

NON-COOPERATIVE GAMES

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Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n -person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communications, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of those concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

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Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior* [1]. This book also contains a theory of n -person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium* point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing “good strategies”.

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion on a special section of this paper.

Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceded by a subtitle indicating the concept defined¹. The non-cooperative idea will be implicit, rather than explicit, below.

Definition 2.1 (Finite Game). For us an **n-person game** will be a set of n **players**, or **positions**, each with an associated finite of **pure strategies**; and corresponding to each player, i , a **pay-off function**, P_i , which maps the set of all n -tuples of pure strategies into the real numbers. When we use the word **n-tuples** we shall always mean a set of n items, with each item associated with a different player.

Definition 2.2 (Mixed Strategy, S_i). A **mixed strategy** of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha}$ with $\sum_{\alpha} c_{i\alpha} = 1$ and $c_{i\alpha} \geq 0$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i . We regard the s_i 's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be regarded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i , and v_i , etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the i th player's α th pure strategy, etc.

Definition 2.3 (Pay-off functions, P_i). The pay-off function, P_i , used in the definition of a finite game above, has a unique extension to the n -tuples of mixed strategies which is linear in the mixed strategy of each player [n -linear]. This extension we shall also denote P_i , writing $P_i(s_1, s_2, \dots, s_n)$.

We shall write \mathfrak{s} or \mathfrak{t} to denote an n -tuple of mixed strategies and if $\mathfrak{s} = (s_1, \dots, s_n)$ then $P_i(\mathfrak{s})$ shall mean $P_i(s_1, s_2, \dots, s_n)$. Such an n -tuple, \mathfrak{s} , will also be regarded as a point in a vector space, which space could be obtained by multiplying together the vector spaces containing the mixed strategies. And the set of all such n -tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation $(\mathfrak{s}; t_i)$ to stand for $(s_1, s_2, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_n)$ where $\mathfrak{s} = (s_1, s_2, \dots, s_n)$. The effect of successive substitutions $((\mathfrak{s}; t_i); v_j)$ we indicate by $(\mathfrak{s}; t_i; v_j)$, etc.

Definition 2.4 (Equilibrium Points). An n -tuple \mathfrak{s} is an **equilibrium point** if and only if for every i

$$P_i(\mathfrak{s}) = \max_{\text{all } v_i \text{'s}} [P_i(\mathfrak{s}; v_i)]. \quad (1)$$

¹We actually use **boldface** for definitions instead, but this note on the subtitles is left in to preserve the wording of the original.

Thus an equilibrium point is an n -tuple \mathfrak{s} such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i **uses** a pure strategy $\pi_{i\beta}$ if $s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha}$ and $c_{i\beta} > 0$. If $\mathfrak{s} = (s_1, s_2, \dots, s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that \mathfrak{s} uses $\pi_{i\alpha}$.

From the linearity of $P_i(s_1, \dots, s_n)$ in S_i ,

$$\max_{\text{all } v_i\text{'s}} [P_i(\mathfrak{s}; v_i)] = \max_{\alpha} [P_i(\mathfrak{s}; \pi_{i\alpha})]. \quad (2)$$

We define $P_{i\alpha}(\mathfrak{s}) = P_i(\mathfrak{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for \mathfrak{s} to be an equilibrium point:

$$P_i(\mathfrak{s}) = \max_{\alpha} P_{i\alpha}(\mathfrak{s}). \quad (3)$$

If $\mathfrak{s} = (s_1, s_2, \dots, s_n)$ and $s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha}$ then $P_i(\mathfrak{s}) = \sum_{\alpha} c_{i\alpha} P_{i\alpha}(\mathfrak{s})$, consequently for (3) to hold we must have $c_{i\alpha} = 0$ whenever $P_{i\alpha}(\mathfrak{s}) < \max_{\beta} P_{i\alpha}(\mathfrak{s})$, which is to say that \mathfrak{s} does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i . So we write

$$\text{if } \pi_{i\alpha} \text{ is used in } \mathfrak{s} \text{ then } P_{i\alpha}(\mathfrak{s}) = \max_{\beta} P_{i\beta}(\mathfrak{s}) \quad (4)$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion (3) for an eq. pt. can be expressed as the equating of two continuous functions on the space of n -tuples \mathfrak{s} the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

I have previously published in [Proc. N. A. S. 36 (1950) 48-49] [2] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings: $\mathfrak{s} \rightarrow \mathfrak{s}'(\mathfrak{s}, 1); \mathfrak{s} \rightarrow \mathfrak{s}'(\mathfrak{s}, 2); \dots$ whose fixed points have an equilibrium point as a limit points. A limit mapping exists, but is discontinuous, and need not have any fixed points.

