

Algebraic Topology Miscellaneous Notes

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Miscellaneous notes for the Fall 2020 graduate section of Algebraic Topology (Math 382C) at UT Austin, taught by Dr. Allcock. The course was loaded with pictures and fancy diagrams, so I didn't \TeX any notes for the lectures themselves. However, I did take some miscellaneous supplementary notes, here they are. Source files: https://git.simonxiang.xyz/math_notes/files.html

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§1 Category Theory

Today we talk about abstract nonsense! These notes will follow Evan Chen's Napkin §60 and May's "A Concise Course in Algebraic Topology" §2.

§1.1 Motivation

Why do we talk about categories? Categories rise from objects (sets, groups, topologies) and maps between them (bijections, isomorphisms, homeomorphisms). Algebraic topology speaks of maps from topologies to groups, which makes maps between categories a suitable tool for us.

Example 1.1. Here are some examples of morphisms between objects:

- A bijective homomorphism between two groups G and H is an isomorphism. What also works is two group homomorphisms $\phi : G \rightarrow H$ and $\psi : H \rightarrow G$ which are mutual inverses, that is $\phi \circ \psi = \text{id}_H$ and $\psi \circ \phi = \text{id}_G$.
- Metric (or topological) spaces X and Y are isomorphic if there exists a continuous bijection $f : X \rightarrow Y$ such that f^{-1} is also continuous.
- Vector spaces V and W are isomorphic if there is a bijection $T : V \rightarrow W$ that's a linear map (aka, T and T^{-1} are linear maps).
- Rings R and S are isomorphic if there is a bijective ring homomorphism ϕ (or two mutually inverse ring homomorphisms).

§1.2 Categories

Definition 1.1 (Category). A category \mathcal{A} consists of

- A class of *objects*, denoted $\text{obj}(\mathcal{A})$.
- For any two objects $A_1, A_2 \in \text{obj}(\mathcal{A})$, a class of *arrows* (also called *morphisms* or *maps* between them). Let's denote the set of arrows by $\text{Hom}_{\mathcal{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \text{obj}(\mathcal{A})$, if $f : A_1 \rightarrow A_2$ is an arrow and $g : A_2 \rightarrow A_3$ is an arrow, we can compose the two arrows to get $h = g \circ f : A_1 \rightarrow A_3$ an arrow, represented in the *commutative diagram* below:

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ & \searrow h & \downarrow g \\ & & A_3 \end{array}$$

The composition operation can be denoted as a function

$$\circ : \text{Hom}_{\mathcal{A}}(A_2, A_3) \times \text{Hom}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Hom}_{\mathcal{A}}(A_1, A_3)$$

for any three objects A_1, A_2, A_3 . Composition must be associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$. In the diagram above, we say h *factors* through A_2 .

- Every object $A \in \text{obj}_{\mathcal{A}}$ has a special *identity arrow* $\text{id}_{\mathcal{A}}$. The identity arrow has the expected properties $\text{id}_{\mathcal{A}} \circ f = f$ and $f \circ \text{id}_{\mathcal{A}} = f$.

Note. We can't use the word "set" to describe the class of objects because of some weird logic thing (there is no set of all sets). But you can think of a class as a set.

From now on, $A \in \mathcal{A}$ is the same as $A \in \text{obj}(\mathcal{A})$. A category is *small* if it has a set of objects, and *locally small* if $\text{Hom}_{\mathcal{A}}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathcal{A}$.

Example 1.2 (Basic Categories). Here are some basic examples of categories:

- We have the category of groups Grp .
 - The objects of Grp are groups.
 - The arrows of Grp are group homomorphisms.
 - The composition of Grp is function composition.
- Describe the category CRing (of commutative rings) in a similar way.
- Consider the category Top of topological spaces, whose arrows are continuous maps between spaces.

- Also consider the category \mathbf{Top}_* of topological spaces with a distinguished basepoint, that is, a pair (X, x_0) , $x_0 \in X$. Arrows are continuous maps $f : X \rightarrow Y$ with $f(x_0) = y_0$.
- Similarly, the category of (possibly infinite-dimensional) vector spaces over a field k \mathbf{Vect}_k has linear maps for arrows. There is even a category \mathbf{FVect}_k of finite-dimensional vector spaces.
- Finally, we have a category \mathbf{Set} of sets, arrows denote any map between sets.

Definition 1.2 (Isomorphism). An arrow $A_1 \xrightarrow{f} A_2$ is an *isomorphism* if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \text{id}_{A_2}$ and $g \circ f = \text{id}_{A_1}$. We say A_1 and A_2 are *isomorphic*, denoted $A_1 \cong A_2$.

