Automatic Control

Solution of LTI dynamical systems

Solution of LTI continuous time systems: problem setup

Linear Time Invariant (LTI) state space representation

In this course we will be mainly concerned with the case of **linear-time-invariant** (LTI) systems:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

State equation

$A \in \mathbb{R}^{n \times n}$: state matrix
$B \in \mathbb{R}^{n \times p}$: input matrix

$$y(t) = C\,x(t) + D\,u(t)$$

Output equation

$C \in \mathbb{R}^{q \times n}$: output matrix
$D \in \mathbb{R}^{q \times p}$: input-output matrix

State response

The solution of the state equation is obtained through the Lagrange equation:

$$x(t) = e^{At}x(0) + \int_{0_-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau = x_{zi}(t) + x_{zs}(t)$$

This result is easily verified by direct substitution in the state equation.

The solution of LTI systems can be split into two independent contributions

$X_{zi}(t)$: **zero-input response**

$X_{zs}(t)$: **zero-state response**

Output response

The output response can be obtained from the output equation

$$y(t) = C\,x(t) + D\,u(t)$$

$$y(t) = Ce^{At}x(0) + C\int_{0_-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = y_{zi}(t) + y_{zs}(t)$$

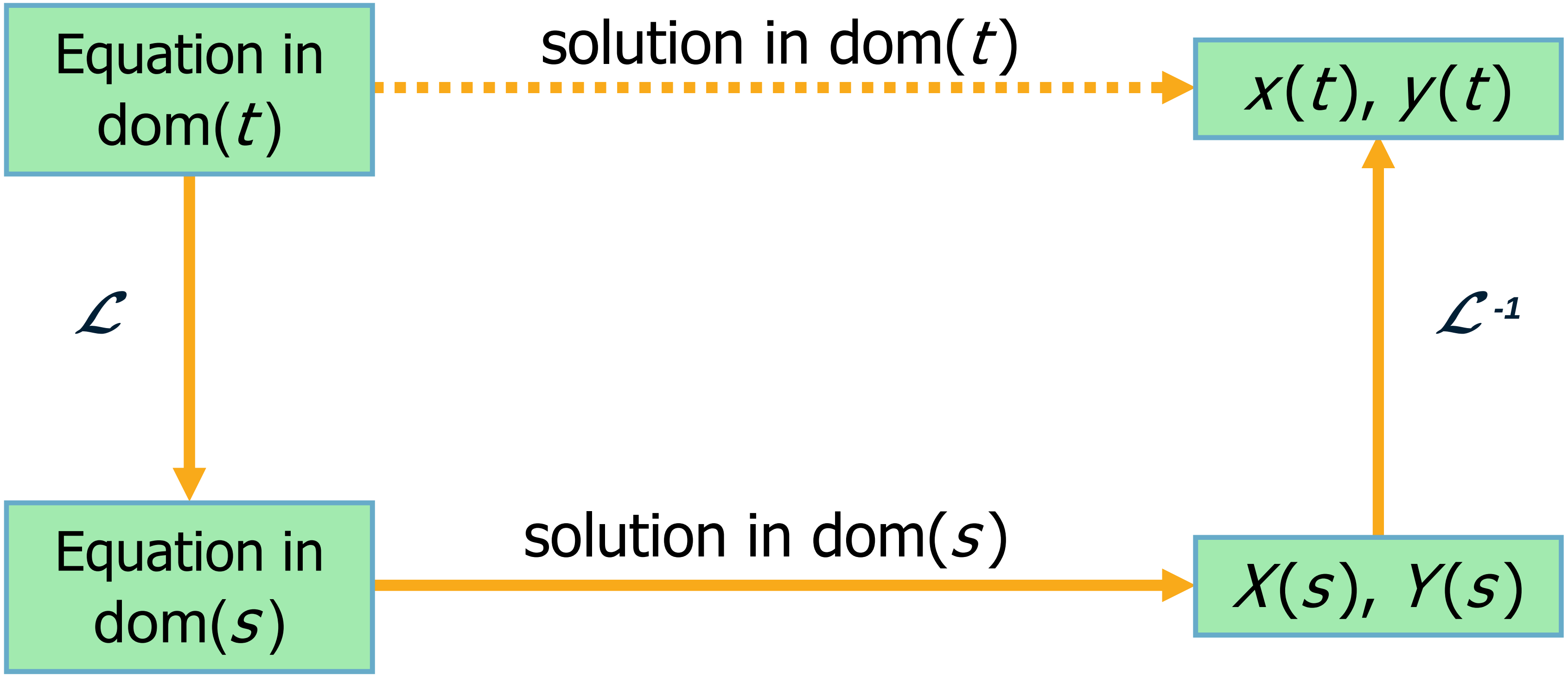
$y_{zi}(t)$: **zero-input output response**

