Introduction to digital control

Introduction

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Mathematical model of a sampler

Spectrum of sampled signals - Aliasing

Signal reconstruction through ideal low pass filter and sampling theorem

Signal reconstruction through ZOH

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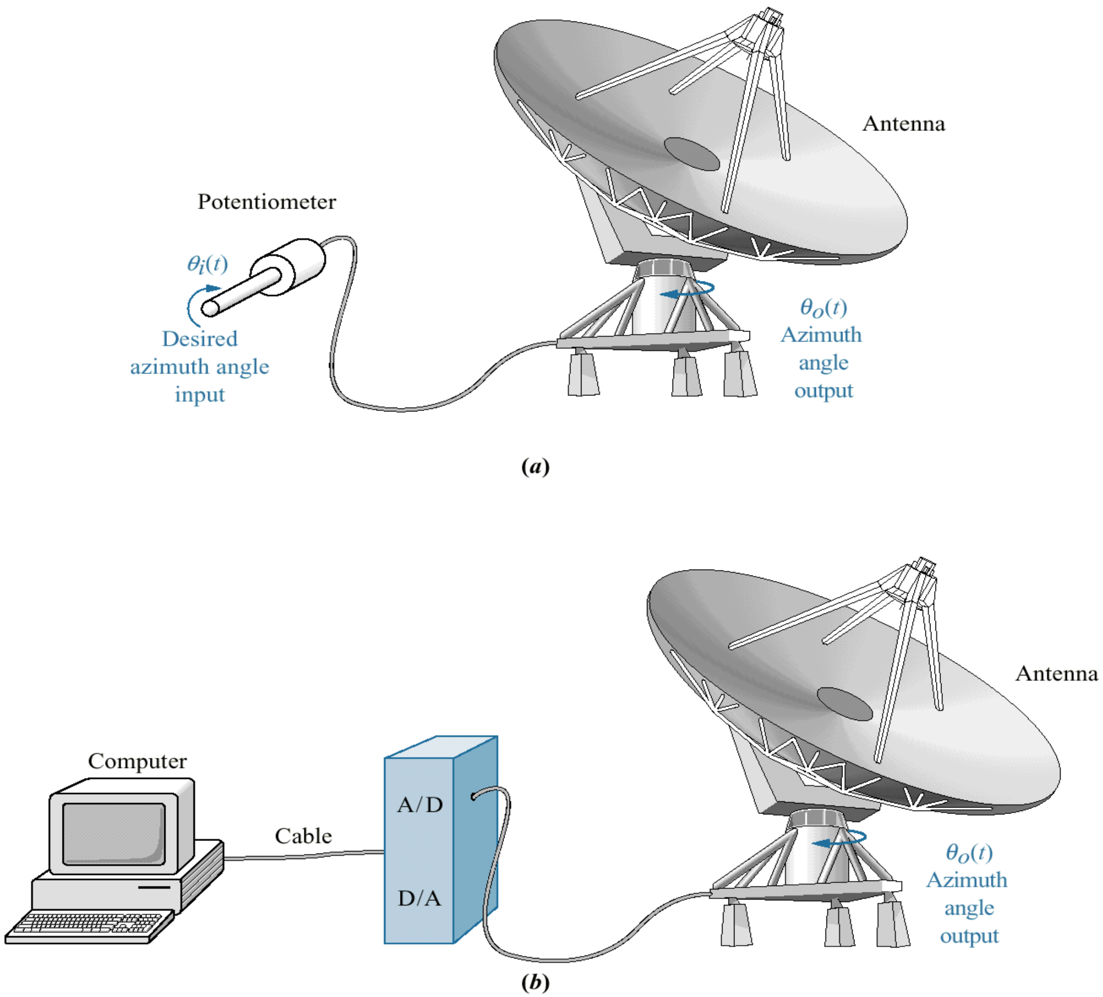
Introduction

What is digital control?

A **control system** is an interconnection of components forming a system configuration that provides a desired system response. When digital computers are placed within the system to modify the system dynamics such that a more satisfactory response is obtained, the system becomes a **digital control system**.

- We refer to systems in which all signals change continuously with time as **continuous-time systems** (or analog systems).
- **Discrete-time systems** have signals that can change values only at discrete instants of time.
- Most control systems use discrete-time and continuous-time variables, and as a result, are called **sampled-data systems**.

Conversion of antenna azimuth position control system from analog control to digital control



Why digital control? Pro’s.