Remark 1.1. In the category \mathbf{Set} , $X \cong Y \iff |X| = |Y|$.

In other fields, we can tell a lot about the structure of an object by looking at maps between them. In category theory, we *only* look arrows, and ignore what the objects themselves are.

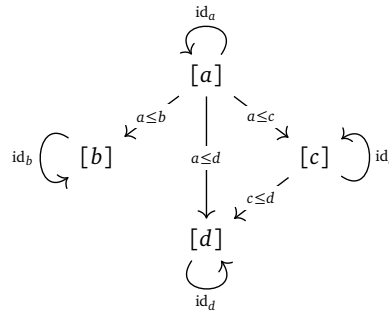
Example 1.3 (Posets are Categories). Let \mathcal{P} be a poset. Then we can construct a category P for it as follows:

- The objects of P are elements of \mathcal{P} .
- We define the arrows of P as follows:
 - For every object $p \in P$, we add an identity arrow id_p , and
 - For any pair of distinct objects $p \leq q$, we add a single arrow $p \rightarrow q$.

There are no other arrows.

- We compose arrows in the only way possible, examining the order of the first and last object.

Here's a figure depicting the category of a poset \mathcal{P} on four objects $\{a, b, c, d\}$ with $a \leq b$ and $a \leq c \leq d$.



Note that no two distinct objects of a poset are isomorphic.

This shows that categories don't have to refer to just structure preserving maps between sets (these are called "concrete categories").

Example 1.4 (Groups as a category with one object). A group G can be thought of as a category \mathcal{G} with one object $*$, all of whos arrows are isomorphisms.

If the universe were structured differently and kids learned category theory before groups, symmetries transforming X into itself would be a natural extension of categories that transform X into other objects, a special case in which all the maps are invertible. Alas, this is not the right timeline.

Example 1.5 (Deriving Categories). We can make categories from other categories!

- Given a category \mathcal{A} , we can construct the *opposite category* \mathcal{A}^{op} , which is the same as \mathcal{A} but with all the arrows reversed.
- Given categories \mathcal{A} and \mathcal{B} , we can construct the *product category* $\mathcal{A} \times \mathcal{B}$ as follows: the objects are pairs (A, B) for $A \in \mathcal{A}$, $B \in \mathcal{B}$, and the arrows from (A_1, B_1) to (A_2, B_2) are pairs

$$(A_1 \xrightarrow{f} A_2, B_1 \xrightarrow{g} B_2).$$

The composition is just pairwise composition, and the identity is the pair of identity functions on A and B .

§1.3 Special objects in categories

Some categories have things called *initial objects*. For example the empty set \emptyset , the trivial group, the empty space, initial element in a poset, etc. More interestingly: the initial object of \mathbf{CRing} is the ring \mathbb{Z} .

Definition 1.3 (Initial object). An *initial object* of \mathcal{A} is an object $A_{\text{init}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{init}}$), there is exactly one arrow from A_{init} to A .

The yang to this yin is the *terminal object*:

Definition 1.4 (Terminal object). A *terminal object* of \mathcal{A} is an object $A_{\text{final}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{final}}$), there is exactly one arrow from A to A_{final} .

For example, the terminal object of \mathbf{Set} is $\{*\}$, \mathbf{Grp} is $\{1\}$, \mathbf{CRing} is the zero ring, \mathbf{Top} is the single point space, and a poset its maximal element (if one exists).

§1.4 Monic and epic maps (todo)

Injectivity and surjectivity don't really make sense when talking about categories. todo

§1.5 Functors

Motivation: maps between categories, objects rising from other objects.

Example 1.6 (Basic Functors). Here are some basic examples of functors:

- Given an algebraic structure (group, field, vector space) we can take its underlying set S : this is a functor from $\mathbf{Grp} \rightarrow \mathbf{Set}$ (or whatever you want to start with).
- If we have a set S , if we consider the vector space with basis S we get a functor $\mathbf{Set} \rightarrow \mathbf{Vect}$.
- Taking the power set of a set S gives a functor $\mathbf{Set} \rightarrow \mathbf{Set}$.
- Given a locally small category \mathcal{A} , we can take a pair of objects (A_1, A_2) and obtain a set $\text{Hom}_{\mathcal{A}}(A_1, A_2)$. This turns out to be a functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{Set}$.

Finally, the most important example (WRT this course):

- In algebraic topology, we build groups like $H_1(X)$, $\pi_1(X)$ associated to topological spaces. All these group constructions are functors $\mathbf{Top} \rightarrow \mathbf{Grp}$.

