Non-cooperative games

A DISSERTATION

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Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communications, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of those concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

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Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior* [**1**]. This books also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium* point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies".

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion on a special section of this paper.

Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceeded by a subtitle indicating the concept defined[1]. The non-cooperative idea will be implicit, rather than explicit, below.

Definition 2.1 (Finite Game)**.** For us an **n-person game** will be a set of n **players**, or **positions**, each with an associated finite of **pure strategies**; and corresponding to each player, i, a **pay-off function**, P_i, which maps the set of all n-tuples of pure strategies into the real numberes. When we use the word **n-tuples** we shall always mean a set of n items, with each item associated with a different player.

Definition 2.2 (Mixed Strategy, s_i)**.** A **mixed strategy** of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ with $\sum_\alpha c_{i\alpha} = 1$ and $c_{i\alpha} \geq 0$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i. We regard the s_i's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be regarded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i, and v_i, etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the ith player's αth pure strategy, etc.

Definition 2.3 (Pay-off functions, P_i)**.** The pay-off function, P_i, used in the definition of a finite game above, has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player [n-linear]. This extension we shall also denote P_i, writing $P_i(s_1, s_2, \cdots, s_n)$.

We shall write $\mathfrak{s}$ or $\mathfrak{t}$ to denote an n-tuble of mixed strategies and if $\mathfrak{s} = (s_1, \cdots, s_n)$ then $P_i(\mathfrak{s})$ shall mean $P_i(s_1, s_2, \cdots, s_n)$. Such an n-tuple, $\mathfrak{s}$, will also be regarded as as point in a vector space, which space could be obtained by multiplying together the vector spaces containing the mixed strategies. And the set of all such n-tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation $(\mathfrak{s}; t_i)$ to stand for $(s_1, s_2, \cdots, s_{i-1}, t_i, s_{i+1}, \cdots, s_n)$ where $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$. The effect of successive substitutions $((\mathfrak{s}_i; t_i); v_j)$ we indicate by $(\mathfrak{s}; t_i; v_j)$, etc.

Definition 2.4 (Equilibrium Points)**.** An n-tuple $\mathfrak{s}$ is an **equilibrium point** if and only if for every i

$$P_i(\mathfrak{s}) = \max_{\text{all } v_i\text{'s}} \left[P_i(\mathfrak{s}; v_i)\right]. \tag{1}$$

[1] We actually use **boldface** for definitions instead, but this note on the subtitles is left in to preserve the wording of the original.

Thus an equilbrium point is an n-tuple $\mathfrak{s}$ such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i **uses** a pure strategy $\pi_{i\beta}$ if $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ and $c_{i\beta} > 0$. If $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that $\mathfrak{s}$ uses $\pi_{i\alpha}$.

From the linearity of $P_i(s_1, \cdots, s_n)$ in S_i,

$$\max_{\text{all } v_i\text{'s}} [P_i(\mathfrak{s}; v_i)] = \max_\alpha [P_i(\mathfrak{s}; \pi_{i\alpha})] . \tag{2}$$

We define $P_{i\alpha}(\mathfrak{s}) = P_i(\mathfrak{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for $\mathfrak{s}$ to be an equilibrium point:

$$P_i(\mathfrak{s}) = \max_\alpha P_{i\alpha}(\mathfrak{s}). \tag{3}$$

If $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ and $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ then $P_i(\mathfrak{s}) = \sum_\alpha c_{i\alpha}P_{i\alpha}(\mathfrak{s})$, consequently for (3) to hold we must have $c_{i\alpha} = 0$ whenever $P_{i\alpha}(\mathfrak{s}) < \max_\beta P_{i\alpha}(\mathfrak{s})$, which is to say that $\mathfrak{s}$ does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i. So we write

$$\text{if } \pi_{i\alpha} \text{ is used in } \mathfrak{s} \text{ then } P_{i\alpha}(\mathfrak{s}) = \max_\beta P_{i\beta}(\mathfrak{s}) \tag{4}$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion (3) for an eq. pt. can be expressed as the equating of two continuous functions on the space of n-tubles $\mathfrak{s}$ the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

I have previously published in [Proc. N. A. S. 36 (1950) 48-49] [**2**] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings: $\mathfrak{s} \to \mathfrak{s}'(\mathfrak{s}, 1)$; $\mathfrak{s} \to \mathfrak{s}'(\mathfrak{s}, 2); \cdots$ whose fixed points have an equilibrium point as a limit points. A limit mapping exists, but is discontinuous, and need not have any fixed points.

