Representation of Mathematical Expressions in LaTeX

Simon Xiang

January 27, 2020

1 Limits

Example 1.1 (Algebraic approach)**.** Consider the limit

$$\lim_{x \to 0} \frac{x}{3 - \sqrt{x+9}}.$$

First, reduce the expression by completing the square, since directly plugging in the value for which the limit converges into the equation will yield $0/0$, which is complete nonsense.

$$\begin{aligned} \frac{x}{3-\sqrt{x+9}} &= \frac{x(3+\sqrt{x+9})}{(3-\sqrt{x+9})(3+\sqrt{x+9})} \\ &= \frac{x(3+\sqrt{x+9})}{9-(\sqrt{x+9})^2} \\ &= \frac{x(3+\sqrt{x+9})}{-x} \\ &= -3-\sqrt{x+9}. \end{aligned}$$

Therefore, we can safely say that the expressions $\frac{x}{3-\sqrt{x+9}}$ and $-3-\sqrt{x+9}$ are equivalent. Hence, the limit can be simply evaluated by plugging in the value that the limit converges to into the second expression for x.

$$\begin{aligned} \lim_{x \to 0} \frac{x}{3-\sqrt{x+9}} &= \lim_{x \to 0} -3-\sqrt{x+9} \\ &= -3-\sqrt{0+9} \\ &= -3-\sqrt{9} \\ &= -6 \end{aligned}$$

Thus

$$\lim_{x \to 0} \frac{x}{3-\sqrt{x+9}} = -6.$$

Example 1.2 (L'hopital's Rule)**.** Assuming knowledge of L'Hopital's Rule, one basic limit that first-year Calculus students are often told to memorize is

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

However, with a working knowledge of derivatives and L'Hopital's rule, we can prove that the value of this limit is indeed 1. First, let us state L'Hopital's rule.

Definition 1. *(L'Hopital's rule).* Let $f(x)$ and $g(x)$ be differentiable functions where $g(x) \neq 0$. If

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L \quad \ni \quad c, L \in \mathbb{R} \cup \{-\infty, +\infty\},$$

and either of

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \quad \lim_{x\to c} |f(x)| = \lim_{x\to c} |g(x)| = \infty$$

are true, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} \quad = \quad \lim_{x\to c} \frac{f'(x)}{g'(x)} \quad = \quad L.$$

If we let $f(x) = \sin(x)$ and $g(x) = x$, then evaluating $\lim_{x\to 0} \sin(x)$ and $\lim_{x\to 0} x$ both yield 0 as the limit, therefore the function is in the perfect form to directly apply L'Hopital's rule. Let us do so.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x} \\ &= \lim_{x\to 0} \frac{\cos(x)}{1} \\ &= \frac{\cos(0)}{1} \\ &= 1 \end{aligned}$$

Therefore we have verified using L'Hopital's rule that the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

is indeed true.

2 Derivatives

Example 2.1 (Chain, product, and quotient rule)**.** In a moment of genius, I realized I would only have to type one example instead of three if I used a derivative problem that involved the product, quotient, *and* the chain rule!

Definition 2. *(Product rule).* Let $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions of x. Then

$$f'(x) \quad = \quad (g'(x) \cdot h(x)) + (h'(x) \cdot g(x)).$$

Definition 3. *(Quotient rule).* Let $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are both differentiable functions of x, and $h(x) \neq 0$. Then

$$f'(x) \quad = \quad \frac{(g'(x) \cdot h(x)) - (h'(x) \cdot g(x))}{[h(x)]^2}.$$

Definition 4. *(Chain rule, standard).* Let $f(x) = g(h(x))$, where $g(x)$ and $h(x)$ are both differentiable functions of x. Then

$$f'(x) \quad = \quad g'(h(x)) \cdot h'(x).$$

Definition 5. *(Chain rule, composition of functions).* Let $F = f \circ g$, then

$$F' \quad = \quad (f' \circ g) \cdot g'.$$

Definition 6. *(Chain rule, Leibniz notation).* If we have a function such that the variable y depends on the variable u, and u depends on the variable x, we can safely say that y depends on x, and thus

