Analysis and properties of Discrete time systems

State Space representation

Z – transform

Solution of DT LTI state equation: time domain

Solution of DT LTI state equation: z-domain

Examples of state and output response computation for LTI systems

Analysis and properties of Discrete time systems

Modal analysis

Internal stability of DT LTI systems

Bibo stability of DT LTI systems

Equilibrium points and linearization

Analysis and properties of Discrete time systems

State Space representation

Discrete time state space system description

We consider the description of systems through difference equations of the form

$$\boxed{x(k+1) = f\left(k, x(k), u(k)\right)}$$ **State equation**

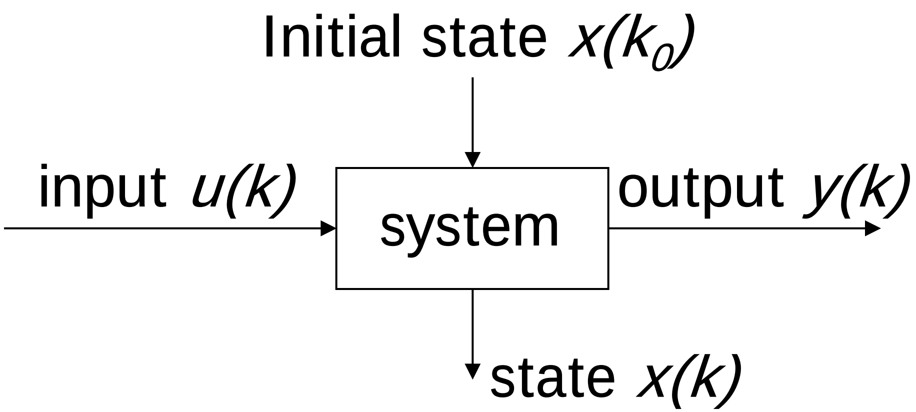
$$\boxed{y(k) = g\left(k, x(k), u(k)\right)}$$ **Output equation**

$x(k) \in \mathbb{R}^n; \quad u(k) \in \mathbb{R}^p$

$y(k) \in \mathbb{R}^q$

Given $u(k)$, $k \geq k_0$; $x(k_0)$

Find $x(k)$ and $y(k)$



Discrete time state space system description

For discrete time dynamical **linear** systems, the time evolution of the system is described by the system of difference equations

$$\boxed{x(k+1) = A(k)\,x(k) + B(k)\,u(k)}$$

State equation

$A(k) \in \mathbb{R}^{n \times n}$ state matrix

$B(k) \in \mathbb{R}^{n \times q}$ input matrix

$$\boxed{y(k) = C(k)\,x(k) + D(k)\,u(k)}$$

Output equation

$C(k) \in \mathbb{R}^{q \times n}$ output matrix

$D(k) \in \mathbb{R}^{q \times p}$

Discrete time state space system description

For discrete time dynamical **linear time invariant** systems, the time evolution of the system is described by a constant coefficient system of difference equations

$$\boxed{x(k+1) = A\,x(k) + B\,u(k)}$$

State equation

$$\boxed{y(k) = C\,x(k) + D\,u(k)}$$

Output equation

with A, B, C, D constant matrices.

Discrete time state space system description

Example: given a system described by the following state-space equations

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + 2u_1(k) \\
x_2(k+1) &= k\,x_1(k) + u_2(k) \\
y(k) &= x_1(k) + 3u_1(k)
\end{aligned}
$$

Evaluate the properties of the mathematical model.

Discrete time state space system description

Example: given a system described by the following state-space equations

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + 2u_1(k) \\
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y(k) &= x_1(k) + 3u_1(k)
\end{aligned}
$$

Evaluate the properties of the mathematical model.

The system is:

- **Dynamic**: the output $y(k)$ depends on the whole history of the signal $u(\cdot)$

Discrete time state space system description

Example: given a system described by the following state-space equations

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + 2u_1(k) \\
x_2(k+1) &= k\,x_1(k) + u_2(k) \\
y(k) &= x_1(k) + 3u_1(k)
\end{aligned}
$$

Evaluate the properties of the mathematical model.

The system is:

- **discrete time**: the system is described by difference equations

Discrete time state space system description

Example: given a system described by the following state-space equations

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + 2u_1(k) \\
x_2(k+1) &= k\,x_1(k) + u_2(k) \\
y(k) &= x_1(k) + 3u_1(k)
\end{aligned}
$$

Evaluate the properties of the mathematical model.

The system is:

- **MIMO**: the system has 1 output and 2 inputs

  (i.e dim(y) = 1, dim(u) = 2)

Discrete time state space system description

Example: given a system described by the following state-space equations

$$\begin{aligned} x_1(k+1) &= x_2(k) + 2u_1(k) \\ x_2(k+1) &= k\,x_1(k) + u_2(k) \\ y(k) &= x_1(k) + 3u_1(k) \end{aligned}$$

Evaluate the properties of the mathematical model.

The system is:

- **finite dimension**: the system has a finite number of state

  (i.e. $\dim(x) = 2$)

Discrete time state space system description

Example: given a system described by the following state-space equations

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + 2u_1(k) \\
x_2(k+1) &= k\,x_1(k) + u_2(k) \\
y(k) &= x_1(k) + 3u_1(k)
\end{aligned}
$$

Evaluate the properties of the mathematical model.

The system is:

- **linear**: the system equations are linear in x and u

Discrete time state space system description

Example: given a system described by the following state-space equations

$$\begin{aligned} x_1(k+1) &= x_2(k) + 2u_1(k) \\ x_2(k+1) &= k\,x_1(k) + u_2(k) \\ y(k) &= x_1(k) + 3u_1(k) \end{aligned}$$

Evaluate the properties of the mathematical model.

The system is:

- **time variant**: the term $k\,x_1(k)$ shows the non-constant coefficient k (the coefficient is time-dependent)

Discrete time state space system description

The system

$$\begin{aligned} x_1(k+1) &= x_2(k) + 2u_1(k) \\ x_2(k+1) &= k\,x_1(k) + u_2(k) \\ y(k) &= x_1(k) + 3u_1(k) \end{aligned}$$

is:

- discrete time
- MIMO
- finite dimension
- Linear
- time variant

Analysis and properties of Discrete time systems

Z – transform

The **Unilateral Z – transform** of the discrete time sequence $f(k): \mathbb{N} \rightarrow \mathbb{R}$ is the function of the complex variable z defined as:

$$F(z) = \mathcal{Z}\{f(k)\} \triangleq \sum_{k=0}^{\infty} f(k)\, z^{-k}, \quad z \in \mathbb{C}$$

Z – transform: main properties

Linearity

- Given $f_1(k)$, $f_2(k)$ two discrete time function having $F_1(z)$ and $F_2(z)$ as Z – transform and c_1 , c_2 real values constants, then the following relation holds:

$$\mathcal{Z}\left\{c_1 f_1(k) + c_2 f_2(k)\right\} = c_1 F_1(z) + c_2 F_2(z)$$

Backward time shift

- Given $f(k)$ a discrete time function having $F(z)$ as Z–transform and a positive integer constant h, then:

$$\mathcal{Z}\left\{f(k-h)\right\} = z^{-h}F(z)$$

Forward time shift

- Given $f(k)$ a discrete time function having $F(z)$ as Z–transform and a positive integer constant h, then:

$$\mathcal{Z}\{f(k+h)\} = z^h\left(F(z) - \sum_{k=0}^{h-1} z^k f(k)\right)$$

- In particular, if $h = 1$:

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$$

Convolution

- Given $f_1(k)$, $f_2(k)$ two discrete time function and the Z – transforms $F_1(z)$, $F_2(z)$ respectively, then the Z – transform of the convolution integral defined by

$$f_1(k) * f_2(k) = \sum_{j=0}^{k} f_1(k-j) \cdot f_2(j) = \sum_{j=0}^{k} f_1(j) \cdot f_2(k-j)$$

is given by:

$$\mathcal{Z}\left\{f_1(k) * f_2(k)\right\} = F_1(z) \cdot F_2(z)$$

Final value theorem

- Given $f(k)$ a discrete time function having $F(z)$ as Z–transform. If

$$(z-1)F(z)$$

has no poles with absolute value equal or greater than 1, then:

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z)$$

Initial value theorem

- Given $f(k)$ a discrete time function having $F(z)$ as Z–transform, then:

$$\lim_{k \to 0} f(k) = \lim_{z \to \infty} F(z)$$

main Z – transforms

$f(k)$	$F(z)$
$\delta(k)$	1
$\varepsilon(k)$	$\dfrac{z}{z-1}$
$\dbinom{k}{\ell} \triangleq \dfrac{k(k-1)\cdots(k-\ell+1)}{\ell!}, \ell > 0$	$\dfrac{z}{(z-1)^{\ell+1}}$

$f(k)$	$F(z)$
a^k	$\dfrac{z}{z-a}$
$\begin{pmatrix} k \\ \ell \end{pmatrix} a^{k-\ell}, \ell > 0$	$\dfrac{z}{(z-a)^{\ell+1}}$
$\sin(\vartheta k), \vartheta \in \mathbb{R}$	$\dfrac{z\sin(\vartheta)}{z^2 - 2\cos(\vartheta)z + 1}$
$\cos(\vartheta k), \vartheta \in \mathbb{R}$	$\dfrac{z(z-\cos(\vartheta))}{z^2 - 2\cos(\vartheta)z + 1}$
$A^k, A \in \mathbb{R}^{n,n}$	$z(zI - A)^{-1}$

Inverse Z – transform

- If $F(z)$ is a rational function, in order to find the inverse of the Z-transform we can evaluate the Heaviside's Partial Fraction Expansion (PFE).
- Elementary Z – transforms usually taken from available tables, are not in the canonical PFE form, since there is always a factor z at the numerator, e.g.:

$$\frac{z}{z-1}, \frac{0.1z}{(z-0.1)^2}$$

- Example: consider the function

$$F(z) = \frac{z}{(z-0.5)(z-0.4)} = \frac{5}{z-0.5} - \frac{4}{z-0.4}$$

$$\mathcal{Z}^{-1}\left\{\frac{5}{z-0.5}\right\} = ??$$

$$\mathcal{Z}^{-1}\left\{\frac{-4}{z-0.4}\right\} = ??$$

There are no terms like $\dfrac{R}{z-a}$ in the tables

To find the inverse of the Z-transform proceed as follows:

- Evaluate the function

$$\tilde{F}(z) = \frac{F(z)}{z}$$

- Evaluate the PFE of $\tilde{F}(z)$ as usual

- Multiply the obtained PFE by z

Example: $$F(z) = \frac{z}{(z-0.5)(z-0.4)}$$

$$\tilde{F}(z) = \frac{F(z)}{z} = \frac{1}{(z-0.5)(z-0.4)} = \frac{10}{z-0.5} - \frac{10}{z-0.4}$$

$$F(z) = z \cdot \tilde{F}(z) = \frac{10z}{z-0.5} - \frac{10z}{z-0.4}$$

From the relation $F(z) = \frac{10z}{z-0.5} - \frac{10z}{z-0.4}$ it is possible to find the inverse Z-transform as:

$$f(k) = \mathcal{Z}^{-1}\{F(z)\} = \mathcal{Z}^{-1}\left\{\frac{10z}{z-0.5} - \frac{10z}{z-0.4}\right\} =$$

$$= \left(10 \cdot 0.5^k - 10 \cdot 0.4^k\right)\varepsilon(k)$$

In summary:

- Inverse Z – transforms are obtained similarly to Inverse Laplace transforms since they both require the evaluation of the PFE in a similar way.
- Also for the Inverse Z – transform, a particular case occurs when the function has complex conjugated poles (i.e.: $a = \sigma + j\omega$, $a^* = \sigma - j\omega$) giving rise to a couple of residues like

$$\frac{Rz}{z-a} + \frac{R^* z}{z-a^*}$$

- In such a case, the Inverse Z – transform is:

$$2|R||a|^k \cos\left((\angle a)k + \angle R\right)$$

Analysis and properties of Discrete time systems

Solution of DT LTI state equation: time domain

Linear Time Invariant (LTI) state space equation

If A, B, C, D are constant, the system is **time-invariant**.

$$\boxed{x(k+1) = A\,x(k) + B\,u(k)}$$

State equation

$A \in \mathbb{R}^{n \times n}$: state matrix

$B \in \mathbb{R}^{n \times p}$: input matrix

$$\boxed{y(k) = C\,x(k) + D\,u(k)}$$

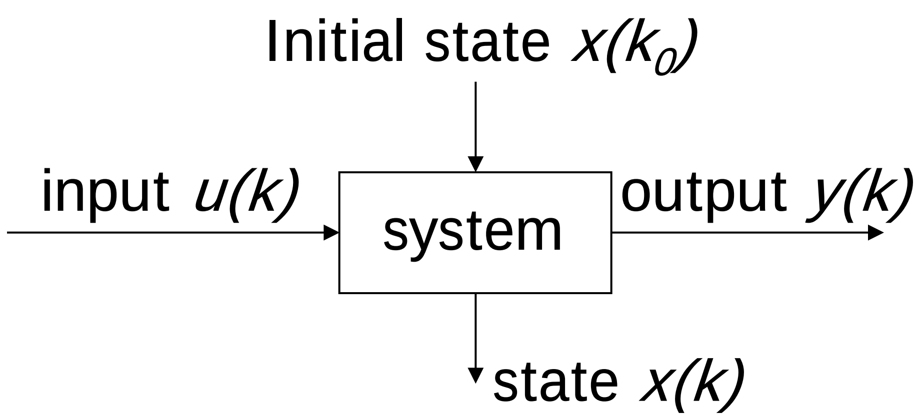
Output equation

$C \in \mathbb{R}^{q \times n}$: output matrix

$D \in \mathbb{R}^{q \times p}$

Given $u(k)$, $k \geq k_0$; $x(k_0)$

Find $x(k)$ and $y(k)$



The solution of the state equation

$$x(k+1) = A\,x(k) + B\,u(k)$$

can be obtained iteratively as:

$$x(1) = A\,x(0) + B\,u(0)$$

$$x(2) = A\,x(1) + B\,u(1) = A^2 x(0) + A\,Bu(0) + B\,u(1)$$

$$x(3) = A\,x(2) + B\,u(2) =$$

$$= A^3 x(0) + A^2 B\,u(0) + A\,B\,u(1) + B\,u(2)$$

$$\vdots$$

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B\,u(i)$$

The solution of the state equation is:

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i) = x_{zi}(k) + x_{zs}(k)$$

Which can be split into two components

$X_{zi}(k)$: **zero-input response**

$X_{zs}(k)$: **zero-state response**

Output response

The output response can be obtained from the output equation

$$y(k) = C\,x(k) + D\,u(k)$$

as:

$$y(k) = C\,A^k x(0) + C\sum_{i=0}^{k-1} A^{k-i-1} B\,u(i) + D\,u(k) = y_{zi}(k) + y_{zs}(k)$$

$y_{zi}(k)$: zero-input response

$y_{zs}(k)$: zero-state response

Analysis and properties of Discrete time systems

Solution of DT LTI state equation: z-domain

Solution by Z – transform

Transformation of the state and the output equations and use of the forward shift property give

$$\begin{cases} x(k+1) = A\,x(k) + B\,u(k) \\ y(k) = C\,x(k) + D\,u(k) \end{cases}$$

$$\Big\downarrow \mathcal{Z}$$

$$\begin{cases} zX(z) - zx(0) = AX(z) + B\,U(z) \\ Y(z) = C\,X(z) + D\,U(z) \end{cases}$$

The transformed state response is given by

$$X(z) = z(zI - A)^{-1}x(0) + (zI - A)^{-1}B\,U(z) = X_{zi}(z) + X_{zs}(z)$$

While the transformed output response is given by

$$Y(z) = z\,C\,(zI - A)^{-1}x(0) + \left\{C\,(zI - A)^{-1}B + D\right\}U(z) = Y_{zi}(z) + Y_{zs}(z)$$

$$Y_{zi}(z) = z\,C\,(zI - A)^{-1}x(0) = H_{zi}(z)\,x(0)$$

$$Y_{zs}(z) = \left\{C\,(zI - A)^{-1}B + D\right\}U(z) = H(z)\,U(z)$$

The matrix transfer function

The Z – transform of the zero-state impulse response is the **matrix transfer function** *H(z)*

$$Y_{zs}(z) = \left\{C\left(zI - A\right)^{-1}B + D\right\}U(z) = H(z)U(z)$$

$$H(z) = \left\{C\left(zI - A\right)^{-1}B + D\right\}$$

$$\boxed{H(z) = C\,\frac{Adj(z\,I - A)}{\det(z\,I - A)}\,B + D}$$

The matrix *Adj(zI-A)* is a matrix of polynomials of order *n-1* at most. Pre- or post-multiplication by a matrix results in weighted sum of such polynomials, so that the numerator of *H(z)* is a polynomial of degree *n-1* at most (if *D=0*). The denominator of *H(z)* is a polynomial of degree *n*.

The system transfer function

If the system has a single input (p=1) and a single output (q=1), we say that it is a SISO system. In that case, the **matrix transfer function** *H(z)* shows a single element, which is called **system transfer function** *H(z)*

$$\boxed{H(z)=\frac{N_H(z)}{D_H(z)}=\frac{b_m z^m+b_{m-1}z^{m-1}+\cdots+b_1 z+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0},\quad m\le n}$$

The system transfer function

$$H(z) = \frac{N_H(z)}{D_H(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}, \qquad m \le n$$

- $m < n \rightarrow$ the transfer function is strictly proper ($b_m = D = 0$)
- $m = n \rightarrow$ the transfer function is proper
- Zeros of *H(z)*: $z \in \mathbb{C}$ such that $H(z) = 0$
- Poles of *H(z)*: $z \in \mathbb{C}$ such that $H(z) = \infty$
- If there is no cancellation between $N_H(z)$ and $D_H(z)$, then the roots of $N_H(z)$ are zeros of *H(z)* and the roots of $D_H(z)$ are poles of *H(z)*

The system transfer function

$$H(z) = \frac{N_H(z)}{D_H(z)} = C\,\frac{\mathrm{Adj}(zI - A)}{\det(zI - A)}B + D = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

- the denominator of $H(z)$ is $\det(zI-A)$
- the roots of $\det(zI-A)$ are the eigenvalues of A
- the roots of $D_H(z)$ are poles of $H(z)$
- All poles of $H(z)$ are eigenvalues of A, the converse is not necessarily true (there could be a cancellation).

Analysis and properties of Discrete time systems

Example of state and output response computation for LTI systems

Example 1

Given the DT LTI dynamical system described by the following equations:

$$x(k+1) = \begin{bmatrix} 3 & 0 \\ -3.5 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(k)$$

find the analytical expression of $x(k)$ and $y(k)$ when

- $u(k) = 2\varepsilon(k)$
- $x(0) = [1 \;\; -2]^T$

Example 1

We will proceed as follows:

- Look for the solution of $X(z)$ in the z – domain
- Find $x(k)$ as the inverse z – transform of $X(z)$
- Find $y(k)$ with the static relation $y(k) = Cx(k)$

Example 1

Solution of $X(z)$ in the z – domain:

$$X(z) = z\left(zI - A\right)^{-1} x(0) + \left(zI - A\right)^{-1} B\,U(z) = X_{zi}(z) + X_{zs}(z)$$

with

$$A = \begin{bmatrix} 3 & 0 \\ -3.5 & -0.5 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\quad U(z) = \frac{2z}{z-1}$$

Example 1

Solution of $X(z)$ in the z – domain:

$$X(z) = z\left(zI - A\right)^{-1}x(0) + \left(zI - A\right)^{-1}B\,U(z) = X_{zi}(z) + X_{zs}(z)$$

Evaluate $(z\,I - A)^{-1}$:

$$(zI - A)^{-1} = \left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -3.5 & -0.5 \end{bmatrix}\right]^{-1} = \begin{bmatrix} z-3 & 0 \\ 3.5 & z+0.5 \end{bmatrix}^{-1} =$$

$$= \underbrace{\frac{1}{(z-3)(z+0.5)}}_{\det(zI-A)} \underbrace{\begin{bmatrix} z+0.5 & 0 \\ -3.5 & z-3 \end{bmatrix}}_{Adj\,(zI-A)} = \begin{bmatrix} \dfrac{1}{z-3} & 0 \\ \dfrac{-3.5}{(z-3)(z+0.5)} & \dfrac{1}{z+0.5} \end{bmatrix}$$

Example 1

Solution of $X(z)$ in the z – domain:

$$X(z) = z\left(zI - A\right)^{-1} x(0) + \left(zI - A\right)^{-1} B\,U(z) = X_{zi}(z) + X_{zs}(z)$$

