Automatic Control

Introduction to LTI systems stability

Modal analysis of LTI systems

Internal stability of LTI systems

Stability of LTI systems: basic concepts

Stability of LTI systems

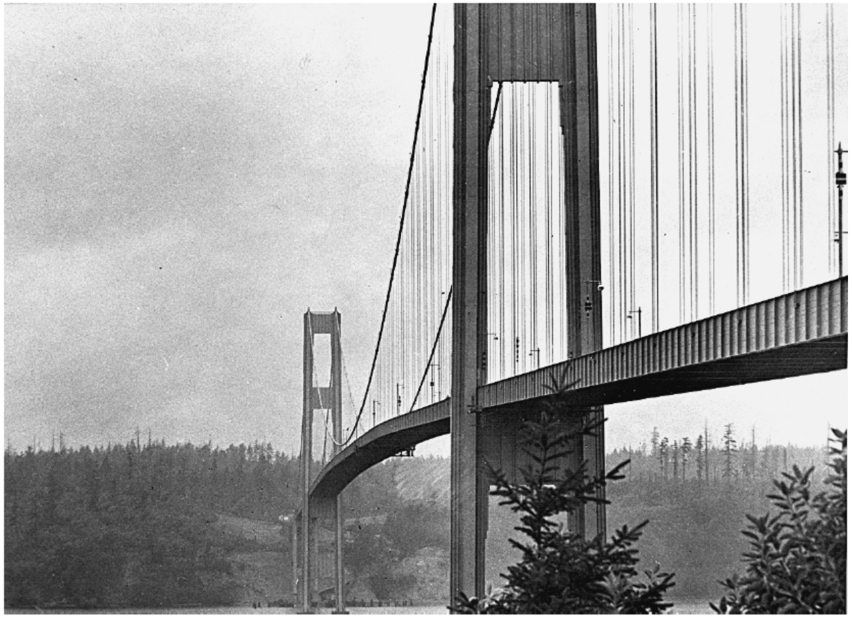
Stability is the most important property of a dynamical system.
It deals with the following problems:

- is the zero input state response $x_{zi}(t)$ bounded for any initial state $x_0 \in \mathbb{R}^n$? → **internal stability**
- is the (zero state) system output response $y(t)$ bounded for any bounded input $u(t)$? → **BIBO stability**

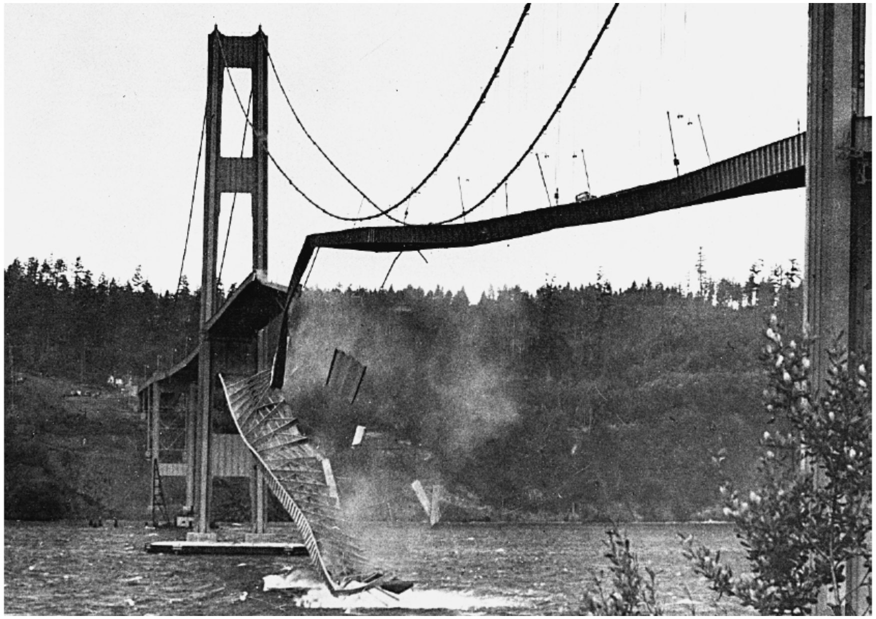
Stability prevents the relevant variables of a physical system to assume arbitrarily large values in the presence of arbitrary initial condition and bounded inputs → system damages

The study of stability involves the introduction of formal definitions and the development of (easy to check) necessary and sufficient conditions

Tacoma bridge (WA), opened to traffic on July 1st, 1940, was found to oscillate whenever the wind blew. On November 7th 1940, a blast of wind produced an oscillation that grew in amplitude until the bridge broke apart



(a)



(b)

Tacoma Narrows Bridge (a) as oscillation begins and (b) at catastrophic failure.

Dorf/Bishop
Modern Control Systems 9/E

Internal stability of LTI systems: formal definition

Internal stability of LTI systems

Definition (Internal stability of LTI system)

An LTI system is **internally stable** if the zero input state response $x_{zi}(t)$ is bounded for any initial state x_0

Definition (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if the zero input response $x_{zi}(t)$ converges to 0, as t → ∞, for any initial state x_0

Definition (Unstability of LTI system)

An LTI system is **unstable** if it is not stable (i.e. if the zero input state response $x_{zi}(t)$ is unbounded)

Internal stability of LTI systems

The study of the internal stability of an LTI system using directly the definition can not be performed, since, in practice, it requires an infinite number of test (i.e. check if the zero input response of the state is bounded for every initial condition x_0 in $\mathbb{R}^n$)

Therefore, it would be nice to obtain necessary and sufficient conditions for stability that can be easily applied for its study

In the following, starting from the analysis of the structure and of the the properties of the zero input state response, such necessary and sufficient conditions will be derived leading to an easy to check stability test

Natural modes of LTI systems

Analysis of the zero input state response

Consider an LTI system described by its state equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^p, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}$$

The state response $x(t)$ in the time interval $[0, t]$ due to the input signal $u(t)$ and the initial condition $x(0)=x_0$ is given by:

$$x(t) = x_{zi}(t) + x_{zs}(t) = e^{At}x(0) + \int_{0_-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$x_{zi}(t)$ is the *zero-input response*
$x_{zs}(t)$ is the *zero-state response*

Analysis of the zero input state response

Let us analyze the structure of the zero-input state response $x_{zi}(t)$ starting from its expression in the Laplace transform domain:

$$x_{zi}(t) = e^{At}x(0) \quad \xrightarrow{\mathcal{L}} \quad X_{zi}(s) = (sI - A)^{-1}x(0)$$

where

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} Adj(sI - A) = \frac{\left[a_{ij}(s)\right]}{p_A(s)}$$

Analysis of the zero input state response

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} Adj(sI - A) = \frac{\left[a_{ij}(s)\right]}{p_A(s)}$$

$p_A(s)$ is the characteristic polynomial of the matrix $A \in \mathbb{R}^{n \times n}$

$$p_A(s) = (s - \lambda_1)^{\mu_1}(s - \lambda_2)^{\mu_2} \cdots (s - \lambda_r)^{\mu_r}$$

$\deg(p_A(s)) = n$

$p_A(s)$ has, in general, $r \le n$ distinct roots

$\text{spec}(p_A) = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ → spectrum of the polynomial

μ_i is the **algebraic multiplicity** of the eigenvalue λ_i

Analysis of the zero input state response

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}, p_A(s) = \det(sI - A) = s^2(s+5), \lambda_1 = 0, \mu_1 = 2, \lambda_2 = -5, \mu_2 = 1$$

$$\Rightarrow \quad (sI - A)^{-1} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s+5 \end{bmatrix}^{-1} = \frac{1}{s^2(s+5)} \begin{bmatrix} a_{11}(s) & a_{12}(s) & a_{13}(s) \\ a_{21}(s) & a_{22}(s) & a_{23}(s) \\ a_{31}(s) & a_{32}(s) & a_{33}(s) \end{bmatrix}$$

Analysis of the zero input state response

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} Adj(sI - A) = \frac{\left[a_{ij}(s)\right]}{p_A(s)}$$

$$a_{ij}(s) = (-1)^{i+j} \alpha_{ji}(s)$$

$\alpha_{ji}(s)$ is the minor of $(sI - A)$, obtained as the determinant of the $(n-1)\times(n-1)$ submatrix that results from deleting row j and column i of $(sI - A)$

Note that: $\deg(a_{ji}(s)) \leq n - 1$

Analysis of the zero input state response

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}, p_A(s) = \det(sI - A) = s^2(s+5), \lambda_1 = 0, \mu_1 = 2, \lambda_2 = -5, \mu_2 = 1$$

$$\Rightarrow \quad (sI - A)^{-1} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s+5 \end{bmatrix}^{-1} = \frac{1}{s^2(s+5)} \begin{bmatrix} s(s+5) & 0 & 0 \\ 0 & s(s+5) & s \\ 0 & 0 & s^2 \end{bmatrix}$$

Analysis of the zero input state response

Thus, the generic entry $f_{i,j}(s)$ of the matrix $(sI - A)^{-1}$ is given by:

$$f_{ij}(s) = \frac{a_{ij}(s)}{p_A(s)}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix} \rightarrow (sI - A)^{-1} = \begin{bmatrix} \frac{s(s+5)}{s^2(s+5)} & 0 & 0 \\ 0 & \frac{s(s+5)}{s^2(s+5)} & \frac{s}{s^2(s+5)} \\ 0 & 0 & \frac{s^2}{s^2(s+5)} \end{bmatrix}$$

Analysis of the zero input state response

$$f_{ij}(s) = \frac{a_{ij}(s)}{p_A(s)}$$

In general, some roots of $p_A(s)$ can be the same as the ones of $a_{ij}(s)$

The common factors to $p_A(s)$ and $a_{ij}(s)$ can be deleted, obtaining:

$$f_{ij}(s) = \frac{a'_{ij}(s)}{q_{ij}(s)}$$

The **minimal polynomial** of A is defined as the least common multiple

$$q_A(s) = l.c.m.\left(q_{ij}(s)\right), \quad i, j = 1, \dots, n$$

$q_A(s)$ can be written as

$$q_A(s) = (s-\lambda_1)^{\mu'_1}(s-\lambda_2)^{\mu'_2}\cdots(s-\lambda_r)^{\mu'_r}$$

