hodge decomposition. The Picard torus is given by $\mathcal{P}\cong\left(S^1\right)^{b_1(X)}$.

9.2 Curvature

We will not get to instantons, but at least we can cover curvature. Observe that $F_{u\cdot\nabla} = uF_\nabla u^{-1}$, but since $u\in U(1)$ (for line bundles), this is just F_∇. So for line bundles, curvature is fully *gauge invariant*. Then

$$F_{\nabla+ia} = F_\nabla - i\,da,$$

so curvature is basically the exterior derivative. Then F is defined on $\mathcal{B}_L$. Suppose that for instance we want to think about the set of gauge orbits of connections that have the same curvature as ∇, or $\{\nabla'\in\mathcal{A}_L \mid F_{\nabla'} = F_\nabla\}/\mathcal{G}_L$; this lies in $P\times\operatorname{im} d^*$. Prescribing curvature is a copy of $\mathcal{P}$, where $\mathcal{P}\times 0\subseteq\mathcal{P}\times\operatorname{im} d^*$.

10 The moduli space of instantons on the line bundle

We will also preview Seiberg-Witten invariants today. Let (X^4,g) be our closed Riemannian oriented 4-manifold, and $L\to X$ our hermitian line bundle. We were talking about our space of connections $\mathcal{A}_L = \{\text{unitary covariant derivatives in } L\}$ which $\mathcal{G} = C^\infty(X,S^1)$ acts on. Then we had our quotient $\mathcal{B}_L = \mathcal{A}_L/\mathcal{G}$; things simplify if we choose a reference connection $\nabla_{\text{ref}}\in\mathcal{A}_L$. We had the Coulomb gauge slice $\mathcal{S}_L = \nabla_{\text{ref}} + i\ker d^*$. Then $\mathcal{S}_L\xrightarrow{\cong}\mathcal{A}_L/\mathcal{G}^0$ (the identity component of $\mathcal{G}$), and $\mathcal{B}_L = \mathcal{S}_L/\pi_0\mathcal{G} = \mathcal{S}_L/H^1(X;\mathbb{Z})$. So $\mathcal{B}_L\cong\left(\mathcal{P} = \frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})}\cong\left(S^1\right)^{b_1(X)}\right)\times(\operatorname{im} d^*)$.

10.1 Instantons

We have $\nabla \in \mathcal{A}_L, F^+_\nabla = 0$. We observed that the existence of such a ∇ implies

$$c_1(L) \in H^2_{\mathbb{Z}} \cap \mathcal{H}^-_{[g]} \subseteq H^2_{\mathrm{DR}}(X).$$

We claimed that the converse holds.

Proof. Say $c_1(:) \in H^2_{\mathbb{Z}} \cap \mathcal{H}^-_{[g]}$. Pick some random connection $\nabla_0 \in \mathcal{A}_L$. We have $\frac{i}{2\pi}\left[F_{\nabla_0}\right] = c_1(L) \in H^2_{\mathbb{Z}}$. By the Hodge theorem, there exists a harmonic 2-form representative for $c_1(L)$, given by $\eta \in \mathcal{H}^2_{[g]}, [\eta] \in c_1(L)$. Then η is anti self dual. We would like to find a covariant derivative ∇ such that $\frac{i}{2\pi}F_\nabla = \eta$, because that is then an instanton. If $\eta - \frac{i}{2\pi}F_{\nabla_0} = da$, then $\nabla = \nabla_0 - 2\pi i a$ has curvature $F_\nabla = F_{\nabla_0} - 2\pi i da = \eta$. ⊠

This most definitely only works for line bundles. Curvature can be entirely captured by the cohomology class (while for higher vector bundles it only captures a portion through the trace), and is mediated by the simple expression da.

10.2 Uniqueness of instantons

The simple reason that instantons are not unique is that they are invariant under the gauge group, so if you have one, you have a whole gauge orbit of them. The question is how it parametrizes the gauge orbit. Let $\mathcal{I}_L \subseteq \mathcal{A}_L$ denote the space of instantons.