Find $X_{zi}(z)$:

$$X_{zi}(z) = z(zI - A)^{-1} x(0) = z \underbrace{\begin{bmatrix} \dfrac{1}{z-3} & 0 \\ \dfrac{-3.5}{(z-3)(z+0.5)} & \dfrac{1}{z+0.5} \end{bmatrix}}_{(zI - A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{x(0)} =$$

$$= z \begin{bmatrix} \dfrac{1}{z-3} \\ \dfrac{-2z+2.5}{(z-3)(z+0.5)} \end{bmatrix}$$

Example 1

Solution of $X(z)$ in the z – domain:

$$X(z) = z\left(zI - A\right)^{-1}x(0) + \left(zI - A\right)^{-1}B\,U(z) = X_{zi}(z) + X_{zs}(z)$$

Find $X_{zs}(z)$:

$$X_{zs}(z) = (zI - A)^{-1}BU(z) = \underbrace{\begin{bmatrix} \dfrac{1}{z-3} & 0 \\ \dfrac{-3.5}{(z-3)(z+0.5)} & \dfrac{1}{z+0.5} \end{bmatrix}}_{(zI - A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{B} U(z) =$$

$$\underset{U(z)=\frac{2z}{z-1}}{\overset{\uparrow}{=}} \begin{bmatrix} \dfrac{1}{z-3} \\ \dfrac{2z-9.5}{(z-3)(z+0.5)} \end{bmatrix} \frac{2z}{z-1} = z \begin{bmatrix} \dfrac{2}{(z-3)(z-1)} \\ \dfrac{4z-19}{(z-3)(z+0.5)(z-1)} \end{bmatrix}$$

Example 1

Solution of $X(z)$ in the z – domain:

$$X(z) = z\left(zI - A\right)^{-1} x(0) + \left(zI - A\right)^{-1} B\, U(z) = X_{zi}(z) + X_{zs}(z)$$

Find $X(z)$:

$$X(z) = X_{zi}(z) + X_{zs}(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = z \begin{bmatrix} \dfrac{z+1}{(z-3)(z-1)} \\ \dfrac{-2z^2 + 8.5z - 21.5}{(z-3)(z+0.5)(z-1)} \end{bmatrix}$$

$$= z \begin{bmatrix} \dfrac{R_1^{(1)}}{z-3} + \dfrac{R_2^{(1)}}{z-1} \\ \dfrac{R_1^{(2)}}{z-3} + \dfrac{R_2^{(2)}}{z+0.5} + \dfrac{R_3^{(2)}}{z-1} \end{bmatrix}$$

Example 1

Find $X_1(z)$:

$$\tilde{X}_1(z) = \frac{X_1(z)}{z} = \frac{z+1}{(z-3)(z-1)} = \frac{R_1^{(1)}}{z-3} + \frac{R_2^{(1)}}{z-1}$$

$$R_1^{(1)} = \lim_{z\to 3}(z-3)\tilde{X}_1(z) = \lim_{z\to 3}(z-3)\frac{z+1}{(z-3)(z-1)} = 2$$

$$R_2^{(1)} = \lim_{z\to 1}(z-1)\tilde{X}_1(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\to \tilde{X}_1(z) = \frac{2}{z-3} - \frac{1}{z-1}$$

$$\to X_1(z) = z\tilde{X}_1(z) = \frac{2z}{z-3} - \frac{z}{z-1}$$

Find $X_2(z)$:

$$\tilde{X}_2(z) = \frac{X_2(z)}{z} = \frac{-2z^2 + 8.5z - 21.5}{(z-3)(z+0.5)(z-1)} = \frac{R_1^{(2)}}{z-3} + \frac{R_2^{(2)}}{z+0.5} + \frac{R_3^{(2)}}{z-1}$$

$$R_1^{(2)} = \lim_{z\to 3}(z-3)\tilde{X}_2(z) = \lim_{z\to 3}(z-3)\frac{-2z^2 + 8.5z - 21.5}{(z-3)(z+0.5)(z-1)} = -2$$

$$R_2^{(2)} = \lim_{z\to -0.5}(z+0.5)\tilde{X}_2(z) = \lim_{z\to -0.5}(z+0.5)\frac{-2z^2 + 8.5z - 21.5}{(z-3)(z+0.5)(z-1)} = -5$$

$$R_3^{(2)} = \lim_{z\to 1}(z-1)\tilde{X}_2(z) = \lim_{z\to 1}(z-1)\frac{-2z^2 + 8.5z - 21.5}{(z-3)(z+0.5)(z-1)} = 5$$

$$\to \tilde{X}_2(z) = -\frac{2}{z-3} - \frac{5}{z+0.5} + \frac{5}{z-1}$$

$$\to X_2(z) = z\tilde{X}_2(z) = -\frac{2z}{z-3} - \frac{5z}{z+0.5} + \frac{5z}{z-1}$$

Example 1

Finally:

$$X(z)=\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}=\begin{bmatrix} \dfrac{2z}{z-3}-\dfrac{z}{z-1} \\ -\dfrac{2z}{z-3}-\dfrac{5z}{z+0.5}+\dfrac{5z}{z-1} \end{bmatrix}$$

$x(k)$ is obtained by applying the inverse of the Z – transform:

$$x(k)=\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}=\begin{bmatrix} 2\cdot 3^k-1 \\ -2\cdot 3^k-5\cdot(-0.5)^k+5 \end{bmatrix}\varepsilon(k)$$

Example 1

Recalling that $y(k) = C\,x(k)$

$$y(k) = Cx(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2\cdot 3^k - 1 \\ -2\cdot 3^k - 5\cdot(-0.5)^k + 5 \end{bmatrix} \varepsilon(k) =$$

$$= \left(4\cdot 3^k + 5\cdot(-0.5)^k - 6\right)\varepsilon(k)$$

Example 2

- Rational transforms can be factored using the same partial fractions approach we used for the Laplace transforms.
- The partial fractions approach is preferred if we want a closed-form solution rather than the numerical solution long division provides.
- Example 2:

$$X(z) = \frac{z^3 + 1}{z^3 - z^2 - z - 2}$$

In this example, the order of the numerator and denominator are the same. For this case, we again use the trick of factoring $X(z)/z$

Example 2

$$D_X(z) = z^3 - z^2 - z - 2 = (z-2)(z+0.5+j0.866)(z+0.5-j0.866)$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z+0.5+j0.866} + \frac{\bar{c}_1}{z+0.5-j0.866} + \frac{c_3}{z-2}$$

$$c_0 = \left[\frac{X(z)}{z}(z)\right]_{z=0} = \frac{1}{-2} = -0.5$$

$$c_1 = \left[\frac{X(z)}{z}\left(z+0.5+j0.866\right)\right]_{z=-0.5-j0.866} = 0.429 + j0.0825$$

$$c_3 = \left[\frac{X(z)}{z}\left(z-2\right)\right]_{z=2} = 0.643$$

Example 2

We can evaluate the inverse using our table of common transforms

$$X(z) = c_0 + \frac{c_1 z}{z + 0.5 + j0.866} + \frac{\bar{c}_1 z}{z + 0.5 - j0.866} + \frac{c_3 z}{z - 2}$$

$$= c_0 + \frac{c_1}{1 + 0.5 + j0.866 z^{-1}} + \frac{\bar{c}_1}{1 + 0.5 - j0.866 z^{-1}} + \frac{c_3}{1 - 2z^{-1}}$$

$$x[k] = c_0 \delta[k] + c_1 (-0.5 - j0.866)^k \varepsilon[k] + \bar{c}_1 (-0.5 + j0.866)^k \varepsilon[k] + c_3 2^k \varepsilon[k]$$

Example 2

The exponential terms can be converted to a single cosine using a magnitude/phase conversion:

$$|p_1| = \sqrt{(0.5)^2 + (0.866)^2} = 1$$

$$\angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3}\ \text{rad}$$

$$|c_1| = \sqrt{(0.429)^2 + (0.0825)^2} = 0.437$$

$$\angle c_1 = \tan^{-1}\frac{0.0825}{0.429} = 0.19\ \text{rad} \quad (10.89°)$$

$$\begin{aligned} x[k] &= c_0\delta[k] + c_1(-0.5 - j0.866)^k\varepsilon[k] + \bar{c}_1(-0.5 + j0.866)^k\varepsilon[k] + c_3 2^k\varepsilon[k] \\ &= c_0\delta[k] + 2|c_1||p_1|\cos(\angle p_1 k + \angle c_1) + c_3(2)^k\varepsilon[k] \\ &= -0.5\delta[k] + 0.874\cos(\frac{4\pi}{3}k + 0.19) + 0.643(2)^k\varepsilon[k] \end{aligned}$$

Analysis and properties of Discrete time systems

Modal analysis

Modal Analysis: a recall

Recall that:

- To evaluate the internal stability property of a dynamical LTI system we must check if the zero input response of the state is bounded for every initial condition x_0 in $\mathbb{R}^n$
- The zero input response is a linear combination of functions called natural modes
- The internal stability properties of an LTI system can be studied through the asympotical convergence characteristics of the natural modes
- Convergence properties of the natural modes of an LTI system depend on the characteristics of the corresponding eigenvalues

Modal Analysis: a recall

The classification of the natural modes of DT systems is similar to the one introduced for CT systems

- A natural mode is convergent if: $\lim_{k\to\infty} |m(k)| = 0$

- A natural mode is bounded if: $\exists\ M \in \mathbb{R} : \forall k \geq 0,\ 0 \leq |m(k)| \leq M < \infty$

- A natural mode is divergent if: $\lim_{k\to\infty} |m(k)| = \infty$

The zero input response $x_{zi}(k)$ of an LTI DT system is described by the following system equation

$$x(k+1) = A\,x(k)$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, is given by:

$$x_{zi}(k) = A^k x(0)$$

with $x(0) \in \mathbb{R}^n$ a known initial state

The Z – transform of the zero input response is computed as

$$X_{zi}(z) = z(zI - A)^{-1}x(0)$$

where

$$z(zI - A)^{-1} = z\frac{1}{\det(zI - A)}Adj(zI - A) = z\frac{\left[a_{ji}(z)\right]}{p_A(z)}$$

- $p_A(z)$ is the characteristic polynomial of the matrix A

$$p_A(z) = (z - \lambda_1)^{\mu_1}(z - \lambda_2)^{\mu_2}\cdots(z - \lambda_r)^{\mu_r}$$