Definition 1.5 (Functors). Let \mathcal{A} and \mathcal{B} be categories. A *functor* F takes every object of \mathcal{A} to an object of \mathcal{B} . In addition, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. Refer to the commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \ni & \begin{array}{c} A_1 \\ \downarrow f \\ A_2 \end{array} & \\ & \begin{array}{c} \xrightarrow{\quad F \quad} \\ \text{.....} \end{array} & \\ & \begin{array}{c} B_1 = F(A_1) \\ \downarrow F(f) \\ B_2 = F(A_2) \end{array} & \in \mathcal{B} \end{array}$$

Functors also satisfy the following requirements:

- Identity arrows get sent to identity arrows, that is, for each identity arrow id_A , we have $F(\text{id}_A) = \text{id}_{F(A)}$.
- Functors respect composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ are arrows in \mathcal{A} , then $F(g \circ f) = F(g) \circ F(f)$.

More precisely, these are covariant functors. A contravariant functor F reverses the direction of arrows, so that F sends $f: A_1 \rightarrow A_2$ to $F(f): F(A_2) \rightarrow F(A_1)$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ instead. A category \mathcal{A} has an opposite category \mathcal{A}^{op} with the same objects and with $\mathcal{A}^{\text{op}}(A_1, A_2) = \mathcal{A}(A_2, A_1)$. A contravariant functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is just a covariant functor $\mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$.

Example 1.7. We have already talked about *free* and *forgetful* functors in Example 1.3: the forgetful functors are functors from spaces to sets (the underlying set of a group) and free functors are from sets to spaces (the basis set forming a vector space).

- Another example of a forgetful functor is a functor $\mathbf{CRing} \rightarrow \mathbf{Grp}$ by sending a ring R to its abelian group $(R, +)$.
- Another example of a free functor is a functor $\mathbf{Set} \rightarrow \mathbf{Grp}$ by taking the free group generated by a set S (who would have known this is free?)

Here is a cool example: functors preserve isomorphism. If two groups are isomorphic, then they must have the same cardinality. In the language of category theory, this can be expressed as such: if $G \cong H$ in \mathbf{Grp} and $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ is the forgetful functor, then $U(G) \cong U(H)$. We can generalize this to *any* functor and category!

Theorem 1.1. *If $A_1 \cong A_2$ are isomorphic objects in \mathcal{A} and $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor then*

$$F(A_1) \cong F(A_2).$$

Proof. Let's go diagram chasing!

$$\begin{array}{ccc} \mathcal{A} \ni & \begin{array}{c} A_1 \\ \uparrow f \\ A_2 \\ \downarrow g \end{array} & \begin{array}{c} B_1 = F(A_1) \\ \uparrow F(f) \\ F(A_2) \\ \downarrow F(g) \end{array} \\ & \begin{array}{c} \text{---} F \text{---} \\ \text{---} F(f) \text{---} \end{array} & \in \mathcal{B} \end{array}$$

The main idea of the proof follows from the fact that functors preserve composition and the identity map. ☒

This is very very useful for us (people who are doing algebraic topology) because functors will preserve isomorphism between spaces (we get that homotopic spaces have isomorphic fundamental groups).

Note. As a meme (or not really, but it's still funny), we can construct the category \mathbf{Cat} whose objects are categories and arrows are functors.

§1.6 Homotopy Categories and Homotopy Equivalence

Let \mathbf{Top}_* be the category of pointed topological spaces. Then the fundamental group gives a functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$. When we have a suitable relation of homotopy between maps in a category \mathcal{C} , we define the homotopy category $\mathbf{Ho}(\mathcal{C})$ to be the category sharing the same objects as \mathcal{C} , but morphisms the homotopy classes of maps. On \mathbf{Top}_* , we require homotopies to map basepoint to basepoint, and we get the homotopy category \mathbf{hTop}_* of pointed spaces.

Homotopy equivalences in \mathcal{C} are isomorphisms in $\mathbf{Ho}(\mathcal{C})$. More concretely, recall that a map $f: X \rightarrow Y$ is a homotopy equivalence if there is a map $g: Y \rightarrow X$ such that both $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. In the language of category theory, we can obtain the analogous notion of a pointed homotopy equivalence. Functors carry isomorphisms to isomorphisms, so then the pointed homotopy equivalence will induce an isomorphism of fundamental groups. This also holds, but less obviously, for the category of non pointed homotopy equivalences.

Theorem 1.2. *If $f: X \rightarrow Y$ is a homotopy equivalence, then*

$$f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$$

is an isomorphism for all $x \in X$.

