Automatic Control

Equilibrium solution of dynamical systems

Linearization of nonlinear dynamical systems

Equilibrium solution of dynamical systems

Equilibrium solution of dynamical systems

A constant state $\bar{x}$ is an **equilibrium solution (or equilibrium state)** of the nonlinear dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = g(x(t), u(t))$$

if, in the presence of the <u>constant</u> input $u(t) = \bar{u}$ and the initial condition $x(0) = \bar{x}$, it results

$$x(t) = \bar{x} \quad \forall t \geq 0$$

- the input $\bar{u}$ is said **equilibrium input**
- the couple $(\bar{x}, \bar{u})$ is said **equilibrium point**
- the output $\bar{y} = g(\bar{x}, \bar{u})$ is said **equilibrium output**

Equilibrium condition

An equilibrium point $(\bar{x},\bar{u})$ of the dynamical system

$$\dot{x}(t) = f(x(t),u(t))$$

satisfies the equilibrium condition:

$$f(\bar{x},\bar{u}) = 0$$

In fact:

$$x(t) = \bar{x}, \forall t \geq 0 \Rightarrow \dot{x}(t) = \dot{\bar{x}} = f(\bar{x},\bar{u}) = 0, \forall t \geq 0$$

Equilibrium condition

For LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

the equilibrium condition

$$f(\bar{x}, \bar{u}) = 0$$

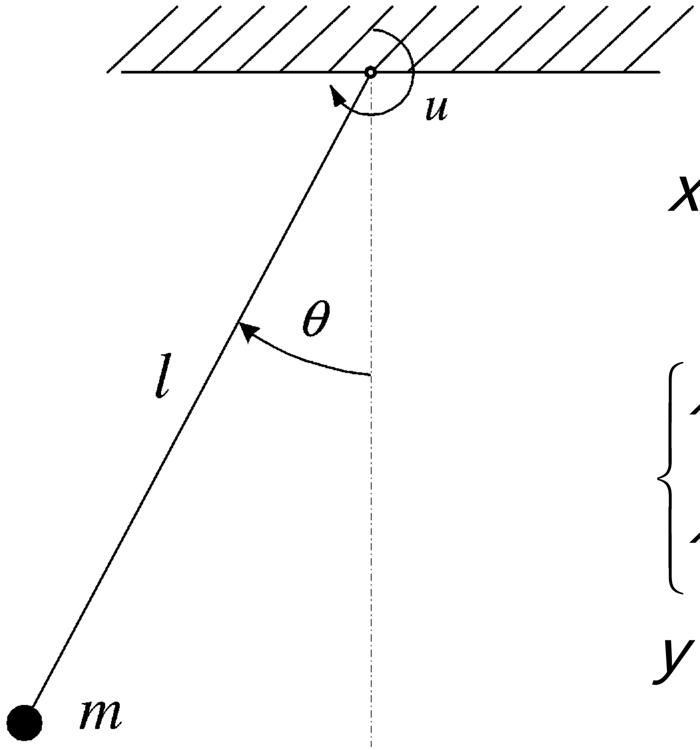
becomes

$$A\bar{x} + B\bar{u} = 0$$

$$B\bar{u} \in \mathrm{Im}(A\bar{x})$$

$$\Rightarrow \bar{x} \underset{\substack{\uparrow \\ \text{if } \det(A) \neq 0}}{=} -A^{-1}B\bar{u}$$

Example: pendulum



$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{g}{l}\sin(x_1(t)) - \dfrac{\beta}{ml^2} x_2(t) + \dfrac{1}{ml^2} u(t) \end{cases}$$

$$y(t) = x_1(t)$$

Example: pendulum

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{g}{l}\sin(x_1(t)) - \dfrac{\beta}{ml^2}x_2(t) + \dfrac{1}{ml^2}u(t) \end{cases}$$

Find all the equilibrium states with respect to the equilibrium input $\bar{u} = 0$

By using the equilibrium condition $\dot{x}(t) = \dot{\bar{x}} = 0 = f(\bar{x}, \bar{u})$, we get

$$\begin{cases} 0 = \bar{x}_2 \\ 0 = -\dfrac{g}{l}\sin\bar{x}_1 - \dfrac{\beta\bar{x}_2}{Ml^2} + \dfrac{\bar{u}}{Ml^2} = -\dfrac{g}{l}\sin\bar{x}_1 \end{cases} \Rightarrow \begin{cases} \sin\bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases}$$

Example: pendulum

- Since at the equilibrium the following relations hold

$$\begin{cases} \sin \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} \bar{x}_1 = k\pi, \quad k = 0, \pm 1, \dots \\ \bar{x}_2 = 0 \end{cases}$$

  there exist infinite equilibrium points of the form:

$$x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \quad k = 0, \pm 1, \dots$$

