Automatic Control

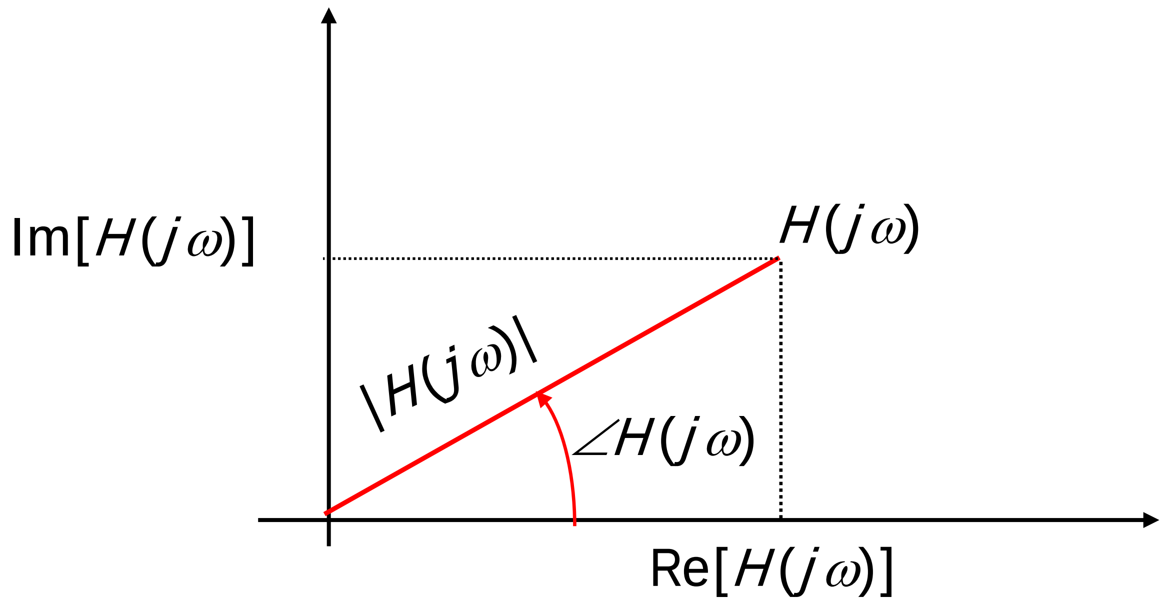
Frequency response tools for analysis and design of feedback control systems

- Part I: Bode diagrams resume

Frequency response graphical representations

Frequency response function

The function $H(j\omega) : \mathbb{R}^+ \to \mathbb{C}$ of the variable $\omega \in \mathbb{R}^+$ is called **frequency response funtion** of the system:



$$H(j\omega) = \mathrm{Re}[H(j\omega)] + \mathrm{Im}[H(j\omega)] \rightarrow$$ **Cartesian representation**

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)} \rightarrow$$ **Polar representation**

Graphical representations of the frequency response

The **frequency response function** of a dynamic system can be graphically represented through:

Bode diagrams $\rightarrow$ representation of $|H(j\omega)|$ and $\angle H(j\omega)$ in function of $\omega \in \mathbb{R}^+$

Polar diagram $\rightarrow$ representation of $\mathrm{Im}[H(j\omega)]$ vs. $\mathrm{Re}[H(j\omega)]$ parameterized in $\omega \in \mathbb{R}^+$

Nichols diagram $\rightarrow$ representation of $|H(j\omega)|$ vs. $\angle H(j\omega)$ parameterized in $\omega \in \mathbb{R}^+$

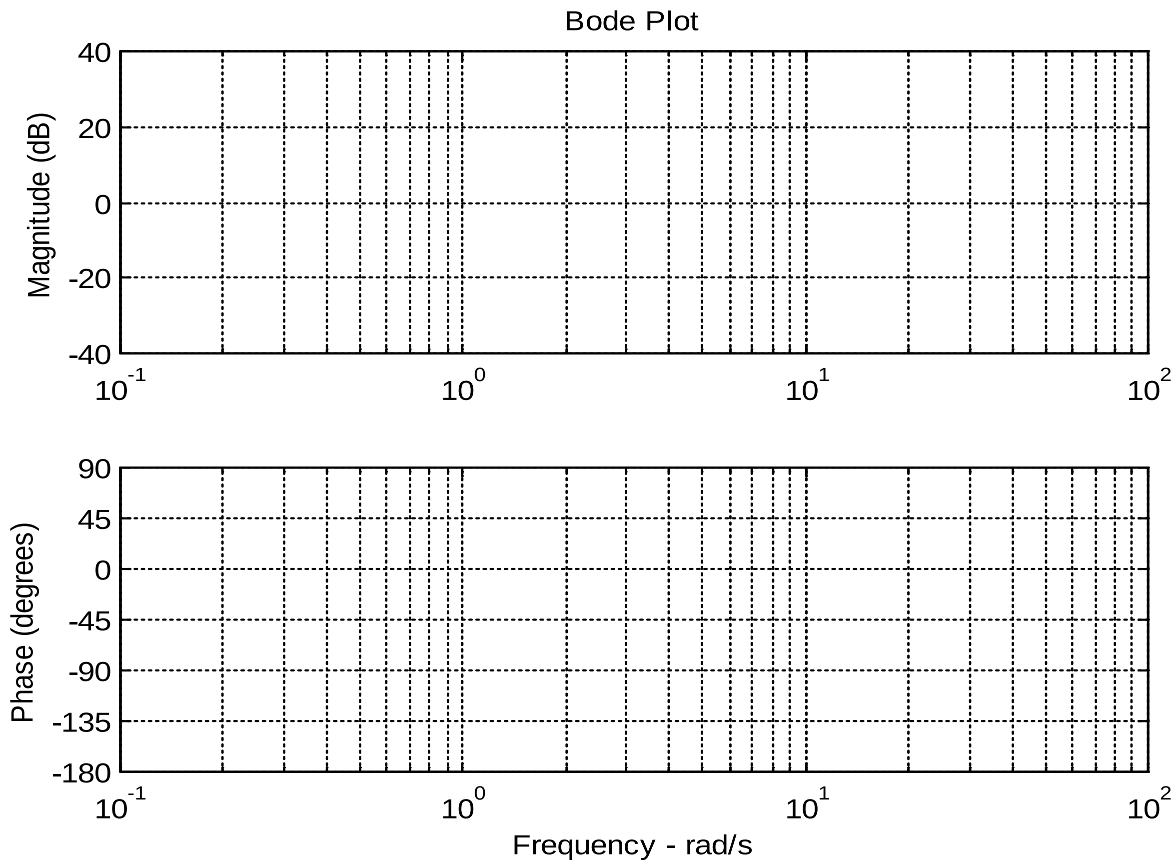
Bode plots: resume

Graphical representations: Bode plots

Bode plots $\rightarrow$ plots of $|H(j\omega)|$ and $\angle H(j\omega)$ in function of $\omega \in \mathbb{R}^+$

- Magnitude Bode diagram $\rightarrow$ $|H(j\omega)|$ in function of ω
  - $|H(j\omega)|$ expressed in dB $|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)|$, linear scale
  - ω expressed in rad/s , logarithmic scale
- Phase Bode diagram $\rightarrow$ $\angle H(j\omega)$ in function of ω
  - $\angle H(j\omega)$ expressed in degrees (°) (or in rad), linear scale
  - ω expressed in rad/s , logarithmic scale

Bode plots $\rightarrow$ representation of $|H(j\omega)|$ and $\angle H(j\omega)$ in function of $\omega \in \mathbb{R}^{+}$



The dc-gain form of system transfer function

$$H(s) = K \frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_m)}{s^r (1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_{n-r})}$$