$y_{zs}(t)$: **zero-state output response**

In order to simplify the computation of the time responses (i.e. avoid to solve differential equations) Laplace transform can be suitably exploited

Sketch of the solution procedure

Computation of $x(t)$ and $y(t)$ through the Laplace transform is obtained according to the following scheme:



Solution by Laplace trasform

In order to simplify the computation of the time responses (i.e. avoid to solve differential equations) Laplace transform can be suitably exploited.

Transformation of the state and the output equations and use of the real differentiation theorem give

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$\Big\downarrow \mathcal{L}$$

$$\begin{cases} sX(s) - x(0) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

Solution by Laplace trasform

The transformed state response is given by

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) = X_{zi}(s) + X_{zs}(s)$$

$$X_{zi}(s) = (sI - A)^{-1}x(0) = H_{x,zi}(s)x(0)$$

$$X_{zs}(s) = (sI - A)^{-1}BU(s) = H_x(s)U(s)$$

While the transformed output response is given by

$$Y(s) = C(sI - A)^{-1}x(0) + \left[C(sI - A)^{-1}B + D\right]U(s) = Y_{zi}(s) + Y_{zs}(s)$$

$$Y_{zi}(s) = C(sI - A)^{-1}x(0) = H_{zi}(s)x(0)$$

$$Y_{zs}(s) = \left[C(sI - A)^{-1}B + D\right]U(s) = H(s)U(s)$$

Some unilateral ($t \geq 0$) Laplace transform pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$\varepsilon(t)$	$\dfrac{1}{s}$
$\dfrac{t^n}{n!}$	$\dfrac{1}{s^{n+1}}$

$f(t)$	$F(s)$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{t^n e^{at}}{n!}$	$\dfrac{1}{(s-a)^{n+1}}$
$\sin(\omega_o t)$	$\dfrac{\omega_o}{s^2+\omega_0^2}$
$\cos(\omega_o t)$	$\dfrac{s}{s^2+\omega_0^2}$

$f(t)$	$F(s)$
e^{At}	$(sI-A)^{-1}$

Solution of LTI continuous time systems: explicit computation

The solution is obtained exploiting the following procedure:

1. Compute $X(s)$, $Y(s)$ in the Laplace variable s domain
2. Derive the Heaviside's partial fraction expansion (PFE) of $X(s)$ and $Y(s)$
3. Compute the coefficient (residues) of the PFE
4. Obtain $x(t)$ and $y(t)$ via Laplace inverse transform of $X(s)$ and $Y(s)$

Computation example 1

Consider the following LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

compute the state response $x(t)$ when $u(t) = 2\varepsilon(t)$ and $x(0) = [2 \;\; 2]^T$

The solution in Laplace domain can be computed as

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

Computation example 1

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, U(s) = \frac{2}{s}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} Adj(sI - A) = \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} =$$

$$= \frac{1}{\underbrace{s^2 + 3s + 2}_{\det(sI-A)}} \underbrace{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}_{Adj(sI-A)} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Computation example 1

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

$$X_{zi}(s) = (sI - A)^{-1} x(0) = \underbrace{\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}}_{(sI - A)^{-1}} \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{x(0)} = \begin{bmatrix} \dfrac{2s+8}{(s+1)(s+2)} \\ \dfrac{2s-4}{(s+1)(s+2)} \end{bmatrix}$$

$$X_{zs}(s) = (sI - A)^{-1} BU(s) = \underbrace{\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}}_{(sI - A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\frac{2}{s}}_{U(s)} = \begin{bmatrix} \dfrac{2(s+3)}{s(s+1)(s+2)} \\ \dfrac{-4}{s(s+1)(s+2)} \end{bmatrix}$$