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$ [1]

Now that we have all the tools in our toolbox, let us begin to evaluate a particularly nasty derivative. Let

$$f(x) \quad = \quad \frac{\sin(x^2)\sqrt{e^x + e^{-x}}}{4^x}.$$

To solve for $f'(x)$, let $g(x) = \sin(x^2)\sqrt{e^x + e^{-x}}$ and $h(x) = 4^x$. Apply the quotient rule to get

$$f'(x) \quad = \quad \frac{\left(\frac{d}{dx}\big(\sin(x^2)\sqrt{e^x + e^{-x}}\big) \cdot 4^x\right) - \left(\frac{d}{dx}(4^x) \cdot sin(x^2)\sqrt{e^x + e^{-x}}\right)}{(4^x)^2}$$

To finish solving out the problem, we will need to compute $\frac{d}{dx}\big(\sin(x^2)\sqrt{e^x + e^{-x}}\big)$ which requires both the product and the chain rule, and $\frac{d}{dx}(4^x)$, which also requires the chain and product rule. Let us compute the easier derivative first, recalling that $e^{\ln(x)} = x$.

$$\overline{f}(x) = 4^x = e^{ln(4^x)}.$$

[1] I omitted doing two problems because of laziness yet defined the chain rule three times. Ironic.

Let $u(x) = ln(4^x) = xln(4)$, and therefore $\overline{f}(u) = e^u$. Set $\overline{f} = y$, and apply the Leibniz definition of the chain rule. Recall that $\frac{d}{dx}e^x = e^x$ to finish solving for $\overline{f}'(x)$.

$$\frac{dy}{du} = e^u, \quad \frac{du}{dx} = ln(4), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot ln(4) = e^{ln(4^x)} ln(4) = 4^x ln(4)$$

Let us define another function $\hat{f}(x) = \sin(x^2)\sqrt{e^x + e^{-x}}$. If we set $\hat{g}(x) = \sin(x^2)$ and $\hat{h}(x) = \sqrt{e^x + e^{-x}}$, we can continue on to apply the product rule. To solve for $\hat{g}'(x)$ and $\hat{h}'(x)$, the chain rule will need to be applied.

$$\begin{aligned}
\hat{f}'(x) &= \hat{g}'(x) \cdot \hat{h}(x) + \hat{h}'(x) \cdot \hat{g}(x) \\
\hat{g}'(x) &= \cos(x^2) \cdot \frac{d}{dx}(x^2) = 2x\cos(x^2) \\
\hat{h}'(x) &= \frac{1}{2\sqrt{e^x + e^{-x}}} \cdot \frac{d}{dx}(e^x + e^{-x}) = \frac{e^x - e^{-x}}{2\sqrt{e^x + e^{-x}}} \\
\hat{f}'(x) &= \big(2x\cos(x^2) \cdot \sqrt{e^x + e^{-x}}\big) + \big(\frac{e^x - e^{-x}}{2\sqrt{e^x + e^{-x}}} \cdot \sin(x^2)\big)
\end{aligned}$$