- $z_1, \ldots, z_m$ → zeros of $H(s)$
- r → poles of $H(s)$ at the origin
- $p_1, \ldots, p_{n-r}$ → poles of $H(s)$
- K → generalized dc-gain $\rightarrow K = \lim_{s \to 0} s^r H(s)$

Example: $H(s) = \dfrac{s+5}{s^2 + 3s + 2} = \dfrac{5(1 + s/5)}{1 \cdot (1+s) \cdot 2 \cdot (1 + s/2)} = \dfrac{5}{2} \dfrac{1 + s/5}{(1+s)(1+s/2)}$

No specific MatLab statement

Consider the dc-gain form of $H(s)$

$$H(s) = K \frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_m)}{s^r (1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_{n-r})}$$

$$\rightarrow K = \lim_{s \to 0} s^r H(s) \quad \text{generalized dc-gain}$$

$$\rightarrow \begin{cases} \text{zeros in } s = z_i \\ \text{poles in } s = p_i \end{cases}$$

$$\begin{cases} r = 0 \rightarrow \text{ no singularities at } s = 0 \\ r > 0 \rightarrow \text{ poles at } s = 0 \\ r < 0 \rightarrow \text{ zeros at } s = 0 \rightarrow K \dfrac{s^r (1-s/z_1)(1-s/z_2)\cdots(1-s/z_{m-r})}{(1-s/p_1)(1-s/p_2)\cdots(1-s/p_n)} \end{cases}$$

Bode plots computation

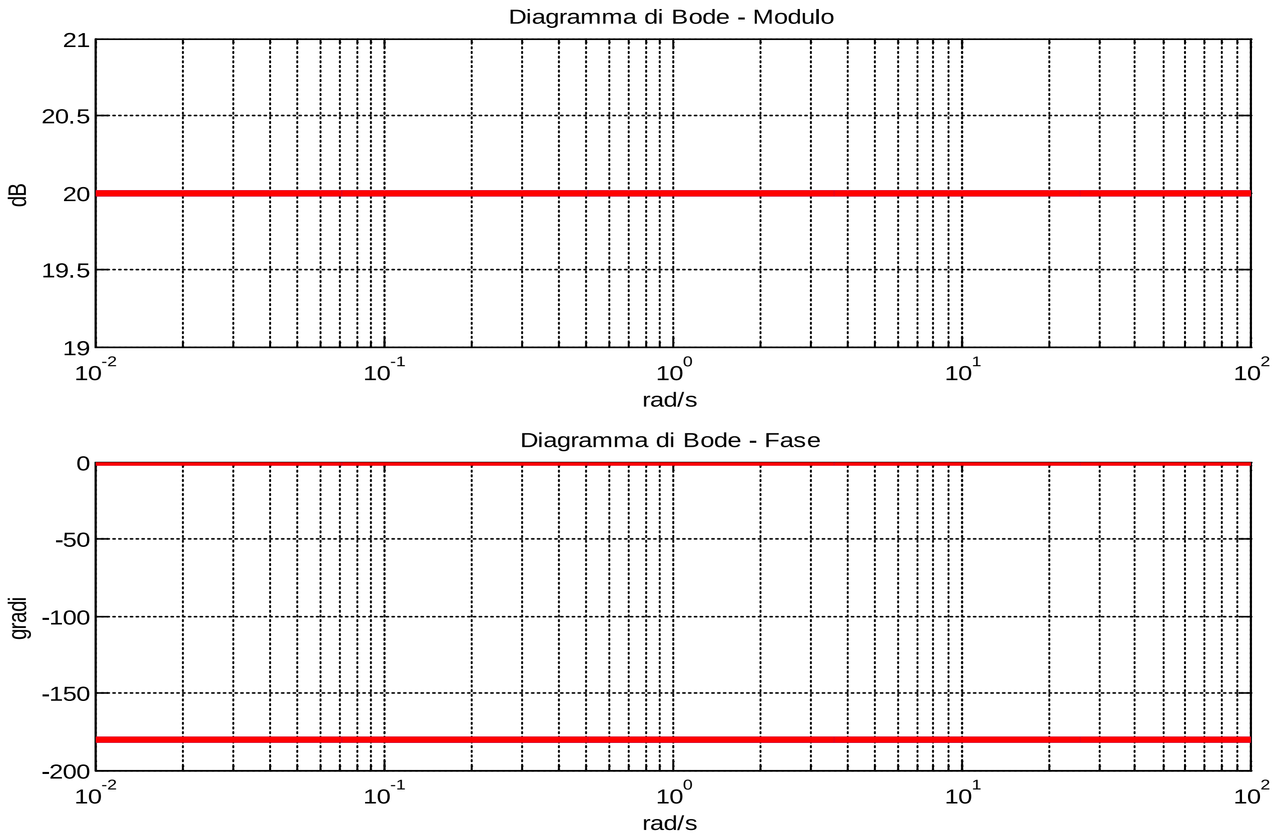
$$H(s) = 10\frac{1+s/2}{s\left(1+s+s^2\right)}$$

Frequency response: $H(j\omega) = \overbrace{10}^{H_0(j\omega)} \frac{\overbrace{1+j\omega/2}^{H_1(j\omega)}}{\underbrace{j\omega}_{H_2(j\omega)}\underbrace{\left(1+j\omega-\omega^2\right)}_{H_3(j\omega)}} = H_0(j\omega)\frac{H_1(j\omega)}{H_2(j\omega)H_3(j\omega)}$

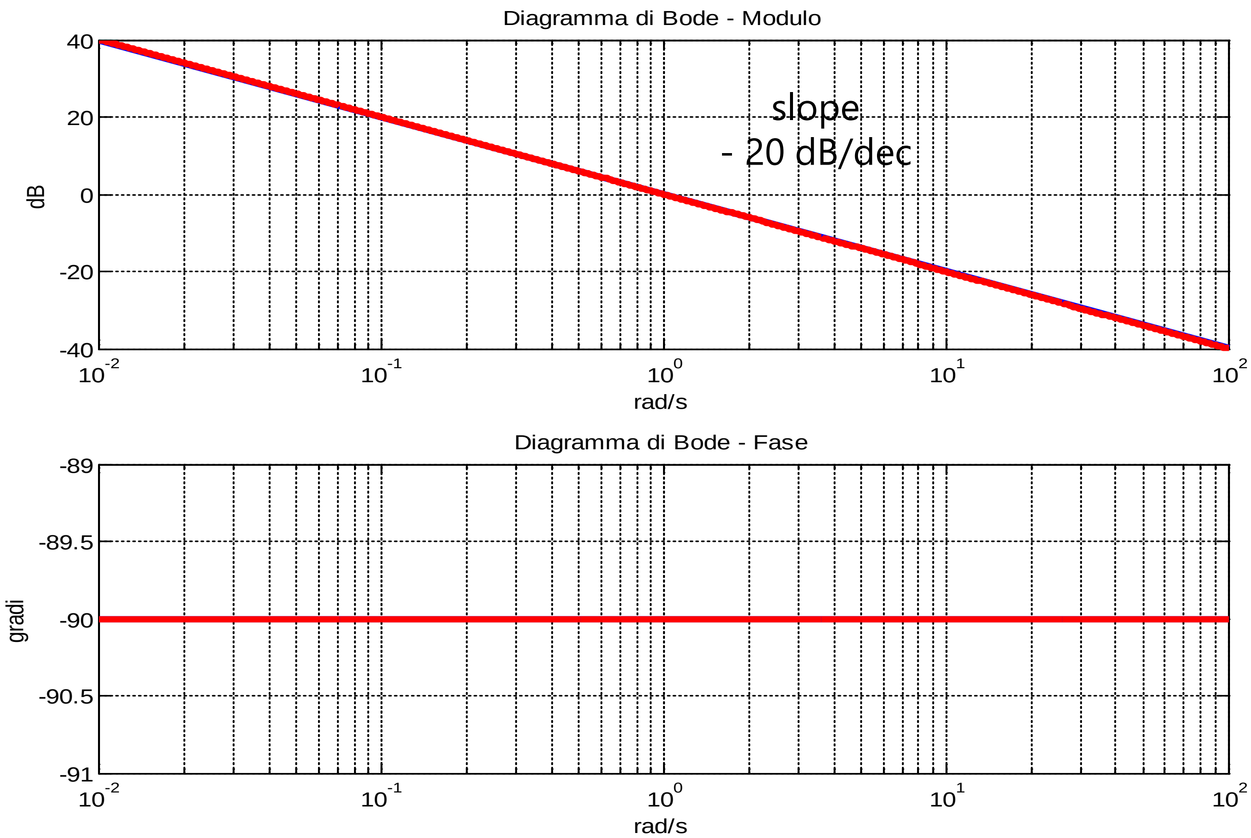
magnitude $\rightarrow \begin{cases} |H(j\omega)|_{\log} = |H_0(j\omega)|_{\log} + |H_1(j\omega)|_{\log} - |H_2(j\omega)|_{\log} - |H_3(j\omega)|_{\log} \\ |H(j\omega)|_{\log} = 20\log_{10}\left(|H(j\omega)|\right) \quad dB \end{cases}$

