Algebraic Topology II Lecture Notes

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Lecture notes for the Spring 2022 graduate section of Algebraic Topology II (Math 392C) at UT Austin, taught by Dr. Perutz. These notes were taken live in class (and so they may contain many errors). Source files: https://git.simonxiang.xyz/math_notes/files.html

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1 How homology determines cohomology additively (and cap products)

The algebra problem is this: take some chain complex (C_*, ∂) a chain complex over R, and $D^* = \bigoplus D^q$, $\delta = \partial^*, D^q \to D^{q+1}$, $D^q = \mathrm{Hom}_p(C_q, R)$, (D^*, δ) a cochain complex.

Problem. How does H_*C determine H^*D? How does the chain homotopy type of C_* lead to the chain homotopy type of D^*?

For complexes of projectives, "quasi-isomorphism type of C_* implies the chain homotopy type". A quasi-isomorphism is a map that induces isomorphisms on homology, which is different from saying the two complexes have the same homology.

Example 1.1. Let $C_* = C_*^{\mathrm{cell}}(\mathbb{RP}^2) = \{0 \to \underset{e_2}{\mathbb{Z}} \xrightarrow{2} \underset{e_1}{\mathbb{Z}} \xrightarrow{0} \underset{e_0}{\mathbb{Z}} \to 0\}$. Then $H_*(C) = \underset{1}{\mathbb{Z}/2} \oplus \underset{0}{\mathbb{Z}}$. Dualizing, we get $D^* = \{0 \leftarrow \underset{\eta^2=2}{\mathbb{Z}} \xleftarrow{2} \underset{\eta^1=1}{\mathbb{Z}} \xleftarrow{0} \underset{\eta^0=0}{\mathbb{Z}} \leftarrow 0\}$. Then

$$\begin{aligned} H^0 &= \mathbb{Z} \quad [\eta^0] \\ H^1 &= 0 \\ H^2 &= \mathbb{Z}/2 \quad [\eta^2] \end{aligned}$$

So the $\mathbb{Z}/2$ moved a degree. Here $H^2 \cong H_1, H^1 = H_2 = 0$.

There is a natural map $\Delta\colon H^n(D^*) \to \mathrm{Hom}(H_nC_*, R)$. Namely, $\beta[\zeta]$ (where ζ is a cocycle representing our cohomology class) is equal to $\{[z] \sim \text{cycle} \mapsto \zeta(z)\}$. This is well-defined. We need to check that boundaries go to zero and coboundaries go to the zero map, and both of these come out from checking the definitions essentially.

Universal coefficients for cohomology. *Let R be a PID and $C_* = \bigoplus_{n\geq 0} C_n$ a non-negatively graded complex of free modules.*[1] *Let M be some R-module. Consider the cochain complex $\mathrm{Hom}_R(C_*, M)$. Then there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}C_*, M) \to H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\beta} \mathrm{Hom}_R(H_nC_*, M) \to 0$$

This sequence splits, but not naturally.

Corollary 1.1. *For spaces X and abelian groups M, there are natural, non-naturally split, short exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}X, M) \to H^n(X, M) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(H_nX, M) \to 0$$

Now we have to say what Ext is. This is very similar to Tor, and both fit into the framework of derived functors. We will be very brief and very sketchy. Let's think about $h = \mathrm{Hom}_R(\cdot, M)$, a functor from $(\mathrm{mod}_R)^{\mathrm{op}} \to \mathrm{mod}_R$ (assuming R is commutative). This functor sends a module $N \mapsto \mathrm{Hom}(N, M)$, and morphism to the natural thing. The assertion is that h is **right exact**, or cokernel preserving. Concretely, if we take a short exact sequence $0 \to S \to N \to Q \to 0$ in mod_R, which we can think of as a short exact sequence $0 \to Q \to N \to S \to 0$ in $(\mathrm{mod}_R)^{\mathrm{op}}$. This leads to an exact sequence of modules

$$\mathrm{Hom}(Q, M) \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(S, M) \to 0$$

The assertion is that this is exact, which is right exactness. To define $\mathrm{Ext}^*_R(N, M)$, we choose a projective resolution $P_* \to N$. Set $\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}(P_*, M))$ which is parallel to what one does for Tor using projective resolutions. The fundamental lemma of homological algebra gives well-definedness.

[1] Over a PID, every projective module is free.

Example 1.2. Let R be a PID. Any N has a two step free resolution. So $\mathrm{Ext}_R^n \equiv 0$ for $n \geq 2$, just as with Tor over a PID.

