SISO feedback control systems characteristics

Feedback control systems configurations & design objectives

Relative Stability

Steady-state response to polynomial reference inputs

Steady-state response to polynomial disturbance inputs

Steady-state frequency response of feedback control systems

SISO feedback control systems characteristics

Steady-state response to sinusoidal disturbance inputs

Sensitivity of control systems to parameter variations

General feedback control systems transient response to step input

Prototype 2° order systems transient response to step input

General feedback control systems frequency response

SISO feedback control systems characteristics

Prototype 2° order systems frequency response

Relations between frequency response and time response of prototype 2° order systems

Constant magnitude loci of *T* and *S*

Relative Stability: Stability margins and the sensitivity peaks

Nichols chart

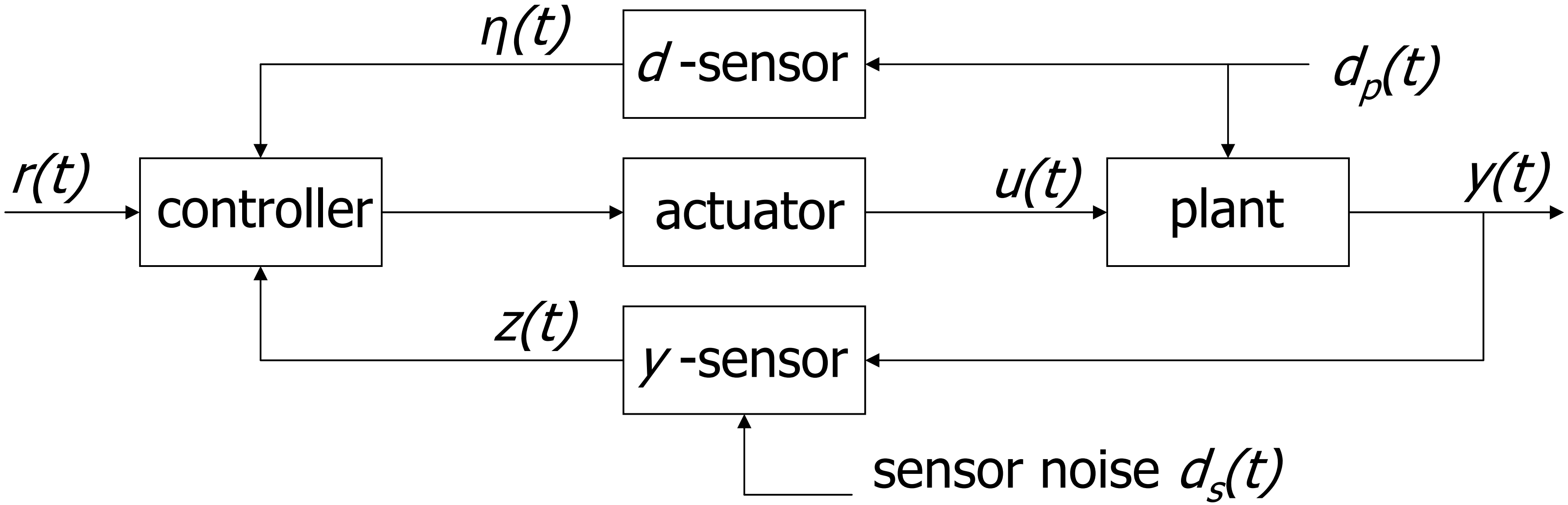
SISO feedback control systems characteristics

Output feedback control systems configurations & design objectives

Output feedback control systems configurations

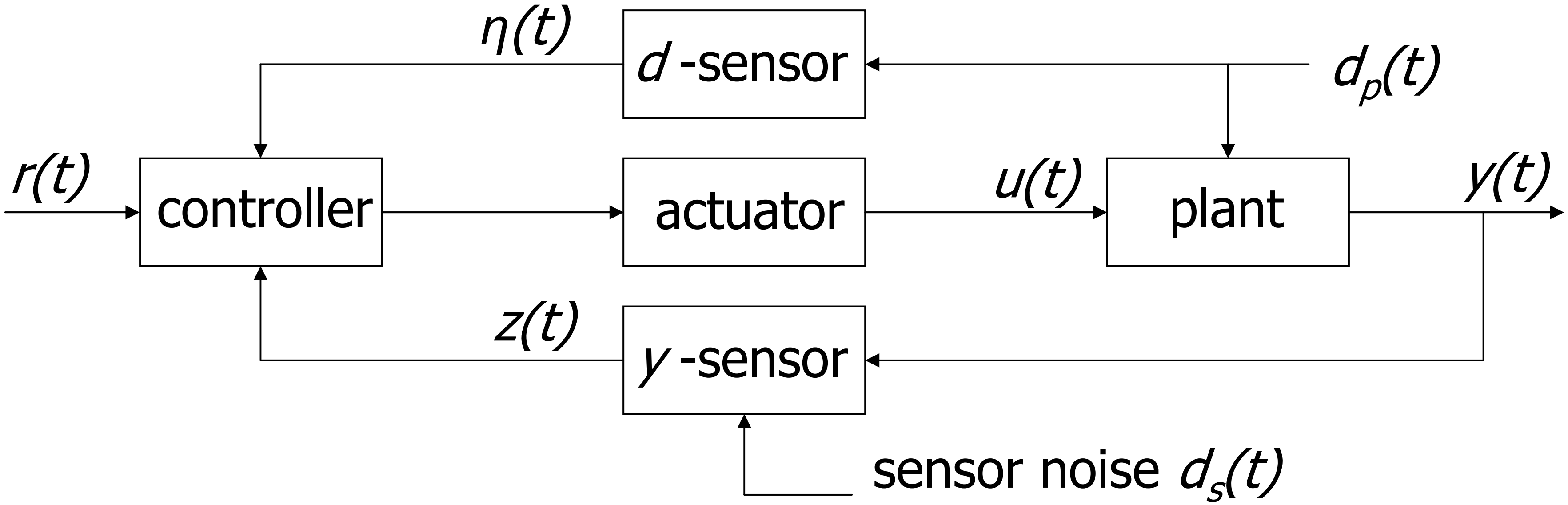
A layout of a quite general feedback control system is represented below. The components of a the control system are:

- The **plant**: is the system to be controlled.
- One or more **sensors**: give information about the plant and, possibly, the disturbances.



Output feedback control systems configurations

- The **controller**: compares the measured values to their desired values and provides the input variables to the plant.
- The **actuator**: provides the input signal $u(t)$ suitably amplified.

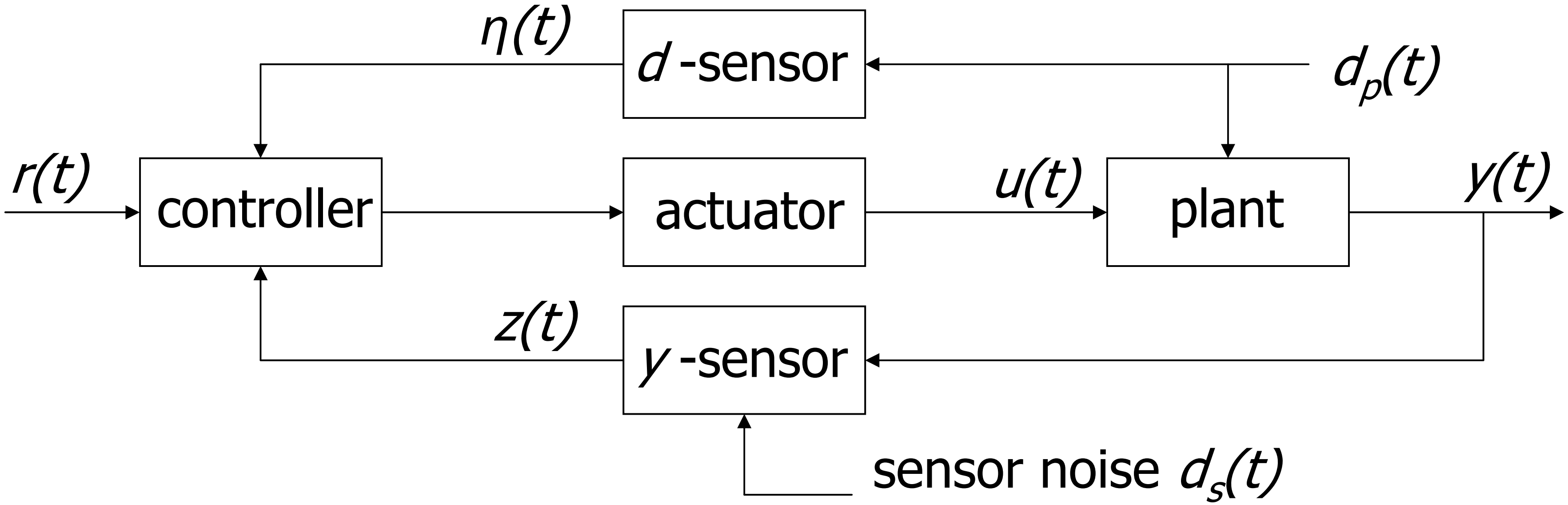


Output feedback control systems configurations

The main signal of interest are:

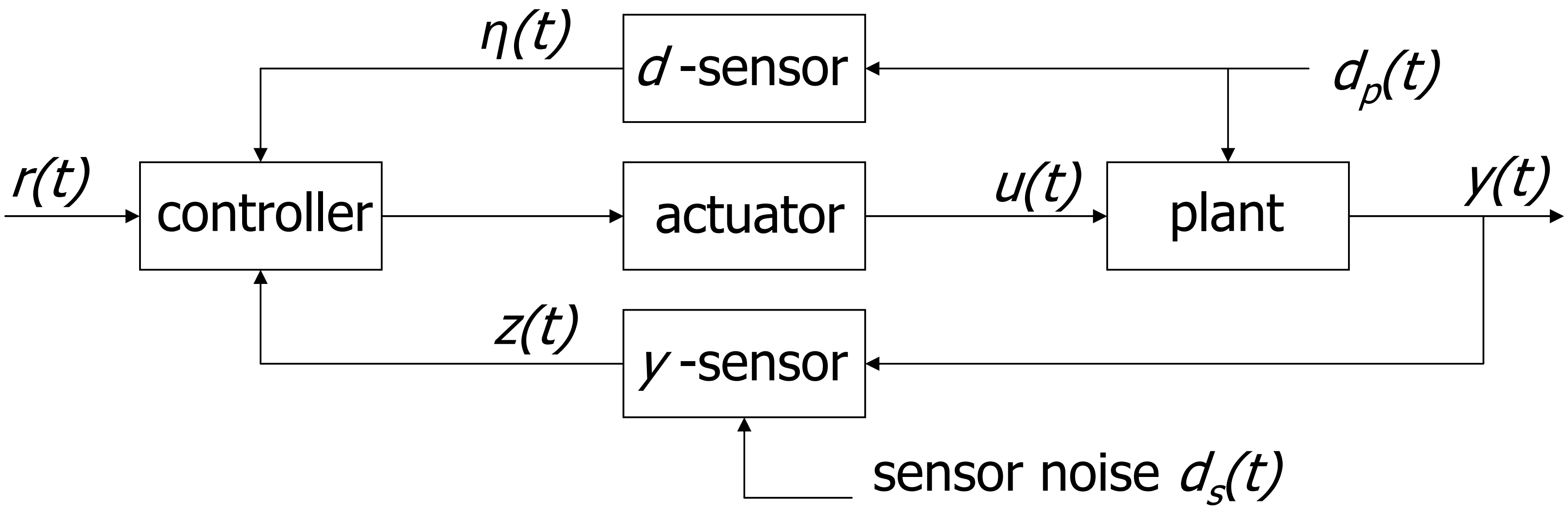
- The **reference signal** $r(t)$: gives the prescribed values of the controlled signal.

- The **input signal** $u(t)$: influences the plant and is a manipulatable variable.



Output feedback control systems configurations

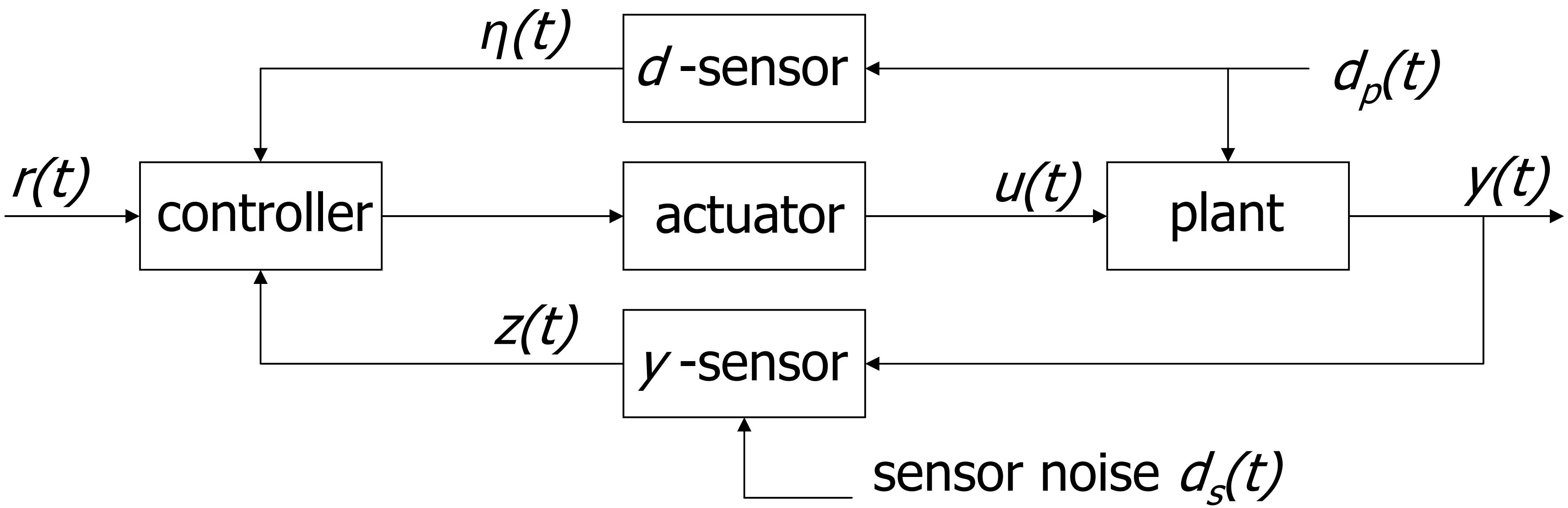
- The **controlled signal** $y(t)$: is the signal under control.
- the **disturbance signal** $d_p(t)$: influences the plant and cannot be manipulated.
- the **sensor noise** $d_s(t)$: affects the measured signal.



Output feedback control systems configurations

- The **measured signal** $z(t)$: is measured by means of a sensor and contains information about the state of the plant.

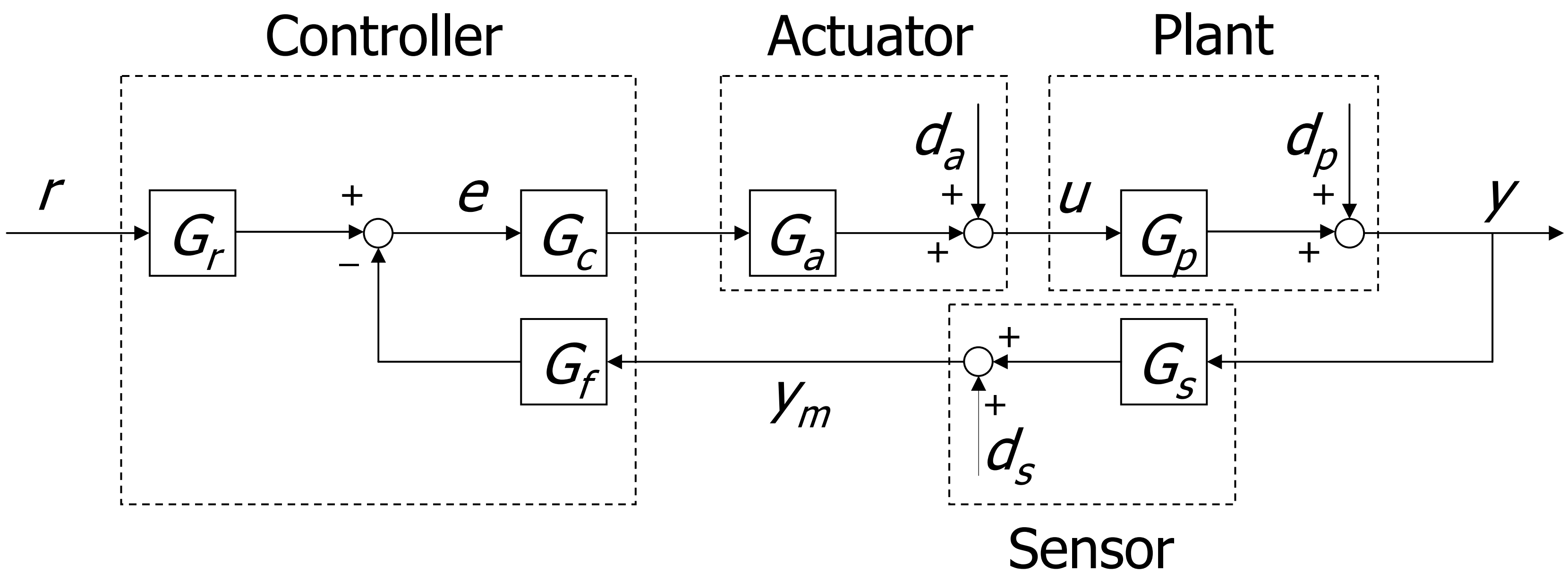
- Possibly, the **measured signal** $\eta(t)$: is measured by means of a sensor and contains information about the plant disturbance.



Output feedback control systems configurations

The given parts of the configuration we are dealing with are:

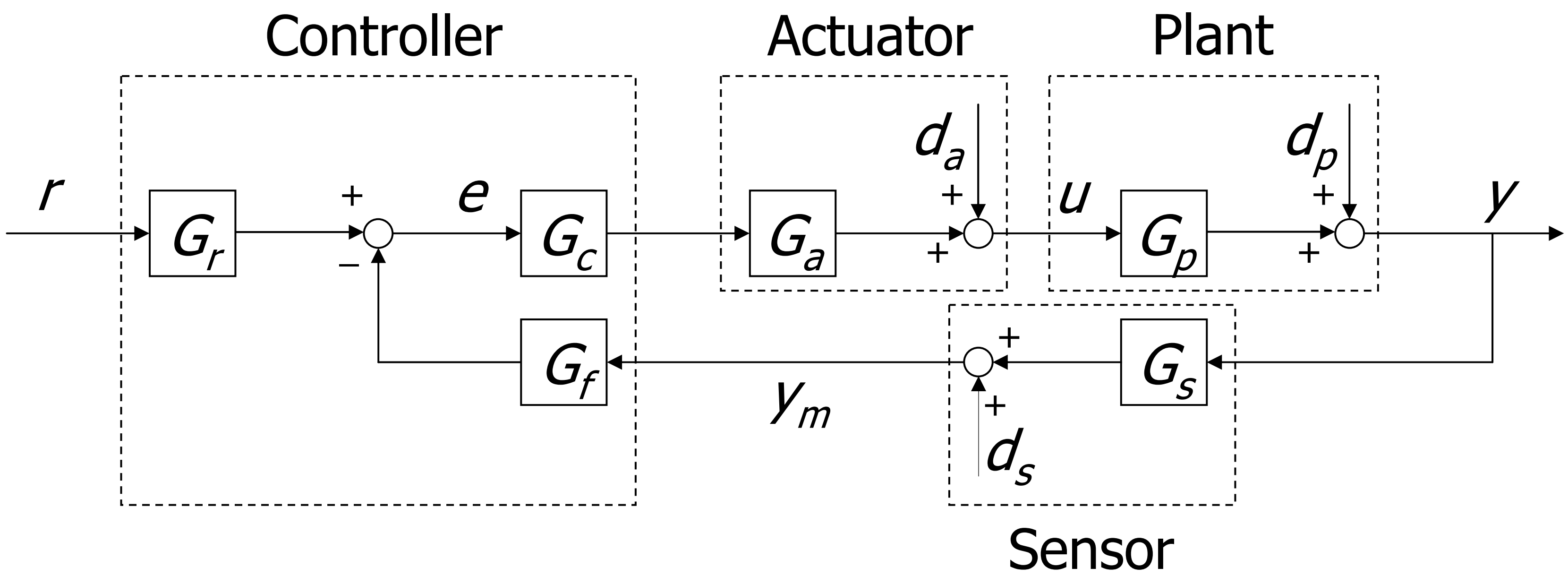
- **plant** G_p with plant disturbance d_p
- **actuator** G_a with actuator disturbance d_a
- **sensor** G_s with sensor noise d_s



Output feedback control systems configurations

The controller to be designed consists of:

- **prefilter** G_r (reference generator)
- **cascade controller** G_c
- **feedback controller** G_f (for 2 DOF or, if constant, for dc gain)

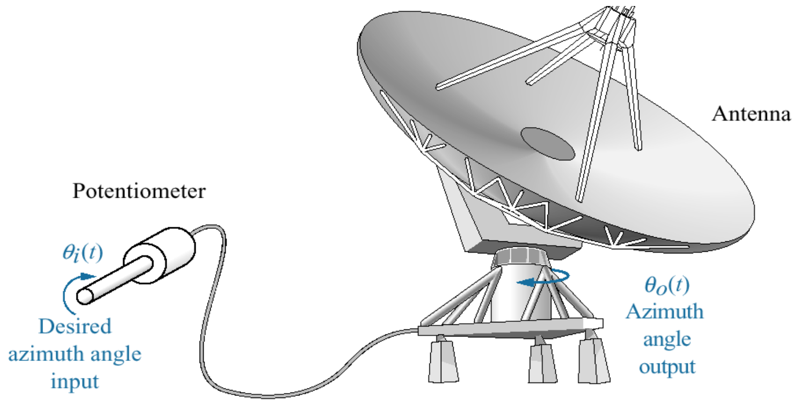


Example: Antenna azimuth position control system

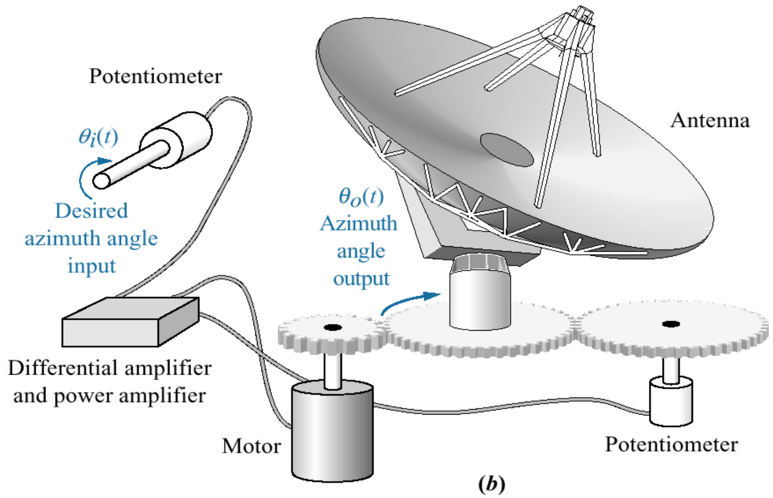
We consider again the problem of controlling the **antenna azimuth position**.

a. system concept

b. detailed layout



(*a*)



(*b*)

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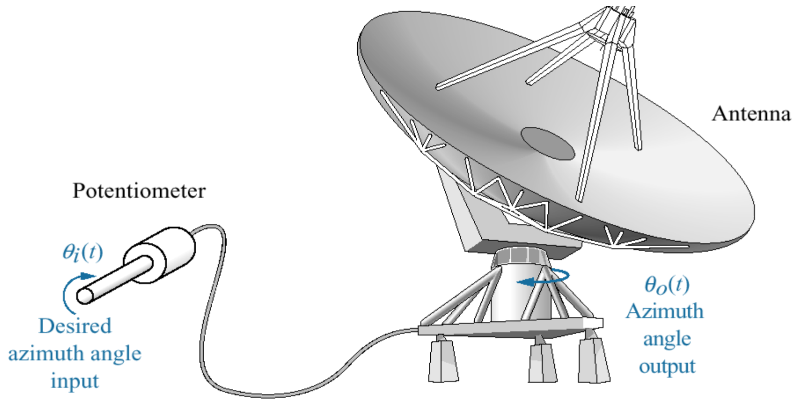
Example: Antenna azimuth position control system

a. system concept

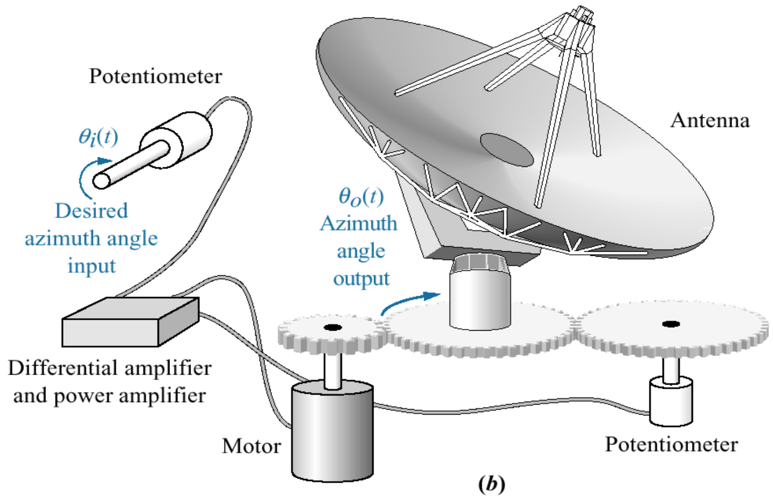
b. detailed layout

The antenna is driven by an electric motor.

The control problem is to command the motor in such a way that the angular position of the antenna tracks the desired angular position.



(a)



(b)

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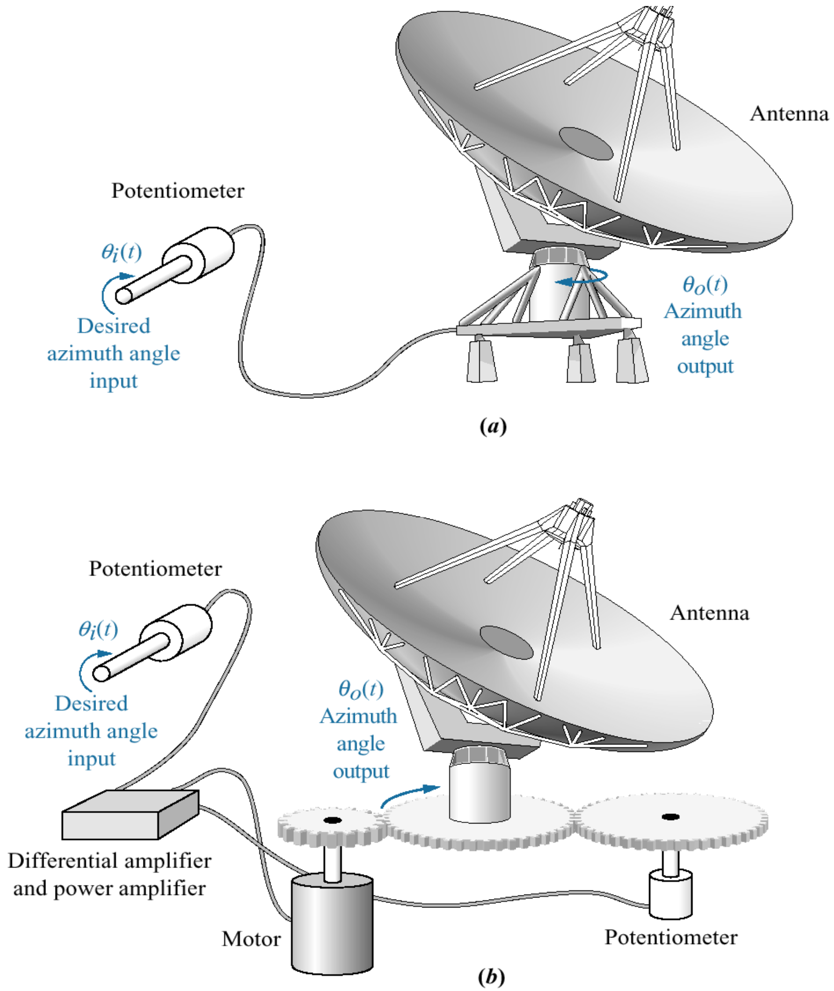
Example: Antenna azimuth position control system

a. system concept

b. detailed layout

The desired angular position is made available as a mechanical angle by manually pointing binoculars in the direction of the desired object.



Example: Antenna azimuth position control system

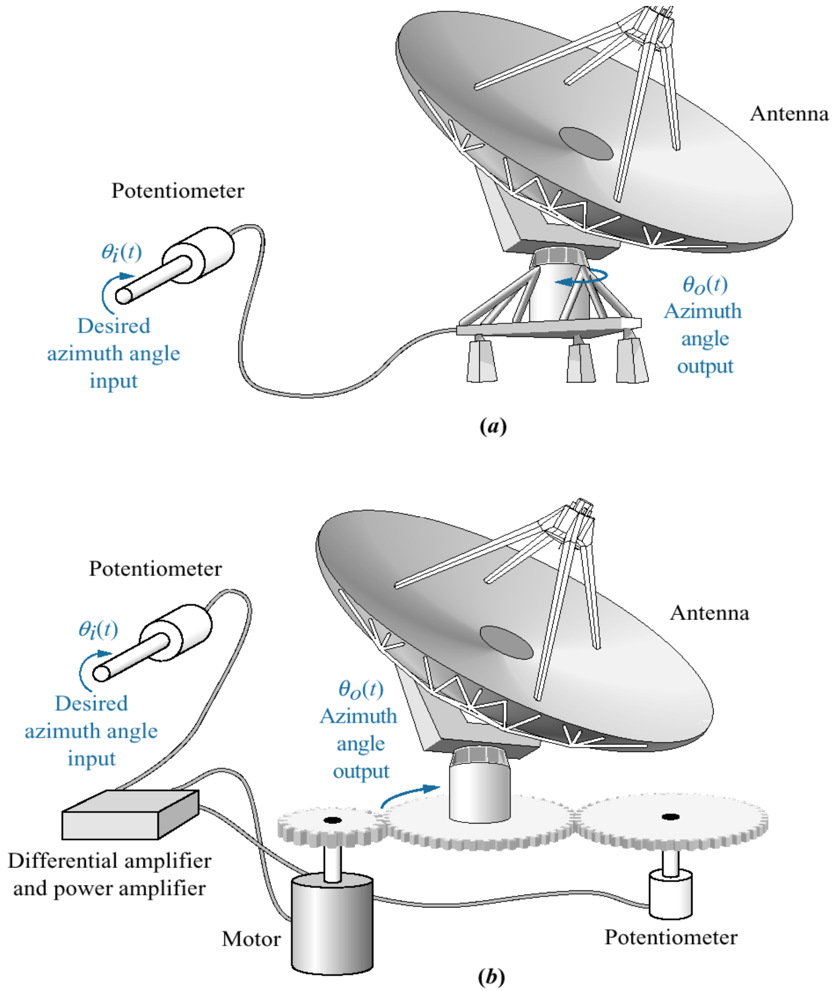
a. system concept

b. detailed layout

The plant consists of the antenna and the motor.

The disturbance is the torque exerted by the wind on the antenna.

The measured variable is the output of a potentiometer mounted on the shaft of the antenna.



Example: Antenna azimuth position control system

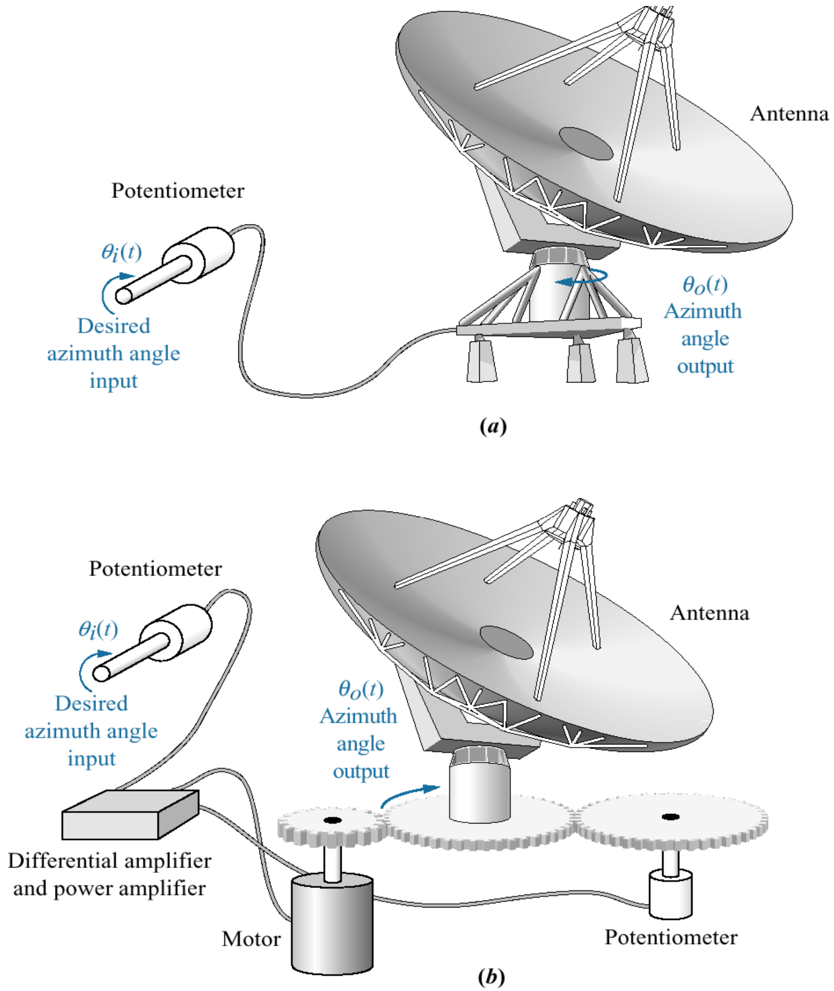
a. system concept

b. detailed layout

In this example, the azimuth angle of the antenna is the controlled variable.

The reference variable is the direction of the object to be tracked.

The input to the plant is the input voltage to the electric motor.



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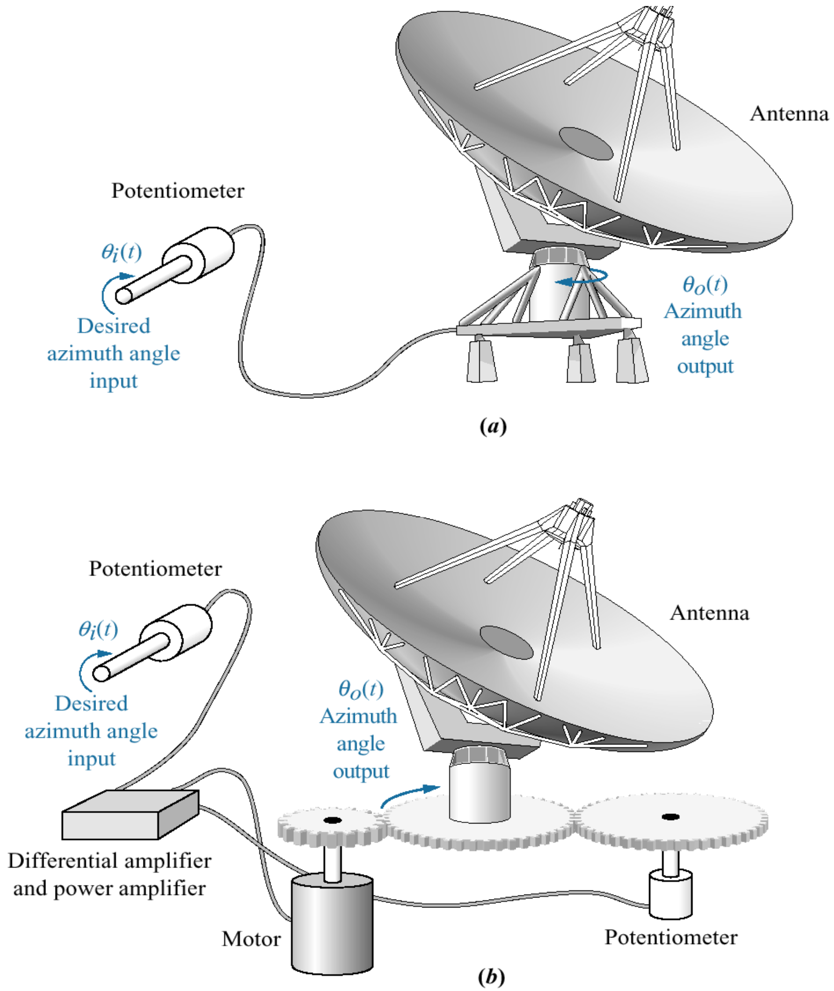
Example: Antenna azimuth position control system

a. system concept

b. detailed layout

Both the azimuth angle of the antenna and the angle of the object to be tracked are converted to electrical variables by means of potentiometers mounted on the shaft of the antenna and the binoculars.

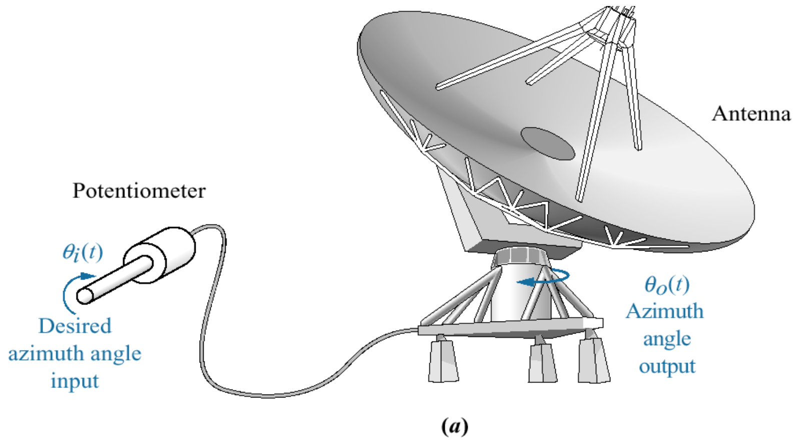


Example: Antenna azimuth position control system

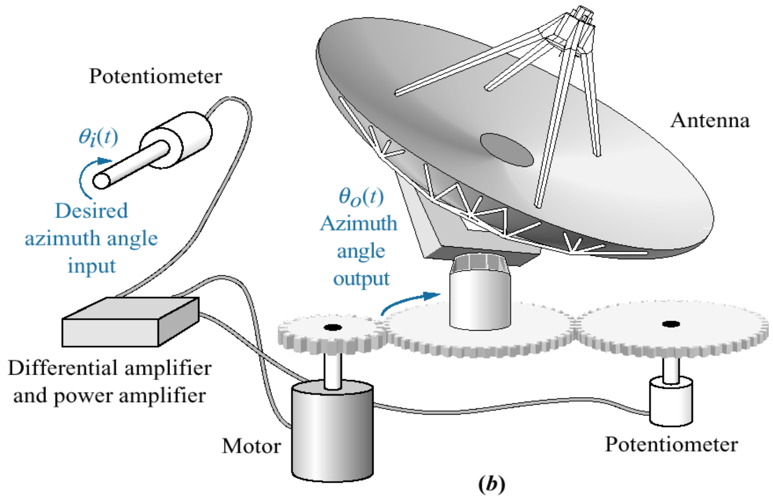
a. system concept

b. detailed layout

The output of the difference amplifier is amplified and provides the input voltage to the electric motor.