Pro’s

Accurate and reliable (analog drift is eliminated)

Accuracy can be controlled by choosing word length

Repeatable

Reduced sensitivity: digital processing is insensitive to component tolerances, aging, environmental conditions, temperature, electromagnetic interference, noise in general, etc

Flexibility can be achieved with software implementations: parameters can be varied in software

Complex algorithms fit into a single chip

Digital information can be encrypted for security

Why digital control? Con's.

Con's

Sampling causes loss of information

Discrete time processing artifacts (aliasing, delay)

A/D and D/A requires mixed-signal hardware

Limited speed of processors

Quantization and round-off errors

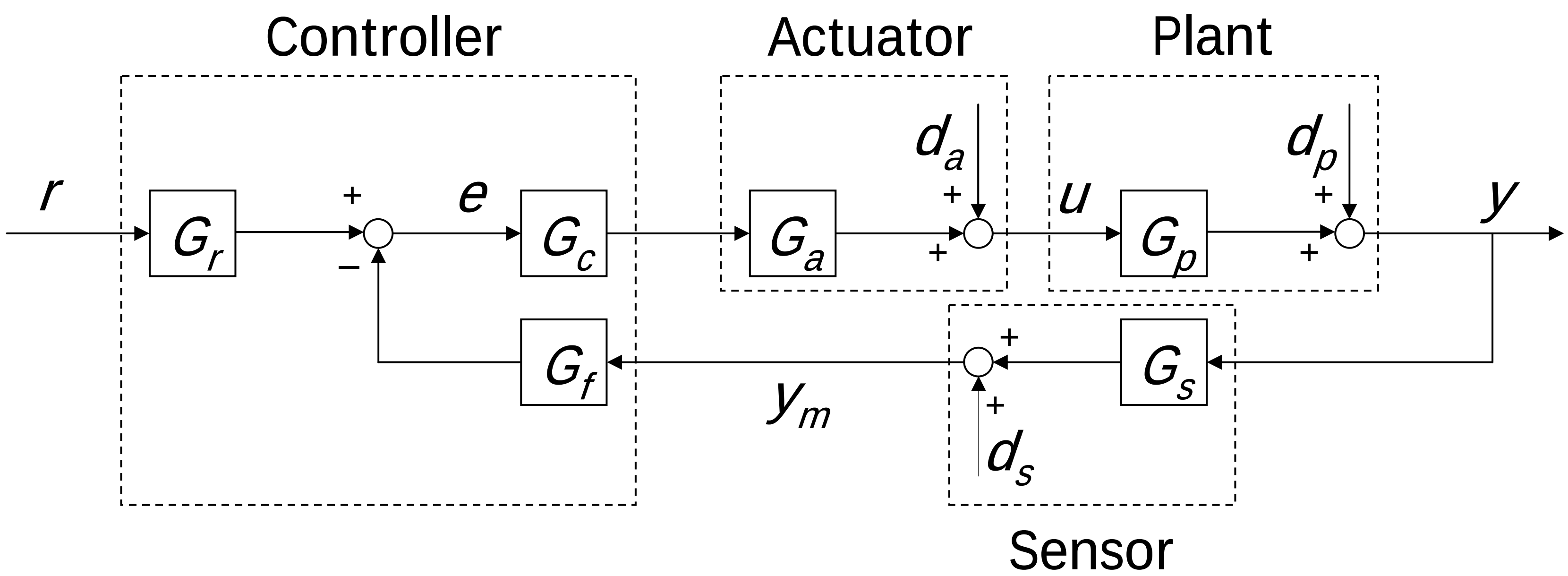
Introduction to digital control

Systems and signals in a digital control structure

Analog output feedback control system configuration

The controller consists of

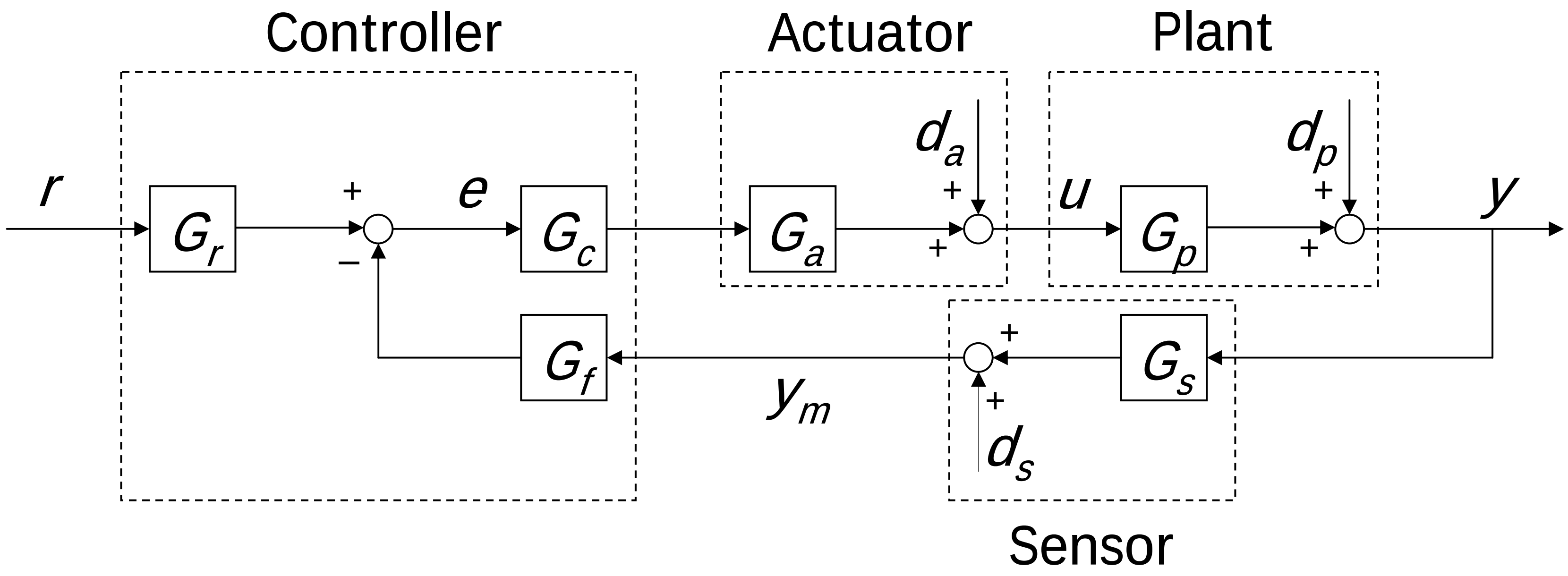
- **prefilter** G_r (reference generator)
- **cascade controller** G_c
- **feedback controller** G_f (for 2 DOF or, if constant, for dc gain)



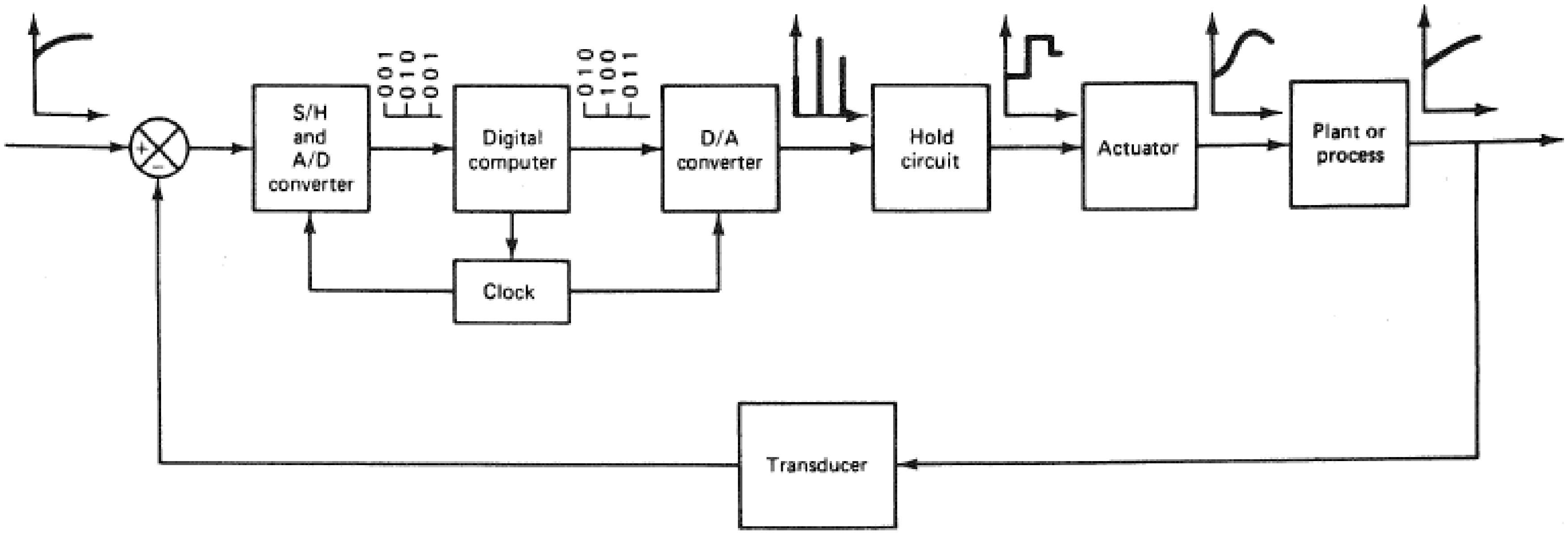
Analog output feedback control system configuration