Proof. Let $g: Y \rightarrow X$ be a homotopy inverse of f . By our homotopy invariance diagram, we see that the composites

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x))$$

and

$$\pi_1(Y, y) \xrightarrow{g_*} \pi_1(X, g(y)) \xrightarrow{f_*} \pi_1(Y, (f \circ g)(y))$$

are isomorphisms determined by paths between basepoints given by chosen homotopies $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. Then in each displayed composite, the first map is a monomorphism and the second is an epimorphism. Taking $y = f(x)$ in the second composite, we see that the second map in the first composite is an isomorphism. Therefore so is the first map, and we are done. ☒

A space X is said to be contractible if it is homotopy equivalent to a point.

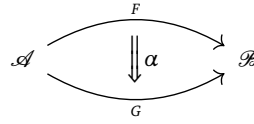
Corollary 1.1. *The fundamental group of a contractible space is zero.*

§1.7 Natural Transformations

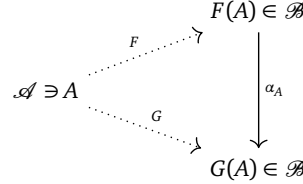
We talked about maps between objects which led to categories, and then maps between categories which lead to functors. Now let's talk about maps between functors, the natural transformation: this is actually not too strange (recall the homotopy, a “deformation” from a map to another map).

In this case, we also want to pull a map (functor) F to another map G by composing a bunch of arrows in the target space \mathcal{B} .

Definition 1.6 (Natural Transformations). Let $F, G: \mathcal{A} \rightarrow \mathcal{B}$ be two functors. A *natural transformation* $\alpha: F \rightarrow G$ denoted



consists of, for each $A \in \mathcal{A}$ an arrow $\alpha_A \in \text{Hom}_{\mathcal{B}}(F(A), G(A))$, which is called the component of α at A . Pictorially, it looks like this:

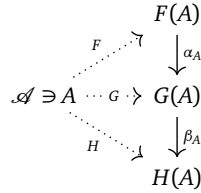


The α_A are subject to the “naturality” requirement such that for any $A_1 \xrightarrow{f} A_2$, the following diagram commutes:

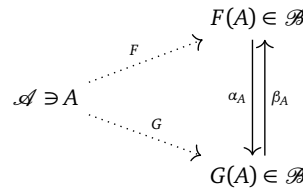
$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(f)} & F(A_2) \\ \alpha_{A_1} \downarrow & & \downarrow \alpha_{A_2} \\ G(A_1) & \xrightarrow{G(f)} & G(A_2) \end{array}$$

The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (like in a homotopy from f to g the point $f(t)$ gets deformed to the point $g(t)$ continuously). Think of f representing the homotopy and the basepoints being $F(A_1), G(A_1)$ to $F(A_2), G(A_2)$.

Natural transformations can be composed. Take two natural transformations $\alpha: F \rightarrow G$ and $\beta: G \rightarrow H$. Consider the following commutative diagram:



We can also construct inverses: suppose α is a natural transformation such that α_A is an isomorphism for each A . Then we construct an inverse arrow β_A in the following way:



We say α is a *natural isomorphism*. Then $F(A) \cong G(A)$ naturally in A (and β is an isomorphism too!) We write $F \cong G$ to show that the functors are naturally isomorphic.

Example 1.8. If $F: \text{Set} \rightarrow \text{Grp}$ is the free functor that sends a set to the free group on such set and $U: \text{Grp} \rightarrow \text{Set}$ is the forgetful functor sending a free group to its generating set, then we have a natural inclusion of $S \hookrightarrow UF(S)$. The functors F and U are left and right adjoint to each other, in the sense that we have a natural isomorphism

$$\text{Grp}(F(S), A) \cong \text{Set}(S, U(A))$$

for a set S and an abelian group A . This expresses the “universal property” of free objects: a map of sets $S \rightarrow U(A)$ extends uniquely to a homomorphism of groups $F(S) \rightarrow A$.

Definition 1.7. Two categories \mathcal{A} and \mathcal{B} are equivalent if there are functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{A}$ and natural isomorphisms $FG \rightarrow \text{Id}$ and $GF \rightarrow \text{Id}$, where the Id are the respective identity functors.