(a)



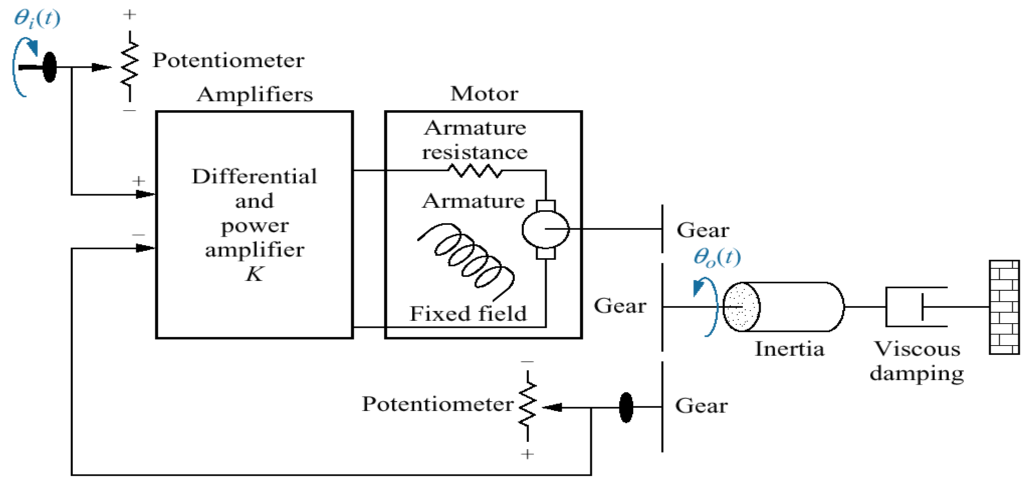
(b)

Example: Antenna azimuth position control system

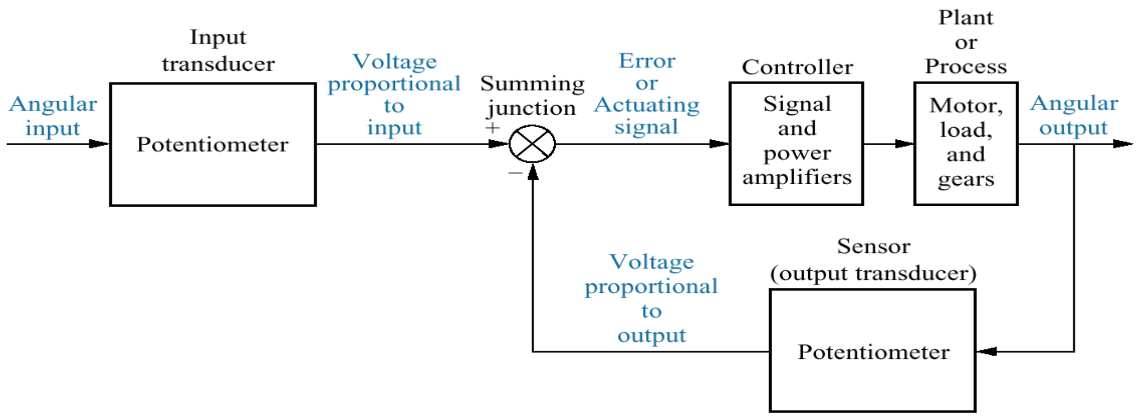
c. schematic

d. functional block diagram

This scheme represents a closed-loop control system.



(c)



(d)

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design objectives: internal stability

In the design of a feedback control system, some objectives have to be kept in mind:

Internal stability - For all bounded disturbances and inputs, the system response at every point inside the control loop must be bounded. The difference between internal stability and input-output stability was established in the previous lessons.

It is possible for a system to be internally unstable and yet to have a stable transfer function, i.e. to be input-output stable. This happens when the system has unstable hidden modes.

design objectives: robust stability

Robust stability - If the plant deviate from a nominal model, then a set of models is suitable for a better representation of the plant.

The set of models could be generated, for example, by letting the model parameters vary over their uncertainty intervals, with each parameter value defining a member of the set.

For a controller design to be acceptable, the feedback control system must be internally stable for every model in the set. This property is referred to as robust stability.

Tracking – A good feedback control systems must provide a satisfactory steady-state and transient tracking. It is not usually possible to have good tracking for all possible reference input. That is the reason why response specifications are normally given in the presence of specific reference signals or classes of signals.

Sometimes specific time functions are specidfied: for example, the set of signals used for typical aircraft maneuvers.

It is common to examine in some detail the response to the unit step, which is used as a test signal in many applications.

design objectives: disturbance attenuation/rejection

Disturbance attenuation/rejection – A further useful property of feedback is that the effect of (external) disturbances is reduced.

A good feedback control systems must also provide a satisfactory steady-state and transient disturbance reduction/rejection. It is not usually possible to have good disturbance attenuation/rejection for all possible disturbances. Disturbance reduction specifications are normally given in the presence of specific disturbance signals or classes of signals.

It is common to examine in some detail the response to polynomial disturbances and sinusoidal disturbances.

SISO feedback control systems characteristics

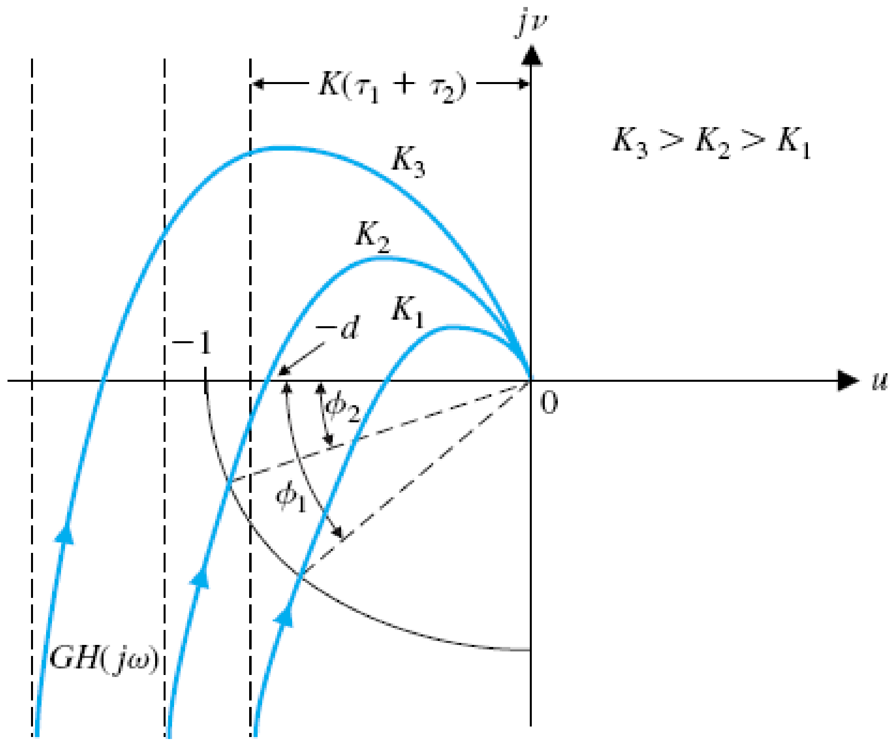
Relative Stability

Relative stability

The Nyquist criterion is defined in terms of the (-1,0) point on the polar plot or the 0dB, -180^0 point on the Bode diagram or log-magnitude-phase diagram. The proximity of the GH(jω) locus to this stability point is a measure of relative stability.

For example, for the three pole system

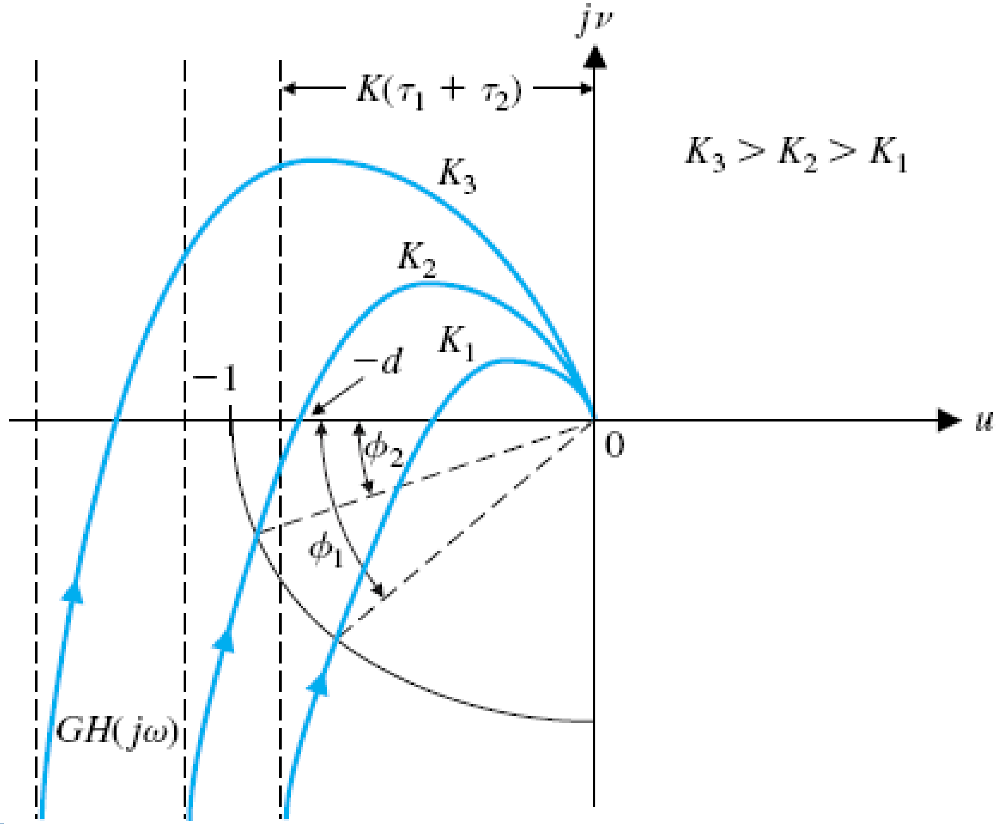
$$GH(s) = \frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$$



Shown in the figure, as K increases, the GH(jω) locus approaches the -1 point and eventually encircles the -1 point when $K=K_3$. We have found the intersection with the u-axis is

$$u = \frac{-K\tau_1\tau_2}{\tau_1 + \tau_2}$$

$$u = -1 \Rightarrow K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$$



As K decreases below this marginal value, the stability increases.

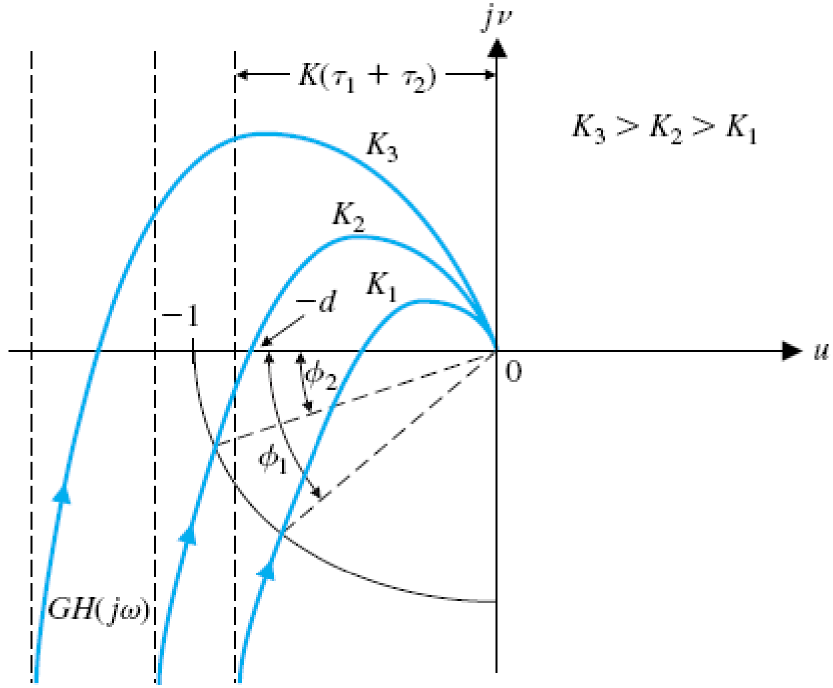
Gain margin – a measure of relative stability

The **Gain margin** is defined as the reciprocal of the gain $|GH(j\omega)|$ at the frequency at which the phase angle reaches -180^0 (that is v=0). The gain margin is a measure of the factor by which the system gain would have to be increased for the $GH(j\omega)$ locus to pass the u=-1 point. For the system shown in the figure, when $K=K_2$, the gain margin is $1/|GH(j\omega)|$ when v=0.

Gain margin can be expressed in dB, i.e., when v=0, gain margin in dB is given as

$$20\log\left(\frac{1}{|GH(j\omega)|}\right) = 20\log\left(\frac{1}{d}\right) =$$

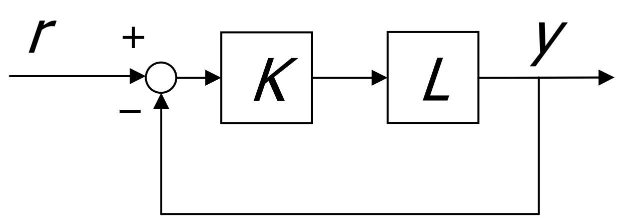
$$= -20\log d \text{ dB}$$

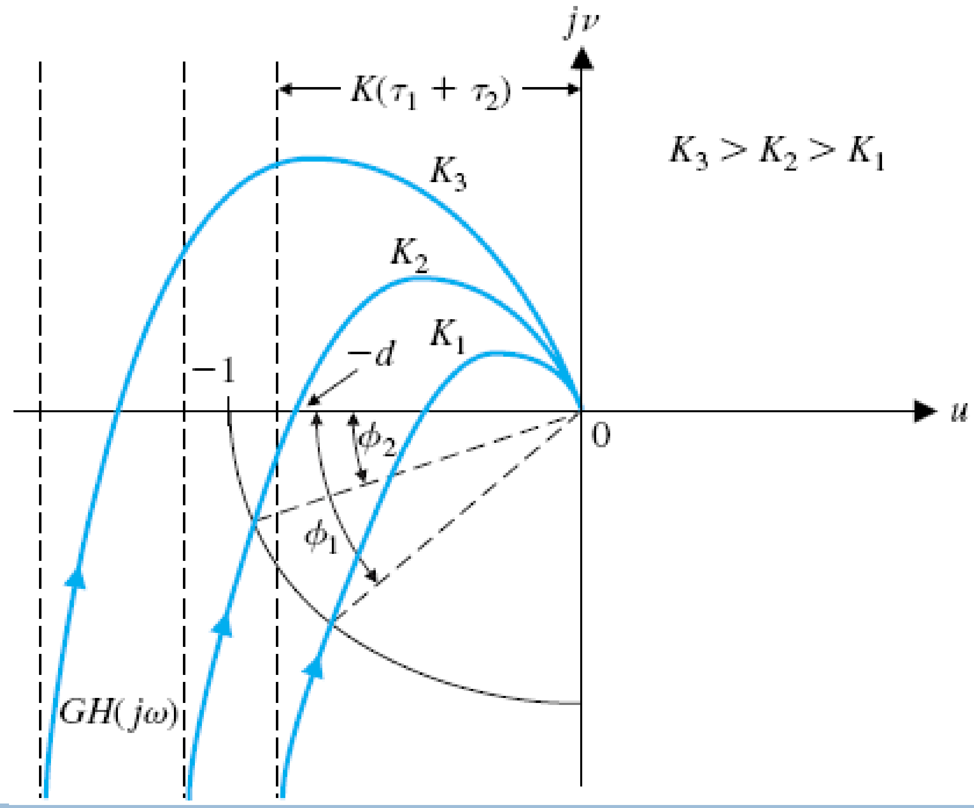


Gain margin – a measure of relative stability

The gain margin is the increase in the system gain when the phase is -180^0 that will result in a marginally stable system with intersection of (-1,0) point on the Nyquist diagram.

The gain margin is the upper bound of the multiplication factors of the loop function that the feedback control system can tolerate without losing the stability property.



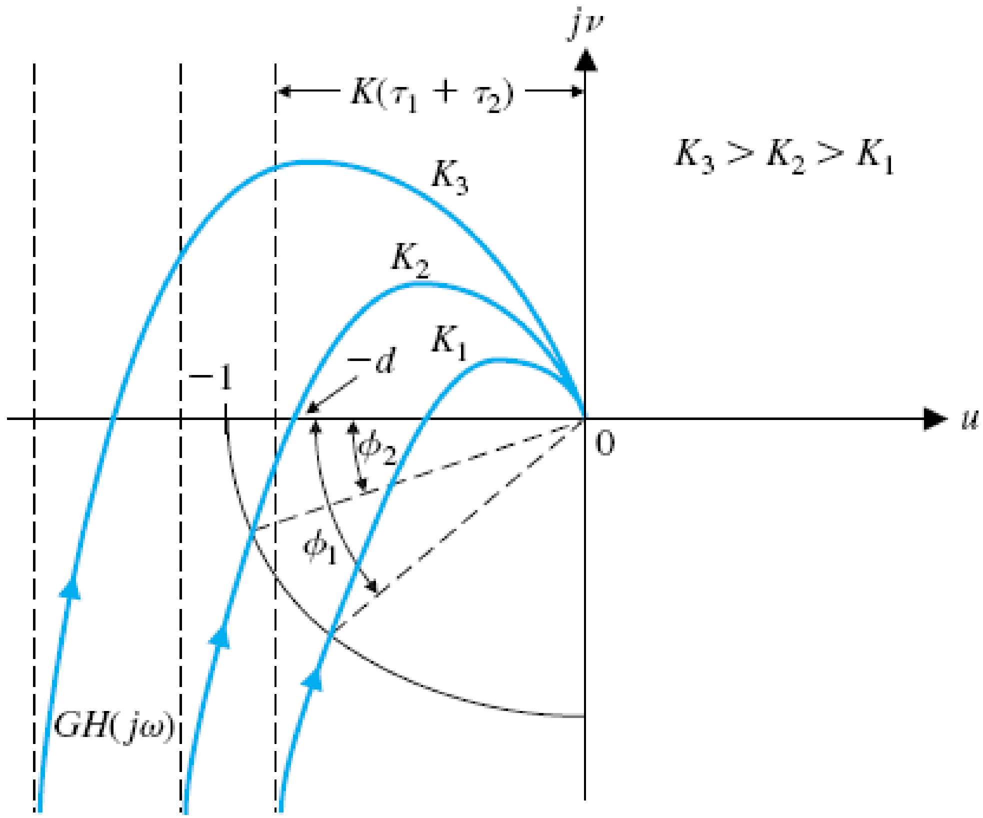


Phase margin – a measure of relative stability

The **Phase margin** is defined as the phase angle through which the GH(jω) locus must be rotated so that the unity |GH(jω)|=1 point will pass through the (-1,0) point in the GH(jω)-plane.

This measure of relative stability is equal to the additional phase lag required before the system becomes unstable.

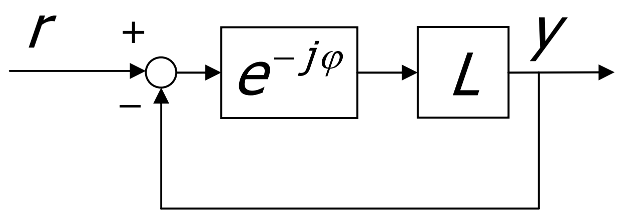
For the Nyquist diagram shown in the figure, when $K=K_2$, phase margin is Φ_2, and $K=K_1$, phase margin is Φ_1.

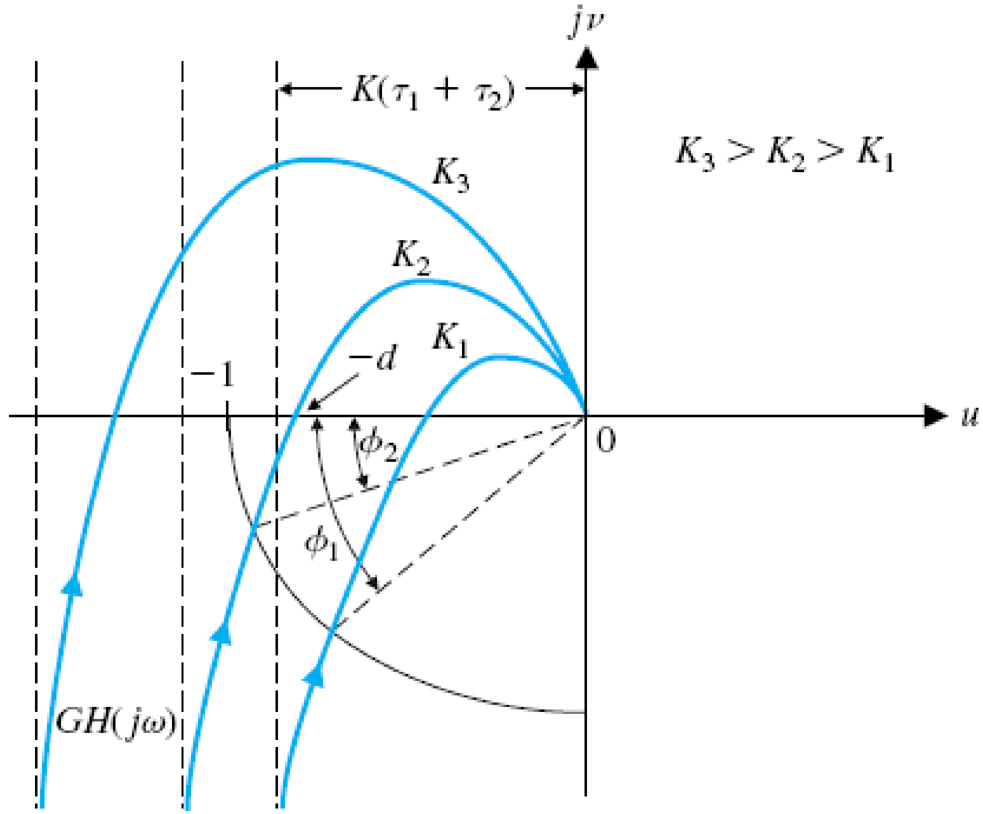


Phase margin – a measure of relative stability

The phase margin is the amount of phase shift of GH(jω) at unity magnitude that will result in a marginally stable system with intersection of (-1,0) point on the Nyquist diagram.

The phase margin is the maximum phase delay of the loop function that the feedback control system can tolerate without losing the stability property.

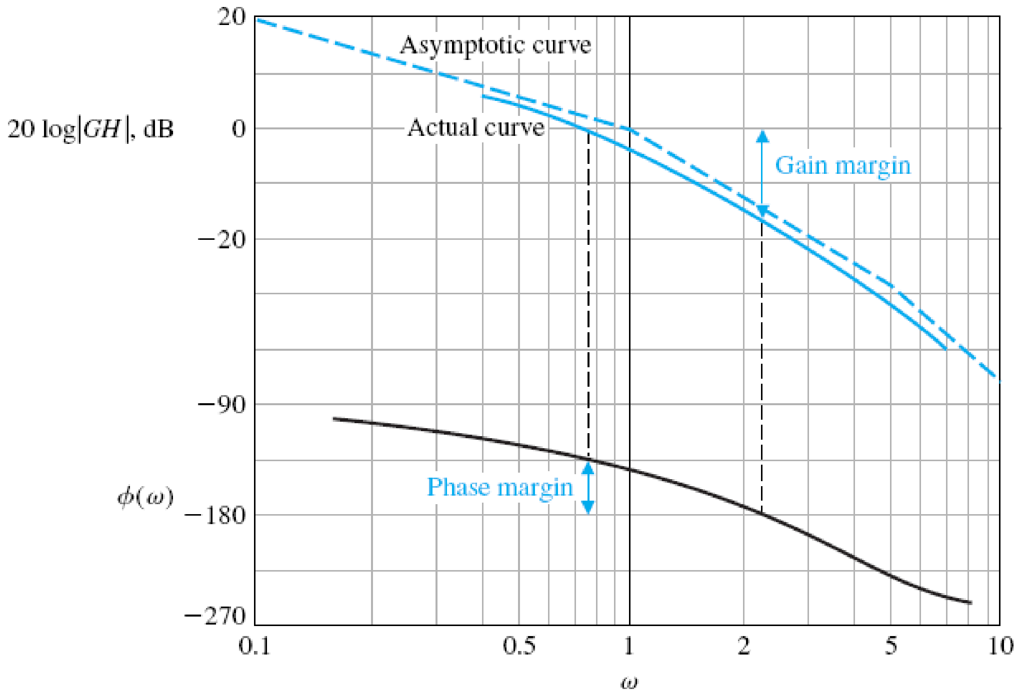




Relative stability determined from Bode diagram

The critical point for stability is u=-1, v=0 point in the GH(jω) plane, which is equivalent to 0dB, -180^0 on the Bode diagram. Gain margin and phase margin can be easily determined from the Bode diagram. Bode diagram of the following system is shown in the figure

$$GH(j\omega) = \frac{1}{j\omega(j\omega+1)(0.2j\omega+1)}$$



- The phase angle is -137^0 when log magnitude is 0dB, so phase margin is $-137^0-(-180^0)=43^0$.
- The log magnitude is -15dB when phase angle is -180^0, so gain margin is 15dB.

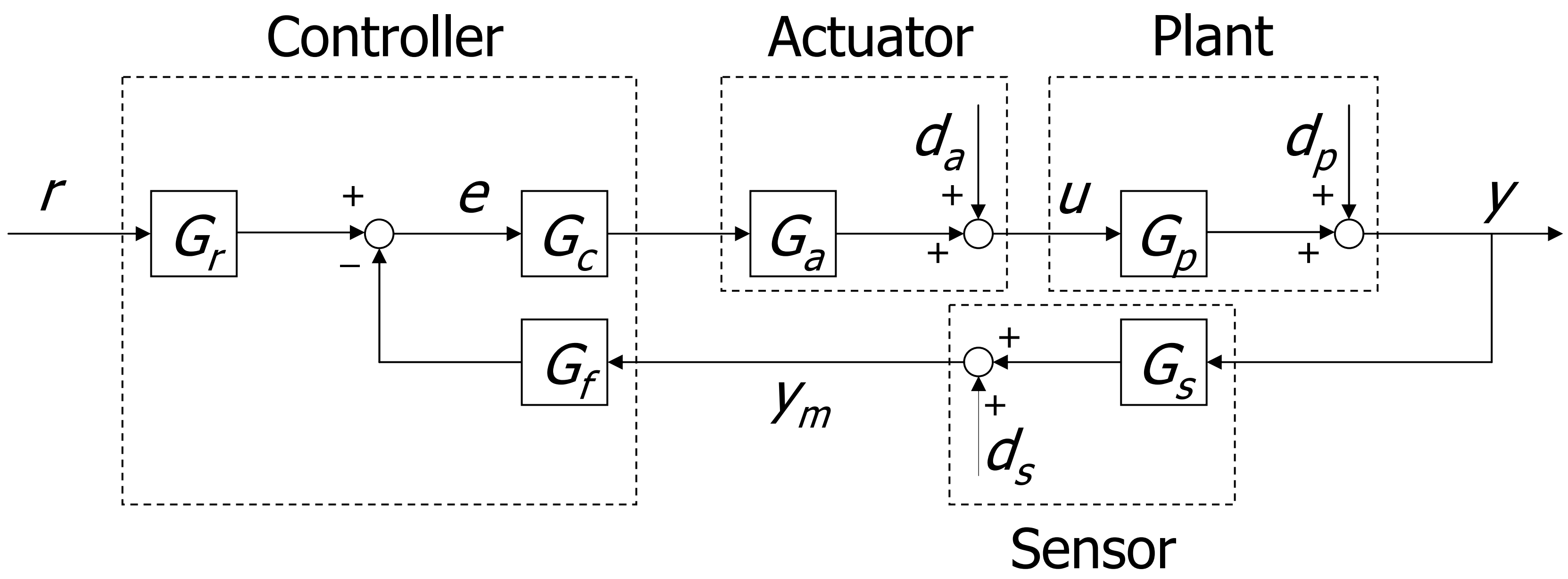
SISO feedback control systems characteristics

Steady-state response to polynomial reference inputs

Introduction

- Errors in a control system can be attributed to many factors (disturbances, imperfections in the system components, changes in the reference inputs, etc...).
- In this section, we shall investigate the steady-state error that is caused by the incapability of a system to follow particular types of inputs.
- Control systems may be classified according to their ability to track polynomial inputs, or in other words, their ability to reach zero steady-state to step-inputs, ramp inputs, parabolic inputs and so on.
- This is a reasonable classification scheme because actual inputs may frequently be considered as combinations of such inputs.

Output feedback control systems configurations



Without loss of generality we assume that

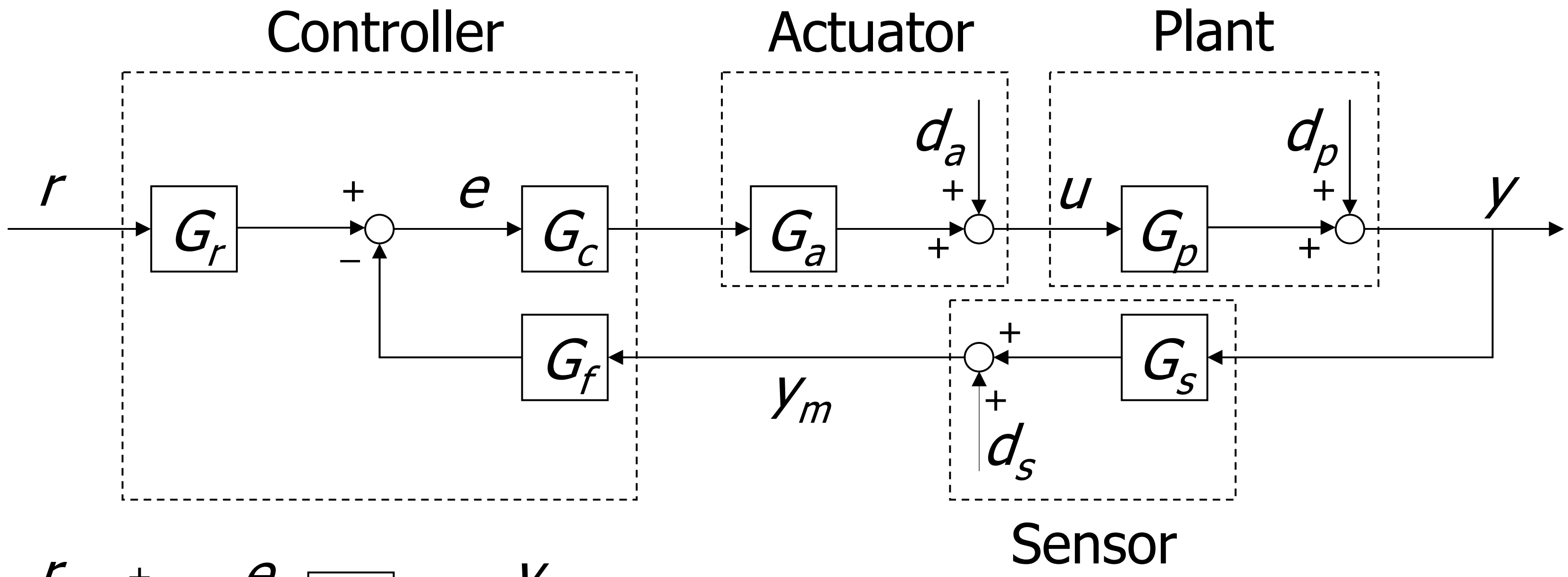
$G_r = 1$

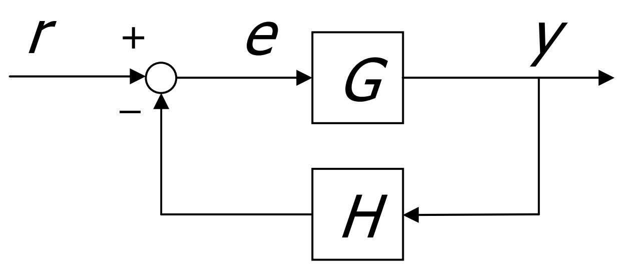
G_a = constant (large bandwidth because of fast dynamics)

G_f = constant (to handle the control system steady-state gain)

G_s = constant (large bandwidth because of fast dynamics)

Output feedback control systems configurations





$$G = G_c G_a G_p = \frac{N_G}{D_G} \qquad H = G_s G_f = \frac{N_H}{D_H}$$

$G_r = 1$
G_a = constant
G_f = constant
G_s = constant

- LTI controllers described by the transfer function

$$G_c(s) = \frac{K_c}{s^\nu} \prod_i \left(\frac{1 + \frac{s}{z_{di}}}{1 + \frac{s}{m_{di} z_{di}}} \right) \prod_j \left(\frac{1 + \frac{s}{m_{ij} p_{ij}}}{1 + \frac{s}{p_{ij}}} \right)$$

will be considered from now on.

- The controller has ν poles at $s = 0$ and its generalized steady-state gain is defined as:

$$\lim_{s \to 0} s^\nu G_c(s) = K_c$$

- The plant has p poles at $s = 0$ and its generalized steady-state gain is defined as:

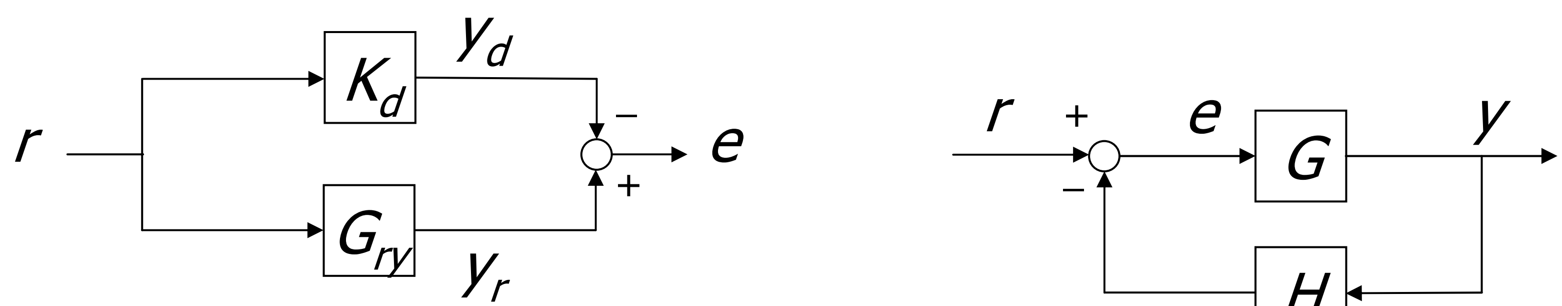
$$\lim_{s \to 0} s^p G_p(s) = K_p$$

Steady-state response to polynomial reference inputs

- Polynomial reference signals

$$r(t) = R_0 \frac{t^h}{h!}\varepsilon(t) \rightarrow r(s) = R_0 \frac{1}{s^{h+1}}$$

are considered here. h is the order of the polynomial signal.



$$G = G_c G_a G_p = \frac{N_G}{D_G} \qquad H = G_s G_f = \frac{N_H}{D_H}$$

Steady-state response to polynomial reference inputs

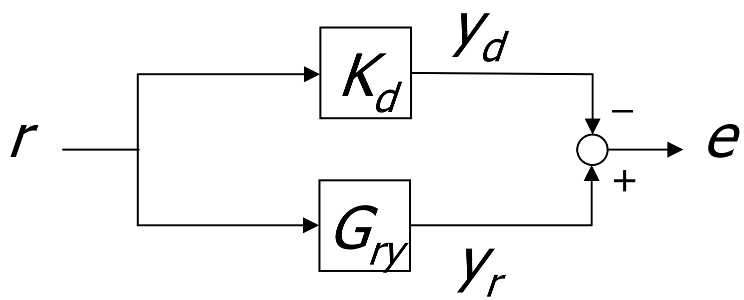
- The **tracking error** is defined as:

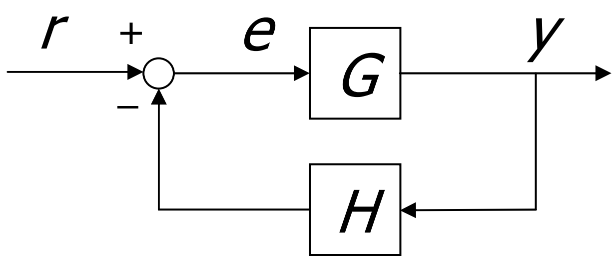
$$e_r(t) = y_r(t) - y_d(t) = y_r(t) - K_d r(t)$$

where y_d is the desired output and K_d is a constant scaling factor (ratio between the desired output and the reference signal)

- The **steady-state tracking error** is defined as:

$$e_r^\infty = \lim_{t \to +\infty} e_r(t)$$





$$G = G_c G_a G_p = \frac{N_G}{D_G} \qquad H = G_s G_f = \frac{N_H}{D_H}$$

Steady-state response to polynomial reference inputs

- *Definition (System type)*
  An LTI feedback control system is a type-h system if and only if its steady-state tracking error due to a polynomial reference input of order h is bounded.

  Examples …

- From now on we assume that $H = 1/K_d$ where K_d is a constant.

  Derive …

Steady-state response to polynomial reference inputs

Let us compute the transfer function between the reference signal and the tracking error:

$$e_r(s) = y_r(s) - K_d r(s) = \frac{G(s)}{1+G(s)H(s)} r(s) - K_d r(s) = \frac{K_d^2}{K_d + G} r(s)$$

$$\Rightarrow G_{re}(s) \equiv \frac{e_r(s)}{r(s)} = \frac{K_d^2 D_G(s)}{K_d D_G(s) + N_G(s)}$$

- *Result (System type and error function)*
  An LTI feedback control system is a type-h system if and only if the transfer function $G_{re}(s)$ has a zero of multiplicity h at $s=0$

- *Result (System type and loop function)*
  An LTI feedback control system is a type-h system if and only if the function $G(s)$ has a pole of multiplicity h at $s=0$

Steady-state response to polynomial reference inputs

- The steady-state tracking error can be computed by means of the **final-value theorem**:

$$e_r^\infty = \lim_{t \to +\infty} e_r(t) = \lim_{s \to 0} s\, e_r(s)$$

- The final-value theorem can be applied if the time-domain limit exists and is bounded, i.e. if all poles of $\{s\, e_r(s)\}$ are in the open left-half of the s-plane, which means that the function $\{s\, e_r(s)\}$ is BIBO stable.

Steady-state response to polynomial reference inputs

- Assume that $\nu + p \geq h$. Application of the final-value theorem leads to:

$$e_r^{\infty} = \lim_{t \to +\infty} e_r(t) = \lim_{s \to 0} s\, e_r(s) = \lim_{s \to 0} s\,(y_r(s) - K_d r(s)) =$$

$$= \lim_{s \to 0} s\, G_{re}(s) r(s) = \lim_{s \to 0} s \frac{K_d^2}{K_d + G(s)} r(s) =$$

$$= \lim_{s \to 0} s \frac{s^{\nu+p} K_d^2}{s^{\nu+p} K_d + K_c K_p G_a} \frac{R_0}{s^{h+1}} = \begin{cases} 0 \leftarrow (\nu + p > h) \\ \dfrac{K_d^2 R_0}{\beta K_d + K_c K_p G_a} \leftarrow (\nu + p = h) \end{cases}$$

where β = 1 (if $\nu + p = 0$) and β = 0 (if $\nu + p > 0$).

Note that the system type is given by $\nu+p$.

Steady-state response to polynomial reference inputs

- Note that the error signal $e_r(t)$ can be seen as the step response of a system with transfer function $G_{re}(s)/s^h$.

- Now let us consider the case $\nu + p < h$. Since $\nu+p$ is the number of zeros at $s=0$ of the function $G_{re}(s)$ it follows that the system described by $G_{re}(s)/s^h$ is NOT BIBO stable.

- Thus, in the case $\nu + p < h$, the time-domain limit is NOT bounded and the final-value theorem can NOT be applied.

- In this case, the steady-state error is unbounded because of instability of $G_{re}(s)/s^h$, i.e.:

$$e_r^\infty = \lim_{t \to +\infty} e_r(t) = \infty \quad (\nu < h - p)$$

Steady-state response to polynomial reference inputs

Input order / System type	Step input (order 0)	Ramp input (order 1)	Parabola input (order 2)
0	$\dfrac{K_d^2 R_0}{K_d + K_p K_c G_a}$	∞	∞
1	0	$\dfrac{K_d^2 R_0}{K_p K_c G_a}$	∞
2	0	0	$\dfrac{K_d^2 R_0}{K_p K_c G_a}$

Steady-state response to polynomial reference inputs

- *Definition (System type)*
  A LTI feedback control system is a type-h system if and only if its steady-state tracking error due to a polynomial reference input of order h is bounded.

- *Result (System type and zero tracking error)*
  The steady-state tracking error of an LTI feedback control system due to a polynomial reference input of order h is zero if and only if the system type is greater than h.