- $a_{ji}(z)$ is the (j,i) minor of ($zI - A$), obtained as the determinant of the (n-1)x(n-1) matrix that results from deleting row i and column j of ($zI - A$)

$$p_A(z) = (z-\lambda_1)^{\mu_1}(z-\lambda_2)^{\mu_2}\cdots(z-\lambda_r)^{\mu_r}$$

$\deg(p_A(z)) = n$

$p_A(z)$ has $r \le n$ distinct roots

$\mathrm{spec}(p_A) = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ → spectrum of the polynomial

μ_i is the algebraic multiplicity of the eigenvalue λ_i

$\deg(a_{ji}(z)) \le n\text{-}1$

The common factors to $p_A(z)$ and $a_{ji}(z)$ can be cancelled, obtaining:

$$f_{ij}(z) = z\,\frac{a'_{ji}(z)}{q_{ij}(z)}$$

The minimal polynomial of A is defined as the least common multiple

$$q_A(z) = (z-\lambda_1)^{\mu'_1}(z-\lambda_2)^{\mu'_2}\cdots(z-\lambda_r)^{\mu'_r} = l.c.m.\left(q_{ij}(z)\right),\ \ i,j = 1,\ldots,n$$

where $\mu'_i \le \mu_i$ is the geometric multiplicity of the eigenvalue λ_i

The generic term $f_{ij}(z)$ of the matrix $z(zI - A)^{-1}$ can be written as

$$f_{ij}(z) = \frac{za'_{ji}(z)}{(z-\lambda_1)^{\mu'_1}(z-\lambda_2)^{\mu'_2}\cdots(z-\lambda_r)^{\mu'_r}}$$

It follows that

$$f_{ij}(z) = \sum_{h=1}^{\mu'_1}\frac{r_{1,h}z}{(z-\lambda_1)^h} + \sum_{k=1}^{\mu'_2}\frac{r_{2,h}z}{(z-\lambda_2)^h} + \cdots + \sum_{h=1}^{\mu'_r}\frac{r_{r,h}z}{(z-\lambda_r)^h} =$$

$$= \sum_{i=1}^{r}\sum_{h=1}^{\mu'_i}\frac{r_{i,h}z}{(z-\lambda_i)^h}$$

The zero-input response is a linear combination of the terms $f_{ij}(z)$

The inverse of the Z – transform of the generic term

$$f_{ij}(z) = \sum_{h=1}^{\mu'_1} \frac{r_{1,h} z}{(z-\lambda_1)^h} + \sum_{h=1}^{\mu'_2} \frac{r_{2,h} z}{(z-\lambda_2)^h} + \cdots + \sum_{h=1}^{\mu'_r} \frac{r_{r,h} z}{(z-\lambda_r)^h}$$

is

$$f_{ij}(k) = \sum_{h=1}^{\mu'_1} r_{1,h} \binom{k}{h-1} \lambda_1^{k-h+1} + \sum_{h=1}^{\mu'_2} r_{2,k} \binom{k}{h-1} \lambda_2^{k-h+1} + \ldots$$

$$\ldots + \sum_{h=1}^{\mu'_r} r_{r,h} \binom{k}{h-1} \lambda_r^{k-h+1}$$

Thus, the zero-input response is a linear combination of:

$$
\begin{matrix}
\lambda_1^{\,k} & \lambda_2^{\,k} & \cdots & \lambda_r^{\,k} \\
k\lambda_1^{\,k-1} & k\lambda_2^{\,k-1} & \cdots & k\lambda_r^{\,k-1} \\
\vdots & \vdots & \cdots & \vdots \\
\begin{pmatrix} k \\ \mu'_1 - 1 \end{pmatrix}\lambda_1^{\,k-\mu'_1+1} & \begin{pmatrix} k \\ \mu'_2 - 1 \end{pmatrix}\lambda_2^{\,k-\mu'_2+1} & \cdots & \begin{pmatrix} k \\ \mu'_r - 1 \end{pmatrix}\lambda_r^{\,k-\mu'_r+1}
\end{matrix}
$$

where

$$
\begin{pmatrix} k \\ \mu'_i - 1 \end{pmatrix}\lambda_i^{\,k-\mu'_i+1} = \frac{k(k-1)\cdots(k-\mu'-2)}{(\mu_i'-1)!}\lambda_i^{\,k-\mu'_i+1}
$$

By considering the *i–th* eigenvalue λ_i ($i = 1,\ldots,r$) with geometric multiplicity μ'_i, the functions $m_{ij}(k)$ ($i = 1,\ldots,r$, $j = 1,\ldots, \mu'_i$)

$$m_{i,0}(k) = \lambda_i^{\,k}, \quad m_{i,1}(k) = k\lambda_i^{\,k-1}, \ldots, \quad m_{i,\mu'_i}(k) = \binom{k}{\mu'_i - 1}\lambda_i^{\,k-\mu'_i+1}$$

are the so-called natural modes of the system.

For each eigenvalue λ_i having geometric multiplicity μ'_i, there are μ'_i natural modes.

For each couple of complex conjugate eigenvalues

$$\lambda = \sigma_0 \pm j\omega_0 = \nu e^{\pm j\theta}$$

with geometric multiplicity μ', the resulting natural modes can be written as:

$$m_0(k) = \begin{cases} \nu^k \cos(\theta k) \\ \nu^k \sin(\theta k) \end{cases}, \qquad m_1(k) = \begin{cases} k\nu^{k-1} \cos(\theta(k-1)) \\ k\nu^{k-1} \sin(\theta(k-1)) \end{cases}, \dots$$

$$\dots, m_{\mu'}(k) = \begin{cases} \begin{pmatrix} k \\ \mu'-1 \end{pmatrix} \nu^{k-\mu'+1} \cos(\theta(k-\mu'+1)) \\ \begin{pmatrix} k \\ \mu'-1 \end{pmatrix} \nu^{k-\mu'+1} \sin(\theta(k-\mu'+1)) \end{cases}$$

Modal analysis: $\lambda \in \mathbb{R}$, $\mu' = 1$

The natural mode λ^k, associated to the real eigenvalue $\lambda \in \mathbb{R}$ with $\mu' = 1$, is:

- **Geometrically convergent** if $|\lambda| < 1$ (e.g.: 0.5^k, $(-0.5)^k$)
- **Bounded** if $|\lambda| = 1$ (e.g.: $1^k = 1$, $(-1)^k$)
- **Geometrically divergent** if $|\lambda| > 1$ (e.g.: 2^k, $(-2)^k$)

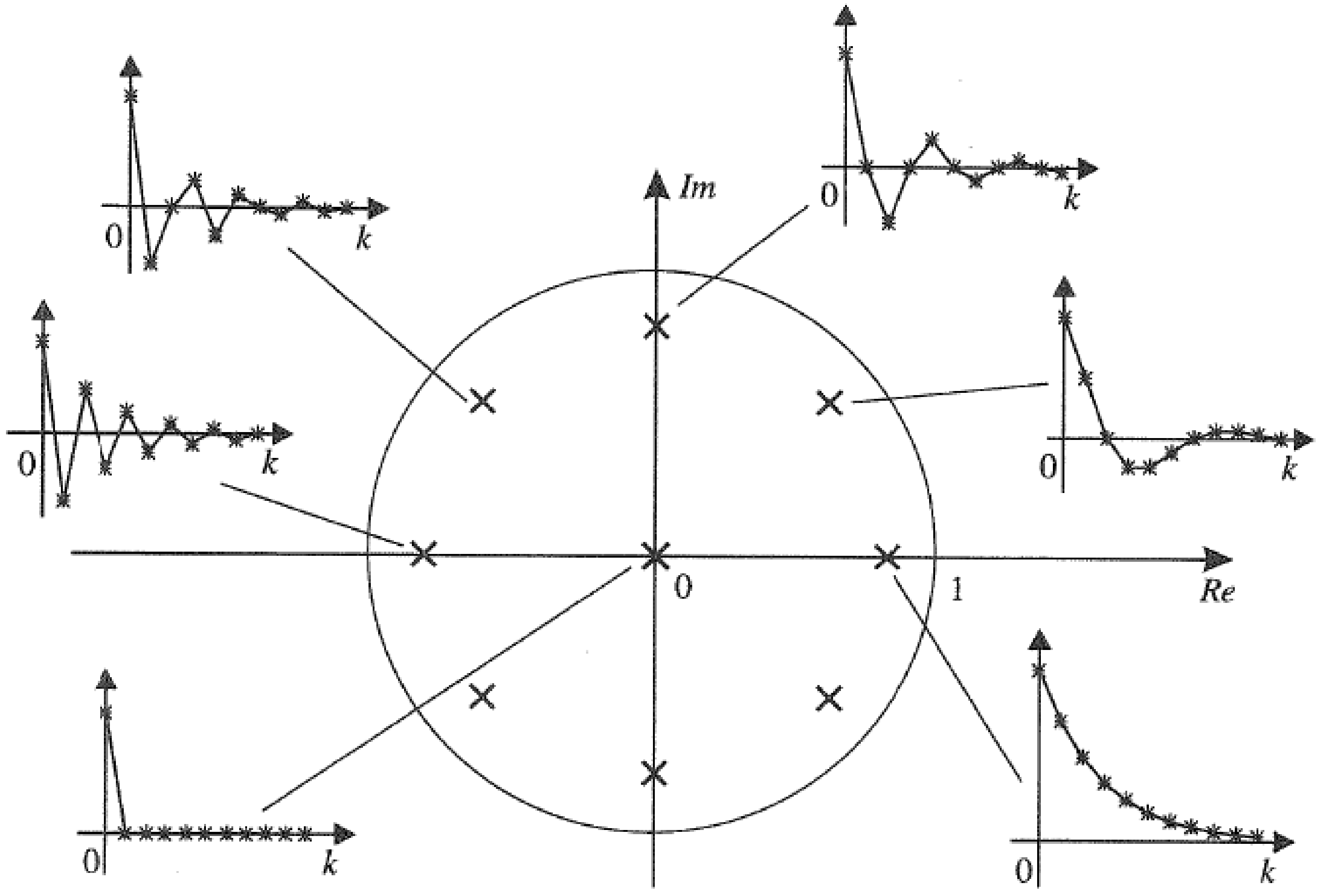
The natural mode associated to an eigenvalue such that $\mathbb{R}e(\lambda) < 0$ provides an alternating sign sequence.