Computation example 1

$$X(s)=\underbrace{(sI-A)^{-1}x(0)}_{X_{zi}(s)}+\underbrace{(sI-A)^{-1}BU(s)}_{X_{zs}(s)}$$

$$X(s)=X_{zi}(s)+X_{zs}(s)=\underbrace{\begin{bmatrix}\dfrac{2s+8}{(s+1)(s+2)}\\ \dfrac{2s-4}{(s+1)(s+2)}\end{bmatrix}}_{X_{zi}(s)}+\underbrace{\begin{bmatrix}\dfrac{2(s+3)}{s(s+1)(s+2)}\\ \dfrac{-4}{s(s+1)(s+2)}\end{bmatrix}}_{X_{zs}(s)}=\begin{bmatrix}\dfrac{2s^2+10s+6}{s(s+1)(s+2)}\\ \dfrac{2s^2-4s-4}{s(s+1)(s+2)}\end{bmatrix}$$

$$X(s)=\begin{bmatrix}X_1(s)\\ X_2(s)\end{bmatrix}=\begin{bmatrix}\dfrac{2s^2+10s+6}{s(s+1)(s+2)}\\ \dfrac{2s^2-4s-4}{s(s+1)(s+2)}\end{bmatrix}=\begin{bmatrix}\dfrac{R_1^{(1)}}{s}+\dfrac{R_2^{(1)}}{s+1}+\dfrac{R_3^{(1)}}{s+2}\\ \dfrac{R_1^{(2)}}{s}+\dfrac{R_2^{(2)}}{s+1}+\dfrac{R_3^{(2)}}{s+2}\end{bmatrix}=\begin{bmatrix}\dfrac{3}{s}+\dfrac{2}{s+1}-\dfrac{3}{s+2}\\ -\dfrac{2}{s}-\dfrac{2}{s+1}+\dfrac{6}{s+2}\end{bmatrix}$$

Computation example 1

$$X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{3}{s} + \dfrac{2}{s+1} - \dfrac{3}{s+2} \\ -\dfrac{2}{s} - \dfrac{2}{s+1} + \dfrac{6}{s+2} \end{bmatrix}$$

$$Re^{at}\varepsilon(t) = \mathcal{L}^{-1}\left\{\frac{R}{s-a}\right\}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 + 2e^{-t} - 3e^{-2t} \\ -2 - 2e^{-t} + 6e^{-2t} \end{bmatrix}\varepsilon(t)$$

Computation example 1: MatLab procedure

- Define the Laplace variabile **s** using **tf** statement

```
>> s=tf('s')
Transfer function:
s
```

- Define the system input and initial condition

```
>> U=2/s, x0=[2;2]
Transfer function:
2
-
s
x0 =
     2
     2
```

Computation example 1: MatLab procedure

- Introduce the system matrices A and B

```
>> A=[0 1;-2 -3], B=[1;0]

A =

     0     1

    -2    -3

B =

     1

     0
```

Computation example 1: MatLab procedure

- Compute $X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s) = (sI - A)^{-1} [x(0) + BU(s)]$

  use statements **`minreal`** and **`zpk`**, in order to simplify and highlights denominator roots respectively

```
>> X=zpk(minreal(inv(s*eye(2)-A)*(B*U+x0)))

 Zero/pole/gain from input to output...

       2 (s+4.303) (s+0.6972)
 #1:   ----------------------
           s (s+2) (s+1)


       2 (s-2.732) (s+0.7321)
 #2:   ----------------------
           s (s+2) (s+1)
```

Computation example 1: MatLab procedure

- For each of the two components of X(s) , compute the PFE using the statements **tfdata** and **residue**

```
>> [num_X1,den_X1]=tfdata(X(1),'v')

num_X1 =

         0    2.0000   10.0000    6.0000

den_X1 =

    1.0000    3.0000    2.0000         0

>> [r1,p1]=residue(num_X1,den_X1)

r1 =

   -3.0000

    2.0000

    3.0000

p1 =

   -2.0000

   -1.0000

         0
```

$$\rightarrow X_1(s) = \frac{3}{s} + \frac{2}{s+1} - \frac{3}{s+2} \qquad \rightarrow x_1(t) = \left(3 + 2e^{-t} - 3e^{-2t}\right)\varepsilon(t)$$