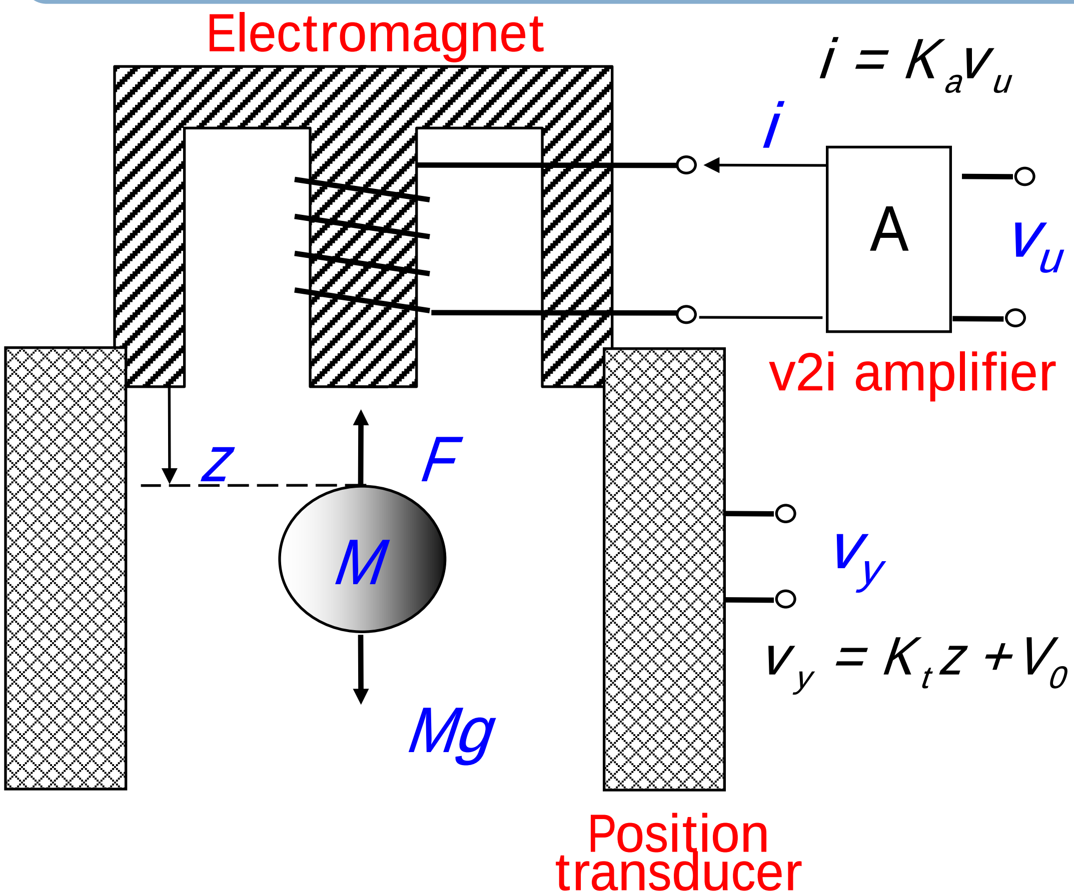
- At the equilibrium,

$$y(t) = \bar{y} = g(\bar{x}, \bar{u}), \forall t \geq 0$$

$$\bar{y} = \bar{x}_1 = k\pi, k = 0, \pm 1, \dots$$

Example: magnetic levitator

Example: magnetic levitator



$z \rightarrow$ ball position

$v_y \rightarrow$ position transducer voltage (ouput)

$v_u \rightarrow$ amplifier input voltage (input)

$$\underbrace{M\ \ddot{z} = Mg - F}_{\text{2nd Newton's law}}$$

$$\underbrace{F = \frac{B_m i^2}{z^2}}_{\text{electromagnetic force}}$$

State space representation

$$M\ddot{z} = Mg - F = Mg - \frac{B_m i^2}{z^2} =$$

$$= Mg - \frac{B_m (K_a v_u)^2}{z^2}$$

$$\ddot{z} = g - \frac{K_1 v_u^2}{z^2}, K_1 = \frac{B_m K_a^2}{M}$$

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$u(t) = v_u(t)$$

$$y(t) = v_y(t) = K_t z(t) + V_0$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g - K_1 \dfrac{u^2}{x_1^2} \end{cases}$$

$$y = K_2 x_1 + K_3, K_2 = K_t, K_3 = V_0$$

Example: magnetic levitator

Compute all the equilibrium points with respect to the input $\bar{u} \neq 0$

By using the equilibrium condition,

$$\dot{x}(t) = \dot{\bar{x}} = 0 = f(\bar{x}, \bar{u})$$

$$\begin{cases} 0 = \bar{x}_2 \\ 0 = g - (K_1)\bar{u}^2 / \bar{x}_1^2 \end{cases} \Rightarrow \begin{cases} \bar{x}_1 = \sqrt{\dfrac{K_1}{g}} |\bar{u}| \\ \bar{x}_2 = 0 \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} \sqrt{\frac{K_1}{g}} |\bar{u}| \\ 0 \end{bmatrix}$$

$$\bar{y} = K_2 \bar{x}_1 + K_3 = K_2 \sqrt{\tfrac{K_1}{g}} |\bar{u}| + K_3$$

Equilibrium and nonlinear electronic devices analysis

Motivating example: nonlinear electronic devices

In order to establish proper operating conditions in electronic components, biasing networks are usually used.

