Automatic Control

BIBO stability of LTI systems

Definition (BIBO stability of LTI system)

A SISO* LTI system is **bounded-input bounded-output** (BIBO) stable if the zero state output response is bounded for all bounded inputs:

$$\forall u_M \in (0,\infty), \quad \exists y_M \in (0,\infty):$$
$$|u(t)| \le u_M,\ \forall t \ge 0 \quad \Rightarrow \quad |y(t)| \le y_M,\ \forall t \ge 0$$

As in the case of internal stability, the study of BIBO stability for an LTI system can not be performed using directly the definition, since, in practice, it requires an infinite number of tests (i.e. check if the output zero state response is bounded for every possible bounded input)

** for simplicity of notation and development, the SISO case is considered here. The results can be extended to MIMO systems. Details can be found in C.T. Chen, "Linear systems analysis and design"*

Thus, in order to to obtain (easy to check) necessary and sufficient conditions for BIBO stability, we consider the expression of the output response $y(t)$ of an LTI system described by the ss representation:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + Du(t) \end{cases} \rightarrow$$

$$\rightarrow y(t) = Ce^{At}x(0) + \int_{0_-}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = y_{zi}(t) + y_{zs}(t)$$

In the presence of zero initial conditions (i.e. x(0) =0) we have:

$$y(t) = y_{zs}(t) = \int_{0_-}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

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$$y(t) = \int_{0_-}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

The convolution integral expression of $y(t)$ above can be also written as:

$$y(t) = \int_{0_-}^{t} \left(Ce^{A(t-\tau)}B + D\delta(t-\tau)\right)u(\tau)d\tau = \int_{0_-}^{t} h(t-\tau)u(\tau)d\tau$$

where the function $h(t) = Ce^{At}B + D\delta(t)$ is the impulse response of the considered LTI system (see, AC_L3.3 p. 6)

in fact, the Laplace transform of $h(t)$ is the transfer function $H(s)$ of the system:

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{Ce^{At}B + D\delta(t)\} = C(sI - A)^{-1}B + D = H(s)$$

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$$y(t) = \int_{0_-}^{t} h(t-\tau)u(\tau)d\tau = \int_{0_-}^{t} h(\tau)u(t-\tau)d\tau$$

According to the definition of BIBO stability

$$\forall u_M \in (0,\infty), \quad \exists y_M \in (0,\infty) : |u(t)| \le u_M, \forall t \ge 0 \quad \Rightarrow \quad |y(t)| \le y_M, \forall t \ge 0$$

in order to ensure the boundedness of the output response, in the presence of bounded inputs, we need to guarantee:

$$|y(t)| = \left| \int_{0_-}^{t} h(\tau)u(t-\tau)d\tau \right| \le y_M < \infty, \forall t \ge 0$$

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$$\left|y(t)\right| \leq \int_{0_-}^{t} \left|h(\tau)\right|\left|u(t-\tau)\right| d\tau \leq y_M < \infty, \forall t \geq 0$$

Thus:

$$\left|y(t)\right| \leq \int_{0_-}^{t} \left|h(\tau)\right|\left|u(t-\tau)\right| d\tau \underset{\substack{\uparrow \\ \left|u(t)\right| \leq u_M}}{\leq} u_M \int_{0_-}^{t} \left|h(\tau)\right| d\tau \leq y_M < \infty$$

$$\rightarrow \int_{0_-}^{t} \left|h(\tau)\right| d\tau \leq \frac{y_M}{u_M} = K < \infty, \forall t \geq 0$$

i.e. the impulse response function $h(t)$ has to be absolutely integrable

Result (BIBO stability of LTI systems)

An LTI system is **BIBO stable** if and only if* there exists a positive constant $K < \infty$ such that

$$\int_0^t |h(\tau)| d\tau \leq K < \infty$$

where $h(t)$ is the impulse response of the system

Remark. Unfortunately, although the stated result requires a single test to study BIBO stability of an LTI system, it can not be efficiently employed in practice since it needs the integration of the impulse response function $h(t)$

→ a simpler necessary and sufficient condition can be obtained by exploiting the structure of $h(t)$ via its Laplace transform

** a proof of the necessity is reported in C.T. Chen, "Linear systems analysis and design"*

BIBO stability of LTI systems

$$h(t) = Ce^{At}B + D\delta(t) \xrightarrow{\mathcal{L}} C(sI - A)^{-1}B + D = H(s)$$