Substitute this new information into the original equation to get

$$\begin{aligned}
f'(x) &= \frac{\big(\hat{f}'(x) \cdot \overline{f}(x)\big) - \big(\overline{f}'(x) \cdot \hat{f}(x)\big)}{(\overline{f}(x))^2} \\
&= \frac{\Bigg[4^x\Bigg(\Big(2x\cos(x^2)\sqrt{e^x + e^{-x}}\Big) + \Big(\frac{e^x - e^{-x}}{2\sqrt{e^x+e^{-x}}} \cdot \sin(x^2)\Big)\Bigg)\Bigg] - \Bigg[ln(4) \cdot 4^x\Big(\sin(x^2) \cdot \sqrt{e^x + e^{-x}}\Big)\Bigg]}{4^{2x}} \\
&= \frac{\Big((2x\cos(x^2)\sqrt{e^x + e^{-x}}\Big) + \Big(\frac{e^x - e^{-x}}{2\sqrt{e^x+e^{-x}}} \cdot \sin(x^2)\Big) - \Big(ln(4) \cdot 4^x\big(\sin(x^2) \cdot \sqrt{e^x + e^{-x}}\big)\Big)}{4^x} \\
&= \frac{2x\cos(x^2)\sqrt{e^x + e^{-x}}}{4^x} + \frac{(e^x - e^{-x})\sin(x^2)}{2 \cdot 4^x\sqrt{e^x + e^{-x}}} - \frac{ln(4)\sin(x^2)\sqrt{e^x + e^{-x}}}{4^x}.
\end{aligned}$$

Example 2.2 (Partial derivatives and the chain rule)**.** The multivariate chain takes on a different, yet familiar form when juxtaposed to its single-variable counterpart.

Definition 7. *(Multivariate chain rule).* Let $z = f(x, y)$, where x and y and both functions of t. If z is differentiable at $x(t)$, $y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

Jumping right into the problem, find $\frac{dy}{dx}$ of $x^2 \tan(5y) + x^3y^3 = 4x - e^{(x^2+y^2)}$. Hold up. This looks awfully familiar to a concept learned in a first-semester Calculus course and nothing like a question that involves partial derivatives! However, we can rewrite this as a multivariate function in the form $z = f(x, y)$, where $y = \hat{f}(x)$. Therefore, if we wanted to find $\frac{dz}{dx}$, we could apply the multivariate chain rule and obtain the following:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{f_x}{f_y}$$

This equation is known as the *Implicit Function Theorem*, and we can apply it here to take the implicit derivative of the given function. Rearrage the equation as such to obtain a zero on one side: $x^2 \tan(5y) + x^3y^3 - 4x + e^{x^2+y^2} = 0$. Now we just have to solve for f_x and f_y and plug it into our newly obtained formula to finish the question.

$$f_x = 2x\tan(5y) + 3x^2y^3 - 4 + 2xe^{(x^2+y^2)}$$
$$f_y = 5x^2\sec^2(5y) + 3y^2x^3 + 2ye^{(x^2+y^2)}$$
$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x\tan(5y) + 3x^2y^3 - 4 + 2xe^{(x^2+y^2)}}{5x^2\sec^2(5y) + 3y^2x^3 + 2ye^{(x^2+y^2)}}$$

3 Integrals

Example 3.1 (Difficult integration)**.** I was going to make separate sections for matrix operations and integration, but in yet another stroke of genius I realized we can just solve integration problems using basic linear algebra! Here we attempt an infamous integral.

$$\int \sqrt{\tan(x)}\, dx$$

The substitution to be used is not immediately clear, and the later part of this integral involves a particularly nasty case of partial fractions, which makes this such a challenging integral. Jumping right into the question, use the substitution $u = \sqrt{\tan(x)}$ to reduce the integral to something that partial fractions can be applied on.

$$u = \sqrt{\tan(x)} \qquad u^2 = \tan(x)$$
$$2u\, du = \sec^2(x)\, dx \qquad dx = \frac{2u\, du}{\sec^2(x)}$$

$$\int \sqrt{\tan(x)}\, dx = \int (u) \cdot \left(\frac{2u\, du}{\sec^2(x)}\right) = \int \frac{2u^2\, du}{\sec^2(x) - 1 + 1} = \int \frac{2u^2\, du}{\tan^2(x) + 1}$$
$$= \int \frac{2u^2\, du}{u^4 + 1}$$

Now the problem remains of how to factor a polynomial in the form $x^4 + 1$, which involves a sneaky complete the square operation to put the polynomial into a form that allows us to apply the difference of squares formula to factor it. Truly, this question is quite difficult and probably should not be assigned as homework for an introductory calculus course (looking at my Cal II professor).