The rest of the control system

- **plant** G_p with plant disturbance d_p
- **actuator** G_a with actuator disturbance d_a
- **sensor** G_s with sensor noise d_s

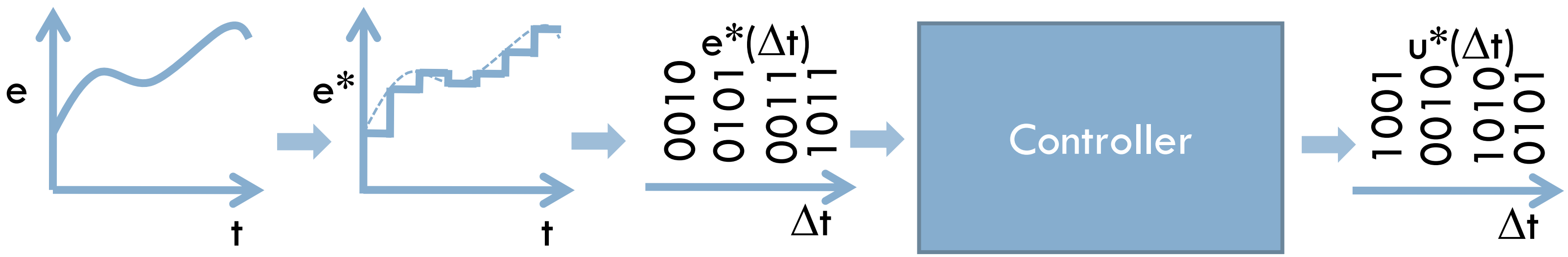


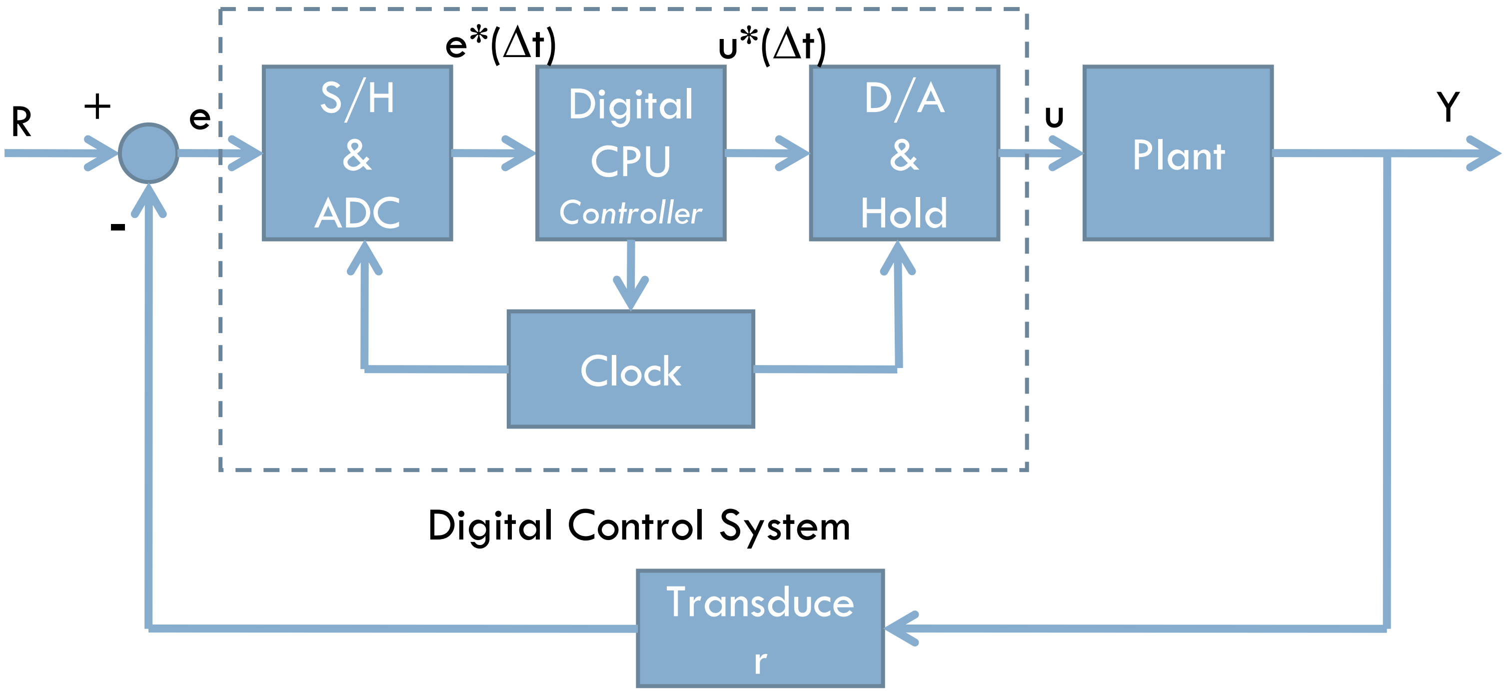
Major components of a digital control system