$H(s)$ is the system transfer function and can be computed as:

$$H(s) = \frac{1}{\underbrace{\det(sI - A)}_{\text{polynomial of degree } n}} \overbrace{C\left[a_{ij}(s)\right]B}^{\text{polynomial of degree } \leq n-1} + \underbrace{D}_{\text{constant}} =$$

$$= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + D, \quad m < n$$

The poles of $H(s)$ are the roots of the polynomial

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Suppose that $H(s)$ has $h \leq n$ distinct poles p_1, ..., p_h of multiplicity n_1, ..., n_h, respectively (note that: $n_1 + \ldots + n_h = n$)

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{(s-p_1)^{n_1}(s-p_2)^{n_2}\cdots(s-p_h)^{n_h}} + D$$

then, by expressing the first term of $H(s)$ through its PFE, we obtain:

$$\begin{aligned} H(s) = {} & \frac{r_{1,1}}{s-p_1} + \frac{r_{1,2}}{(s-p_1)^2} + \ldots + \frac{r_{1,n_1}}{(s-p_1)^{n_1}} + \\ & + \frac{r_{2,1}}{s-p_2} + \frac{r_{2,2}}{(s-p_2)^2} + \ldots + \frac{r_{2,n_2}}{(s-p_2)^{n_2}} + \ldots \\ & + \ldots + \frac{r_{h,1}}{s-p_h} + \frac{r_{h,2}}{(s-p_h)^2} + \ldots + \frac{r_{h,n_h}}{(s-p_h)^{n_h}} + D \end{aligned}$$

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Thus:

$$h(t) = r_{1,1}e^{p_1 t} + r_{1,2}te^{p_1 t} \ldots + r_{1,n_1}\frac{t^{n_1-1}}{(n_1-1)!}e^{p_1 t} +$$

$$+r_{2,1}e^{p_2 t} + r_{2,2}te^{p_2 t} \ldots + r_{2,n_2}\frac{t^{n_2-1}}{(n_2-1)!}e^{p_2 t} + \ldots$$

$$+r_{h,1}e^{p_h t} + r_{h,2}te^{p_h t} \ldots + r_{h,n_h}\frac{t^{n_h-1}}{(n_h-1)!}e^{p_h t} + D\delta(t)$$

→ $h(t)$ is a linear combination of the system natural modes plus a Dirac's delta function

$$h(t) = r_{1,1}e^{p_1 t} + r_{1,2}te^{p_1 t} \ldots + r_{1,n_1}\frac{t^{n_1-1}}{(n_1-1)!}e^{p_1 t} +$$

$$+r_{2,1}e^{p_2 t} + r_{2,2}te^{p_2 t} \ldots + r_{2,n_2}\frac{t^{n_2-1}}{(n_2-1)!}e^{p_2 t} + \ldots$$

$$+r_{h,1}e^{\lambda_h t} + r_{h,2}te^{p_h t} \ldots + r_{r,n_h}\frac{t^{n_h-1}}{(n_h-1)!}e^{p_h t} + D\delta(t)$$

$$\int_{0_-}^{t} \left|h(\tau)\right| d\tau \le K < \infty, \forall t \ge 0$$

→ Note that, according to the properties of the δ function, it holds

$$\int_{0_-}^{t} \left|D\delta(\tau)\right| d\tau \le D < \infty, \forall t \ge 0$$

then, $h(t)$ is absolutely integrable if all the natural modes are convergent (i.e. if all the distinct poles p_1, ..., p_h of $H(s)$ have strictly negative real part)

Result (BIBO stability of LTI system)

An LTI system is **BIBO stable** if and only if* all the poles of the transfer function have strictly negative real parts

Examples:

$$H_1(s) = \frac{(s-5)}{(s+4)(s+300)} \rightarrow \text{ BIBO stable}$$

$$H_2(s) = \frac{(s+2)}{(s-5)(s+100)} \rightarrow \text{ BIBO unstable}$$

$$\left.\begin{aligned} H_3(s) &= \frac{5}{s} \\ H_4(s) &= \frac{s}{(s^2+5)} \end{aligned}\right\} \rightarrow \text{ BIBO unstable}$$

* *a proof of the necessity is reported in C.T. Chen, "Linear systems analysis and design"*

Internal vs. BIBO stability of LTI systems

Two different concepts for LTI systems stability have been introduced:

- internal stability
- BIBO stability

Now, we want to study their relationships

Recall that:

- internal stability depends on the system eigenvalues
- BIBO stability depends on the system poles

Thus, in order to study the relationship between internal and BIBO stability, we need to investigate the relationship between the system eigenvalues and the system poles

Internal vs. BIBO stability of LTI systems

The system eigenvalues are the solution of the characteristic polynomial $\det(sI - A)$ of the A matrix of the state space representation:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

