Automatic Control

Stability of equilibrium solutions of nonlinear systems

Stability analysis of equilibrium solutions using linearized models

Stability of an equilibrium solution

Equilibrium stability: introduction

Intuitively and somewhat crudely speaking, we say an equilibrium point is **stable** if all solutions which start near $\bar{x}$ (meaning that the initial conditions are in a neighborhood of $\bar{x}$) remain near $\bar{x}$ for all time.

The equilibrium point $\bar{x}$ is said to be **asymptotically stable** if it is stable and, furthermore, all solutions starting near $\bar{x}$ tend towards $\bar{x}$ as $t \to \infty$.

Nonetheless, it is intuitive that a pendulum has a locally stable equilibrium point when the pendulum is hanging straight down and an unstable equilibrium point when it is pointing straight up. If the pendulum is damped, the stable equilibrium point is locally asymptotically stable.

Equilibrium stability: introduction

Therefore, in order to study the equilibrium properties of an **equilibrium solution** $(\bar{x}, \bar{u})$ of a given dynamical system

$$\dot{x}(t) = f(x(t), u(t))$$

we need to introduce a **perturbation of the equilibrium solution** $x_p(t)$ obtained as the solution to $\dot{x}(t) = f(x(t), u(t))$ in the presence of:

- the same equilibrium input $\bar{u}$
- a perturbed initial condition x_0 which lies in some suitable neighborhood of $\bar{x}$

Equilibrium stability: introduction

Thus, equilibrium stability deals with the analysis of the behavior of the perturbed solution $x_p(t)$

In particular, it studies if the time course of $x_p(t)$

- evolves inside a bounded neighborhood of $\bar{x}$ (→ **stability**)
- asymptotically converges to $\bar{x}$ (→ **asymptotic stability**)
- evolves far away from $\bar{x}$ (**unstability**)

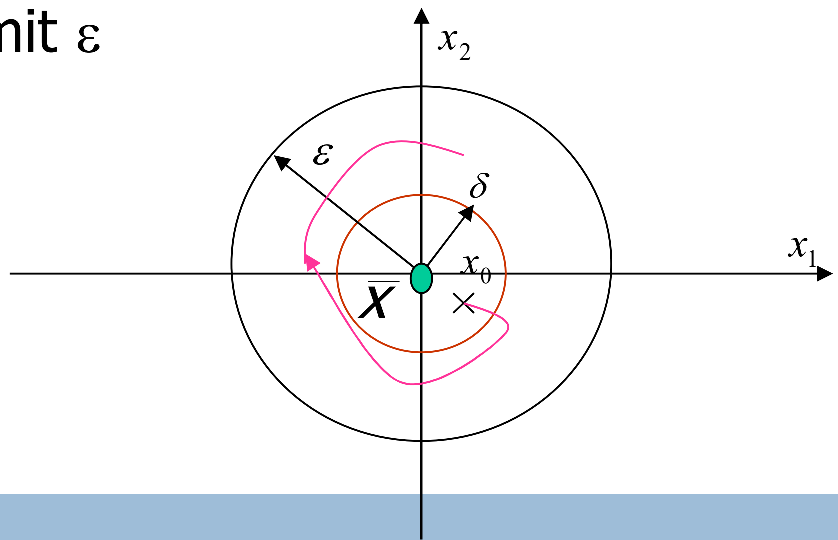
Equilibrium stability: definition

The equilibrium state $\bar{x}$ is :

stable if, $\forall\ \varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x_0 : \left\| x_0 - \bar{x} \right\| \le \delta \Rightarrow \left\| x_p(t) - \bar{x} \right\| \le \varepsilon, \quad \forall t \ge 0$$

Roughly speaking, $\bar{x}$ is **stable** if we are able to select a bound δ on initial condition that will result in trajectories that remain within a given arbitrarily small finite limit ε



Equilibrium instability: definition

The equilibrium state $\bar{x}$ is :

unstable if the perturbed solution $x_p(t)$ does not satisfy the stability condition

