Stability of feedback control systems

Summarizing Internal stability and BIBO stability

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Stability of feedback control systems

Summarizing Internal stability and BIBO stability

Internal stability of LTI systems

Consider an LTI system described in terms of state equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x \in R^n, u \in R^p, y \in R^q$$
$$y(t) = Cx(t) + Du(t)$$

The state response *x(t)* in the time interval *[0, t]* due to the input signal u(t) and the initial condition $x(0)=x_0$ is given by

$$x(t) = x_{zs}(t) + x_{zi}(t)$$

- $x_{zs}(t)$ is the *zero-state response*
- $x_{zi}(t)$ is the *zero-input response*

Internal stability of LTI systems

Definition (Internal stability of LTI system)

An LTI system is **internally stable** if the zero input response $x_{zi}(t)$ is bounded for any initial state x_0

Definition (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if the zero input response $x_{zi}(t)$ converges to 0 for any initial state x_0

Definition (Instability of LTI system)

An LTI system is **unstable** if it is not stable

Recalling that the zero input response is a linear combination of the system natural modes, we have the following results:

Result (Internal stability of LTI system)

An LTI system is **internally stable** if and only if all the natural modes are bounded (convergence is a special case of boundedness)

Result (Asymptotic stability of LTI system)

An LTI system is **asymptotically stable** if and only if all the natural modes are convergent

Result (Instability of LTI system)

An LTI system is **unstable** if and only if there exists at least one divergent mode

Internal stability of LTI systems

The natural mode properties depend on the real part of the system eigenvalues.

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A, then

Result (Internal stability of LTI systems)

- An LTI system is **internally stable** if and only if:

  $\mathbb{R}e(\lambda_i(A)) \leq 0$, $i = 1, \ldots, n$ <u>AND</u> $\mu'(\lambda_j(A)) = 1$ for all the eigenvalues such that $\mathbb{R}e(\lambda_j(A)) = 0$ ($\mu'(\cdot)$ is the geometric multiplicity)

In plain words, an LTI system is **internally stable** if and only if <u>both</u> the following conditions are met:
(a) There are no eigenvalues of A with positive real part
(b) The eigenvalues of A with null real part are simple roots of the minimal polynomial of A (geometric multiplicity = 1).

Internal stability of LTI systems

Result (Internal stability of LTI systems)

- An LTI system is **asymptotically stable** if and only if:

  $\mathbb{R}e(\lambda_i(A)) < 0,\ i = 1, \ldots, n$

In plain words, An LTI system is **asymptotically stable** if and only if all eigenvalues of A have strictly negative real parts.

Result (Internal stability of LTI systems)

- An LTI system is **unstable** if and only if:

  $\exists i : \mathbb{R}e(\lambda_i(A)) > 0$ <u>OR</u> $\mathbb{R}e(\lambda_i(A)) \leq 0$, $i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) > 1$ for all the eigenvalues such that $\mathbb{R}e(\lambda_j(A)) = 0$ ($\mu'(\cdot)$ is the geometric multiplicity)

In plain words, an LTI system is **unstable** if and only if <u>one</u> the following conditions are met:
(a) There is at least one eigenvalue of A with positive real part
(b) The eigenvalues of A with null real part <u>are not</u> simple roots of the minimal polynomial of A (geometric multiplicity > 1).

Definition (BIBO stability of LTI system)

An LTI system is **bounded-input bounded-output** *(BIBO)* stable if the zero state output response $y_{zs}(t) = Cx_{zs}(t) + Du(t)$ is bounded for all bounded inputs, i.e.,

$$\forall \bar{u} \in (0,\infty), \quad \exists \bar{y} \in (0,\infty):$$

$$\|u(t)\| \leq \bar{u}, \forall t \geq 0 \quad \Rightarrow \quad \|y(t)\| \leq \bar{y}, \forall t \geq 0$$

Result (BIBO stability of LTI system)

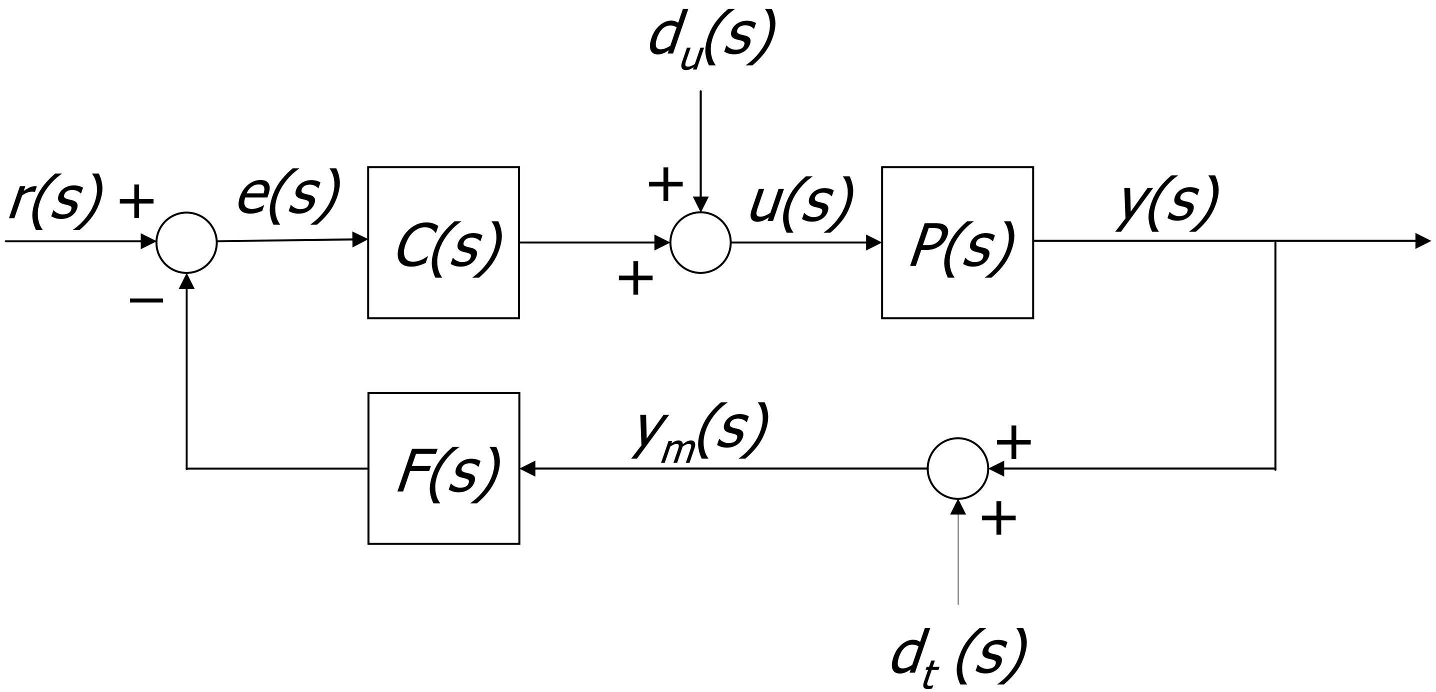
An LTI system is **BIBO stable** if and only if all poles of the transfer function $H(s) = C(sI-A)^{-1}B+D$ have strictly negative real part.

Stability of feedback control systems

Internal stability of feedback control systems

Internal stability of feedback systems: Introduction

Let us consider the following control system:



Internal stability of feedback systems : Introduction

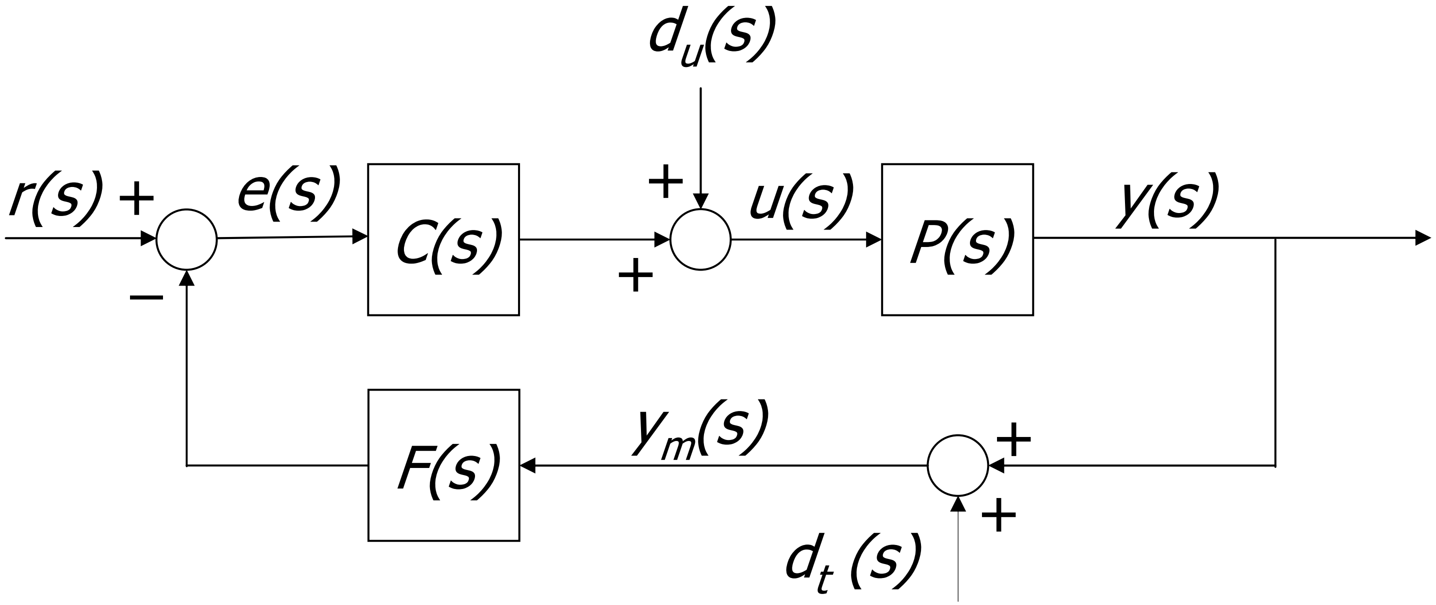
The multivariable transfer function $M(s)$ from inputs signals r, d_u and d_t to ouputs signals e, u ed y_m is given by:

$$\begin{bmatrix} e \\ u \\ y_m \end{bmatrix} = \frac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} r \\ d_u \\ d_t \end{bmatrix}$$

Define

- the **loop function** as $L(s) = P(s)C(s)F(s)$
- the **sensitivity function** as $S(s) = [1+L(s)]^{-1}$
- the **complementary sensitivity function** as $T(s) = 1-S(s)$

Internal stability of feedback systems: Introduction



The feedback control system is said to be **well-posed** iff

$$\lim_{s\to\infty}\{1+P(s)C(s)F(s)\}\neq 0$$

Then, all the scalar transfer functions appearing in $M(s)$ exist (i.e. have denominator $\neq 0$ when s tends to ∞) and are proper tf.

Internal stability of feedback systems

Let us assume that the plant *P(s)* to be controlled is stabilizable by input u and detectable through output y.

Definition (Internal stability of a feedback system)

The feedback system is **internally stable** if and only iff the signals e, u and y_m are bounded for any possible choice of bounded signals r, d_u and d_t (i.e., if and only if all the transfer functions in $M(s)$ are proper and BIBO stable).

Result (Internal stability of feedback systems)
The closed loop system considered here is internally stable if and only if the following conditions are met:
(1) all roots of the equation $1 + L(s) = 0$ have real part < 0
(2) there are no cancellations in $Re[s] \geq 0$ when the product PCF is formed.

Remarks:
- No proof is given
- Condition (1) follows from the fact that poles of all the transfer functions in $M(s)$ are roots of the equation $1+L(s) = 0$;
- the meaning of condition (2) can be intuitively derived from the discussion on pole zero cancellation, and better understood through a simple example.

Internal stability of feedback systems: Example

Example: assume F(s) = 1

$$C(s) = \frac{s-1}{s+2}, P(s) = \frac{1}{(s-1)(s+2)} \rightarrow L(s) = \frac{s-1}{s+2}\frac{1}{(s-1)(s+2)} = \frac{1}{(s+2)^2}$$

$$T(s) = \frac{1}{(s+2)^2+1} \qquad S(s) = \frac{(s+2)^2}{(s+2)^2+1}$$

$$C(s)S(s) = \frac{(s-1)(s+2)}{(s+2)^2+1} \qquad P(s)S(s) = \frac{(s+2)}{(s-1)[(s+2)^2+1]}$$

$$1+L(s) = 1+\frac{1}{(s+2)^2} = \frac{s^2+2s+5}{(s+2)^2}$$

Remark: This example shows that, because of the pole-zero unstable cancellations, not all the transfer function in *M(s)* are BIBO stable, althought all the roots of $1+L(s)=0$ are stable.

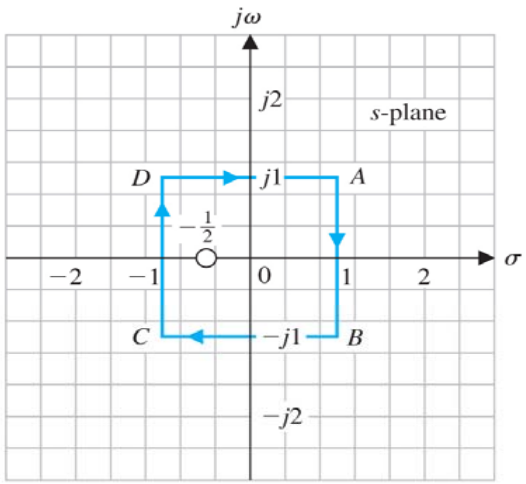
Stability of feedback control systems

Contour Mapping: Cauchy’s Encirclement Theorem

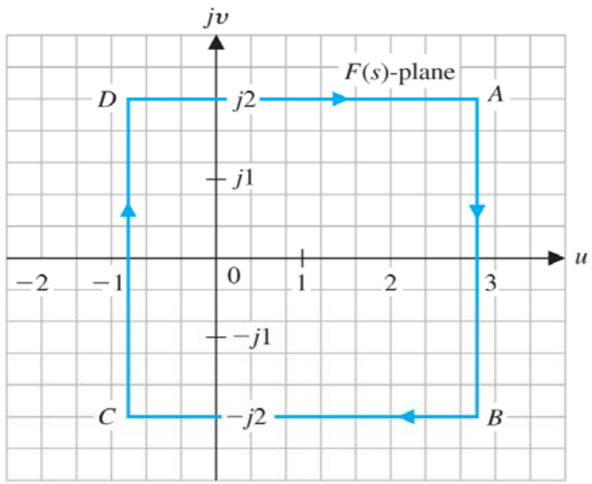
Contour Mapping: Example 1

A *contour map* is a contour or trajectory in a plane that is mapped or translated into another plane by a relation $F(s) = u + jv$

Clockwise traversal is assumed to be positive. The area within a contour to the right of the traversal is considered to be the area enclosed by the contour.



(a)



(b)

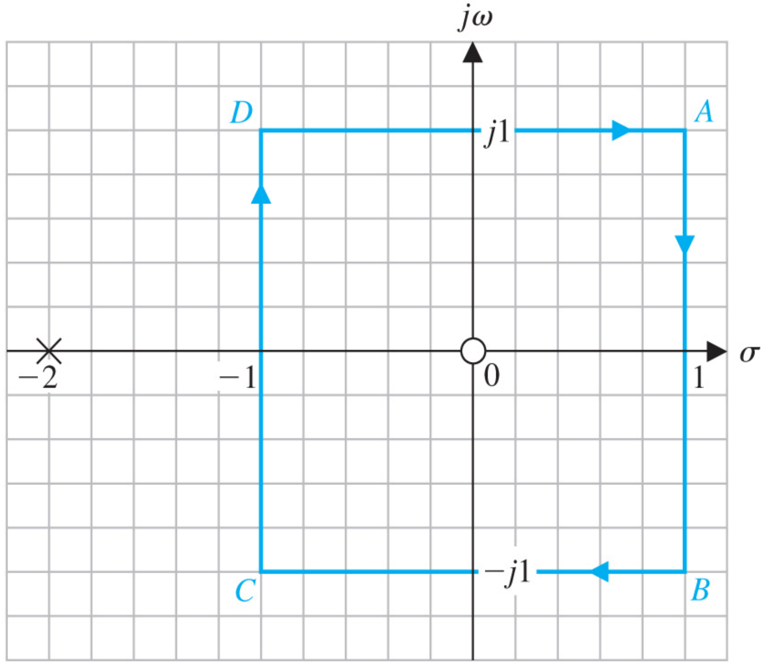
Example: $F(s) = 2s + 1$ and unit square s-plane contour *ABCD.*

$F(s) = u + jv = 2s + 1 = 2(\sigma + j\omega) + 1 = (2\sigma + 1) + j\,(2\omega)$

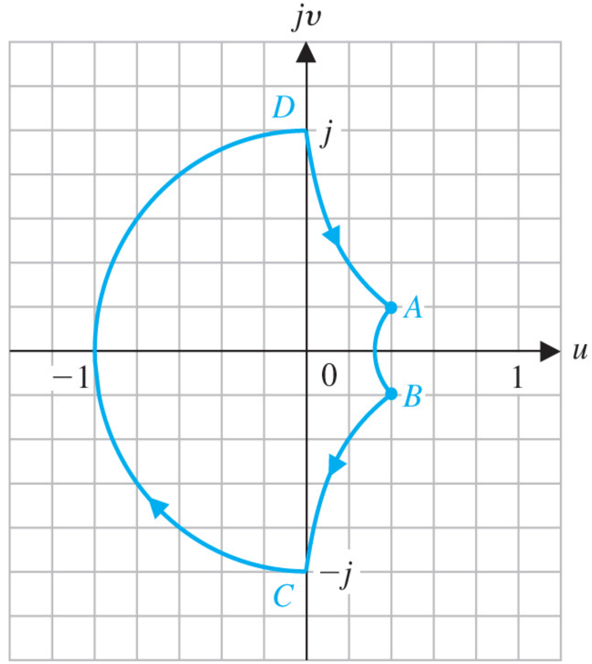
Result: Contour of identical form, twice as large, shifted one unit to right. A closed contour in the s-plane results in a closed contour in the $F(s)$ plane.