The system poles are the values of $s \in \mathbb{C}$ such that $H(s) = \infty$

i.e. they are the roots of denominator polynomial $D_H(s)$ of the transfer function $H(s)$:

$$H(s) = \frac{N_H(s)}{D_H(s)}$$

Internal vs. BIBO stability of LTI systems

In order to study the relationship between eigenvalues and poles of an LTI system, we recall that:

$$H(s) = C\frac{Adj(sI-A)}{\det(sI-A)}B + D = \frac{1}{\underbrace{\det(sI-A)}_{\text{polynomial of degree } n}} \overbrace{C\left[\; a_{ij}(s) \;\right]B}^{\text{polynomial of degree } \le n-1} + \underbrace{D}_{\text{constant}} =$$

$$= \frac{\overbrace{\overbrace{C\left[a_{ij}(s)\right]B}^{\text{polynomial of degree } \le n-1} + D\det(sI-A)}^{\text{polynomial of degree } \le n}}{\underbrace{\det(sI-A)}_{\text{polynomial of degree } n}} = \frac{N_H(s)}{D_H(s)}$$

Then, one can argue that $D_H(s) = \det(sI - A)$ and hence the system eigenvalues coincide with the system poles. However...

Internal vs. BIBO stability of LTI systems

... consider the following example:

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0 & -1 \end{bmatrix} x(t)$$

whose eigenvalues are +1 and -1 (→ system internally unstable)

Then, compute the transfer function as:

$$H(s) = C(sI - A)^{-1}B = \frac{-3\cancel{(s-1)}}{\cancel{(s-1)}(s+1)} = \frac{-3}{s+1}$$

Note that, in $H(s)$, the common factor $(s-1)$ at numerator and denominator can be simplified

Thus the value $s = -1$ is the unique system pole (→ system BIBO stable)

Minimality of LTI systems

The facts introduced by the just considered example can be generalized and formalized as:

- the poles of an LTI system are the roots of the denominator polynomial $D_H(s)$ of the transfer function $H(s)$ obtained after the simplification of all the possible common factors of numerator and denominator (this simplification is improperly referred to as **zero-pole cancellation**, or **pole-zero cancellation**)

- the system poles are a subset of system the eigenvalues
- the transfer function, through its poles, carries a subset of the natural modes of the system

A system without zero-pole cancellations in the computation of the transfer function is said in **minimal form** or simply **minimal**

Minimality of LTI systems

The consequence of the points discussed above is that, in general, <u>internal stability and BIBO stability</u> <u>are not equivalent properties</u>

Therefore, a BIBO stable representation if no assumptions are made on its minimality properties may hinder the presence of unstable natural modes (→ problems in the control design phase)

In this course, we will concern with minimal transfer functions if not differently specified

Indeed, connection (i.e. series, parallel, feedback) of minimal systems may lead to nonminimal tf (this problem will be analyzed later)

Minimality of LTI systems

A necessary and sufficient condition for minimality is the following

Result (minimality of LTI systems)

An LTI system of the form $\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$ is minimal iff,

the two following matrices,

$$M_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \quad M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Are such that $\rho(M_C) = \rho(M_O) = n$, where $n = \dim(x)$

M_C and M_O are referred to as controllability and observability matrices respectively

Minimality of LTI systems: example 1

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} x(t)$$

$$M_R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & -3 \end{bmatrix}, \quad M_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \rho(M_R) = \rho(M_O) = n = 2$$

$$H(s) = C(sI - A)^{-1}B = \frac{s+13}{(s-1)(s+1)}$$

Minimality of LTI systems: example 2

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0 & -1 \end{bmatrix} x(t)$$

$$M_R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & -3 \end{bmatrix}, \quad M_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \rho(M_R) = n = 2, \quad \rho(M_O) = 1 < n$$

$$H(s) = C(sI - A)^{-1}B = \frac{-3\cancel{(s-1)}}{\cancel{(s-1)}(s+1)} = \frac{-3}{s+1}$$

The controllability matrix M_C of an LTI system can be computed in MatLab through the function `ctrb`

```
>> A=[1 1;0 -1]; B=[2;3];

>> M_C=ctrb(A,B)

M_C =

     2     5

     3    -3
```

The rank r of M_C can be computed using the function `rank`

```
>> r=rank(M_C)

r =

     2
```

The observability matrix M_O of an LTI system can be computed in MatLab through the function `obsv`

```
>> A=[1 1;0 -1]; C=[0 -1];

>> M_O=obsv(A,C)

M_O =

     0    -1

     0     1
```

The rank o of M_O can be computed using the function `rank`

```
>> o=rank(M_O)

o =

     1
```

Resume: internal vs. BIBO stability of LTI systems

Internal stability		BIBO stability
1) Asimptotically stable	$\Rightarrow$	1) Stable
2) Stable (marginally) 2) Unstable 3) Unstable	$\Leftarrow$	

In general, the inverse implications do not hold (unless minimality of the system is assumed)