Computation example 1: MatLab procedure

- For each of the two components of X(s) , compute the PFE using the statements **`tfdata`** and **`residue`**

```
>> [num_X2,den_X2]=tfdata(X(2),'v')

num_X2 =

         0    2.0000   -4.0000   -4.0000

den_X2 =

    1.0000    3.0000    2.0000         0

>> [r2,p2]=residue(num_X2,den_X2)

r2 =

    6.0000

   -2.0000

   -2.0000

p2 =

   -2.0000

   -1.0000

         0
```

$$\rightarrow X_2(s) = -\frac{2}{s} - \frac{2}{s+1} + \frac{6}{s+2} \qquad \rightarrow x_2(t) = \left(-2 - 2e^{-t} + 6e^{-2t}\right)\varepsilon(t)$$

Computation example 2

Consider the following LTI system:

$$\dot{x}(t) = \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

compute the output response $y(t)$ when $u(t) = \varepsilon(t)$ and $x(0) = [1 \ \ 1]^T$

The solution in Laplace domain can be computed as

$$Y(s) = \underbrace{C(sI - A)^{-1} x(0)}_{Y_{zi}(s)} + \underbrace{\left[C(sI - A)^{-1} B + D\right] U(s)}_{Y_{zs}(s)}$$

Computation example 2

$$Y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{Y_{zi}(s)} + \underbrace{\left[C(sI-A)^{-1}B+D\right]U(s)}_{Y_{zs}(s)}$$

$$A=\begin{bmatrix}-3 & 2\\ -2 & -3\end{bmatrix}, B=\begin{bmatrix}1\\0\end{bmatrix}, C=\begin{bmatrix}0 & 1\end{bmatrix}, D=[0], x(0)=\begin{bmatrix}1\\1\end{bmatrix}, U(s)=\frac{1}{s}$$

$$(sI-A)^{-1}=\begin{bmatrix}s+3 & -2\\ 2 & s+3\end{bmatrix}^{-1}=\frac{1}{\underbrace{s^2+6s+13}_{\det(sI-A)}}\underbrace{\begin{bmatrix}s+3 & 2\\ -2 & s+3\end{bmatrix}}_{Adj(sI-A)}=$$

$$=\begin{bmatrix}\dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13}\\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13}\end{bmatrix}$$

Computation example 2

$$Y(s) = \underbrace{C\left(sI - A\right)^{-1} x(0)}_{Y_{zi}(s)} + \underbrace{\left[C\left(sI - A\right)^{-1} B + D\right] U(s)}_{Y_{zs}(s)}$$

$$Y_{zi}(s) = C(sI - A)^{-1} x(0) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13} \\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x(0)} = \frac{s+1}{s^2+6s+13}$$

$$Y_{zs}(s) = \left[C(sI - A)^{-1} B + D\right] U(s) \underset{\substack{\uparrow \\ D=0}}{=} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13} \\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\frac{1}{s}}_{U(s)} = \frac{-2}{s^3+6s^2+13s}$$

$$Y(s) = Y_{zi}(s) + Y_{zs}(s) = \frac{s^2+s-2}{s^3+6s^2+13s} = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)}$$

Computation example 2

$$Y(s) = Y_{zi}(s) + Y_{zs}(s) = \frac{s^2+s-2}{s^3+6s^2+13s} = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)}$$

Note that in the denominator of Y(s) there is a factor of the form

$(s - \sigma_0 - j\omega_0)\ (s - \sigma_0 + j\omega_0)$ leading to two complex conjugate roots with $\sigma_0 = -3$ and $\omega_0 = 2$

The PFE is:

$$Y(s) = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)} = \frac{R_1}{s+3-2j} + \frac{R_1^*}{s+3+2j} + \frac{R_2}{s} =$$

$$= \frac{0.57..+0.38..j}{s+3-2j} + \frac{0.57..-0.38..j}{s+3+2j} - \frac{0.15..}{s}$$

Please also note that R_1 is the residue associated with the complex root having positive imaginary part (this is important for the inverse transform procedure)

Computation example 2

A reminder on the inverse Laplace transform in the presence of complex conjugate roots.