§1.8 The Yoneda lemma (todo)

Definition 1.8 (The functor category). The *functor category* of two categories \mathcal{A} and \mathcal{B} , denoted $[\mathcal{A}, \mathcal{B}]$ is defined as follows:

- The objects of $[\mathcal{A}, \mathcal{B}]$ are (covariant) functors $F: \mathcal{A} \rightarrow \mathcal{B}$, and
- The morphisms are natural transformations $\alpha: F \rightarrow G$.

todo

§1.9 Limits and colimits (todo)

Let \mathcal{D} be a small category and \mathcal{C} be any category. A \mathcal{D} -shaped diagram in \mathcal{C} is a functor $F: \mathcal{D} \rightarrow \mathcal{C}$. A morphism $F \rightarrow F'$ of \mathcal{D} -shaped diagrams is a natural transformation, and we have the category $\mathcal{D}[\mathcal{C}]$ of \mathcal{D} shaped diagrams in \mathcal{C} . Any object C of \mathcal{C} determines the constant diagram

§2 Homology

The big boy has arrived. These notes will follow Hatcher §2.1.

Remark 2.1. This is something I heard even before I enrolled in this course. The homotopy groups are easy to define, but impossible to compute and work with. The homology groups take a lot of work to define, but the resulting groups are much nicer and easier to work with.



The fundamental group is a cool tool when dealing with low-dimensional spaces (the pride and joy of UT Austin), but it doesn't do well with higher dimensional spaces, for example, it can't distinguish between the n -spheres S^n for $n \geq 2$. We can get rid of this limitation by considering the higher homotopy groups $\pi_n(X)$, which are defined in terms of maps from the n -dimensional cube I^n and homotopies $I^n \times I \rightarrow X$ of such maps. Cool things about higher homotopy groups: for X a CW complex, $\pi_n(X)$ only depends on the $(n+1)$ -skeleton, and $\pi_i(S^n) = 0$ for $i < n$ and \mathbb{Z} for $i = n$, as expected. However, the drawback is that they're extremely difficult to compute in general—take the “simple” task of computing $\pi_i(S^n)$ for $i > n$.

Enter the homology groups $H_n(X)$. Similar to $\pi_n(X)$, $H_n(X)$ for X a CW complex depends only on the $(n+1)$ -skeleton, and for the spheres $H_i(S^n) \simeq \pi_i(S^n)$ for $1 \leq i \leq n$, but the homology groups have the advantage in that $H_i(S^n) = 0$ for $i > n$. However, everything has a price. How exactly do we define these so called homology groups? We start by motivating, then doing simplicial homology, before moving onto singular homology. Most efficient method for computing homology groups is called cellular homology. We'll also talk about Mayer-Vietoris sequences, the analogue of the van Kampens for the fundamental group.

Something interesting about homology: most of the time we only use the basic properties of homology, not the definition itself. So we could almost invoke an axiomatic approach, which will happen soon. We could also skip the algebra and talk about geometry, but then Dr. Brand would be unhappy (and so would I), so we'll approach it with a mix of the two (talk about intuition first then state the axioms later).

§2.1 The big idea of homology

Issues with homotopy groups: things get really wacky because S^2 has no cells of dimension greater than 2, but some (infinitely many) of the higher homotopy groups $\pi_n(S^2)$ are nontrivial. (*god shattering star noises*) However, homology groups are (directly) related to cell structures, in that you can regard them as an algebraization of how cells of dimension n attach to cells of dimension $n-1$.

Imagine a circle with two antipodal points x and y , with four arrows a, b, c, d drawn in the direction from x to y , which we'll denote by X_1 .

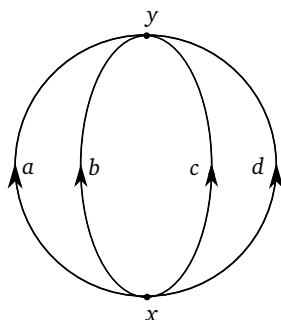


Figure 1: The graph X_1 , consisting of two vertices and four edges.

Usually loops are nonabelian, so suppose we abelianize the loops. That is, the loops ab^{-1} and $b^{-1}a$ are “the same circle” (but with a different starting point), so we'll just say they're equal. Formally (not really), rechoosing the basepoint just permutes the letters cyclically, so by abelianizing we can cast off our silly worries about the basepoint. So we make the transition from loops (chosen basepoint) \longrightarrow cycles (no chosen basepoint).

Now we abelian, and all the cool abelian groups use additive notation. So a cycle looks something like $a - b + c - d$ now, a linear combination of edges with integer coefficients. We'll call these linear combinations **chains** of edges. We can decompose these into cycles by several ways, eg $(a - c) + (b - d) = (a - d) + (b - c)$, so it's better just to say cycles are any LC of edges st at least one decomposition makes geometric sense. When is a chain a cycle? Cycles are distinguished by the fact that they enter and exit a vertex the same amount of times. So for an arbitrary chain $ka + lb + mc + nd$, it enters y about $k + l + m + n$ times (one for each thing) and enters x (or leaves it) $-k - l - m - n$ times. So if we want $ka + lb + mc + nd$ to be a cycle, we just need to require $k + l + m + n = 0$.