Many electronic devices that process time-varying signals (AC), also require steady-state current or steady-state voltage (DC) to operate correctly.

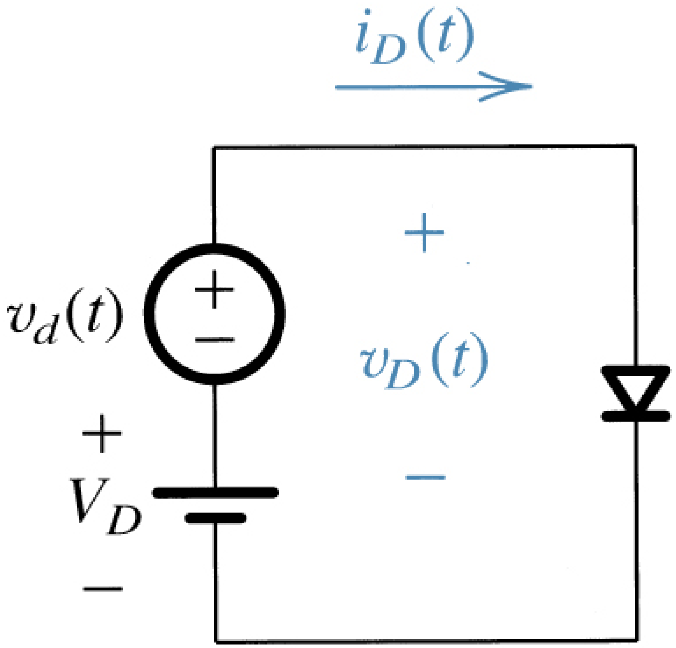
The AC signal is superimposed on DC bias current or voltage.

The operating point of a device is also known as bias point, quiescent point, or Q-point.

The <u>operating point</u> (OP) of a device is the steady-state voltage or steady-state current at a specified terminal of an active device with no input AC signal applied.

OP <u>can be seen as an equilibrium point</u> of the device

The operating point (also known as bias point), is determined by the steady-state voltage V_D .



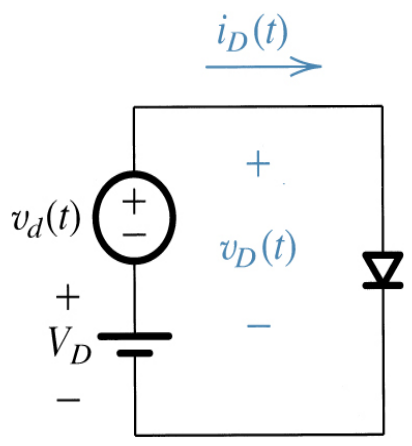
$$i_D(t) = I_s e^{v_D(t)/(\eta V_T)}$$

The AC signal superimposed on DC bias voltage is the time-varying voltage $v_d(t)$.

Example: diode model - small signal approximation

We may be interested in looking for a model in terms of a resistor whose value is the reciprocal of the slope of the "*i-v*" curve.

We can do this by means of a linearization around bias point V_D, where the time varying signal $v_d(t)$ is superimposed.



$i_D(t) = I_s e^{v_D(t)/(\eta V_T)}$

Current I_D due to V_D is, approximatively, given by:

$$i_D(t) = I_s e^{v_D(t)/(\eta V_T)} \to I_D = I_s e^{V_D/(\eta V_T)}$$

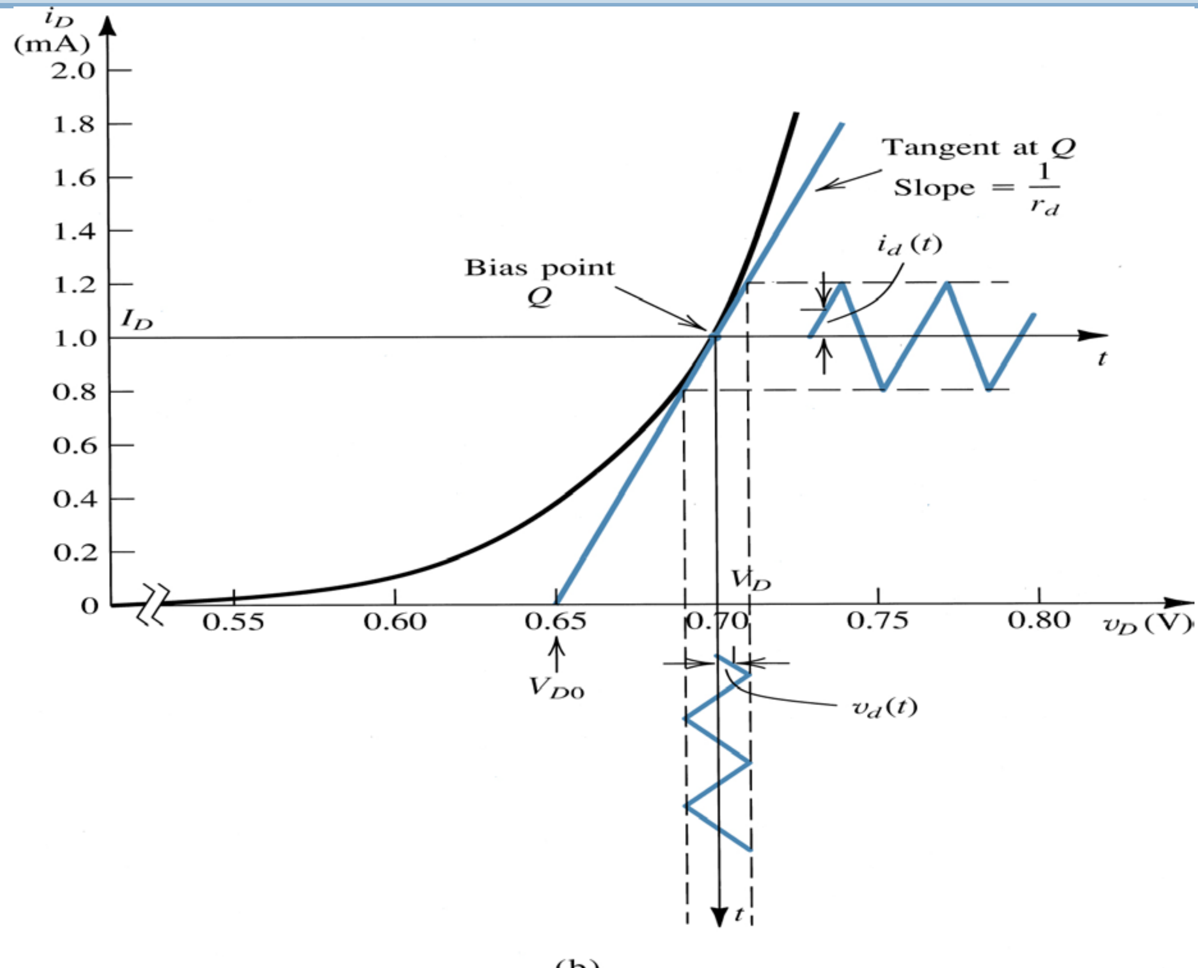
$$v_D(t) = V_D + v_d(t) \Rightarrow i_D(t) = I_s e^{(V_D + v_d(t))/(\eta V_T)} =$$

$$= I_s e^{V_d/(\eta V_T)} \cdot e^{v_d(t)/(\eta V_T)} = I_D \cdot e^{v_d(t)/(\eta V_T)}$$

$$\frac{v_d(t)}{\eta V_T} << 1 \to e^{v_D(t)/(\eta V_T)} \underset{\uparrow}{\approx} 1 + \frac{v_d(t)}{\eta V_T} \to i_D(t) \approx I_D \cdot \left(1 + \frac{v_d(t)}{\eta V_T}\right)$$

Taylor polynomial degree 1

Example: diode model - small signal approximation

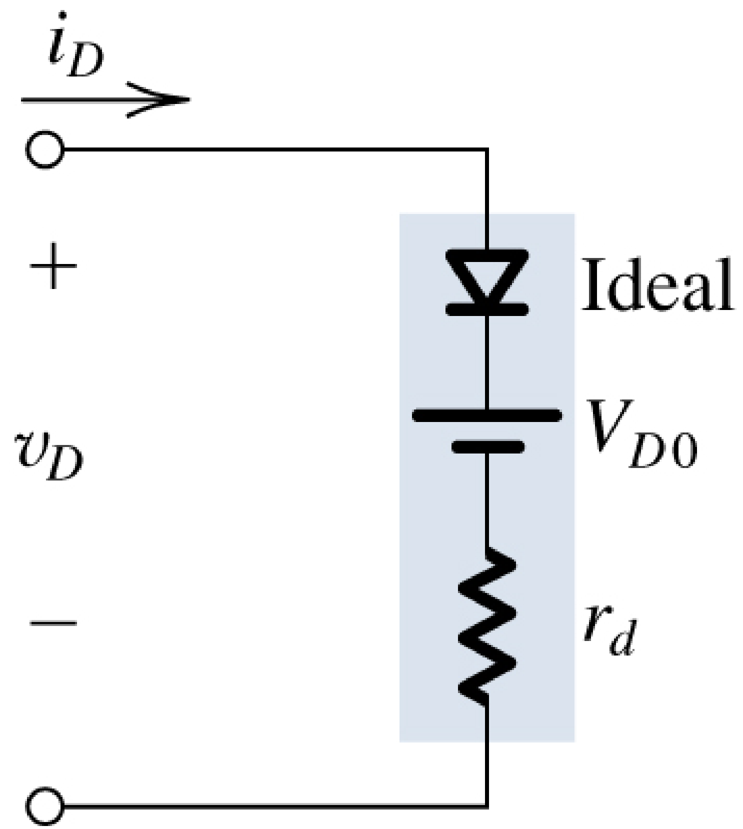


If the AC signal $v_d(t)$ is "small", the diode current $i_D(t)$ can be approximated as the sum of the DC bias signal and the superimposed AC signal