$\mu'_i \leq \mu_i$ is the **geometric multiplicity** of the eigenvalue λ_i

$\rightarrow$ $\deg(q_A(s)) \leq \deg(p_A(s)) = n$

Analysis of the zero input state response

$$\Rightarrow \quad (sI - A)^{-1} = \begin{bmatrix} \dfrac{s(s+5)}{s^2(s+5)} & 0 & 0 \\ 0 & \dfrac{s(s+5)}{s^2(s+5)} & \dfrac{s}{s^2(s+5)} \\ 0 & 0 & \dfrac{s^2}{s^2(s+5)} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{1}{s} & 0 & 0 \\ 0 & \dfrac{1}{s} & \dfrac{1}{s(s+5)} \\ 0 & 0 & \dfrac{1}{s+5} \end{bmatrix} \Rightarrow q_A(s) = s(s+5)$$

Analysis of the zero input state response

Computing the PFE to the generic entry $f_{i,j}(s)$ of $(sI - A)^{-1}$:

$$f_{ij}(s) = \frac{a'_{ij}(s)}{(s-\lambda_1)^{\mu'_1}(s-\lambda_2)^{\mu'_2}\cdots(s-\lambda_r)^{\mu'_r}}$$

we obtain:

$$f_{ij}(s) = \frac{r_{1,1}}{s-\lambda_1} + \frac{r_{1,2}}{(s-\lambda_1)^2} + \ldots + \frac{r_{1,\mu'_1}}{(s-\lambda_1)^{\mu'_1}} +$$

$$+ \frac{r_{2,1}}{s-\lambda_2} + \frac{r_{2,2}}{(s-\lambda_2)^2} + \ldots + \frac{r_{2,\mu'_2}}{(s-\lambda_2)^{\mu'_2}} + \ldots$$

$$+ \ldots + \frac{r_{r,1}}{s-\lambda_r} + \frac{r_{r,2}}{(s-\lambda_r)^2} + \ldots + \frac{r_{r,\mu'_r}}{(s-\lambda_r)^{\mu'_r}}$$

The inverse Laplace transform of the generic term $f_{ij}(s)$ is ... →

Analysis of the zero input state response

$$f_{ij}(t) = r_{1,1}e^{\lambda_1 t} + r_{1,2}te^{\lambda_1 t} \dots + r_{1,\mu'_1}\frac{t^{\mu'_1-1}}{(\mu'_1-1)!}e^{\lambda_1 t}$$

$$+r_{2,1}e^{\lambda_2 t} + r_{2,2}te^{\lambda_2 t} \dots + r_{2,\mu'_2}\frac{t^{\mu'_2-1}}{(\mu'_2-1)!}e^{\lambda_2 t} + \dots$$

$$+r_{r,1}e^{\lambda_r t} + r_{r,2}te^{\lambda_r t} \dots + r_{r,\mu'_r}\frac{t^{\mu'_r-1}}{(\mu'_r-1)!}e^{\lambda_r t}$$

Analysis of the zero input state response

Summing up, we have:

$$X_{zi}(s) = (sI - A)^{-1} x(0) = \begin{bmatrix} f_{11}(s) & f_{12}(s) & \cdots & f_{1n}(s) \\ f_{21}(s) & \ddots & f_{ij}(s) & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ f_{n1}(s) & \cdots & \cdots & f_{nn}(s) \end{bmatrix} x(0)$$

$$\rightarrow x_{zi}(t) = e^{At} x(0) = \begin{bmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & \ddots & f_{ij}(t) & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ f_{n1}(t) & \cdots & \cdots & f_{nn}(t) \end{bmatrix} x(0)$$

Thus, each component of the zero-input state response is a linear combination of the following functions:

$$\lambda_1 \rightarrow m_{1,0}(t) = e^{\lambda_1 t}, m_{1,1}(t) = te^{\lambda_1 t}, \ldots, m_{1,\mu'_1}(t) = \frac{t^{\mu'_1 - 1}}{(\mu'_1 - 1)!} e^{\lambda_1 t}$$

$$\lambda_2 \rightarrow m_{2,0}(t) = e^{\lambda_2 t}, m_{2,1}(t) = te^{\lambda_2 t}, \ldots, m_{2,\mu'_2}(t) = \frac{t^{\mu'_2 - 1}}{(\mu'_2 - 1)!} e^{\lambda_2 t}$$

$$\ldots$$

$$\lambda_r \rightarrow m_{r,0}(t) = e^{\lambda_r t}, m_{r,1}(t) = te^{\lambda_r t}, \ldots, m_{r,\mu'_r}(t) = \frac{t^{\mu'_r - 1}}{(\mu'_r - 1)!} e^{\lambda_r t}$$

All the functions $m_{lk}(t)$, l = 1,..., r, k = 1, ..., μ'_r are referred to as the system **natural modes**

In particular, considering the i–th distinct eigenvalue λ_i ($i = 1,\ldots,r$) with geometric multiplicity μ'_i, the functions

$$\lambda_i \rightarrow m_{i,0}(t) = e^{\lambda_i t}, m_{i,1}(t) = te^{\lambda_i t}, \ldots, m_{i,\mu'_i}(t) = \frac{t^{\mu'_i - 1}}{(\mu'_i - 1)!} e^{\lambda_i t}$$

are referred to as the **natural modes** associated with the eigenvalue λ_i

For each eigenvalue λ_i with geometric multiplicity μ'_i, there are μ'_i natural modes

Natural modes of LTI systems

For each couple of complex conjugate eigenvalues $\lambda = \sigma_0 \pm j\omega_0$ with geometric multiplicity μ', the resulting natural modes can be written as:

$$m_0(t) = \begin{cases} e^{(\sigma_0 + j\omega_0)t} \\ e^{(\sigma_0 - j\omega_0)t} \end{cases} \rightarrow e^{\sigma_0 t}\cos(\omega_0 t + \varphi),$$

$$m_1(t) = \begin{cases} te^{(\sigma_0 + j\omega_0)t} \\ te^{(\sigma_0 - j\omega_0)t} \end{cases} \rightarrow te^{\sigma_0 t}\cos(\omega_0 t + \varphi)$$

$$\ldots$$

$$m_{\mu'}(t) = \begin{cases} \dfrac{t^{\mu'-1}}{(\mu'-1)!} e^{(\sigma_0 + j\omega_0)t} \\ \dfrac{t^{\mu'-1}}{(\mu'-1)!} e^{(\sigma_0 - j\omega_0)t} \end{cases} \rightarrow \frac{t^{\mu'-1}}{(\mu'-1)!} e^{\sigma_0 t}\cos(\omega_0 t + \varphi)$$

Reminder: internal stability of LTI systems

Definition (Internal stability of LTI system)

An LTI system is **internally stable** if the zero input state response $x_{zi}(t)$ is bounded for any initial state x_0

Definition (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if the zero input response $x_{zi}(t)$ converges to 0, as $t \rightarrow \infty$, for any initial state x_0

Definition (Unstability of LTI system)

An LTI system is **unstable** if it is not stable (i.e. if the zero input state response $x_{zi}(t)$ is unbounded)

Natural modes of LTI systems

Conclusions:

the zero input state response is a linear combination of the natural modes of the system

internal stability properties of an LTI system can be studied through the boundedness characteristics and asymptotic behavior (i.e. for $t \rightarrow \infty$) of the natural modes

it is important to classify natural modes according to:

- bounded time course
- asymptotic behavior

- A natural mode is said convergent if:

$$\lim_{t\to\infty} \left| m(t) \right| = 0$$

- A natural mode is said bounded if: $\exists\ M \in \mathbb{R} : \forall t \geq 0,$

$$0 \leq \left| m(t) \right| \leq M < \infty$$

- A natural mode is said divergent if:

$$\lim_{t\to\infty} \left| m(t) \right| = \infty$$

Modal analysis of LTI systems

Convergence and boundedness properties of the natural modes of an LTI system depend on the characteristics of the associated eigenvalues

The study of the convergence of the natural modes on basis of the eigenvalues characteristics is referred to as
modal analysis of the system

Thus, the modal analysis tool allows us to link...

- ... characteristics of the system eigenvalues to...
- ... convergence and boundedness properties of the natural modes to...
- ... convergence and boundedness of the zero input response (i.e. internal stability)

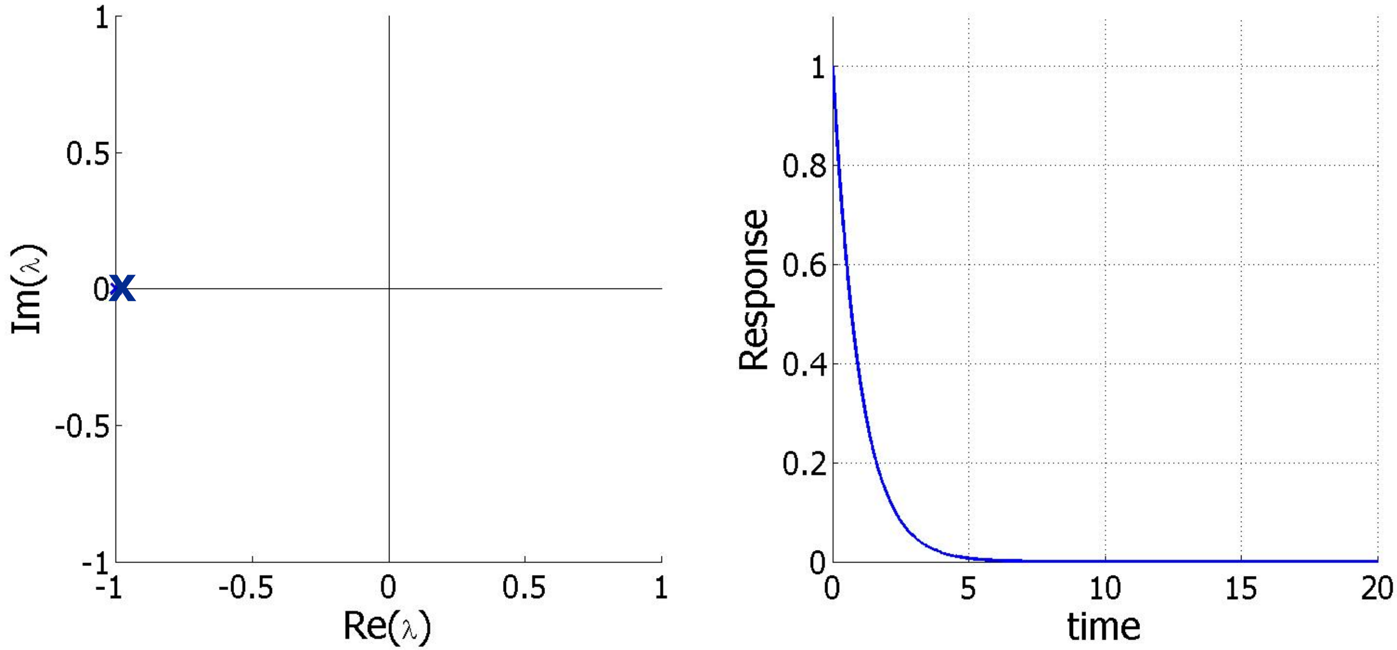
In order to perform the system modal analysis, we first need to study the convergence and boundedness properties of each natural mode