To generalize this, let C_1 be the free abelian group with a basis set $\{a, b, c, d\}$ (edges), and C_0 be the free abelian group with basis $\{x, y\}$ (vertices). Elements of C_1 are chains of edges, and elements of C_0 are linear combinations of

vertices. Define a homomorphism $\partial : C_1 \rightarrow C_0$ by sending each basis element to $y - x$, then $\partial(ka + lb + mc + nd) = (k + l + m + n)y - (k + l + m + n)x$, so cycles are precisely $\ker \partial$. It can be seen that $a - b$, $b - c$, and $c - d$ form a basis for $\ker \partial$, so every cycle in X_1 is a unique linear combination of these three elts. Basically, X_1 has three “holes”, the three gaps in between the four edges.

Now let's attach a 2-cell to X_1 to get X_2 , as seen below.

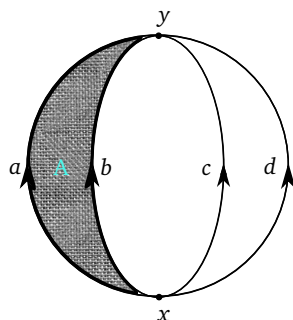


Figure 2: X_1 with a 2-cell attached, denoted X_2 . Have you ever seen a 2-cell that looks like cloth?

The 2-cell is attached along the cycle $a - b$, forming the 2-skeleton X_2 . Now the cycle is trivial (homotopically), which suggest we form a quotient by factoring out the subgroup generated by $a - b$. For example, $a - c$ and $b - c$ are now equivalent, since they're homotopic in X_2 . Algebraically, we define a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the infinite cyclic group generated by A , and $\partial_2(A) = a - b$. ∂_1 is the boundary homomorphism, defined earlier. We are interested in $\ker \partial_1 / \text{im } \partial_2$, that is, the 1-dimensional cycles modulo the boundaries (multiples of $a - b$). Remember, factor groups collapse everything we don't like to the identity. This quotient group is the **homology group** $H_1(X_2)$. If we were to talk about X_1 , since it has no 2-cells C_2 is simply zero, so $H_1(X_1) = \ker \partial_1 / \text{im } \partial_2 = \ker \partial_1$, which is free abelian on three generators. $H_1(X_2)$ is free abelian on two generators ($b - c$ and $c - d$), which expresses the geometric observation that there are two holes remaining after filling one of them in with the 2-cell A .

Let's go farther. Add another 2-cell to the pre-existing 2-cell A , to get the 3-complex X_3 .

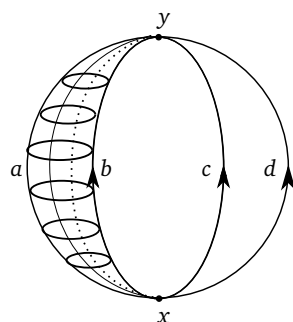


Figure 3: The 3-complex X_3 , formed by attaching a 2-cell to X_2 .

This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B , and the boundary homomorphism $\partial_2 : C_2 \rightarrow C_1$ sends A, B to $a - b$. $H_1(X_3) = \ker \partial_1 / \text{im } \partial_2 = H_1(X_2)$, but now ∂_2 has a nontrivial kernel (the infinite cyclic group generated by $A - B$). We view $A - B$ as a 2d cycle generating $H_2(X_3) = \ker \partial_2 \simeq \mathbb{Z}$. The second homology detects the 2d “hole” in X_3 .

Unfortunately the diagrams will have to stop now, but let's go even farther and make the complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere by A and B , creating a chain group C_3 generated by C . The boundary homomorphism $\partial_3 : C_3 \rightarrow C_2$ that sends C to $A - B$ should be seen as the boundary of C , similar to how $a - b$ is the boundary of A . Now we have a sequence of boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and $H_2(X_4) = \ker \partial_2 / \text{im } \partial_3$ is now trivial. $H_3(X_4) = \ker \partial_3 = 0$, note that $H_1(X_4) = H_1(X_3) \simeq \mathbb{Z} \times \mathbb{Z}$, so this is the only homology group of X_4 that isn't trivial.