Example: diode model - small signal approximation

For small changes around the bias point *Q*, the equivalent diode model for small variation is given by:

$$r_d = 1 / \left[\frac{\partial i_D}{\partial v_D} \right]_{i_D = I_D}$$



Where V_{D0} is the intercept of the tangent on the v_D axis

$$i_D(t) = \frac{1}{r_d}\left(v_D(t) - v_{D0}\right)$$

Linearization of dynamical nonlinear systems

Linearization of a nonlinear system

Real world systems are, in general, nonlinear

Anyway, their behavior can be approximated in the neighbourhood of a given solution (e.g. an equilibrium solution) through suitable linear models known as **linearized models**

Objective: compute a linear dynamical model able to accurately approximate the behavior of a nonlinear system in the neighbourhood of an equilibrium solution

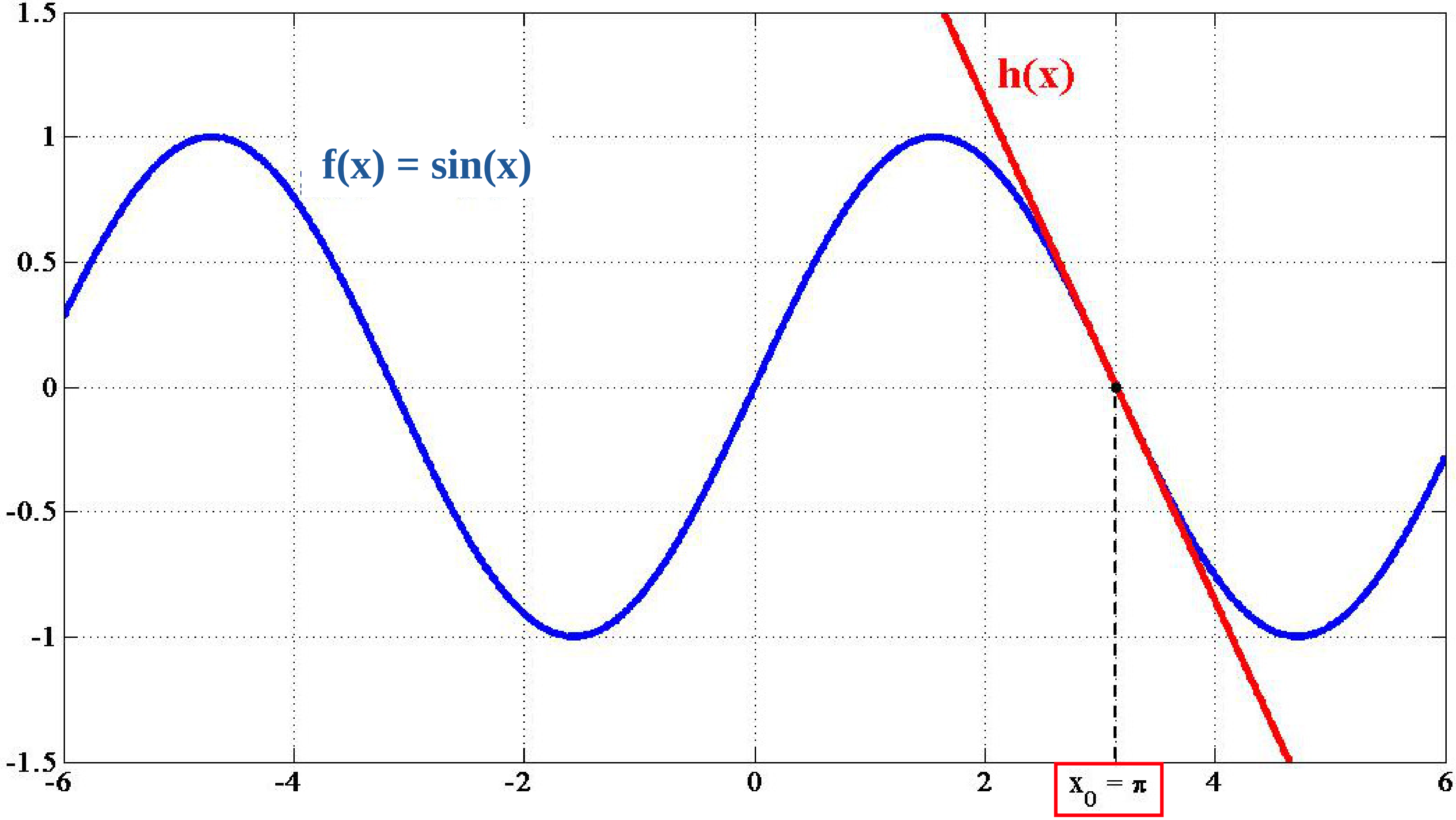
Idea: use a similar procedure as done in scalar function approximation by means of Taylor series expansions