$$\begin{aligned} u^4+1 \quad &= \quad u^4+2u^2+1-2u^2 \quad = \quad (u^2+1)-(\sqrt{2}u)^2 \\ &= (u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1) \end{aligned}$$

Now we have to decompose $\frac{2u^2}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)}$ by partial fractions which will leave a rather troublesome system of linear equations, thankfully we can solve it by simply inverting the resultant matrix to solve for the vector x, which can be represented as such: $A\vec{x}=\vec{b} \qquad \vec{x}=A^{-1}\vec{b}$

$$\frac{2u^2}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)} = \frac{Au+B}{(u^2+\sqrt{2}u+1)} + \frac{Cu+D}{(u^2-\sqrt{2}u+1)}$$

$$= (Au+B)(u^2-\sqrt{2}u+1)+(Cu+D)(u^2+\sqrt{2}u+1)$$

$$= (Au^3-\sqrt{2}Au^2+Au)+(Bu^2-\sqrt{2}Bu+B)+(Cu^3+\sqrt{2}Cu^2+Cu)+(Du^2+\sqrt{2}Du+D)$$

$$= u^3(A+C)+u^2(B-\sqrt{2}A+\sqrt{2}C+D)+u(A-\sqrt{2}B+\sqrt{2}D+C)+(B+D)$$

$$\text{Let} \quad u=0, \quad \text{then} \quad \boxed{(B+D)=0}$$

$$\frac{2u^2}{u} = \frac{u^3(A+C)+u^2(B-\sqrt{2}A+\sqrt{2}C+D)+u(A-\sqrt{2}B+\sqrt{2}D+C)}{u}$$

$$2u = u^2(A+C)+u(B-\sqrt{2}A+\sqrt{2}C+D)+(A-\sqrt{2}B+\sqrt{2}D+C)$$

$$\text{Let} \quad u=0, \quad \text{then} \quad \boxed{(A-\sqrt{2}B+\sqrt{2}D+C)=0}$$

$$\frac{2u}{u} = \frac{u^2(A+C)+u(B-\sqrt{2}A+\sqrt{2}C+D)}{u}$$

$$2 = u(A+C)+(B-\sqrt{2}A+\sqrt{2}C+D)$$

$$\text{Let} \quad u=0, \quad \text{then} \quad \boxed{(B-\sqrt{2}A+\sqrt{2}C+D)=2}$$

$$2 = u(A+C)+2$$

$$(A+C) = \frac{0}{u} = 0$$

$$\boxed{(A+C)=0}$$

To complete the partial fraction decomposition, all that's left is to solve the resultant system of linear equations. Arrange the equations as such to make it easier to set up the matrix equation:

$$\begin{array}{lllllllll} A & + & 0 & + & C & + & 0 & = & 0 \\ -\sqrt{2}A & + & B & + & \sqrt{2}C & + & D & = & 2 \\ A & + & -\sqrt{2}B & + & C & + & \sqrt{2}D & = & 0 \\ 0 & + & B & + & 0 & + & D & = & 0 \end{array}$$

Now rewrite this system as a matrix and invert the matrix to finish solving for A, B, C and D.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b}, \qquad \vec{x} = A^{-1}\vec{b}$$

Now we could have solved this system with row reduction, but I am an avid hater of row reduction, and so we invert the matrix to solve the system instead. By extension, we're not calculating this inverse with the $[A \,|\, I] \xrightarrow{\hspace{1em} row\, reduction \hspace{1em}} [I \,|\, A^{-1}]$ method, but using the method involving determinants and the adjugate matrix instead, where

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

In order to calculate the determinant of the matrix, perform a cofactor expansion along the first row, then along the third row for the resultant two matrices.