Proposition 10.1. *The subspace $\mathcal{I}_L/\mathcal{G} \subseteq \mathcal{B}_L$ is isomorphic to the Picard torus $\mathcal{P}$.*

Proof. Say $\nabla \in \mathcal{I}$. Recall that $d^+ = \frac{1}{2}(\mathrm{id} + *) \circ d \colon \Omega^1 \to \Omega^+_g$. Then $\nabla + ia \in \mathcal{I}_L \iff d^+ a = 0$. We need to look at $\ker d^+$. Recall the self duality complex $\mathcal{E}^*$, given by:

$$0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to 0,$$

where

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^+ \to 0$$

by definition. Computing the cohomology, $H^1(\mathcal{E}) = \frac{\ker d^+}{\operatorname{im} d} = \frac{\ker d}{\operatorname{im} d} = H^1_{\mathrm{DR}}(X)$. For $\mathcal{I}_L \cap \mathcal{S}_L = \nabla + i(\ker d^+ \cap \ker d^*) = \nabla = i\mathcal{H}^1_g = \nabla + iH^1_{\mathrm{DR}}(X)$. Then

$$\mathcal{I}_L/\mathcal{G} = (\mathcal{I}_L \cap \mathcal{S}_L)/H^1(X;\mathbb{Z}) = [\nabla] + \underbrace{\frac{i}{2\pi} \cdot \frac{H^1_{\mathrm{DR}}(X)}{H^1(X;\mathbb{Z})}}_{\mathcal{P}}$$

where $\mathcal{I}_L/\mathcal{G}$ sits in $\mathcal{B}_L = \mathcal{A}_L/\mathcal{G} = \mathcal{S}_L/(H^1(X;\mathbb{Z}) = \pi_0\mathcal{G})$. ⊠

We have $\mathcal{I}_L/\mathcal{G} = (\mathcal{I}_L \cap \mathcal{S}_L)/\pi_0\mathcal{G}$ cut out as a manifold. Then F is defined on $\mathcal{A}_L$, $F(\nabla) = F^+_\nabla$. We are interested in $F^{-1}(0)/\mathcal{G}$. Instead look at $F'(\nabla) = (F(\nabla), d^*(\nabla - \nabla_{\mathrm{ref}})$, where $(F')^{-1}(0) \subseteq \mathcal{S}_L$. We have

$$\ker DF' = \ker(d^+ \oplus d^*) = \mathcal{H}^1,$$

while

$$\operatorname{coker} DF' = \operatorname{coker}(d^+ \oplus d^*) = \Omega^+/\operatorname{im} d \oplus \operatorname{coker} d^* = \mathcal{H}^+ \oplus \mathbb{R}.$$

The cokernel is not zero, but has constant rank by the constant rank theorem, a corollary of the inverse function theorem. $F'^{-1}(0)$ is a *clean* level set, i.e., submanifold of a domain. This is true in finite dimensions, or when your spaces are Banach spaces; we need to address this! We will fix this later using Sobolev spaces.

10.3 Preview of Seiberg-Witten theory

On any oriented manifold M, there is a set $\mathrm{Spin}^c(M)$, which is the set of spin^c-structures on M modulo isomorphism. When it's non-empty, it's a torsor for $H^2(M;\mathbb{Z})$. The set $\mathrm{Spin}^c(M)$ is natural under oriented diffeomorphism. In dimension 4, , $\mathrm{Spin}^c(X) \neq \emptyset$. Given $\mathfrak{s} \in \mathrm{Spin}^c$ a spin^c-structure, we get a pair of rank 2 hermitian vector bundles $\mathbb{S}^+ \to X$, $\mathbb{S}^- \to X$ and a bundle map $T^*X \xrightarrow{\rho} \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-)$ which satisfies certain properties (a Clifford relation). These are the positive and negative spinor bundles. There is a notion of a *spin connection* in $\mathbb{S}^+$, which induce a unitary connection in the line bundle $\Lambda^2\mathbb{S}^+$, where $\nabla \mapsto \nabla^t$. This correspondence turns out to be a bijection. The **Seiberg-Witten configuration space** $\mathcal{C}$ is given by

$$\mathcal{C} = \{\text{spin connections in } \mathbb{S}^+\} \times \Gamma(\mathbb{S}^+).$$

