Calculus III Review

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July 5, 2020

1 Vectors

Vectors are elements of something called a *vector space*, which in essence are subsets of $\mathbb{R}^n$. A typical vector can be notated several ways:

$$\langle a, b, c\rangle \quad = \quad a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}} \tag{1}$$

are the two most typical examples of how they're notated. $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are called the *unit vectors* of $\mathbb{R}^3$. This is because every vector is just the unit vectors multiplied by some scalar quantity, hence the notation. In fact, the entirety of $\mathbb{R}^3$ can be expressed through adding and multiplying the unit vectors, something known as a *linear combination* of $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

The vector $\vec{PQ}$ representing the distance betweent two points $P, Q \in \mathbb{R}^3$ where $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is equal to

$$\langle q_1 - p_1,\, q_2 - p_2,\, q_3 - p_3\rangle, \tag{2}$$

where P is the base of the vector and Q is the tip of the vector. A vector in the form $\langle a, b, c\rangle$ is the same as the vector between $(0, 0, 0)$ and (a, b, c).

Where the vector is placed in $\mathbb{R}^3$ is irrelevant. Let $P = (1, 2, 3)$ and $Q = (4, 6, 5), (a, b, c) = (3, 4, 2)$. Then $\vec{PQ}$ is equivalent to $\langle a, b, c\rangle$, even though they lie in different "parts" of the plane.

The length or magnitude of a vector is defined by

$$|\vec{PQ}| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}. \tag{3}$$

This formula can be thought of as the distance between two far corners of a cube, where one of the edges of a right triangle is $\sqrt{a^2 + b^2}$ and the other is c. This intuition is helpful when we eventually talk about the arc length of curves.

A vector divided by its length will give a vector pointing in the same direction as the original vector, but having a length of one. This is called the *directional unit vector* of a vector $\vec{v}$, and is given by

$$\frac{\vec{v}}{|v|}. \tag{4}$$