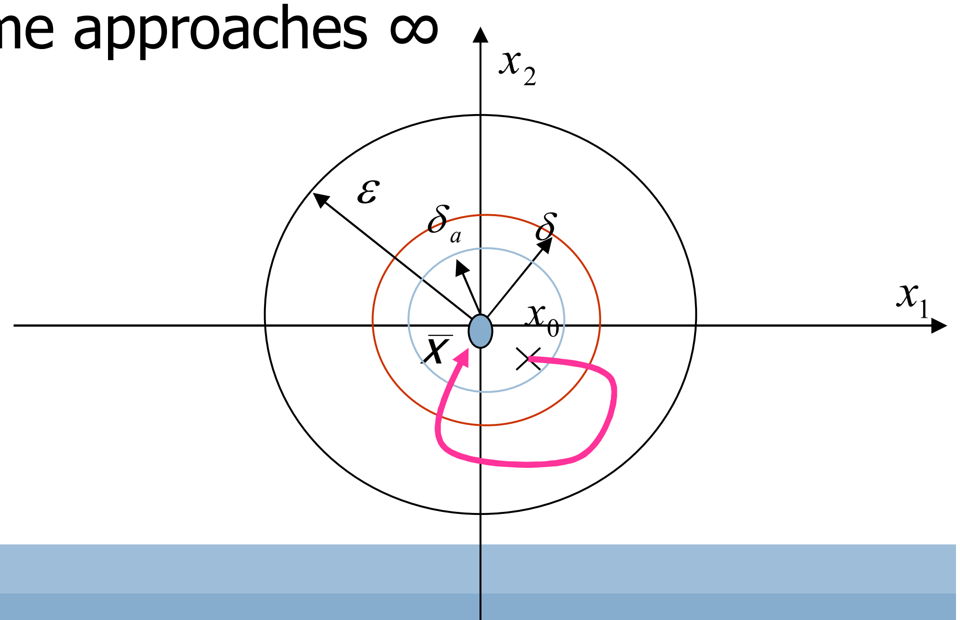
Equilibrium asymptotic stability: definition

The equilibrium state $\bar{x}$ is :