Modal analysis: $\lambda \in \mathbb{C}$, $\mu' = 1$

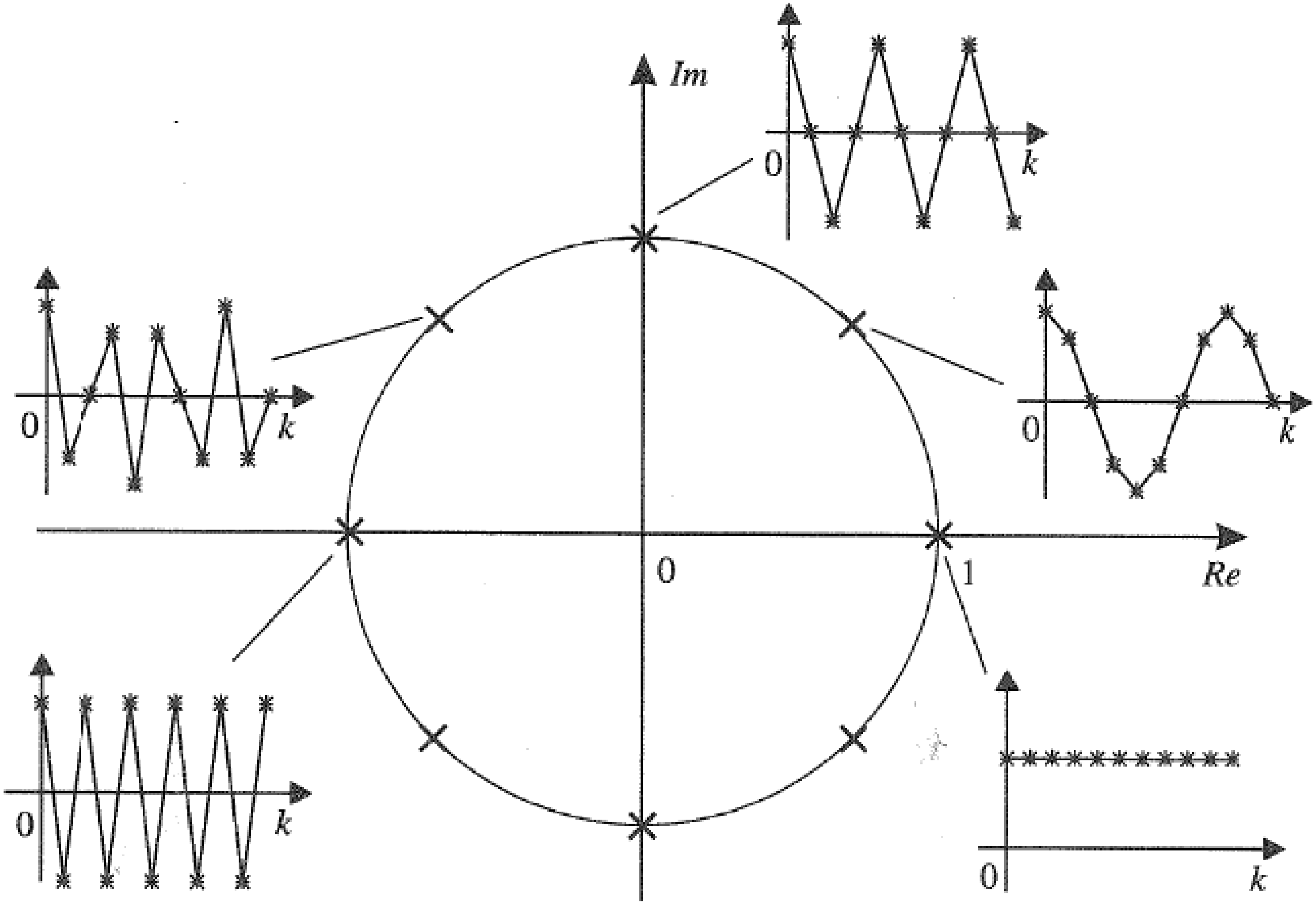
The natural modes $\nu^k \cos(\theta k)$, $\nu^k \sin(\theta k)$, associated to a couple of complex conjugated eigenvalues $\lambda = \sigma \pm j\omega = \nu e^{\pm j\theta}$ with $\mu' = 1$, is:

- **Geometrically convergent** if $|\lambda| = \nu < 1$ (e.g.: $0.5^k \sin(k)$)
- **Bounded (oscillating)** if $|\lambda| = \nu = 1$, $\mathrm{Arg}(\lambda) = \theta \neq 0$ (e.g.: $\sin(5k)$)
- **Geometrically divergent** if $|\lambda| = \nu > 1$ (e.g.: $1.5^k \sin(k)$)

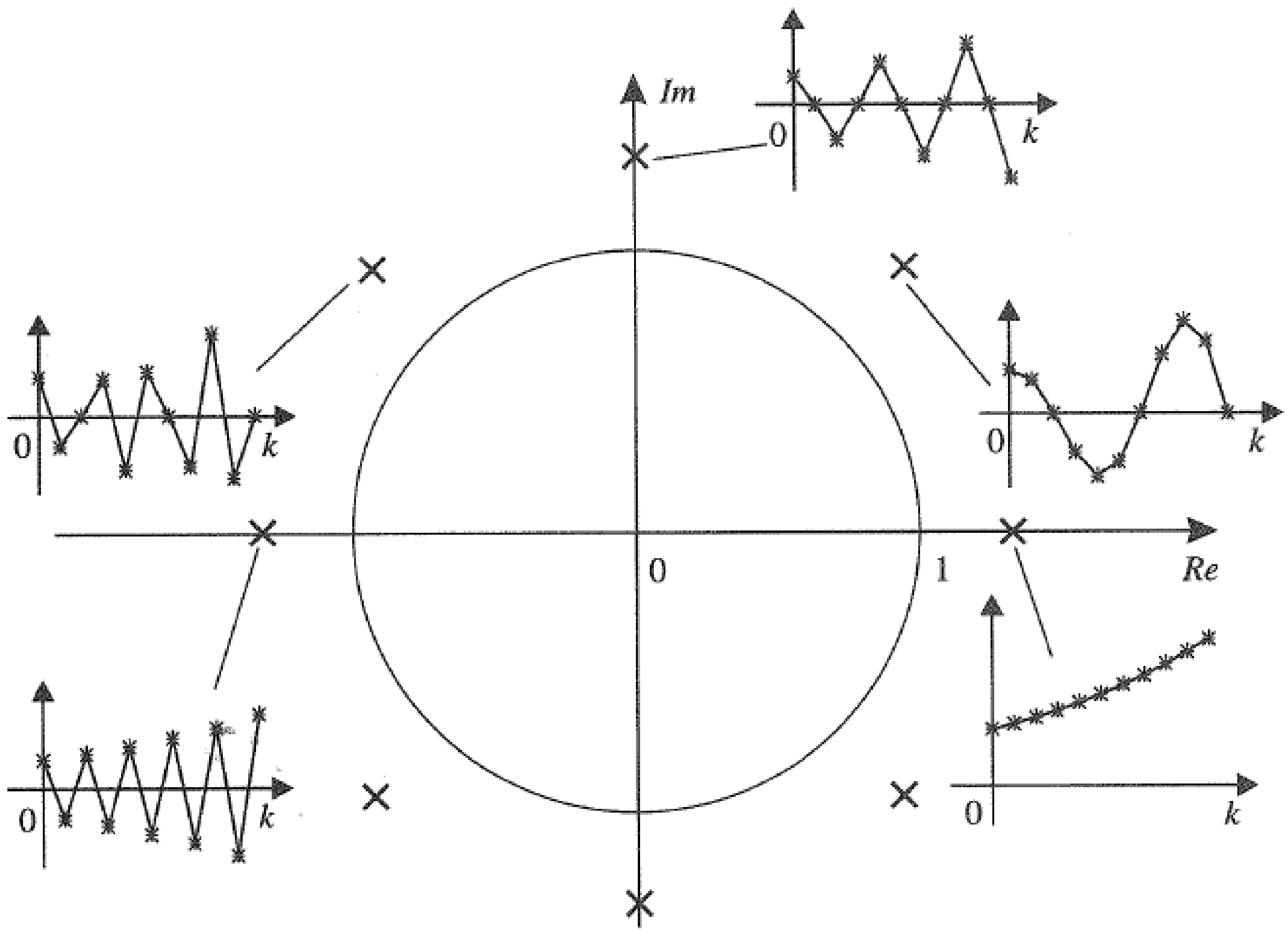
Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\mu' = 1$



Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\mu' = 1$



Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\mu' = 1$



Modal analysis: $\lambda \in \mathbb{R}$, $\mu' > 1$

The μ' natural modes of the form

$k\,(k-1)\,\ldots\,(k-\mu'-2)\,\lambda^{k-\mu'+1}, \ldots, k\,\lambda^{k-1}, \lambda^{k}$,

associated to a real eigenvalue λ with $\mu' > 1$ are:

- **Geometrically convergent** if $|\lambda| < 1$

  (e.g.: $k \cdot 0.5^{k-1}$, $k \cdot (-0.5)^{k-1}$)

- **Polynomially divergent** if $|\lambda| = 1$

  (e.g.: $k\,1^{k-1} = k$)

- **Geometrically divergent** if $|\lambda| > 1$

  (e.g.: $k \cdot 1.5^{k-1}$, $k \cdot (-1.5)^{k-1}$)

Modal analysis: $\lambda \in \mathbb{C}$, $\mu' > 1$

The μ' natural modes associated to a couple of complex conjugated eigenvalues $\lambda = \sigma \pm j\omega = \nu e^{\pm j\theta}$ with $\mu' > 1$ are:

- **Geometrically convergent** if $|\lambda| = \nu < 1$

  (e.g.: $k\,0.5^{k-1}\sin(k-1)$)