Contour Mapping: Example 2

F(s) = s / (s + 2) and unit square s-plane contour *ABCD*



(a)

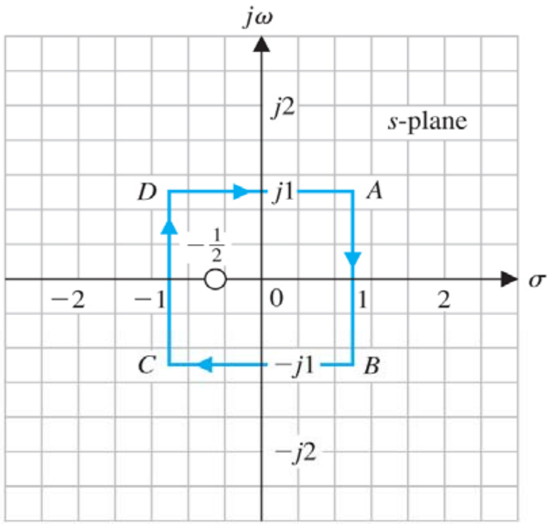


(b)

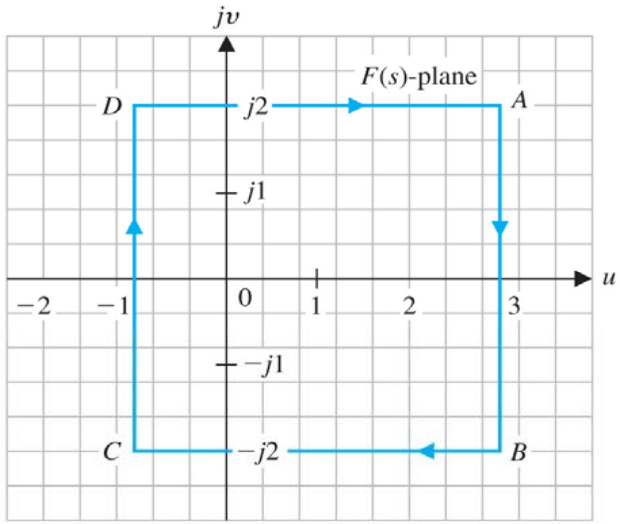
	A		B		C		D	
$s = \sigma + j\omega$	$1 + j1$	$1 + j0$	$1 - j1$	$0 - j1$	$-1 - j1$	$-1 + j0$	$-1 + j1$	$0 + j1$
$F(s) = u + jv$	$(4 + j2) / 10$	$1/3 + j0$	$(4 - j2) / 10$	$(1 - j2) / 5$	$0 - j1$	$-1 + j0$	$0 + j1$	$(1 + j2) / 5$

Cauchy's Encirclement Theorem

If a contour Γ_s in the s-plane, traversed in the clockwise direction, encircles Z zeros and P poles of $F(s)$ but does not pass through any of these poles or zeros of $F(s)$, the corresponding contour Γ_F in the *F(s)-plane* encircles the origin of the s-plane $N = Z - P$ times in the clockwise direction. Example: $F(s) = 2s + 1 = 2(s + ½)$

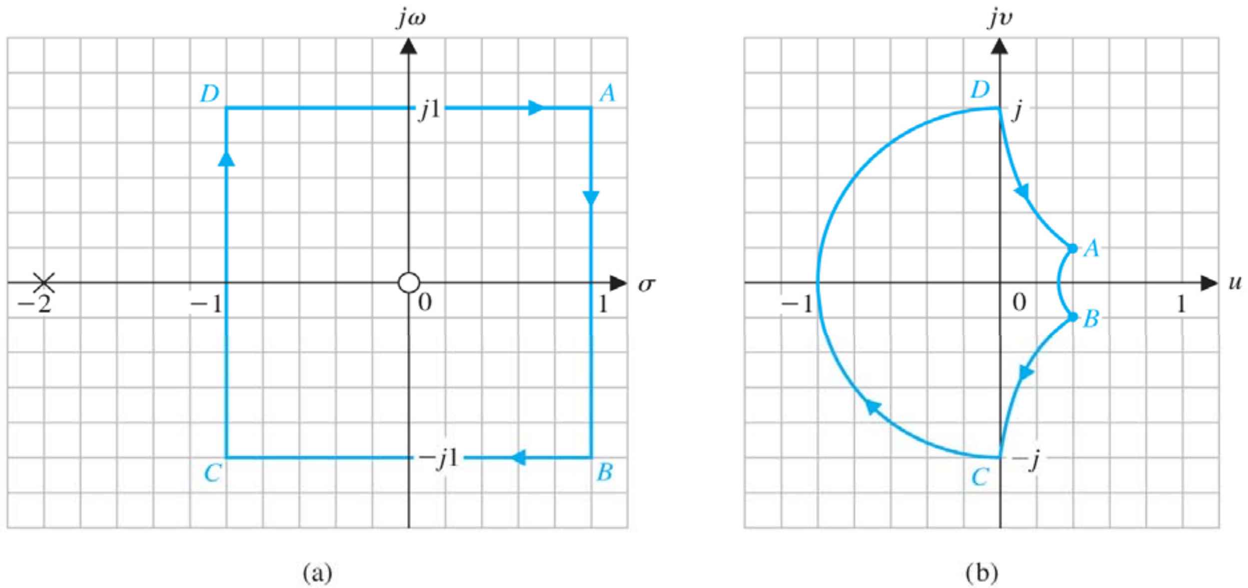


(a)



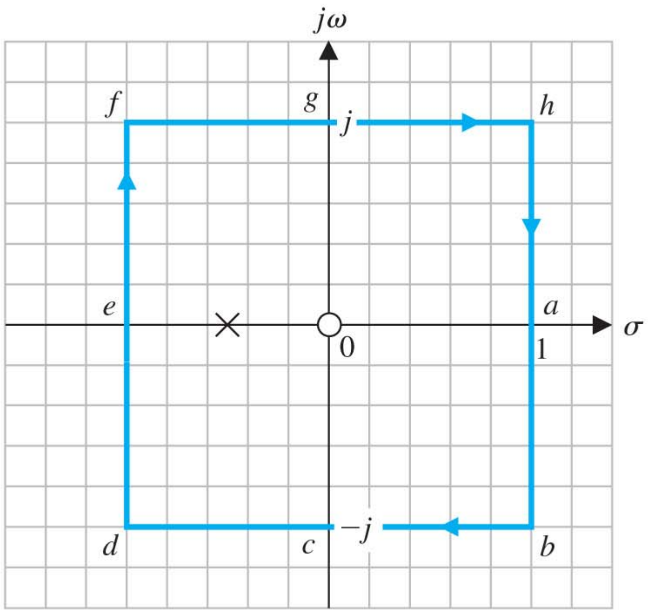
(b)

If a contour Γ_s in the s-plane, traversed in the clockwise direction, encircles Z zeros and P poles of $F(s)$ but does not pass through any of these poles or zeros of $F(s)$, the corresponding contour Γ_F in the *F(s)-plane* encircles the origin of the s-plane $N = Z - P$ times in the clockwise direction. Example: $F(s) = s / (s + 2)$