phase $\angle H(j\omega) = \angle H_0(j\omega) + \angle H_1(j\omega) - \angle H_2(j\omega) - \angle H_3(j\omega)$

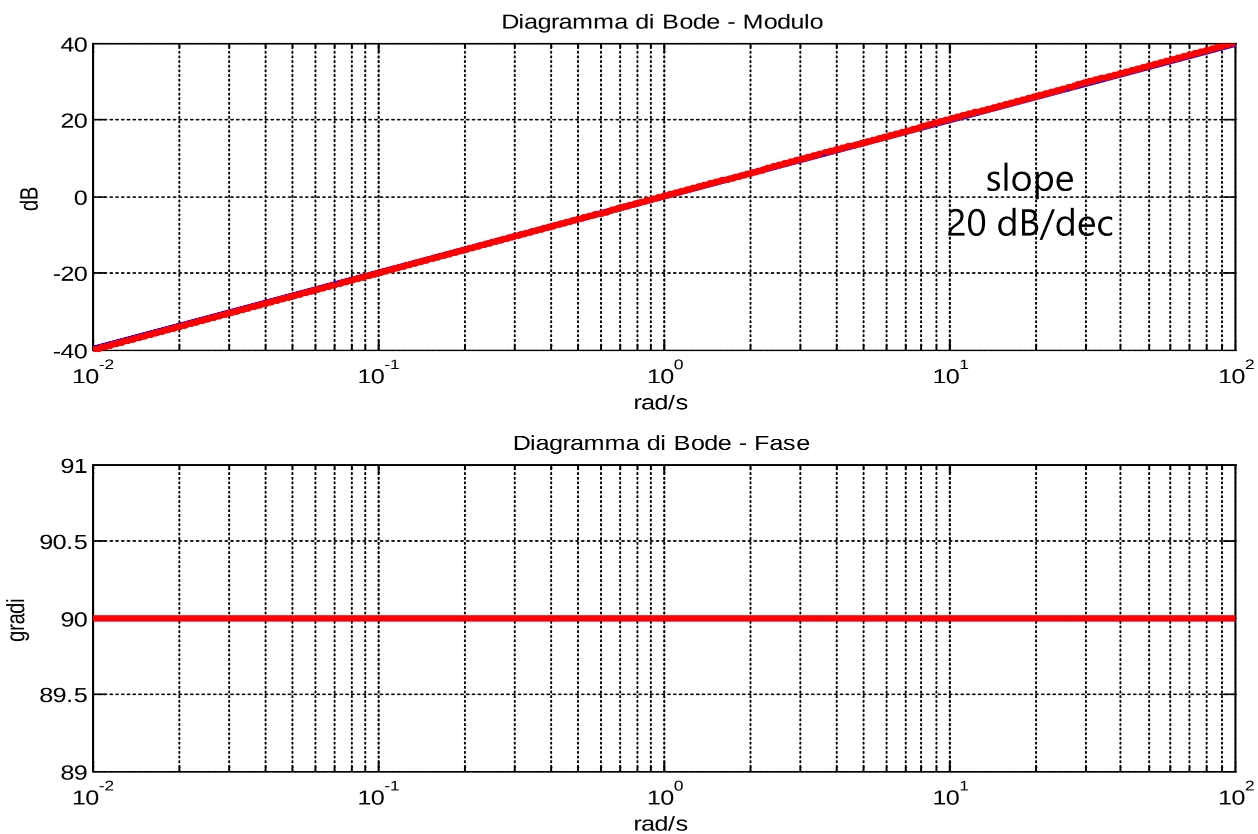
Costant gain $H(j\omega) = K$



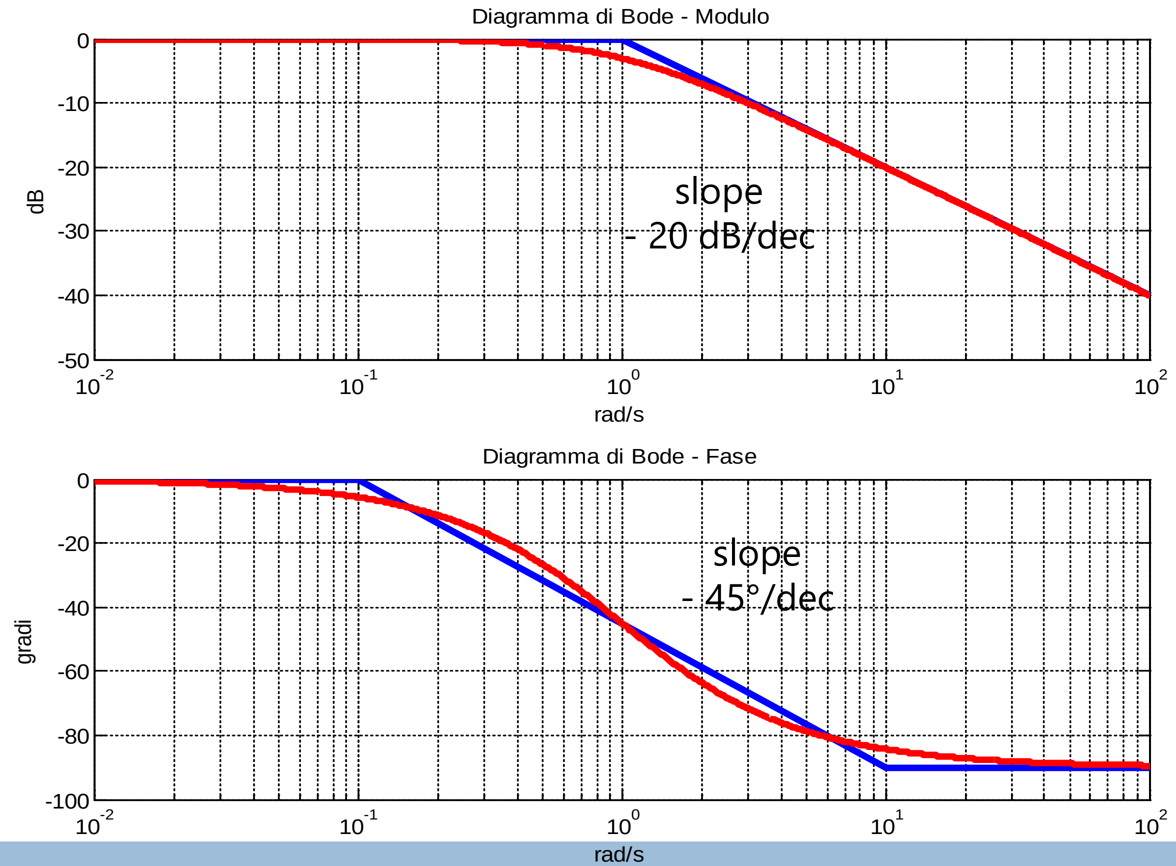
Pole at the origin $H(j\omega) = \dfrac{1}{j\omega}$