You can pretty much see where this is going. For a cell complex X , we have chain groups $C_n(X)$ free abelian with basis the n -cells of X , with boundary homomorphisms $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$, by which we define the homology group $H_n(X) = \ker \partial_n / \text{im } \partial_{n+1}$. So what's the problem? It's how to define ∂_n in general— for $n = 1$ this is easy, it's the vertex head minus the one at the tail. For $n = 2$, it still isn't hard per say, if the cell is attached on a loop of edges, just take the cycle of edges, keeping in mind orientation. This is much trickier for higher dimension cells, even with restrictions to polyhedral cells and nice attaching maps we still have to worry about orientation and stuff.

So what do we do? Use triangles, of course. We can subdivide arbitrary polyhedra into certain special types of polyhedra called simplices (what we talked about in class day 1), so there isn't any loss of generality (but there is a loss of efficiency). This gives rise to our more basic **simplicial homology**, which deals with cell complexes from simplices. However, we are still quite limited in what we can do.

So, what do we really do this time? Make things less simple, and make your life difficult by considering the collection of all possible continuous maps of simplices into a space X (wow). The chain groups $C_n(X)$ are tremendously large, but the quotients $H_n(X) = \ker \partial_n / \text{im } \partial_{n+1}$, the **singular homology groups**, are much smaller and easier to work with¹. For example, in the examples above the singular homology groups coincide with the ones computed from cellular chains. Furthermore (as we will see later), singular homology lets us define these nice cellular homology groups for *all* cell complexes, which solves the issue of how to define boundary maps for cellular chains.

§2.2 The structure of Δ -complexes

I have a feeling we're gonna be typing a lot of \Delta's. So basically, the only thing cool kids talk about is singular homology, but it's kinda complicated so we gotta talk about the inferior version for those who have the brain capacity of a literal ape², simplicial homology, first. We talk about simplicial homology in the domain of Δ -complexes. Take the standard fundamental polygons with orientation for \mathbb{T}^2 , \mathbb{RP}^2 , and the Klein bottle K . Cut the squares in half with a diagonal to get two triangles, from here we can get the original shape by identifying in pairs. We can do this with any n -gon, decomposing it into $n - 2$ base triangles. So we can make any closed surface from triangles, furthermore, we could also make a larger class of spaces that aren't surfaces by allowing more than two edges to be glued together at the same time.

The idea of a Δ -complex is to generalize these constructions to n -dimensions. The n -dimensional triangle is the n -simplex, the smallest convex set in \mathbb{R}^m containing $n + 1$ points v_0, \dots, v_n that don't lie in a hyperplane of dimension less than n , where by "hyperplane" we mean the set of solutions to a system of linear equations. We could also just say that the difference vectors $v_1 - v_0, \dots, v_n - v_0$ are LI. The v_i are **vertices** of the simplex, and the simplex itself is $[v_0, \dots, v_n]$.

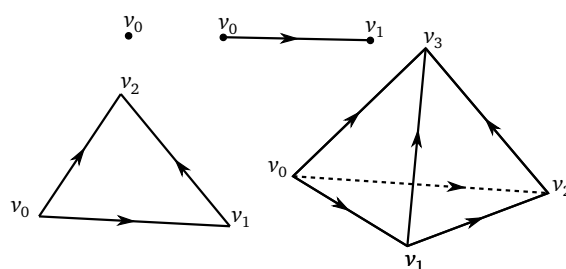


Figure 4: The 0-simplex to the 3-simplex, respectively (with ordered vertices and oriented edges).

For example, we have the standard n -simplex given by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\},$$

whose vertices are the unit vectors along the coordinate axes. Think of this as taking the unit vectors, and drawing a triangle from each of their endpoints. This works because the difference vectors are LI. For homology, orientation of vertices is really important, so n -simplex really means n -simplex with an ordering on its vertices. Ordering the vertices will determine an orientation on its subsimplices, as can be seen in Figure 4. This also determines a canonical linear homeomorphism from the standard n -simplex Δ^n onto any other simplex $[v_0, \dots, v_n]$ that preserves the order of the vertices, given by

$$(t_0, \dots, t_n) \mapsto \sum_i t_i v_i.$$

We say the coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i \in [v_0, \dots, v_n]$. Deleting a vertex of a n -simplex yields something that spans an $(n - 1)$ -simplex, called a **face** of $[v_0, \dots, v_n]$. We'll adopt the following convention: *The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.* That sounds reasonable enough. We say the union of all faces of Δ^n is the **boundary** of Δ^n , written $\partial \Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is equal to $\Delta^n \setminus \partial \Delta^n$, the interior of Δ^n .