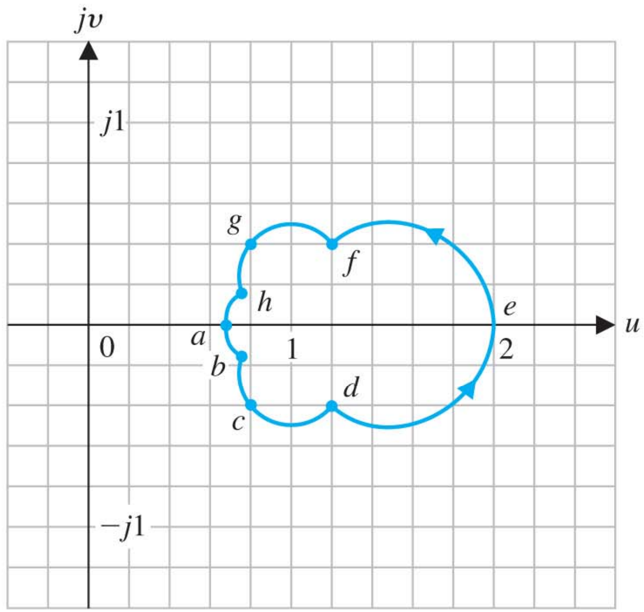


Cauchy's Encirclement Theorem

F(s) = s / (s + ½) and unit square s-plane contour *ABCD*



(a)



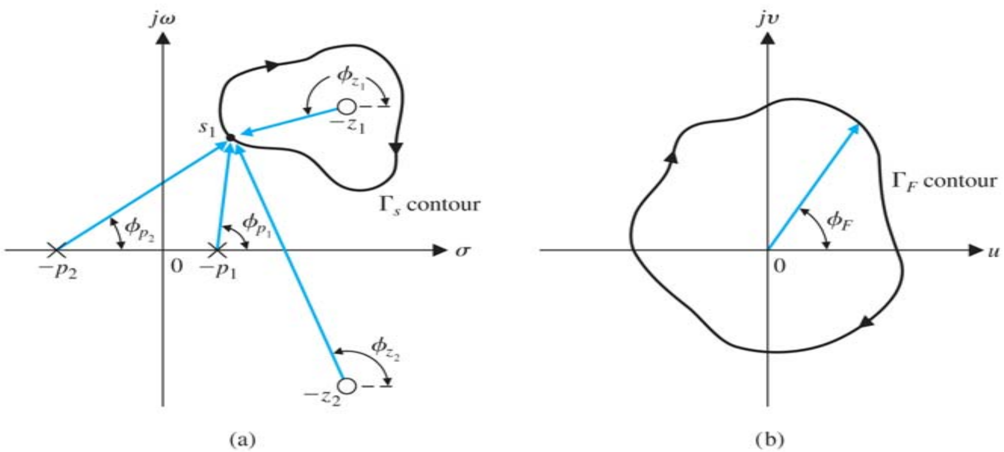
(b)

The contour in the s-plane encircles the zero s=0 and the pole s=-1/2 of F(s). The contour in the F(s)-plane does not encircle the origin. $N = Z - P = 1 - 1 = 0$

Cauchy's Encirclement Theorem

$$F(s) = (s + z_1)\,(s + z_2) \,/\, (s + p_1)\,(s + p_2)$$
$$F(s) = |F(s)|\;\underline{\;/s + z_1 + /s + z_2 - /s + p_1 - /s + p_2)}$$

$$F(s) = |F(s)|\;\underline{\;/\,\Phi_{z1} + \Phi_{z2} - \Phi_{p1} - \Phi_{p2}}$$



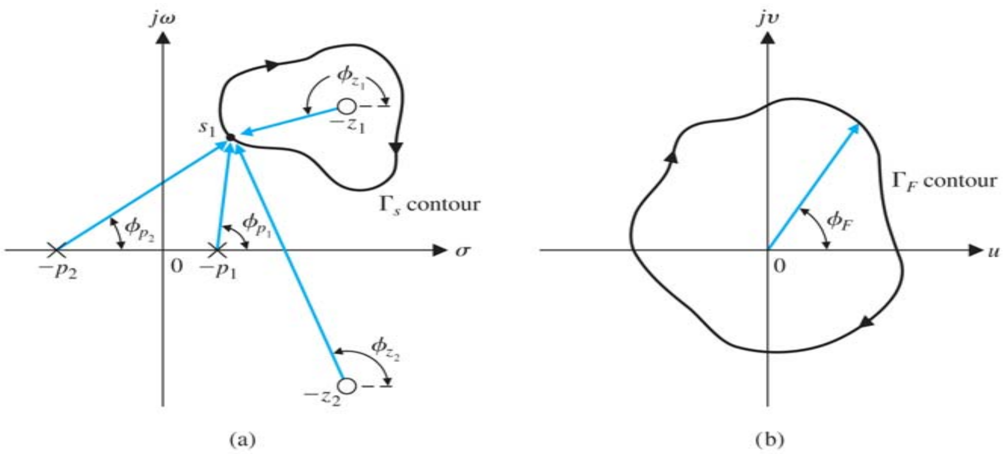
As s traverses Γ_s, the net change in angle of $F(s)$ due to z_2, p_1, and p_2 *is* 0^o, but the net change in angle of $F(s)$ due to the encircled z_1 is a full 360^o clockwise

Cauchy's Encirclement Theorem

$$F(s) = (s + z_1)\,(s + z_2)\,/\,(s + p_1)\,(s + p_2)$$

$$F(s) = |F(s)|\ \underline{/s + z_1 + /s + z_2 - /s + p_1 - /s + p_2)}$$

$$F(s) = |F(s)|\ \underline{/\,\Phi_{z1} + \Phi_{z2} - \Phi_{p1} - \Phi_{p2}}$$

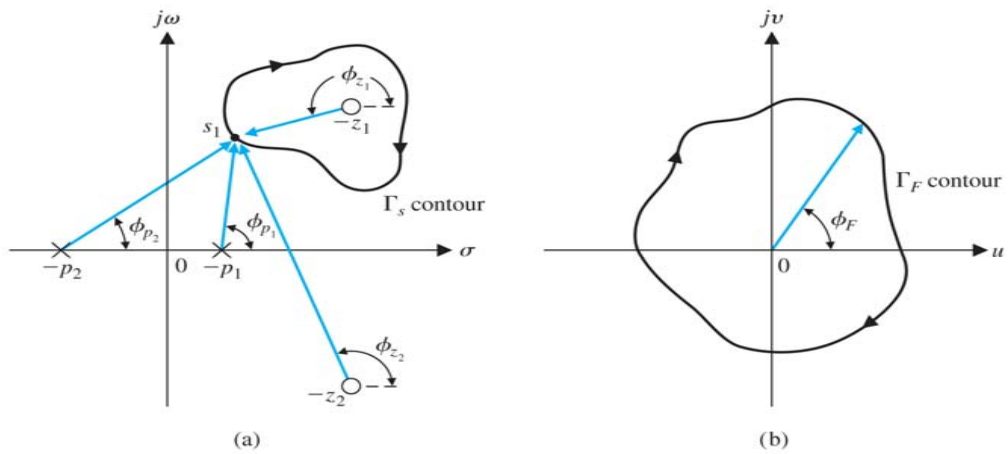


So, as Γ_s is completely traversed, the net angle increase of $F(s)$ is $360\,^o$ with *1 zero* encircled. With Z zeros encircled, the net angle increase in $F(s)$ is $Z(360\,^o) = 2\pi Z$ radians. With Z zeros and P poles encircled, the net angle increase of $F(s)$ is $2\pi Z - 2\pi P$ radians