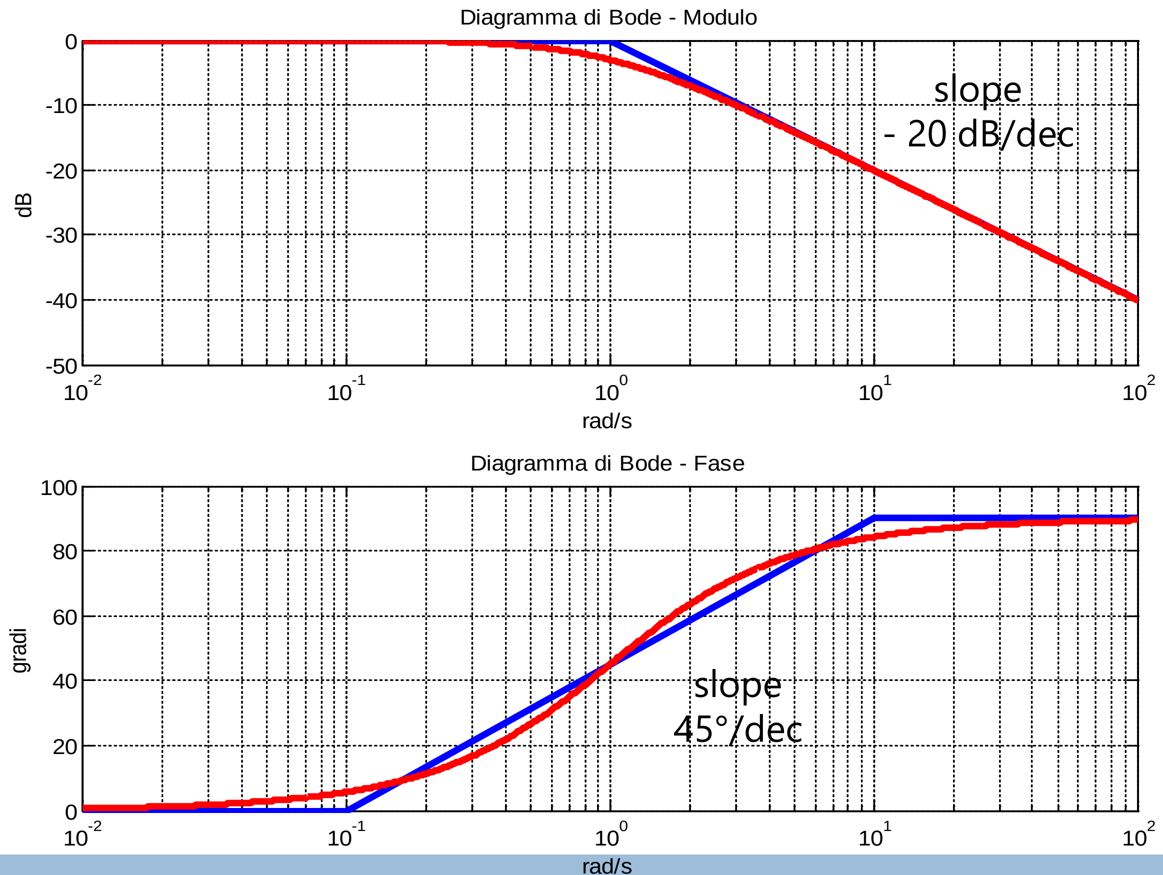
Zero at the origin $H(j\omega) = j\omega$



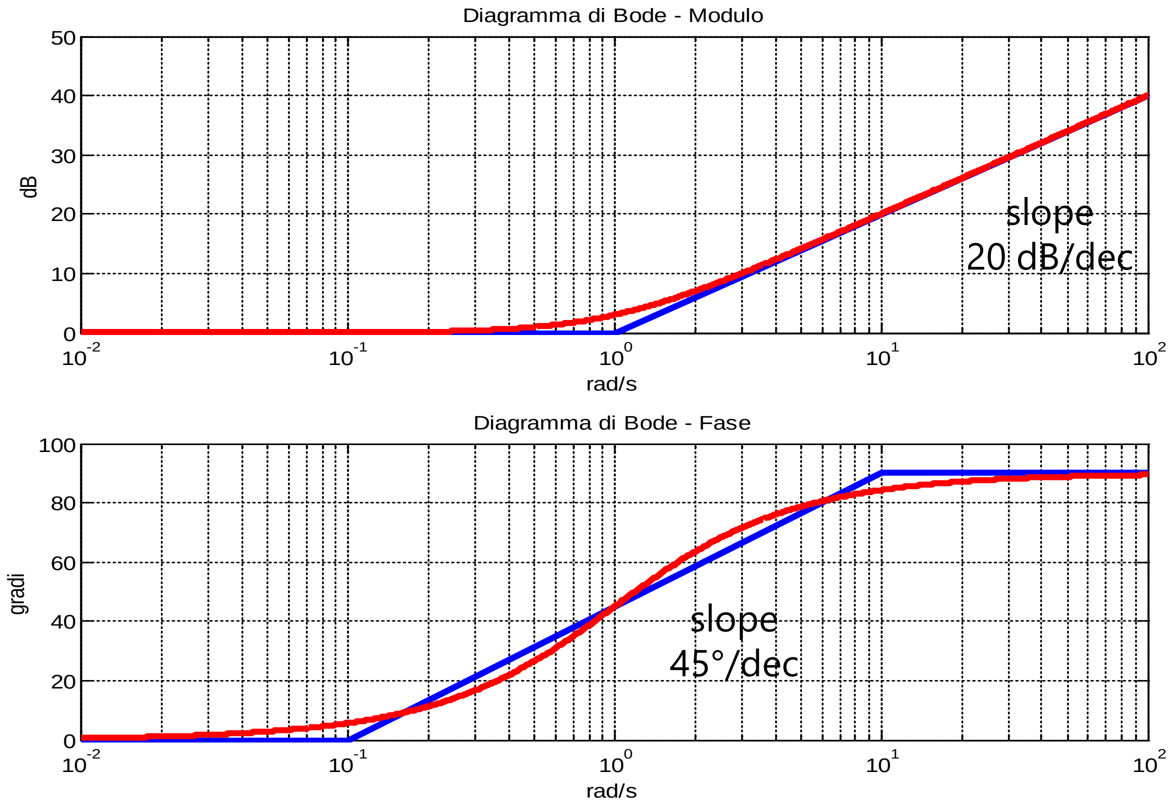
Real negative pole $H(j\omega)=\dfrac{1}{1-j\dfrac{\omega}{p}} \underset{Example\ p=-1}{\overset{\uparrow}{=}} \dfrac{1}{1+j\omega}$