This study is performed by considering the natural modes corresponding to:

- real and complex eigenvalues with unitary geometric multiplicity (i.e. $\mu' = 1$)
- real and complex eigenvalues with nonunitary geometric multiplicity (i.e. $\mu' > 1$)

The natural mode $e^{\lambda t}$, associated with a real eigenvalue $\lambda \in \mathbb{R}$ with $\mu' = 1$, is:

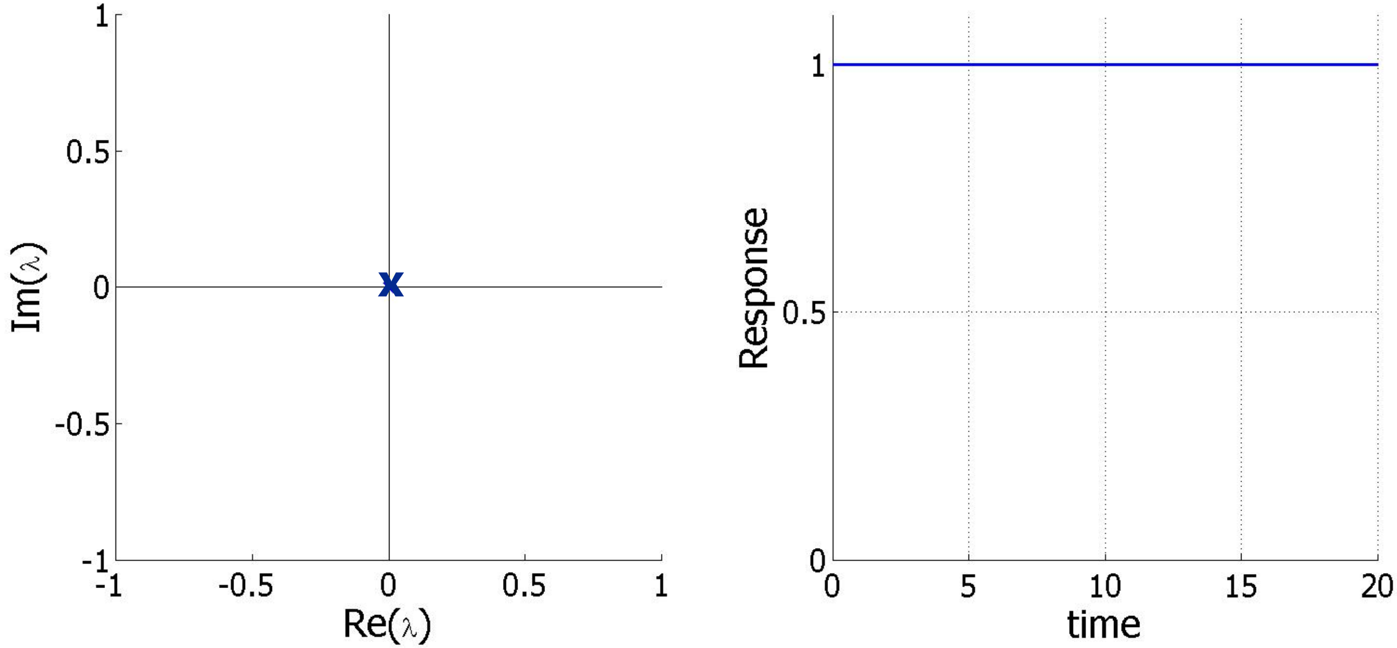
- **Esponentially convergent** if $\mathbb{R}\mathrm{e}(\lambda) < 0$, e.g. e^{-t}:



λ on the complex plane (left) and natural mode time response (right)

The natural mode $e^{\lambda t}$, associated with a real eigenvalue $\lambda \in \mathbb{R}$ with $\mu' = 1$, is:

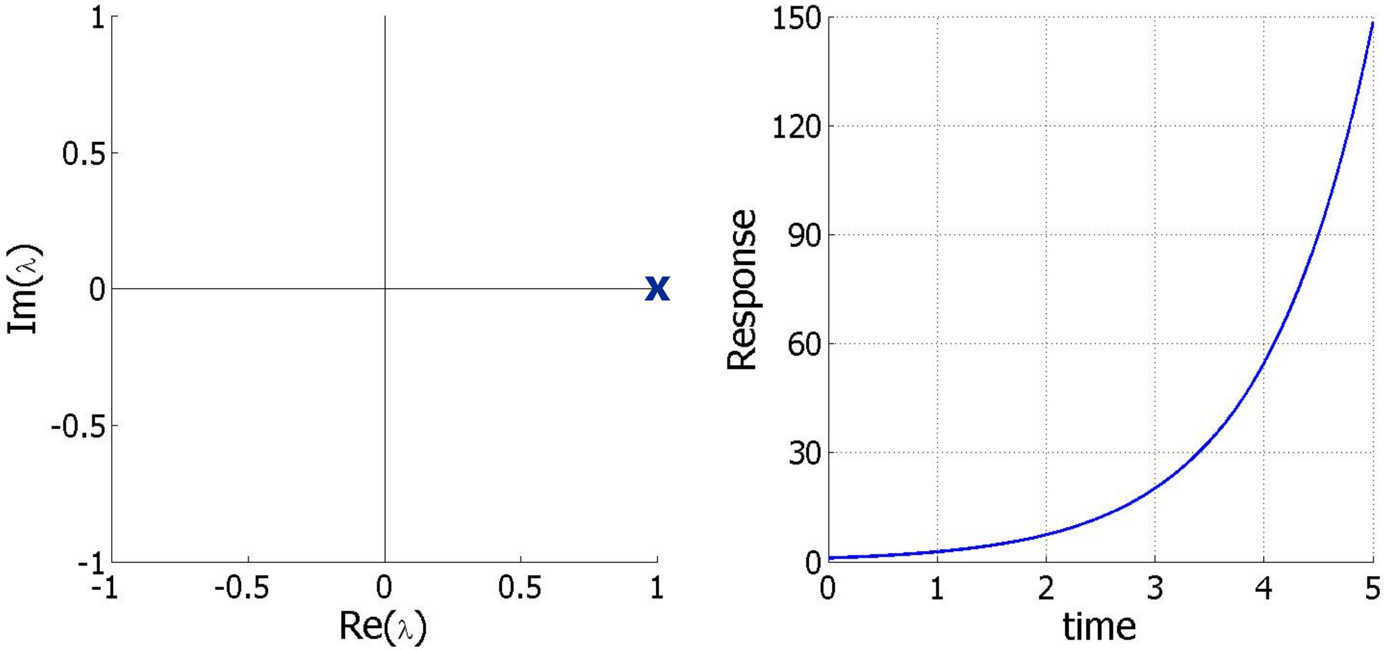
- **Bounded (constant)** if $\mathbb{R}e(\lambda) = 0$, e.g. $e^{-0t} = 1$:



λ on the complex plane (left) and natural mode time response (right)

The natural mode $e^{\lambda t}$, associated with a real eigenvalue $\lambda \in \mathbb{R}$ with $\mu' = 1$, is:

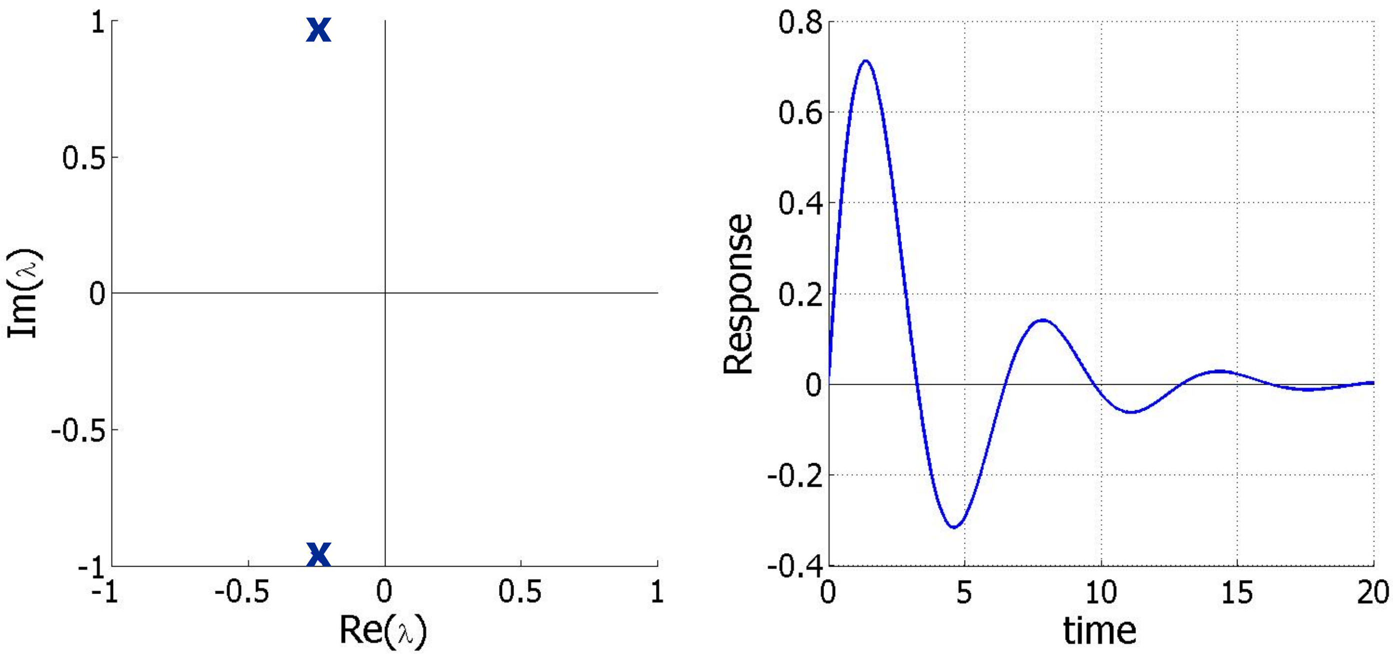
- **Esponentially divergent** if $\mathbb{R}\mathrm{e}(\lambda) > 0$, e.g. e^{t}:



λ on the complex plane (left) and natural mode time response (right)

The natural modes $e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, associated with two complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with $\mu' = 1$ are:

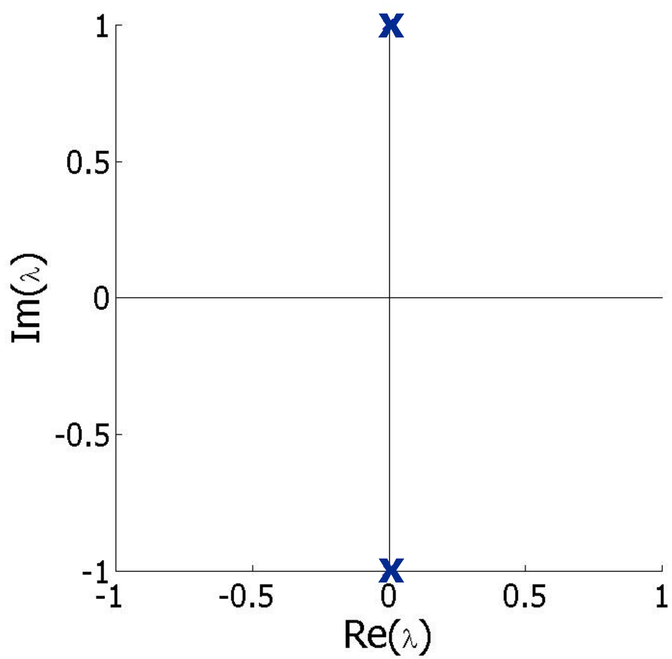
- **Esponentially convergent** if $\mathbb{R}e(\lambda) = \sigma < 0$, e.g. $e^{-t}\cos(0.1t)$:

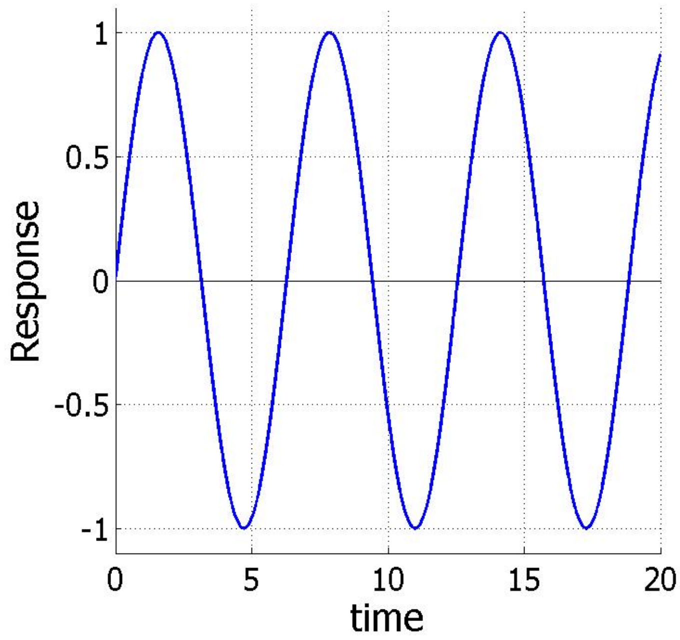


λ on the complex plane (left) and natural mode time response (right)

The natural modes $e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, associated with two complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with $\mu' = 1$ are:

- **Bounded (oscillating)** if $\mathbb{R}\text{e}(\lambda) = \sigma = 0$, $\mathbb{I}\text{m}(\lambda) = \omega \neq 0$, e.g. $\cos(0.1t)$:

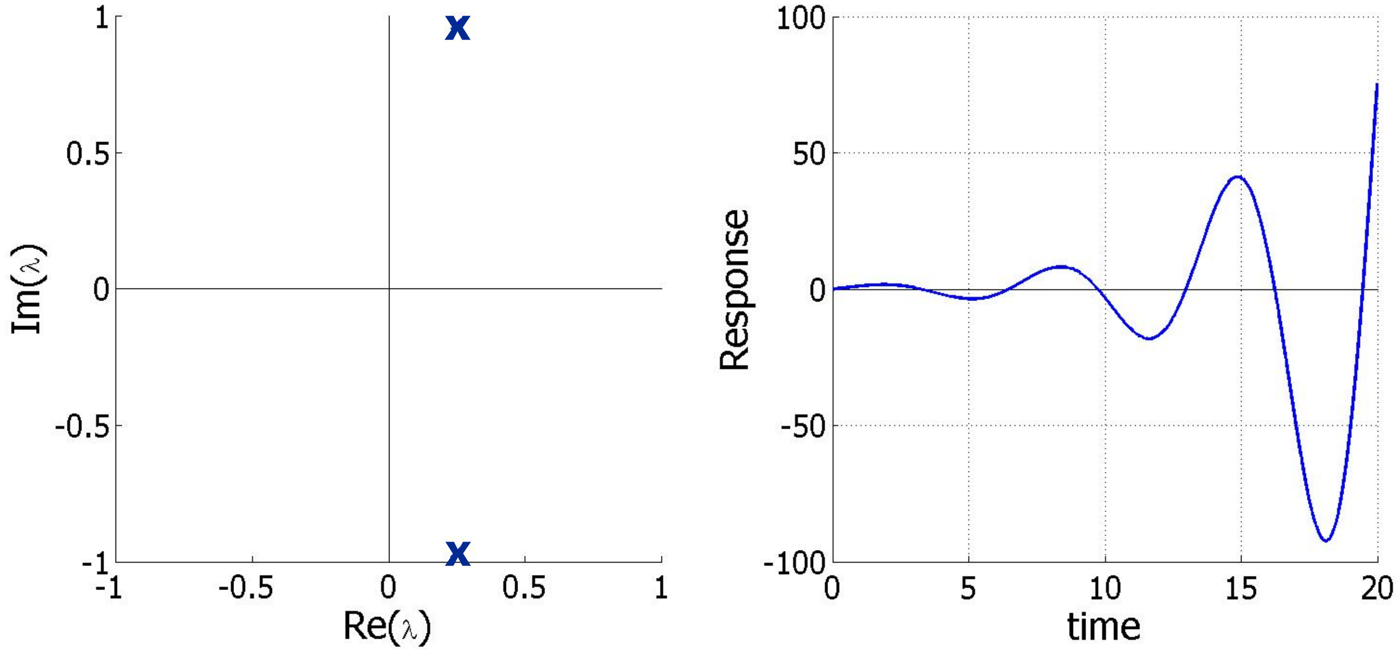




λ on the complex plane (left) and natural mode time response (right)

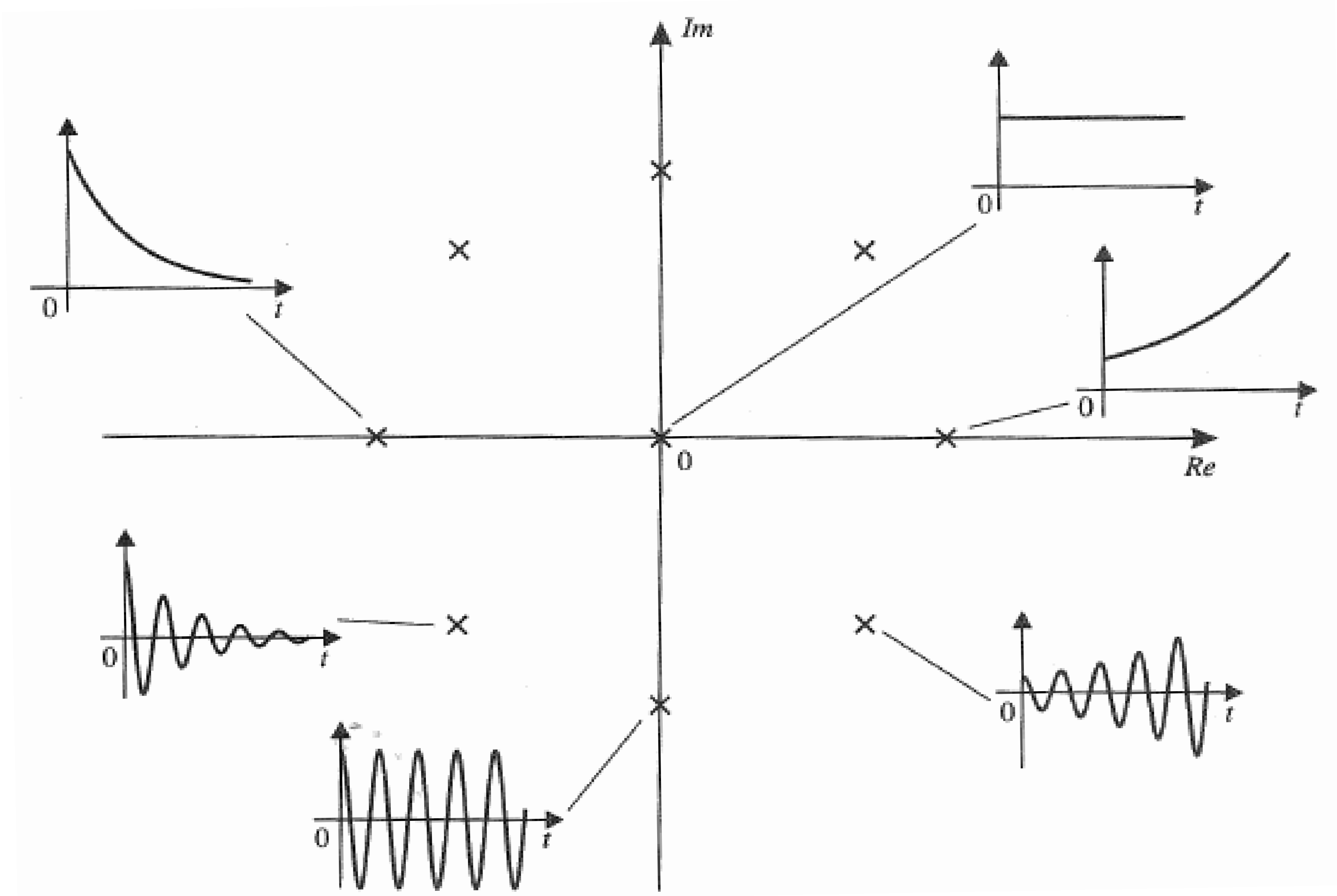
The natural modes $e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, associated with two complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with $\mu' = 1$ are:

- **Esponentially divergent** if $\mathbb{R}\mathrm{e}(\lambda) = \sigma > 0$, e.g. $e^{t}\cos(0.1t)$:



λ on the complex plane (left) and natural mode time response (right)

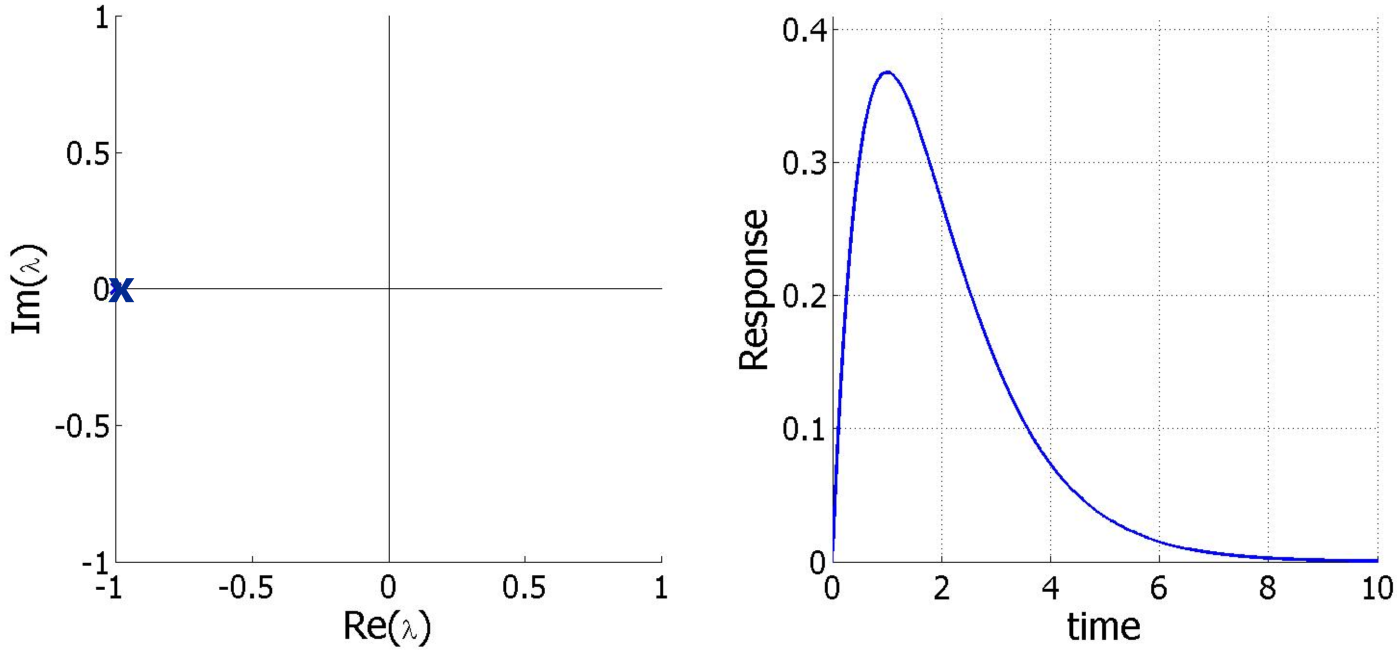
Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\mu' = 1$



Modal analysis: $\lambda \in \mathbb{R}$, $\mu' > 1$

The μ' natural modes of the form $t^{\mu'-1}e^{\lambda t}$, ..., $t\,e^{\lambda t}$, associated with a real eigenvalue λ with $\mu' > 1$ are:

- **Esponentially convergent** if $\mathbb{R}\mathrm{e}(\lambda) < 0$, e.g. $t\,e^{-t}$:

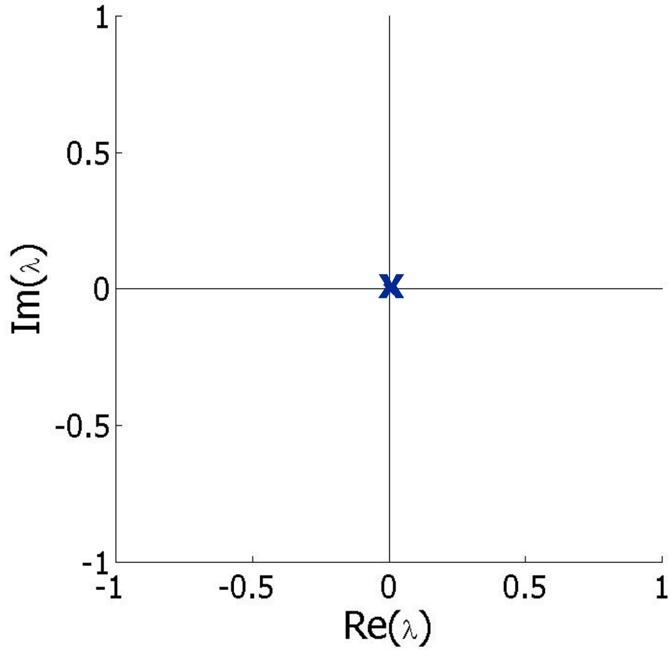


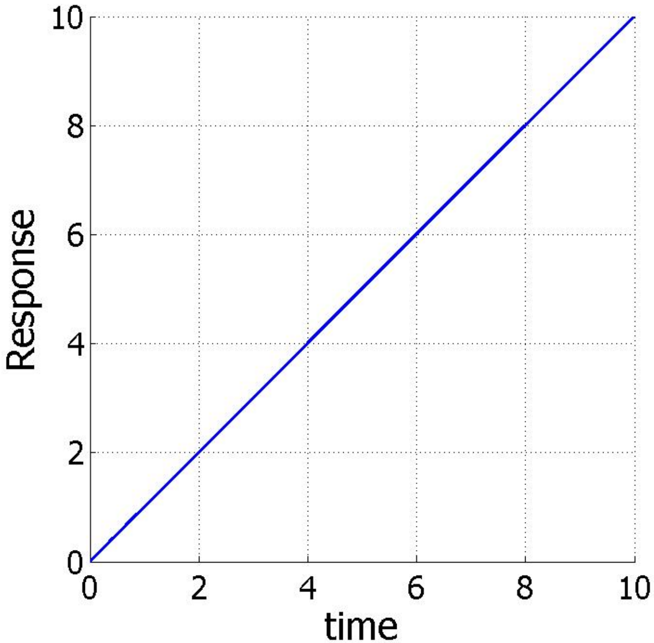
λ on the complex plane (left) and natural mode time response (right)

Modal analysis: $\lambda \in \mathbb{R}$, $\mu' > 1$

The μ' natural modes of the form $t^{\mu'-1}e^{\lambda t}$, ..., $t\,e^{\lambda t}$, associated with a real eigenvalue λ with $\mu' > 1$ are:

- **Polynomially divergent** if $\mathbb{R}e(\lambda) = 0$, e.g. $t\,e^{0t} = t$:



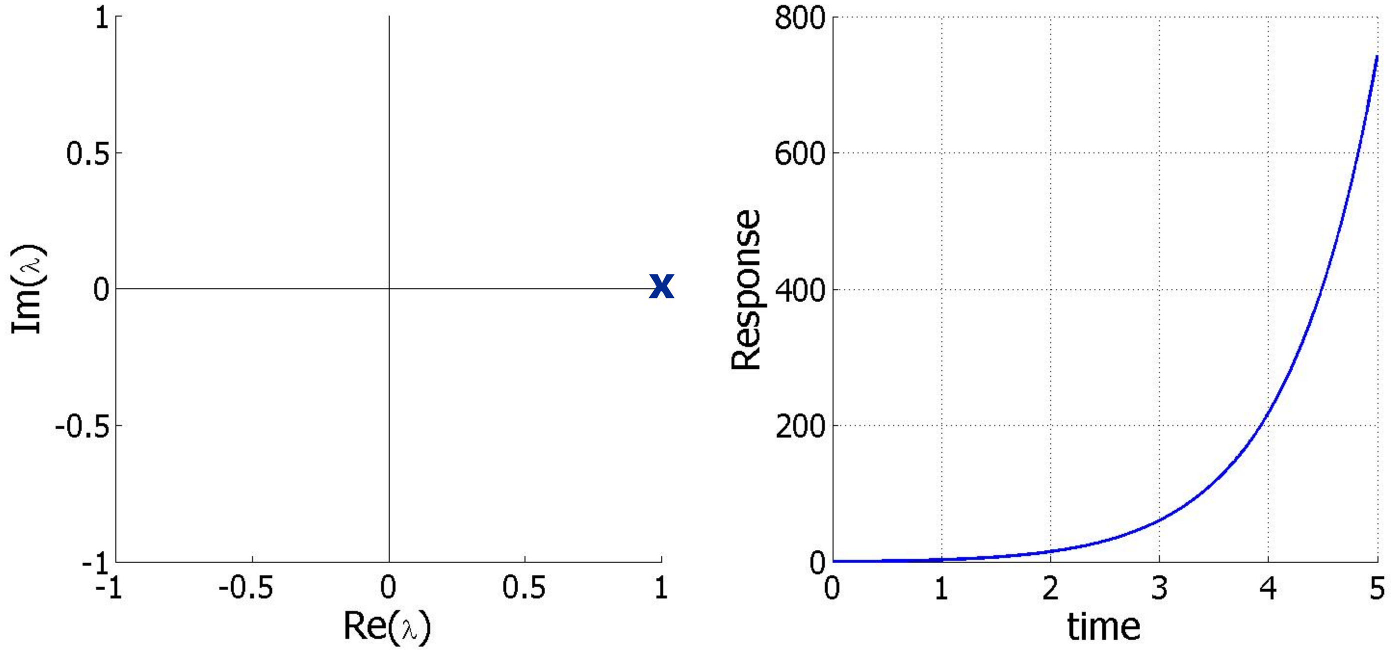


λ on the complex plane (left) and natural mode time response (right)