Theorem 3.1. *Every finite game has an equilibrium point.*

Proof. Using our standard notation, let \mathfrak{s} be an n -tuple of mixed strategies, and $P_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the

others use their respective mixed strategies in \mathfrak{s} . For each integer λ we define the following continuous functions of \mathfrak{s} :

$$\begin{aligned} q_i(\mathfrak{s}) &= \max_{\alpha} P_{i\alpha}(\mathfrak{s}), \\ \phi_{i\alpha}(\mathfrak{s}, \lambda) &= P_{i\alpha}(\mathfrak{s}) - q_i(\mathfrak{s}) + \frac{1}{\lambda}, \text{ and} \\ \phi_{i\alpha}^+(\mathfrak{s}, \lambda) &= \max[0, \phi_{i\alpha}(\mathfrak{s}, \lambda)]. \end{aligned}$$

Now $\sum_{\alpha} \phi_{i\alpha}^+(\mathfrak{s}, \lambda) \geq \max_{\alpha} \phi_{i\alpha}^+(\mathfrak{s}, \lambda) = \frac{1}{\lambda} > 0$ so that $c'_{i\alpha}(\mathfrak{s}, \lambda) = \frac{\phi_{i\alpha}^+(\mathfrak{s}, \lambda)}{\sum_{\beta} \phi_{i\beta}^+(\mathfrak{s}, \lambda)}$ is continuous. Define $s'_i(\mathfrak{s}, \lambda) = \sum_{\alpha} \pi_{i\alpha} c'_{i\alpha}(\mathfrak{s}, \lambda)$ and $\mathfrak{s}'(\mathfrak{s}, \lambda) = (s'_1, s'_2, \dots, s'_n)$. Since all the operations have preserved continuity, the mapping $\mathfrak{s} \rightarrow \mathfrak{s}'(\mathfrak{s}, \lambda)$ is continuous; and since the space of n -tuples, \mathfrak{s} , is a cell, there must be a fixed point for each λ . Hence there will be a subsequence λ_{μ} , converging to \mathfrak{s}^* , where \mathfrak{s}_{μ} is fixed under the mapping $\mathfrak{s} \rightarrow \mathfrak{s}'(\mathfrak{s}, \lambda_{(\mu)})$.

Now supposed \mathfrak{s}^* were not an equilibrium point. Then if $\mathfrak{s}^* = (s_1^*, \dots, s_n^*)$ some component s_i^* must be non-optimal against the others, which means s_i^* uses some pure strategy $\pi_{i\alpha}$ which is non-optimal [see (4)]. This means that $P_{i\alpha}(\mathfrak{s}^*) < q_i(\mathfrak{s}^*)$ which justifies writing $P_{i\alpha}(\mathfrak{s}^*) - q_i(\mathfrak{s}^*) < -\epsilon$.

From continuity, if μ is large enough,

$$\left| [P_{i\alpha}(\mathfrak{s}_{\mu}) - q_i(\mathfrak{s}_{\mu})] - [P_{i\alpha}(\mathfrak{s}^*) - q_i(\mathfrak{s}^*)] \right| < \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{\lambda_{(\mu)}} < \frac{\epsilon}{2}.$$

Adding, $P_{i\alpha}(\mathfrak{s}_{\mu}) - q_i(\mathfrak{s}_{\mu}) + \frac{1}{\lambda_{(\mu)}} < 0$ which is simply $\phi_{i\alpha}(\mathfrak{s}_{\mu}, \lambda_{(\mu)}) < 0$, whence $\phi_{i\alpha}^+(\mathfrak{s}_{\mu}, \lambda_{(\mu)}) = 0$, whence $c'_{i\alpha}(\mathfrak{s}_{\mu}, \lambda_{(\mu)}) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in \mathfrak{s}_{μ} since $\mathfrak{s}_{\mu} = \sum_{\alpha} \pi_{i\alpha} c'_{i\alpha}(\mathfrak{s}_{\mu}, \lambda_{(\mu)})$, because \mathfrak{s}_{μ} is a fixed point.

