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ECE300 Communication Theory
Problem Set II: Baseband-Bandpass and Probability

September 10, 2021

1. **Baseband-Bandpass Energy Relation**

   Let $x(t)$ be a bandpass signal with baseband equivalent $x_{BB}(t)$ corresponding to carrier frequency f_c. Recall the energy of a signal $s(t)$ is defined as:

   $$\mathcal{E}_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

   If $\mathcal{E}_{bp}$ is the energy of the bandpass signal $x(t)$ and $\mathcal{E}_{bb}$ is the energy of the baseband signal $x_{BB}(t)$, provide a *graphical argument* to show that:

   $$\mathcal{E}_{bp} = \frac{1}{2}\mathcal{E}_{bb}$$

   That is, use the conceptual "graphs" of the baseband and bandpass spectra to establish this relationship.

   Remark: You may recall that the power associated with a phasor V has a one-half factor associated with it, $\frac{1}{2}|V|^2$. This is essentially the same idea.

   Remark: For this reason, the "energy" of a baseband signal is sometimes defined with the 1/2 built in, i.e., something like $\frac{1}{2}\int_{-\infty}^{\infty} |s(t)|^2 \, dt$, but we won't do that. Also, suppose we have a phasor $3 + j4$ which we associate with $3\cos(2\pi f_c t) - 4\sin(2\pi f_c t)$. We may consider associating this with the complex number $3 + j4$, and then it seems the power of the signal should be $\frac{1}{2}(3^2 + 4^2)$. If we connect this to a constellation concept though, and $3 + j4$ is a point in the constellation, the energy of the symbol should be $3^2 + 4^2$. How do we reconcile these ideas? What happened to the 1/2? Recall that the constellation point $3 + j4$ can be viewed as the point $(3, 4)$ in $\mathbb{R}^2$, corresponds to a real signal of the form $3\phi_1(t) + 4\phi_2(t)$ where ϕ_1, ϕ_2 are orthonormal. The energy then is indeed $3^2 + 4^2$. The point is ϕ_1, ϕ_2 are not simply $\cos, \sin$. Recall ϕ_1, ϕ_2 are *scaled* so that $\int |\phi_i|^2 = 1$, so we don't have any nasty 1/2 factors and such flying around! Indeed, whether the underlying signals are viewed at baseband or bandpass, we interpret our constellation diagrams as being based on an underlying orthonormal basis and the energy of a vector $\vec{s}$ is ALWAYS $\|\vec{s}\|^2 = |s_1|^2 + \cdots + |s_N|^2$. No random 1/2 factors flying around!

2. **Chapman-Kolmogorov Equation for Markov Processes**

   A *discrete-time Markov process* is a sequence of random variables $X_1, X_2, X_3, \cdots$ such that for each $n \geq 1$:

   $$f(x_{n+1}|x_n, x_{n-1}, \cdots, x_1) = f(x_{n+1}|x_n)$$

   In words, given the present (x_n), the future (x_{n+1}) is independent of the past (x_k for $k < n$). Verify the *Chapman-Kolmogorov equation* that extends the so-called one-step transition probability distribution function $f(x_{n+1}|x_n)$ to two time steps:

   $$f(x_{n+2}|x_n) = \int_{x_{n+1}=-\infty}^{\infty} f(x_{n+2}|x_{n+1}) f(x_{n+1}|x_n) \, dx_{n+1} \tag{1}$$

This generalizes: for $k \geq 2$:

$$f(x_{n+k}|x_n) = \int_{x_{n+k-1}=-\infty}^{\infty} \cdots \int_{x_{n+1}=-\infty}^{\infty} f(x_{n+k}|x_{n+k-1}) \cdots f(x_{n+1}|x_n)\, dx_{n+k-1} \cdots dx_{n+1}$$

but *I'm only asking you to verify (1).* **Hint:** As a first step, show:

$$f(x_{n+2}|x_{n+1}, x_n) = f(x_{n+2}|x_{n+1})$$

3. **Gaussian Mixture Distribution**

   Let μ_i, σ_i, $1 \leq i \leq M$, be a fixed set of constants. Let Θ be a discrete random variable taking on values θ_i, $1 \leq i \leq M$, with respective probabilities π_i, $1 \leq i \leq M$. Assume $X|\Theta = \theta_i \sim N(\mu_i, \sigma_i^2)$. This means the conditional distribution of X given $\Theta = \theta_i$ is Gaussian with mean μ_i and variance σ_i^2.

   (a) Find the (unconditional) pdf of X, $f(x)$. The distribution of X is called a *Gaussian mixture.*

   (b) Is the conditional distribution $\Theta|X$ discrete or continuous? Write the pmf $p(\theta|x)$ or pdf $f(\theta|x)$, whichever is appropriate. This must be an EXPLICIT formula, though it does not have to be simplified.