Modal analysis: $\lambda \in \mathbb{R}$, $\mu' > 1$

The μ' natural modes of the form $t^{\mu'-1}e^{\lambda t}$, ..., $t\,e^{\lambda t}$, associated with a real eigenvalue λ with $\mu' > 1$ are:

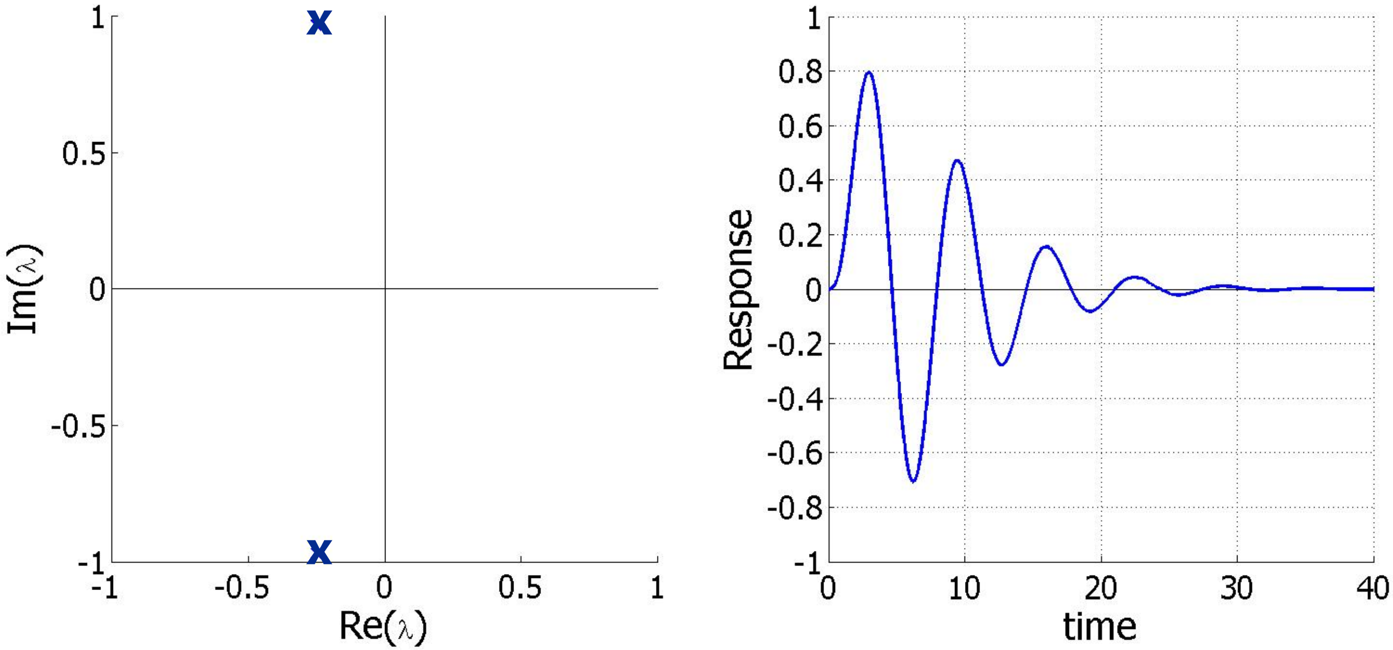
- **Esponentially divergent** if $\mathbb{R}\mathrm{e}(\lambda) > 0$, e.g. $t\,e^{t}$:



λ on the complex plane (left) and natural mode time response (right)

The μ' natural modes associated with a couple of complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with $\mu' > 1$ are:

- **Esponentially convergent** if $\mathbb{R}e(\lambda) = \sigma < 0$, e.g. $t\,e^{-t}\cos(0.1t)$:

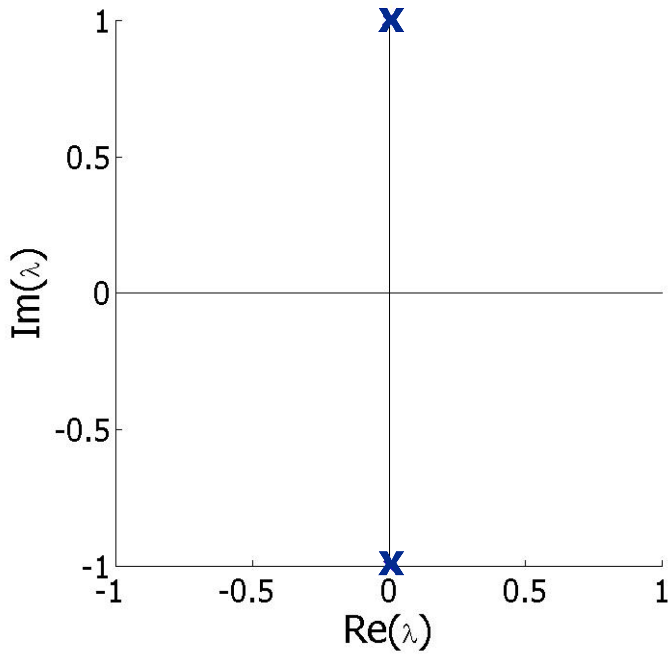


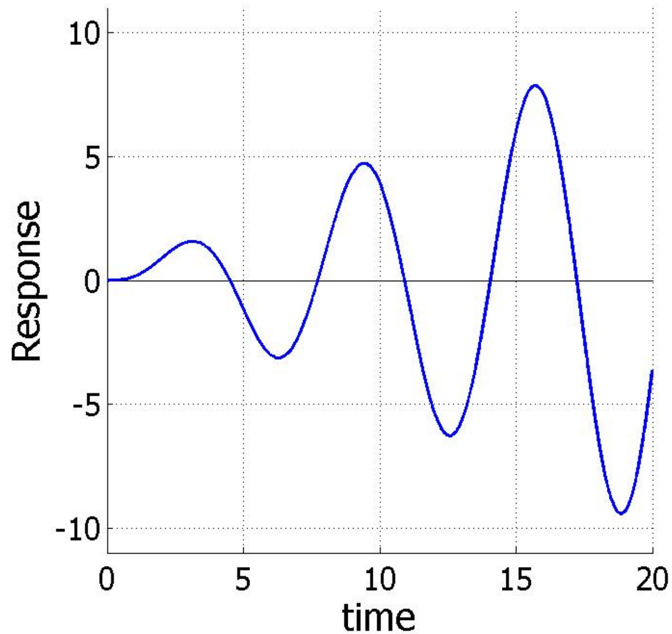
λ on the complex plane (left) and natural mode time response (right)

Modal analysis: $\lambda \in \mathbb{C}$, $\mu' > 1$

The μ' natural modes associated with a couple of complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with $\mu' > 1$ are:

- **Polynomially divergent** if $\mathbb{R}\mathrm{e}(\lambda) = \sigma = 0$, e.g. $t\cos(0.1t)$:



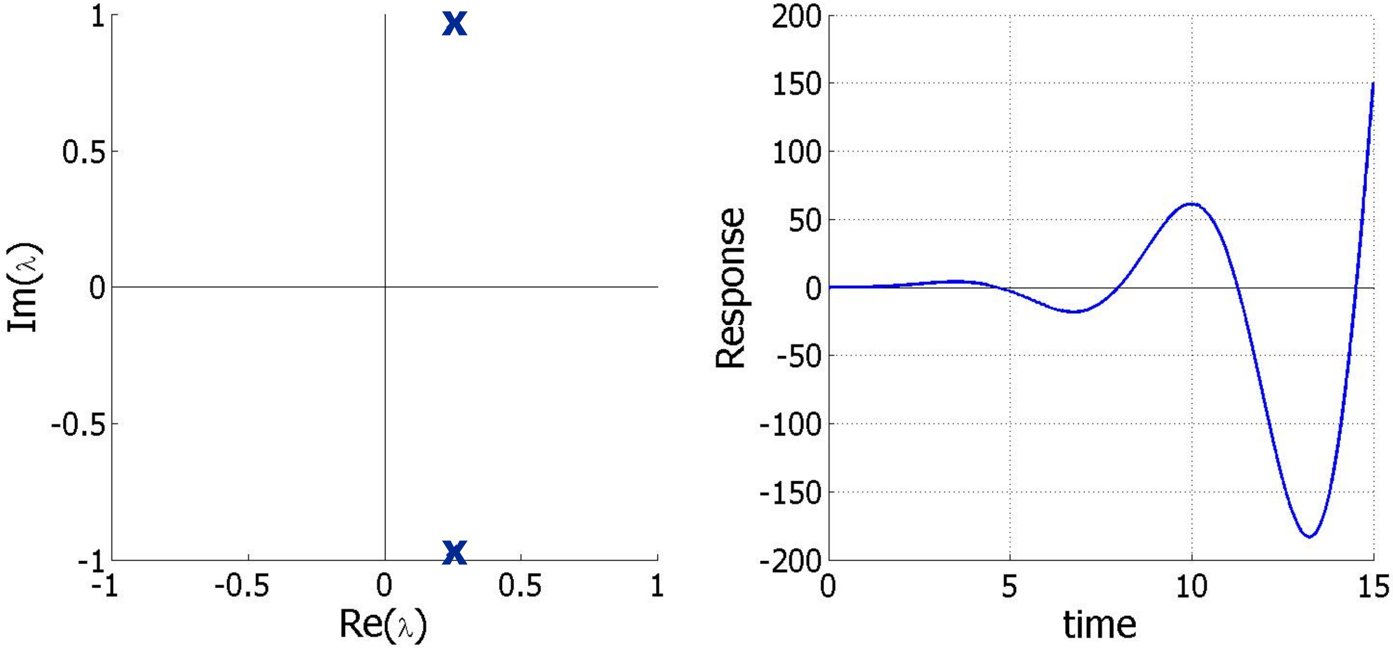


λ on the complex plane (left) and natural mode time response (right)