And since $\mathfrak{s}_{\mu} \rightarrow \mathfrak{s}^*$, $\pi_{i\alpha}$ is not used in \mathfrak{s}^* , which contradicts our assumption. Hence \mathfrak{s}^* is indeed an equilibrium point. \boxtimes

Symmetries of Games

An **automorphism**, or **symmetry**, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

If two strategies belong to a single player they must go into two strategies belonging to a single player. Thus if ϕ is the permutation of the pure strategies it induces a permutation ψ of the players.

Each n -tuple of pure strategies is therefore permuted into another n -tuple of pure strategies. We may call χ the induced permutation of these n -tuples. Let ξ denote an n -tuple of pure strategies and $P_i(\xi)$ the pay-off to player i when the n -tuple ξ is employed. We require that if

$$j = i^{\psi} \quad \text{then} \quad P_j(\xi^{\chi}) = P_i(\xi)$$

which completes the definition of a symmetry.

The permutation ϕ has a unique linear extension to the mixed strategies. If $s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha}$ we define $(s_i)^\phi = \sum_{\alpha} c_{i\alpha} (\pi_{i\alpha})^\phi$. The extension of ϕ to the mixed strategies clearly generates an extension of χ to the n -tuples of mixed strategies. We shall also denote this by χ .

We define a **symmetric n -tuple** \mathfrak{s} of a game by $\mathfrak{s}^\chi = \mathfrak{s}$ for all χ 's, it being understood that χ means a permutation derived from a symmetry ϕ .

Theorem 4.1. *Any finite game has a symmetric equilibrium point.*

Proof. First we note that $s_{i0} = \frac{\sum_{\alpha} \pi_{j\alpha}}{\sum_{\alpha} 1}$ has the property $(s_{i0})^\phi = s_{j0}$ where $j = i^\psi$, so that the n -tuple $\mathfrak{s}_0 = (s_{10}, s_{20}, \dots, s_{n0})$ is fixed under any χ ; hence any game has at least one symmetric n -tuple.

If $\mathfrak{s} = (s_1, \dots, s_n)$ and $\mathfrak{t} = (t_1, \dots, t_n)$ are symmetric then $\frac{\mathfrak{s} + \mathfrak{t}}{2} = (\frac{s_1 + t_1}{2}, \dots, \frac{s_n + t_n}{2})$ is so too because $\mathfrak{s}^\chi = \mathfrak{s} \iff s_j = (s_i)^\phi$ where $j = i^\psi$, hence

$$\frac{s_j + t_j}{2} = \frac{(s_i)^\phi + (t_i)^\phi}{2} = \left(\frac{s_i + t_i}{2} \right)^\phi, \quad \text{hence} \quad \left(\frac{\mathfrak{s} + \mathfrak{t}}{2} \right)^\chi = \frac{\mathfrak{s} + \mathfrak{t}}{2}.$$

This shows that the set of symmetric n -tuples is a convex subset of the space of n -tuples since it is obviously closed.

Now observe that for each λ the mapping $\mathfrak{s} \rightarrow \mathfrak{s}'(\mathfrak{s}, \lambda)$ used in the proof of existence theorem was intrinsically defined. Therefore, if $\mathfrak{s}_2 = \mathfrak{s}'(\mathfrak{s}_1, \lambda)$ and χ is an automorphism of the game we will have $\mathfrak{s}_2^\chi = \mathfrak{s}'(\mathfrak{s}_1^\chi, \lambda)$. If \mathfrak{s}_1 is symmetric $\mathfrak{s}_1 = \mathfrak{s}_1^\chi$ and therefore $\mathfrak{s}_2^\chi = \mathfrak{s}'(\mathfrak{s}_1, \lambda) = \mathfrak{s}_2$. Consequently this mapping maps the set of symmetric n -tuples to itself.