The inverse Laplace transform of the following PFE (considered as a whole):

$$\frac{R}{s-\sigma_0-j\omega_0}+\frac{R^*}{s-\sigma_0+j\omega_0}$$

is given by:

$$2|R|e^{\sigma_0 t}\cos(\omega_0 t+\arg(R))\varepsilon(t)$$

$$|R|=\sqrt{\mathrm{Re}^2(R)+\mathrm{Im}^2(R)},\arg(R)=\arctan\frac{\mathrm{Im}(R)}{\mathrm{Re}(R)}$$

Computation example 2

$$Y(s)=\frac{R_1}{s+3-2j}+\frac{R_1^*}{s+3+2j}+\frac{R_2}{s}=\frac{0.57..+0.38..j}{s+3-2j}+\frac{0.57..-0.38..j}{s+3+2j}-\frac{0.15..}{s}$$

$$\sigma_0=-3,\omega_0=2$$

$$R_1=0.57..+0.38..j$$

$$|R_1|=\sqrt{(0.57..)^2+(0.38..)^2}=0.69..$$

$$\arg(R_1)=\arctan\left(\frac{0.38..}{0.57..}\right)=0.58..\text{rad}$$

$$R_2=0.15..$$

$$y(t)=\left(2|R_1|e^{\sigma_0 t}\cos(\omega_0 t+\arg(R_1))+R_2\right)\varepsilon(t)$$

$$y(t)=\left(\underbrace{1.38}_{2|R_1|}e^{\overset{\sigma_0}{\overset{\downarrow}{-3}}\,t}\cos(\overset{\omega_0}{\overset{\downarrow}{2}}\,t+\underbrace{0.58}_{\arg(R_1)})\underbrace{-0.15}_{R_2}\right)\varepsilon(t)$$

Computation example 2: MatLab procedure

- Define the Laplace variabile **s** using **tf** statement

```
>> s=tf('s')
Transfer function:
s
```

- Define the system input and initial condition

```
>> U=1/s, x0=[1;1]
Transfer function:
1
-
s
x0 =
     1
     1
```

Computation example 2: MatLab procedure

- Introduce the system matrices A, B and C

```
>> A=[-3 2;-2 -3], B=[1;0], C=[0 1]
A =
    -3     2
    -2    -3
B =
     1
     0
C =
     0     1
```

Computation example 2: MatLab procedure

- Compute $Y(s) = C(sI - A)^{-1} x(0) + C(sI - A)^{-1} BU(s) = C(sI - A)^{-1}\left[x(0) + BU(s)\right]$

  use statements **`minreal`** and **`zpk`**, in order to simplify and highlights denominator roots respectively

```
>> Y=zpk(minreal(C*inv(s*eye(2)-A)*(B*U+x0)))

Zero/pole/gain:
   (s+2) (s-1)
------------------
s (s^2  + 6s + 13)
```

Computation example 2: MatLab procedure

- For Y(s) , compute the PFE using the statements **tfdata** and **residue**

```
>> [num_Y,den_Y]=tfdata(Y,'v')
num_Y =
         0    1.0000    1.0000   -2.0000
den_Y =
    1.0000    6.0000   13.0000
>> [r,p]=residue(num_Y, den_Y)
r =
   0.5769 + 0.3846i
   0.5769 - 0.3846i
  -0.1538
p =
  -3.0000 + 2.0000i
  -3.0000 - 2.0000i
        0
```

$$\rightarrow Y(s) = \frac{0.57.. + 0.38..j}{s+3-2j} + \frac{0.57.. - 0.38..j}{s+3+2j} - \frac{0.15..}{s}$$

Computation example 2: MatLab procedure

- Compute magnitude and phase of the residue corresponding to the roots with positive imaginary part

```
>> M=abs(r(1)), 2*M

M =

    0.6934

ans =

    1.3868

>> phi=angle(r(1))

phi =

    0.5880
```

$$\rightarrow y(t) = \left(1.38e^{-3t}\cos(2t + 0.58) - 0.15\right)\varepsilon(t)$$