Then $\mathcal{G} = C^\infty(X, U(1))$ acts on $\mathcal{C}$, where $u \cdot (\nabla, \phi) = (u \cdot \nabla, u\phi)$, and $u \cdot \nabla^t = \nabla^t - 2(du)u^{-1}$. Because of the passage from spinor line bundles to line bundles we have to tweak our original equation and add a 2. The action of $\mathcal{G}$ on $\mathcal{C}$ is *free* except where $\phi \equiv 0$. Then $\mathcal{C}^*$ is local where $\phi \not\equiv 0$, and

$$\mathcal{B}^* \subseteq \mathcal{C}^*/\mathcal{G} \cong \mathcal{P} \times (\operatorname{im} d^*) \times \frac{(\Gamma(\mathbb{S}^+) \setminus \{0\})}{U(1)}.$$

We write down a gauge invariant equation on $\mathcal{C}$, which gives rise to a moduli space of solutions $\mathcal{M} \subseteq \mathcal{C}/\mathcal{G}$. When $\mathcal{M}$ avoids the locus $\phi \equiv 0$, then $\mathcal{M} \subseteq \mathcal{B}^*$. It turns out that it carries a fundamental homology class $[\mathcal{M}]$ in the homology $H_*(\mathcal{B}^*)$. This class is the **Seiberg-Witten invariant** for the specified spin^c-structure.

11 Preview of Seiberg-Witten theory

We continue our hurried overview from last time.

11.1 The Seiberg-Witten equations

Let (X^4, g) be a closed, oriented Riemannian manifold, and let $\mathrm{Spin}^c(X)$ be the set of isomorphism classes of spin^c structures, which acts freely and transitively on $H^2(X;\mathbb{Z})$. For $\mathfrak{s} \in \mathrm{Spin}^c(X)$, it leads to a spinor bundle $\mathbb{S}^\pm \to X$, which is a rank 2 and hermitian vector bundle. Clifford mulitplication is given by $T^*X \xrightarrow{\rho} \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-)$, and the configuration space $\mathcal{C} = \{\text{spin connections } \nabla \text{ in } \mathbb{S}^+\} \times \Gamma(X, \mathbb{S}^+)$ where $\Gamma(X, \mathbb{S}^+)$ consists of C^∞ sections of $\mathbb{S}^+$. The set of spin connections is identified with $U(1)$ connections in $\det \mathbb{S}^+ = \Lambda^2\mathbb{S}^+$. The gauge group is given by $\mathcal{G} = C^\infty(X, U(1))$, and define $\mathcal{B} = \mathcal{C}/\mathcal{G}$. A $\mathcal{G}$-action on $\mathcal{C}$ is free except at "irreducible configurations", $(\nabla, 0) \in \Gamma(\mathbb{S}^+)$. The set of irreducible configurations $\mathcal{C}^* \to \mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ which has the same homotopy type of $\mathcal{P} = \frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})} \times \mathbb{CP}^\infty$, $\Gamma(\mathbb{S}^+)/U(1)$. Then

$$\begin{aligned} H_* B^* &= H_*(\mathcal{P} \times \mathbb{CP}^\infty) \\ &= H_*\mathcal{P} \otimes H_*(\mathbb{CP}^\infty) \quad \text{(plus torsion terms)} \\ &= \Lambda^* H^1(X;\mathbb{Z}) \otimes \mathbb{Z}[U] \quad \text{(deg 2)} \end{aligned}$$

There were then the Seiberg-Witten equations; for a pair (∇, ϕ), this induces a connection ∇^t in $\Lambda^2\mathbb{S}^+$, and therefore is a configuration in $\mathcal{C}$. The equations are as follows:

- **Dirac equation:** $D_\nabla \phi = 0$, where D is the *Dirac operator* (a linear first order differential operator taking sections $\Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$). This is an affine bilinear equation in (∇, ϕ).
- **Curvature equation:** The Clifford equation ρ induces an endomorphism $\rho \colon \Lambda^+ \to \mathfrak{su}(\mathbb{S}^+)$. Then apply ρ to get $\rho(F_{\nabla^t} = (\phi \otimes \phi^*)_0$ (the subscript 0 means trace free).

- We also impose a Coulomb gauge fix on ∇^t, where $d^*(\nabla^t - \nabla^t_{\mathrm{ref}}) = 0$.

These three equations together are **elliptic**. That implies that their linearization at a solution is **Fredholm**, which is to say it has finite dimensional kernel and cokernel.