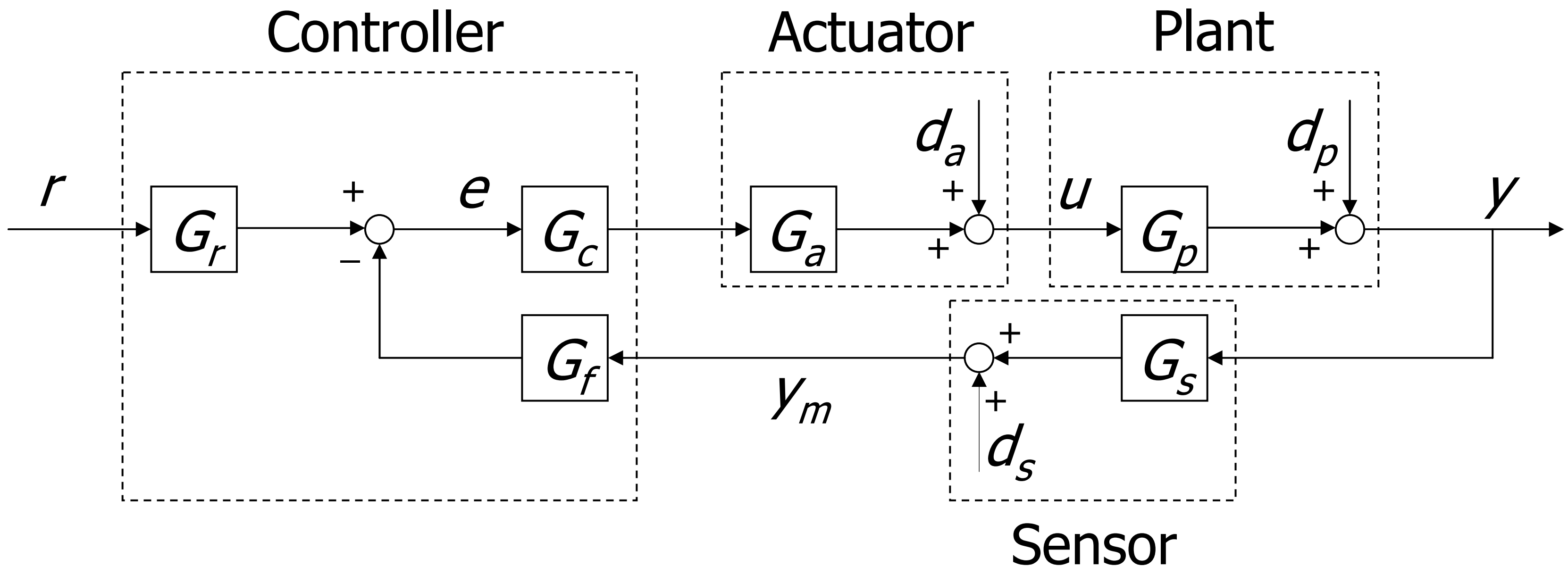
Example: steady-state error in antenna azimuth position control system.

Example: steady-state error in op-amp non-inverting amplifier.

SISO feedback control systems characteristics

Steady-state response to polynomial disturbance inputs

Output feedback control systems configurations



Without loss of generality we assume that

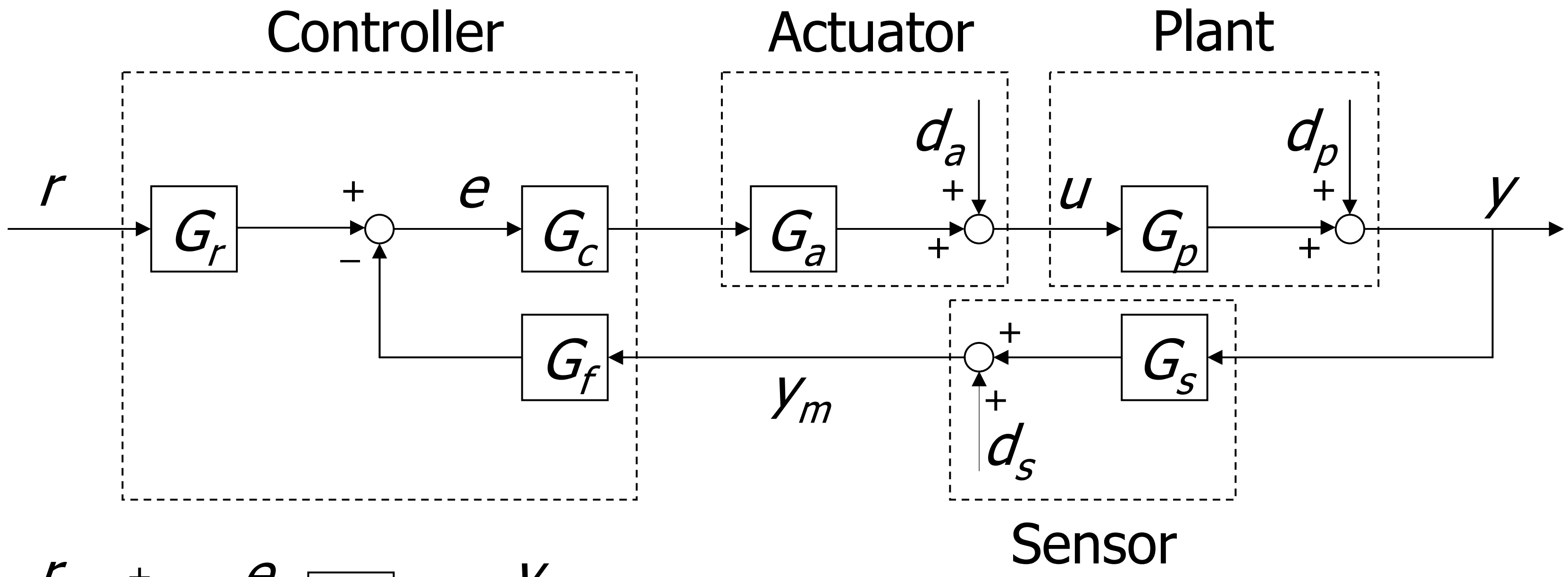
$G_r = 1$

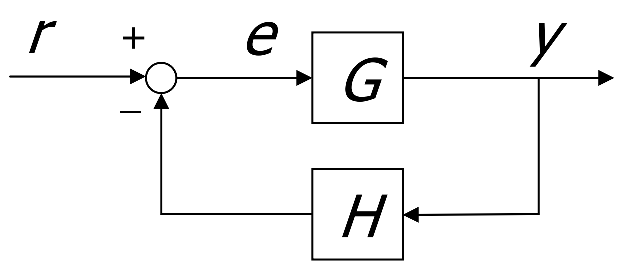
G_a = constant (large bandwidth because of fast dynamics)

G_f = constant (to handle the control system steady-state gain)

G_s = constant (large bandwidth because of fast dynamics)

Output feedback control systems configurations





$$G = G_c G_a G_p = \frac{N_G}{D_G} \qquad H = G_s G_f = \frac{N_H}{D_H}$$

$G_r = 1$
G_a = constant
G_f = constant
G_s = constant

- LTI controllers described by the generic transfer function

$$G_c(s) = \frac{K_c}{s^\nu} \prod_i \left(\frac{1 + \dfrac{s}{z_{di}}}{1 + \dfrac{s}{m_{di} z_{di}}} \right) \prod_j \left(\frac{1 + \dfrac{s}{m_{ij} p_{ij}}}{1 + \dfrac{s}{p_{ij}}} \right)$$

will be considered from now on.

- The controller has ν poles at $s = 0$ and its generalized steady-state gain is defined as:

$$\lim_{s \to 0} s^\nu G_c(s) = K_c$$

- The plant has p poles at $s = 0$ and its generalized steady-state gain is defined as:

$$\lim_{s \to 0} s^p G_p(s) = K_p$$

Steady-state response to polynomial disturbance inputs

- Polynomial disturbance signals of the kind

$$d(t) = D_0 \frac{t^h}{h!}\varepsilon(t) \rightarrow d(s) = D_0 \frac{1}{s^{h+1}}$$

  are here considered. *h* is called the order of the polynomial signal.

- The **output error** due to the generic disturbance *d(t)* is the contribution of the disturbance on the ouput *y(t)*, i.e.:

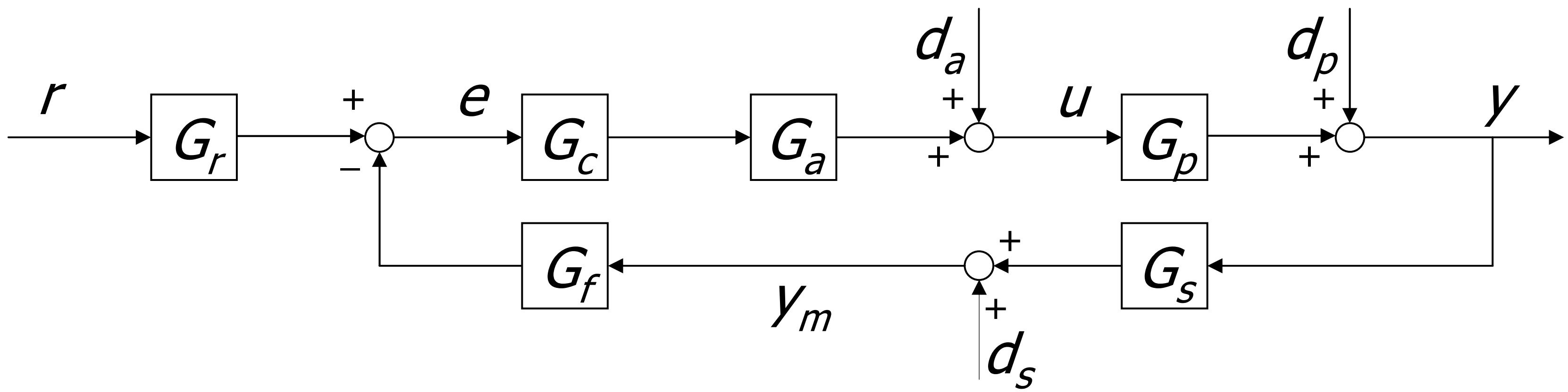
$$e_d(t) = y_d(t)$$

- The **steady-state output error** due to the generic disturbance *d(t)* is defined as:

$$e_d^\infty = \lim_{t\to+\infty} e_d(t)$$

Steady-state response to polynomial disturbance d_p

Let us compute, for example, the output error due to the disturbance d_p :



$$e^{d_p}(s) = y^{d_p}(s) = S(s)d_p(s)$$

- The steady-state output error due to d_p can be computed by means of the **final-value theorem** (can be applied if and only if the time-domain limit exists and is bounded).

Steady-state response to disturbance d_p

- Assume that $\nu + p \geq h$. Assume that G_f and G_s are constant gain.
- Application of the final-value theorem leads to:

$$e_\infty^{d_p} = \lim_{t \to +\infty} e^{d_p}(t) = \lim_{s \to 0} s e^{d_p}(s) = \lim_{s \to 0} s y^{d_p}(s) =$$

$$= \lim_{s \to 0} s S(s) d_p(s) = \lim_{s \to 0} s \frac{1}{1 + L(s)} d_p(s) =$$

$$= \lim_{s \to 0} s \frac{s^{\nu+p}}{s^{\nu+p} + K_c K_p G_a G_f G_s} \frac{D_{p0}}{s^{h+1}} = \begin{cases} 0 \leftarrow (\nu + p > h) \\ \dfrac{D_{p0}}{\beta + K_c K_p G_a G_f G_s} \leftarrow (\nu + p = h) \end{cases}$$

where $\beta = 1$ (for $\nu + p = 0$) and $\beta = 0$ ($\nu + p > 0$).

Steady-state response to disturbance d_p

- Note that the error signal $e^d(t)$ can be interpreted as the step response of a system with transfer function $S(s)/s^h$.

- Now let us consider the case $\nu + p < h$. Since $\nu+p$ is the number of zeros at $s=0$ of the function $S(s)$ it follows that the system described by $S(s)/s^h$ is NOT BIBO stable.

- Thus, in the case $\nu + p < h$, the time-domain limit is NOT bounded and the final-value theorem can NOT be applied.

- The steady-state error is unbounded, due to instability of $S(s)/s^h$, i.e.:

$$e_\infty^{d_p} = \lim_{t \to +\infty} e^{d_p}(t) = \infty \qquad (\nu + p < h)$$

SISO feedback control systems characteristics

Frequency response of feedback control systems

frequency response of a control system

- Before we address the problem of deriving the response of a feedback control system to the class of sinusoidal disturbances, we introduce the frequency response of a control system. More precisely:

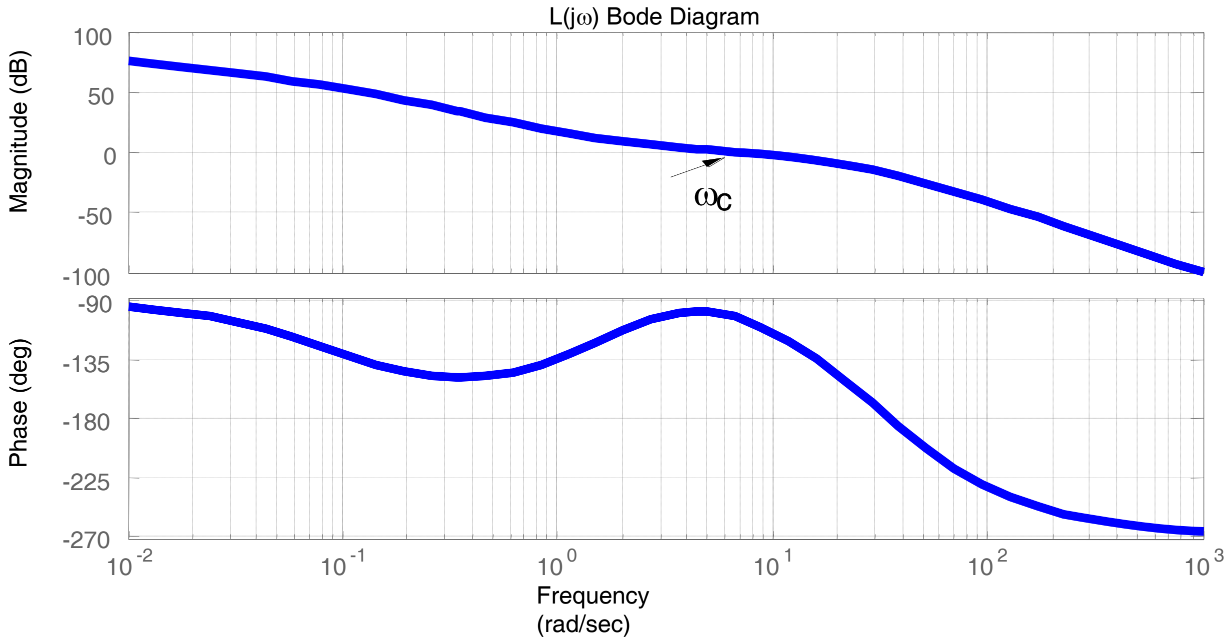
  - the frequency response $L(j\omega)$ of the **loop function** $L(s)$

  - the frequency response $S(j\omega)$ of the **sensitivity function** $S(s) = [1+L(s)]^{-1}$

  - the frequency response $T(j\omega)$ of the **complementary sensitivity function** $T(s) = 1-S(s)$

Frequency response of the loop function

A typical plot of $L(j\omega)$ is reported below



The main parameter is the **crossover frequency ω_c**:

$$\omega_c : |L(j\omega_c)| = 1 \quad (|L(j\omega_c)|_{dB} = 0)$$

The crossover frequency ω_c

In a control system, the **crossover frequency ω_c** of the loop function discriminates the low and high frequency ranges:

$\omega \ll \omega_c$ → **low frequency** range

$\omega \gg \omega_c$ → **high frequency** range

$\omega \approx \omega_c$ → **medium frequency** range

Low frequency range, $\omega \ll \omega_c$, typically $|L(j\omega)| \gg 1$

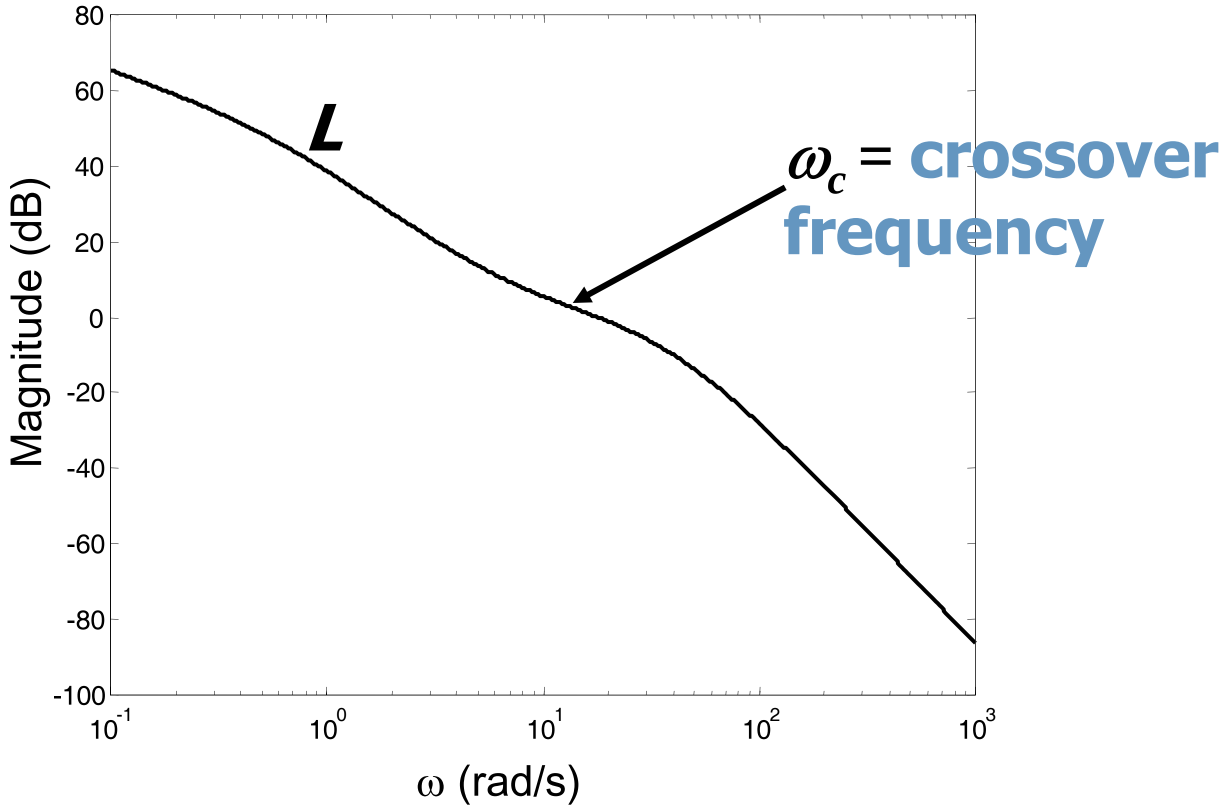
High frequency range, $\omega \gg \omega_c$, typically $|L(j\omega)| \ll 1$

Phase behavior of $L(j\omega)$ ($\angle L(j\omega)$) at medium frequency determines the stability characteristics of the closed loop system.

Approssimate relations among $L(j\omega)$, $T(j\omega)$, $S(j\omega)$

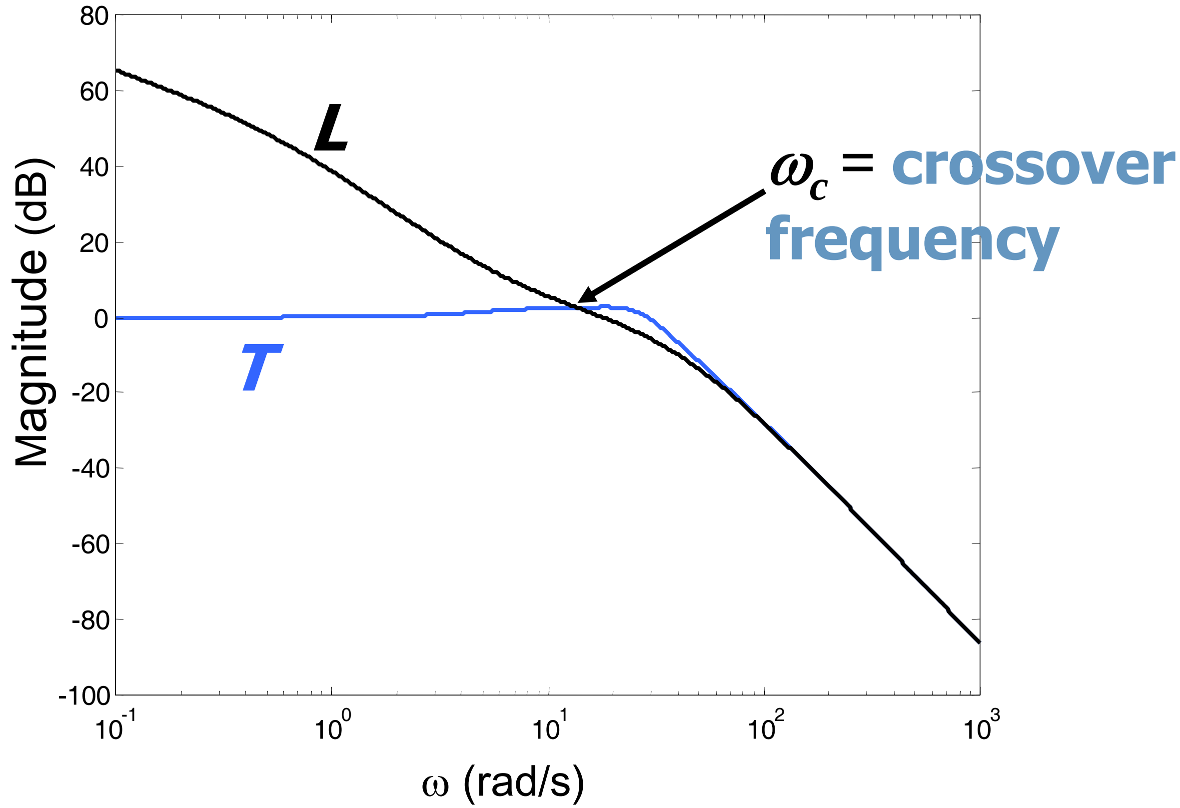
Typical plot of $|L(j\omega)|$:

- low frequency ($\omega << \omega_c$), $|L(j\omega)| >> 1$
- high frequency ($\omega >> \omega_c$), $|L(j\omega)| << 1$



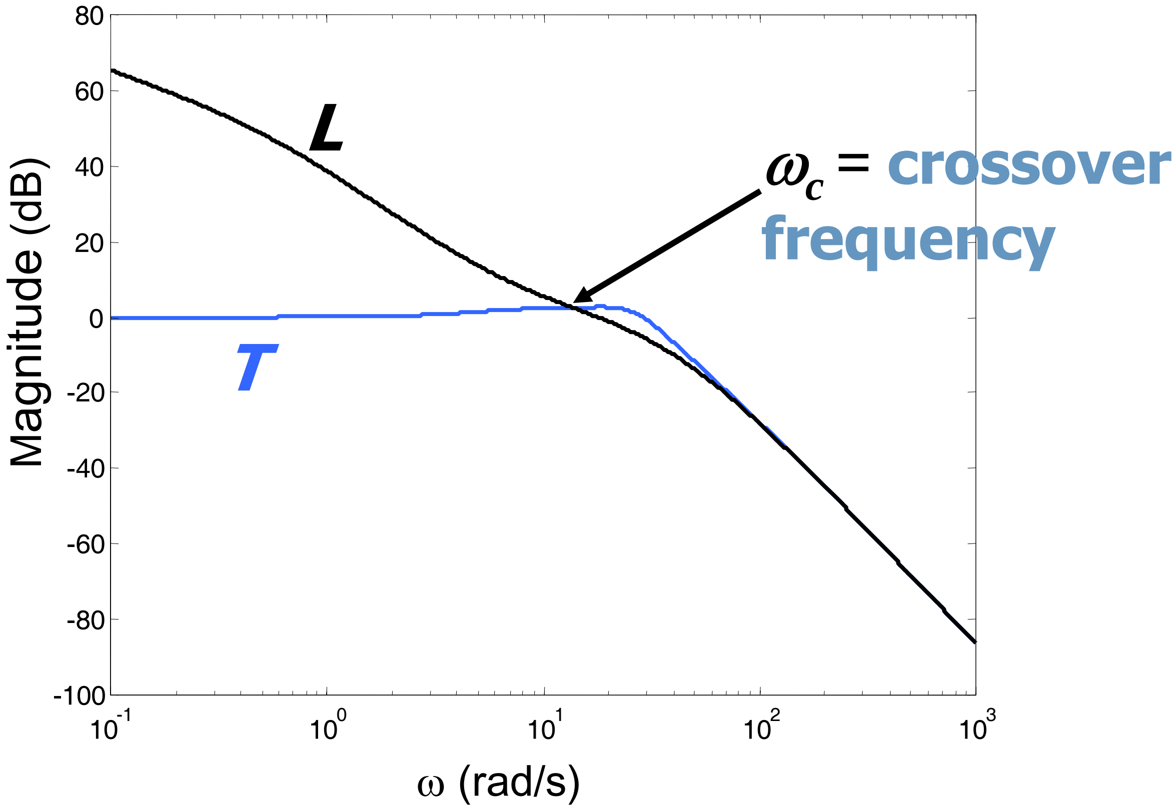
Approssimate relations between $L(j\omega)$ and $T(j\omega)$

$$\underset{\omega \ll \omega_c}{\text{low frequency}} \rightarrow \quad |T(j\omega)| = \left| \frac{L(j\omega)}{1 + L(j\omega)} \right| \underset{|L(j\omega)|>>1}{\overset{\uparrow}{\approx}} 1$$



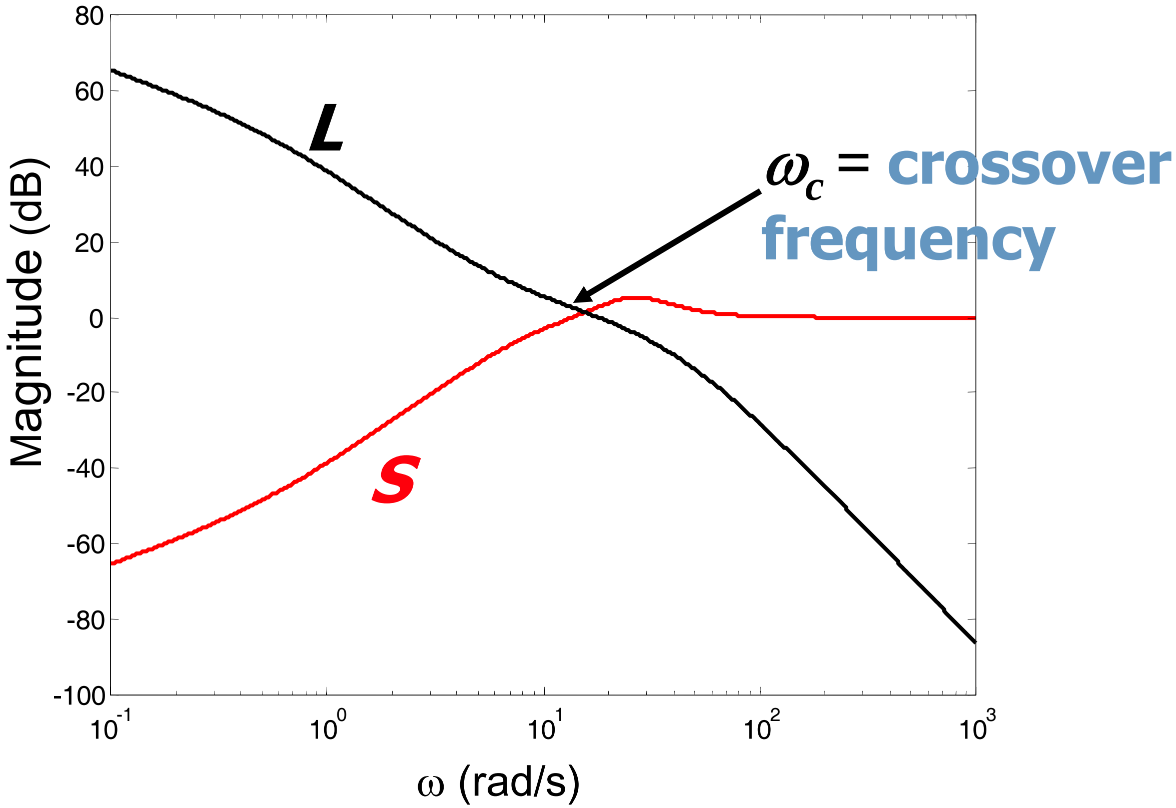
Approssimate relations between $L(j\omega)$ and $T(j\omega)$

$$\text{high frequency}_{\omega \gg \omega_c} \rightarrow |T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \underset{|L(j\omega)|<<1}{\underset{\uparrow}{\approx}} |L(j\omega)|$$



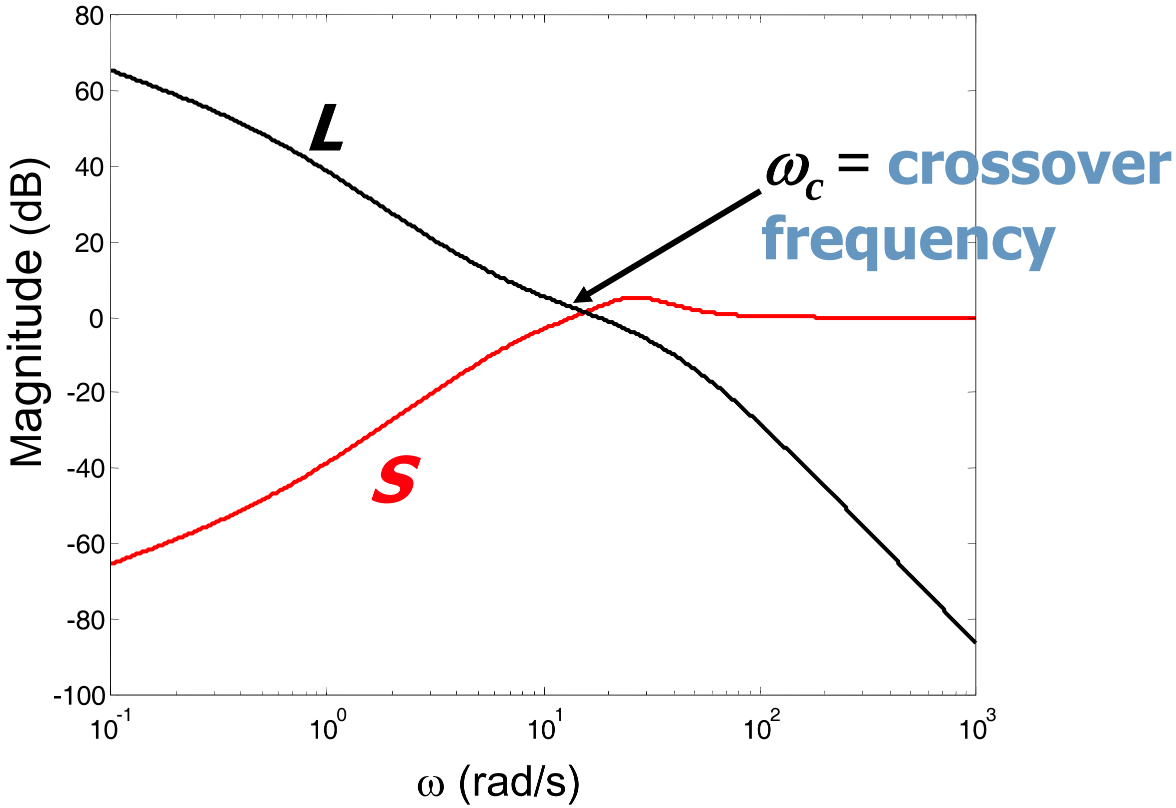
Approssimate relations between $L(j\omega)$ and $S(j\omega)$

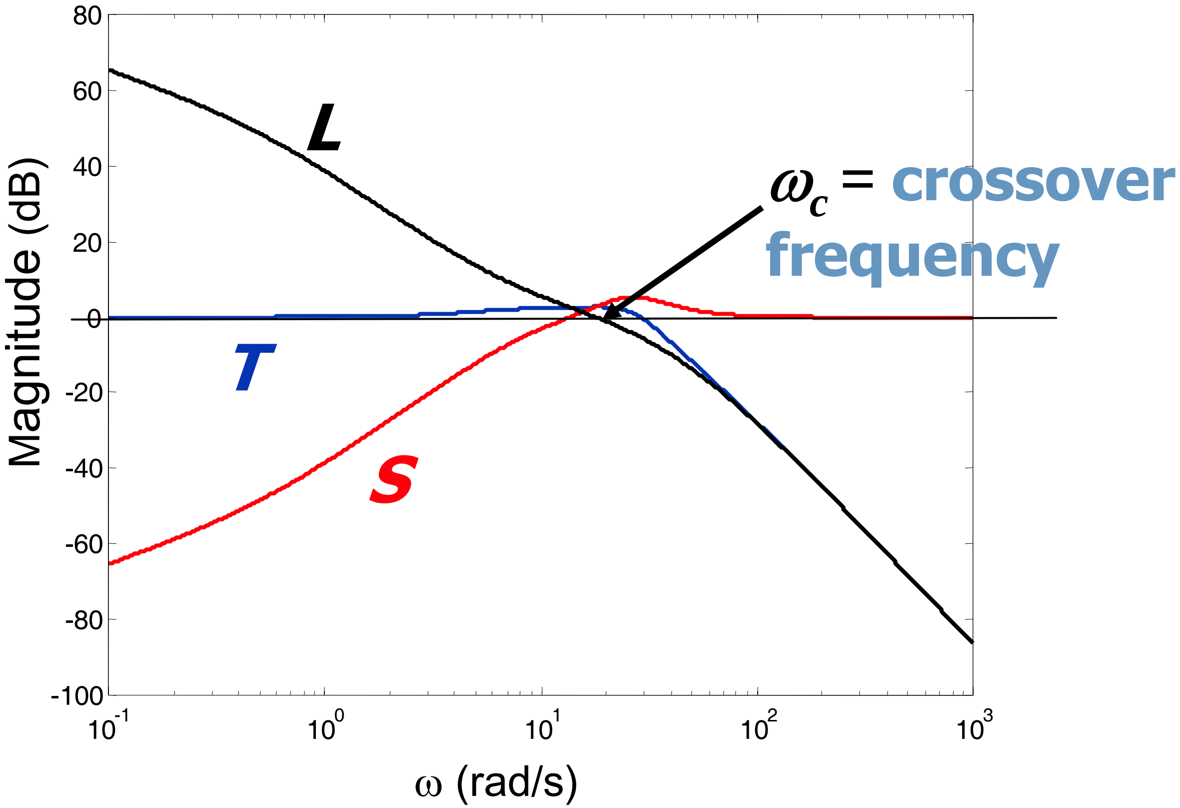
$$\text{low frequency}_{\omega \ll \omega_c} \rightarrow |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \underset{|L(j\omega)| >> 1}{\underset{\uparrow}{\approx}} \frac{1}{|L(j\omega)|}$$



Approssimate relations between $L(j\omega)$ and $S(j\omega)$

$$\text{high frequency}_{\omega \gg \omega_c} \rightarrow |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \underset{|L(j\omega)|<<1}{\overset{\uparrow}{\approx}} 1$$




80
60
40
20
0
-20
-40
-60
-80
-100
Magnitude (dB)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω (rad/s)
L
T
S
ω_c = crossover frequency

SISO feedback control systems characteristics

Steady-state response to sinusoidal disturbance inputs

Steady-state response to sinusoidal disturbances

- Here the problem of attenuating the effect of the sensor noise d_s is considered.

- The focus is restricted to the class of sinusoidal signals, i.e.:

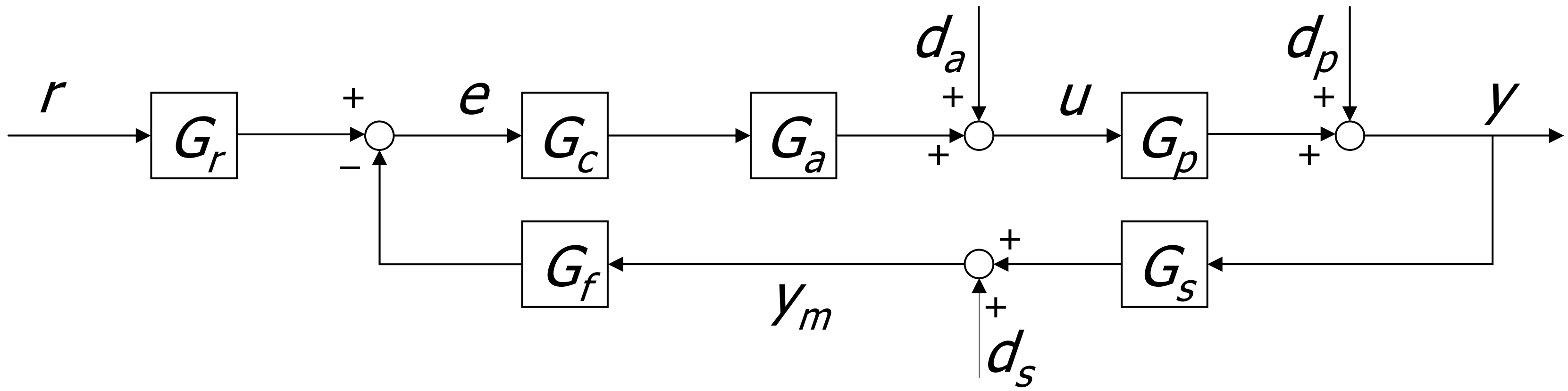
$$d_s = a_s \sin(\omega_s t) \quad \forall \omega_s \geq \omega_s^- \qquad \text{given } a_s \text{ and } \omega_s^-$$

- The output error at steady-state is required to be bounded by a given constant:

$$\left| e_{d_s}^{\infty} \right| = \left| y_{d_s}^{\infty} \right| \leq \rho_s \qquad \text{given } \rho_s > 0$$

Design constraints on $T(j\omega)$ due to d_s

- It is easily shown that such a specification leads to a frequency domain constraint on the complementary sensitivity function $T(s)$



$$\left|e_{d_s}^{\infty}\right| = \left|y_{d_s}^{\infty}\right| = \left|a_s \left|T(j\omega_s)\right| \frac{1}{G_s} \sin(\omega_s t + \varphi_s)\right| \leq a_s \left|T(j\omega_s)\right| \frac{1}{G_s} \leq \rho_s \Rightarrow$$

$$\Rightarrow \left|T(j\omega_s)\right| \leq \frac{\rho_s G_s}{a_s} \qquad \forall \omega_s \geq \omega_s^-$$

Design constraints on the loop function due to d_s

- A design constraint on $L(j\omega)$ is obtained by considering that the inequality derived in the previous slide requires that $|T(j\omega_s)|$ is small for $\forall \omega_s \geq \omega_s^-$.

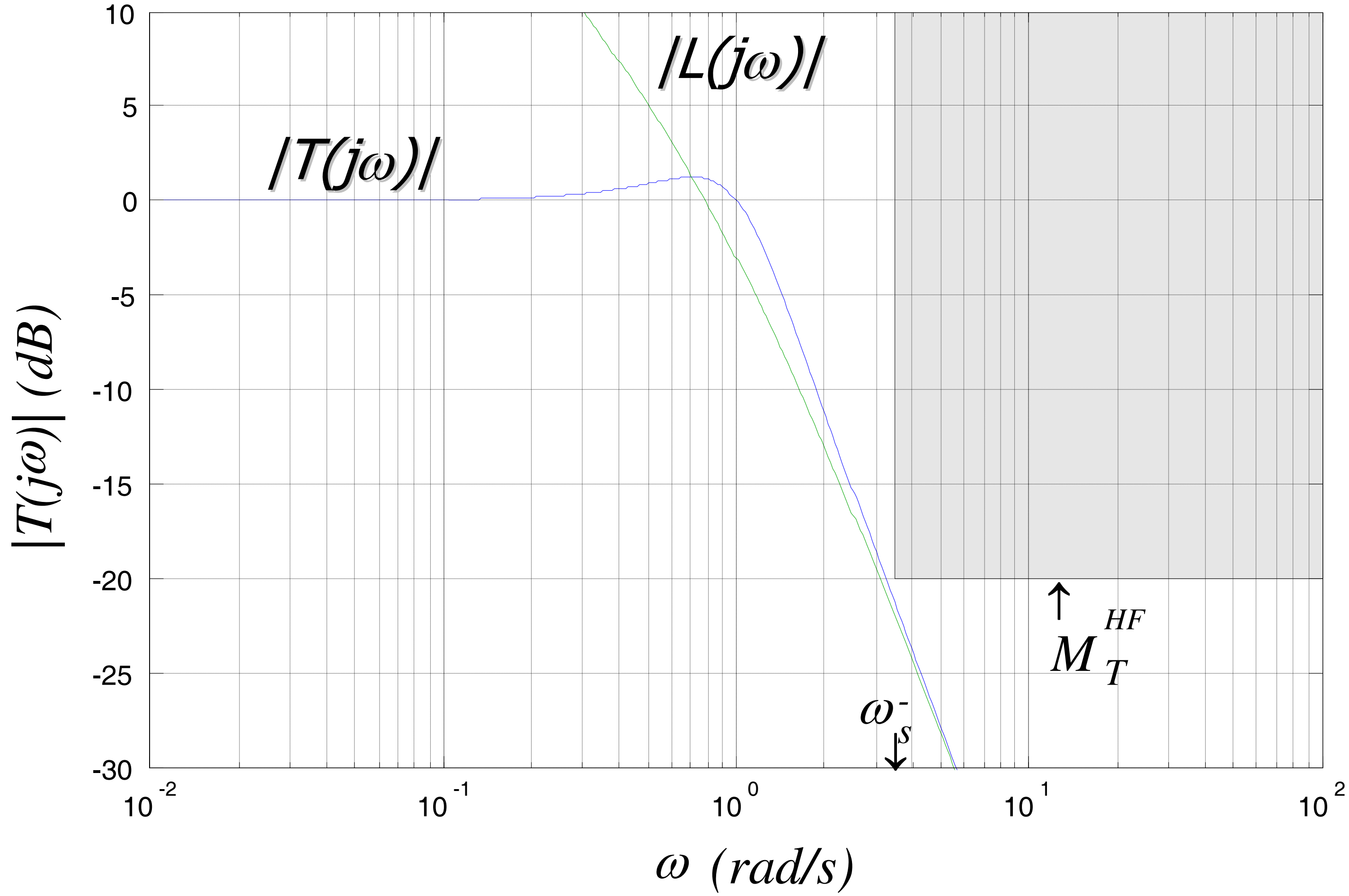
- Since

$$|T(j\omega)| \approx |L(j\omega)| \quad \text{for} \quad |L(j\omega)| << 1$$

we get:

$$\left|e_{d_s}^{\infty}\right| \leq \rho_s \Rightarrow |L(j\omega_s)| \leq \frac{\rho_s G_s}{a_s} = M_T^{HF} \qquad \forall \omega_s \geq \omega_s^-$$

Design constraints on the loop function due to d_s



Steady-state response to sinusoidal disturbances

- Here the problem of attenuating the effect of the output disturbance d_p is considered.