Theorem 3.1. *Every finite game has an equilibrium point.*

Proof. Using our standard notation, let $\mathfrak{s}$ be an n-tuple of mixed strategies, and $P_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the others use their respective mixed strategies in $\mathfrak{s}$. For each integer λ we define the following continuous functions of $\mathfrak{s}$:

$$q_i(\mathfrak{s}) = \max_{\alpha} P_{i\alpha}(\mathfrak{s}),$$
$$\phi_{i\alpha}(\mathfrak{s}, \lambda) = P_{i\alpha}(\mathfrak{s}) - q_i(\mathfrak{s}) + \frac{1}{\lambda}, \text{ and}$$
$$\phi^{+}_{i\alpha}(\mathfrak{s}, \lambda) = \max\left[0, \phi_{i\alpha}(\mathfrak{s}, \lambda)\right].$$

Now $\sum_\alpha \phi^{+}_{i\alpha}(\mathfrak{s}, \lambda) \geq \max\limits_{\alpha} \phi^{+}_{i\alpha}(\mathfrak{s}, \lambda) = \frac{1}{\lambda} > 0$ so that $c'_{i\alpha}(\mathfrak{s}, \lambda) = \frac{\phi^{+}_{i\alpha}(\mathfrak{s},\lambda)}{\sum_\beta \phi^{+}_{i\beta}(\mathfrak{s},\lambda)}$ is continuous. Define $s'_i(\mathfrak{s}, \lambda) = \sum_\alpha \pi_{i\alpha} c'_{i\alpha}(\mathfrak{s}, \lambda)$ and $\mathfrak{s}'(\mathfrak{s}, \lambda) = (s'_1, s'_2, \cdots, s'_n)$. Since all the operations have preserved continuity, the mapping $\mathfrak{s} \to \mathfrak{s}'(\mathfrak{s}, \lambda)$ is continuous; and since the space of n-tuples, $\mathfrak{s}$, is a cell, there must be a fixed point for each λ. Hence there will be a subsequence λ_μ, converging to $\mathfrak{s}^*$, where $\mathfrak{s}_\mu$ is fixed under the mapping $\mathfrak{s} \to \mathfrak{s}'(\mathfrak{s}, \lambda_{(\mu)})$.

Now supposed $\mathfrak{s}^*$ were not an equilibrium point. Then if $\mathfrak{s}^* = (s^*_1, \cdots, s^*_n)$ some component s^*_i must be non-optimal against the others, which means s^*_i uses some pure strategy $\pi_{i\alpha}$ which is non-optimal [see (4)]. This means that $P_{i\alpha}(\mathfrak{s}^*) < q_i(\mathfrak{s}^*)$ which justifies writing $P_{i\alpha}(\mathfrak{s}^*) - q_i(\mathfrak{s}^*) < -\epsilon$.

From continuity, if μ is large enough,

$$\Big| \left[P_{i\alpha}(\mathfrak{s}_\mu) - q_i(\mathfrak{s}_\mu)\right] - \left[P_{i\alpha}(\mathfrak{s}^*) - q_i(\mathfrak{s}^*)\right] \Big| < \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{\lambda_{(\mu)}} < \frac{\epsilon}{2}.$$

Adding, $P_{i\alpha}(\mathfrak{s}_\mu) - q_i(\mathfrak{s}_\mu) + \frac{1}{\lambda_{(\mu)}} < 0$ which is simply $\phi_{i\alpha}(\mathfrak{s}_\mu, \lambda_{(\mu)}) < 0$, whence $\phi^{+}_{i\alpha}(\mathfrak{s}_\mu, \lambda_{(\mu)}) = 0$, whence $c'_{i\alpha}(\mathfrak{s}_\mu, \lambda_{(\mu)}) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in $\mathfrak{s}_\mu$ since $\mathfrak{s}_\mu = \sum_\alpha \pi_{i\alpha} c'_{i\alpha}(\mathfrak{s}_\mu, \lambda_{(\mu)})$, because $\mathfrak{s}_\mu$ is a fixed point.

And since $\mathfrak{s}_\mu \to \mathfrak{s}^*$, $\pi_{i\alpha}$ is not used in $\mathfrak{s}^*$, which contradicts our assumption. Hence $\mathfrak{s}^*$ is a fixed point. $\boxtimes$

References

[1] von Neumann, Morgenstern, "Theory of Games and Economic Behavior", Princeton University Press, 1944.

[2] J. F. Nash, Jr., "Equilibrium Points in N-Person Games", *Proc. N. A. S.* 36 (1950) 48-49. `https://www.jstor.org/stable/88031`.

Acknowledgements