Some facts:

- $\mathrm{Ext}_0^R(N,M) = \mathrm{Hom}_R(N,M)$.
- $\mathrm{Ext}_R^n(R^m,M) = 0$ for all $n > 0$, R free (don't need a PID for this)
- $\mathrm{Ext}_R^1(R/(x),M)$ is equal to? We have a free resolution

$$0 \to R \xrightarrow{x} R \to R/x \to 0$$

  Then $\mathrm{Ext}_R^1(R/x,M) = H^1(\underset{0}{M} \xrightarrow{x} \underset{1}{M}) = \mathrm{coker}(x : M \to M) = M/xM$. Once again we need not R be a PID, but this is most useful over PIDs, since modules of PIDs look like $R^m \oplus \bigoplus R/x_i$, $x_i \in R$.
- $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m \supset \mathbb{Z}) \cong \mathbb{Z}/m$. Hence for A a finitely generated abelian group,

$$\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Z}) = A_{\mathrm{tors}} \text{ since } A \cong \mathbb{Z}^m \oplus \underbrace{\bigoplus_i \mathbb{Z}/k_i}_{A_{\mathrm{tors}}}.$$

  This is also true without the finitely generated assumption which we can prove with some limiting argument. Hence from universal coefficients, there exists a non-natural isomorphism between $H^n(X) \cong \mathrm{Hom}(H_nX,\mathbb{Z}) \oplus H_{n-1}(X)_{\mathrm{tors}}$. In a sense this gives us complete understanding of first cohomology, since $H^1X \cong \mathrm{Hom}(H_1X,\mathbb{Z})$ is free abelian. This is isomorphic to $\mathrm{Hom}(\pi_1^{\mathrm{ab}}X,\mathbb{Z}) \cong \mathrm{Hom}(\pi_1X,\mathbb{Z})$. (This is also the same as homotopy classes of maps $[X,S^1]$).

  This also tells us that $H^2X \cong \mathrm{Hom}(H_2X,\mathbb{Z}) \oplus H_1(X)_{\mathrm{tors}}$, e.g. $H^2(\mathbb{R}\mathrm{P}^2) \cong H_1(\mathbb{R}\mathrm{P}^2)_{\mathrm{tors}} = \mathbb{Z}/2$, recovering our earlier direct calculation.

Example 1.3. We calculated way back that $H_*(B\mathbb{Z}/n) \cong \underset{0}{\mathbb{Z}}\ \underset{1}{\mathbb{Z}/n}\ \underset{2}{0}\ \underset{3}{\mathbb{Z}/n}\ \underset{4}{0}\ \cdots$, which is 2-periodic. So $H^*(\beta\mathbb{Z}/n) \cong \underset{0}{\mathbb{Z}}\ \underset{1}{0}\ \underset{2}{\mathbb{Z}/n}\ \underset{3}{0}\ \underset{4}{\mathbb{Z}/n} \cdots$ by universal coefficients, that is the $\mathbb{Z}/n$'s have shifted up one degree.

Looking at cohomology with real coefficients $H^n(X;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_nX,\mathbb{R})$, we have seen that for manifolds M, this is just de Rham cohomology or $H^n(M;\mathbb{R}) \simeq H_{DR}(M)$. Then $H^n(X,\mathbb{Z}) \to H^n(X;\mathbb{R})$, where the image is a copy of $H^n(X;\mathbb{Z})$/torsion. Namely the de Rham cohomology of a manifold contains the integer cohomology mod torsion, which sits as a lattice $H^n(X;\mathbb{Z})/\mathrm{tors} \hookrightarrow H^n_{DR}(M)$. This is the lattice of classes of closed n-forms α such that

$$\int z^*\alpha \in \mathbb{Z} \text{ for all smooth cycles } z.$$

These closed forms have "integral period" per say. For example, we can think of the n-torus as V/L where $V \cong \mathbb{R}^n$ and L is a lattice inside V. Then $H^*(V/L,\mathbb{R}) \subset H^*_{DR}(V/) \cong \bigwedge^* L^V \subset \bigwedge^* V^V$.

2 Cap and slant products, plus Poincaré duality

Previously:

$$\boxed{H^* \text{ is naturally a graded (unital) ring}}$$

The big idea of today is the following:

$$\boxed{H_{-*}(X) \text{ is naturally a graded (unital) module over } H^*X}$$

Recall the cup product $\smile$ on H^* comes from

- dualized $E-Z$ Zilber map, where $E-Z=\lambda\colon S_*(X\times Y)\to S_*X\otimes S_*Y$. Use $\lambda^\vee\colon S^*X\otimes S^*Y\to S^*(X\times Y)$
- diagonal $X\xrightarrow{\text{diag}}X\times X$

The **cap product** on the other hand comes from $H^pX\otimes H_nX\to H_{n-p}X, c\otimes x\mapsto c\smile x$. It is built from the same ingredients. Some formal properties: it makes homology a module over cohomology, where $(b\smile c)\frown x=b\frown(c\frown x)$, $1\frown x=x$. For $f: X\to Y$, for $c\in H^*Y$ take the pullback class $f^*c\frown x$ for $x\in H_*X$. Then

$$f_*(f^*c\frown x)=c\frown f_*x.$$

This is also called the "projection formula". There is some reason for this nomenclature but Dr. Perutz doesn't remember why.