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \left(1 \cdot \begin{vmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{vmatrix}\right) + \left(-1 \cdot \begin{vmatrix} -\sqrt{2} & 1 \\ 1 & \sqrt{2} \end{vmatrix} + 1 \cdot \begin{vmatrix} -\sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{vmatrix}\right)$$

$$= (1 + 3) + (3 + 1)$$
$$= 8$$

To compute the adjugate matrix, we would need to calculate fourteen more 3x3 determinants for each coordinate of the matrix to complete the matrix of minors. However, such computations are trivial and left to the reader as an exercise. Thus, we have the matrix of minors as such:

$$\begin{bmatrix} 4 & -2\sqrt{2} & 4 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & -2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ -2\sqrt{2} & 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

Multiply the matrix of minors by the "checkerboard" pattern to find the matrix of cofactors, then take the transpose of the resulting matrix to get the adjugate matrix of A.

$$\begin{bmatrix} 4 & -2\sqrt{2} & 4 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & -2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ -2\sqrt{2} & 4 & 2\sqrt{2} & 4 \end{bmatrix} \implies \begin{bmatrix} 4 & 2\sqrt{2} & 4 & -2\sqrt{2} \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 0 \\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 4 & -2\sqrt{2} & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2\sqrt{2} & 4 & -2\sqrt{2} \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 0 \\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 4 & -2\sqrt{2} & 4 \end{bmatrix}^T = \begin{bmatrix} 4 & -2\sqrt{2} & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 4 \end{bmatrix}$$

Now that we have both the adjugate matrix and the determinant, we can divide the adjugate matrix by the determinant of A to get A^{-1}.

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{\begin{bmatrix} 4 & -2\sqrt{2} & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 4 \end{bmatrix}}{8}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

From here, all that remains is to multiply the inverted matrix by the vector $\vec{b}$ to obtain the solution vector $\vec{x}$.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} \\ 0 \end{bmatrix} \cdot 2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad A = -\frac{1}{\sqrt{2}} \qquad B = 0 \qquad C = \frac{1}{\sqrt{2}} \qquad D = 0$$

$$\frac{Au+B}{(u^2+\sqrt{2}u+1)} + \frac{Cu+D}{(u^2-\sqrt{2}u+1)} = \frac{-\frac{1}{\sqrt{2}}u+0}{(u^2+\sqrt{2}u+1)} + \frac{\frac{1}{\sqrt{2}}u+)}{(u^2-\sqrt{2}u+1)}$$

$$= \boxed{\frac{u}{\sqrt{2}(u^2-\sqrt{2}u+1)} - \frac{u}{\sqrt{2}(u^2+\sqrt{2}u+1)}}$$

To finish the problem, simply integrate the newly obtained expression and plug back in $\sqrt{\tan(x)}$ for u. Integrating this expression requires splitting the integral so that the left half can be integrated with a simple u-sub and the right half with a complete the square, which looks like this: $u = \frac{1}{2}(2u - \sqrt{2}) + \frac{\sqrt{2}}{2}$ for the left portion and $u = \frac{1}{2}(2u + \sqrt{2}) - \frac{\sqrt{2}}{2}$ for the right.

$$\int \left(\frac{u}{\sqrt{2}(u^2 - \sqrt{2}u + 1)} - \frac{u}{\sqrt{2}(u^2 + \sqrt{2}u + 1)} \right) du$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{\frac{1}{2}(2u - \sqrt{2}) + \frac{\sqrt{2}}{2}}{\sqrt{2}(u^2 - \sqrt{2}u + 1)} \right) du - \frac{1}{\sqrt{2}} \int \left(\frac{\frac{1}{2}(2u + \sqrt{2}) - \frac{\sqrt{2}}{2}}{\sqrt{2}(u^2 + \sqrt{2}u + 1)} \right) du$$

$$= \frac{1}{2\sqrt{2}} \int \left(\frac{2u - \sqrt{2}}{(u^2 - \sqrt{2}u + 1)} \right) du + \frac{1}{2} \int \left(\frac{1}{(u^2 - \sqrt{2}u + 1)} \right) du$$

$$- \frac{1}{2\sqrt{2}} \int \left(\frac{2u + \sqrt{2}}{(u^2 + \sqrt{2}u + 1)} \right) du + \frac{1}{2} \int \left(\frac{1}{(u^2 + \sqrt{2}u + 1)} \right) du$$