There are the reducible solutions, where $(\nabla, \phi)\phi \equiv 0$. The Dirac equation becomes vacuous, and $Fs^+_{\nabla^t} = 0$, i.e., ∇^t is an *instanton* in $\det \mathbb{S}^+$. If $b^+ > 0$ and $\det \mathbb{S}^+$ is a non-trivial line bundle, then there do not exist instantons for generic conformal structures g (proved in class modulo the proof of the Hodge theorem). Then all solutions to the Seiberg-Witten equations are *irreducible*. So we have the **Seiberg-Witten moduli space** $\mathcal{M}_{\mathfrak{s},g} \subseteq \mathcal{B}^*$. The nice situation is when the linearization of the SW equations at any solution is surjective, or coker $= 0$. This is the the transverse case, and in this case, $\mathcal{M}_{\mathfrak{s},g}$ is a submanifold of $\mathcal{B}^*$. In general, one proves a generic transversality theorem; for generic metrics g, the nice situation applies. The dimension of the moduli space is given by

$$\dim \mathcal{M}_{\mathfrak{g},g} = \dim \ker \mathcal{D}_{(\nabla,\phi)} = \operatorname{index} \mathcal{D}_{(\nabla,f)} = \frac{1}{4}\left(c_1(\Lambda^2\mathbb{S}^+)^2[X] - 2\chi(X) - 3\tau(X)\right) \in \mathbb{Z}$$

where $\mathcal{D}$ is the linearization of SW, and the index is dim ker − dim coker. The index is much better behaved, and can be computed by topological formulas like the Atiyah-Singer index theorem, which leads to the formulation above. Here $c_1(\Lambda^2\mathbb{S}^+)$ is the characteristic vector. Then a Seiberg-Witten miracle happens; $\mathcal{M}_{\mathfrak{s},g}$ is compact. Towards the end of the course we talk about the proof of this fact. It is also orientable; an orientation for $H^0_{\mathrm{DR}}(X) \oplus H^1_{\mathrm{DR}}(X) \oplus \mathcal{H}^+_{\mathcal{G}}(X)$ determines an orientation for $\mathcal{M}_{\mathfrak{s},g}$ (we call this a "homology orientation"). Then we have a fundamental homology class $[\mathcal{M}_{\mathfrak{s},g}] \in H_{d(\mathfrak{s})}(\mathcal{B}^*)$, and $\mathrm{SW}_X(\mathfrak{s})$ is essentially $[\mathcal{M}_{\mathfrak{s},g}] \in (\Lambda^* H^1 \otimes \mathbb{Z}[U])$. In many cases $d(\mathfrak{s}) = 0$, and $\mathcal{M}_{\mathfrak{s},g}$ is an oriented 0-manifold, or a signed set of points. Then $\mathrm{SW}_X(\mathfrak{s}) = \#\mathcal{M}_{\mathfrak{s},g}$.

What justifies calling this a Seiberg-Witten *invariant*? We have a parameter here, the metric. So we need to check metric invariance. Consider $\mathcal{M}_{\mathfrak{s},g_0}$ versus $\mathcal{M}_{\mathfrak{s},g_1}$. Pick a generic path $\{g_t\}$ of metrics. Then the parametric moduli space is given by $\mathcal{D} = \{(t \in [0,1], [\nabla, \phi] \in \mathcal{M}_{\mathfrak{s},g_t}\}$. This gives a cobordism from $\mathcal{M}_{\mathfrak{s},g_0} \to \mathcal{M}_{\mathfrak{s},g_1}$;

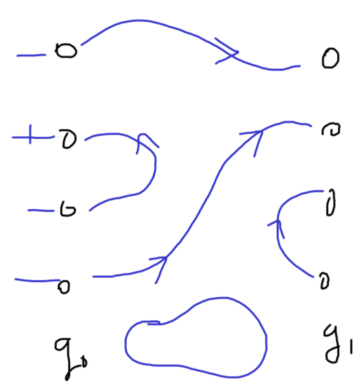

Figure 2: The cobordism of moduli spaces.