- The focus is restricted to the class of sinusoidal signals, i.e.:

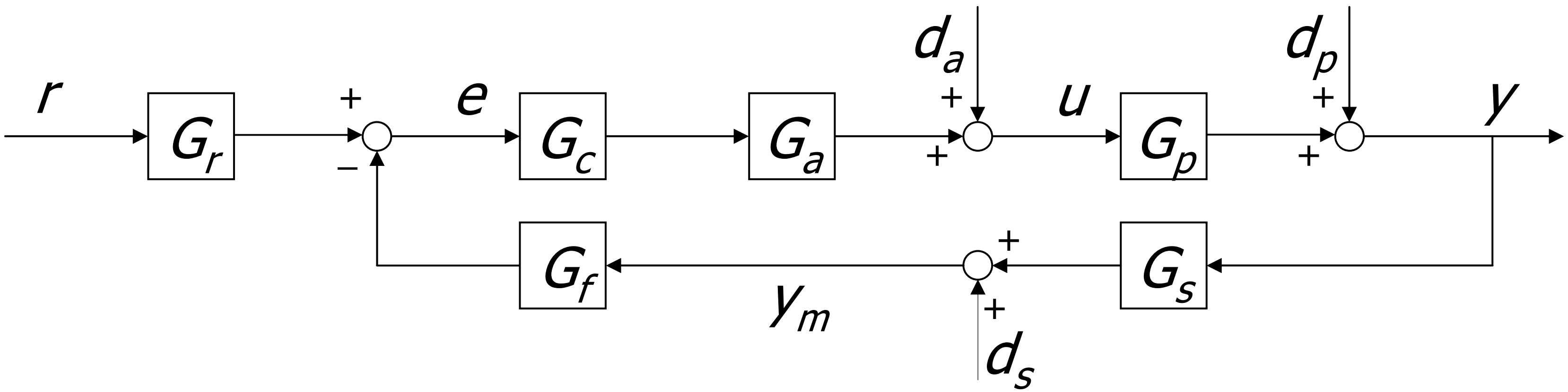
$$d_p = a_p \sin(\omega_p t) \quad \forall \omega_p \le \omega_p^+ \qquad \text{given } a_p \text{ and } \omega_p^+$$

- The output error at steady-state is required to be bounded by a given constant:

$$\left|e_{d_p}^\infty\right| = \left|y_{d_p}^\infty\right| \le \rho_p \qquad \text{given } \rho_p > 0$$

Design constraints on *S(jω)* due to d_p

- It is easily shown that such a specification leads to a frequency domain constraint on the sensitivity function *S(s)*:



$$\left|e_{d_p}^{\infty}\right| = \left|y_{d_p}^{\infty}\right| = \left|a_p \left|S(j\omega_p)\right| \sin(\omega_p t + \varphi_p)\right| \le a_p \left|S(j\omega_p)\right| \le \rho_p \Rightarrow$$

$$\Rightarrow \left|S(j\omega_p)\right| \le \frac{\rho_p}{a_p} = M_S^{LF} \qquad \forall \omega_p \le \omega_p^{+}$$

Design constraints on the loop function due to d_p

- A design constraint on $L(j\omega)$ is obtained by considering that the inequality derived in the previous slide requires that $|S(j\omega_t)|$ is small for $\forall\omega_p \leq \omega_p^+$.

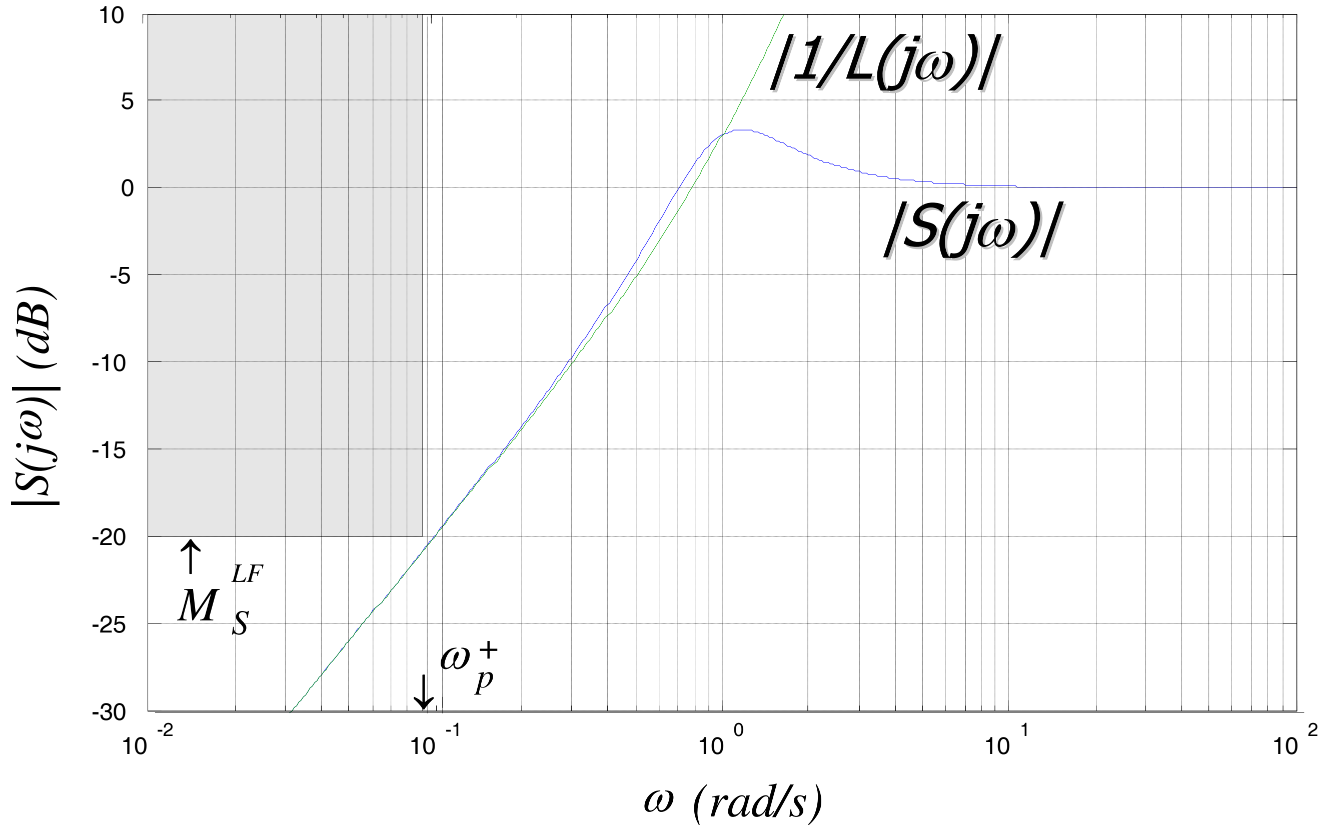
- Since

$$\left|S(j\omega)\right| \approx \left|\frac{1}{L(j\omega)}\right| \quad for \quad \left|S(j\omega)\right| << 1$$

we get:

$$\left|e_{d_p}^{\infty}\right| \leq \rho_p \Rightarrow \left|L(j\omega_p)\right| \geq \frac{a_p}{\rho_p} \quad \forall\omega_p \leq \omega_p^+$$

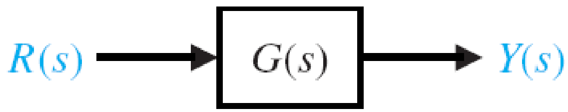
Design constraints on the loop function due to d_p



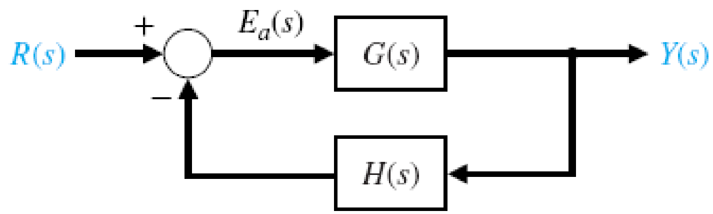
SISO feedback control systems characteristics

Sensitivity of control systems to parameter variations

Sensitivity of control systems to parameter variations



Open-loop system



Closed-loop system

A process, represented by $G(s)$, whatever its nature, is subject to a changing environment, aging, ignorance of the exact values of the process parameters, and the natural factors that affect a control process.

For open-loop systems, all these errors and changes will affect the output (parameter variations directly affect the output).

For closed-loop systems, the change in the output is sensed and fed back such that the effect of the unwanted change is reduced.

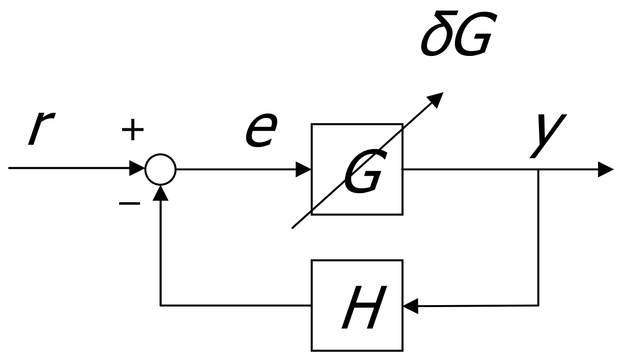
Sensitivity to parameter variations in G

System sensitivity of the control system to parameter variations in G, is defined as the ratio of the percentage change in the system transfer function to the percentage change of G

$$S_G^{G_{ry}} = \frac{\partial G_{ry} / G_{ry}}{\partial G / G} \qquad G_{ry}(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$S_G^{G_{ry}} = \frac{\partial G_{ry}}{\partial G}\frac{G}{G_{ry}} = \frac{1}{(1+GH)^2}\frac{G}{G/(1+GH)}$$

$$S_G^{G_{ry}} = \frac{1}{1 + G(s)H(s)}$$



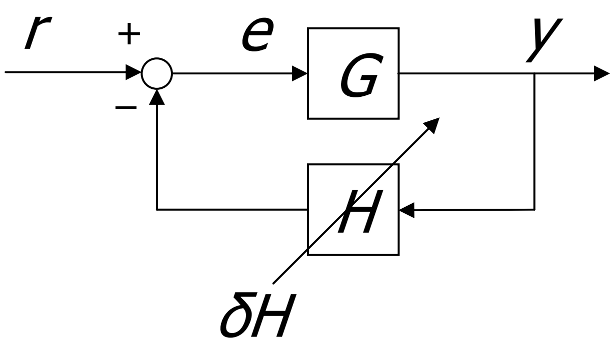
Sensitivity to parameter variations in H

System sensitivity of the control system to parameter variations in H, is defined as the ratio of the percentage change in the system transfer function to the percentage change of H

$$S_H^{G_{ry}} = \frac{\partial G_{ry} / G_{ry}}{\partial H / H} \qquad G_{ry}(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$S_H^{G_{ry}} = \frac{\partial G_{ry}}{\partial H}\frac{H}{G_{ry}} = -\frac{G^2}{(1+GH)^2}\frac{H}{G/(1+GH)}$$

$$S_G^{G_{ry}} = -\frac{G(s)H(s)}{1+G(s)H(s)}$$

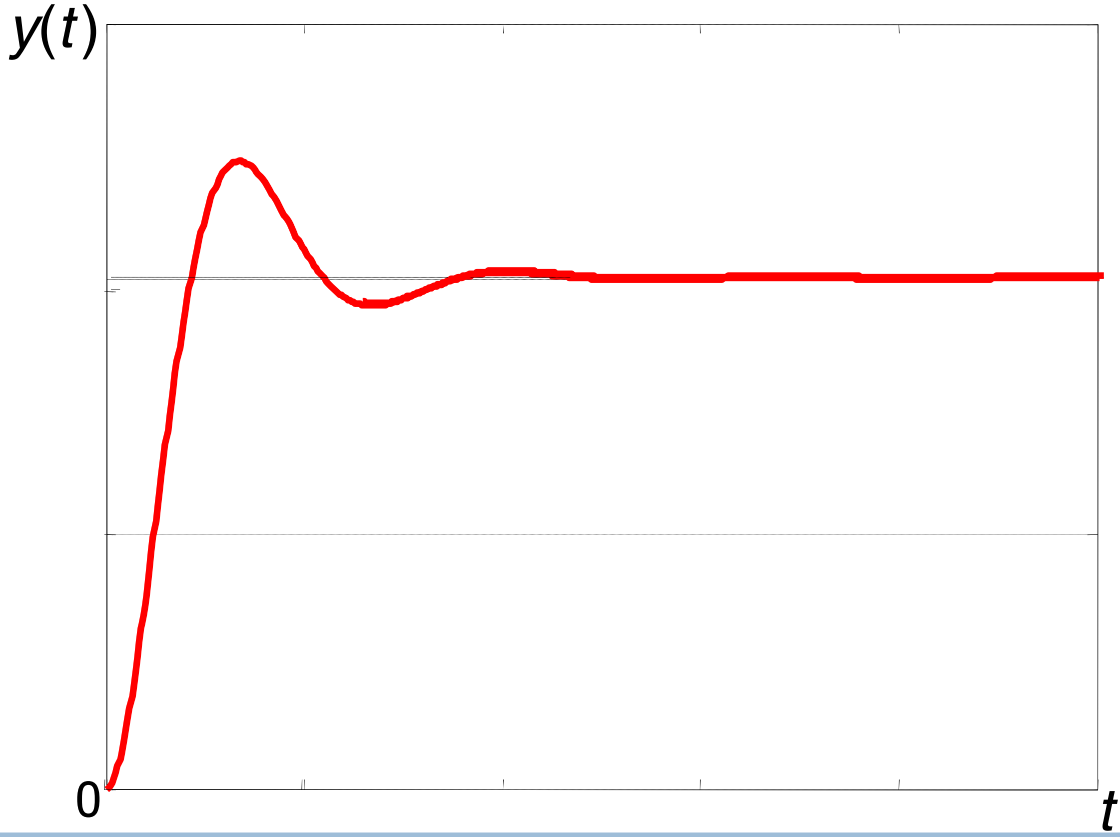


SISO feedback control systems characteristics

General feedback control systems transient response to step input

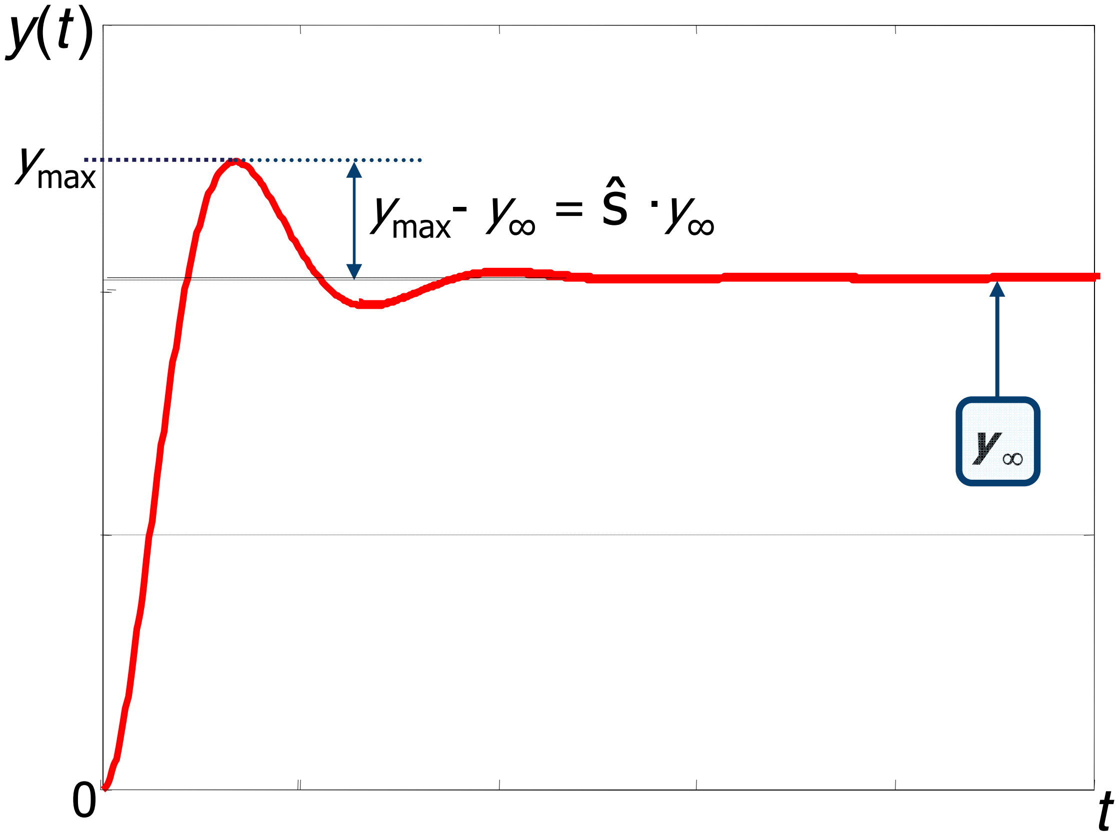
Step response of a control system

- In order to define the transient response requirements we consider the output response of the controlled system to a step input. A typical plot of such a response is reported below



Maximum overshoot

- The **maximum overshoot $\hat{s}$** is defined as $$\hat{s} = \frac{y_{max} - y_{\infty}}{y_{\infty}}$$



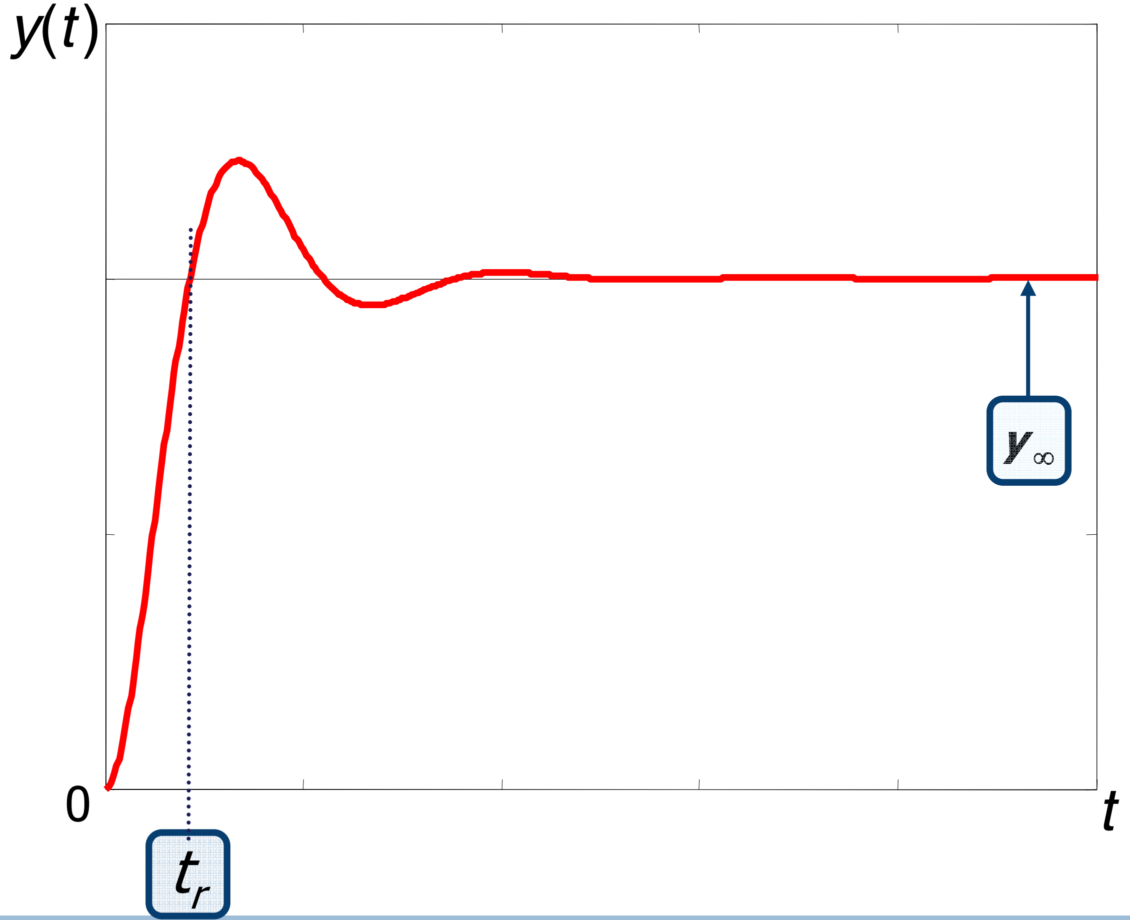
- The quantity $\hat{s}$ can be also expressed in percentual terms $\hat{s}_{\%}$

$$\hat{s}_{\%} = 100 \cdot \hat{s}$$

- In practice, the same symbol $\hat{s}$ is used to indicate $\hat{s}_{\%}$

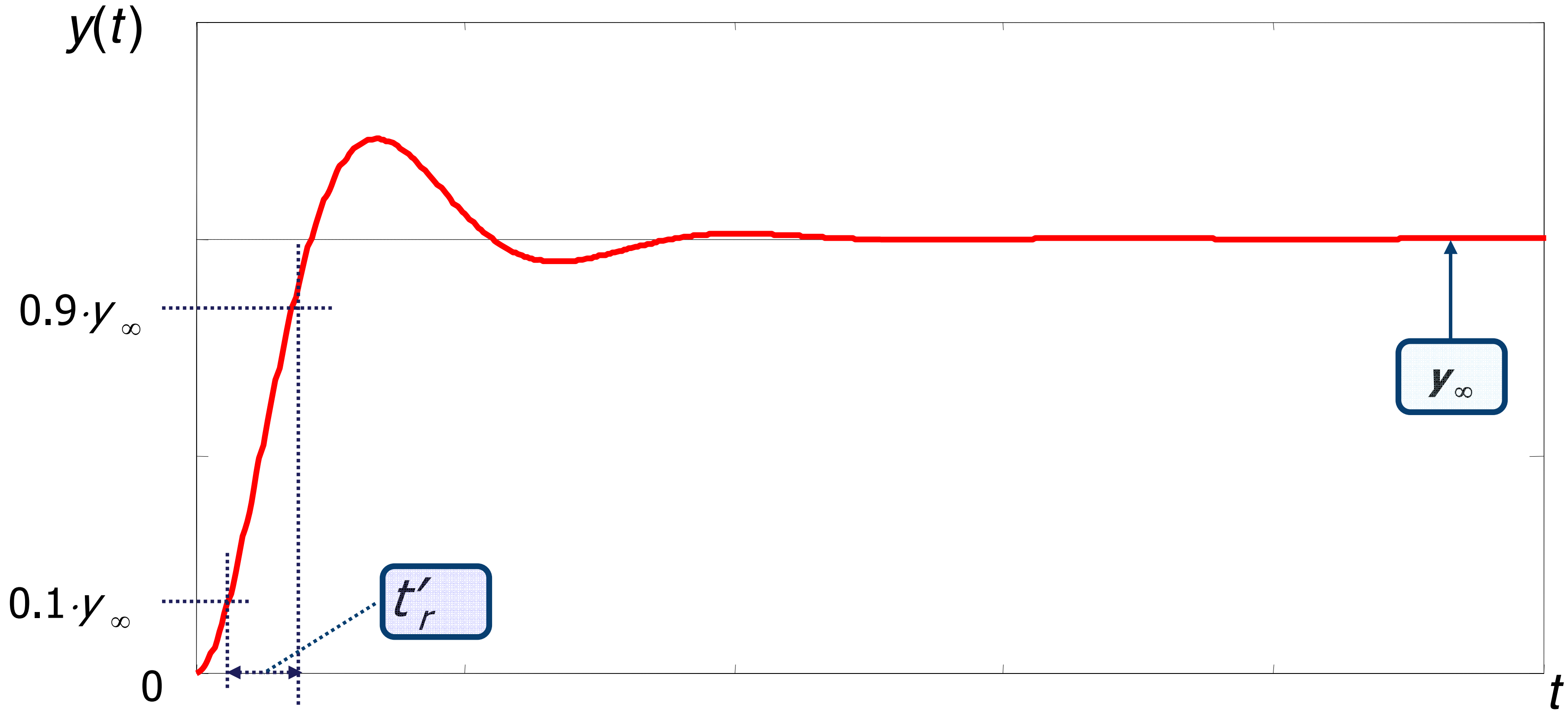
Rise time

- The **rise time** t_r is the time required for the response to rise from 0% to 100% of its final value (steady state value → $y = y_\infty$)

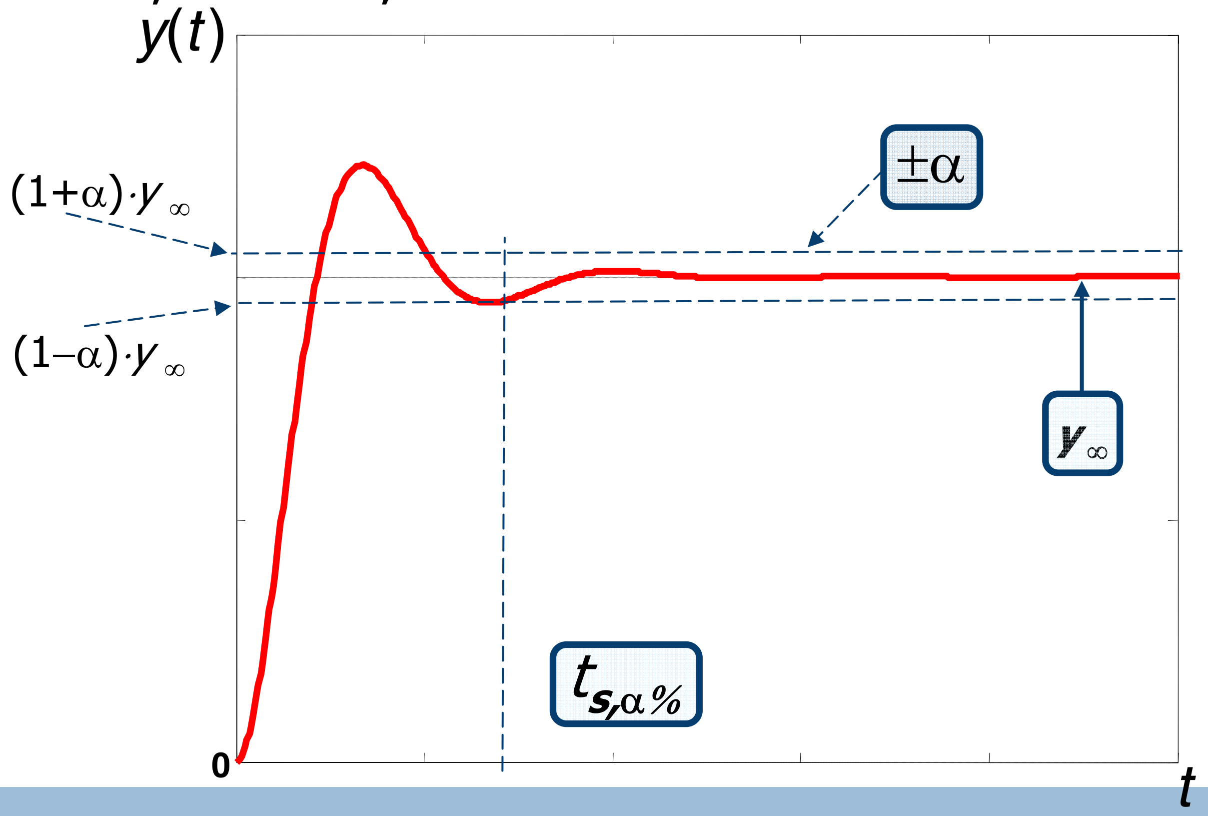


10%-90% rise time

- The **10% ÷ 90% rise time t'_r** is the time required for the response to rise from 10% to 90% of its final value (steady state value → $y = y_\infty$)



- The **settling time** $\pm\,\alpha$ ***100 %,** $t_{s,\alpha\%}$ is the amount of time required for the step response to reach and stay within a range about $\pm\ \alpha*100$ % of the steady-state value y_∞. Typical values of α are: α=0.01, α=0.02, α=0.05.



Transient time response requirements

- The following indices are used to describe the transient performance of a control system:
  - Maximum overshoot $\hat{S}$
  - Rise time t_r
  - Settling time $t_{a,\alpha\%}$

- Transient performance requirements can be expressed as inequalities of the form

$$\hat{s} \leq \bar{\hat{s}} \quad t_r \leq \bar{t}_r \quad t_{s,\alpha\%} \leq \bar{t}_{s,\alpha\%}$$

- Example

$$\hat{s} \leq 10\% \quad t_r \leq 0.5\, s \quad t_{s,1\%} \leq 1.5\, s$$

SISO feedback control systems characteristics

Prototype 2° order system transient response to step input

Step response of prototype 2nd order control system

For a prototype 2nd order control system of the form:

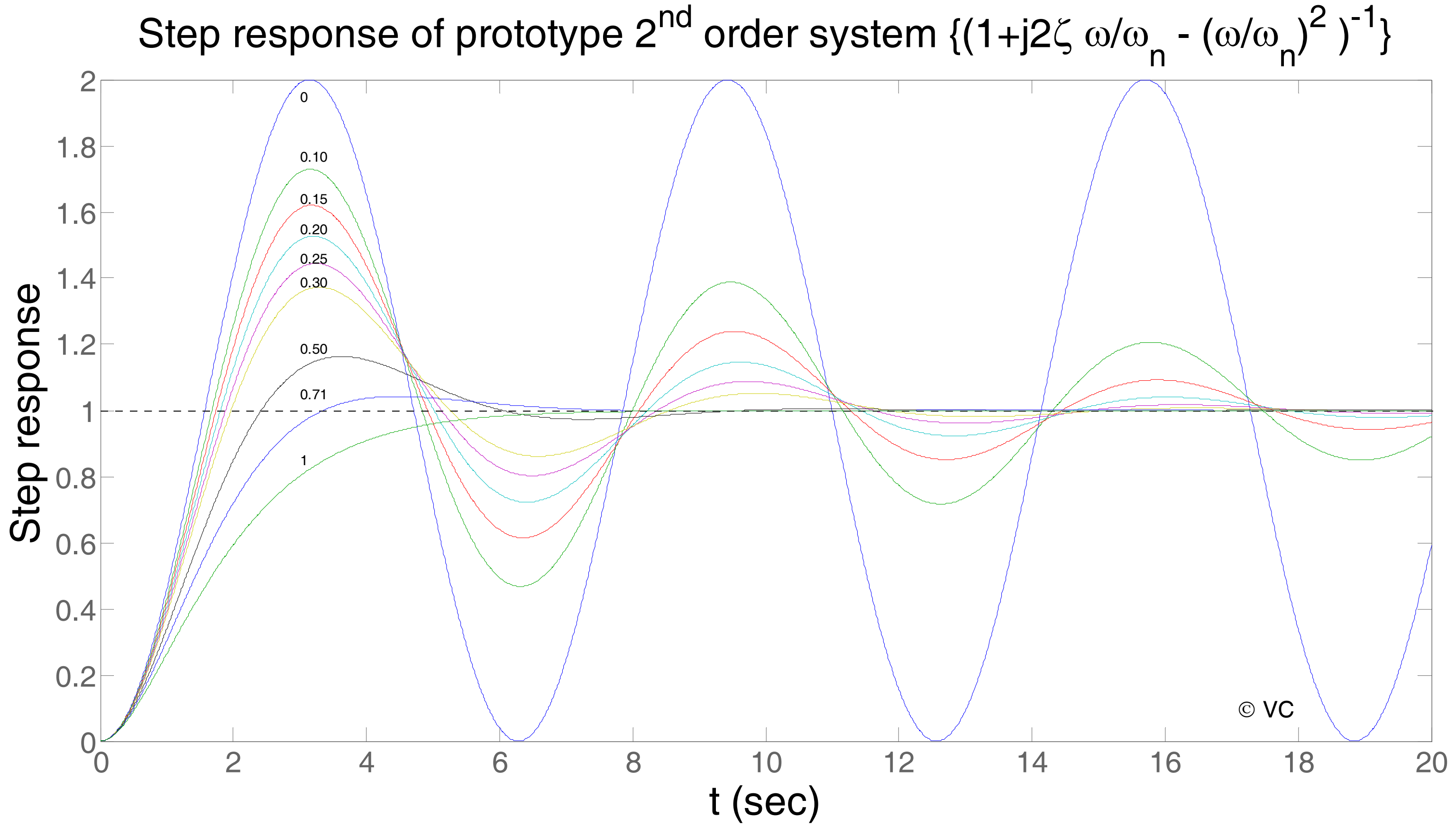
$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

the unit step response is given by

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\omega_n t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right], t \geq 0$$

If **$0 < \zeta < 1$** (complex poles): The system is **underdamped**
If **$\zeta = 1$** (equal real poles): the system is **critically damped**

Step response of prototype 2nd order control system



An underdamped system with $0.5 < \zeta < 0.8$ gets close to the final value more rapidly than a critically damped system.

Step response of prototype 2nd order control system

- For a prototype 2nd order control system of the form:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

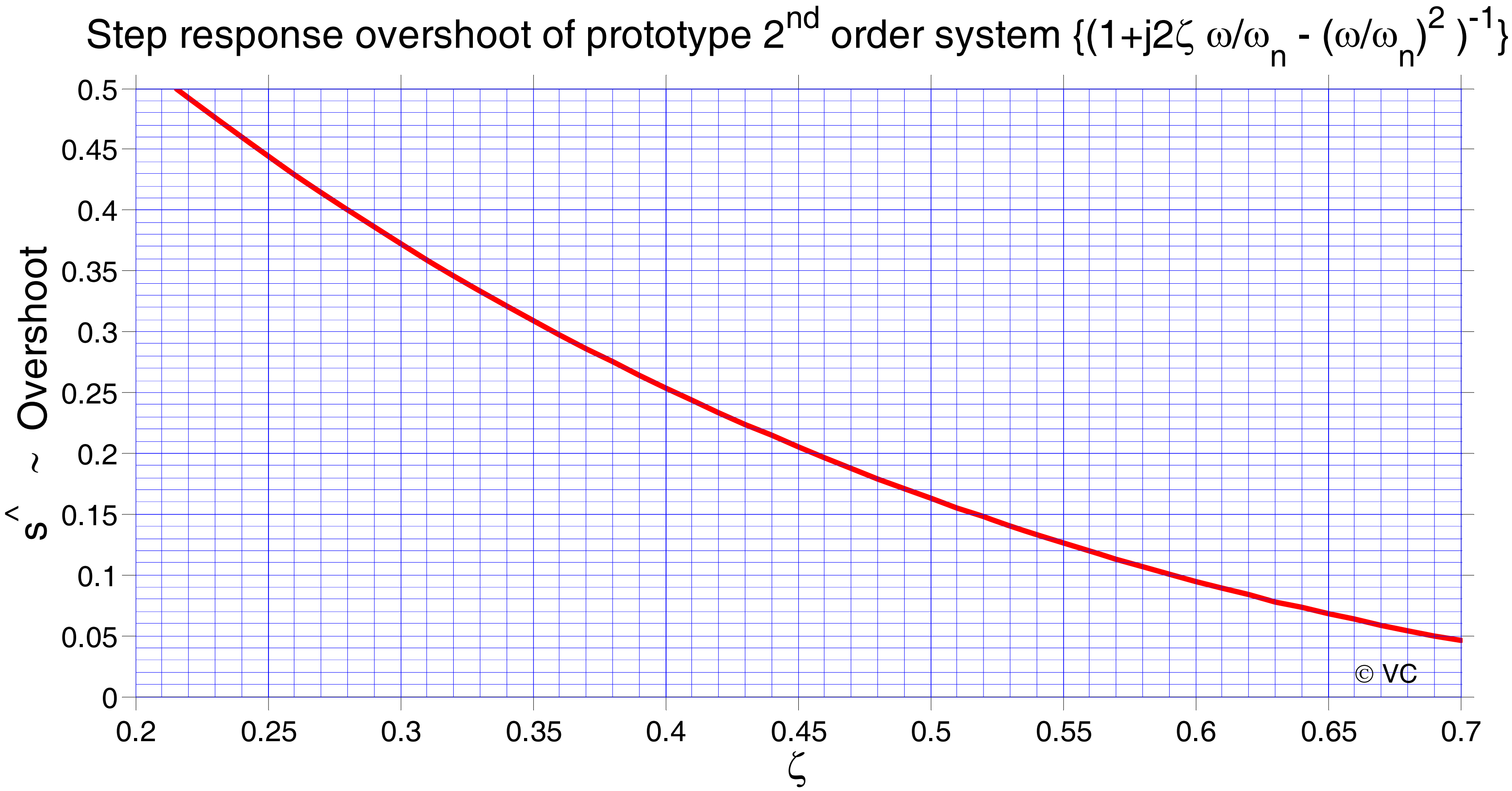
the transient response indices $\hat{s}$, t_r e $t_{s,\alpha\%}$ can be expressed as functions of parameters ζ ed ω_n:

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = f_{\hat{s}}(\zeta)$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right) = f_{t_r}(\zeta, \omega_n)$$

$$t_{s,\alpha\%} = -\frac{\ln\alpha}{\omega_n\zeta} = f_{t_s}(\zeta, \omega_n, \alpha)$$

Overshoot of prototype 2nd order control system



SISO feedback control systems characteristics

General feedback control systems frequency response

frequency response of a control system

- In order to define the frequency response requirements of a control system we consider:

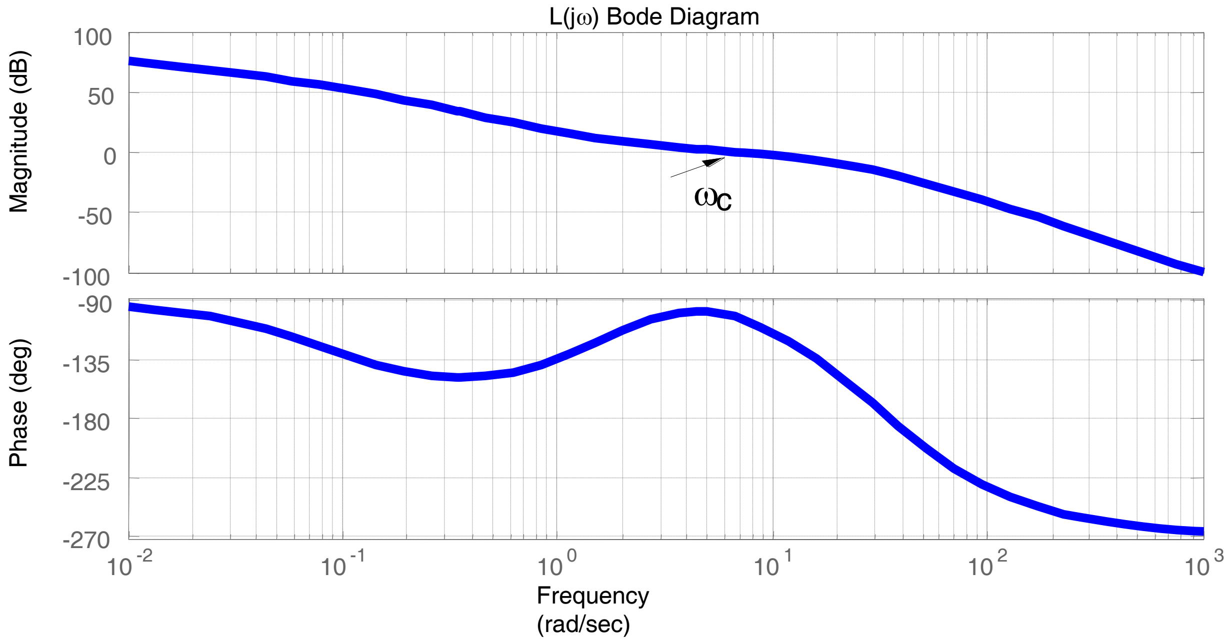
  - the frequency response $L(j\omega)$ of the **loop function** $L(s)$