Since this set is a cell there must be a symmetric fixed point of \mathfrak{s}_λ . And, as in the proof of the existence theorem we could obtain a limit point \mathfrak{s}^* which would have to be symmetric. \boxtimes

Solutions

We define here solutions, strong solutions, and sub-solutions. A non-cooperative game does not always have a solution, but when it does the solution is unique. Strong solutions are solutions with special properties. Sub-solutions always exist and have many of the properties of solutions, but lack uniqueness.

S_i will denote a set of mixed strategies of player i and \mathcal{S} a set of n -tuples of mixed strategies.

Definition 5.1 (Solvability). A game is solvable if its set, \mathcal{S} , of equilibrium points satisfies the condition

$$(\mathfrak{t}; v_i) \in \mathcal{S} \text{ and } \mathfrak{s} \in \mathcal{S} \implies (\mathfrak{s}; v_i) \in \mathcal{S} \text{ for all } i\text{'s.} \quad (5)$$

This is called the **interchangeability** condition. The **solution** of a solvable game is its set, \mathcal{S} , of equilibrium points.

Definition 5.2 (Strong Solvability). A game is **strongly solvable** if it has a solution, \mathcal{S} , such that for all i 's

$$j \in \mathcal{S} \text{ and } P_i(j; v_i) = P_i(j) \implies (j; v_i) \in \mathcal{S}$$

and then \mathcal{S} is called a **strong solution**.

Definition 5.3 (Equilibrium Strategies). In a solvable game set S_i be the set of all mixed strategies s_i such that for some t the n -tuple $(t; s_i)$ is an equilibrium point. [s_i is the i th component of some equilibrium point.] We call S_i the set of **equilibrium strategies** of player i .

Definition 5.4 (Sub-solutions). If \mathcal{S} is a subset of the set of equilibrium points of a game and satisfies condition (5); and if \mathcal{S} is maximal relative to this property then we call \mathcal{S} a **sub-solution**.

For any sub-solution \mathcal{S} we define the i th **factor set**, S_i , as the set of all s_i 's such that \mathcal{S} contains $(t; s_i)$ for some t .

Note that a sub-solution, when unique, is a solution; and its factor sets are the sets of equilibrium strategies.

Theorem 5.1. *A sub-solution, \mathcal{S} , is the set of all n -tuples (s_1, s_2, \dots, s_n) such that each $s_i \in S_i$ where S_i is the i th factor set of \mathcal{S} . Geometrically, \mathcal{S} is the product of its factor sets.*

Proof. Consider such an n -tuple (s_1, \dots, s_n) . By definition $\exists t_1, t_2, \dots, t_n$ such that for each i $(t_i; s_i) \in \mathcal{S}$. Using the condition (5) $n - 1$ times we obtain successively $t_1; s_1; s_2) \in \mathcal{S}, \dots, (t_1; s_1; s_2; s_3; \dots; s_n) \in \mathcal{S}$ and the last is simply $(s_1, s_2, \dots, s_n) \in \mathcal{S}$, which we needed to show. \square

Theorem 5.2. *The factor sets S_1, S_2, \dots, S_n of a sub-solution are closed and convex as subsets of the mixed strategy space.*

Proof. It suffices to show two things:

- (a) if s_i and $s'_i \in S_i$ then $s_i^* = (s_i + s'_i)/2 \in S_i$;
- (b) if $s_i^\#$ is a limit point of S_i then $s_i^\# \in S_i$.

Let $t \in \mathcal{S}$. Then we have $P_j(t; s_i) \geq P_j(t; s_i; v_j)$ and $P_j(t; s_i) \geq P_j(t; s'_i; v_j)$ for any v_j , by using the criterion of (1) for an eq. pt. Adding these inequalities, using the linearity of $P_j(s_1, \dots, s_n)$ in S_i , and dividing by 2, we get $P_j(t; s_i^*) \geq P_j(t; s_i^*; v_j)$ since $s_i^* = (s_i + s'_i)/2$. From this we know that $(t; s_i^*)$ is an eq. pt. for any $t \in \mathcal{S}$. If the set of all such eq. pts. $(t; s_i^*)$ is added to \mathcal{S} the augmented set clearly satisfies condition (5), and since \mathcal{S} was to be maximal it follows that $s_i^* \in S_i$.