We need $\mathcal{P}$ cut out transversely insides $[0,1] \times \mathcal{B}^*$, avoiding reducible solutions for all g_t. If $b^+ > 1$, generic paths of metrics have no instantons. If $b^+ = 1$, we could encounter instantons in the path. This is the outcome:

- If $b^+X > 1$, we get $\mathrm{SW}_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$, and

$$\mathrm{SW}_X(\mathfrak{s}) = \begin{cases} \langle [\mathcal{M}_{\mathfrak{s},g}], U^{d((s)/2} \rangle & \text{if } d(\mathfrak{s}) \in 2\mathbb{Z}, \\ 0 & \text{if } d(\mathfrak{s}) \text{ is odd.} \end{cases}$$

- If $b^+X = 1$, $\mathrm{SW_X}$ depends on additional data.

- If $b^+X = 0$, we have no meaningful invariant.

We still need to talk about spin geometry and elliptic operators. Then we will get into Seiberg-Witten specific things like the compactness of the moduli space.

12 Elliptic operators and their symbols

Last time, for vector bundles $E \to M, F \to M$ vector bundles, we discussed the operator $\mathcal{D}\colon \Gamma(E) \to \Gamma(F)$. Then $\mathcal{D}$ is a 1st order operator iff $\mathcal{D} = L \circ j^1$. For every $f \in C^\infty(M)$, we have the zeroth order $[\mathcal{D}, f]$, where $j^1 s \in \Gamma(J^1 E)$, $L\colon J^1E \to F$. So that's a fancy way of saying 1st order operators only depend on first order derivatives. The symbol

$$\frac{\mathcal{D}(E,F)_1}{\mathcal{D}(E,F)_0} \xrightarrow[\sigma^1]{\cong} \Gamma(\mathrm{Hom}(T^*M \otimes E, F))$$

sends for $\xi_x \in T_x^*M$, $\sigma^1_{\mathcal{D}}(\xi_x)\colon E_x \to F_x$, where $\xi_x = (df)_x$, $\sigma^1_{\mathcal{D}}(\xi_x) = [\mathcal{D}, f]_x$.

Example 12.1. For the exterior derivative $d\colon \Omega^k_M \to \Omega^{k+1}_M$, we have $\sigma^1_d(\xi_x)\colon \Lambda^k_x \to \Lambda^{k+1}_x$, since $[d, f] = df \wedge -$. So the symbol expresses that ξ_x is wedged with something else.

Example 12.2. Consider the Dirac operator, with the trivial $\mathbb{C}^2$ bundle on $\mathbb{C}^3$. Then

$$\mathcal{D} = \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} + \sigma_3 \frac{\partial}{\partial x_3}.$$

The matrices are then given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

These are matrices chosen with the property that $\sigma_k^2 = -I$, and $\sigma_j\sigma_k + \sigma_k\sigma_j = 0$ if $j \neq k$. Then the symbol $\sigma^1_{\mathcal{D}}(dx_j) = \sigma_j$.

The reason why this operator was introduced is because it squares to the geometer's Laplacian, or $\mathcal{D}^2 = -\sum \frac{\partial^2}{\partial x_j^2}$ (minus of the Laplacian).

Example 12.3 (Formal adjoints)**.** Suppose we have the formal adjoints E, F with hermitian metrics (for example bundles) or Euclidian if they're real. Let g be a Riemannian metric on M. We have a first order operator $\mathcal{D}\colon \Gamma(E) \to \Gamma(F)$. Then a formal adjoint $\Gamma(E) \xleftarrow{\mathcal{D}^*} \Gamma(F)$ goes in the opposite direction, and is characterized by the fact that $\langle \mathcal{D}s, s' \rangle = \langle s, \mathcal{D}^* s' \rangle$ for every $s \in \Gamma(E)$, $s' \in \Gamma(F)$. These are L^2 inner products, or $\int_M (Ds, s') d\mu|_g$. Then if you think for a little bit, the following two expressions

$$\langle t, [f, \mathcal{D}]s \rangle_{L^2(F)} = \langle [D^*, f]t, s \rangle_{L^2(E)}$$

are the same. This tells us that the symbol of a formal adjoint applied to some cotangent vector is the same as the symbol of $\mathcal{D}$ applied to ξ, the taking the adjoint operator. In other words,

$$\langle t, [f, \mathcal{D}]s \rangle_{L^2(F)} = \langle [D^*, f]t, s \rangle_{L^2(E)} \implies \sigma^1_{\mathcal{D}^*}(\xi) = -(\sigma^1_{\mathcal{D}}(\xi))^T.$$