A function $f(x): \mathbb{R} \to \mathbb{R}$ can be approximated in a small neighborhood of $x_0 \in \mathrm{dom}(f(x))$ with radius δx, by a Taylor series defined as:

$$f(x) = f(x_0 + \delta x) =$$
$$= f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} \delta x + \frac{1}{2!}\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0} \delta x^2 + \ldots$$

The linear approximation of $f(x): \mathbb{R} \to \mathbb{R}$ in a small neighborhood of x_0 is the first order truncation of the Taylor series expansion

$$f(x) = f(x_0 + \delta x) \cong f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} \delta x = h(x)$$



The smaller is the radius of the neighborhood of x_0 , the better is the approximation accuracy provided by the linear function $h(x)$

Linearization of a nonlinear system: motivation

Problem: given the nonlinear dynamical system:

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$y(t) = g\big(x(t), u(t)\big)$$

compute the solution in the presence of an input of the form:

$$u(t) = \bar{u} + \delta u(t), \ \|\delta u(t)\| \ll \|\bar{u}\|$$

and of the following initial condition

$$x(0) = \bar{x} + \delta x(0), \ \|\delta x(0)\| \ll \|\bar{x}\|$$

where $\left(\bar{x}, \bar{u}\right)$ is a known equilibrium point of the considered system

Linearization of a nonlinear system: motivation

In order to compute the required solution, the following Cauchy problem has to be solved:

$$\dot{x}(t) = f\big(x(t), u(t)\big), u(t) = \bar{u} + \delta u(t), x(0) = \bar{x} + \delta x(0)$$

Indeed, the solution of a nonlinear Cauchy problem(*) is, in general, a hard task (the superposition principle does not hold)

A viable approach is to compute an approximation $\tilde{x}(t)$ of the solution $x(t)$ as:

$$\tilde{x}(t) = \bar{x} + \delta x(t) \approx x(t)$$

(*) The considered Cauchy problem admits a unique solution if $f(x,u)$ is Lipschitz continuous in both x and u.

Linearization of a nonlinear system: preliminaries

$$\tilde{x}(t) = \bar{x} + \delta x(t) \approx x(t)$$

In the solution we are looking for, the term $\delta x(t)$ is obtained solving a suitable LTI system of the form

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t)$$

in the presence of the input $\delta u(t)$ and of the initial condition $\delta x(0)$

Such a linear system will be derived through a linearization procedure applied at the nonlinear state description:

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$
$$y(t) = g(x(t), u(t))$$

in the neighbourhood of the equilibrium point $\left(\bar{x}, \bar{u}\right)$

→ **linearized system**

Definition of system perturbations

In order to compute the **linearized system**, we need to define the concept of perturbation of a system wrt an equilibrium solution

$$\delta x(t) = x(t) - \bar{x} = \textbf{state perturbation} \in \mathbb{R}^n$$
$$\Rightarrow \quad x(t) = \bar{x} + \delta x(t)$$

$$\delta u(t) = u(t) - \bar{u} = \textbf{input perturbation} \in \mathbb{R}^p$$
$$\Rightarrow \quad u(t) = \bar{u} + \delta u(t)$$

$$\delta y(t) = y(t) - \bar{y} = \textbf{output perturbation} \in \mathbb{R}^q$$
$$\Rightarrow \quad y(t) = \bar{y}(t) + \delta y(t)$$

Next step: formally derive the differential equation which describes the dynamic behavior of the state perturbation $\delta x(t)$

Computing the system perturbation

The dynamic behavior of the state perturbation $\delta x(t)$ is governed by the differential equation:

$$\delta\dot{x}(t) = \dot{x}(t) - \dot{\bar{x}} = f(x(t),u(t)) - 0 = f(x(t),u(t))$$

Supposing that the vector valued function $f(x(t),u(t))$ can be expanded in Taylor series in a neighbourhood of $(\bar{x},\bar{u})^{(*)}$, we have:

$$f(x(t),u(t)) = f(\bar{x}+\delta x(t),\bar{u}+\delta u(t)) =$$

$$= f(\bar{x},\bar{u}) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}}(x(t)-\bar{x}) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}}(u(t)-\bar{u}) + \ldots =$$

$$\underset{\substack{\uparrow\\ f(\bar{x},\bar{u})=0}}{=} \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}}\delta x(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}}\delta u(t) + \ldots$$

(*) This is true if $f(x,u)$ is Lipschitz continuous in both x and u.