To attack (b) note that the n -tuple $(t; s_i^\#)$, where $t \in \mathcal{S}$ will be a limit point of the set of n -tuples of the form $(t; s_i)$ where $s_i \in S_i$, since $s_i^\#$ is a limit point of S_i . But this set is a set of eq. pts. and hence any point in its closure is an eq. pt., since the set of all eq. pts. is closed [see pg. 3]. Therefore $(t; s_i^\#)$ is an eq. pt. and hence $s_i^\# \in S_i$ from the same argument as for s_i^* . \square

Definition 5.5 (Values). Let \mathcal{S} be the set of equilibrium points of a game. We define

$$v_i^+ = \max_{\mathfrak{s} \in \mathcal{S}} [P_i(\mathfrak{s})], \quad v_i^- = \min_{\mathfrak{s} \in \mathcal{S}} [P_i(\mathfrak{s})].$$

If $v_i^+ = v_i^-$ we write $v_i = v_i^+ = v_i^-$. v_i^+ is the **upper value** to player i of the game; v_i^- the **lower value**; and v_i the **value**, if it exists.

Values will obviously have to exist if there is but one equilibrium point.

One can define **associated values** for a sub-solution by restricting \mathcal{S} to the eq. pts. in the sub-solution and then using the same defining equations as above.

A two-person zero-sum game is always solvable in the sense defined above. The sets of equilibrium strategies S_1 and S_2 are simply the sets of “good” strategies. Such a game is not generally strongly solvable; strong solutions exist only when there is a “saddle point” in *pure* strategies.

Geometrical Form of Solutions

In the two-person zero-sum case it has been shown that the set of “good” strategies of a player is a convex polyhedral subset of his strategy spaces. We shall obtain the same result for a player’s set of equilibrium strategies in any solvable game.

Theorem 6.1. *The sets S_1, S_2, \dots, S_n of equilibrium strategies in a solvable game are polyhedral convex subsets of the respective mixed strategy spaces.*

Proof. An n -tuple \mathfrak{s} will be an equilibrium point if and only if for every i

$$P_i(\mathfrak{s}) = \max_{\alpha} P_{i\alpha}(\mathfrak{s}) \tag{6}$$

which is condition (3). An equivalent condition is for every i and α

$$P_i(\mathfrak{s}) - P_{i\alpha}(\mathfrak{s}) \geq 0. \tag{7}$$

Let us now consider the form of the set S_j of the equilibrium strategies, s_j , of player j . Let t be any equilibrium point, then $(t; s_j)$ will be an equilibrium point if and only if $s_j \in S_j$, from Theorem (5.1). We now apply condition (7) to $(t; S_j)$, obtaining

$$s_j \in S_j \iff \text{for all } i, \alpha \quad P_i(t; s_j) - P_{i\alpha}(t; s_j) \geq 0. \tag{8}$$

Since P_i is n -linear and t is constant these are a set of linear inequalities of the form $F_{i\alpha}(s_j) \geq 0$. Each such inequality is either satisfied for all s_j or for those lying on and to one side of some hyperplane passing through the strategy simplex. Therefore, the complete set [which is finite] of conditions will all be satisfied simultaneously on some convex polyhedral set of player j 's strategy simplex. [Intersection of half-spaces.]

As a corollary we may conclude that S_k is the convex closure of a finite set of mixed strategies [vertices]. \square

Simple Examples

These are intended to illustrate the concepts defined in the paper and display special phenomena which occur in the game.

The first player has the roman letter strategies and the pay-off to the left, etc.

Example 6.1.

5	a	α	-3
-4	a	β	10
-5	b	α	5
3	b	β	-4

Weak Solutions: $(\frac{9}{16}a + \frac{7}{16}b, \frac{7}{17}\alpha + \frac{10}{17}\beta)$.
 $v_1 = -\frac{5}{17}, v_2 = +\frac{1}{2}$

Example 6.2.

1	a	α	1
-10	a	β	10
10	b	α	-10
-1	b	β	-1

Strong Solution: (b, β) .
 $v_1 = v_2 = -1$

References

- [1] von Neumann, Morgenstern, “Theory of Games and Economic Behavior”, Princeton University Press, 1944.
- [2] J. F. Nash, Jr., “Equilibrium Points in N -Person Games”, *Proc. N. A. S.* 36 (1950) 48-49. <https://www.jstor.org/stable/88031>.

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