Example 12.4. We can compute the symbol of $d^*\colon \Omega^k_M \to \Omega^{k-1}_M$; its symbol is simply given by $\sigma^1_{d^*}(\xi) = -\sigma^1_d(\xi)^* = -(\xi \wedge -)^*$. We just have to work out what the wedging operation is, which turns out to be pretty simple. Here we use the metric on forms induced by the Riemannian metric. So by a little bit of algebra,

$$\sigma^1_{d^*}(\xi_x) = -i(\xi_x^\flat), \quad \Lambda^k_x \to \Lambda^{k-1}_x, \ \xi^\flat \in T_xM, \ \xi_x = g(\xi^\flat, -).$$

We have not yet seen a formula for d^*, but in a sense this gives one.

Definition 12.1. The first order operator $\mathcal{D} \in \mathcal{D}_1(E,F)$ is called **elliptic** if $\sigma^1_{\mathcal{D}}(\xi_x)\colon E_x \to F_x$ is a vector space isomorphism for all $x \in M$ and for all $(\xi_x \neq 0) \in T^*_x M$.

Example 12.5. The Dirac operator $\mathcal{D}$ over $\mathbb{R}^3$ is elliptic since for $\sigma^1_{\mathcal{D}}(\xi_x)\colon \mathbb{C}^2 \to \mathbb{C}^2$, we have $\sigma^1_{\mathcal{D}}(\xi)^2 = -|\xi|^2 I$. This follows from the commutation relations between the matrices.

Example 12.6. The exterior derivative takes k-forms to $k+1$-forms; this cannot be elliptic since the domain and codomain have different dimensions. But if we add it to the codimension and think of it as an operator on forms of arbitrary (mixed) degree, this has a chance of being elliptic.

$$d \oplus d^*\colon \Omega^*_M \to \Omega^*_M$$

The symbol $\sigma^1_{d\oplus d^*}(\xi_x)\colon \Lambda^*_x \to \Lambda^*_x$ is given by $\sigma^1(\xi) = (\xi \wedge -) - i(\xi^\flat)$. It is not immediately obvious that this is an isomorphism, but you can check readily that $\sigma^1_{d\oplus d^*}(\xi)^2 = -|\xi|^2 \operatorname{id}$.

Definition 12.2. A 1st order operator on $:\Gamma(E) \to \Gamma(E)$ is called a **Dirac operator** if its symbol $\sigma^1_{\mathcal{D}}(\xi)^2 = -|\xi|^2 \operatorname{id}$.

Example 12.7. The **quantum Dirac operator** on $\mathbb{R}^3$ is $d \oplus d^*$.

These examples are formally self adjoint, or $\mathcal{D}^* = \mathcal{D}$ (along with all operators of geometric interest). For a Dirac operator, $\sigma^1_{\mathcal{D}^*} = \sigma^1_{\mathcal{D}}$, so $\mathcal{D} - \mathcal{D}^*$ is a 0th order operator. Some Dirac operators are $\mathbb{Z}/2$-graded, i.e. $E = E^{\mathrm{ev}} \oplus E^{\mathrm{odd}}$, where $\mathcal{D}\colon \Gamma(E^{\mathrm{ev}}) \leftrightarrows \Gamma(E^{\mathrm{odd}})$.

Example 12.8. Let (X^4, g) be a compact oriented Riemannian manifold, then

$$d^* \oplus d^+\colon \Omega^1_X \to \Omega^0_X \oplus \Omega^+_g, \quad \alpha \mapsto d^*\alpha \oplus (d\alpha)^+.$$

The symbol $\sigma^1(\xi)\colon \Lambda^1_x \to \Lambda^0_x \oplus \Lambda^+_x$ is given by $\sigma^1(\xi) = -i(\xi^\flat) \oplus (\xi \wedge -)^+$. A quick check with an orthonormal basis on $\mathbb{R}^4$ shows that if $\xi \neq 0$, then $\sigma^1(\xi)$ is an isomorphism.