Modal analysis: $\lambda \in \mathbb{C}$, μ' >1

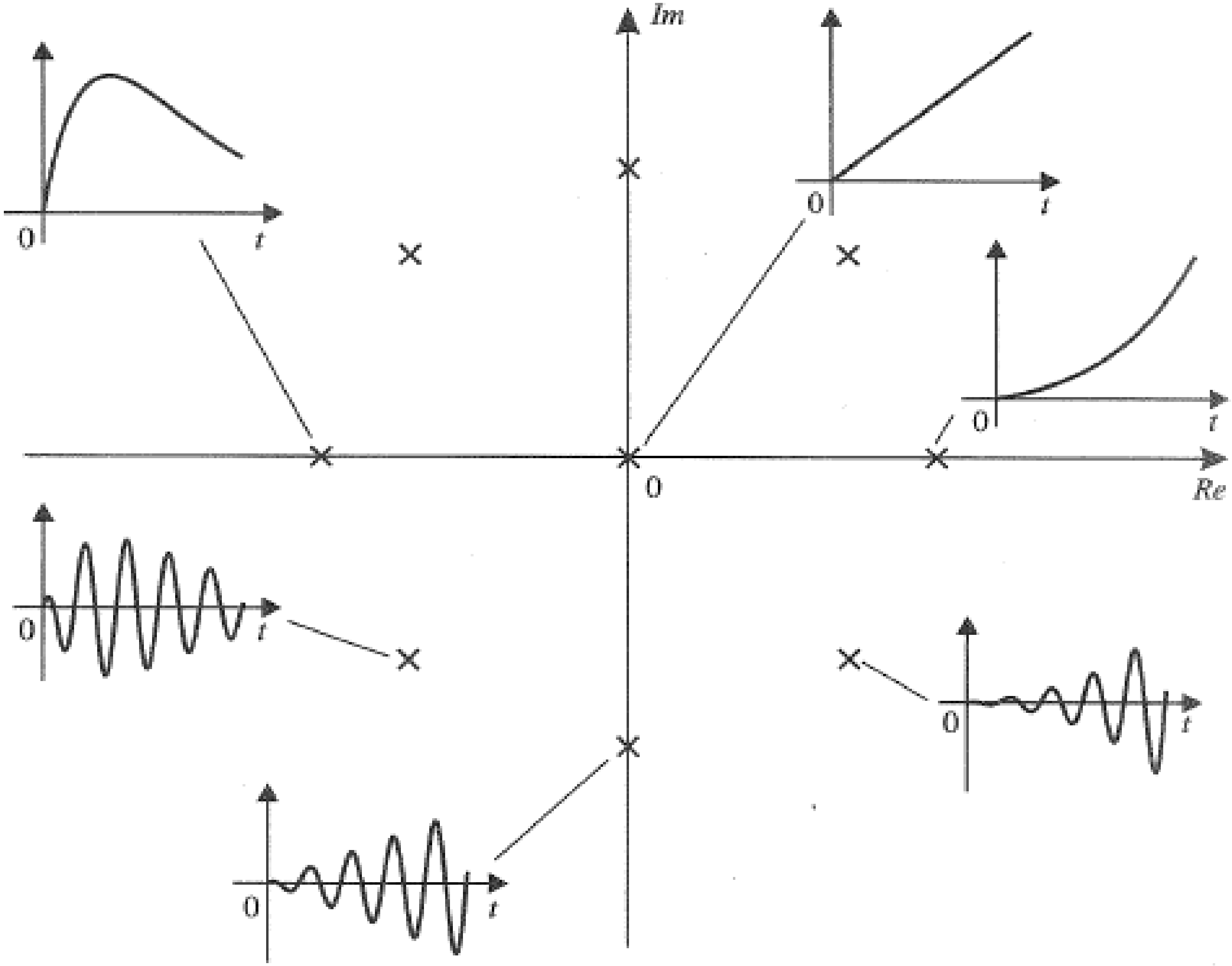
The μ' natural modes associated with a couple of complex conjugated eigenvalues $\lambda = \sigma \pm j\omega$ with μ' > 1 are:

- **Esponentially divergent** if $\mathbb{R}e(\lambda) = \sigma > 0$, e.g. $t\, e^{t} \cos(0.1t)$:



λ on the complex plane (left) and natural mode time response (right)

Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\mu' = 2$



Modal analysis: synthetic resume

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A then

- The natural mode associated with eigenvalue λ_i is **bounded** if:

  $\mathbb{R}\text{e}(\lambda_i(A)) = 0$ and $\mu'(\lambda_i(A)) = 1$

- The natural mode associated with eigenvalue λ_i is **convergent** if:

  $\mathbb{R}\text{e}(\lambda_i(A)) < 0$

- The natural mode associated with eigenvalue λ_i is **divergent** if:

  $\mathbb{R}\text{e}(\lambda_i(A)) > 0$ OR $\mathbb{R}\text{e}(\lambda_i(A)) = 0$ and $\mu'(\lambda_i(A)) > 1$

The time constant τ

For a convergent natural mode associated with an eigenvalue λ having $\mathbb{R}\mathrm{e}(\lambda) < 0$ the **time constant** τ is defined as:

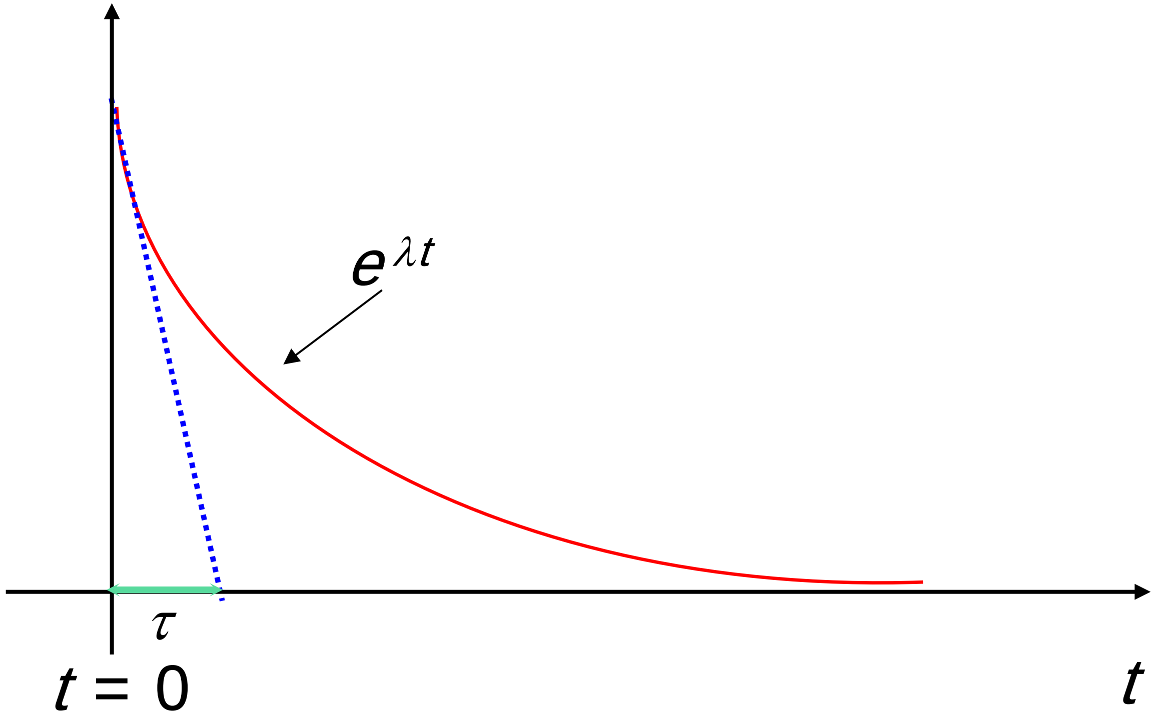
$$\tau = \left| \frac{1}{\mathbb{R}\mathrm{e}(\lambda)} \right|$$

The time constant τ is a measure of the "convergence speed" of the mode

As an example, the mode e^{-2t} ($\tau = 0.5$ s) converges to zero faster than the mode e^{-t} ($\tau = 1$ s)

Graphical interpretation of the time constant τ

The **time constant** can be graphically evaluated by considering the intersection with the time axis of the line tangent to the function $e^{\lambda t}$ ($e^{\sigma t}$) passing through the point having $t = 0$



- Computation of the eigenvalues of a square matrix using MatLab

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

- Use the statement **eig**

```
A =

     0     1

     4     0

>> eig(A)

ans =

    2.0000

   -2.0000
```

Example 1

Perform the modal analysis of the LTI system having state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix}$$

The system eigenvalues are:

$$\lambda_1 = -2, \mu_1 = 2, \lambda_2 = -0.2, \mu_2 = 1, \lambda_3 = -0.3, \mu_3 = 1,$$

Example 1

The corresponding natural modes are:

$$\lambda_1 = -2, \mu_1 = 2 \rightarrow \begin{cases} e^{-2t} \rightarrow \text{exponentially convergent} \\ te^{-2t} \rightarrow \text{exponentially convergent} \end{cases}$$

$$\lambda_2 = -0.2, \mu_2 = 1 \rightarrow e^{-0.2t} \rightarrow \text{exponentially convergent}$$

$$\lambda_3 = -0.3, \mu_3 = 1 \rightarrow e^{-0.3t} \rightarrow \text{exponentially convergent}$$