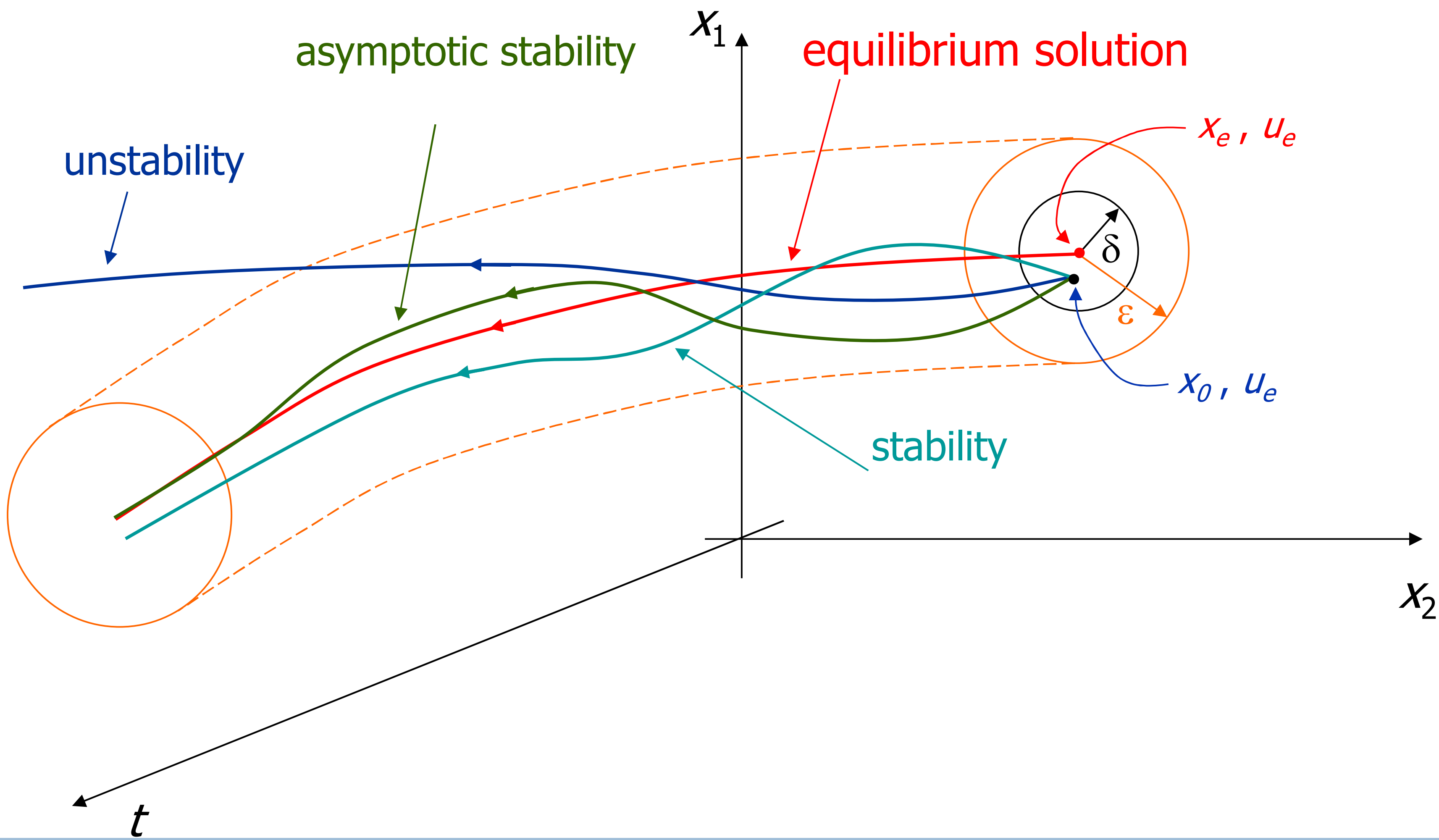
asymptotically stable if it is **stable** and δ can be chosen such that

$$\lim_{t\to\infty} \left\| x_p(t) - \bar{x} \right\| = 0$$

Roughly speaking, $\bar{x}$ is **asymptotically table** if it is stable and if the state approaches $\bar{x}$ as time approaches ∞



Equilibrium stability: resume



Equilibrium stability: conditions

The rigorous and deep study of equilibrium stability, introduced at the end of the XIX century by the Russian mathematician A. M. Lyapunov, is out of the scopes of a basic Automatic Control course

Anyway, simply to check, though not exhaustive, sufficient conditions for the study of stability of the equilibrium of nonlinear system will be provided

Equilibrium stability: LTI systems

For the special case of LTI systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$

we have the following

Result (stability of the equilibrium of LTI systems)

- **internal stability** $\Leftrightarrow$ **stability** of every equilibrium solution $\bar{x}$ of the LTI system
- **asymptotic stability** $\Leftrightarrow$ **asymptotic stability** of every equilibrium solution $\bar{x}$ of the LTI system
- **unstability** $\Leftrightarrow$ **unstability** of every equilibrium solution $\bar{x}$ of the LTI system

Equilibrium stability: LTI systems

Remark:

- for LTI systems, internal stability is a global property which applies to every solution (not only equilibria) and thus to the whole system
- for nonlinear systems, stability is a local property of a particular solution (e.g. an equilibrium)

Stability of an equilibrium solution through the linearized model

Linearized system and equilibrium stability: Result

Let $(\bar{x}, \bar{u})$ be an equilibrium point of the dynamical system

$$\dot{x}(t) = f(x(t), u(t))$$

where $f(\cdot) : \mathrm{D} \to \mathrm{R}^n$ is a C^1 function and D is a neighborhood of $\bar{x}$