12.1 Higher order operators and the Laplacian

We have nth order operators $\mathcal{D}\colon \Gamma(E) \to \Gamma(F)$ vector spaces $\mathcal{D}_n(E,F)$, where $\mathcal{D} \in \mathcal{D}_n(E,F) \iff [\mathcal{D}, f] \in \mathcal{D}_{n-1}(E,F)$ for every $f \in C^\infty(M)$. Once again there is a story with jets; this is equivalent to saying that $\mathcal{D} = L \circ j^n$ for some $L\colon J^n E \to F$, where $J^n E$ is the bundle of n-jets of sections of E. Sections have the same n-jets at x iff they have the same Taylor expansion to some order in the coordinates at x. There is an isomorphism $\sigma^n\colon \frac{\mathcal{D}_n(E,F)}{\mathcal{D}_m(E,F)} \to \Gamma(\operatorname{Sym}^n(T^*X) \otimes E, F)$, which means degree n homogeneous polynomial functions on a vector space $T^*_x M$. As in the first order case, we can set up this isomorphism abstractly, or write it down in terms of commutators and functions. The formula is that for $\xi \in T^*_x M$, $\xi = (df)_*$

$$\sigma^n_{\mathcal{D}}(\underbrace{\xi, \cdots, \xi}_{n})\colon E_x \to F_x, \quad \frac{1}{n!}[\cdots[[\mathcal{D}, f], f]\cdots f]$$

for n copies of f, which is a linear map. This actually respects composition, that is, for $D_1 \in \mathcal{D}_m(E,F)$, $D_2 \in \mathcal{D}_n(F,G)$, their composite $D_2 \circ D_1 \in \mathcal{D}_{m+n}(E,G)$ with symbol given by $\sigma^{m+1}_{D_2 \circ D_1} = \sigma^n_{D_2} \circ \mathcal{D}^m_{D_1}$.

13 Spinors, continued

We had k a field, (V,q) a quadratic k-space (vector space with a quadratic form on it). We build from that the Clifford algebra $\mathrm{Cl}(V,q) = \mathrm{Cl}^0 \oplus \mathrm{Cl}^1$, a $\mathbb{Z}/2$-graded associative unital k-algebra. If $\dim V = n$, then $\dim \mathrm{Cl}(V,q) = 2^n$.

We began on spinors; a **spinor module** over some extension field K is a $\mathbb{Z}/2$-graded K-vector space $S = S^+ \oplus S^-$, with a representation $\rho\colon \mathrm{Cl}(V,q)\otimes K \to \mathrm{sEnd}(S)$, where

$$\mathrm{Cl}^0 \mapsto \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \quad \mathrm{Cl}^0 \mapsto \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$

The spinor condition is that ρ is actually an isomorphism of $\mathbb{Z}/2$-graded K-algebras.

13.1 Construction of spinor modules

Say $(V_k, q_k) \cong (L \oplus L^\vee, \mathrm{ev})$ with $\lambda \in L, \mu \in L^\vee$. Then $\mathrm{ev}(\lambda+\mu) = \mu(\lambda)$. If q is non-degenerate of even rank and K is algebraically closed ($\mathbb{C}$), fixing the rank all non-degenerate forms are isomorphic. Set $S = \Lambda^* L^\vee = \Lambda^{\mathrm{ev}} L^\vee \oplus \Lambda^{\mathrm{odd}} L^\vee$. When $\dim L = m$, we have

$$\dim \mathrm{Cl}(V) = 2^{2m} = (2^m)^2,$$

so we want $\dim S = 2^m$ where $(S^+ = \Lambda^{\mathrm{ev}} L^\vee) \oplus (S^- = \Lambda^{\mathrm{odd}} L^\vee)$. We have that $\lambda \in L$ leads to an "annihilation operator" $a(\lambda) = \iota_\lambda\colon \Lambda^k L^\vee \to \Lambda^{k-1} L^\vee$, and $\mu \in L^\vee$ leads to a "creation operator" $c(\mu) = \mu \wedge -\colon \Lambda^k L^\vee \to \Lambda^{k+1} L^\vee$. The formula is that

$$\rho\colon L \oplus L^\vee \to \mathrm{End}^1(S), \quad \rho(\lambda,\mu) = c(\mu) - a(\lambda).$$