Example 2

Perform the modal analysis of the LTI system having state matrix

$$A = \begin{bmatrix} -0.5 & 5 & 0 & 0 \\ -5 & -0.5 & 0 & 0 \\ -6 & 0 & 0 & 1 \\ -3 & 1 & 3 & -2 \end{bmatrix}$$

Compute the time constant of the convergent modes

The system eigenvalues are

$\lambda_{1,2} = -0.5 \pm 5j$, $\lambda_3 = 1$, $\lambda_4 = -3$

Example 2

The corresponding natural modes are:

$$\lambda_{1,2} = -0.5 \pm 5j,\ \mu_{1,2} = 1 \rightarrow e^{-0.5t}\cos(5t+\varphi) \rightarrow \text{ exponentially convergent}$$

$$\lambda_2 = 1, \mu_2 = 1 \rightarrow e^{t} \rightarrow \text{ exponentially divergent}$$

$$\lambda_3 = -3, \mu_3 = 1 \rightarrow e^{-3t} \rightarrow \text{ exponentially convergent}$$

The time constants of the convergent modes are:

$$\lambda_{1,2} = -0.5 \pm 5j,\ \tau_{1,2} = \left|\frac{1}{\mathbb{R}\mathrm{e}(\lambda_{1,2})}\right| = \left|\frac{1}{-0.5}\right| = 2\ \mathrm{s}$$

$$\lambda_3 = -3, \tau_3 = \left|\frac{1}{\mathbb{R}\mathrm{e}(\lambda_3)}\right| = \left|\frac{1}{-3}\right| = 0.\overline{3}\ \mathrm{s}$$

Internal stability of LTI systems: practical results

Internal stability of LTI systems

Definition (Internal stability of LTI system)

An LTI system is **internally stable** if the zero input response $x_{zi}(t)$ is bounded for any initial state x_0

Recalling that the zero input response is a linear combination of the natural modes of the system →

Result (Internal stability of LTI system)

An LTI system is **internally stable** if and only if all the natural modes are bounded (convergence is a special case of boundedness)

Internal stability of LTI systems

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Recalling that the boundedness of the time behavior of the natural modes depends on the system eigenvalue characteristics →

Result (Internal stability of LTI system)

An LTI system is **internally stable** if and only if all system eigenvalues have nonpositive (i.e. ≤ 0) real part and those with null real part have unitary geometric multiplicity

Asymptotic stability of LTI systems

Definition (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if the zero input response $x_{zi}(t)$ converges to 0, as $t \rightarrow \infty$, for any initial state x_0

Recalling that the zero input response is a linear combination of the natural modes of the system $\rightarrow$

Result (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if and only if all the natural modes are convergent

Asymptotic stability of LTI systems

Result (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if and only if all the natural modes are convergent

Recalling that the convergence of the natural modes depends on the system eigenvalue characteristics →

Result (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if and only if all system eigenvalues have strictly negative (i.e. < 0) real part

Definition (Unstability of LTI system)

An LTI system is **unstable** if it is not stable (i.e. the zero input state response $x_{zi}(t)$ is unbounded)

Recalling that the zero input response is a linear combination of the natural modes of the system →

Result (Unstability of LTI system)

An LTI system is **unstable** if and only if there exists at least one divergent mode

Unstability of LTI systems

Result (Unstability of LTI system)

An LTI system is **unstable** if and only if there exists at least one divergent mode

Recalling that the unboundedness of the time behavior of the natural modes depends on the system eigenvalue characteristics →

Result (Unstability of LTI system)

An LTI system is **unstable** if and only if

- either there is at least one eigenvalue having strictly positive (i.e. > 0) real part

- or all system eigenvalues have nonpositive (i.e. ≤ 0) real part and those with null real part have geometric multiplicity greater than 1

Internal stability of LTI systems: formal resume

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A then

Result (Internal stability of LTI systems)

- An LTI system is **internally stable** if and only if:

  $\mathbb{R}\mathrm{e}(\lambda_i(A)) \leq 0,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) = 1$ for all the eigenvalues such that $\mathbb{R}\mathrm{e}(\lambda_j(A)) = 0$ ($\mu'(\cdot)$ is the geometric multiplicity)

- An LTI system is **asymptotically stable** if and only if:

  $\mathbb{R}\mathrm{e}(\lambda_i(A)) < 0,\ i = 1, \ldots, n$

- An LTI system is **unstable** if and only if:

  $\exists i : \mathbb{R}\mathrm{e}(\lambda_i(A)) > 0$ OR $\mathbb{R}\mathrm{e}(\lambda_i(A)) \leq 0,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) > 1$ for all the eigenvalues such that $\mathbb{R}\mathrm{e}(\lambda_j(A)) = 0$ ($\mu'(\cdot)$ is the geometric multiplicity)

Internal stability of LTI systems: examples

Example 1

Analyse the internal stability properties of the LTI system having state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix}$$

The system eigenvalues are: $\left\{ \lambda_i(A) \right\} = \left\{ -2, -2, -0.2, -0.3 \right\}$

Thus, since all the eigenvalues have strictly negative real part, the system is asymptotically stable

Example 2

Analyse the internal stability properties of the LTI system having state matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

The characteristic polynomial is $\lambda^2(\lambda+5)$ thus the system eigenvalues are:

$$\lambda_1 = 0 \rightarrow \mu_1 = 2$$

$$\lambda_2 = -5 \rightarrow \mu_2 = 1$$

Thus, in order to study internal stability, the geometric multiplicity of the eigenvalue $\lambda_1 = 0$. To this end, the minimal polynomial has to be computed...

Example 2

Let us compute the inverse of the matrix *(λI-A)*

$$(\lambda I-A)^{-1}=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & -1\\ 0 & 0 & \lambda+5\end{bmatrix}^{-1}=\frac{1}{\lambda^2(\lambda+5)}\begin{bmatrix}\lambda(\lambda+5) & 0 & 0\\ 0 & \lambda(\lambda+5) & \lambda\\ 0 & 0 & \lambda^2\end{bmatrix}=$$

$$=\begin{bmatrix}\frac{\lambda(\lambda+5)}{\lambda^2(\lambda+5)} & 0 & 0\\ 0 & \frac{\lambda(\lambda+5)}{\lambda^2(\lambda+5)} & \frac{\lambda}{\lambda^2(\lambda+5)}\\ 0 & 0 & \frac{\lambda^2}{\lambda^2(\lambda+5)}\end{bmatrix}=\begin{bmatrix}\frac{1}{\lambda} & 0 & 0\\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda(\lambda+5)}\\ 0 & 0 & \frac{1}{\lambda+5}\end{bmatrix}\Rightarrow q_A(\lambda)=\lambda(\lambda+5)$$

Minimal polynomial of A
= least common denominator of the terms in $(\lambda I\text{-}A)^{-1}$

the system is **internally stable** (since $\lambda=0$ is a simple root of the minimal polynomial)

MatLab can be used to compute $(sI - A)^{-1}$

```
>> A=[0 0 0;0 0 1;0 0 -5];
>> A1=minreal(zpk(inv(s*eye(3)-A)));
```

In order to explore A1, the following statements can be used:

→ 1st column of A1

```
>> A1(1:3,1)
Zero/pole/gain from input to output...
      1
 #1:  -
      s
 #2:  0
 #3:  0
```

→ 2nd column of A1

```
>> A1(1:3,2)
Zero/pole/gain from input to output...
 #1:  0
      1
 #2:  -
      s
 #3:  0
```

→ 3rd column of A1

```
>> A1(1:3,3)
#1:  0
         1
#2:  -------
      s (s+5)
        1
 #3:  -----
      (s+5)
```

Example 3

Analyse the internal stability properties of the LTI system having state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The characteristic polynomial is $(\lambda^2 + 1)(\lambda + 1)$ thus the system eigenvalues are:

$$\lambda_{1,2} = \pm j \rightarrow \mu_{1,2} = 1$$

$$\lambda_{2} = -1 \rightarrow \mu_{2} = 1$$

There are two eigenvalues with null real part and multiplicity 1 and one eigenvalue with strictly negative real part, thus, the system is internally stable.

Example 4

Discuss wrt the real parameter k the internal stability properties of the LTI system having state matrix

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -k & -1 \end{bmatrix}, k \in \mathbb{R}$$

First of all it can be observed that the given matrix A is block diagonal, then the eigenvalues of A are such that

$$\lambda(A) = \lambda([-2]) \bigcup \lambda\left(\begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix}\right)$$

Example 4

The submatrix

$$\begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix}$$

is in the lower companion form, thus its characteristic polynomial is

$$p(\lambda) = \lambda^2 + \lambda + k$$

Then, according to the Descartes' rule of signs applied to 2nd degree polynomials we have:

- if $k > 0$, $p(\lambda)$ has two roots λ_1 λ_2 such that $Re\{\lambda_1\} < 0$ and $Re\{\lambda_2\} < 0$
- if $k < 0$, $p(\lambda)$ has two roots λ_1 λ_2 such that $Re\{\lambda_1\} < 0$ and $Re\{\lambda_2\} > 0$
- if $k = 0$, $p(\lambda)$ has two roots λ_1 λ_2 such that $Re\{\lambda_1\}<0$ and $Re\{\lambda_2\} = 0$

Example 4

$$\lambda(A) = \lambda([-2]) \bigcup \lambda\left(\begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix}\right)$$

Conclusions:

- if $k > 0$, A has two eigenvalues $\lambda_1\ \lambda_2$ such that Re{λ_1} < 0 and Re{λ_2} < 0 while the third is $\lambda_3 = -2$ → the system is asymptotically stable
- if $k < 0$, A has two eigenvalues $\lambda_1\ \lambda_2$ such that Re{λ_1} < 0 and Re{λ_2} > 0 while the third is $\lambda_3 = -2$ → the system is instable
- if $k = 0$, A has two eigenvalues $\lambda_1\ \lambda_2$ such that Re{λ_1}<0 and Re{λ_2} = 0 while the third is $\lambda_3 = -2$ → the system is stable