  - the frequency response $S(j\omega)$ of the **sensitivity function** $S(s) = [1+L(s)]^{-1}$

  - the frequency response $T(j\omega)$ of the **complementary sensitivity function** $T(s) = 1-S(s)$

Frequency response of the loop function

A typical plot of $L(j\omega)$ is reported below



The main parameter is the **crossover frequency ω_c:**

$$\omega_c : |L(j\omega_c)| = 1 \quad (|L(j\omega_c)|_{dB} = 0)$$

In a control system, the **crossover frequency ω_c** of the loop function discriminates the low and high frequency ranges:

$\omega << \omega_c$ → **low frequency** range

$\omega >> \omega_c$ → **high frequency** range

$\omega \approx \omega_c$ → **medium frequency** range

Low frequency range, $\omega << \omega_c$, typically $|L(j\omega)| >> 1$

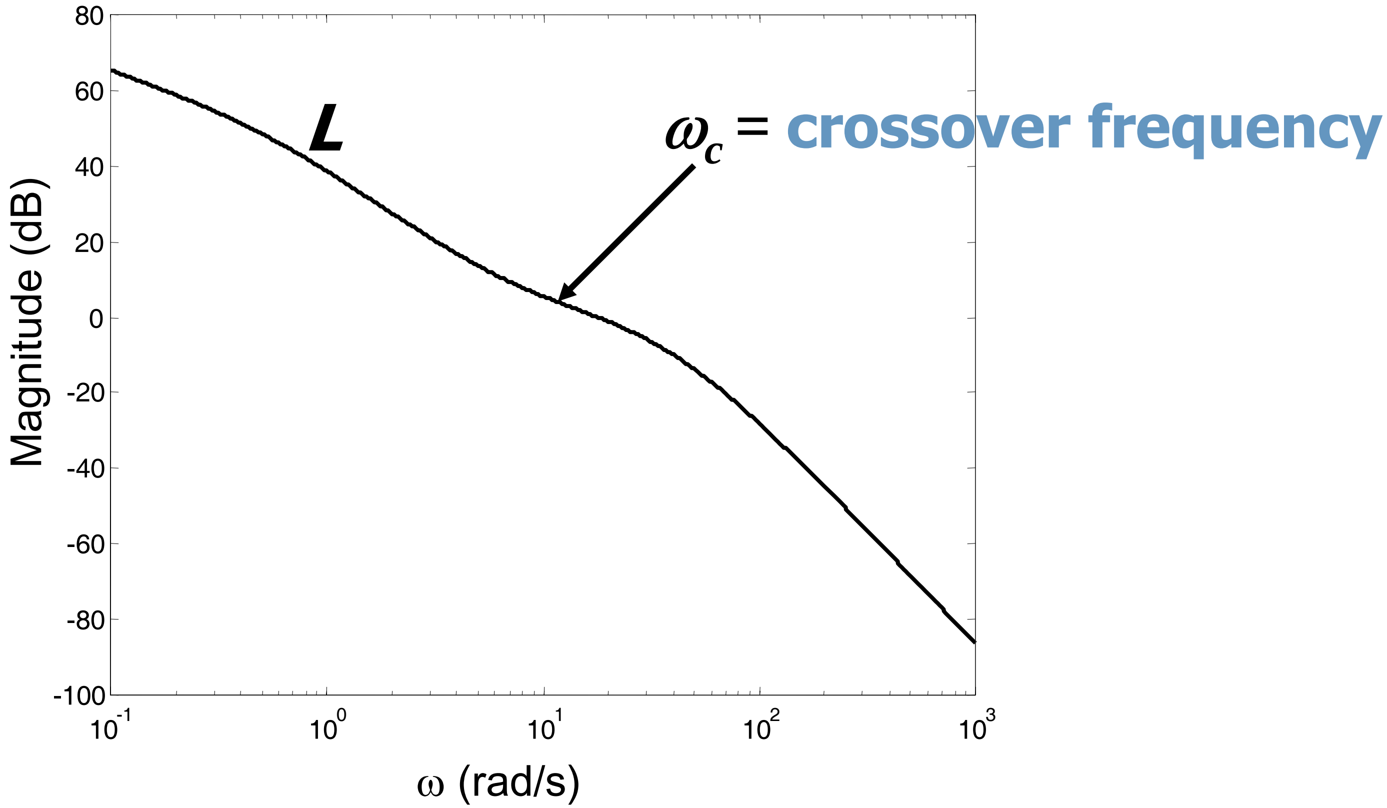
High frequency range, $\omega >> \omega_c$, typically $|L(j\omega)| << 1$

Phase behavior of $L(j\omega)$ ($\angle L(j\omega)$) at medium frequency determines the stability characteristics of the closed loop system

Approssimate relations among $L(j\omega)$, $T(j\omega)$, $S(j\omega)$

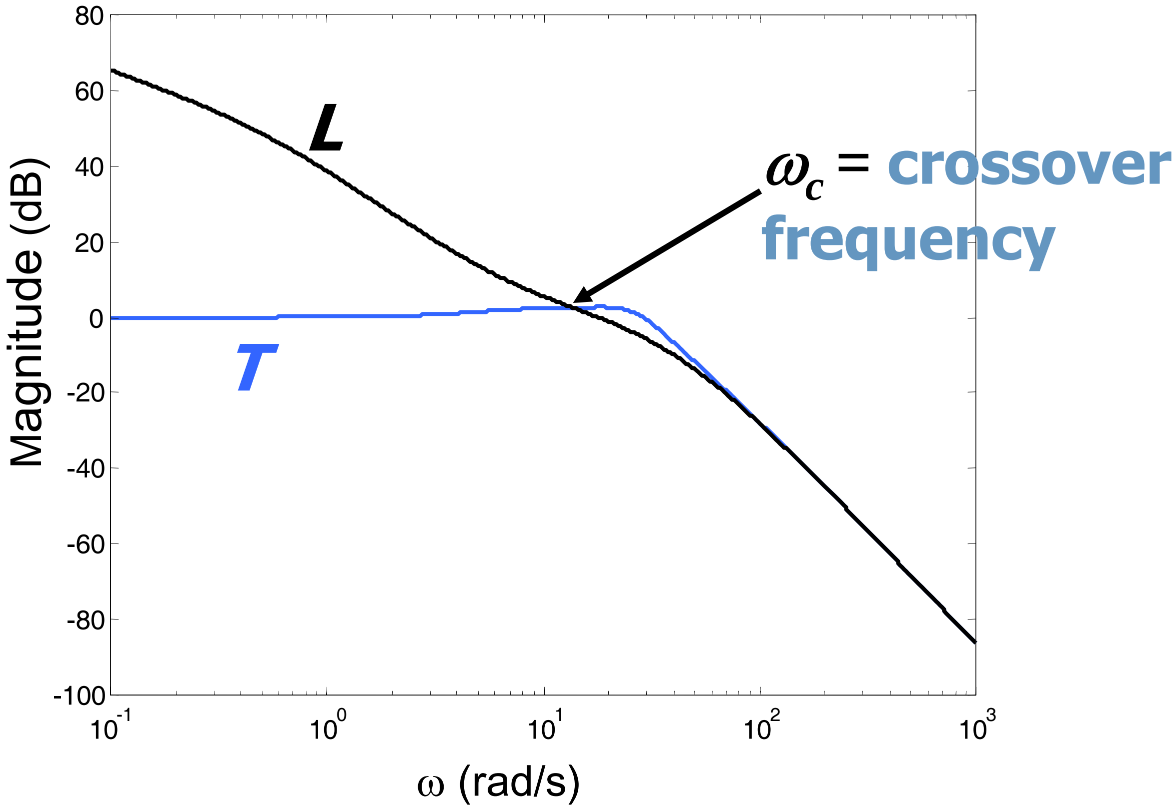
Typical plot of $|L(j\omega)|$:

- low frequency ($\omega << \omega_c$), $|L(j\omega)| >> 1$
- high frequency ($\omega >> \omega_c$), $|L(j\omega)| << 1$



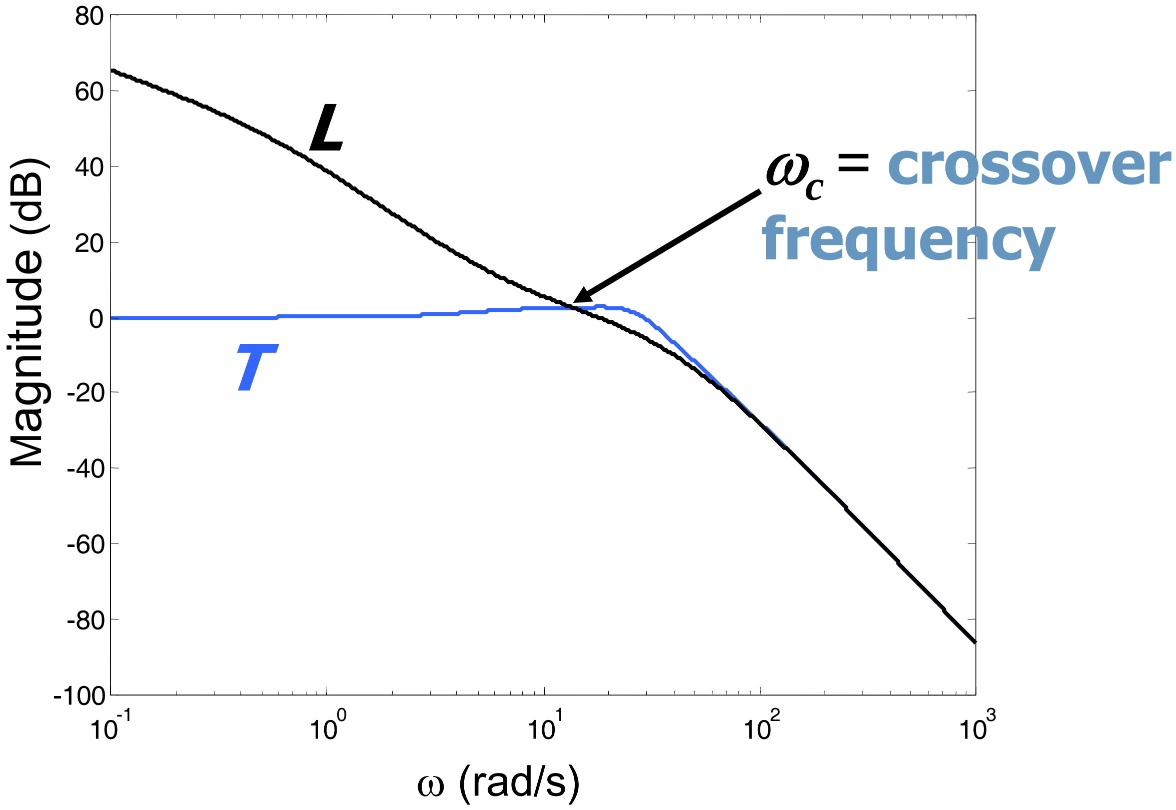
Approssimate relations between $L(j\omega)$ and $T(j\omega)$

$$\underset{\omega \ll \omega_c}{\text{low frequency}} \rightarrow \quad |T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \underset{|L(j\omega)|>>1}{\underset{\uparrow}{\approx}} 1$$



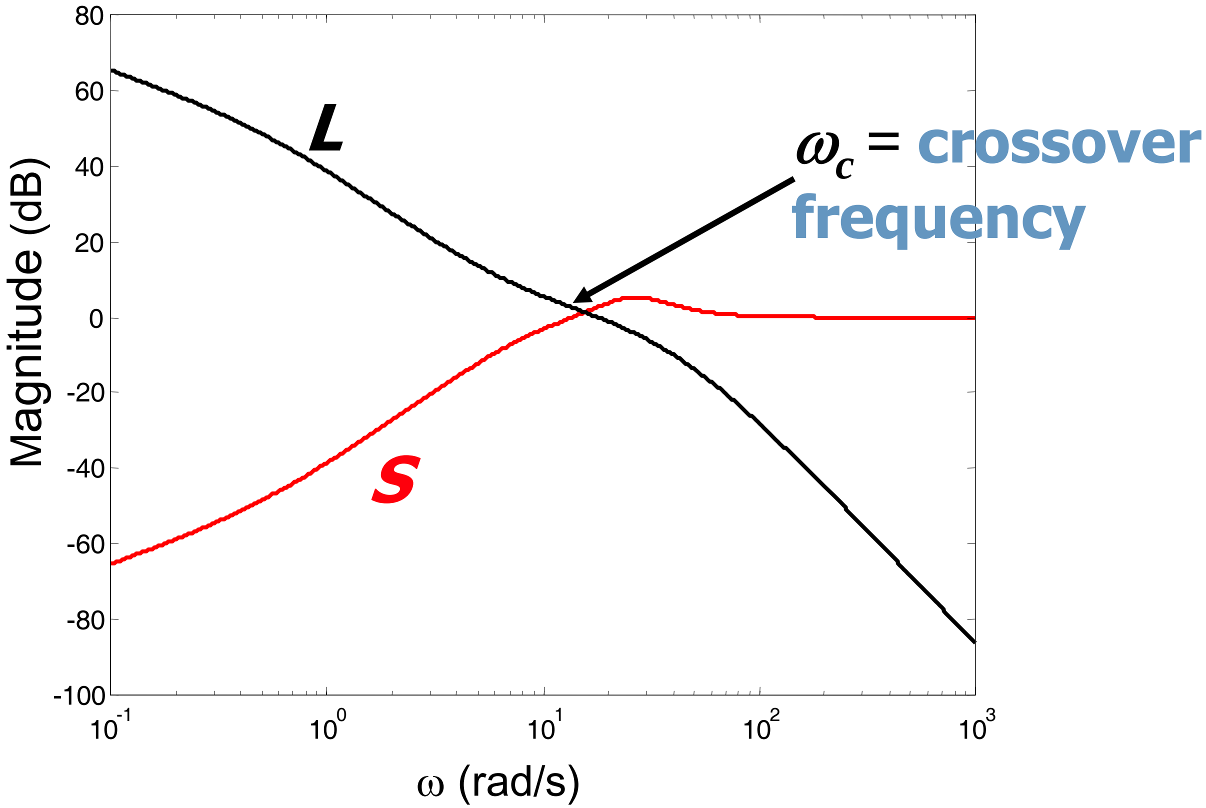
Approssimate relations between $L(j\omega)$ and $T(j\omega)$

$$\underset{\omega \gg \omega_c}{\text{high frequency}} \rightarrow |T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \underset{|L(j\omega)|<<1}{\underset{\uparrow}{\approx}} |L(j\omega)|$$



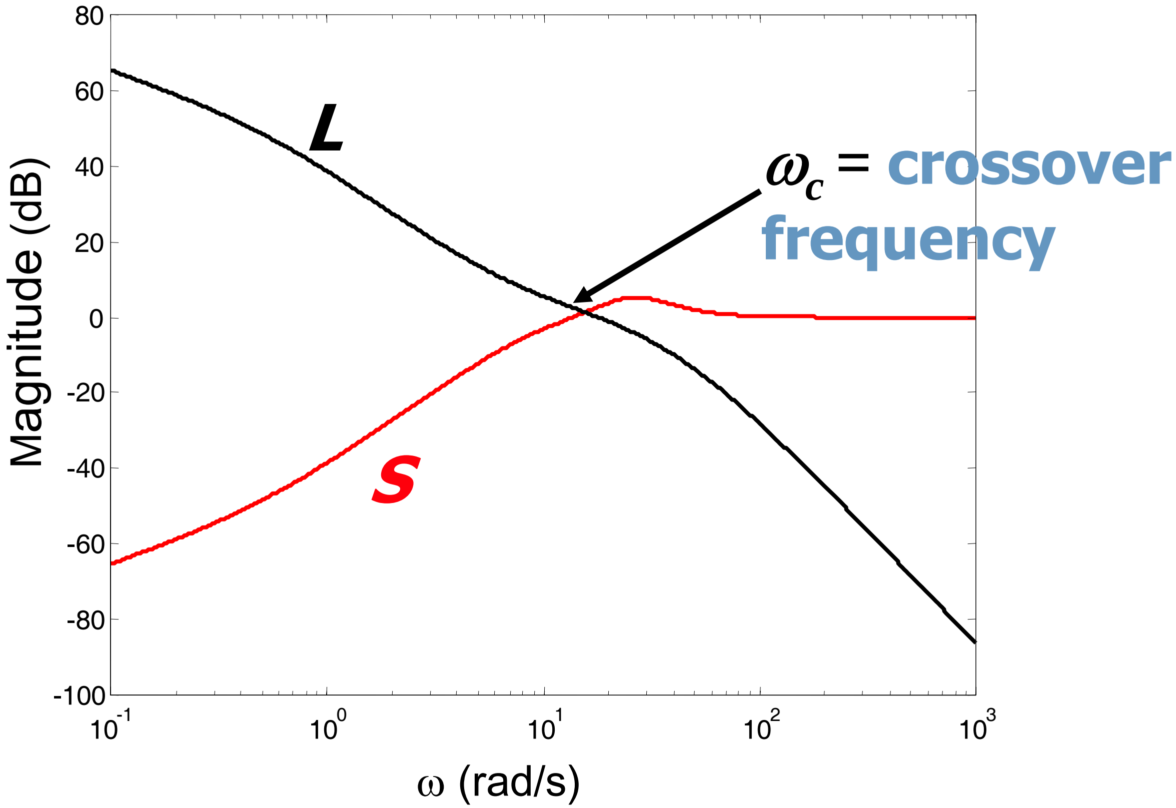
Approssimate relations between $L(j\omega)$ and $S(j\omega)$

$$\text{low frequency}_{\omega \ll \omega_c} \rightarrow |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \underset{|L(j\omega)|>>1}{\underset{\uparrow}{\approx}} \frac{1}{|L(j\omega)|}$$

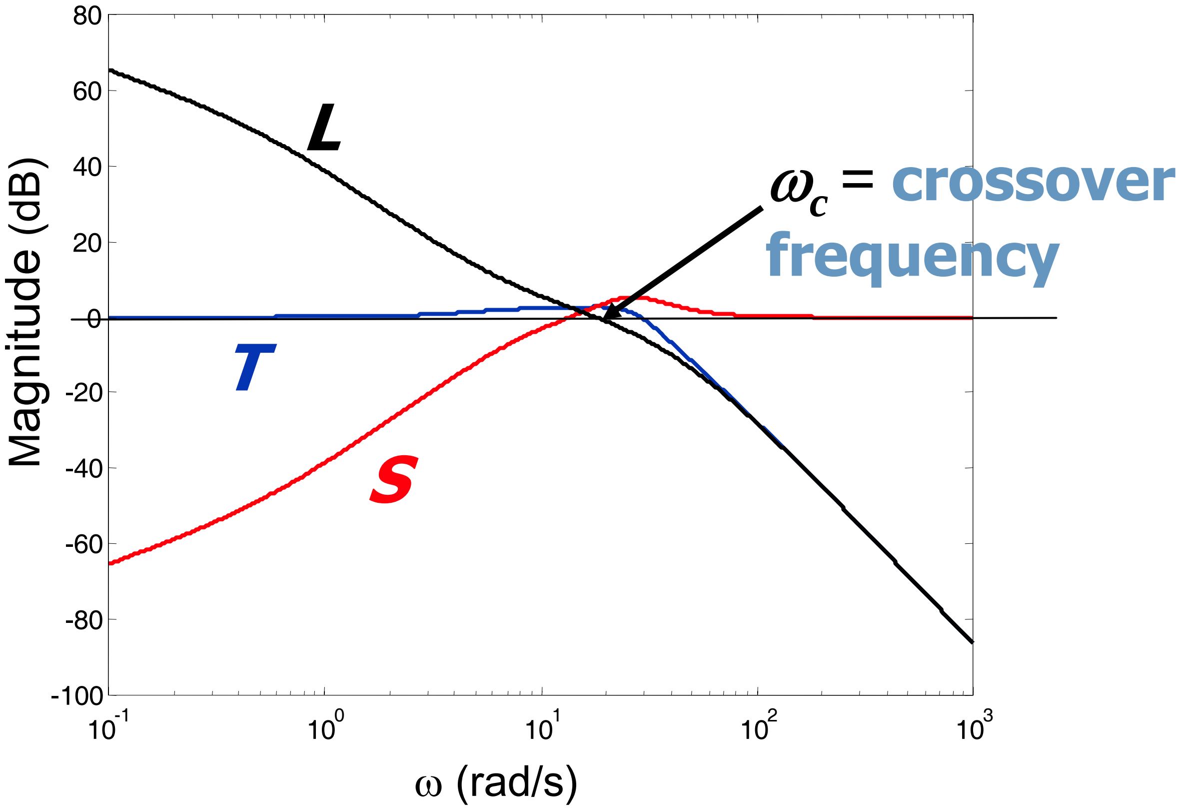


Approssimate relations between $L(j\omega)$ and $S(j\omega)$

$$\underset{\omega \gg \omega_c}{\text{high frequency}} \rightarrow |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \underset{|L(j\omega)|<<1}{\underset{\uparrow}{\approx}} 1$$



Resume



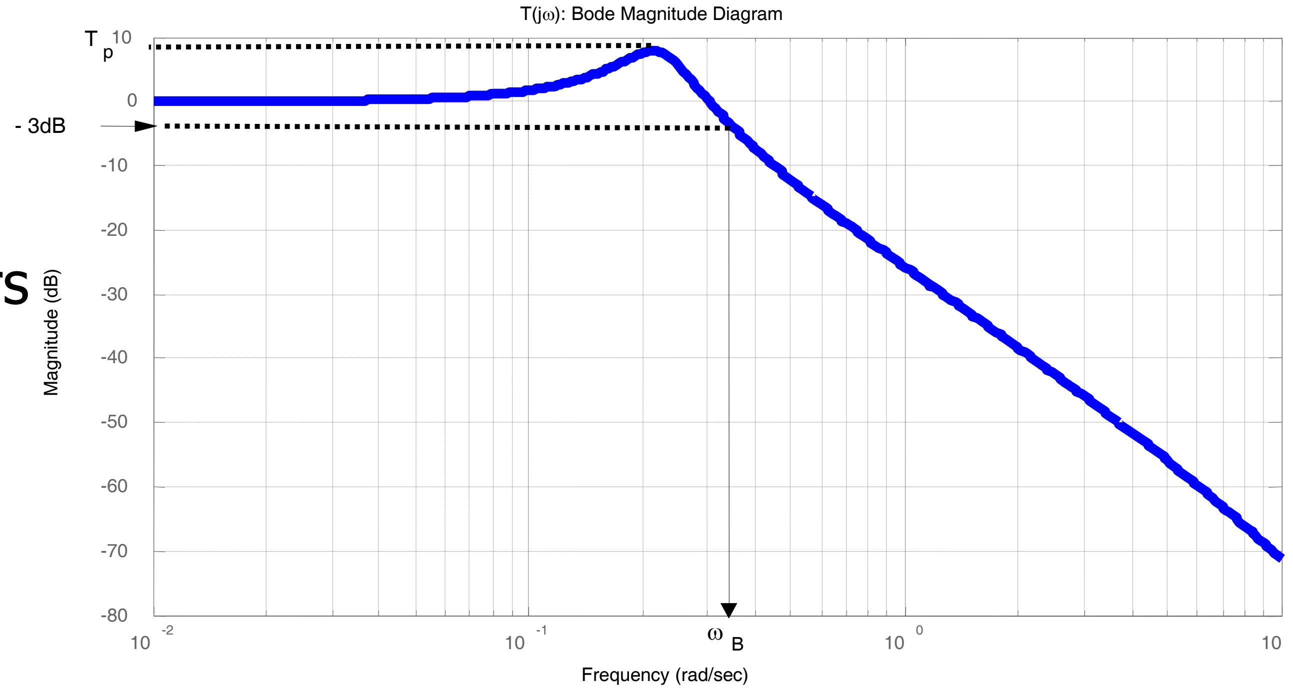
80
60
40
20
0
-20
-40
-60
-80
-100
Magnitude (dB)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω (rad/s)
L
T
S
ω_c = crossover frequency

Frequency response of the complementary sensitivity

A typical plot of the magnitude of $T(j\omega)$ is reported

The main parameters are:

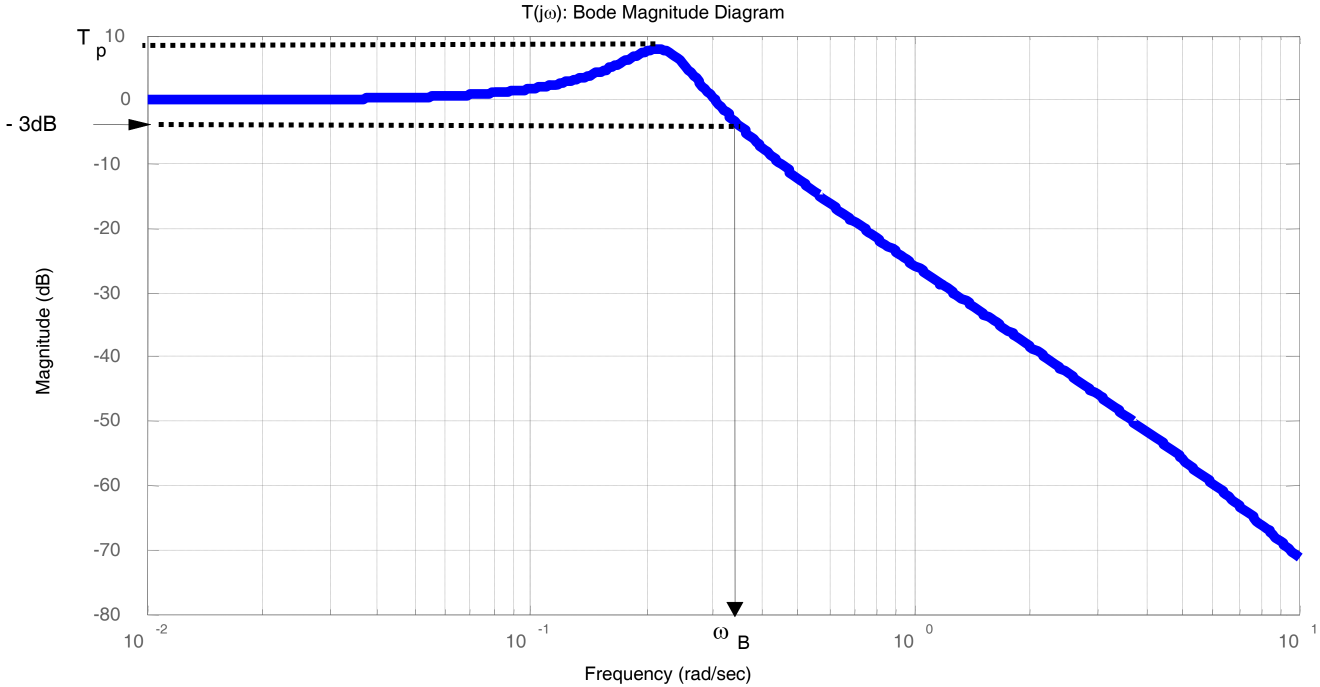
- **Resonance peak T_p**
- **Bandwidth ω_B**



$$T_p = \frac{\max_{\omega} |T(j\omega)|}{|T(j0)|} \rightarrow T_p\big|_{dB} = \max_{\omega} |T(j\omega)|_{dB} - |T(j0)|_{dB}$$

$$\omega_B : |T(j\omega_B)| = \frac{\sqrt{2}}{2} |T(j0)| \rightarrow |T(j\omega_B)|_{dB} = |T(j0)|_{dB} - 3dB$$

Properties of $|T(j\omega)|$



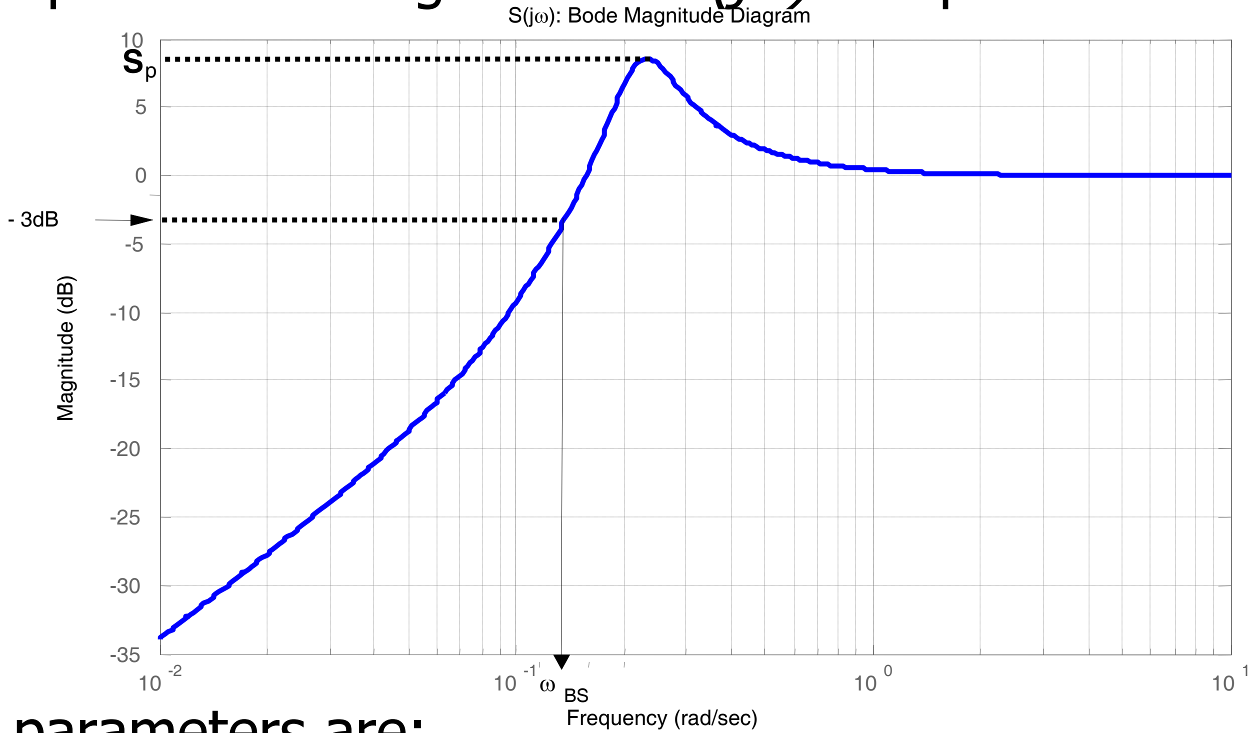
Low frequencies range → $|T(j\omega)| \approx 1$

High frequencies range → $|T(j\omega)| << 1$

Medium frequencies range → T_p

Frequency response of the sensitivity function

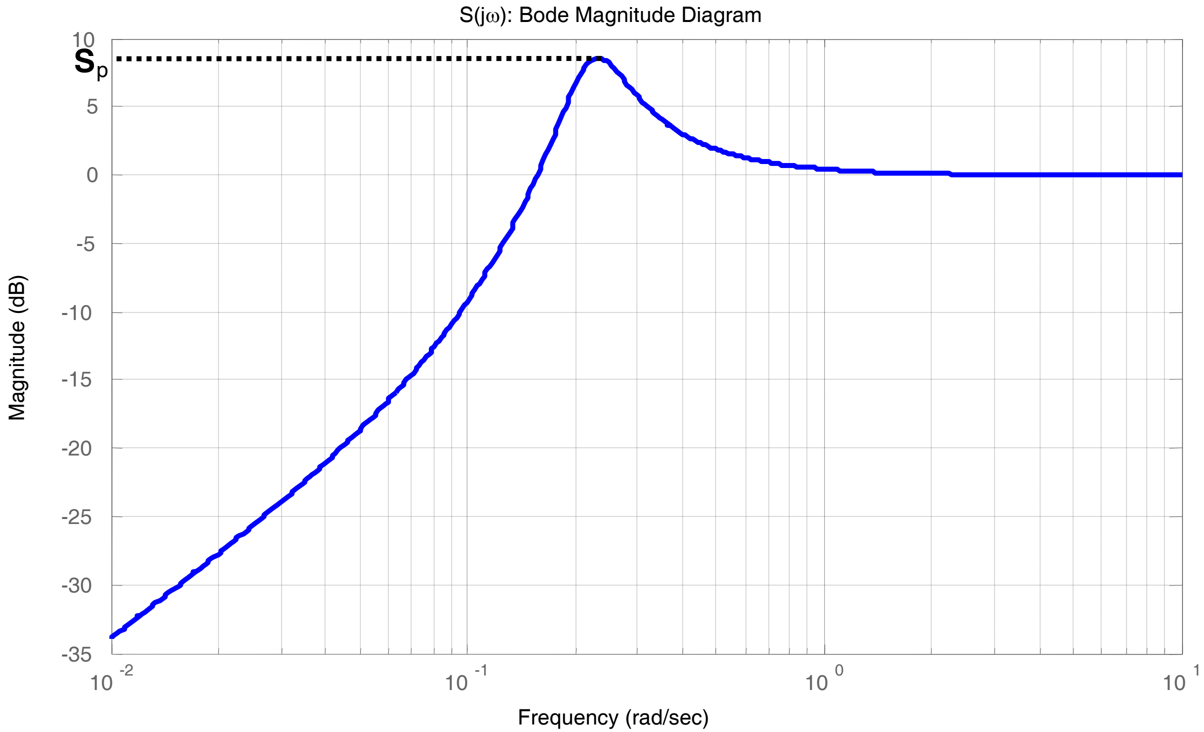
A typical plot of the magnitude of $S(j\omega)$ is reported below



The main parameters are:

- **Resonance peak S_p :** $S_p = \max_{\omega} |S(j\omega)| \rightarrow S_p|_{dB} = \max_{\omega} |S(j\omega)|_{dB}$
- **Bandwidth ω_{BS}:** $\omega_{BS} : |S(j\omega_{BS})| = \frac{\sqrt{2}}{2} \rightarrow |S(j\omega_{BS})|_{dB} = -3dB$

Properties of $|S(j\omega)|$



Low frequencies range → $|S(j\omega)| << 1$

High frequencies range → $|S(j\omega)| \approx 1$

Medium frequencies range → S_p

SISO feedback control systems characteristics

Prototype 2° order systems frequency response

Frequency response of prototype 2nd order systems

- For a prototype 2nd order control system of the form:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

- The corresponding loop function can be computed as:

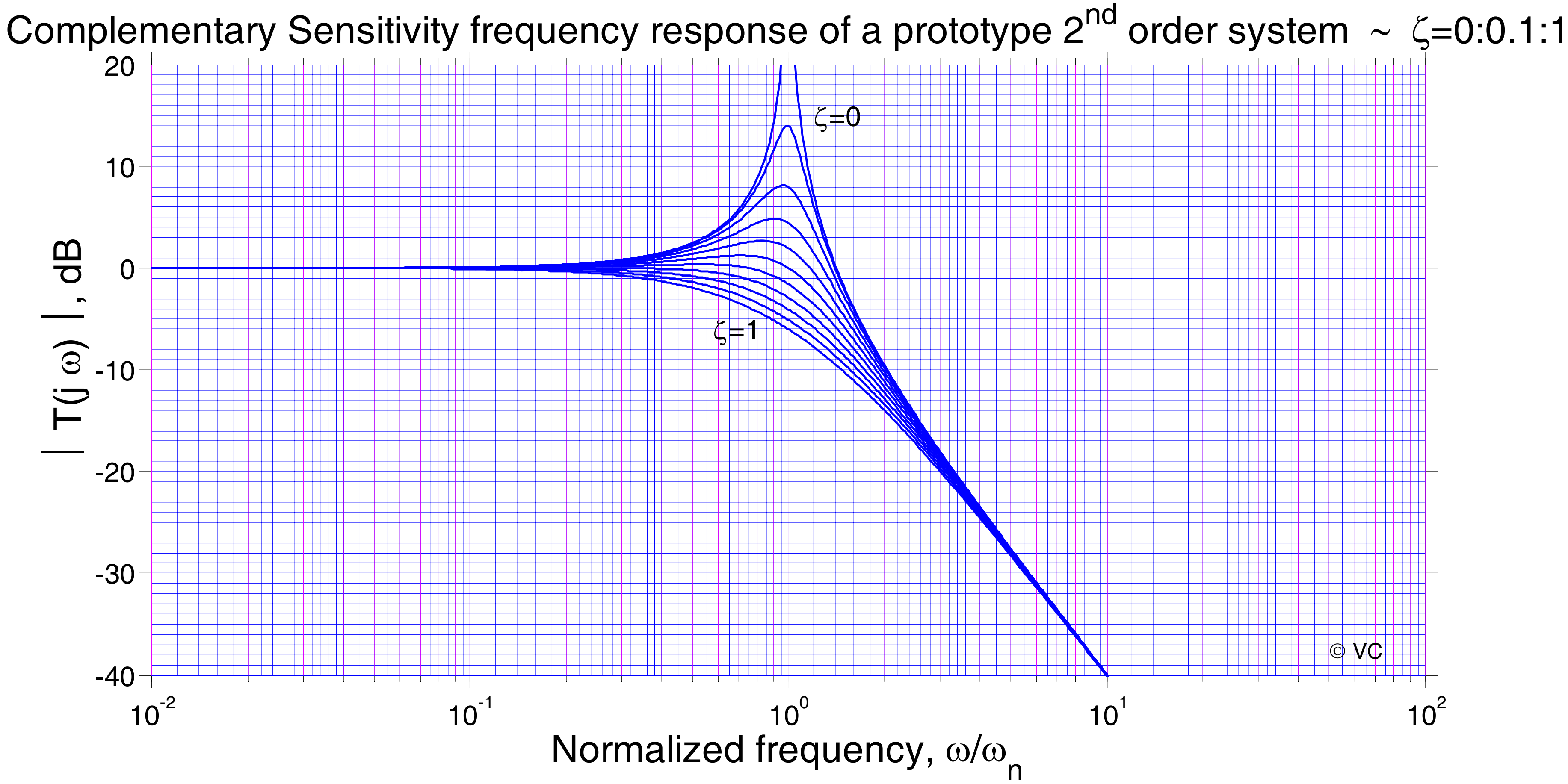
$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}} \quad \underset{\uparrow}{\Rightarrow} \quad L(s) = \frac{\omega_n/(2\zeta)}{s\left(1 + \frac{s}{2\zeta\omega_n}\right)}$$

$$L(s) = \frac{T(s)}{1 - T(s)}$$

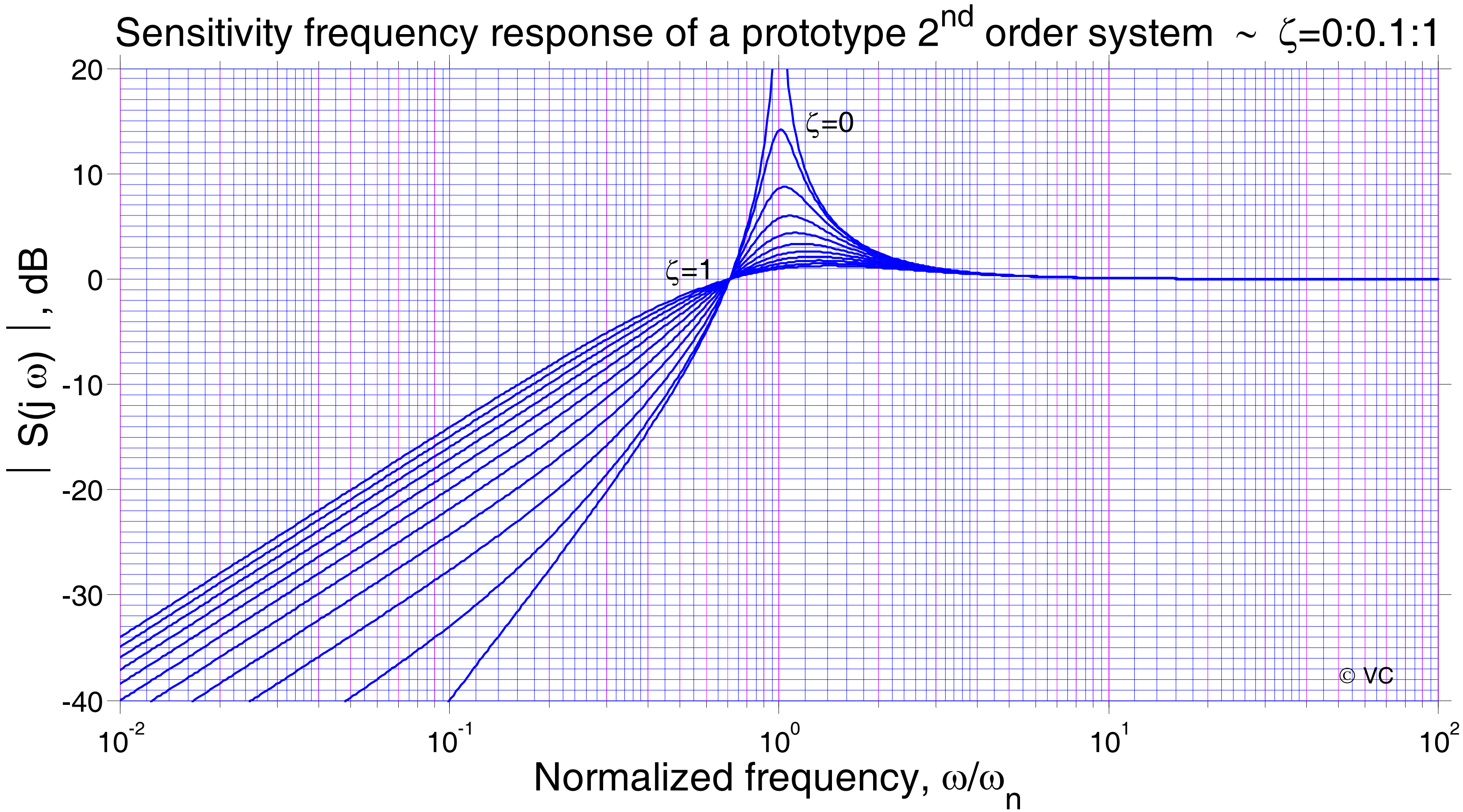
- In this case, the crossover frequency ω_c of $L(j\omega)$ can be expressed as a function of parameters ζ ed ω_n:

$$\omega_c = \omega_n \sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2} = f_{\omega_c}(\zeta, \omega_n)$$