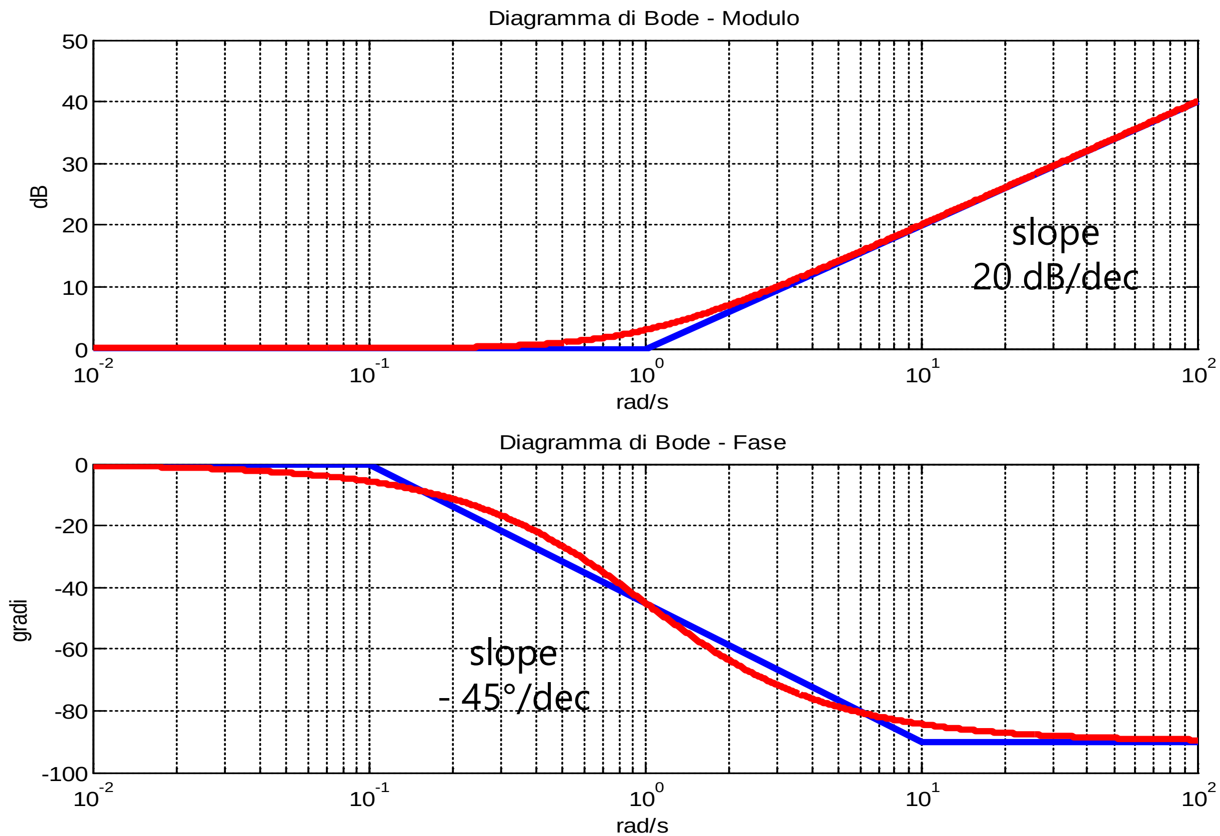
Real positive pole $H(j\omega) = \dfrac{1}{1 - j\dfrac{\omega}{p}} \underset{\substack{\uparrow \\ \text{Example } p=+1}}{=} \dfrac{1}{1 - j\omega}$



Real negative zero $H(j\omega) = 1 - j\dfrac{\omega}{z} \underset{\underset{z=-1}{\uparrow}}{=} 1 + j\omega$



Real positive zero $H(j\omega) = 1 - j\dfrac{\omega}{z} \underset{\substack{\uparrow \\ \text{Example } z=1}}{=} 1 - j\omega$



Bode plots computation

Complex conjugate negative poles $H(s) = \dfrac{1}{1 + 2\dfrac{\zeta}{\omega_n}s + \dfrac{s^2}{\omega_n^2}} = \underset{\text{Example } \zeta=0.5,\, \omega_n=1}{\overset{}{\underset{\uparrow}{=}}} \dfrac{1}{1+s+s^2}$

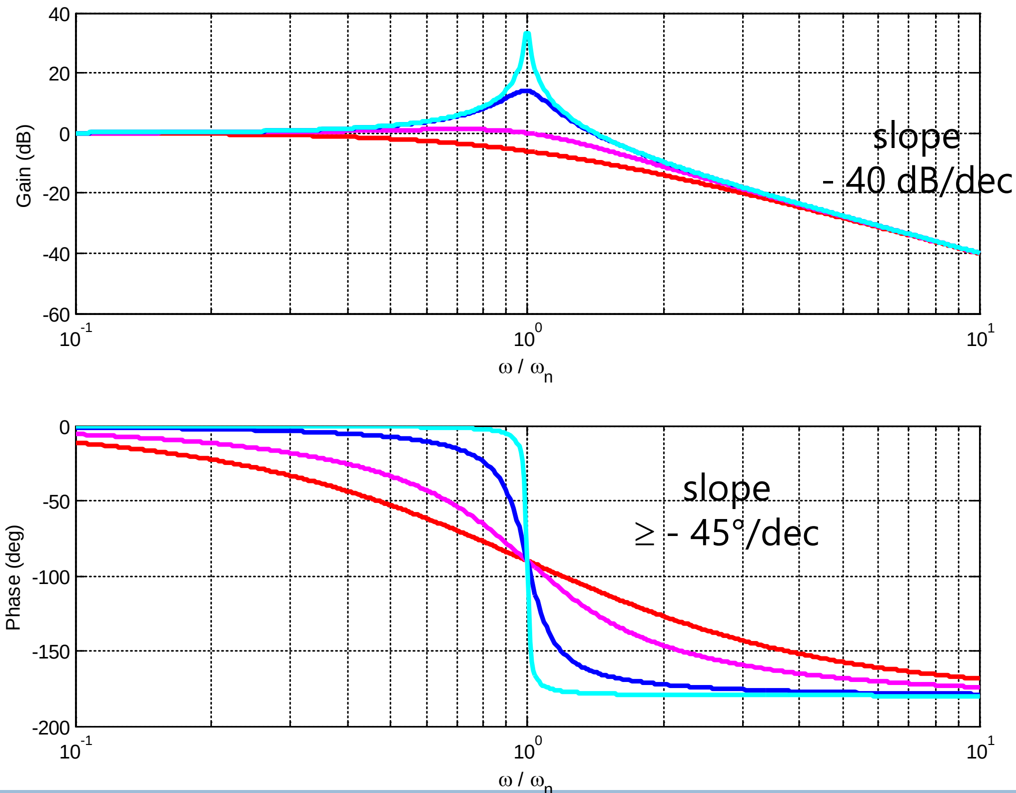
- For 0 < ζ < 0.7 we have a peak amplitude:

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

at the frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

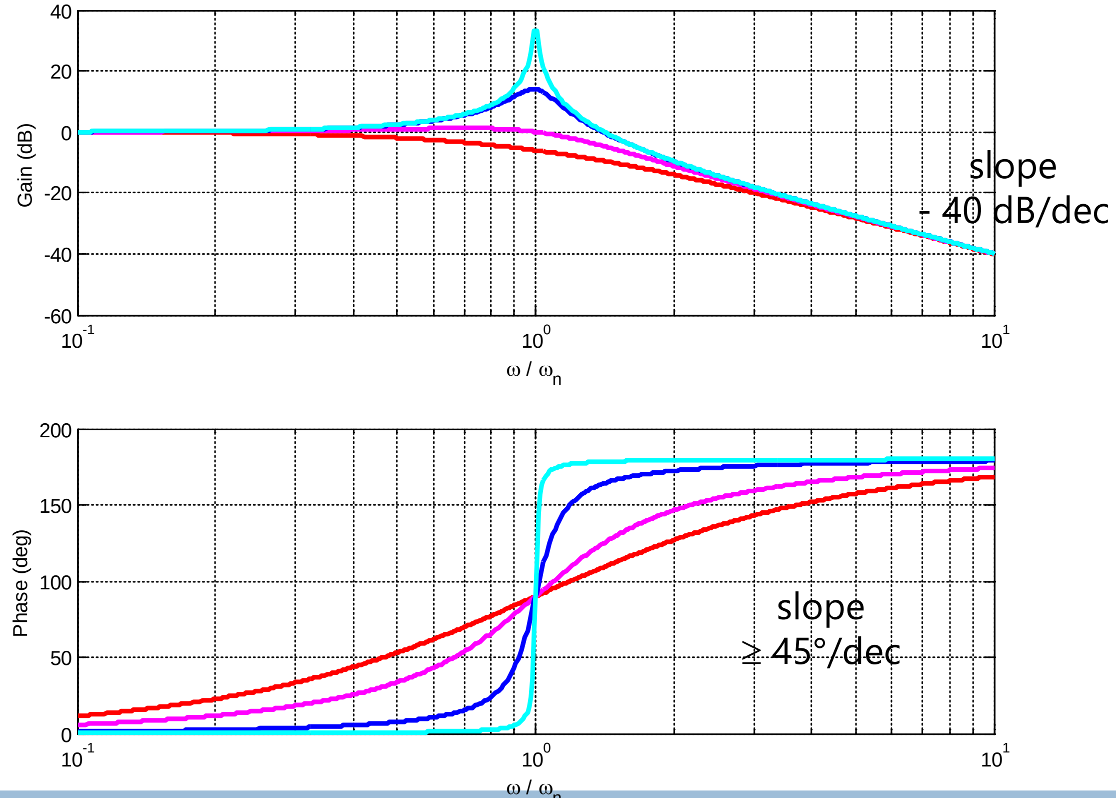
ζ = 0.01 0.1 0.5 1



Complex conjugate positive poles

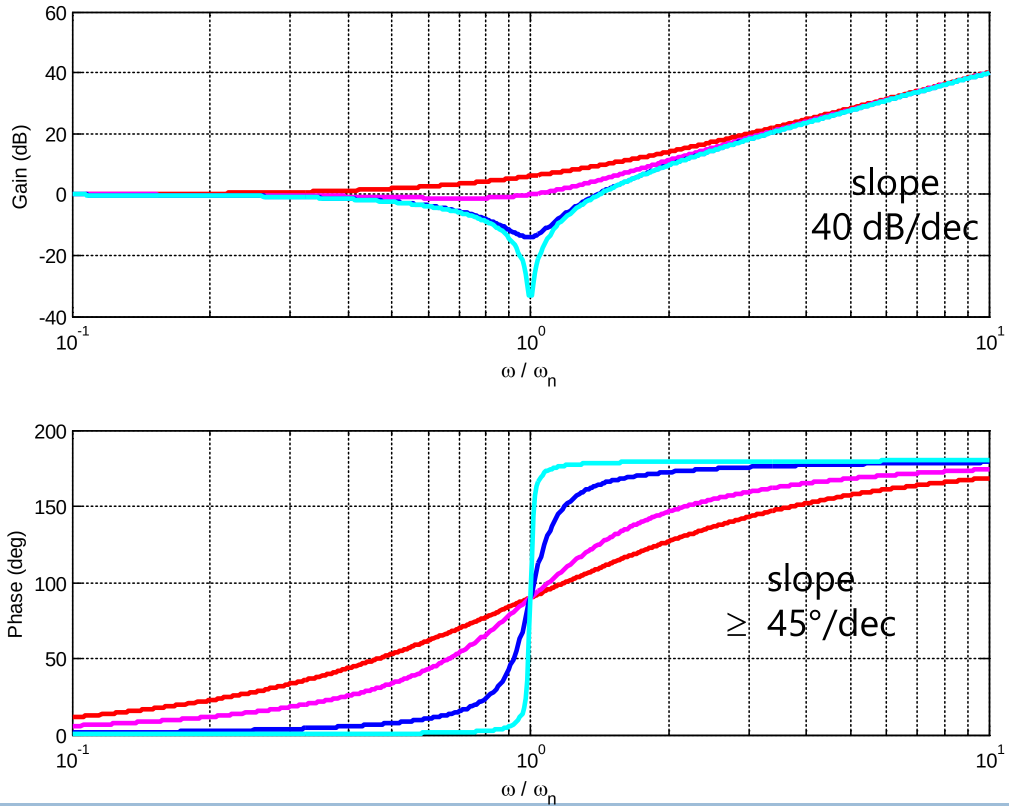
ζ = -0.01 -0.1 -0.5 -1

$$H(s) = \frac{1}{1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} = \underset{\text{Example } \zeta=-0.5,\,\omega_n=1}{\overset{\uparrow}{=}} \frac{1}{1 - s + s^2}$$



Complex conjugate negative zeros $H(s) = 1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2} \underset{\substack{\uparrow \\ Example\ \zeta=0.5,\omega_n=1}}{=} 1 + s + s^2$

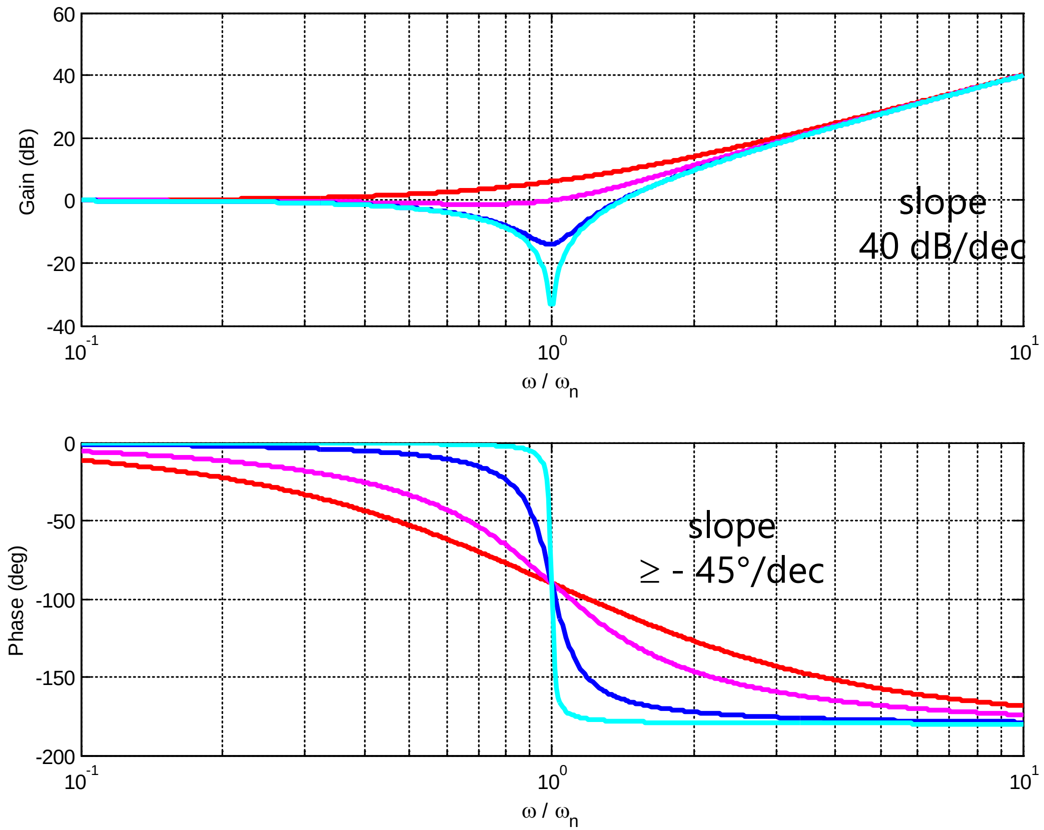
ζ = 0.01 0.1 0.5 1



Complex conjugate positive zeros $H(s) = 1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2} \underset{\uparrow}{=} 1 - s + s^2$