Let

$$A = \left. \frac{\partial f(x)}{\partial x} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

Then:

- the equilibrium $\bar{x}$ is asymptotically stable if $\mathrm{Re}(\lambda_i(A)) < 0,\ i = 1, \ldots, n$
- the equilibrium $\bar{x}$ is unstable if $\mathrm{Re}(\lambda_i(A)) > 0$, for some i

<u>Remark</u>: If $\mathrm{Re}(\lambda_i(A)) \leq 0,\ i = 1, \ldots, n$ no conclusion can be drawn about stability properties of the equilibrium $\bar{x}$

Linearized system and equilibrium stability: Resume

Eigenvalues $\lambda_i(A)$ of the linearized system	Stability properties of the equilibrium
$\forall i : \Re e(\lambda_i(A)) < 0$	**Asymptotic stability**
$\exists i : \Re e(\lambda_i(A)) > 0$	**Unstability**
$\forall i : \Re e(\lambda_i(A)) \leq 0$	**No conclusion can be drawn**

Example: pendulum

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\bar{x}_1 & -\frac{\beta}{Ml^2} \end{bmatrix} \qquad \left(\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \bar{u} = 0 \right)$$

$$k \text{ even} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\beta}{Ml^2} \end{bmatrix}, \; k \text{ odd} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\beta}{Ml^2} \end{bmatrix}$$

Two cases are considered: $\beta > 0$ and $\beta = 0$

Example: pendulum

Case: $\beta > 0$

$$k \text{ even (e.g. } \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\beta}{Ml^2} \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 + \frac{\beta}{Ml^2}\lambda + \frac{g}{l}$$

→ all the coefficients of the characteristic polynomial are positive

→ both the eigenvalues of the linearized system have strictly negative real part (Descartes' rule of signs)

→ all the equilibrium states of the form $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, k \text{ even}$ are asymptotically stable

Example: pendulum

Case: $\beta > 0$

$$k \text{ odd (e.g. } \bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\beta}{Ml^2} \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 + \frac{\beta}{Ml^2}\lambda - \frac{g}{l}$$

→ there is a variation in the sign of the coefficients of the characteristic polynomial

→ there is an eigenvalues of the linearized system with strictly positive real part (Descartes' rule of signs)

→ all the equilibrium states of the form $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, k \text{ odd}$ are unstable

Example: pendulum

Case: $\beta = 0$

$$k \text{ even (e.g. } \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & 0 \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 + \frac{g}{l} \Rightarrow \lambda_{1,2} = \pm j\sqrt{\frac{g}{l}}$$

→ both the eigenvalues of the linearized system have null real part

→ no conclusion can be drawn on the stability properties of the equilibrium states of the form: $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, k$ even

(using other analysis methods, stability can be proved)

Example: pendulum

Case: $\beta = 0$

$$k \text{ odd (e.g. } \bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 - \frac{g}{l} \Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{g}{l}}$$

→ there is an eigenvalue of the linearized system with strictly positive real part

→ all the equilibrium states of the form $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, k$ odd are unstable

Example: magnetic levitator

$$A = \begin{bmatrix} 0 & 1 \\ \frac{2g}{|\bar{u}|}\sqrt{\frac{g}{K_1}} & 0 \end{bmatrix} \rightarrow a = \frac{2g}{|\bar{u}|}\sqrt{\frac{g}{K_1}} > 0 \rightarrow A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$$

$$p_A(A) = \det(\lambda I - A) = \lambda^2 - a$$

$$\Rightarrow \lambda_{1,2}(A) = \pm\sqrt{a} \Rightarrow \mathbf{Re}(\lambda_1) = \sqrt{a} > 0$$

$\Rightarrow$ the levitator equilibrium states $\bar{x} = \begin{bmatrix} \sqrt{\frac{k_i}{Mg}}|\bar{u}| \\ 0 \end{bmatrix}, \bar{u} \neq 0$ are unstable