Computing the system perturbation

... $f(x(t),u(t))$ can be then approximated by its Taylor expansion truncated at the linear term:

$$f\left(x(t),u(t)\right) \cong \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\tilde{x}\\u=\tilde{u}}} \delta x(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\tilde{x}\\u=\tilde{u}}} \delta u(t)$$

Therefore, the approximate behavior of the state perturbation $\delta x(t)$ can be computed through the solution to the following linear differential equation

$$\delta \dot{x}(t) \cong \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta x(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta u(t) =$$

$$= A\delta x(t) + B\,\delta u(t)$$

Computing the system perturbation

Note that:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} = \left.\begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{bmatrix}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \in \mathbb{R}^{n\times n}$$

: Jacobian of f wrt x

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} = \left.\begin{bmatrix} \partial f_1/\partial u_1 & \dots & \partial f_1/\partial u_p \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial u_1 & \cdots & \partial f_n/\partial u_p \end{bmatrix}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \in \mathbb{R}^{n\times p}$$

: Jacobian of f wrt u

Computing the system perturbation

A similar procedure can be applied for the computation of $\delta y(t)$:

$$\delta y(t) = y(t) - \bar{y} = g(x(t),u(t)) - g(\bar{x},\bar{u}) \cong$$
$$\cong \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta x(t) + \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta u(t) =$$
$$= C\,\delta x(t) + D\delta u(t)$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \left.\begin{bmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial g_q/\partial x_1 & \cdots & \partial g_q/\partial x_n \end{bmatrix}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \in \mathbb{R}^{q\times n}$$: Jacobian of g wrt x

$$D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \left.\begin{bmatrix} \partial g_1/\partial u_1 & \cdots & \partial g_1/\partial u_p \\ \vdots & \ddots & \vdots \\ \partial g_q/\partial u_1 & \cdots & \partial g_q/\partial u_p \end{bmatrix}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \in \mathbb{R}^{q\times p}$$: Jacobian of g wrt u

Linearized system

The linearized system is then given by:

$$\delta\dot{x}(t) = A\,\delta x(t) + B\,\delta u(t), \qquad \delta x(0) = x(0) - \bar{x}$$
$$\delta y(t) = C\,\delta x(t) + D\,\delta u(t), \qquad \delta u(t) = u(t) - \bar{x}$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\; B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\; C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\; D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

Linearization

$$\begin{cases} \dot{x}_1 = x_2 & = f_1(x,u) \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{\beta x_2}{Ml^2} + \dfrac{u}{Ml^2} & = f_2(x,u) \end{cases}$$

$$y = x_1 \quad = g(x,u)$$

$$\left(\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \bar{u} = 0\right)$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos\bar{x}_1 & -\dfrac{\beta}{Ml^2} \end{bmatrix}$$

$$k \text{ even} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & -\dfrac{\beta}{Ml^2} \end{bmatrix}, \; k \text{ odd} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{l} & -\dfrac{\beta}{Ml^2} \end{bmatrix}$$

Linearization

$$\begin{cases} \dot{x}_1 = x_2 & = f_1(x,u) \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{\beta x_2}{Ml^2} + \dfrac{u}{Ml^2} & = f_2(x,u) \end{cases}$$
$$y = x_1 \qquad = g(x,u)$$

$$\left(\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \bar{u} = 0\right)$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 \\ \dfrac{1}{Ml^2} \end{bmatrix}$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial g}{\partial u} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 \end{bmatrix}$$

Linearization

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x,u) \\ \dot{x}_2 = g - K_1 \dfrac{u^2}{x_1^2} = f_2(x,u) \end{cases}$$

$$y = K_2 x_1 + K_3 = g(x,u)$$

$$\left(\bar{x} = \begin{bmatrix} \sqrt{\frac{K_1}{g}} |\bar{u}| \\ 0 \end{bmatrix}, \bar{u} \neq 0 \right)$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 & 1 \\ \frac{2K_1\bar{u}^2}{\bar{x}_1^3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{|\bar{u}|}\sqrt{\frac{g}{K_1}} & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 \\ -\frac{2K_1|\bar{u}|}{\bar{x}_1^2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2g}{|\bar{u}|} \end{bmatrix}$$

Example: magnetic levitator

Linearization

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x,u) \\ \dot{x}_2 = g - K_1 \dfrac{u^2}{x_1^2} = f_2(x,u) \end{cases}$$

$$y = K_2 x_1 + K_3 = g(x,u)$$

$$\left(\bar{x} = \begin{bmatrix} \sqrt{\dfrac{K_1}{g}} |\bar{u}| \\ 0 \end{bmatrix}, \bar{u} \neq 0 \right)$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} K_2 & 0 \end{bmatrix}$$

$$D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial g}{\partial u} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 \end{bmatrix}$$

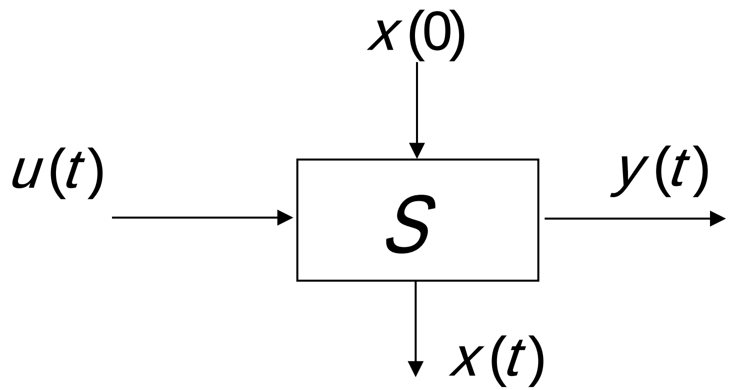
Resume via block diagram representation

Nonlinear system:

$$S:\begin{cases}\dot{x}(t)=f\big(x(t),u(t)\big)\\ y(t)=g\big(x(t),u(t)\big)\end{cases}$$