Frequency response of prototype 2nd order systems



Frequency response of prototype 2nd order systems



Frequency response of prototype 2nd order systems

- The resonance peak T_p, and the bandwidth ω_B of the complemenatry sensitivity function:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

are functions of parameters ζ ed ω_n:

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = f_{T_p}(\zeta)$$

$$\omega_B = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} = f_{\omega_B}(\zeta, \omega_n)$$

Frequency response of prototype 2nd order systems

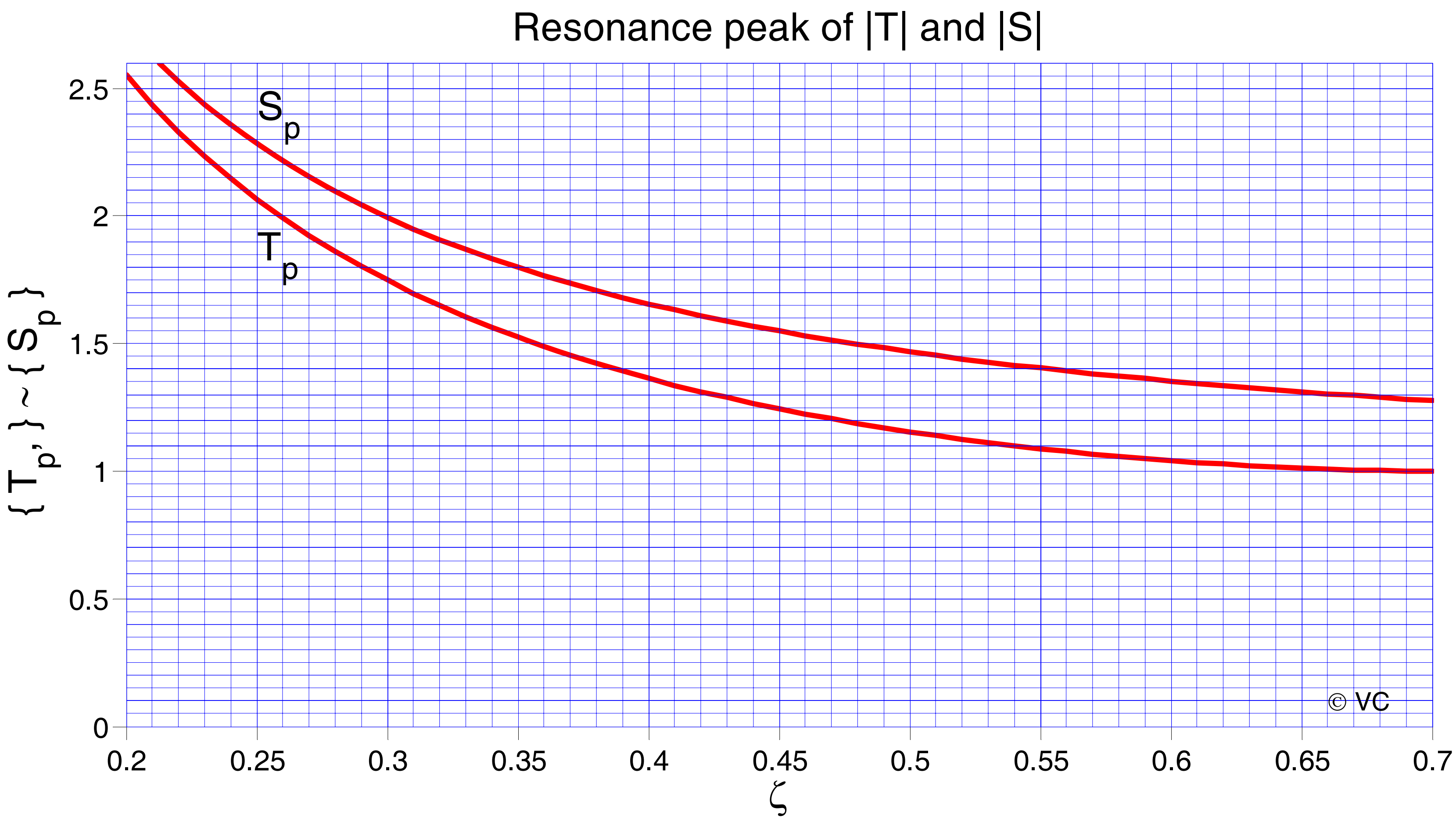
- The resonance peak S_p of the sensityivity function:

$$S(s) = 1 - T(s) = 1 - \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} = \frac{s\left(\frac{2\zeta}{\omega_n} + \frac{s}{\omega_n^2}\right)}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}}$$

is given by the following function of parameters ζ ed ω_n:

$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1} = f_{S_p}(\zeta)$$

Frequency response of prototype 2nd order systems



SISO feedback control systems characteristics

Relations between frequency response and time response of prototype 2° order systems

Frequency response and time response relations

- In order to define the relations between frequency response and time response we consider the expressions of the relevant indices obtained for the 2nd order prototype model:

Time response

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right)$$

$$t_{s,\alpha\%} = -\frac{\ln\alpha}{\omega_n\zeta}$$

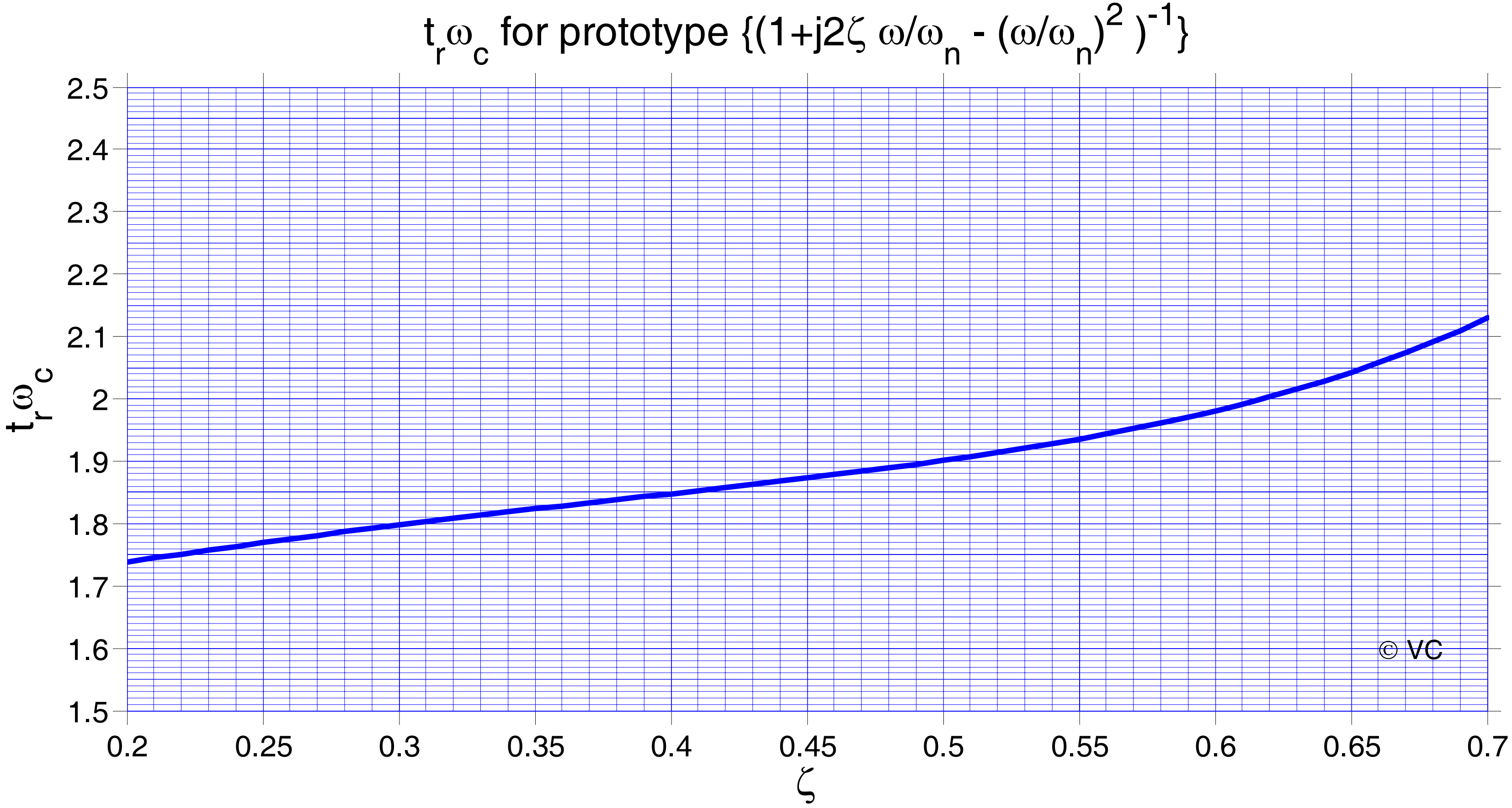
Frequency response

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

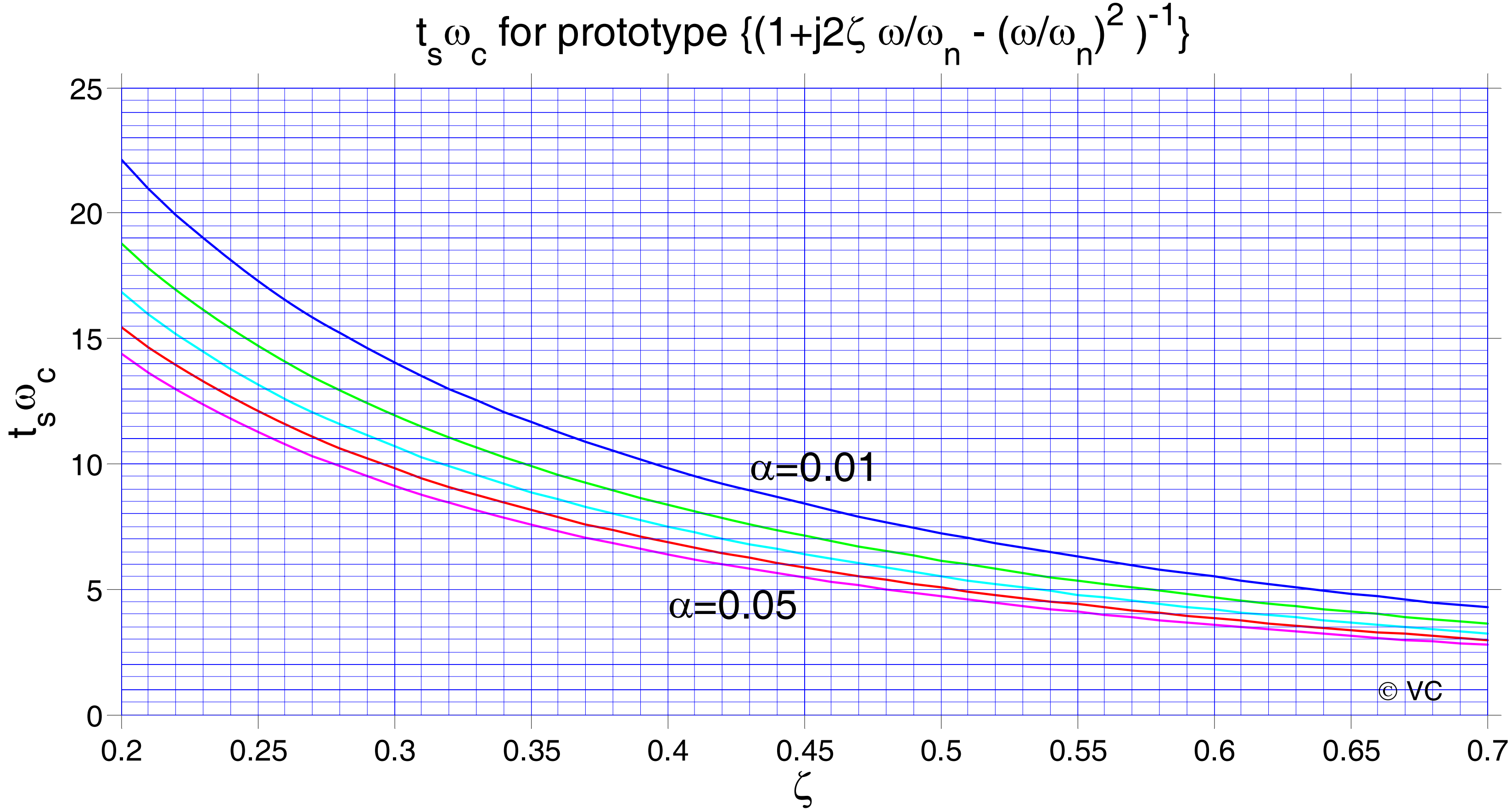
$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1}$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$$

Frequency/time relations for prototype 2nd order syst



Frequency/time relations for prototype 2nd order syst



Requirements translation

- The objective is to translate time domain requirements into frequency domain requirements to be exploited in the controller design

- In particular we would like to translate requirements on $\hat{s}$, t_r e $t_{s,\alpha\%}$ into requirements on T_p, S_p and ω_c

- In order to show the translation procedure a practical example will be considered

Requirements translation: example

- Consider the following time response requirements:

$$\hat{s} \le 10\%$$

$$t_r \le 0.5\ s$$

$$t_{s,1\%} \le 1.5\ s$$

Requirements translation: example

- First, we consider the maximum overshoot requirement:

$$\hat{s} \leq 10\%$$

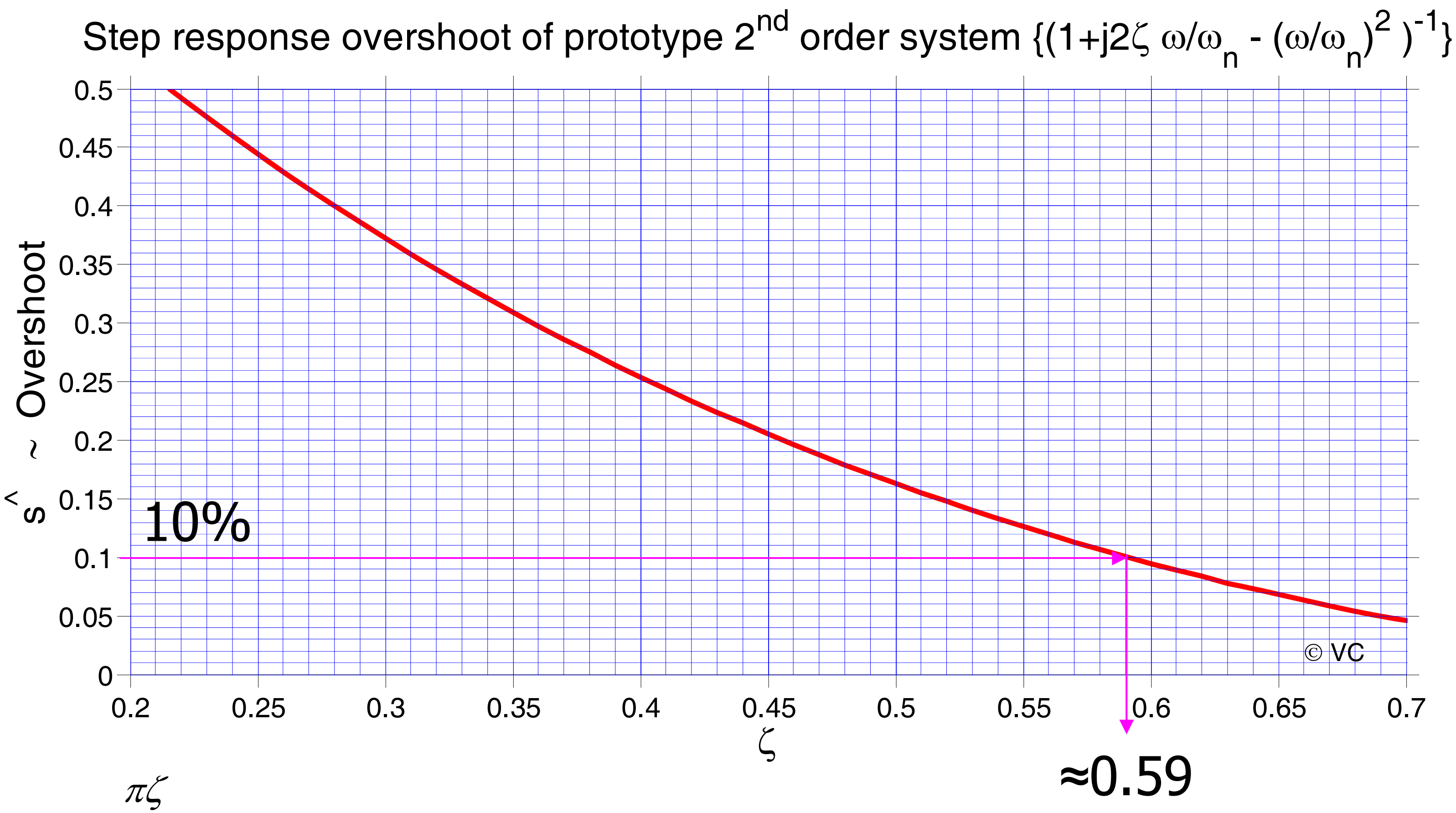
- Recalling that:

$$\hat{s} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}}$$

- It is possible to obtain a requirement on the minimum damping coefficient:

$$\zeta \geq \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}} \underset{\hat{s}=0.1}{\overset{\uparrow}{=}} 0.59$$

Graphical procedure



$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Requirements translation: example

- By using the obtained minimum damping coefficient: $\zeta \geq 0.59$

and recalling that:

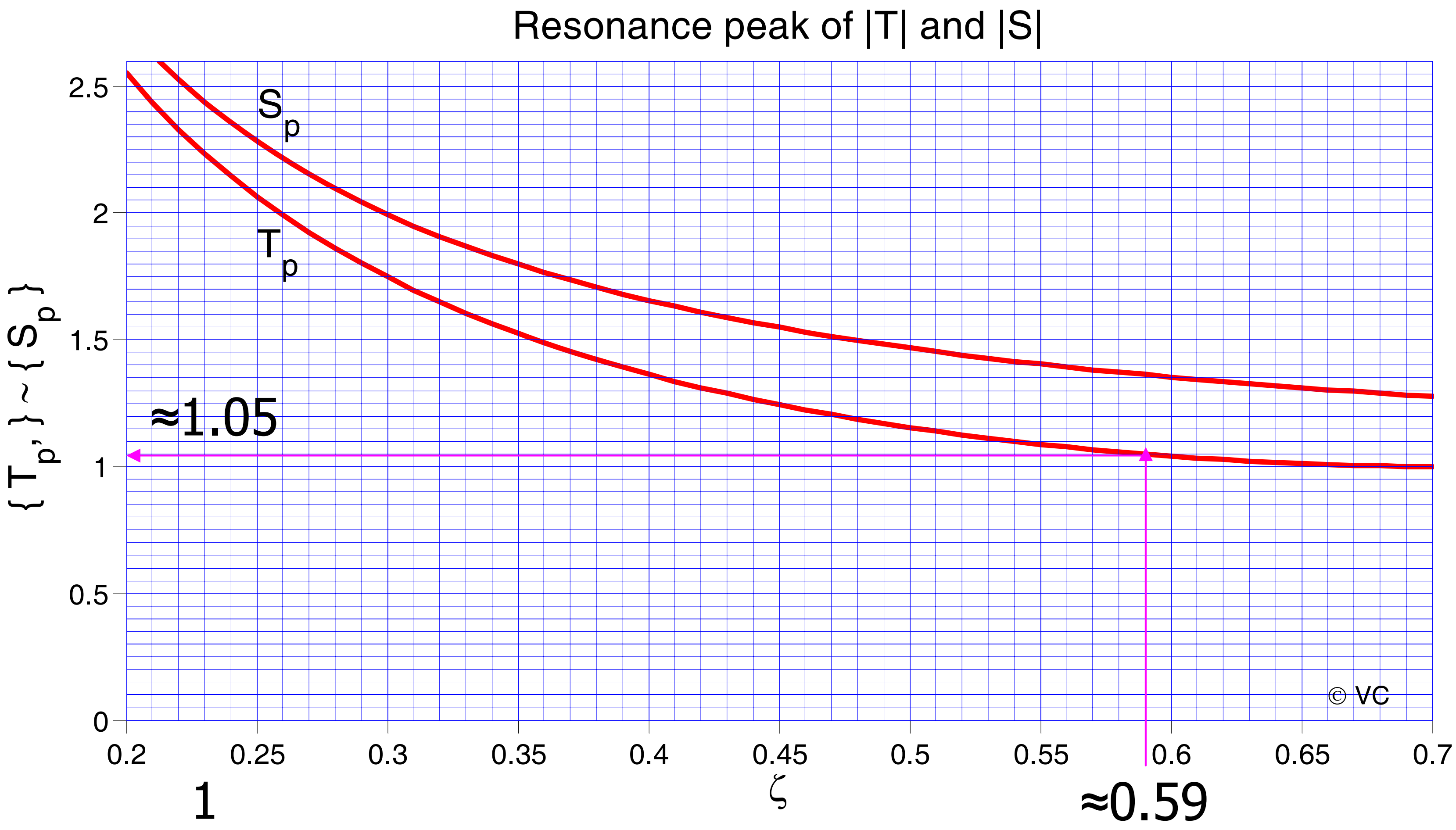
$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2\text{-}1}$$

- Requirements on the resonance peak of the complementary sensitivity (T_p) and sensititivity (S_p) functions can be derived:

$$T_p \leq \frac{1}{2\zeta\sqrt{1-\zeta^2}} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 1.0496 = 0.42\,\mathrm{dB}$$

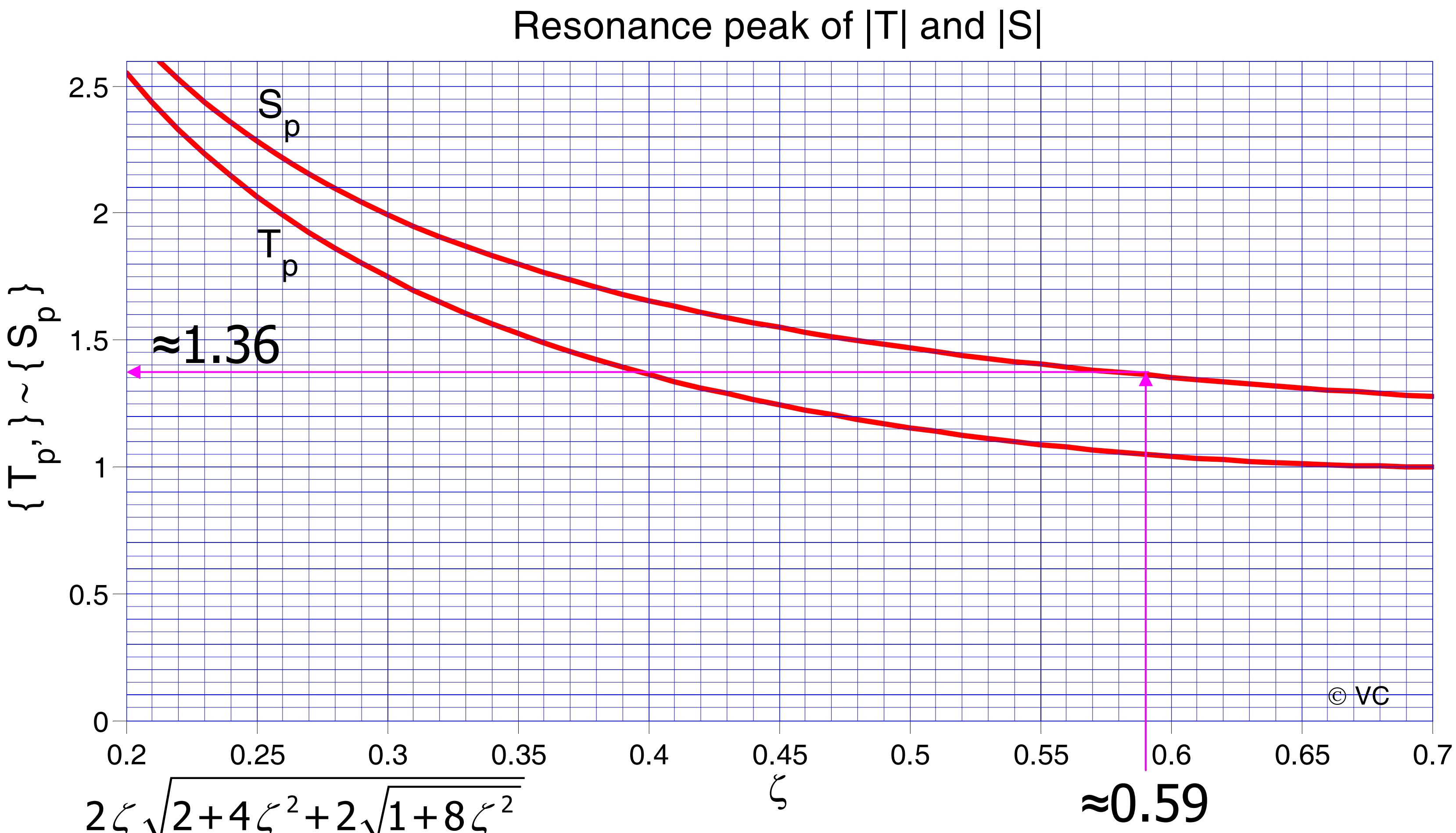
$$S_p \leq \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2\text{-}1} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 1.3622 = 2.68\,\mathrm{dB}$$

Graphical procedure



$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Graphical procedure



$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1}$$

Requirements translation: example

- Let us now consider the rise time requirement

$$t_r \leq 0.5\ s$$

- Recalling that:

$$t_r = \frac{1}{\omega_n \sqrt{1-\zeta^2}} \left(\pi - \arccos(\zeta)\right)$$

$$\omega_c = \omega_n \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

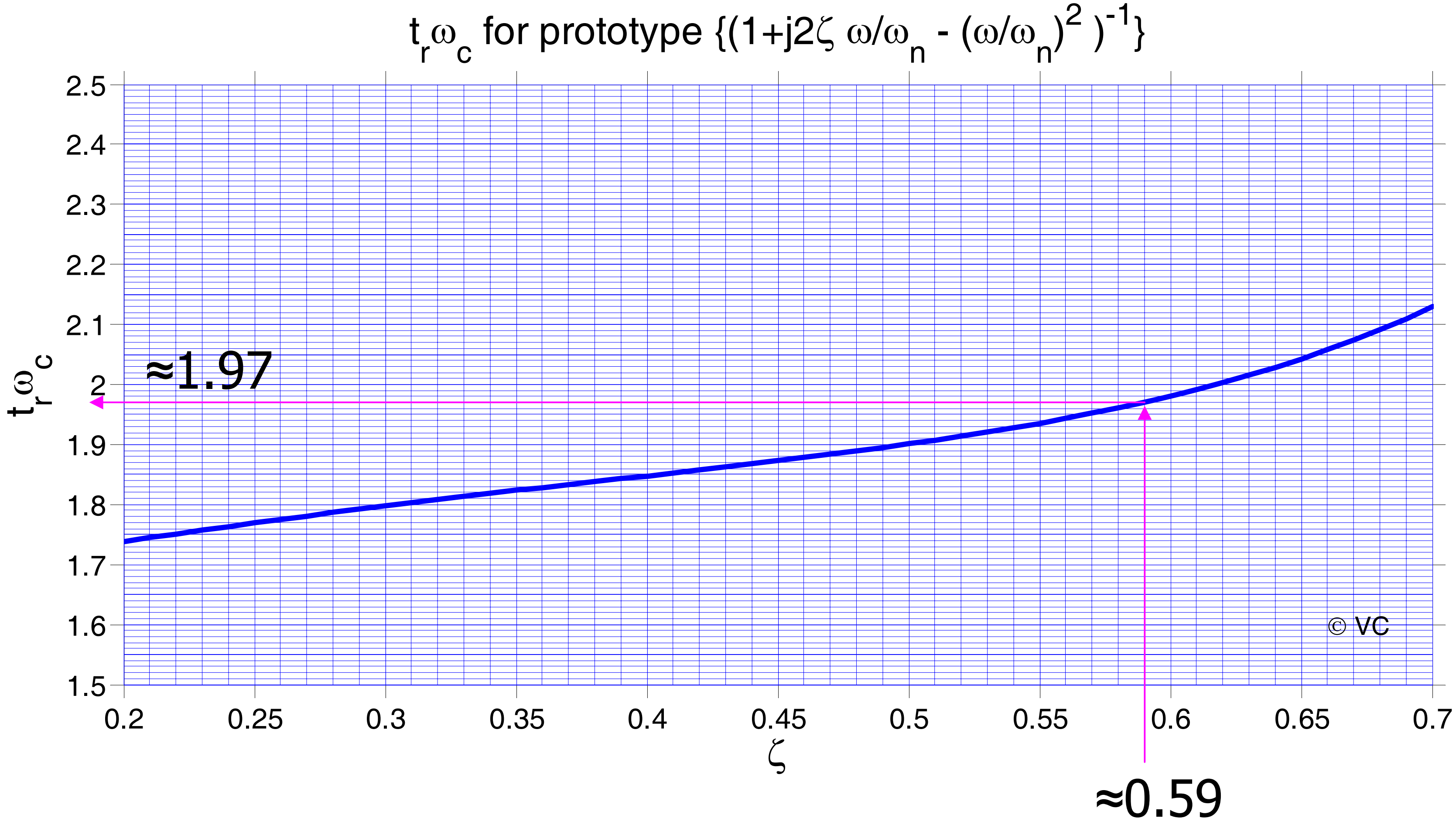
- we obtain:

$$t_r \cdot \omega_c = \frac{1}{\sqrt{1-\zeta^2}} \left(\pi - \arccos(\zeta)\right) \cdot \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2} \underset{\zeta=0.59}{\underset{\uparrow}{=}} 1.9708$$

- Therefore:

$$\omega_c \geq \frac{1.9708}{t_r} \underset{t_r=0.5}{\underset{\uparrow}{=}} 3.9417 \text{ rad/s}$$

Graphical procedure



Requirements translation: example

- A similar procedure can be used for the settling time requirement

$$t_{s,1\%} \le 1.5\ s$$

- Recalling that:

$$t_{s,\alpha\%} = -\frac{\ln\alpha}{\omega_n\zeta} \underset{\alpha=0.01}{\overset{\uparrow}{=}} \frac{4.6052}{\omega_n\zeta}$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$$

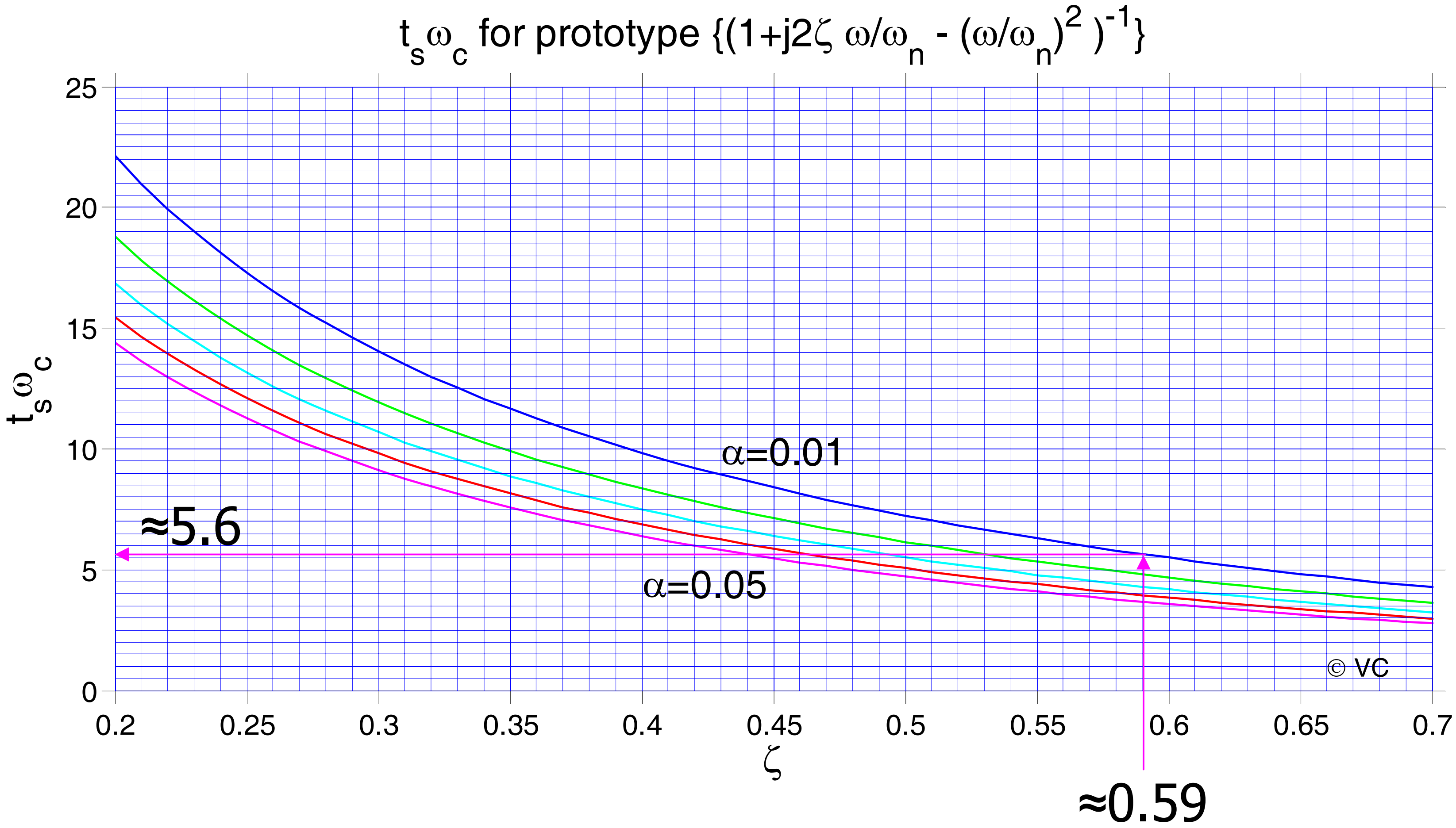
- we obtain:

$$t_{s,\alpha\%}\cdot\omega_c = \frac{4.6052}{\zeta}\cdot\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 5.6409$$

- then:

$$\omega_c \ge \frac{5.6409}{t_{s,1\%}} \underset{t_{s,1\%}=1.5}{\overset{\uparrow}{=}} 3.7606\ \text{rad/s}$$

Graphical procedure



Requirements translation: example

- The translation of the rise time and the settling time requirements leads to two different values of the crossover frequency.

- The crossover frequency value to be employed for the design is chosen in the following way

$$\omega_{c,des} \geq \max(\underset{\substack{\uparrow \\ t_r}}{\omega_c}, \underset{\substack{\uparrow \\ t_{s,1\%}}}{\omega_c}) \geq 3.9147 \Rightarrow \omega_{c,des} = 4 \text{ rad/s}$$

SISO feedback control systems characteristics

Constant magnitude loci of T and S

Closed loop frequency response from Nyquist plot?

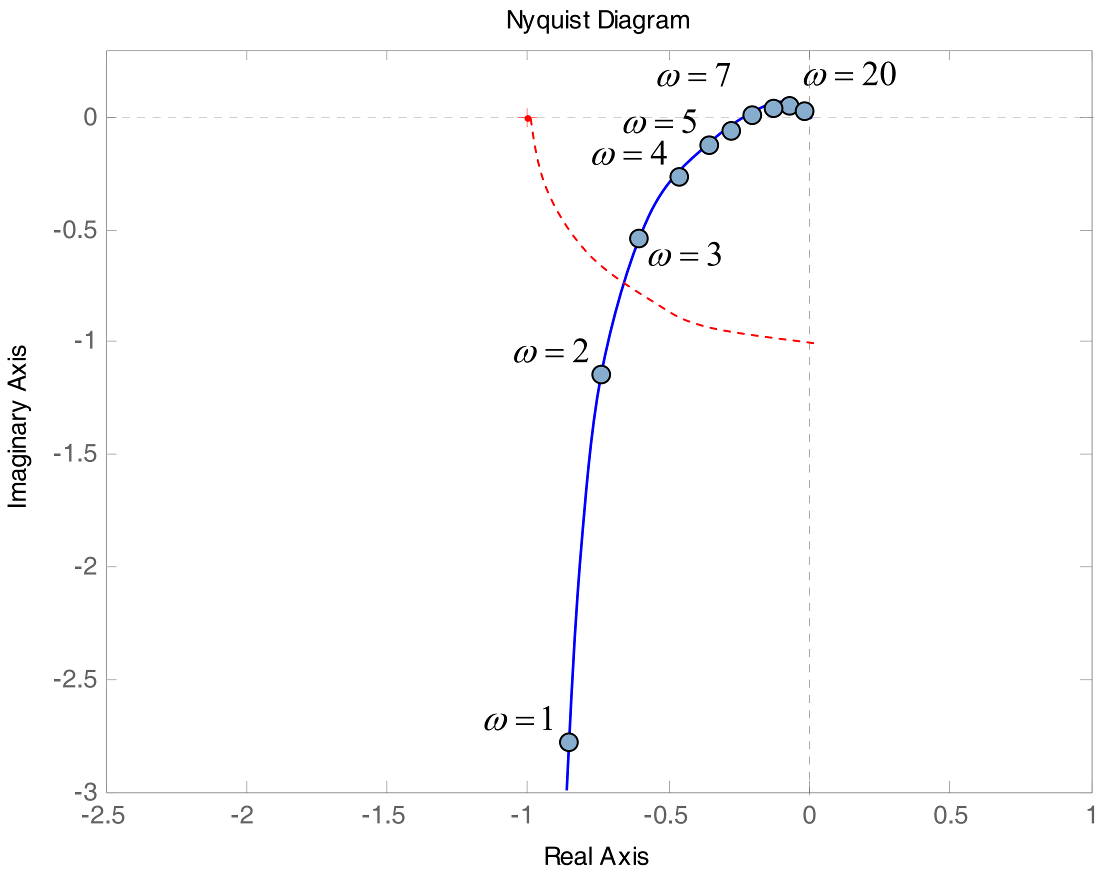
Is it possible to derive the frequency response of the complementary sensitivity

$$T(j\omega) = \frac{L(j\omega)}{1 + L(j\omega)}$$

on the basis of the loop frequency response $L(j\omega)$?

Closed loop frequency response from Nyquist plot?

Let $L(s) = \dfrac{150}{s(s+5)(s+10)}$



$$T(j1) = ? \qquad \omega = 1$$

$$G(j1)G_c(j1) = -0.86 - 2.8j$$

$$T(j1) = \frac{L(j1)}{1 + L(j1)} = \frac{-0.86 - 2.8j}{1 - 0.86 - 2.8j}$$

$$T(j1) = 1.04\angle -20$$

$$T(j2) = ? \quad T(j3) = ? \quad T(j\omega) = ?$$

Closed loop frequency response from Nyquist plot?