One can check that the Clifford relations lead to an isomorphism $\rho\colon \mathrm{Cl}(L\oplus L^\vee) \to \mathrm{sEnd}(S)$. The easy case is where $\dim L = 1$. Then $L \oplus L^\vee$ with $e \in L$ and $e^\vee \in L^\vee$ has basis $1, e, e^\vee, ee^\vee$, for the Clifford algebra, where $S = K \oplus L^\vee$. Then

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(e) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \rho(e^\vee) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \rho(ee^v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which are isomorphic to $\mathrm{sEnd}(S)$. In general, say $L = L_1 \oplus \cdots \oplus L_m$ is a sum of lines, where $\mathrm{Cl}(L\oplus L^\vee) \cong \widetilde{\bigotimes}_i \mathrm{Cl}(L_i \oplus L_i^\vee)$, and $\Lambda^* L^\vee \cong \widetilde{\bigotimes} \Lambda^* L_i^\vee$, $\rho = \bigotimes \rho_i$ isomorphisms. For example, $\mathrm{Cl}(\mathbb{R}^4, |\cdot|^2) \otimes \mathbb{C} \xrightarrow{\cong} \mathrm{sEnd}(\Lambda^* \mathbb{C}^2)$, where $S^+ = \Lambda^0 \oplus \Lambda^2, S = \Lambda^1$.

So far, we have the Clifford algebras and spinor modules. The next thing is the rigidity of spinors, and from this we get a projective (scalar ambiguity) action of $O(V)$ on S, or alternatively, on the Clifford algebra $\mathrm{Cl}(V,q)$ itself by invertible automorphisms. Differentiate to get a projective action of the Lie algebra of $O(V)$ on $\mathrm{Cl}(V,q)$ by inner derivations. This can be made concrete, or there is a formula. The spin groups allow us to resolve the scalar ambiguity, and in some sense that is what spin groups are for.

13.2 Rigidity over $\mathbb{C}$

The spinor module $S = S^+ \oplus S^-$ and its parity reversed version $\Pi S = S^- \oplus S^+$ are the only indecomposable representations. Every finite dimensional $\mathbb{Z}/2$-graded representation of $\mathrm{Cl}(V,q)$ is isomorphic to $S^{\oplus r} \oplus (\Pi S)^{\oplus s}$. Why? It is a standard fact from algebra is that the endormorphism of a vector space has S as its unique indecomposable (left) module. Since $\mathrm{Cl}(V,q) \cong \mathrm{End}(S)$ (as an ungraded algebra), any module is a sum of copies of S. We need to account for the $\mathbb{Z}/2$-grading. The reason is the following; the even Clifford algebra $\mathrm{Cl}^0(V,q) \cong \mathrm{End}(S^+) \times \mathrm{End}(S^-)$, so its center $Z = \mathbb{C} \times \mathbb{C}$ reads off the decomposition of S into $S^+ \oplus S^-$ by looking at extension spaces of central involutions. This corresponds to a central involution $\omega \in \mathrm{Cl}^0(V,q)$, where $\omega^2 = 1$. Given a Clifford module T which is indecomposable, we know that $T \cong S$ in an ungraded sense. We read off the $+,-$ parts from the eigenspaces of ω. Now $\omega = e_1 e_1^\vee \cdots e_m e_m^\vee$.

Now we come to the projective action. For $g \in O(V)$, this implies that $\mathrm{Cl}(g) \in \mathrm{Aut}\mathrm{Cl}(V,q)$, $v_1 \cdots v_k \mapsto g v_1 \cdots g v_k$. We have the action $\rho\colon \mathrm{Cl}(V,q) \to \mathrm{sEnd}(S)$, but we can also look at the representation $\rho \circ \mathrm{Cl}(g)$. By rigidity, these two representations are isomorphic, that is, there exists some $\overline{g}\colon S \to S$ so that $\rho \circ \mathrm{Cl}(g) \circ \overline{g} = \overline{g} \circ \mathrm{Cl}(g)$. We have that $\overline{g}$ is defined up to scalars, that is, we get a representation $\Theta\colon g \to \mathrm{Aut}(S)/K^\times \mapsto \overline{g}$. The claim is that $[\rho \circ \mathrm{Cl}(g)] \circ \overline{g} = \overline{g} \circ \rho$.