Cauchy's Encirclement Theorem

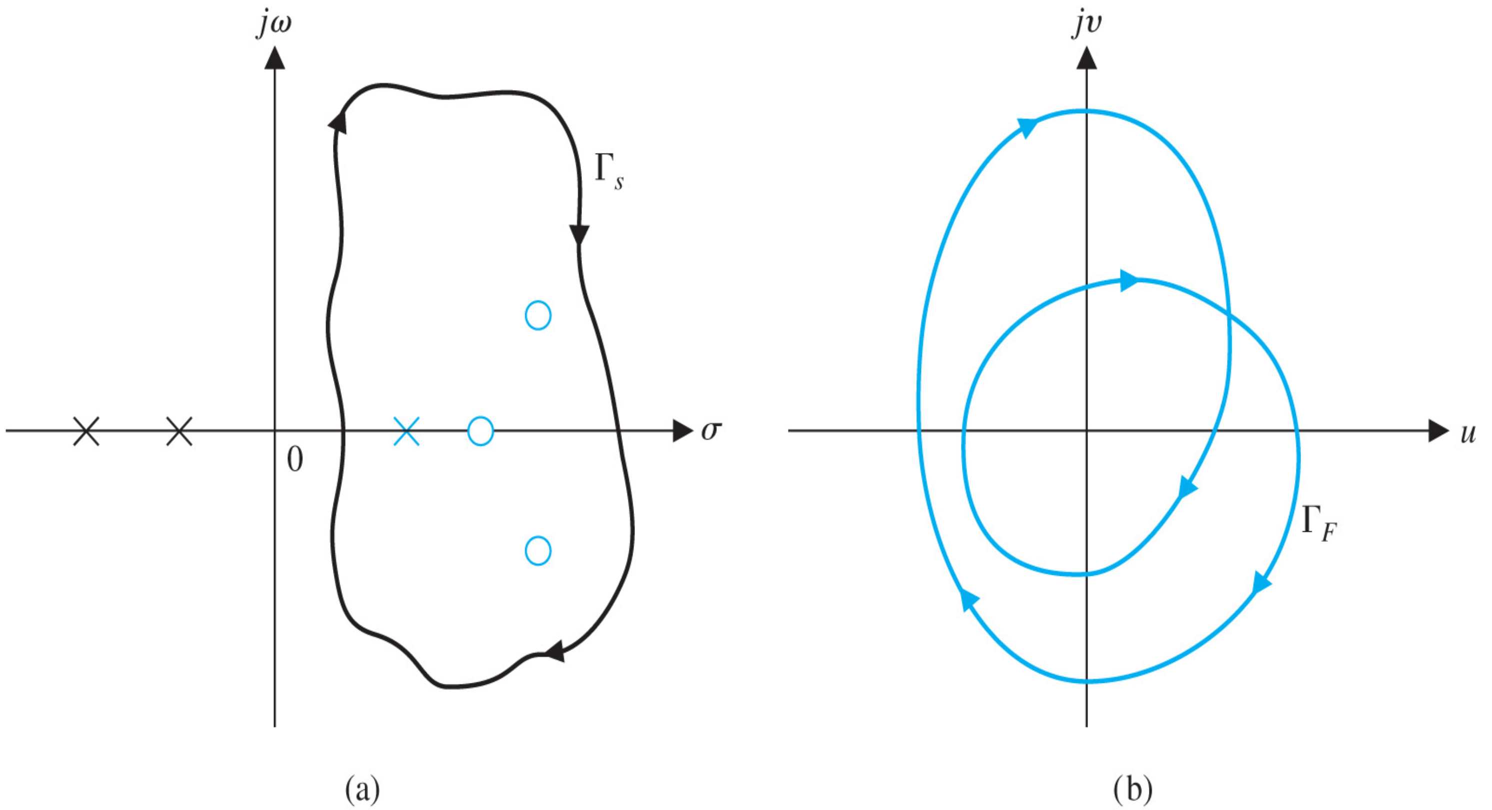
$$F(s) = (s + z_1)\,(s + z_2)\,/\,(s + p_1)\,(s + p_2)$$
$$F(s) = |F(s)|\ \underline{/s + z_1 + /s + z_2 - /s + p_1 - /s + p_2)}$$

$$F(s) = |F(s)|\ \underline{/\,\Phi_{z1} + \Phi_{z2} - \Phi_{p1} - \Phi_{p2}}$$



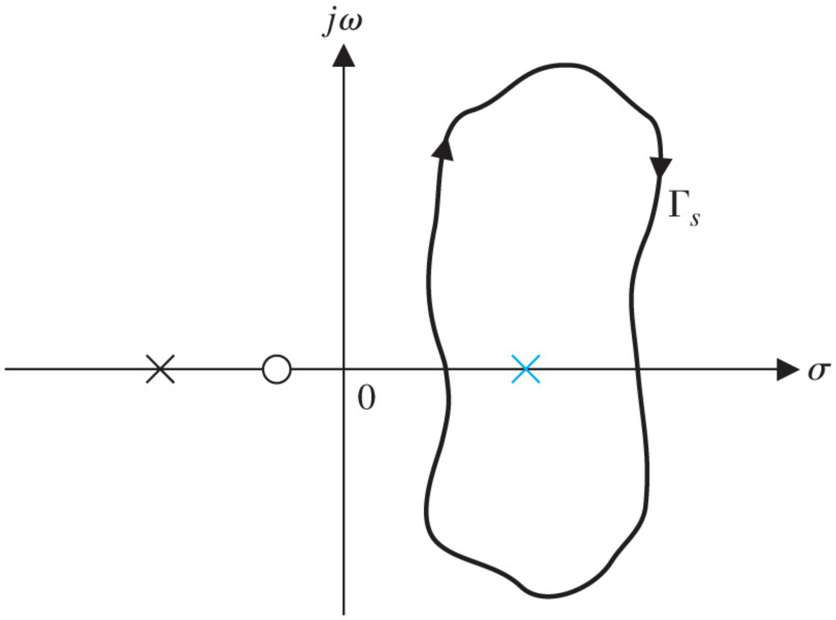
So if N is the number of net encirclements, $2\pi N = 2\pi Z - 2\pi P$ radians, or $N = Z - P$

Cauchy's Encirclement Theorem: Example 1

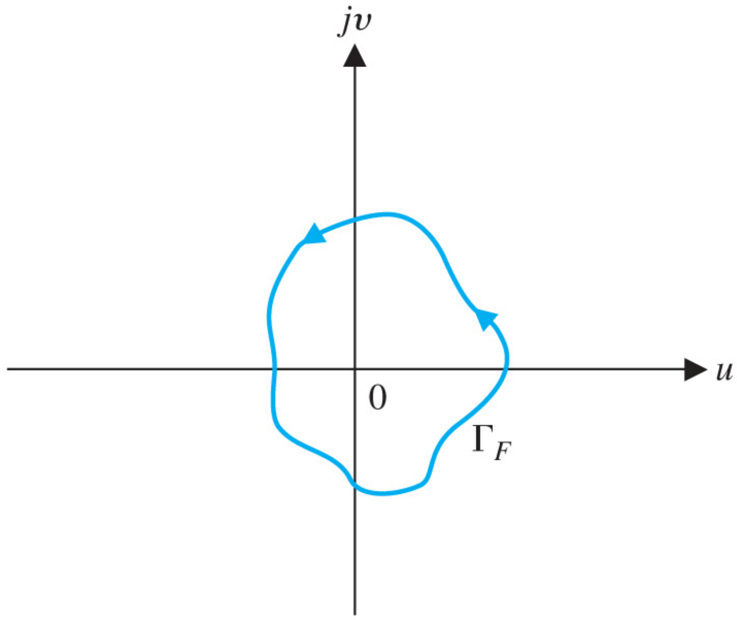


$$N = Z - P = 3 - 1 = 2$$

Cauchy's Encirclement Theorem: Example 2



(a)



(b)

$$N = Z - P = 0 - 1 = -1$$

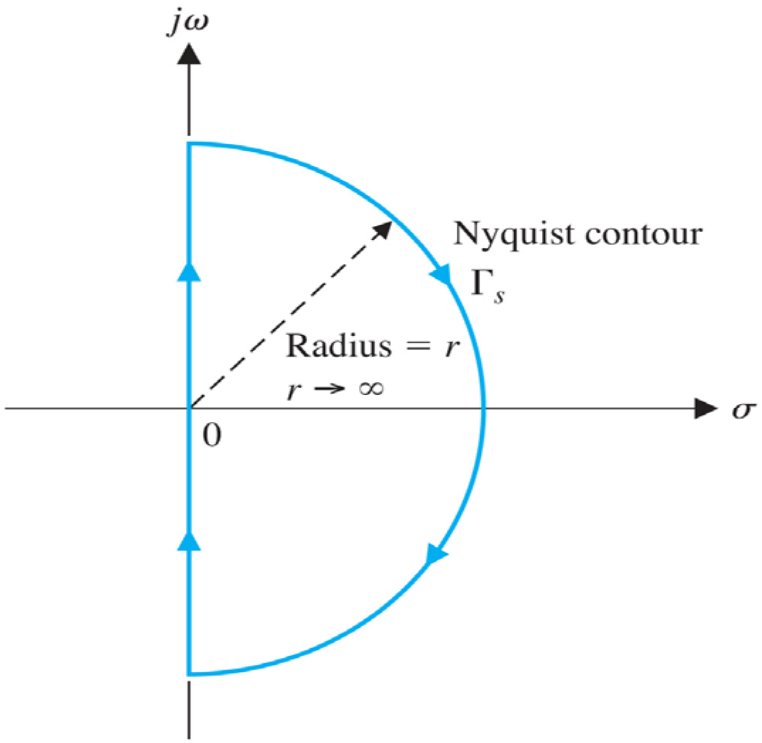
Stability of feedback control systems

Nyquist stability criterion

Relationship to System Stability

Characteristic Equation:

$F(s) = \Delta(s) = 1 + L(s) = K\,(s + z_1)(s + z_2) \,/\, (s + p_1)\,(s + p_2) = 0$



A stable system requires all roots (zeros) of $\Delta(s)$ be in the LH (left half) s-plane. If we choose contour Γ_s to encircle the entire RHP (right half plane), the Nyquist Contour, we can plot Γ_F (a polar plot) and determine whether $F(s)$ has any zeros in the RHP by using Cauchy's encirclement theorem: $N = Z - P$.
Then, the number of zeros of (1+L(s)) within the RHP (i.e.. unstable zeros) is

$$Z = N + P$$

Nyquist Criterion: preview

Issue: $L(s)$ is generally available in factored form
$[e.g.\ L(s) = G(s)H(s)]$,
but $F(s) = 1 + L(s) = \Delta(s)$ generally is not.

