Topology II

Dozent

Mitschrift

Version
git: (None)
kompiliert: 11. April 2022 10:19

Contents

A Projective modules **3**

A Projective modules

fix this

Definition 1. Let R be a ring. An R-module P is projective if one of the following equivalent conditions holds:

(i) The functor $\mathrm{Hom}_R(P, -)$ is exact.

(ii) For every surjection $\mu\colon M \twoheadrightarrow N$ and every morphism $p\colon P \to N$ lifts to a morphism $\hat{p}\colon P \to M$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\hat{p}"'] \arrow[dr, "p"] & \\ M \arrow[rr, "\mu"'] & & N \end{tikzcd}$$

(iii) Every short exact sequence

$$0 \to L \xrightarrow{\lambda} M \xrightarrow{\mu}\xrightarrow{P}\to 0$$

splits.

(iv) P is a direct summand of a free module, that is, there exists a module K and a free module F, such that $F \cong P \oplus K$.

Lemma 2. Let P_1 and P_2 be projective modules. Then, the following hold:

1) $P_1 \oplus P_2$ and $P_1 \otimes_R P_2$ are projective modules.

2) Every direct summand of P_1 is projective.

3) For every R-module M, there exists a projective resolution

$$\ldots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

Lemma 3. Let P be finitely generated R-module. Then P is projective if and only if P is a direct summand of R^n for some $n \in \mathbb{N}$.

Lemma 4. Let P be an R-module. Then, the following hold:

1) Let P be free. Then P is projective.

2) Let P be projective. Then P is flat.

3) Let P be flat. If R is noetherian and P is finitely generated, then P is projective.

Definition 5. A chain complex $C_\bullet$ is called **projective** if all C_n are projective modules.