Example $\zeta = -0.5, \omega_n = 1$

ζ = -0.01 -0.1 -0.5 -1



Bode plots properties resume

$$H(s) = K\frac{(1-s/z_1)(1-s/z_2)\cdots(1-s/z_m)}{s^r(1-s/p_1)(1-s/p_2)\cdots(1-s/p_{n-r})}$$

Magnitude and phase for ω = 0⁺:

$$|H(j0^+)| = \begin{cases} |H(j0)| = K,\ r = 0 \\ \infty,\ r > 0 \rightarrow \text{ poles at } 0 \\ 0,\ r < 0 \rightarrow \text{ zeros at } 0 \end{cases}$$

$$\angle H(j0^+) = r \cdot (-90^\circ) - \begin{cases} 180^\circ \text{ if } K < 0 \\ 0^\circ \text{ if } K \geq 0 \end{cases}$$

Bode plots properties resume

$$H(s) = K\,\frac{(1-s/z_1)(1-s/z_2)\cdots(1-s/z_m)}{s^r\,(1-s/p_1)(1-s/p_2)\cdots(1-s/p_{n-r})}$$

Magnitude and phase for $\omega \to \infty$:

$$|H(j\infty)| = \begin{cases} -\infty\big|_{dB} = 0,\ n > m \\ K\,\dfrac{\prod_{i=1}^{m} 1/z_i}{\prod_{j=1}^{n-r} 1/p_j},\ n = m \end{cases}$$

$$\angle H(j\infty) = (n^p_{\le 0} + n^z_{>0})\cdot(-90^\circ) + (n^p_{>0} + n^z_{\le 0})\cdot 90^\circ - \begin{cases} 180^\circ \text{ if } K < 0 \\ 0^\circ \text{ if } K \ge 0 \end{cases}$$

$n^p_{\le 0}$ = n° poles with $\mathrm{Re}(.) \le 0$, $n^z_{>0}$ = n° zeros with $\mathrm{Re}(.) > 0$

$n^p_{>0}$ = n° poles with $\mathrm{Re}(.) > 0$, $n^z_{\le 0}$ = n° zeros with $\mathrm{Re}(.) \le 0$

Bode plots properties resume

$$H(s) = K \frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_m)}{s^r (1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_{n-r})}$$

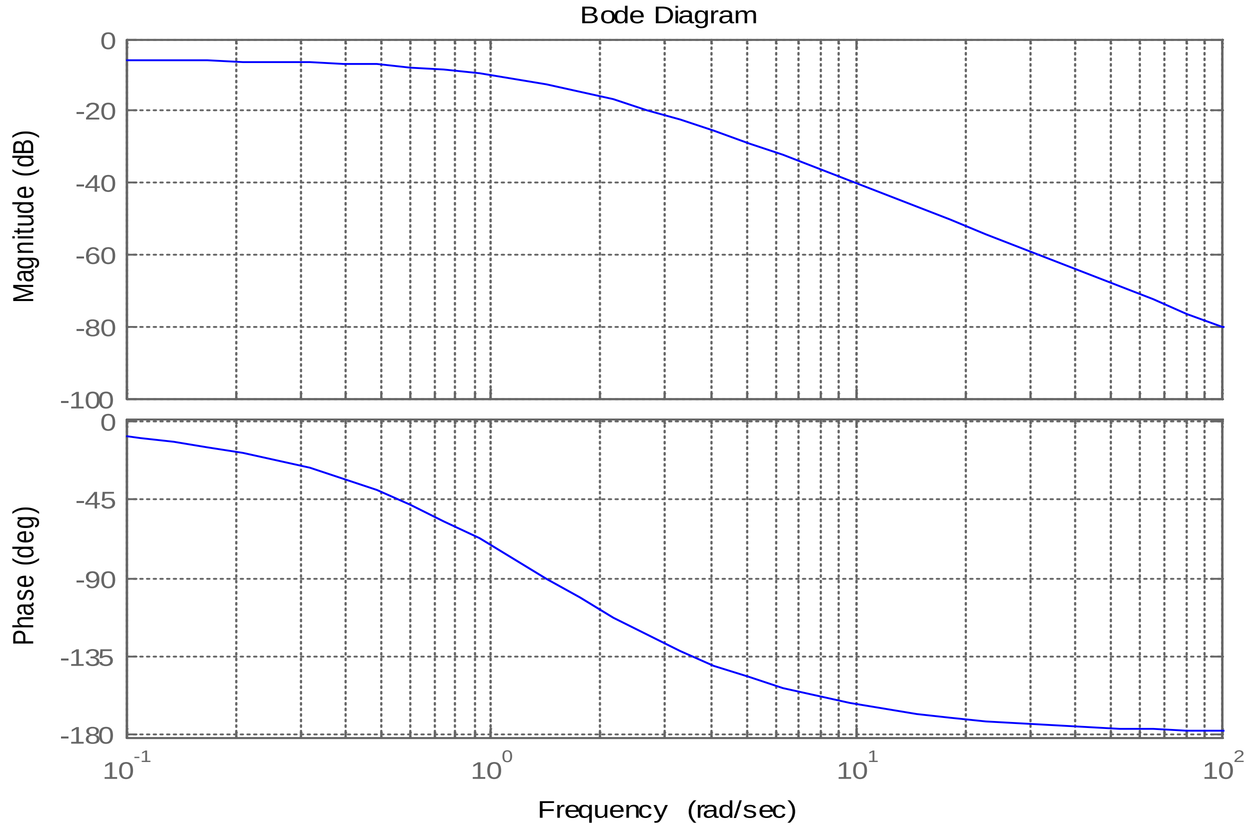
Magnitude and phase for $0 < \omega < \infty$: they depend on the interactions between the tf singularities and on their mutual locations:

- each pole with negative, positive or null real part yields a magnitude slope decrease of –20 dB/dec
- each zero with negative, positive or null real part yields a magnitude slope increase of +20 dB/dec
- each pole with negative or null real part yields a phase lag of -90°
- each pole with positive real part yields a phase lead of +90°
- each zero with negative or null real part yields a phase lead of +90°
- each zero with positive real part yields a phase lag of -90°