Since $L(s) = F(s) - 1$, we can map the Nyquist Contour Γ_s in the s-plane through the function $L(s)$ to the $L(s)$-*plane*, and the number of clockwise encirclements of the origin in the $F(s)$ plane becomes the number of clockwise encirclements of the point $-1 + j\,0$ in the $L(s)$ plane.

Nyquist Criterion: preview

Hence the Nyquist Criterion:

A feedback control system is stable if and only if the contour Γ_L in the $L(s)$ plane does not encircle the point $(-1, 0)$ when the number of poles of $L(s)$ in the RHP is zero ($P = 0$).

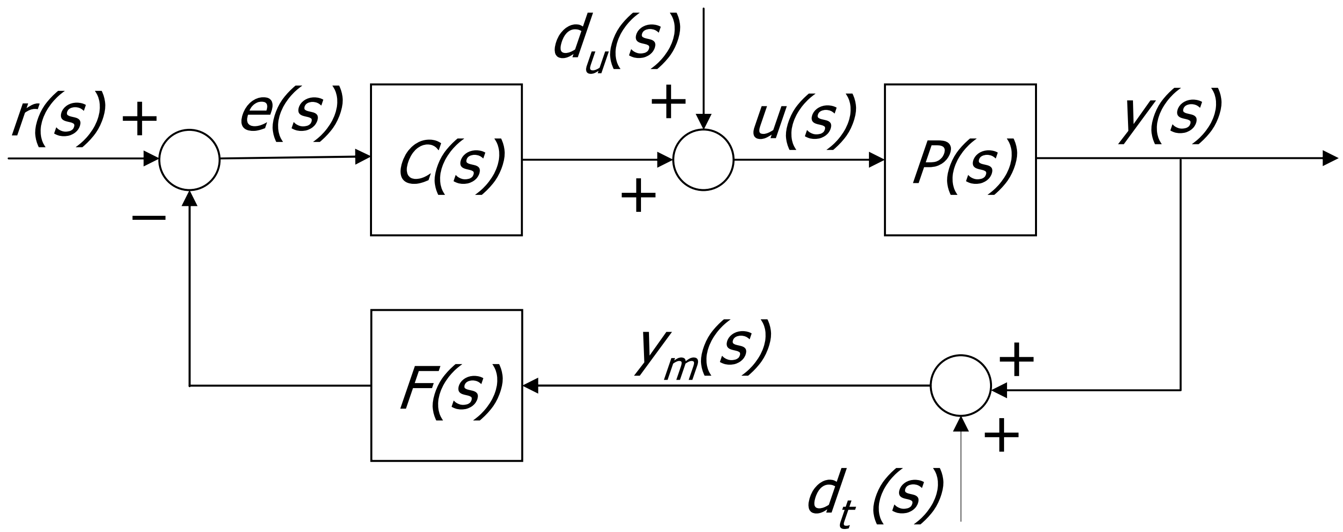
A feedback control system is stable if and only if, for the contour Γ_L in the $L(s)$ plane, the number of counterclockwise encirclements of the point $(-1, 0)$ is equal to the number of poles of $L(s)$ ($P \neq 0$) with positive real parts.

Recall *Result 1 (Internal stability of feedback systems)* : The closed loop system considered here is stable if and only if the following conditions are met:

(1) all roots of the equation $1 + L(s) = 0$ have real part < 0.

(2) there are no cancellations in $Re[s] \geq 0$ when the product PCF is formed.

In this part we present a result, for checking condition (1), referred to as the Nyquist stability criterion.



Recall *Result 2 (Cauchy's argument principle)*

- Let $F(s)$ be a rational function of $s = \sigma + j\omega$
- Let Γ_s be a closed curve on the plane of the variable s
- Assume that Γ_s does not cross any pole or zero of $F(s)$

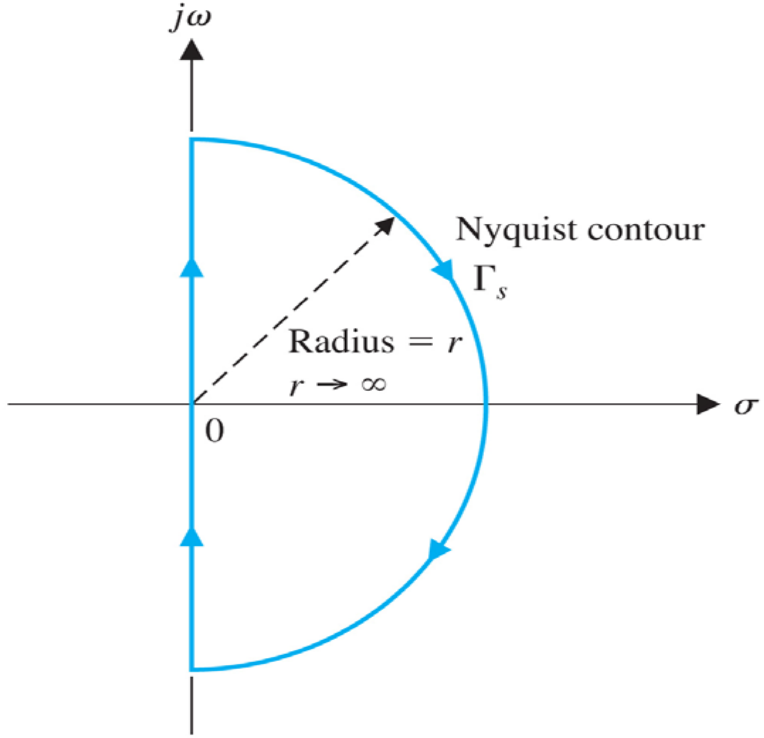
Then, the set given by $F(\Gamma_s)=\{F(s) : s \in \Gamma_s\}$, obtained moving the variable s clockwise along Γ_s, is a closed curve of the complex plane $Re\{F\}+jIm\{F\}$ which

- does not cross the origin
- encircles the origin $N = Z - P$ times clockwise, where
  - ✓ Z is the number of zeros of $F(s)$ encircled by Γ_s
  - ✓ P is the number of poles of $F(s)$ encircled by Γ_s

Remark: $N < 0$ means counterclockwise encirclements.

Application of the Cauchy's argument principle to the case

- $F(s) = 1 + L(s)$
- Γ_s is the Nyquist's contour →



leads to the a fundamental result for the study of feedback system stability called Nyquist's Theorem.

Result 3 (Nyquist's Theorem)

Assume that the Nyquist plot of $L(j\omega)$ does not cross the real axis at point -1+j0.
The number P_{cl} of roots of the equation $1 + L(s) = 0$ with positive real part is given by:

$$P_{cl} = P_{ol} + N$$

- P_{ol} is the number of poles of $L(s)$ with real part > 0
- N is the number of encirclements of the *Nyquist plot* of $L(j\omega)$ around the point -1+j0, computed as the difference between the number of clockwise encirclements and number of counterclockwise encirclements

If the Nyquist plot crosses the real axis at point -1+j0, the following result can be used to study the roots of the equation $1+L(s) = 0$.

