Algebraic Topology Homework

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This is my homework for the Fall 2020 section of Algebraic Topology (Math 382C) at UT Austin with Dr. Allcock. The course follows *Algebraic Topology* by Hatcher. Source files: `https://git.simonxiang.xyz/math_notes/files.html`

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§1.1 Problem 1

Problem. *An n-dimensional* manifold with boundary *means a Hausdorff space M, such that every $x \in M$ has a neighborhood U such that the pair (U, x) is homeomorphic to either $(\mathbb{R}^n, 0)$ or $\big(\mathbb{R}^{n-1} \times [0, \infty), 0\big)$, where in both cases 0 means $(0, \ldots, 0)$. We call x an interior or boundary point according to which of these holds. Note that this is* not *the usual use of "interior" and "boundary" from point-set topology. The set of boundary points is written ∂M.*

Assume that ∂M is compact. Prove that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence.

You may use without proof the fact that no point can be both an interior and a boundary point. Also, the 2-dimensional case is enough to give a complete understanding. Finally, a hint: chain together a sequence of homotopies, being careful that the result makes sense and is continuous.

Remarks: informally, I think of $M \setminus \partial M$ as a sort of deformation-retract of M. But it is easy to see that if $\partial M \neq \emptyset$ then M does not actually deformation retract to $M \setminus \partial M$. Also, without the extra hypotheses, the only solution I know uses something you probably have not seen: topological dimension, which lets you build an open cover with good overlap properties.

Solution. We want to show that the inclusion $M \setminus \partial M \to M$ is a homotopy equivalence, that is, it is one of the continuous maps f or g such that $f \circ g$ is homotopic to ι_M and $g \circ f$ is homotopic to $\iota_{M \setminus \partial M}$. ■

§1.2 Problem 2

Problem (A "bad" group action)**.** *Let $X = \mathbb{R}^2 \setminus \{0\}$ where 0 is the origin. Let G be the group of homeomorphisms of X generated by the transformation $(x, y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.*

(a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).

(b) Prove that G's action on X satisfies the hypothesis of the theorem from class about $\pi_1(X/G) \cong G$, namely: every $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for every $g \in G \setminus \{1\}$.

(c) Prove that Y is a manifold, except for the fact that it is not *Hausdorff.*

(When working on a theorem involving a group action, if I wonder whether some hypothesis can be omitted, checking it for this single example usually reveals the answer.)

Solution. (The condition for an orbit to be discrete comes from the Wikipedia page for a discrete group.)

(a) Let $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, $G(x, y)$ be the orbit of (x, y). We want to show that the singleton containing the identity $\{(x, y)\}$ is open, a sufficient condition for the orbit to be discrete. We know the next two subsets of the orbit "closest" to $\{(x, y)\}$ are the singletons $\{(2x, y/2)\}$ and $\{(x/2, 2y)\}$, generated by the given homeomorphism and its inverse. Let $\varepsilon = {}^1\!/\!_4 \min\{x, y\}$. So take an open set $B((x, y), \varepsilon)$ around (x, y): this doesn't intersect the other two sets, and $B((x, y), \varepsilon) \cap \{(x, y)\} = \{(x, y)\}$. Therefore $\{(x, y)\}$ is open, and orbits in this group action are discrete.

(b) The neighborhood $B((x, y), \varepsilon)$ from the previous part does the trick: the minimum possible distance from one singleton subset to another is $\min\{x, y\}/2$, so taking $\varepsilon = {}^1\!/\!_4 \min\{x, y\}$ ensures that two open $g(B((x, y), \varepsilon))$'s won't intersect (given $g \in G, g \neq 1$). This fulfills the condition from class.

(c) We want to show that Y is Hausdorff.

■

§1.3 Problem 9 Chapter 0

Problem. *Show that a retract of a contractible space is contractible.*

Solution. Let A be a retract of a contractible space X. Then there exists a homotopy of X onto a point, and a retract of X onto A: denote this retract with f, and the homotopy encoding the contraction as $H : X \times I \to X$, where $f|_A = \mathrm{id}\, A$, $H(x, 0) = \mathrm{id}_X$, $H(x, 1) = \{x_0\}$. Consider the homotopy $H' = f \circ H|_A$ from $A \times I \to A$. Then this homotopy is continuous since f and H are continuous, and is a deformation retraction onto a point since $H'(x, 0) = \mathrm{id}_A, H'(x, 1) = \{x_0\}$. Therefore A is contractible. ■

§1.4 Problem 20

Problem. *Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.*

Solution. We can contract the intersecting disk of the Klein bottle to itself, so the resulting structure resembles S^2/S^0 (the sphere with two points identified), which is homotopy equivalent to $S^1 \vee S^2$ by Example 0.8. Counting the boundary of the intersecting disk itself, this forms another S^1 identified with the rest of the bottle (now $S^1 \vee S^2$) at a point. Therefore the self-intersecting Klein bottle in $\mathbb{R}^3$ has the homotopy type of $S^1 \vee S^1 \vee S^2$. ■

§1.5 Problem 17 Section 1.1

Problem. *Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$ (whoops, attempted this one last week).*

Solution. Informal idea: Take the first S^1 and twist it like a pretzel n times. Then fold these loops onto the second S^1, a retraction. This is an infinite family of nonhomotopic retractions.

Formal idea: We can express $S^1 \vee S^1$ as a unit circles centered at $(-2, 0)$ and the origin (the left and right unit circles, respectively) wedged together at the point $(-1, 0)$. Then we have an infinite family of retractions $R = \{r_n \mid n \in \mathbb{Z}\}$, where each r_n is defined as

$$r_n : (\cos\theta - 2, \sin\theta) \mapsto (e^{ni\theta}).$$

(Is there a way to write the left side in terms of e? My algebra is lacking). These retractions are the identity on the right circle and are non-homotopic because they correspond to different elements of the fundamental group $\pi_1(S^1) = \mathbb{Z}$. ■

§1.6 Problem 18

Problem. *Using Lemma* 1.15*, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:*

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1*-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.*

§1.7 Problem 20

Problem. *Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]*

§1.8 Problem 2 Section 1.2

Problem. *Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.*

§1.9 Problem 4

Problem. *Let $X \subseteq \mathbb{R}^3$ be the finite union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.*