Let $v = u^2 - \sqrt{2}u + 1, \quad dv = (2u - \sqrt{2})\, du \qquad \text{and} \qquad w = u^2 + \sqrt{2}u + 1, \quad dw = (2u + \sqrt{2})\, du$

$$= \frac{1}{2\sqrt{2}} \int \left(\frac{dv}{v} \right) + \frac{1}{2} \int \left(\frac{1}{(u - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \right) du - \frac{1}{2\sqrt{2}} \int \left(\frac{dw}{w} \right) + \frac{1}{2} \int \left(\frac{1}{(u + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \right) du$$

$$\text{Let } \hat{v} = u - \frac{\sqrt{2}}{2}, \quad d\hat{v} = du \qquad \text{and} \qquad \hat{w} = u + \frac{\sqrt{2}}{2}, \quad d\hat{w} = du$$

$$= \frac{1}{2\sqrt{2}} \ln(v) + \frac{1}{2} \int \left(\frac{d\hat{v}}{\hat{v}^2 + \frac{1}{2}} \right) - \frac{1}{2\sqrt{2}} \ln(w) + \frac{1}{2} \int \left(\frac{d\hat{w}}{\hat{w}^2 + \frac{1}{2}} \right) + C$$

$$= \frac{1}{2\sqrt{2}} \ln(v) - \frac{1}{2\sqrt{2}} \ln(w) + \frac{1}{2} \int 2 \left(\frac{d\hat{v}}{(\sqrt{2}\hat{v})^2 + 1} \right) + \frac{1}{2} \int 2 \left(\frac{d\hat{v}}{(\sqrt{2}\hat{w})^2 + 1} \right) + C$$

$$= \frac{1}{2\sqrt{2}} \ln(v) - \frac{1}{2\sqrt{2}} \ln(w) + \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\hat{v}) + \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\hat{w}) + C$$

The integral is almost finished! All that remains is to plug back in the dummy variables used to obtain our final answer.

$$\frac{1}{2\sqrt{2}} \ln(v) - \frac{1}{2\sqrt{2}} \ln(w) + \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\hat{v}) + \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\hat{w}) + C$$

$$= \frac{1}{2\sqrt{2}} \ln(u^2 - \sqrt{2}u + 1) - \frac{1}{2\sqrt{2}} \ln(u^2 + \sqrt{2}u + 1) +$$

$$\frac{\sqrt{2}}{2} \arctan \left(\sqrt{2} \left(u - \frac{\sqrt{2}}{2} \right) \right) + \frac{\sqrt{2}}{2} \arctan \left(\sqrt{2} \left(u + \frac{\sqrt{2}}{2} \right) \right) + C$$

$$= \frac{1}{2\sqrt{2}} \ln \left(\tan(x) - \sqrt{2\tan(x)} + 1 \right) - \frac{1}{2\sqrt{2}} \ln \left(\tan(x) + \sqrt{2\tan(x)} + 1 \right) +$$

$$\frac{\sqrt{2}}{2} \arctan \left(\sqrt{2\tan(x)} - 1 \right) + \frac{\sqrt{2}}{2} \arctan \left(\sqrt{2\tan(x)} + 1 \right) + C$$

$$= \boxed{\frac{\ln \left(\tan(x) - \sqrt{2\tan(x)} + 1 \right) - \ln \left(\tan(x) + \sqrt{2\tan(x)} + 1 \right) + 2\arctan \left(\sqrt{2\tan(x)} - 1 \right) + 2\arctan \left(\sqrt{2\tan(x)} + 1 \right)}{2\sqrt{2}} + C}$$