Result 4

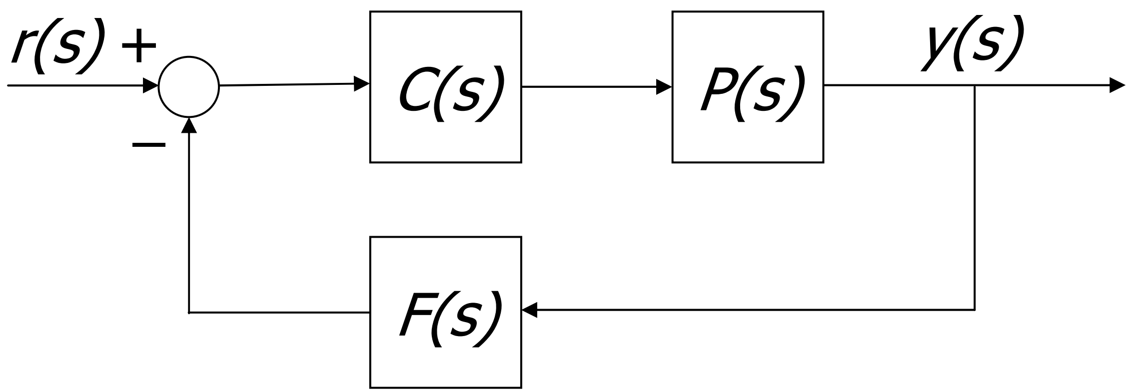
The Nyquist plot of the function $L(j\omega)$ crosses the real axis at point $-1+j0$ if and only if the equation $1 + L(s) = 0$ has imaginary roots.

Nyquist's criterion and BIBO stability

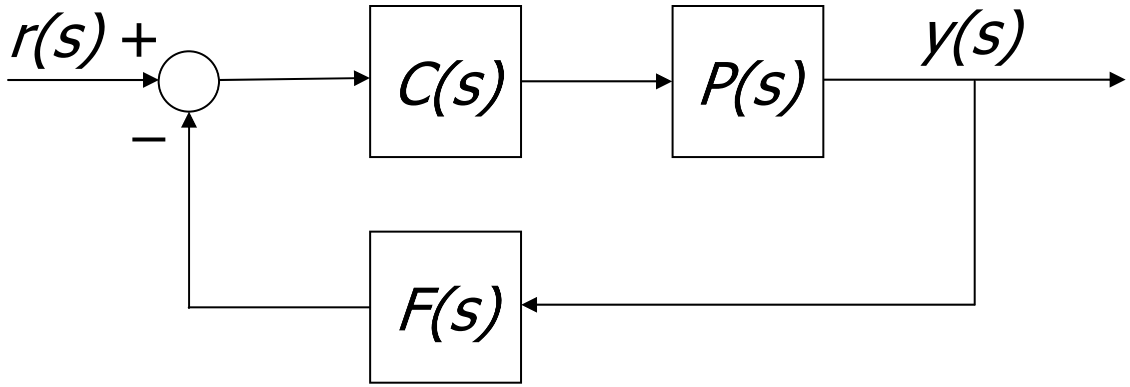
The following result shows that the Nyquist's theorem can be used to study the BIBO stability of the feedback system below.

Result 5 (BIBO stability of feedback systems) - Assume that $F^{-1}(s)$ is BIBO stable. The following 3 conditions are equivalent

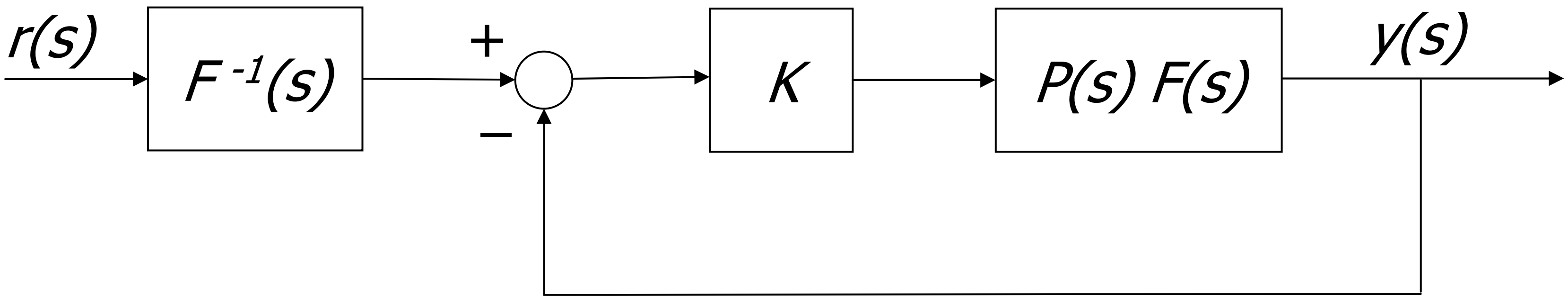
1. $N = -P_{ol}$
2. All roots of the equation $1+L(s) = 0$ have negative real part
3. The feedback system below is BIBO stable



Gain variation effects on feedback stability



Let us consider the case where $C(s) = K$, K real, and consider the equivalent description for the feedback system



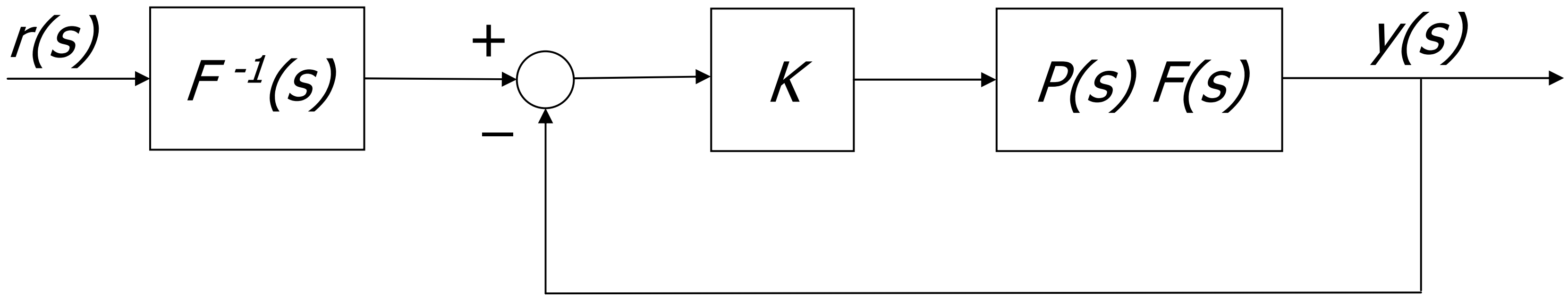
Gain variation effects on feedback stability

Result 6

Assume that $F^{-1}(s)$ is BIBO stable and $C(s) = K$.

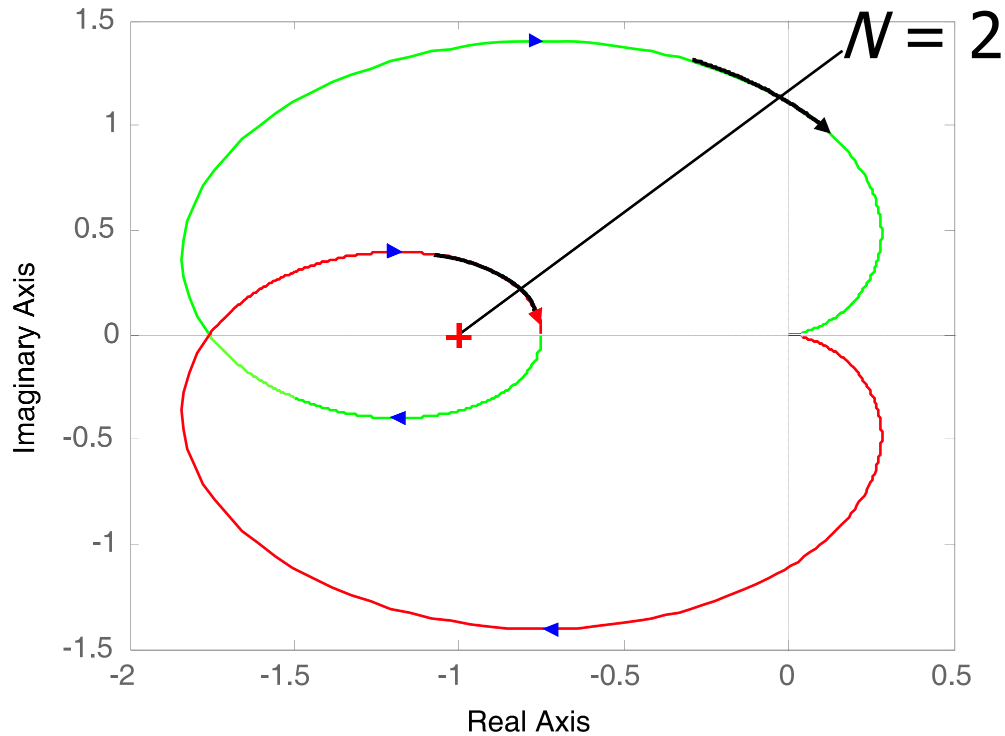
The number of poles with positive real part of the input output transfer function of the feedback system is given by $P_{cl} = P_{ol} + N$ where:

1. P_{ol} is the number of poles of $L(s)$ with real part > 0
2. N is the number of encirclements of the *Nyquist plot* of $L(j\omega)$ around the point -1/K+j0.