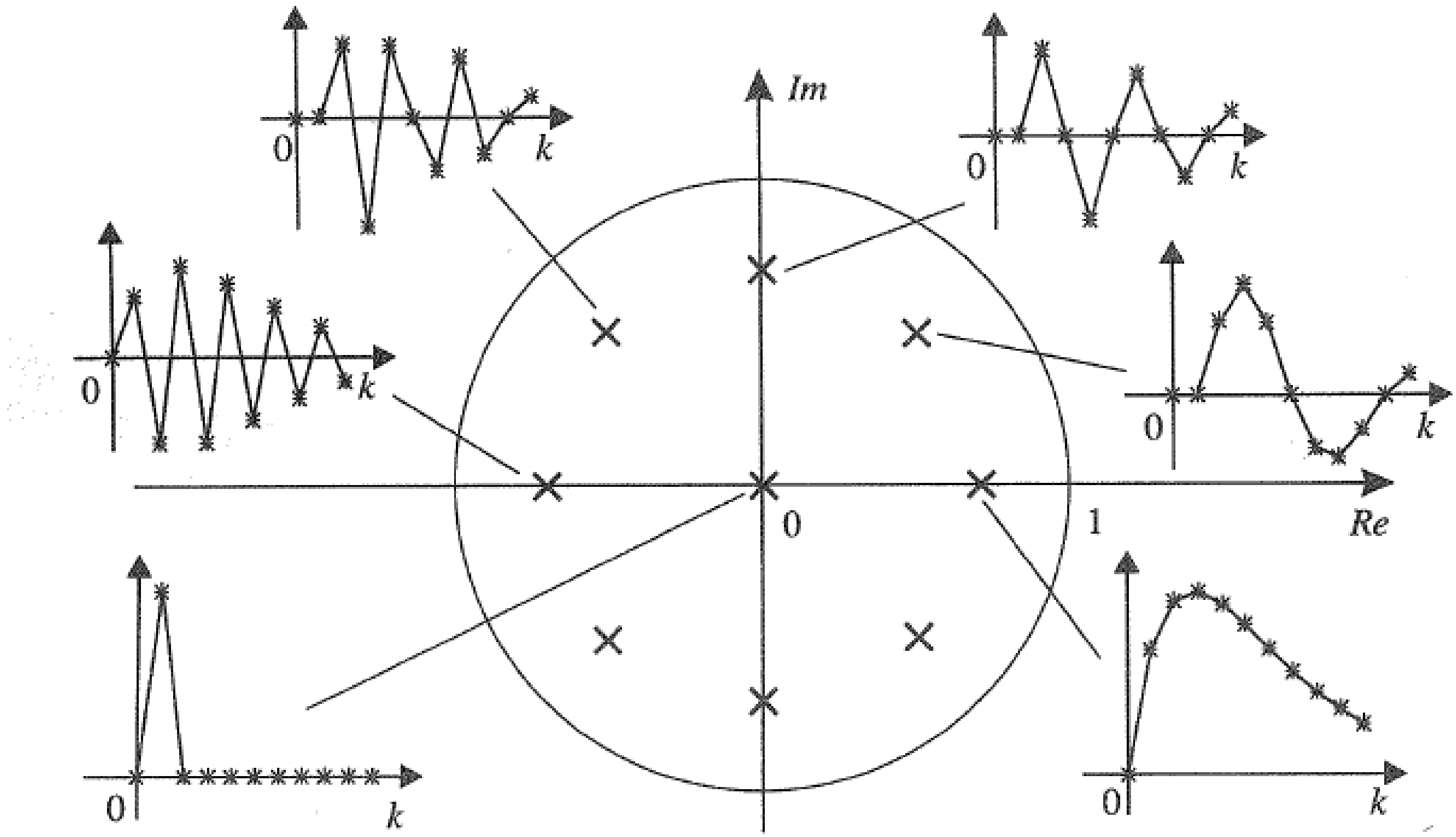
- **Polynomially divergent** if $|\lambda| = \nu = 1$, $\mathrm{Arg}(\lambda) = \theta \neq 0$

  (e.g.: $k\,1^{k-1}\sin(k-1)$)

- **Geometrically divergent** if $|\lambda| = \nu > 1$

  (e.g.: $k\,1.5^{k-1}\sin(k-1)$)

Modal analysis graphical resume: $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{C}$, μ'>1



Modal analysis: example 1

Given a dynamical DT LTI system characterized by the following matrix

$$A = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

evaluate the convergence properties of the natural modes.

$$A = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the matrix is diagonal, the eigenvalues are the entries on the diagonal

$$\lambda_1 = 0.1,\ \lambda_2 = -2,\ \lambda_3 = -0.4,\ \lambda_4 = 0$$

Every eigenvalue is real and has unitary multiplicity, thus the associated modes are in the form λ^k :

$$\lambda_1 \rightarrow 0.1^k,\ \lambda_2 \rightarrow (-2)^k,\ \lambda_3 \rightarrow (-0.4)^k,\ \lambda_4 \rightarrow 0^k = \delta(k)$$

Modal analysis: example 1

$$A = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\lambda_1 \rightarrow 0.1^k$: geometrically convergent
- $\lambda_2 \rightarrow (-2)^k$: geometrically divergent
- $\lambda_3 \rightarrow (-0.4)^k$: geometrically convergent
- $\lambda_4 \rightarrow 0^k = \delta(k)$: geometrically convergent

Modal analysis: example 2

Given a dynamical LTI DT system characterized by the following matrix

$$A = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

evaluate the convergence properties of the natural modes.

Modal analysis: example 2

$$A = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix is block diagonal, then the eigenvalues are the eigenvalues of main diagonal square blocks matrices

$$\lambda_{1,2} = -0.5 \pm 0.5j = \frac{\sqrt{2}}{2} e^{\pm j\frac{3}{4}\pi}, \; \lambda_3 = -1, \; \lambda_4 = 3$$

There are a couple of complex conjugated eigenvalues and two real eigenvalues, all with unitary multiplicity.

$$A = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The natural modes are:

- $\lambda_{1,2} \rightarrow \left(\frac{\sqrt{2}}{2}\right)^k \cos\left(\frac{5}{4}\pi k\right), \left(\frac{\sqrt{2}}{2}\right)^k \sin\left(\frac{5}{4}\pi k\right)$

  $\rightarrow |\lambda_{1,2}| < 1$: geometrically convergent
- $\lambda_3 \rightarrow (-1)^k \rightarrow |\lambda_3| = 1$: bounded
- $\lambda_4 \rightarrow 3^k \rightarrow |\lambda_4| > 1$: geometrically divergent

Analysis and properties of Discrete time systems

Internal stability of DT LTI systems

Internal stability of DT LTI systems

The internal stability property of a discrete time LTI system is related to the eigenvalues of the matrix A.

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A then

Result (Internal stability of LTI systems)

- An LTI system is **internally stable** if and only if all the natural modes are bounded or convergent:

  $|\lambda_i(A)| \leq 1,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) = 1$ for all the eigenvalues such that $|\lambda_j(A)| = 1$ ($\mu'(\cdot)$ is the geometric multiplicity)

- An LTI system is **asymptotically stable** if and only if all the natural modes are convergent:

  $|\lambda_i(A)| < 1,\ i = 1, \ldots, n$

- An LTI system is **unstable** if and only if there exist at least one divergent mode:

  $\exists i : |\lambda_i(A)| > 1$ OR $|\lambda_i(A)| \leq 1,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) > 1$ for all the eigenvalues such that $|\lambda_j(A)| = 1$ ($\mu'(\cdot)$ is the geometric multiplicity)

Internal stability: example 1

Given the DT LTI system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

evaluate the internal stability property when the matrix A has the following eigenvalues $\lambda_i(A)$

$$\{\lambda_i(A)\} = \{-2, -0.4, -0.2, -0.1\}$$

The internal stability property of a DT LTI system is related to the absolute value of the eigenvalues:

$$\{|\lambda_i(A)|\} = \{2, 0.4, 0.2, 0.1\}$$

The eigenvalue $\lambda_1(A) = -2$, $|\lambda_1| = 2 > 1$, makes the system unstable.

Internal stability: example 2

Given the DT LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

evaluate the internal stability property when the matrix A has the following eigenvalues $\lambda_i(A)$

$$\left\{ \lambda_i(A) \right\} = \left\{ -1, -0.5, 0, 0.5 \right\}$$

The internal stability property of a DT LTI system is related to the absolute value of the eigenvalues:

$$\left\{ \left| \lambda_i(A) \right| \right\} = \left\{ 1, 0.5, 0, 0.5 \right\}$$

Since all the eigenvalues are such that $|\lambda_i| \leq 1$ and only one has absolute value equal to 1, the system is stable.

Internal stability: example 3

Given the DT LTI system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

evaluate the internal stability property when the matrix A has the following eigenvalues $\lambda_i(A)$

$$\{\lambda_i(A)\} = \{-0.5 \pm 0.1j,\ 0.4 \pm 0.2j\}$$

The internal stability property of a DT LTI system is related to the absolute value of the eigenvalues:

$$\{|\lambda_i(A)|\} = \left\{\sqrt{0.5^2 + 0.1^2} = \sqrt{0.26}, \sqrt{0.26}, \sqrt{0.4^2 + 0.2^2} = \sqrt{0.2}, \sqrt{0.2}\right\}$$

Since all the eigenvalues are such that $|\lambda_i| < 1$, the system is asymptotically stable.

Internal stability: example 4

Given the DT LTI system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

evaluate the internal stability property when the matrix A has the following eigenvalues $\lambda_i(A)$

$$\{\lambda_i(A)\} = \{-1, \pm j, 0\}$$

The internal stability property of a DT LTI system is related to the absolute value of the eigenvalues:

$$\{|\lambda_i(A)|\} = \{1, 1, 1, 0\}$$

Since all the eigenvalues are such that $|\lambda_i| \leq 1$ and the ones with absolute value 1 have unitary multiplicity, the system is stable.

Internal stability: example 5

Given a DT LTI system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

where the matrix A is the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix}$$

Evaluate the internal stability property of the system

Internal stability: example 5

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix}$$

In order to study the internal stability property, we have to evaluate the eigenvalues of the matrix A.

Since A is a block diagonal matrix, then the eigenvalues of A are the eigenvalues of the main diagonal blocks square matrices.

Internal stability: example 5

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix} = \begin{bmatrix} A_1 & 0_{2\times 2} \\ 0_{2\times 2} & A_2 \end{bmatrix}$$

In order to evaluate the internal stability property, we have to evaluate the eigenvalues of the matrix A.