A **Δ -complex** structure on a space X is a collection of maps $\sigma_\alpha: \Delta^n \rightarrow X$, with n depending on the index α , such that:

1. The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is onto, and each point of X is in the image of exactly one restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta: \Delta^{n-1} \rightarrow X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear order-preserving homeomorphism.
3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α .

¹For reasonably "nice" spaces X , of course.

²The book simply says "primitive" version, so I used my imagination a little bit.

A consequence of (3) is that X can be built as a quotient space of a collection of disjoint simplices Δ_α^n , one for each $\sigma_\alpha: \Delta^n \rightarrow X$, the quotient space obtained by identifying each face of a Δ_α^n with the Δ_β^{n-1} corresponding to the restriction σ_β of σ_α to the face in question. You can think of this as basically cell complexes, attaching 0-simplices (cells) to 1-simplices and 2-simplices, and so on.

In general, we can make Δ -complexes from collections of disjoint simplices by identifying various subspaces spanned by subsets of the vertices, with identifications performed by the canonical order-preserving linear homeomorphisms. Note that if we think of a Δ -complex X as a quotient space of disjoint simplices, then X must be Hausdorff. Each restriction $\sigma_\alpha|_{\Delta^n}$ is a homeomorphism onto its image by condition (3), which is an open simplex in X . Then these open simplices are the cells e_α^n of a CW complex structure on X with the σ_α 's as characteristic maps (we won't use this fact yet).

§2.3 Simplicial homology

Goal: define simplicial homology groups of a Δ -complex X . Let $\Delta_n(X)$ be the free abelian group with basis the open n -simplices e_α^n of X . Formally, we can write elements of $\Delta_n(X)$ as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in \mathbb{Z}$, called **n -chains**. We could also write $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha: \Delta^n \rightarrow X$ is the characteristic map of e_α^n , with image the closure of e_α^n . Such a sum can be thought of as a finite collection, or 'chain', of n -simplices in X .

Take a look at $\partial[v_0, v_1] = [v_1] - [v_0]$, $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$, and $\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$. Naïvely, one might assume the boundary of an n -simplex to be the sum of the faces delete a point, denoted by $[v_0, \dots, \hat{v}_i, \dots, v_n]$ where v_i is the vertex to be deleted. However, note the signs to take orientations into account, it just happens that they work out based on the position of v_i . So we have

$$\partial[v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n].$$

Keeping this in mind, let's define a **boundary homomorphism** $\partial_n: \Delta_n(X) \rightarrow \Delta_{n-1}(X)$ for X a general Δ -complex by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

Lemma 2.1. The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.

Proof. Note that

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}.$$

Then after switching i and j in the second term, it becomes the negative of the first. Alternate proof from Dr. Allcock: note that $\partial\sigma := \sum_{i=0}^n (-1)^i \sigma \circ [v_0, \dots, \hat{v}_i, \dots, v_n]$. Then

$$\partial\partial\sigma = \sum_{i=0}^n (-1)^i \partial(\sigma \circ [v_0, \dots, \hat{v}_i, \dots, v_n]),$$

which distributes because C_{n-1} is free on $\{\text{singular } (n-1)\text{-simplex}\}$. So defining any function $\{\text{singular } (n-1)\text{-simplex}\} \rightarrow C_{n-2}$ extends to a \mathbb{Z} -linear map $C_{n-1} \rightarrow C_{n-2}$. Then

$$\partial\partial\sigma = \sum_{i=0}^n (-1)^i \left(\sum_{j=0}^{i-1} \sigma \circ [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] (-1)^j + \sum_{j=i+1}^n \sigma \circ [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n] (-1)^{j-1} \right),$$

which is equal to zero by cancellation³. □

What we have here is a sequence of homomorphisms of abelian groups

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for all n . This is called a **chain complex**. Note that we've extended the sequence to 0, with $\partial_0 = 0$. The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\text{im } \partial_{n+1} \subseteq \ker \partial_n$, so we can define the **n^{th} homology group** of the chain complex as $H_n = \ker \partial_n / \text{im } \partial_{n+1}$. Elements of $\ker \partial_n$ are called **cycles** and elements of $\text{im } \partial_{n+1}$ are called **boundaries**. Elements of H_n are cosets of $\text{im } \partial_{n+1}$, called **homology classes**. Two cycles representing the same homology class are said to be **homologous**, that is, their difference is a boundary. When $C_n = \Delta_n(X)$, the homology group $\ker \partial_n / \text{im } \partial_{n+1}$ will be denoted by $H_n^\Delta(X)$ and called the **n^{th} simplicial homology group** of X .

³The proof from Dr. Allcock was for singular homology, but the idea is the same.