How to compute N: example 1

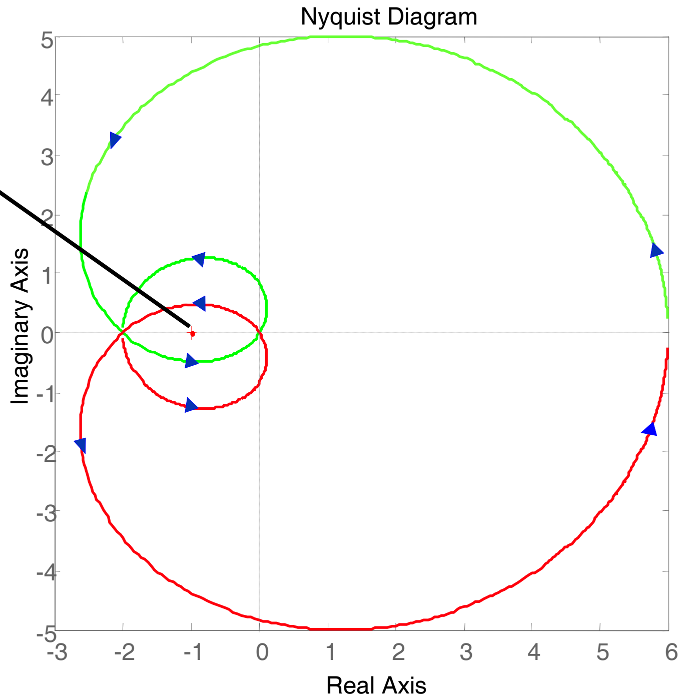
N can be easily computed by counting the number of clockwise intersection between the Nyquist diagram and a generic half-line with origin in the critical point (-1,j0). Each counterclockwise intersection contributes -1 to the total sum.



How to compute N: example 2

N can be easily computed by counting the number of clockwise intersection between the Nyquist diagram and a generic half-line with origin in the critical point (-1,j0). Each counterclockwise intersection contributes -1 to the total sum.

$N = -3$

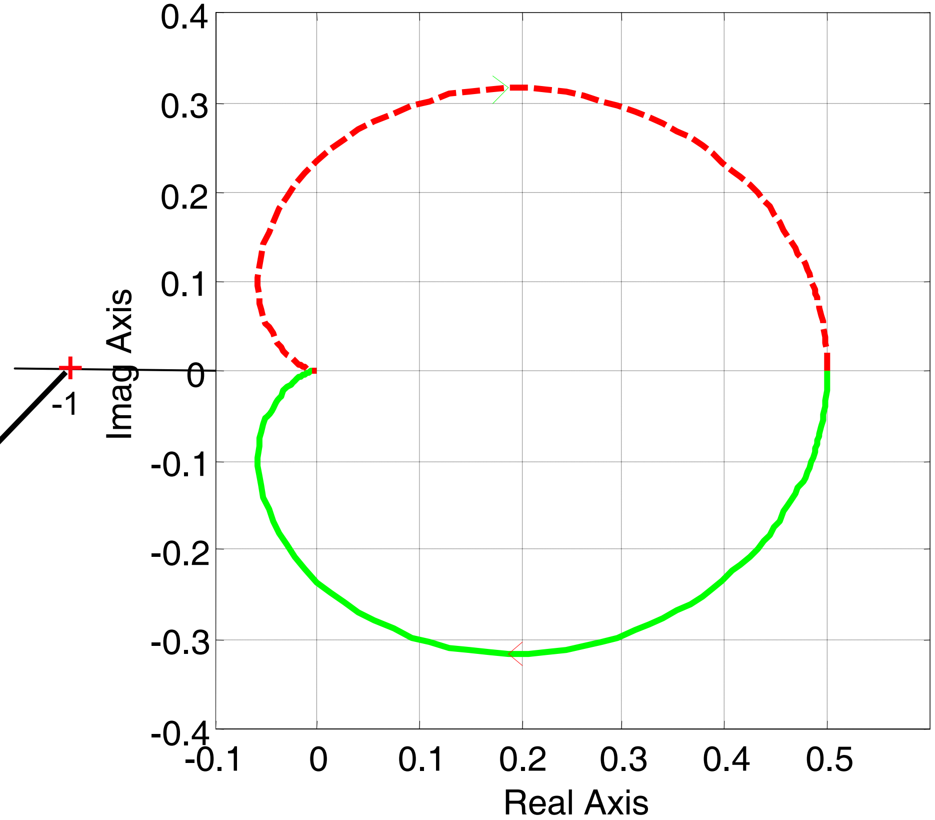


Nyquist Theorem: example 1

$$L(s) = \frac{1}{s^2 + 3s + 2}$$

$P_{ol} = 0$

$N = 0$



$P_{cl} = P_{ol} + N = 0 \rightarrow$ stable feedback system

Nyquist Theorem: example 2

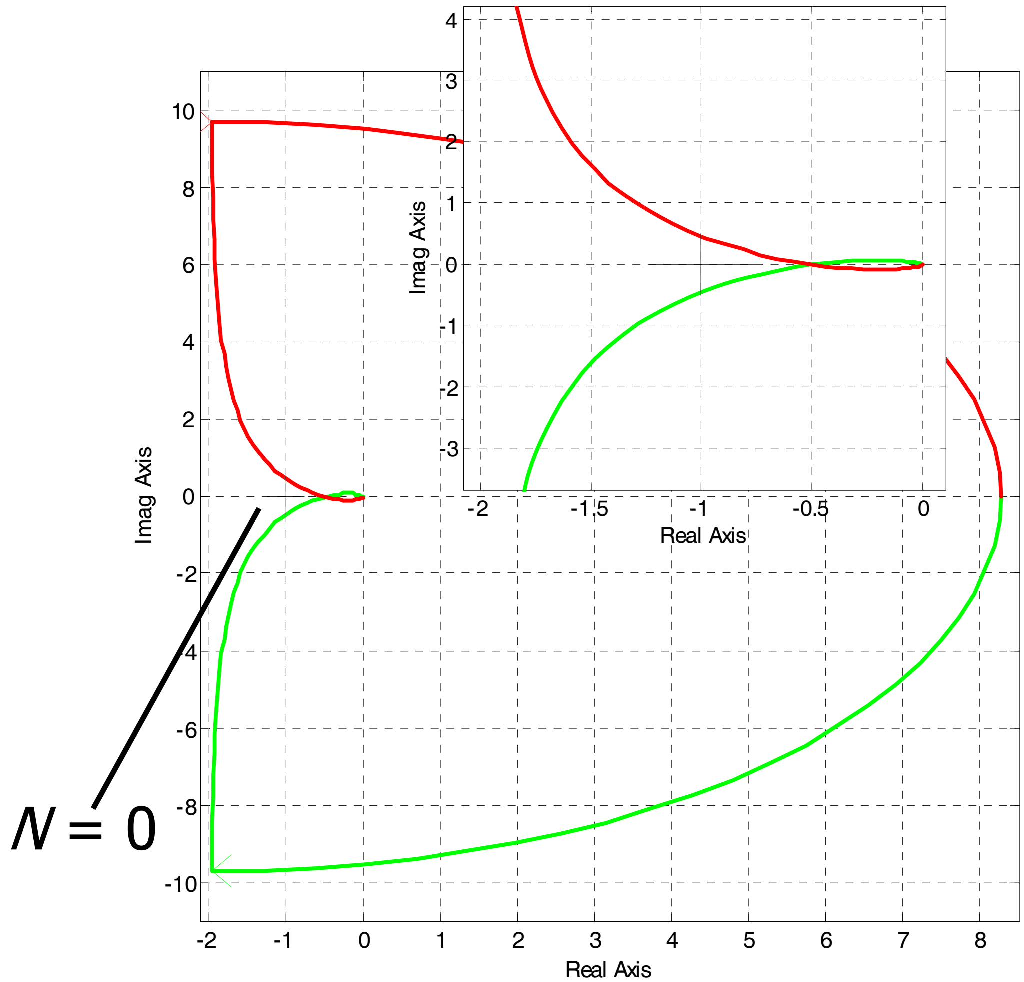
$$L(s) = \frac{1}{s(1+s)^2}$$

$P_{ol} = 0$

$N=0$

$P_{cl} = P_{ol} + N = 0$

→ stable feedback system



Nyquist Theorem: example 3

$$L(s) = \frac{1}{s^2(1+s)^2}$$

$P_{ol} = 0$

$P_{cl} = P_{ol} + N = 2$

→ unstable feedback system

$N = 2$