- In order to answer the question, recall that, for a fixed value of ω, $T(j\omega)$, $L(j\omega)$ are complex numbers

- Let us indicate such complex numbers with T and L respectively

- The relation between T and L is:

$$T = \frac{L}{1 + L}$$

T and L can be expressed as: $T = Me^{j\phi}$ $L = u + jv$

M_T constant magnitude loci

- We have:

$$T = \frac{L}{1+L} = \frac{u+jv}{1+u+jv} = Me^{j\phi} \rightarrow \begin{cases} M^2 = \dfrac{u^2+v^2}{(1+u)^2+v^2} \\ \phi = \tan^{-1}\left(\dfrac{v}{u}\right) - \tan^{-1}\left(\dfrac{v}{1+u}\right) \end{cases}$$

- We aim at deriving the following locus of points on the plane (u, v)

$$\frac{u^2+v^2}{(1+u)^2+v^2} = M_T^2 = \text{const.} \quad \rightarrow$$ **constant magnitude locus**

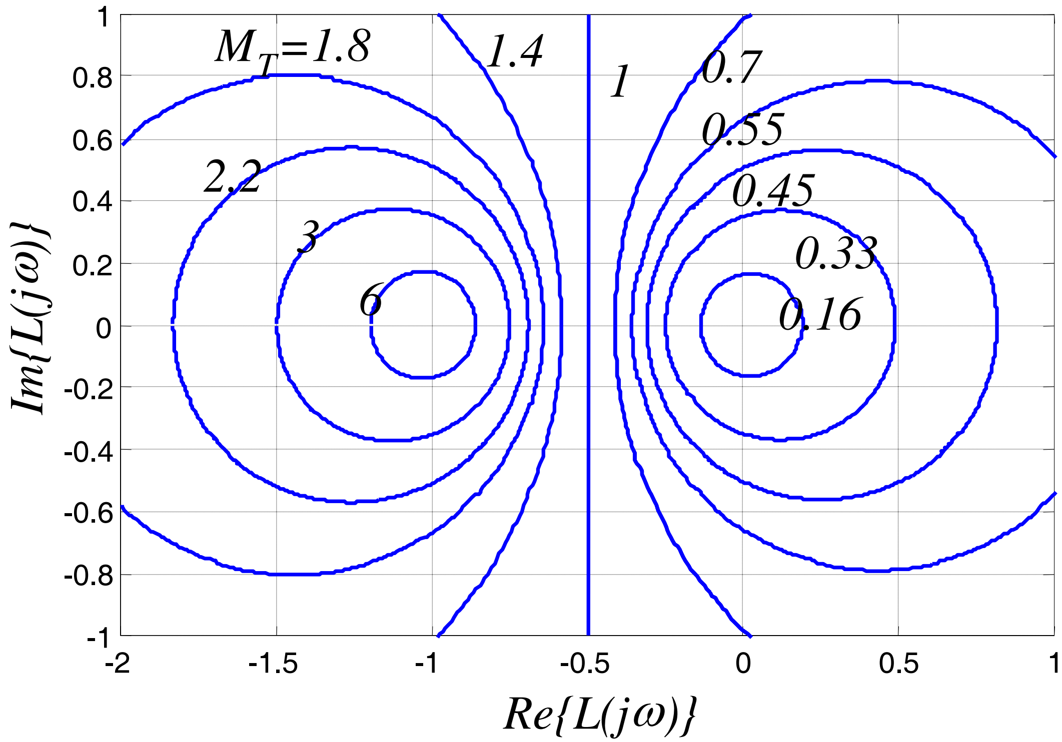
$$\left(u - \frac{M_T^2}{1-M_T^2}\right)^2 + v^2 = \left(\frac{M_T}{1-M_T^2}\right)^2 \rightarrow \text{Eq. of a circle} \begin{cases} C = \left(\dfrac{M_T^2}{1-M_T^2}, 0\right) \\ r = \left|\dfrac{M_T}{1-M_T^2}\right| \end{cases}$$

$\downarrow$

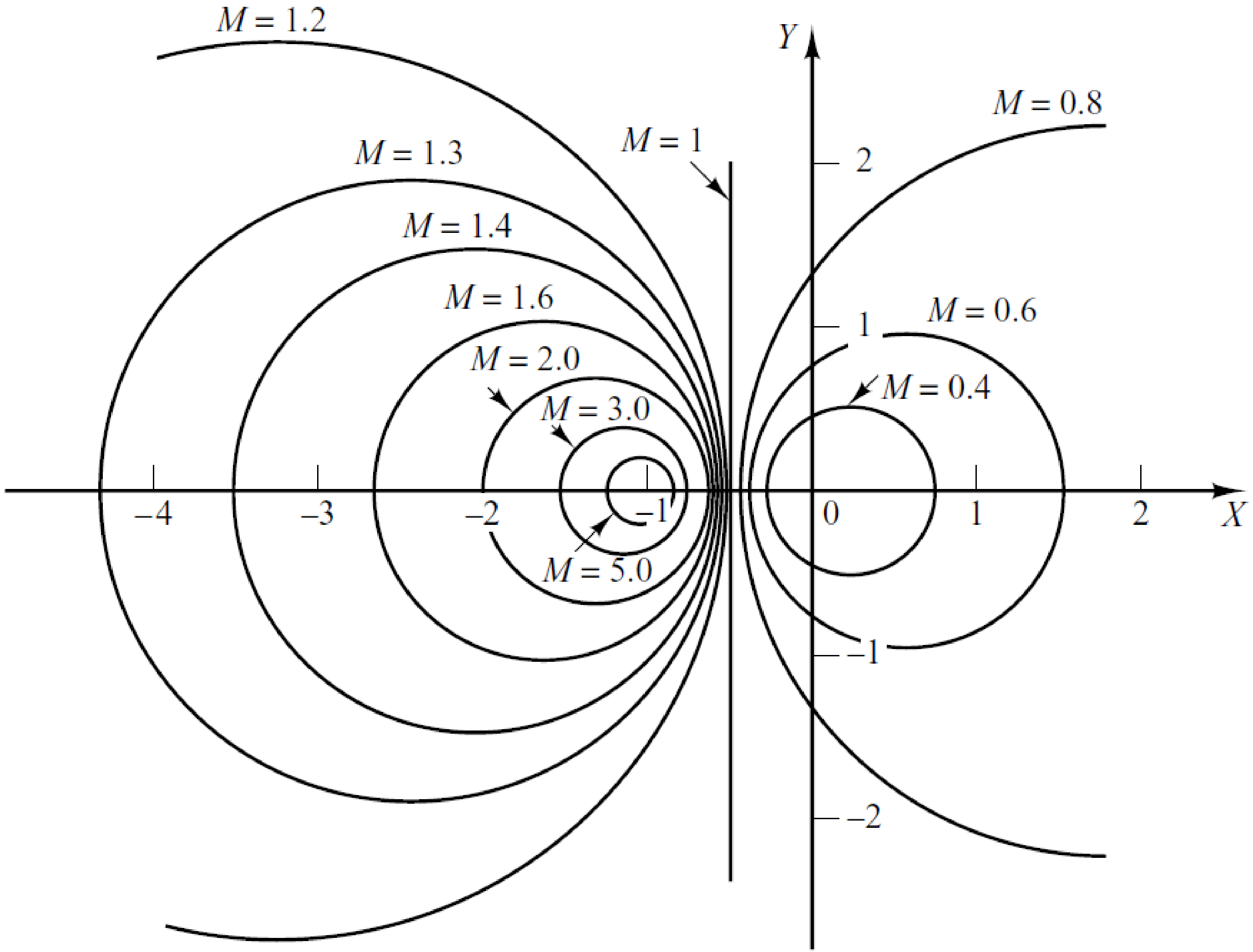
M_T circle

M_T constant magnitude loci

- Circles $M_T > 1$ lay on the left of the line $u = \mathbb{R}e[L(j\omega)] = -0.5$ and their radius decreases for increasing values of M_T
- Circle $M_T = 1$ degenerates into the vertical line $u = -0.5$
- Circles $M_T < 1$ lay on the right of the line $u = \mathbb{R}e[L(j\omega)] = -0.5$ and their radius decreases for decreasing values of M_T

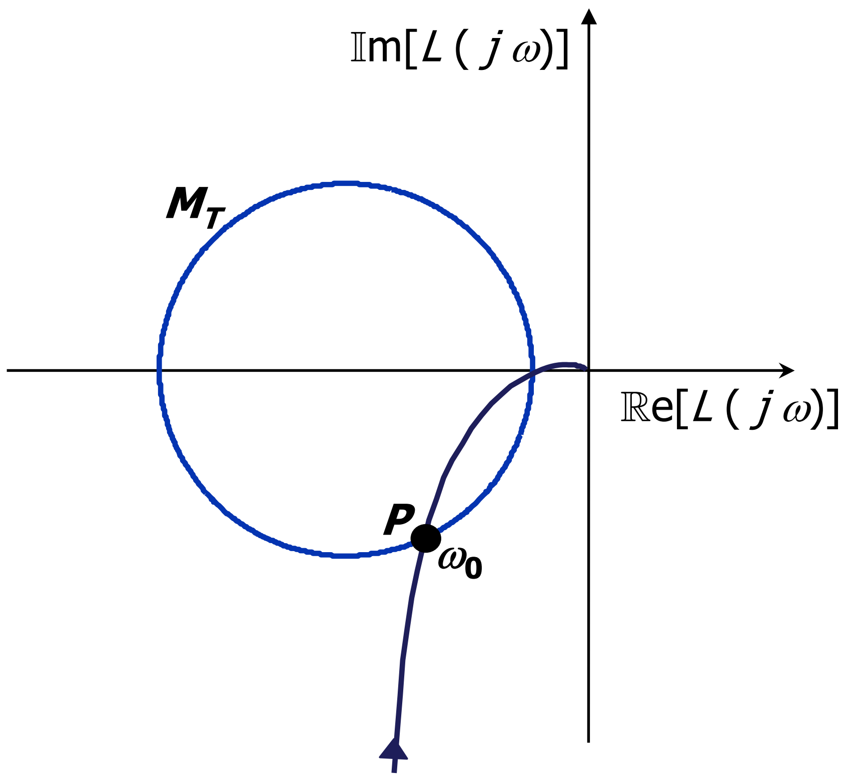


M_T constant magnitude loci



Deriving $|T(j\omega)|$ from $L(j\omega)$

- Let us draw on the plane $(u, v) = (\mathbb{R}\mathrm{e}[L(j\omega)], \mathbb{I}\mathrm{m}[L(j\omega)])$ the generic M_T circle together with the polar plot of $L(j\omega)$



- For a fixed frequency ω_0 we have:
  $P = (\mathbb{R}\mathrm{e}[L(j\omega_0)], \mathbb{I}\mathrm{m}[L(j\omega_0)])$
- moreover P belongs to the considered M_T circle →

$$\frac{u^2+v^2}{(1+u)^2+v^2} =$$

$$= \frac{\mathrm{Re}(L(j\omega_0))^2 + \mathrm{Im}(L(j\omega_0))^2}{(1+\mathrm{Re}(L(j\omega_0)))^2 + \mathrm{Im}(L(j\omega_0))^2} =$$

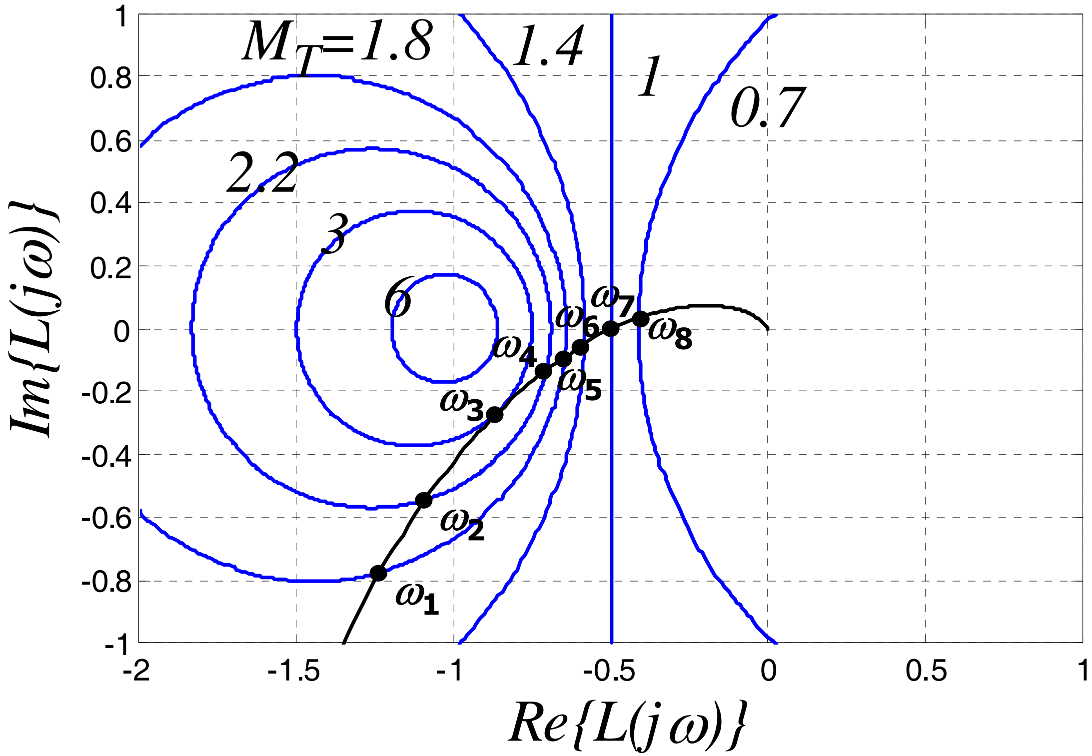
$$= M_T^2 = \left|T(j\omega_0)\right|^2 \Rightarrow \left|T(j\omega_0)\right| = M_T$$

Deriving $|T(j\omega)|$ from $L(j\omega)$

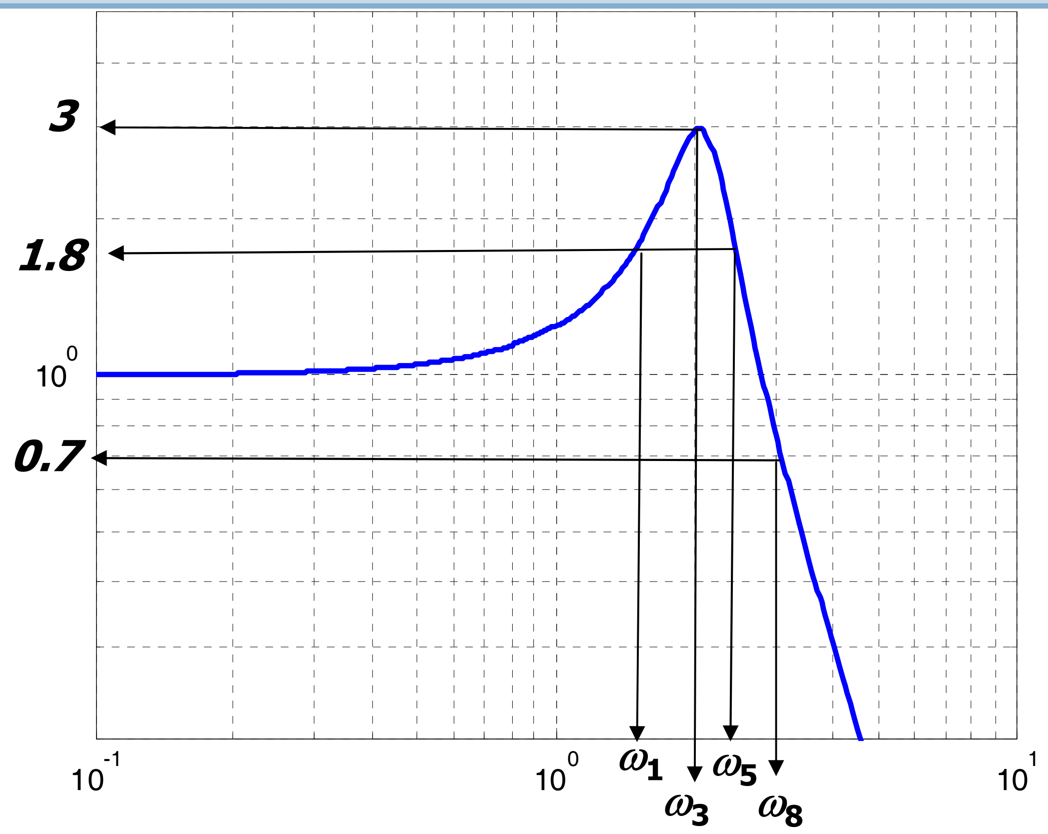
The superposition of a family of M_T circles on the plane

$$(u, v) = (\mathbb{R}\mathrm{e}[L(j\omega)], \mathbb{I}\mathrm{m}[L(j\omega)])$$

makes up a curvilineal coordinate system from which we can derive the plot of $|T(j\omega)|$ through the intersections of the polar plot of $L(j\omega)$

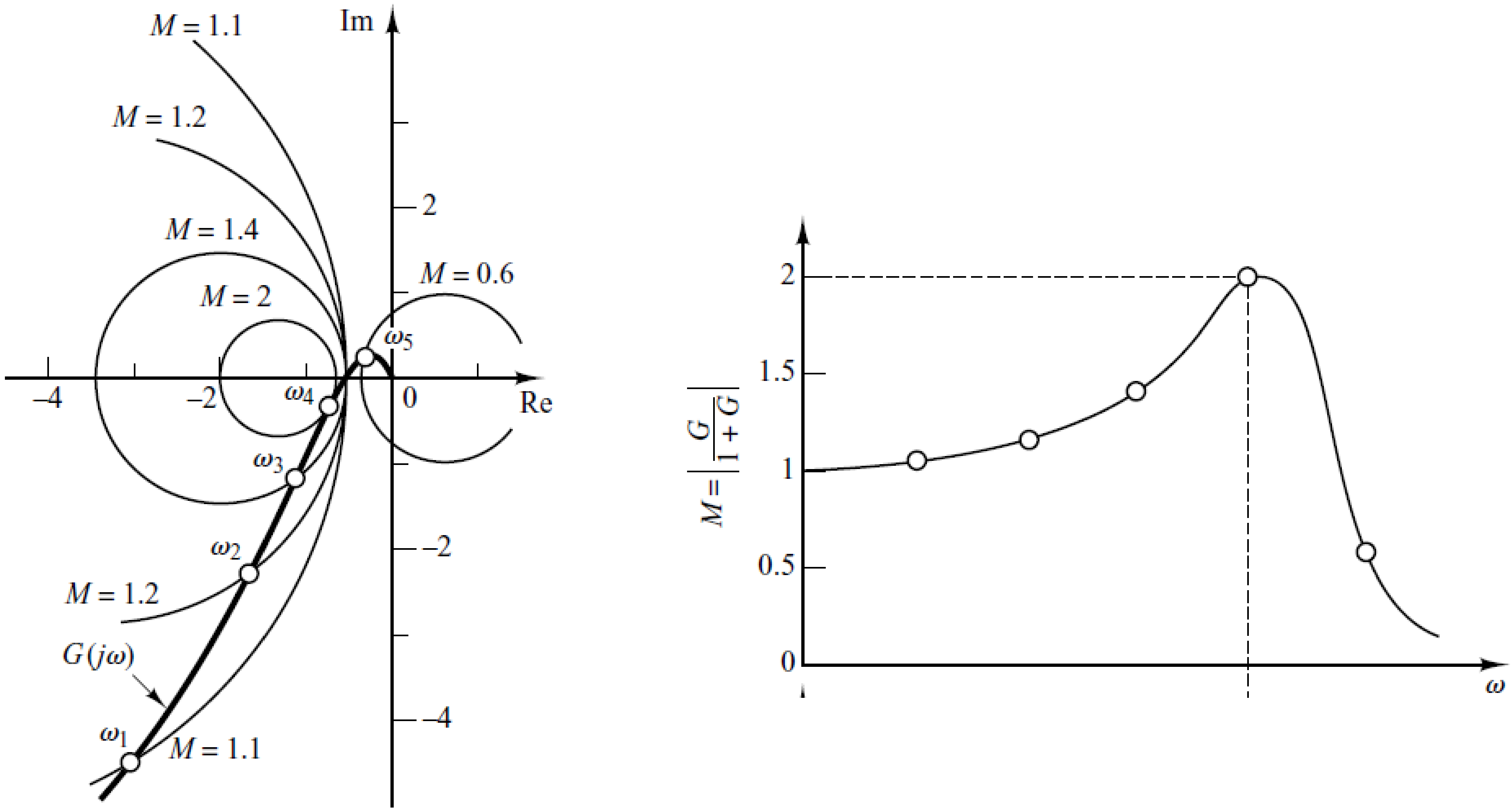


Deriving $|T(j\omega)|$ from $L(j\omega)$



- The values of resonance peak of $|T(j\omega)|$ is the value of the M_T circles which results tangent to the polar diagram of $L(j\omega)$

Deriving $|T(j\omega)|$ from $L(j\omega)$



M_S constant magnitude loci

- In a similar way, it is possible to derive the frequency response of the sensitivity function

$$S(j\omega) = \frac{1}{1 + L(j\omega)}$$

through the plot of $L(j\omega)$

- In this case the constant magnitude locus on the plane $(u, v) = (\mathbb{R}\mathrm{e}[L(j\omega)], \mathbb{I}\mathrm{m}[L(j\omega)])$ can be derived as

$$\frac{1}{(1+u)^2 + v^2} = M_S^2 = \text{const.} \rightarrow$$ **constant magnitude locus**

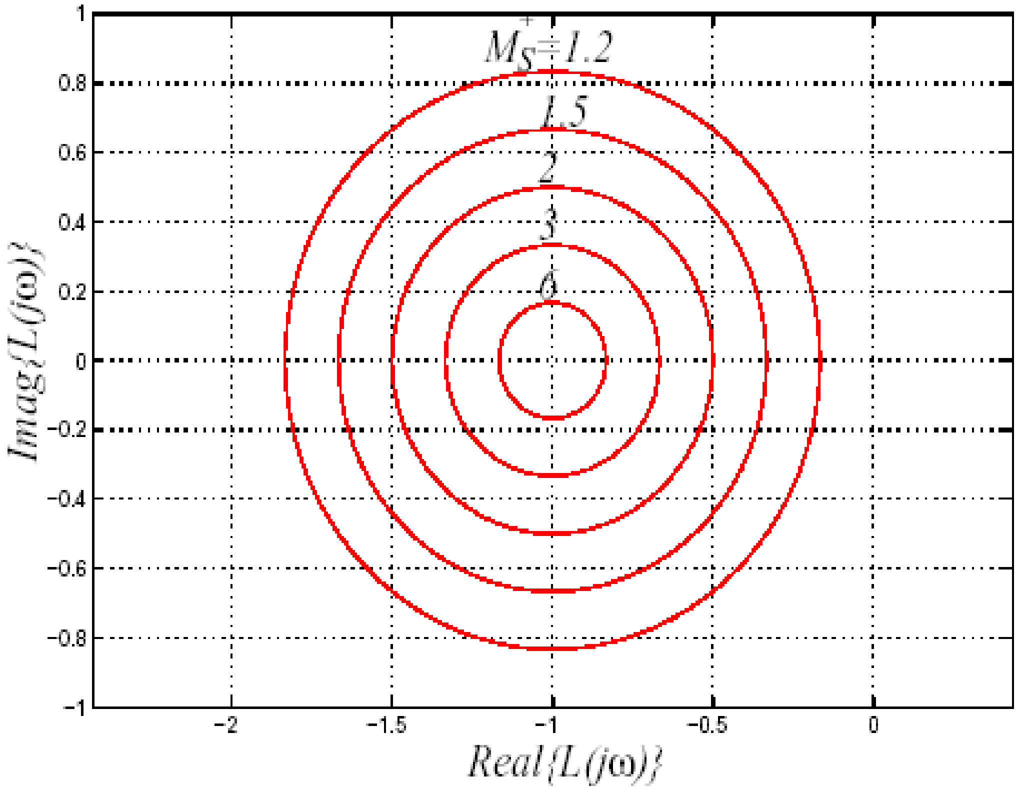
$$(u-1)^2 + v^2 = \frac{1}{M_S^2} \rightarrow \text{Eq. of a circle} \begin{cases} C = (-1, 0) \\ r = \dfrac{1}{M_S} \end{cases}$$

$\downarrow$

M_S circle

M_S constant magnitude loci

- All the M_S circles have the same center (-1, 0) and radius value which decreases for increasing values of M_S



- The superposition of a family of M_S circles on the plane (u, v), makes up a curvilineal coordinate system from which we can derive the plot of $|S(j\omega)|$ through the intersections of the polar plot of $L(j\omega)$

SISO feedback control systems characteristics

Relative Stability: Stability margins and the sensitivity peak

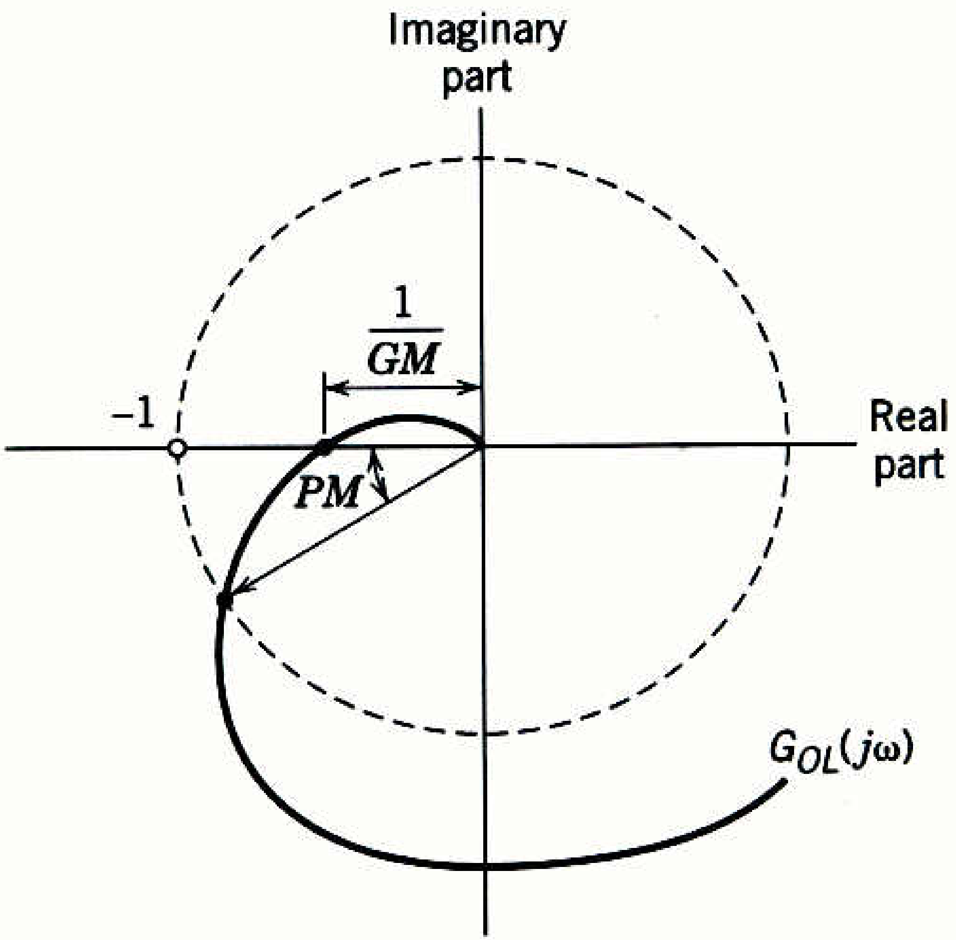
Stability margins

In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).

Gain and phase margins on a Nyquist plot

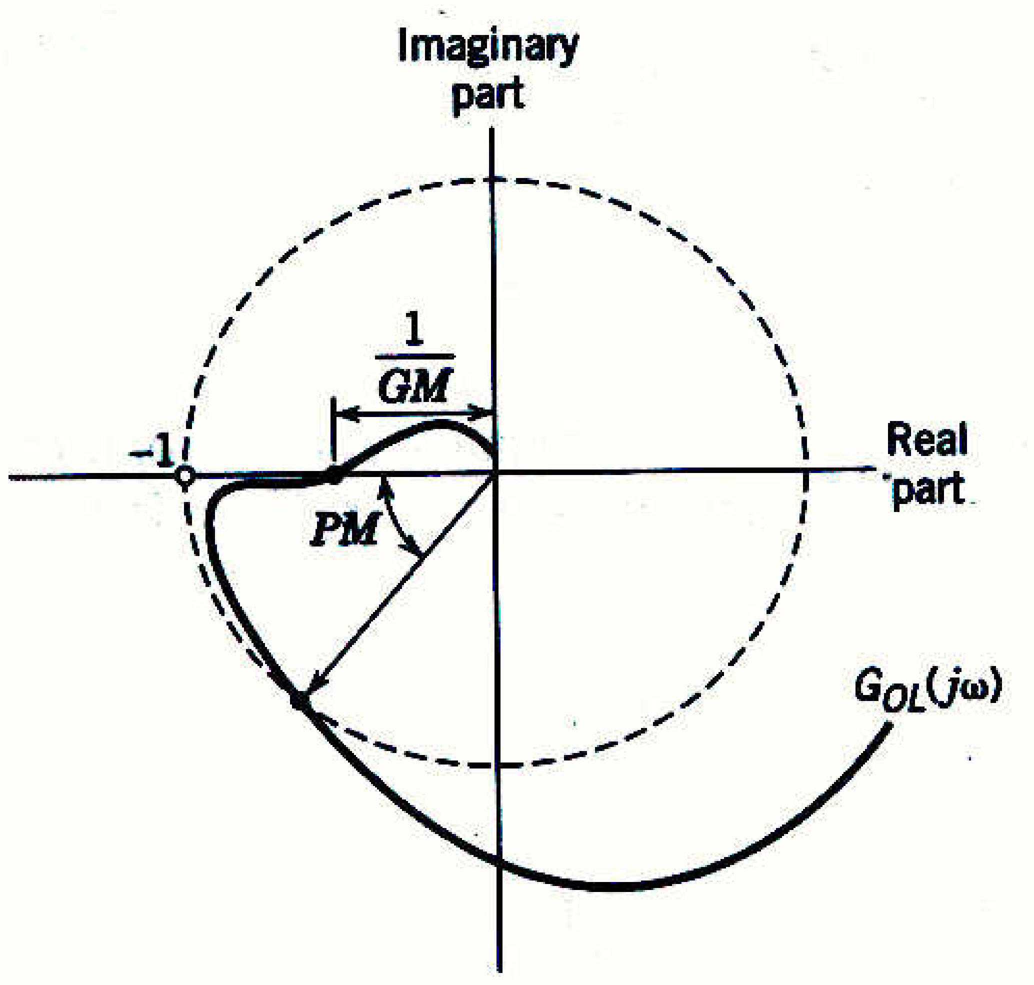
The phase margin and the gain margin provide a measure of relative stability.

The gain margin tells how much the gain has to be increased before the closed loop system becomes unstable and the phase margin tells how much the phase lag has to be increased to make the closed loop system unstable.



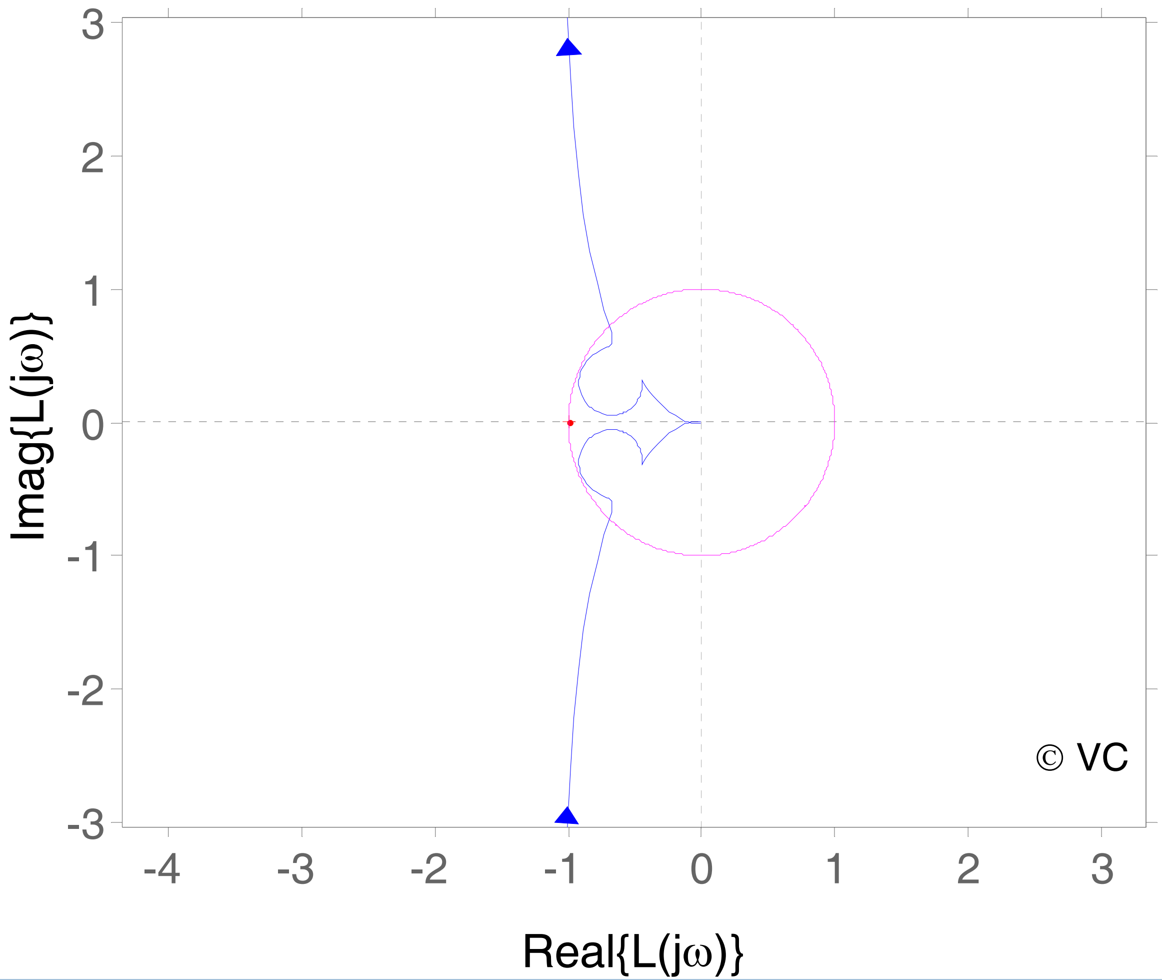
Example 1: Gain and phase margins on a Nyquist plot

Nyquist plot where the gain and phase margins are misleading.

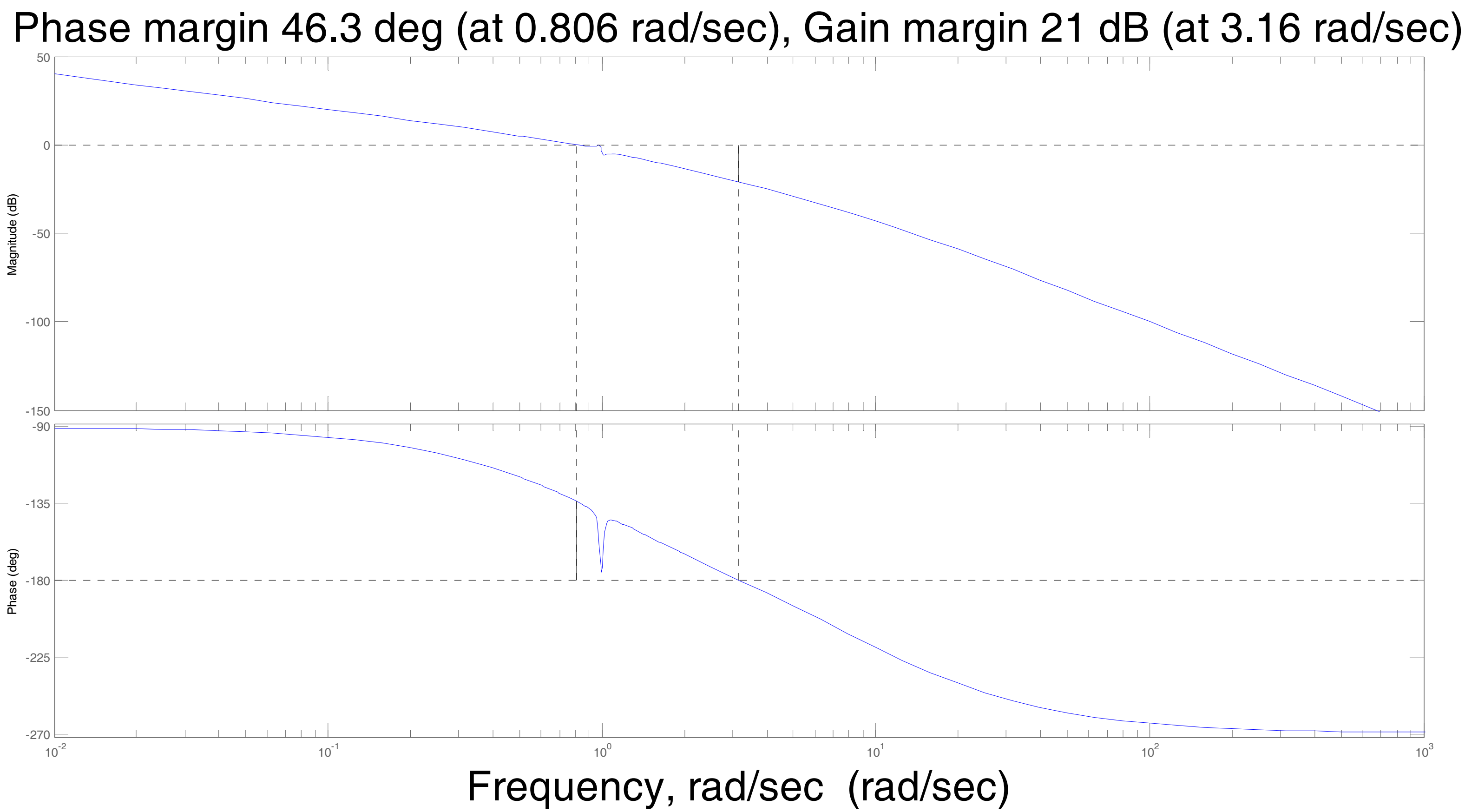


Example 2: Gain and phase margins on a Nyquist plot

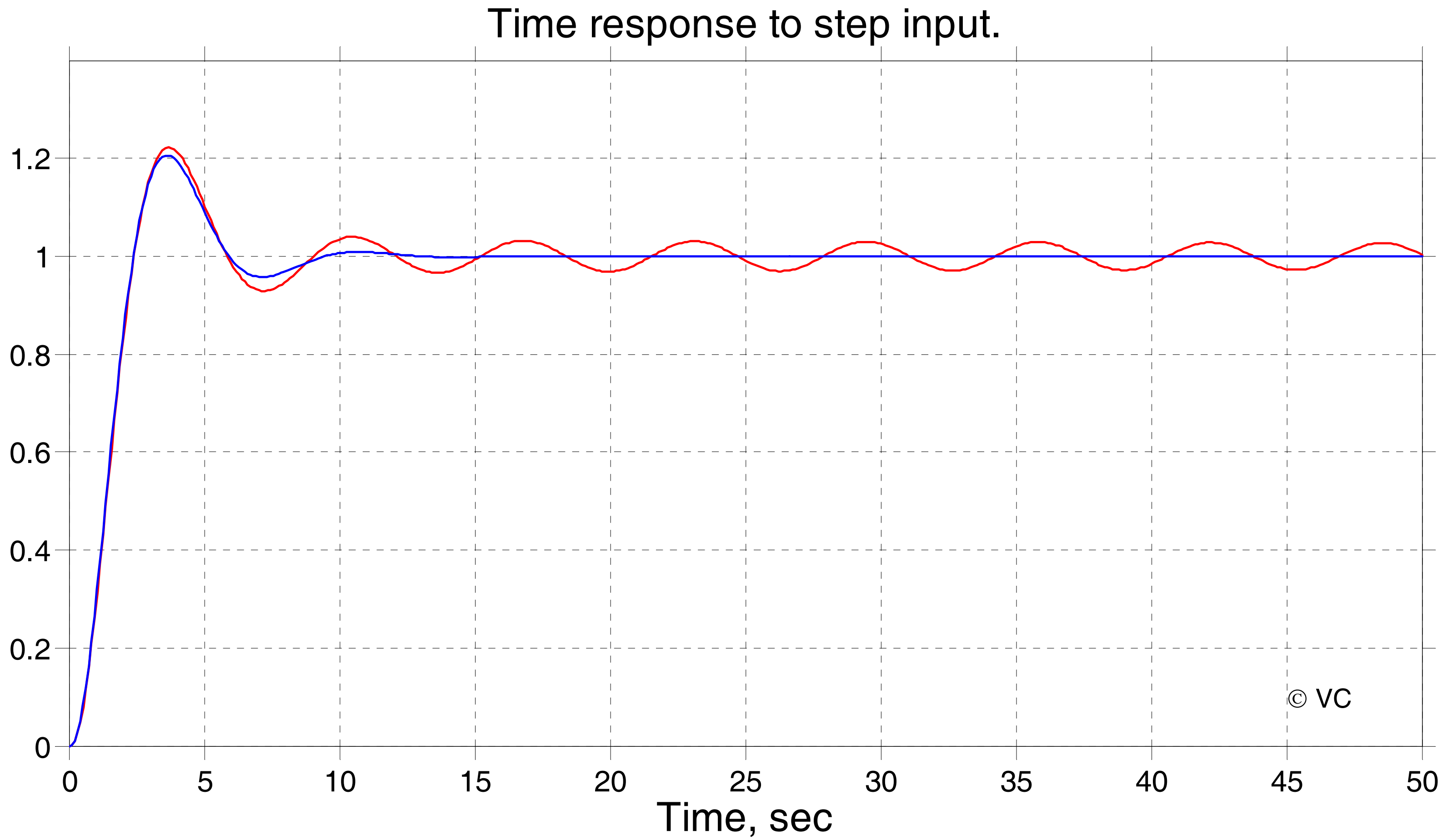
Nyquist plot where the gain and phase margins are misleading.