Since A is a block diagonal matrix, then the eigenvalues of A are the eigenvalues of the main diagonal blocks square matrices.

$$\{ \lambda_i(A) \} = \{ \lambda_i(A_1) \} \bigcup \{ -0.2, -0.3 \}$$

$$p.c.(A_1) = \det(\lambda I - A_1) = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$$\Rightarrow \{ \lambda_i(A) \} = \{ -2, -2, -0.2, -0.3 \}$$

Internal stability: example 5

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix}$$

The eigenvalues of A are:

$$\{\lambda_i(A)\} = \{-2, -2, -0.2, -0.3\}$$

The absolute value of the eigenvalues are:

$$\{|\lambda_i(A)|\} = \{2, 2, 0.2, 0.3\}$$

The eigenvalues $\lambda_1 = \lambda_2 = -2$ have $|\lambda_1| = |\lambda_2| = 2 > 1$, therefore the system is unstable.

Analysis and properties of Discrete time systems

Bibo stability of DT LTI systems

BIBO stability of LTI systems

Definition (BIBO stability of TD LTI systems)

An LTI system is **bounded-input bounded-output** *(BIBO)* stable if the zero state output response

$$y_{zs}(k) = C\, x_{zs}(k) + D\, u(k)$$

is bounded for all bounded inputs.

Result (BIBO stability of LTI system)

A discrete-time LTI system is **BIBO stable** if and only if the absolute value of the poles of the transfer function

$$H(z) = C\,(z\,I - A\,)^{-1}\,B + D$$

is strictly less than 1.

Consider the DT system described by the following transfer function

$$G(z) = \frac{3z^3 + 2z^2 + 5}{(z-2)(z^2 + z + 0.6)}$$

evaluate the BIBO stability property.

The denominator of the transfer function $G(z)$ can be written as:

$$(z-2)(z+0.5-j0.5916)(z+0.5+j0.5916)$$

Since the transfer function has one pole $p = 2$ such that $|p| = 2 > 1$, the system is BIBO unstable

Analysis and properties of Discrete time systems

Equilibrium points and linearization

Discrete time state space system description

We consider the description of systems through difference equations of the form

$$x(k+1) = f\left(k, x(k), u(k)\right)$$ **State equation**

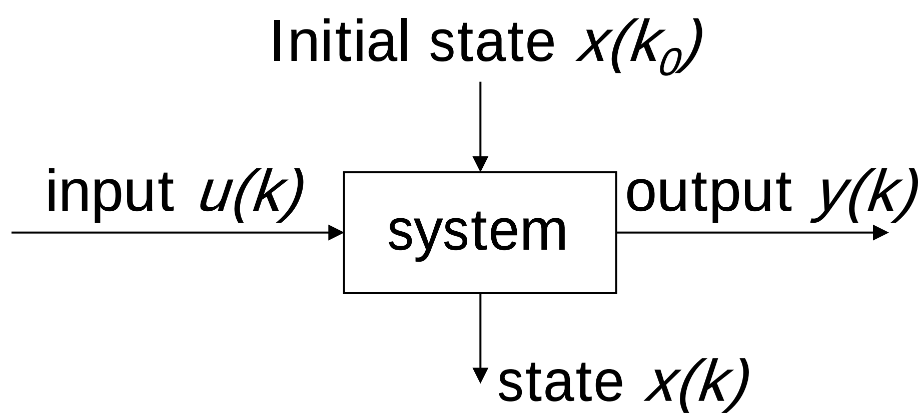
$$y(k) = g\left(k, x(k), u(k)\right)$$ **Output equation**

$x(k) \in \mathbb{R}^n; \quad u(k) \in \mathbb{R}^p$

$y(k) \in \mathbb{R}^q$

Given $u(k)$, $k \geq k_0$; $x(k_0)$

Find $x(k)$ and $y(k)$



Equilibrium point definition (recall)

An equilibrium point of a system is a particular solution to the differential equations such that:

- the system input is constant $\boxed{u(k) = \bar{u} \in \mathbb{R}^p, \forall k \geq 0}$
- the system state remains constant to its initial value

$$\boxed{x(k) = x(k = 0) = \bar{x} \in \mathbb{R}^n, \forall k \geq 0}$$

- the system output is constant $\boxed{y(k) = \bar{y} \in \mathbb{R}^q, \forall k \geq 0}$

Definition:

Suppose $u(k) = \bar{u} \in \mathbb{R}^p$. $\bar{x}$ is said to be an **equilibrium point** for a dynamical system if $f\left(k, \bar{x}, \bar{u}\right) = \bar{x}$

Equilibrium point condition

For a given finite dimension, MIMO, nonlinear time invariant system

$$x(k+1) = f\left(x(k), u(k)\right)$$
$$y(k) = g\left(x(k), u(k)\right)$$

the equilibrium states corresponding to a constant equilibrium input $u(k) = \bar{u}, \forall k \geq 0$, are the constant values of the system state $x(k) = \bar{x}, \forall k \geq 0$ which satisfy the condition

$$x(k+1) = x(k) = \bar{x},\ u(k) = \bar{u}, \forall k \geq 0$$

The equilibrium states are obtained by solving the following equation:

$$\boxed{f\left(\bar{x}, \bar{u}\right) = \bar{x}}$$

To each equilibrium point corresponds the equilibrium output:

$$\boxed{\bar{y} = g\left(\bar{x}, \bar{u}\right)}$$

Equilibrium point condition

By considering a finite dimension, MIMO, linear time invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

for a given equilibrium input $\bar{u}$, the equilibrium states are the values of $\bar{x}$ which satisfy the following condition

$$x(k+1) = x(k) = \bar{x} = A\bar{x} + B\bar{u}, \forall k \geq 0$$

Therefore, for an LTI system, the equilibrium state are obtained by solving the linear system of equations

$$\boxed{(I - A)\bar{x} = B\bar{u}}$$

The equilibrium output is:

$$\boxed{\bar{y} = C\bar{x} + D\bar{u}}$$

Equilibrium point condition

By looking at the structure of the matrix $(I - A)$ it is possible to distinguish two cases:

- If there exists the inverse of $(I - A)$, i.e. $\det(I - A) \neq 0$, then the equilibrium state is the unique point satisfying the relation

$$\bar{x} = (I - A)^{-1} B \bar{u}$$

  and there is a unique equilibrium output defined as

$$\bar{y} = (C(I - A)^{-1} B + D)\bar{u}$$

- If the matrix $(I - A)$ is singular, i.e. $\det(I - A) = 0$, then there can be zero or infinite equilibrium states, depending on the matrices A, B and the considered input.

Equilibrium point: example

Consider the discrete time LTI system described by the following matrices A and B :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Compute all the equilibrium states corresponding to the input $\bar{u}$

Equilibrium point: example

Consider the discrete time LTI system described by the following matrices A and B :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Compute all the equilibrium states corresponding to the input $\bar{u}$

From the state equation, at the equilibrium, we have:

$$\bar{x} = A\bar{x} + B\bar{u} \Rightarrow \begin{cases} \bar{x}_1 = \bar{x}_1 + \bar{u} \\ \bar{x}_2 = \bar{x}_1 \end{cases} \Rightarrow \begin{cases} \bar{u} = 0 \\ \bar{x}_2 = \bar{x}_1 \end{cases}$$

- If $\bar{u} = 0$ there exist infinite equilibrium states in the form

$$\bar{x} = \begin{bmatrix} c \\ c \end{bmatrix}, c \in \mathbb{R}$$

Equilibrium point: example

Consider the discrete time LTI system described by the following matrices A and B :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Compute all the equilibrium states corresponding to the input $\bar{u}$

From the state equation, at the equilibrium, we have:

$$\bar{x} = A\bar{x} + B\bar{u} \Rightarrow \begin{cases} \bar{x}_1 = \bar{x}_1 + \bar{u} \\ \bar{x}_2 = \bar{x}_1 \end{cases} \Rightarrow \begin{cases} \bar{u} = 0 \\ \bar{x}_2 = \bar{x}_1 \end{cases}$$

- If $\bar{u} \neq 0$ there are no equilibrium states

Equilibrium point: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \end{cases}$$
$$y(k) = x_1(k)x_2(k)$$

Compute all the equilibrium states $\bar{x}$ and the equilibrium outputs $\bar{y}$ corresponding to the input $\bar{u} = 0.5$

Equilibrium point: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \end{cases}$$
$$y(k) = x_1(k)x_2(k)$$

Compute all the equilibrium states $\bar{x}$ and the equilibrium outputs $\bar{y}$ corresponding to the input $\bar{u} = 0.5$

By imposing the equilibrium condition

$$x(k+1) = x(k) = \bar{x} = f(\bar{x}, \bar{u}), \ \forall k \geq 0$$

we have

$$\begin{cases} \bar{x}_1 = \bar{x}_1\bar{u} + \bar{x}_1\bar{x}_2 \\ \bar{x}_2 = -\bar{x}_2\bar{u} + 3\bar{x}_2^2 \end{cases} \Rightarrow \begin{cases} \bar{x}_1(1 - \bar{u} - \bar{x}_2) = \bar{x}_1(0.5 - \bar{x}_2) = 0 \\ \bar{x}_2(1 + \bar{u} - 3\bar{x}_2) = \bar{x}_2(1.5 - 3\bar{x}_2) = 0 \end{cases}$$

Equilibrium point: example

At the equilibrium the following relations hold:

$$\begin{cases} \bar{x}_1(0.5-\bar{x}_2)=0 \\ \bar{x}_2(1.5-3\bar{x}_2)=0 \end{cases}$$

The second equation has the solutions:

$$\bar{x}_2^{(a)}=0 \quad \vee \quad \bar{x}_2^{(b)}=0.5$$

- If $\bar{x}_2=\bar{x}_2^{(a)}=0$ then, from the first equation,

$$\bar{x}_1=\bar{x}_1^{(a)}=0 \Rightarrow \bar{x}=\bar{x}^{(a)}=\begin{bmatrix}0\\0\end{bmatrix}$$

- If $\bar{x}_2=\bar{x}_2^{(b)}=0.5$ then, from the first equation

$$\bar{x}=\bar{x}^{(b)}=\begin{bmatrix}c\\0.5\end{bmatrix},\ \forall c\in\mathbb{R}$$

Equilibrium point: example

The equilibrium output is given by:

$$y(k) = \bar{y} = g(\bar{x}, \bar{u}) = \bar{x}_1 \bar{x}_2, \ \forall k \geq 0$$

Then:

- If $\bar{x} = \bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\bar{y} = \bar{y}^{(a)} = \bar{x}_1^{(a)} \bar{x}_2^{(a)} = 0$$