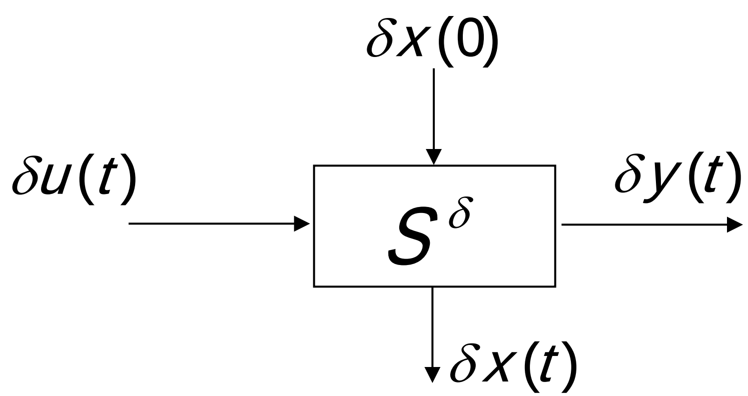
$$u(t)=\bar{u}+\delta u(t)$$

$$x(0)=\bar{x}+\delta x(0)$$

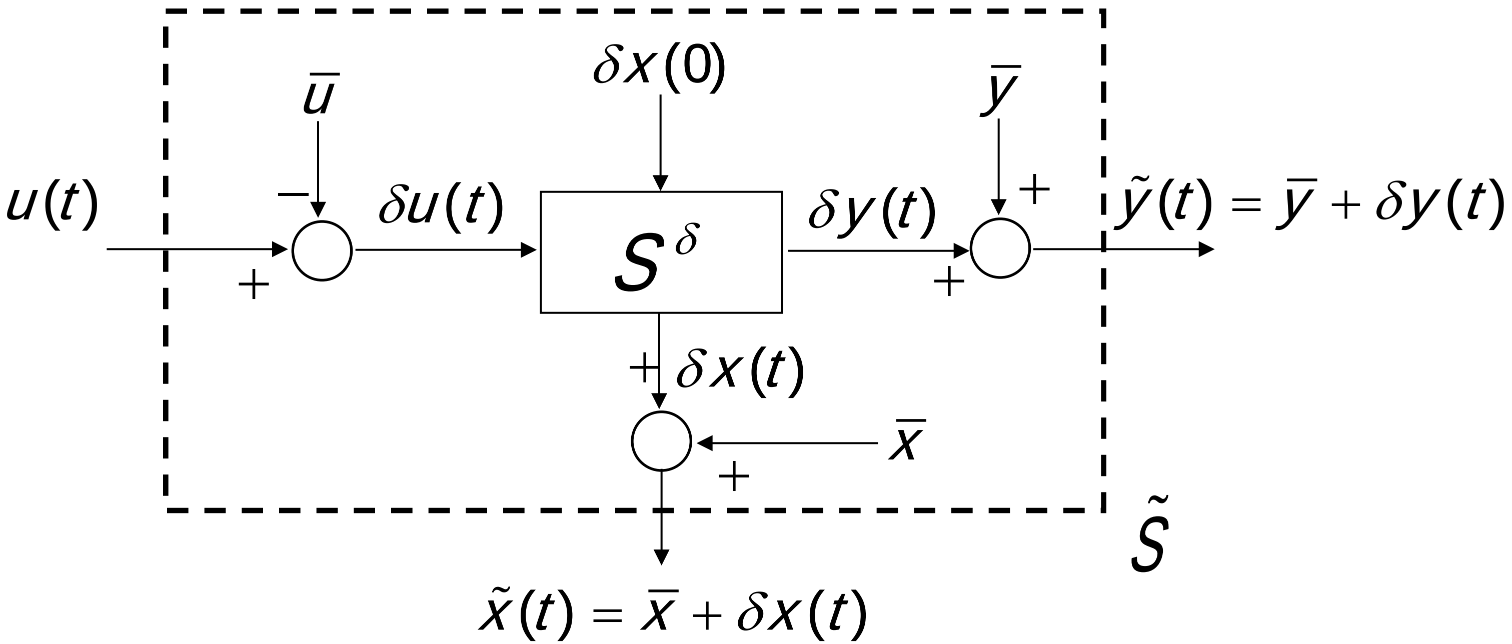


Linearized system:

$$S^{\delta}:\begin{cases}\delta\dot{x}(t)=A\delta x(t)+B\delta u(t)\\ \delta y(t)=C\delta x(t)+D\delta u(t)\end{cases}$$



Approximating system $\tilde{S}$



$$u(t) = \bar{u} + \delta u(t)$$

$$S^{\delta}: \begin{cases} \delta \dot{x}(t) = A\delta x(t) + B\delta u(t) \\ \delta y(t) = C\delta x(t) + D\delta u(t) \end{cases}$$