Bode diagrams with MatLab

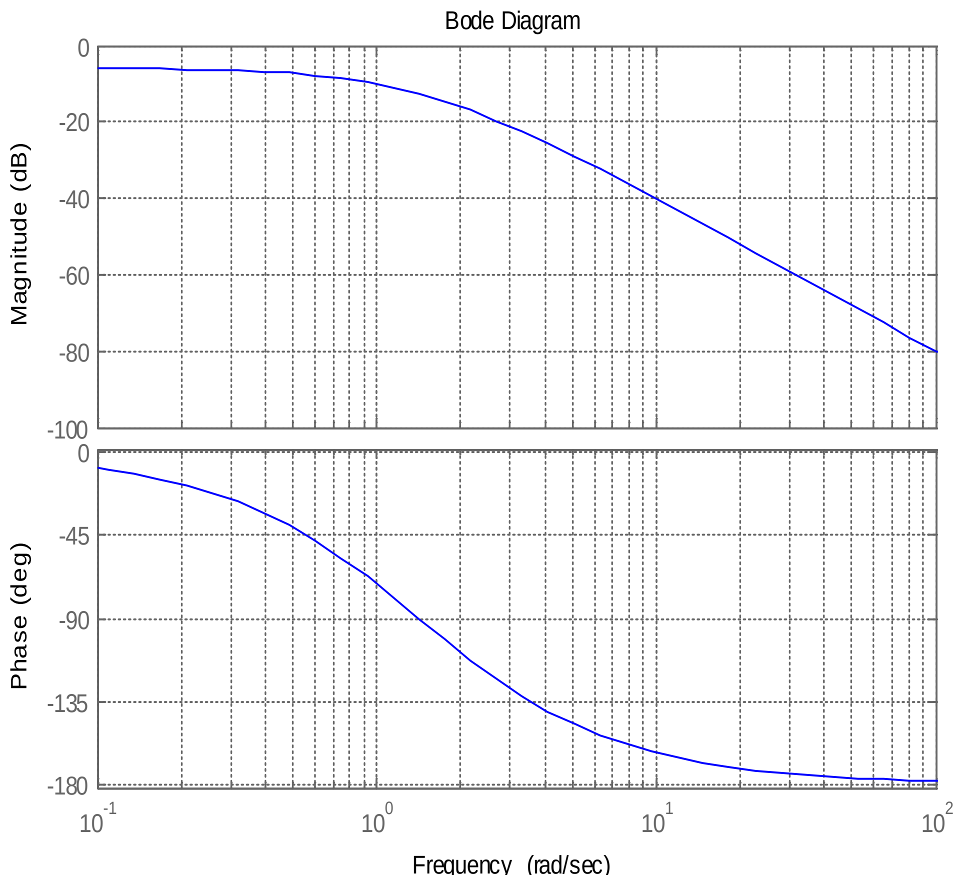
- Statement **bode**

```
>> s=tf('s')
Transfer function:
s
>> H=1/(s^2+3*s+2)
 Transfer function:
      1
-------------
s^2 + 3 s + 2
>> figure, bode(H)
```



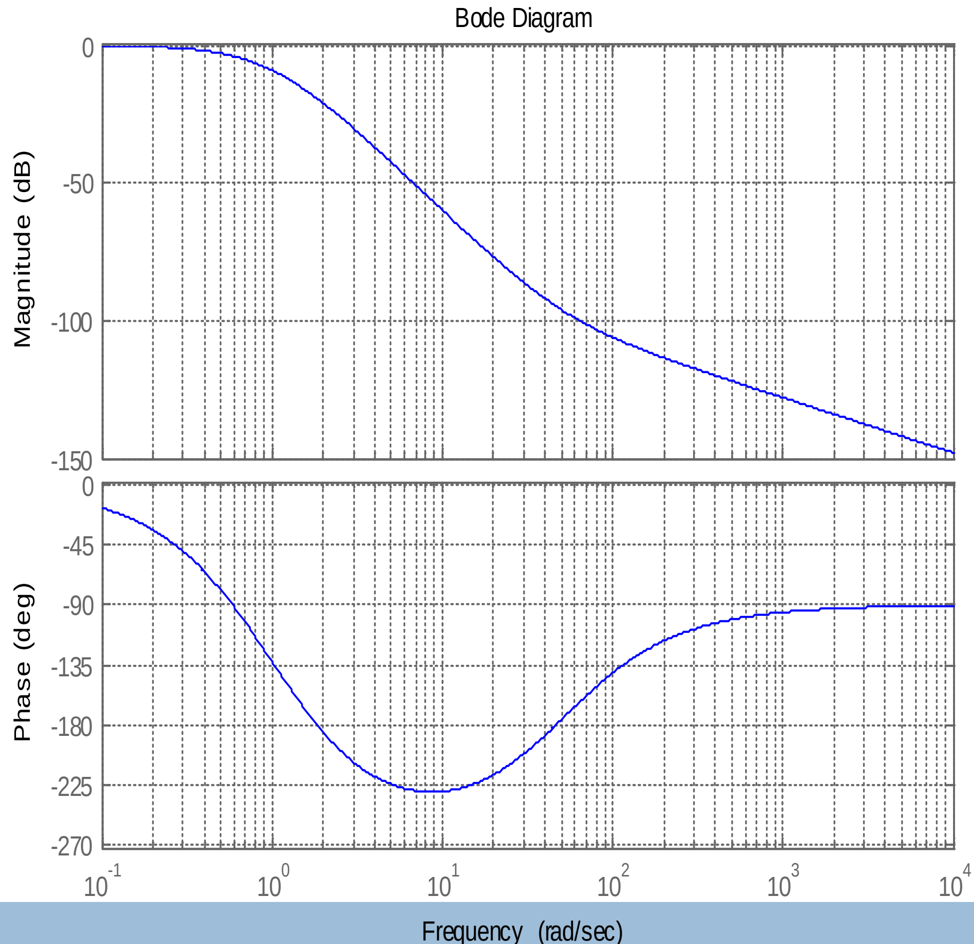
Bode diagrams: example 1

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$



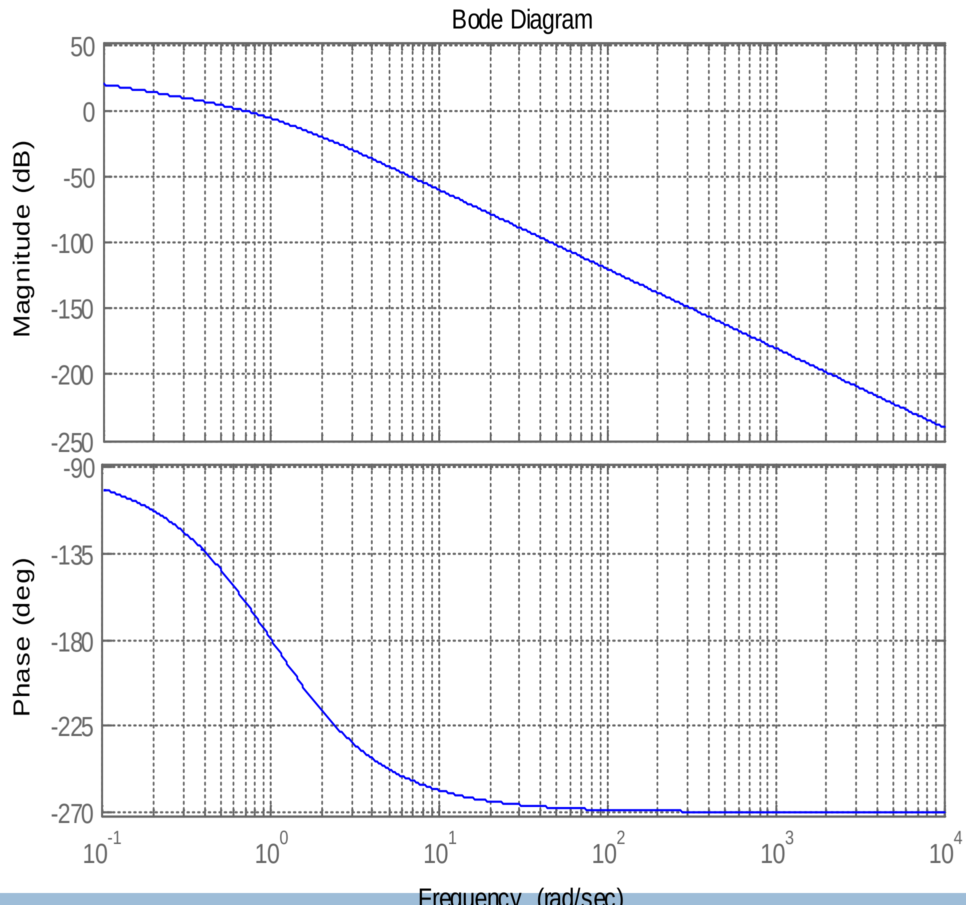
Bode diagrams: example 2

$$H(s) = \frac{(1 + s/50)^2}{(1 + s)^3}$$



Bode diagrams: example 3

$$H(s) = \frac{1}{s(1+s)^2}$$



Bode diagrams: example 4

$$H(s) = \frac{1}{s^2(1+s)}$$