- If $\bar{x} = \bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, c \in \mathbb{R}$

$$\bar{y} = \bar{y}^{(b)} = \bar{x}_1^{(b)} \bar{x}_2^{(b)} = 0.5c$$

System equation linearization

Like in the continuous time domain, in a neighborhood of an equilibrium point $(\bar{x}, \bar{u}, \bar{y})$, the trajectory of the nonlinear system

$$x(k+1) = f\big(x(k), u(k)\big)$$
$$y(k) = g\big(x(k), u(k)\big)$$

can be approximated by

$$x(k) = \bar{x} + \delta x(k),\ u(k) = \bar{u} + \delta u(k),\ y(k) = \bar{y} + \delta y(k)$$

with $\delta x(k)$ and $\delta y(k)$ the solutions to the linearized system equations

$$\delta x(k+1) = A\,\delta x(k) + B\,\delta u(k), \qquad \delta x(k=0) = x(k=0) - \bar{x}$$
$$\delta y(k) = C\,\delta x(k) + D\,\delta u(k)$$
$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\ B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\ C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}},\ D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

System equation linearization: example

Consider the discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \quad = f_2(x,u) \end{cases}$$

$$y(k) = x_1(k)x_2(k) \quad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \qquad = f_2(x,u) \end{cases}$$
$$y(k) = x_1(k)x_2(k) \qquad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $A = \left.\dfrac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} \bar{u}+\bar{x}_2 & \bar{x}_1 \\ 0 & -\bar{u}+6\bar{x}_2 \end{bmatrix}$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \quad = f_2(x,u) \end{cases}$$

$$y(k) = x_1(k)x_2(k) \quad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $A = \begin{bmatrix} \bar{u} + \bar{x}_2 & \bar{x}_1 \\ 0 & -\bar{u} + 6\bar{x}_2 \end{bmatrix} \Rightarrow \begin{cases} \text{if } \bar{x} = \bar{x}^{(a)} \Rightarrow A = A^{(a)} = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \\ \text{if } \bar{x} = \bar{x}^{(b)} \Rightarrow A = A^{(b)} = \begin{bmatrix} 1 & c \\ 0 & 2.5 \end{bmatrix} \end{cases}$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \qquad = f_2(x,u) \end{cases}$$
$$y(k) = x_1(k)x_2(k) \qquad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $B = \dfrac{\partial f(x,u)}{\partial u}\bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}\Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} \bar{x}_1 \\ -\bar{x}_2 \end{bmatrix}$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \qquad = f_2(x,u) \end{cases}$$

$$y(k) = x_1(k)x_2(k) \qquad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $$B = \begin{bmatrix} \bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} \Rightarrow \begin{cases} \text{if } \bar{x} = \bar{x}^{(a)} \Rightarrow B = B^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \text{if } \bar{x} = \bar{x}^{(b)} \Rightarrow B = B^{(b)} = \begin{bmatrix} c \\ -0.5 \end{bmatrix} \end{cases}$$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \qquad = f_2(x,u) \end{cases}$$
$$y(k) = x_1(k)x_2(k) \qquad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} \bar{x}_2 & \bar{x}_1 \end{bmatrix}$$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \qquad = f_2(x,u) \end{cases}$$
$$y(k) = x_1(k)x_2(k) \qquad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $C = \begin{bmatrix} \bar{x}_2 & \bar{x}_1 \end{bmatrix} \Rightarrow \begin{cases} \text{if } \bar{x} = \bar{x}^{(a)} \Rightarrow C = C^{(a)} = \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \text{if } \bar{x} = \bar{x}^{(b)} \Rightarrow C = C^{(b)} = \begin{bmatrix} 0.5 & c \end{bmatrix} \end{cases}$

System equation linearization: example

Consider the following discrete time nonlinear system described by the following state space equations:

$$\begin{cases} x_1(k+1) = x_1(k)u(k) + x_1(k)x_2(k) = f_1(x,u) \\ x_2(k+1) = -x_2(k)u(k) + 3x_2^2(k) \quad = f_2(x,u) \end{cases}$$
$$y(k) = x_1(k)x_2(k) \quad = g(x,u)$$

compute the matrices of the linearized system in a neighborhood of the equilibrium points

$$\left(\bar{x}^{(a)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{u} = 0.5\right) \text{ e } \left(\bar{x}^{(b)} = \begin{bmatrix} c \\ 0.5 \end{bmatrix}, \bar{u} = 0.5\right), c \in \mathbb{R}$$

- $D = \left.\dfrac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \left.\left[\dfrac{\partial g}{\partial u}\right]\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = [0]$

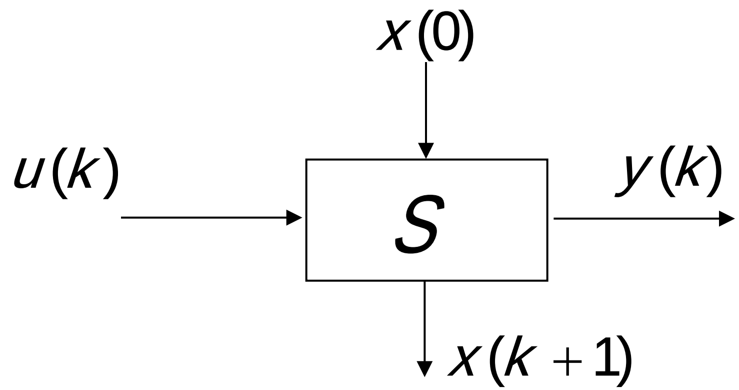
Resume via block diagram representation

Nonlinear system:

$$S:\begin{cases} x(k+1) = f\big(x(k),u(k)\big) \\ y(k) = g\big(x(k),u(k)\big) \end{cases}$$

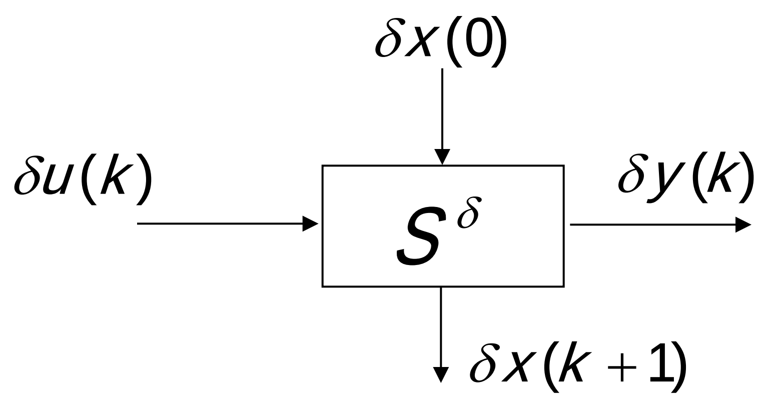
$$u(k) = \bar{u} + \delta u(k)$$

$$x(0) = \bar{x} + \delta x(0)$$



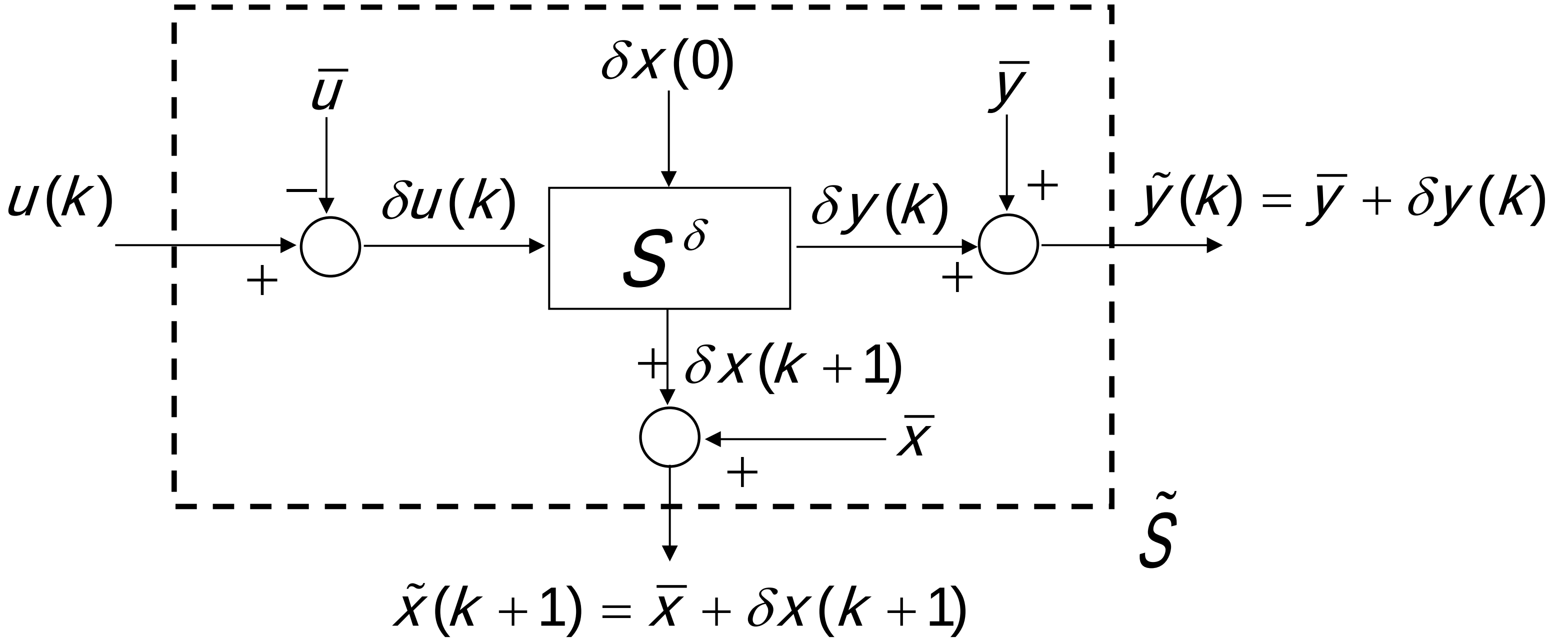
Linearized system:

$$S^{\delta}:\begin{cases} \delta k(k+1) = A\,\delta x(k) + B\,\delta u(k) \\ \delta y(k) = C\,\delta x(k) + D\,\delta u(k) \end{cases}$$



Resume via block diagram representation

Approximating system $\tilde{S}$



$$u(k) = \bar{u} + \delta u(k) \qquad S^{\delta}: \begin{cases} \delta x(k+1) = A\delta x(k) + B\delta u(k) \\ \delta y(k) = C\delta x(k) + D\delta u(k) \end{cases}$$