Construction of the cap product

We start with $S^*X\otimes S_{-*}X\xrightarrow{\text{id}\otimes\text{diag}_*}S^*X\otimes S_{-*}(X\times X)$. We send this second factor to a tensor product $S_{-*}X\otimes S_{-*}X$ by the Eilenberg-Zilber quasi-isomorphism. Then evaluate cochains on chains by definition to get $\mathbb{Z}\otimes S_{-*}=S_{-*}X$. This defines $\frown\colon S^*X\otimes S_{-*}X\to S_{-*}X$. In summary,

$$\frown\colon S^*X\otimes S_{-*}X\xrightarrow{\text{id}\otimes\text{diag}_*}S^*X\otimes S_{-*}(X\times X)\xrightarrow{\text{id}\otimes\lambda}S^*X\otimes S_{-*}X\otimes S_{-*}X\xrightarrow{\text{ev}\otimes\text{id}}\mathbb{Z}\otimes S_{-*}X=S_{-*X}$$

For $c=[\phi]\in H^pX, x=[z]\in H_nX$, $c\frown x=[\phi]\frown[z]=[\phi\frown z]$. Here $S_{-*}X$ means a cochain complex of degree n, $S_{-n}X$, cohomology equals ∂.

Construction of the slant product

This is a map $H_p(X)\otimes H^n(X\times Y)\to H^{n-p}(Y), x\otimes c\mapsto c/x$. This is not quite division (it is bilinear), but it resembles division in the following sense; for $a\in H^*X, b\in H^*Y$, this leads to $a\times b\in H^*(X\times)$ (where $a\times b=\text{pr}_x^*a\smile\text{pr}_y^*b$). Then $(a\times b)/x=a(x)b$ for $x\in H_pX$, and this is how it resembles division.

Let's construct this. We start with $S_{-*}X\otimes S^*(X\times Y)\xrightarrow{\text{id}\otimes\kappa^\vee}S_{-*}X\otimes S^*X\otimes S^*Y$ where $\kappa\colon S_*X\otimes S_*Y\to S_*(X\times Y)$ is the Eilenberg (something) map. Then evaluate like in the cap product. In short,

$$-/-\colon S_{-*}X\otimes S^*(X\times Y)\xrightarrow{\text{id}\otimes\kappa^\vee}S_{-*}X\otimes S^*X\otimes S^*Y\xrightarrow{\text{ev}\otimes\text{id}}S^*Y$$

What is the point of the slant product? Let the algebraic variety V be the module of vector bundles on V, called $\mathcal{M}$. Then $V\times\mathcal{M}$ carries a natural H^* chain c, and $H_*V\to H^*M, x\mapsto c/x$. This is used a lot in the theory of algebraic vector bundles, gauge theory, etc. We will soon use it to construct Poincaré duality.

About the evaluation map. We have been slightly sloppy with degrees, denoting everything with a $*$, but this should clear things up. Precisely, $\text{ev}\colon S_mX\otimes S^nX\to\mathbb{Z}$, where

$$\text{ev}(z,\phi)=\begin{cases}\phi(z), & \text{if } m=n\\ 0, & \text{otherwise.}\end{cases}$$

Manifolds

Let's talk about topological n-manifolds M. Some useful notation: for $K\subseteq M$, set $H_p(M\mid K)=H_p(M,M\setminus K)$. These groups are **contravariant** with respect to K, that is, $K\overset{i}{\hookrightarrow}L\subseteq M$ implies $i_*\colon H_*(M\mid L)\to H_p(M\mid K)$. The important case is that for $x\in M$, we have groups $\text{or}_x=H_n(M\mid x)=H_n(M,M\setminus\{x\})$. This is local homology, and it turns out they control orientation of x.

Say $x \in U \subseteq M$ homeomorphic to $\mathbb{R}^n$, our chart. Then there is a map $H_n(M \mid x) \xleftarrow{\text{excision}} H_n(U \mid x)$ by inclusion, and excision tells us this is an isomorphism. So the rest of the manifold is irrelevant to local homology, and we only need to pay attention to a neighborhood. On the other hand, $\phi : (U, x) \xrightarrow{\cong} (\mathbb{R}^n, 0)$, which is a copy of $\mathbb{Z}$ in the nth degree. What is the generator of this group $\mathbb{Z}$? It is just a single simplex $\sigma : |\Delta^n| \hookrightarrow \mathbb{R}^n$ with 0 in its interior.

Note also that a choice of (ordered) basis for $\mathbb{R}^n$ determines a generator. Only the orientation of this basis matters. For example, permuting basis vectors changes the generator of $H_n(\mathbb{R}^n \mid 0)$ via sign of permutation. This motivates the following definition.

Definition 2.1. An **orientation** for M^n is a coherent family of isomorphisms $\omega_x : H_n(M \mid x) \xrightarrow{\cong} \mathbb{Z}$ for $x \in \mathbb{Z}$.

What does "coherent" mean here? And how does this topological definition match the top form definition for smooth manifolds?