Example 2: Phase margin and Gain margin, Bode plot



Example 2: Time response to step input



Robust stability margins from the Sensitivity Function

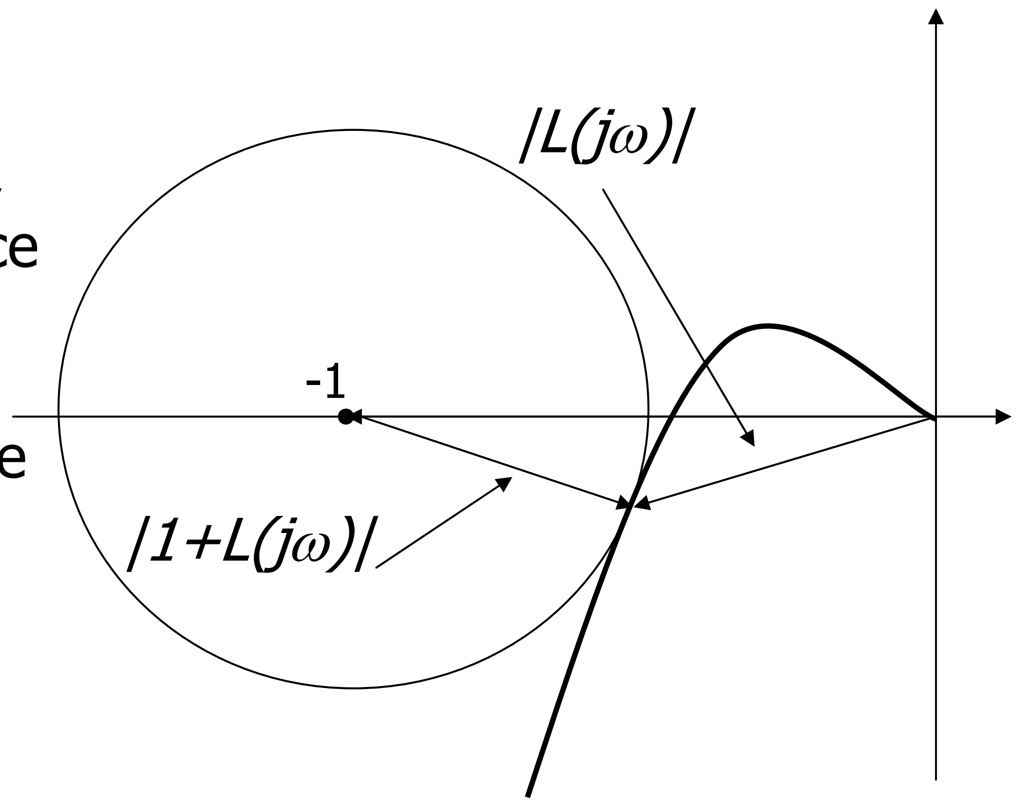
Now we show that the maximum amplitude of the Sensitivity function, that is S_p, can be used to obtain simple bounds on both the gain margin and the phase margin of stable closed loop systems.

We recall that S_p is the maximum value of $|S(j\omega)|$ for all frequencies

$$S_p \triangleq \max_{\omega} |S(j\omega)|$$

Robust stability margins from the Sensitivity Function

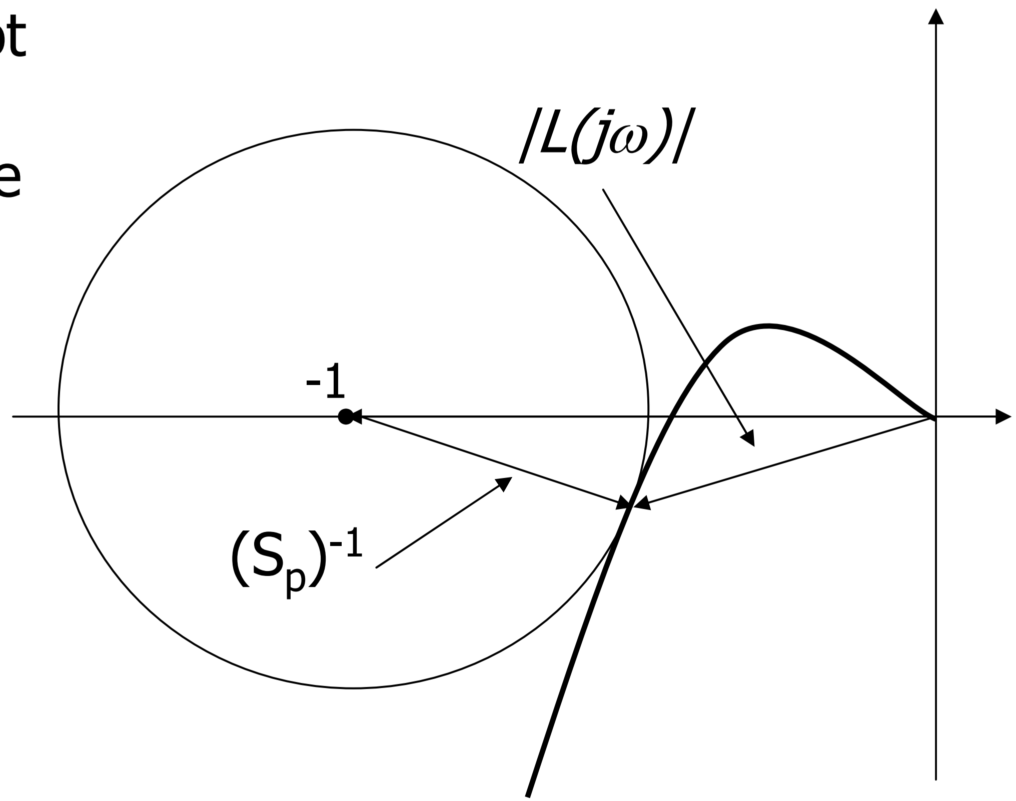
We observe that the frequency response magnitude of $(1+L(s))$, is the length of a vector drawn from -1 to $L(j\omega)$, that is $|1+L(j\omega)|$ is the distance from the loop function to the critical point -1. On the other hand, $L(j\omega)$ does not penetrate the constant magnitude M_s circle with radius $|1+L(j\omega)| = (S_p)^{-1}$, i.e., the inverse of the maximum amplitude of the frequency response of the Sensitivity function.



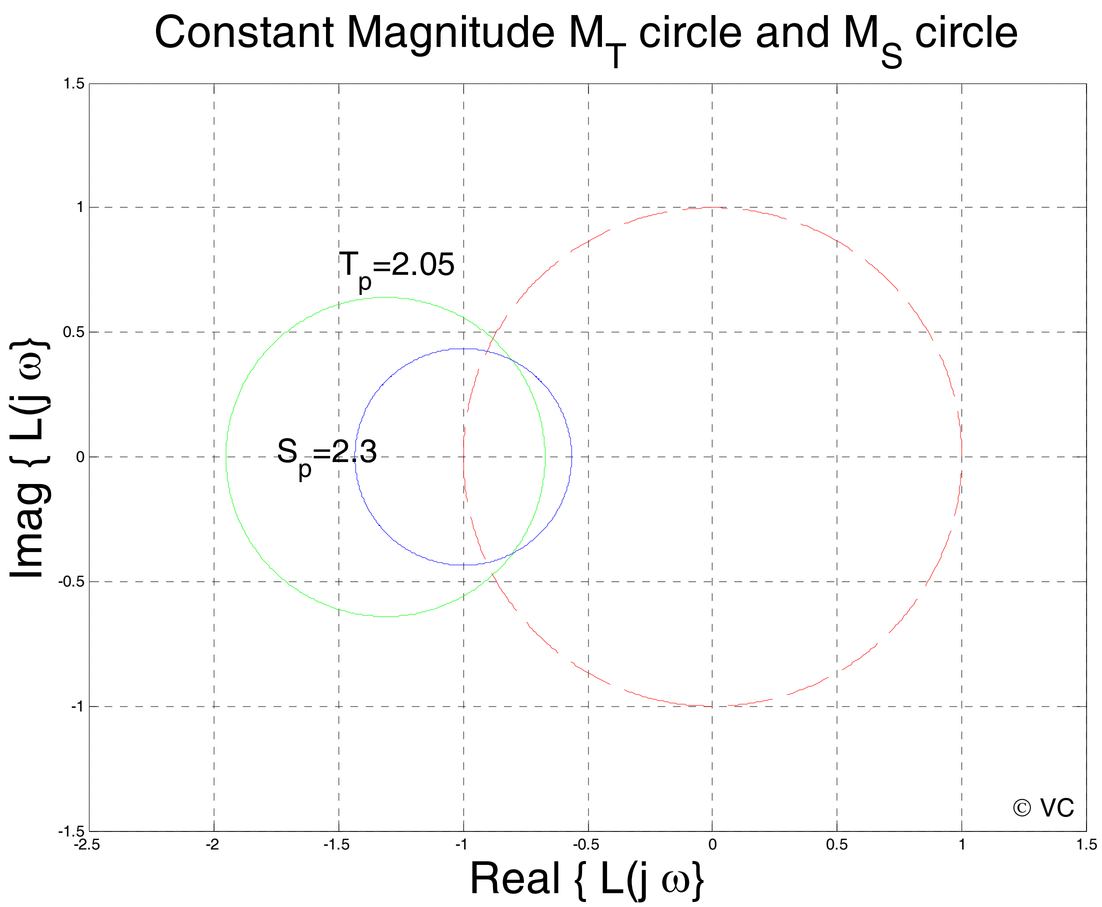
Robust stability margins from the Sensitivity Function

Note that $(S_p)^{-1}$ (hence S_p) can be obtained from the polar plot of $L(j\omega)$ by continuously increasing the radius of a circle centerd at -1 until it just contacts the $L(j\omega)$ contour.

The radius of the resulting circle is equal to $(S_p)^{-1}$.



Robust stability margins from S_p and T_p



Robust stability margins from S_p and T_p

Minimum bounds on both the gain margin and the phase margin can be expressed directly as functions of S_p and T_p.

Recall that S_p is the maximum value of $|S(j\omega)|$ for all frequencies:

$$S_p \triangleq \max_{\omega} |S(j\omega)|$$

T_p is the maximum value of $|T(j\omega)|$: for all frequencies:

$$T_p \triangleq \max_{\omega} |T(j\omega)|$$

Robust stability margins from S_p and T_p

It is easy to show that S_p and T_p are related to the gain and phase margins.

$$\mathrm{GM} \geq \frac{S_p}{S_p - 1}, \qquad \mathrm{PM} \geq 2\sin^{-1}\left(\frac{1}{2S_p}\right)$$

$$\mathrm{GM} \geq 1 + \frac{1}{T_p}, \qquad \mathrm{PM} \geq 2\sin^{-1}\left(\frac{1}{2T_p}\right)$$

SISO feedback control systems characteristics

Nichols chart

Nichol’s Chart (Log-magnitude vs Phase Plot)

Another approach to graphically portraying the frequency-response characteristics is to use the log-magnitude-versus-phase plot.

Which is a plot of the logarithmic magnitude in decibels versus the phase angle for a frequency range of interest.

In the manual approach the log-magnitude-versus-phase plot can easily be constructed by reading values of the log magnitude and phase angle from the Bode diagram.

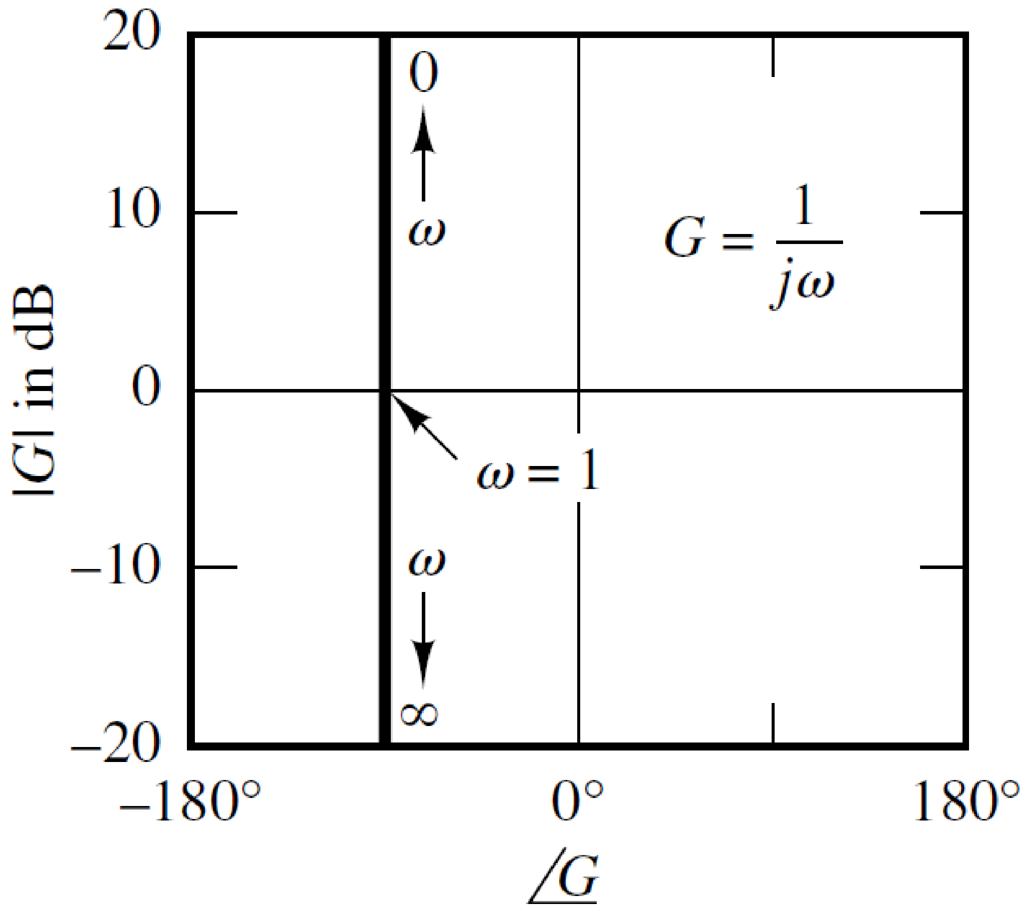
Advantages of the log-magnitude-versus-phase plot are that the relative stability of the closed-loop system can be determined quickly and that compensation can be worked out easily.

Nichol's Chart of simple transfer functions

$$G(j\omega)=\frac{1}{j\omega}$$

$$G(j\omega)=\frac{1}{\omega}\angle-90^\circ$$

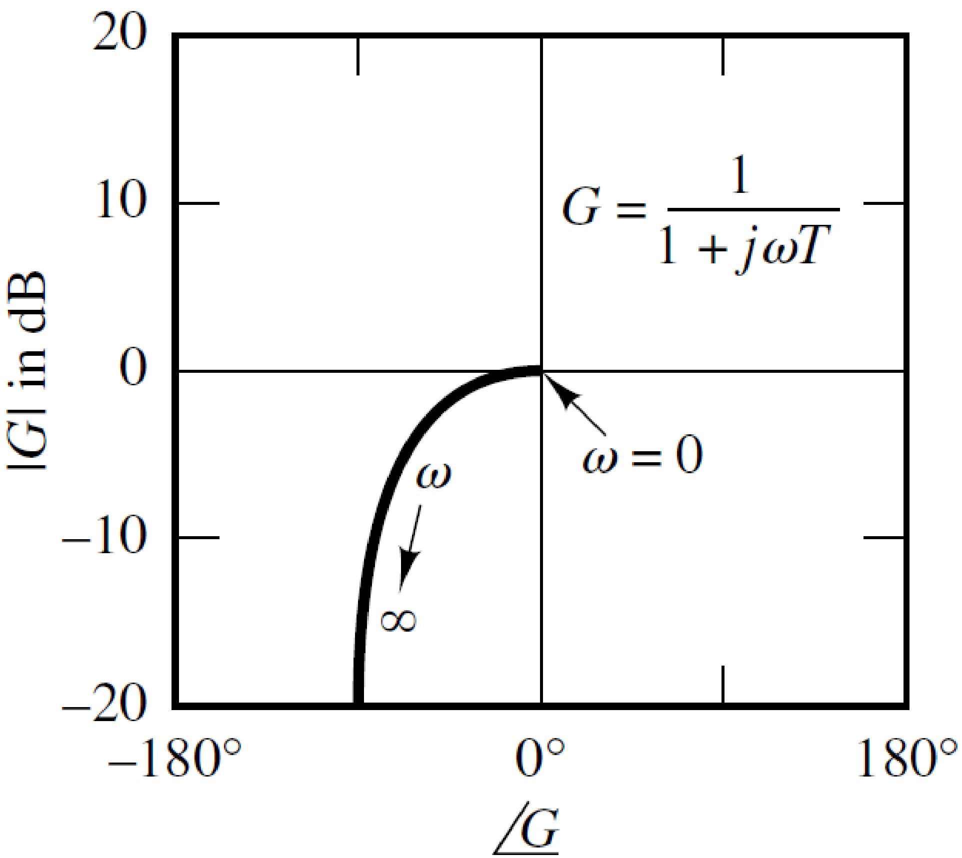
ω	dB	φ
0	∞	-90°
0.5	6	-90°
1	0	-90°
2	-6	-90°
∞	-∞	-90°



Nichol's Chart of simple transfer functions

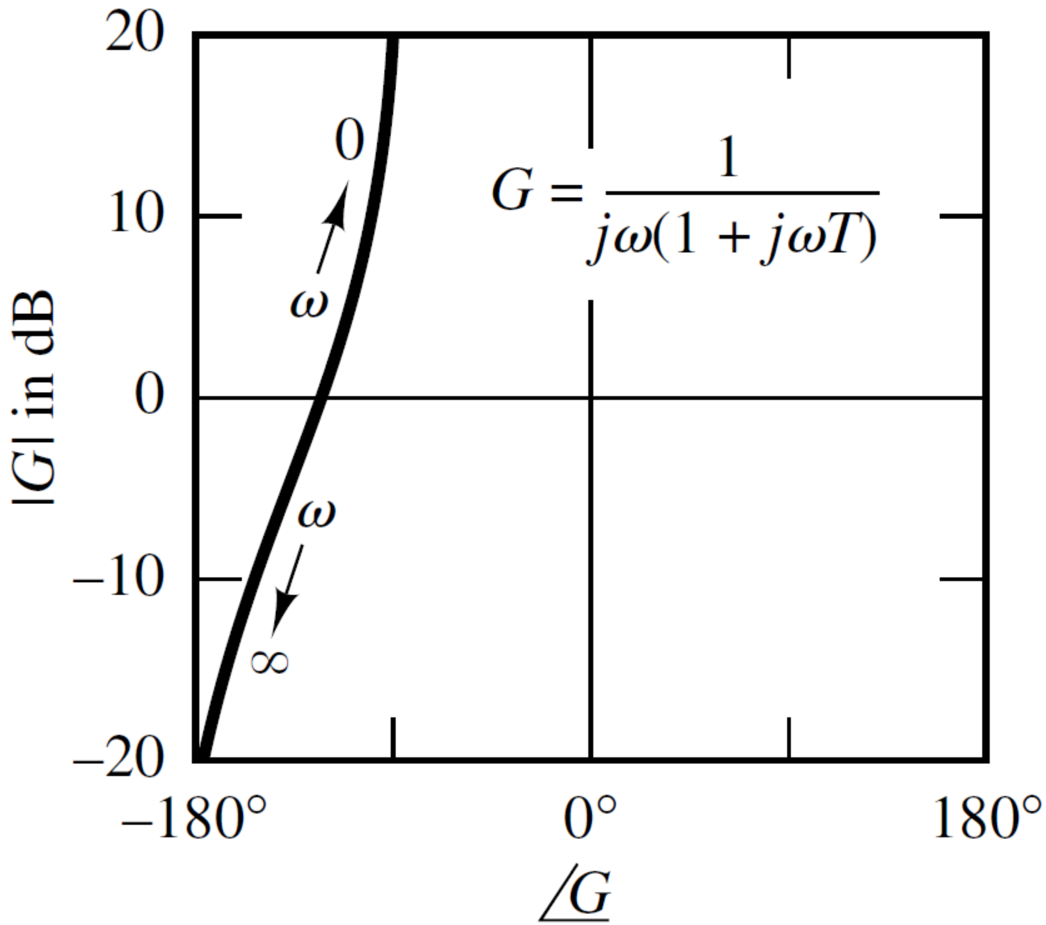
$$G(j\omega) = \frac{1}{j\omega T + 1}$$

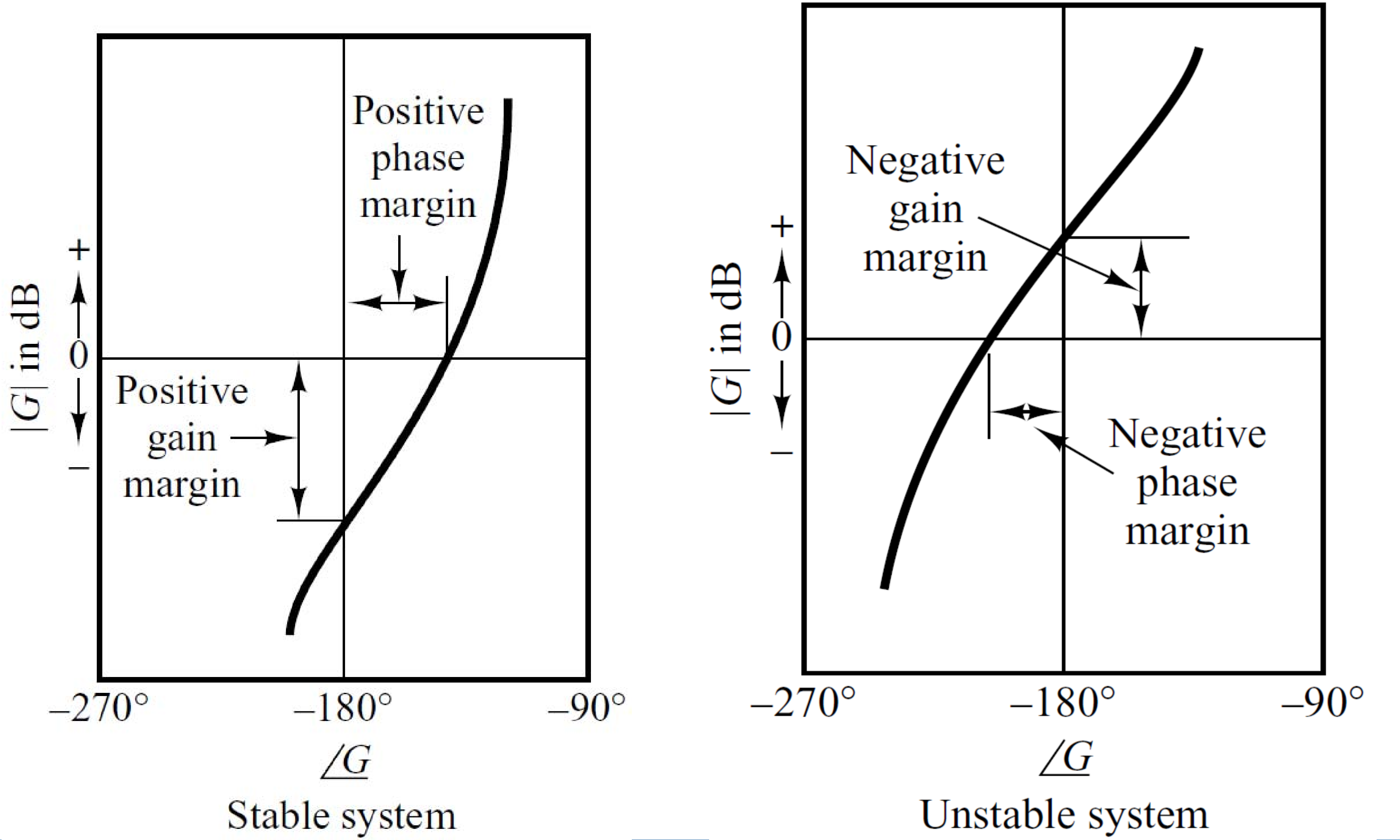
$$G(j\omega) = \frac{1}{\sqrt{T^2\omega^2 + 1}} \angle -\tan^{-1}(\omega T)$$



Nichol's Chart of simple transfer functions

$$G(j\omega) = \frac{1}{j\omega(j\omega T + 1)}$$





Positive
phase
margin
Positive
gain
margin
|G| in dB
+
0
–
–270°
–180°
–90°
∠G
Stable system
Negative
gain
margin
Negative
phase
margin
Unstable system

Constant magnitude loci on the Nichols plane

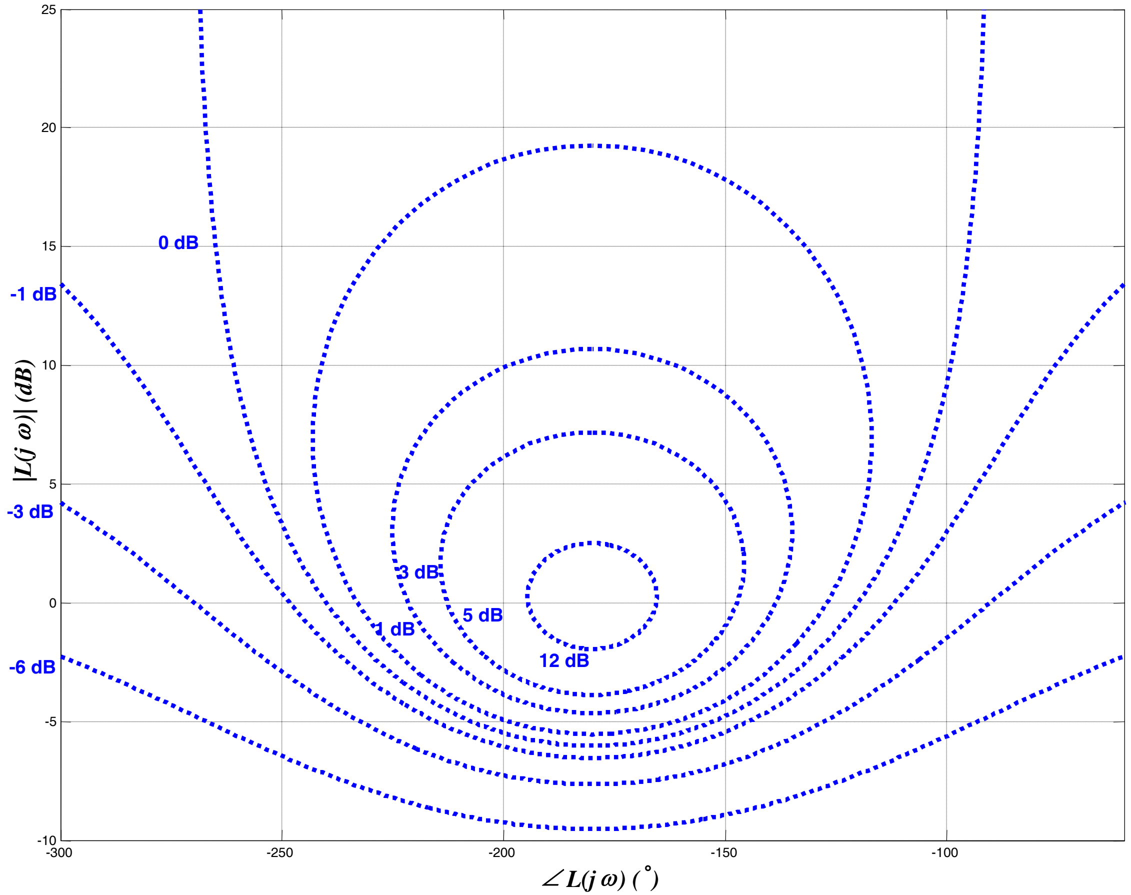
- The derivation of the constant magnitude loci M_T and M_S of the complementary sensitivity and the sensitivity functions can be performed also on the plane

$$(p, q) = (\angle L(j\omega)), |L(j\omega)|_{\mathrm{dB}}) \rightarrow \text{Nichols plane}$$

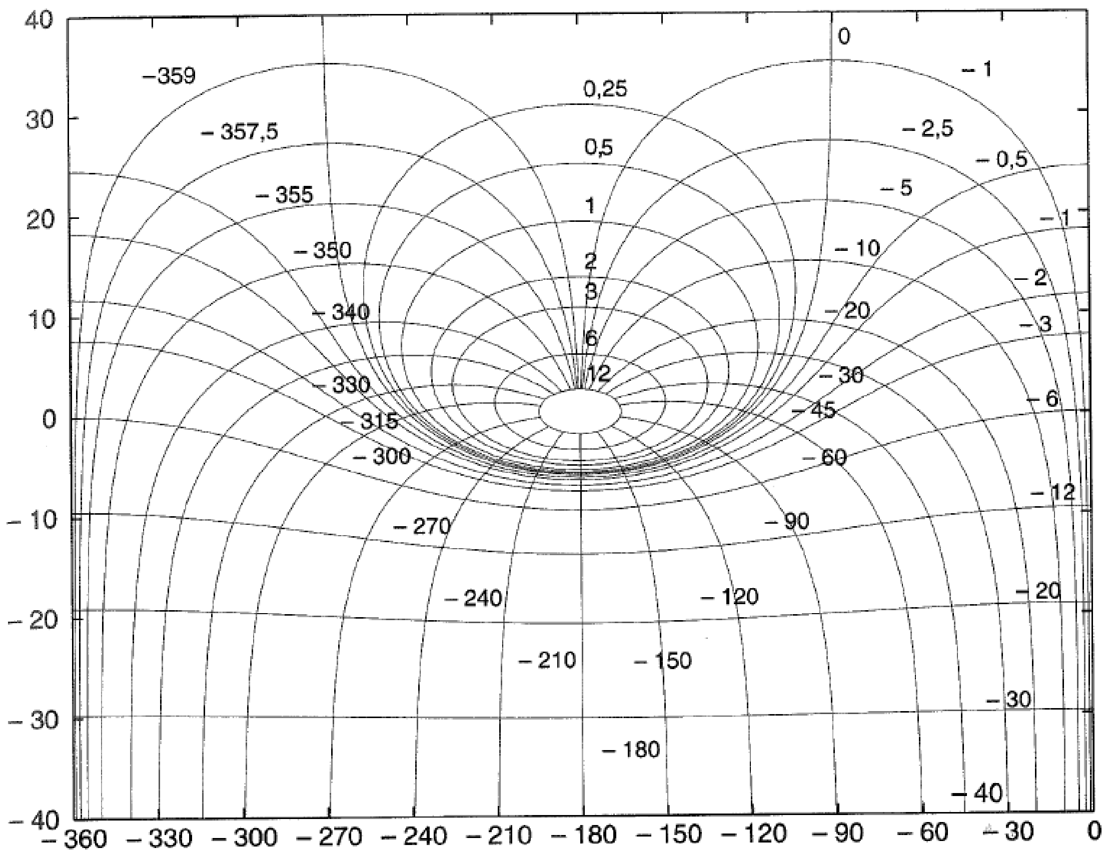
- On the Nichols plane the constant magnitude loci M_T e M_S are:
  - closed curves for magnitude values greater than 1
  - concave open curves for magnitude values less or equal to 1

- The superposition of a family of M_T and M_S constant magnitude loci on the plane (p, q), makes up a curvilineal coordinate system which helps to derive the plot of $|T(j\omega)|$ and $|S(j\omega)|$ through the intersections of the Nichols plot of $L(j\omega)$

M_T constant magnitude loci on the Nichols plane



Nichols chart



Example

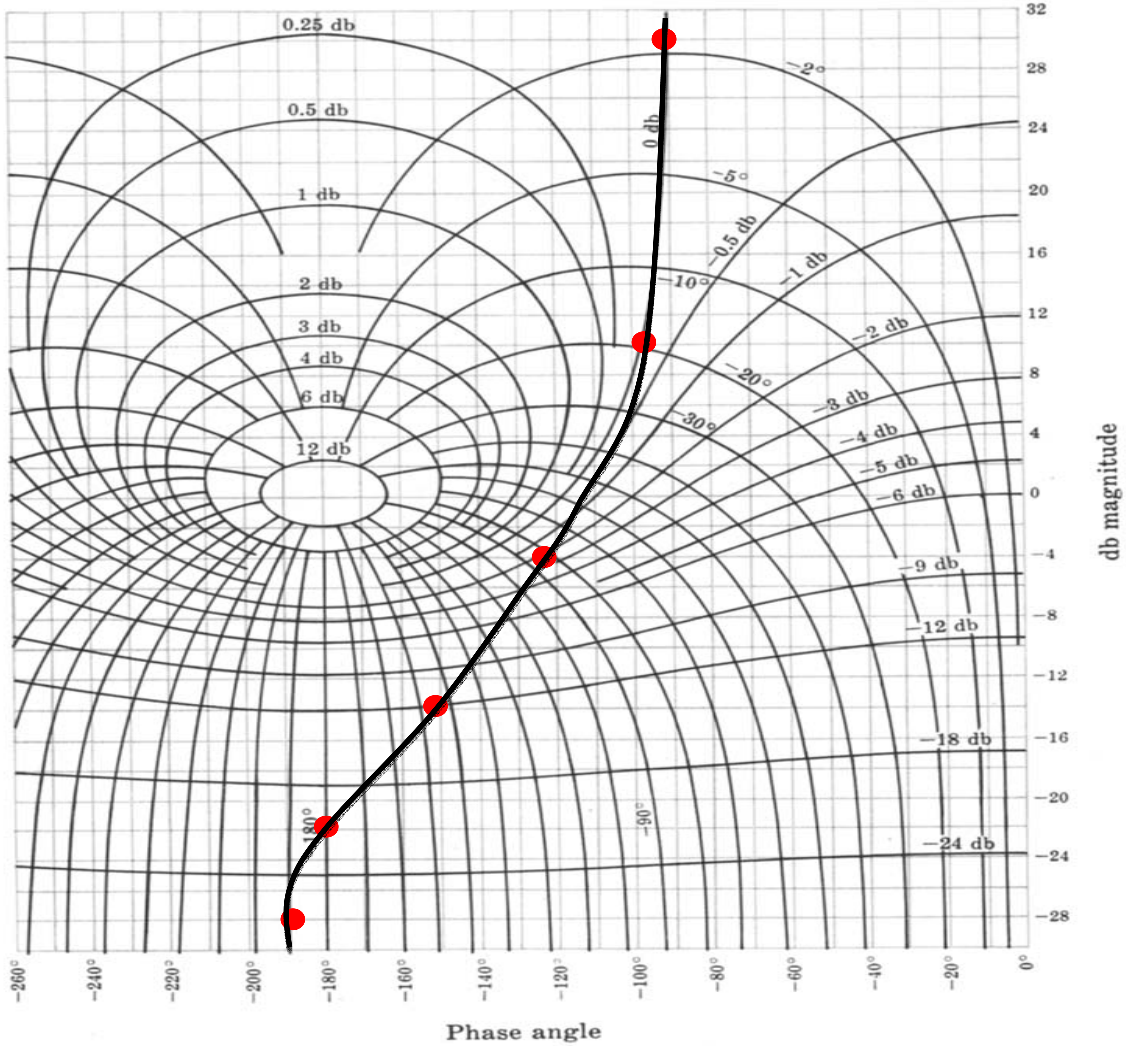
Draw the Nichol's plot of following open loop transfer function and obtain the Gain Margin and Phase Margin.

$$G(s) = \frac{1}{s(s+1)(s+3)}$$

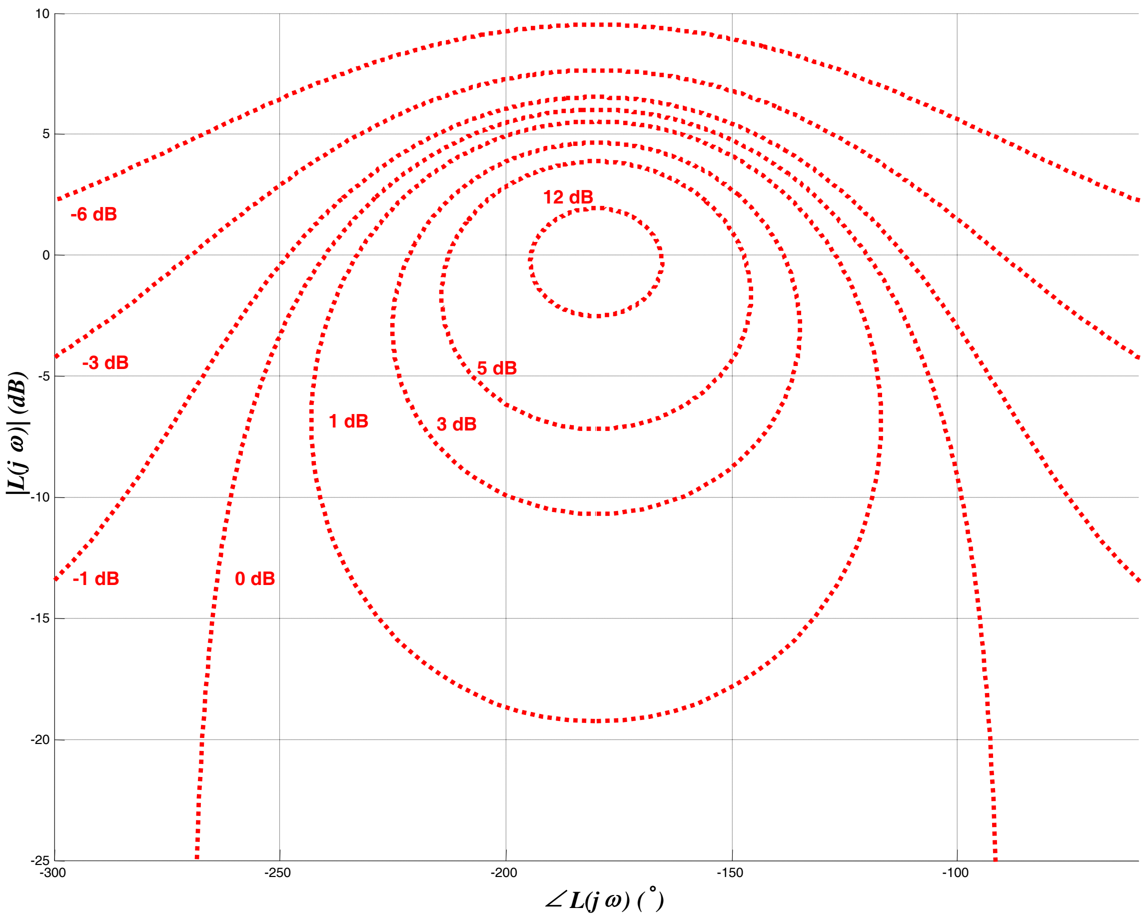
$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+3)}$$

Example

ω	$\left|G(j\omega)\right|_{db}$	$\angle G(j\omega)$
0.01	30	-90°
0.1	10.3	-97.5°
0.5	-4.4	-125°
1	-14	-153°
2	-22	-180°
10	-26	-189°



M_S constant magnitude loci on the Nichols plane



M_T and M_S constant loci on the Nichols plane