Block diagram of a digital control system showing signals in binary or graphic form.

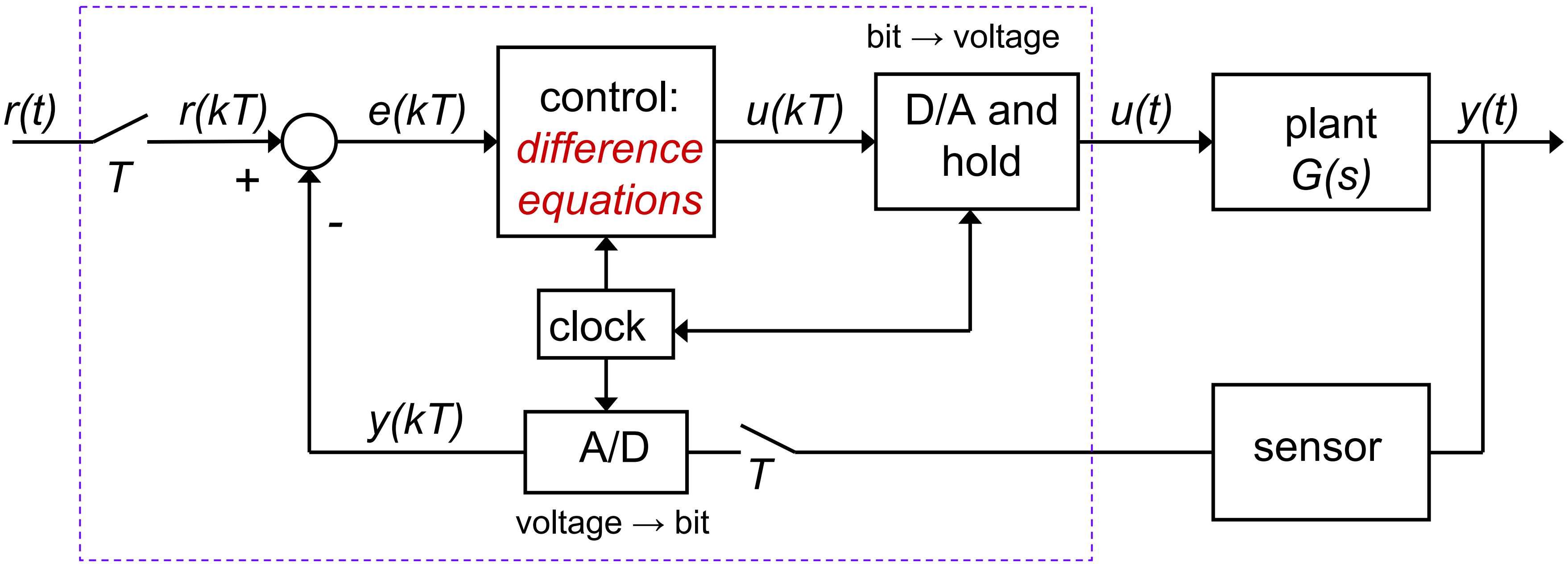
Major components of a digital control system





Major components of a digital control system

In digital control, the compensation function (or control law) is performed by a digital computer.



digital controller

Introduction to digital control

Mathematical model of a sampler

Analog to digital conversion

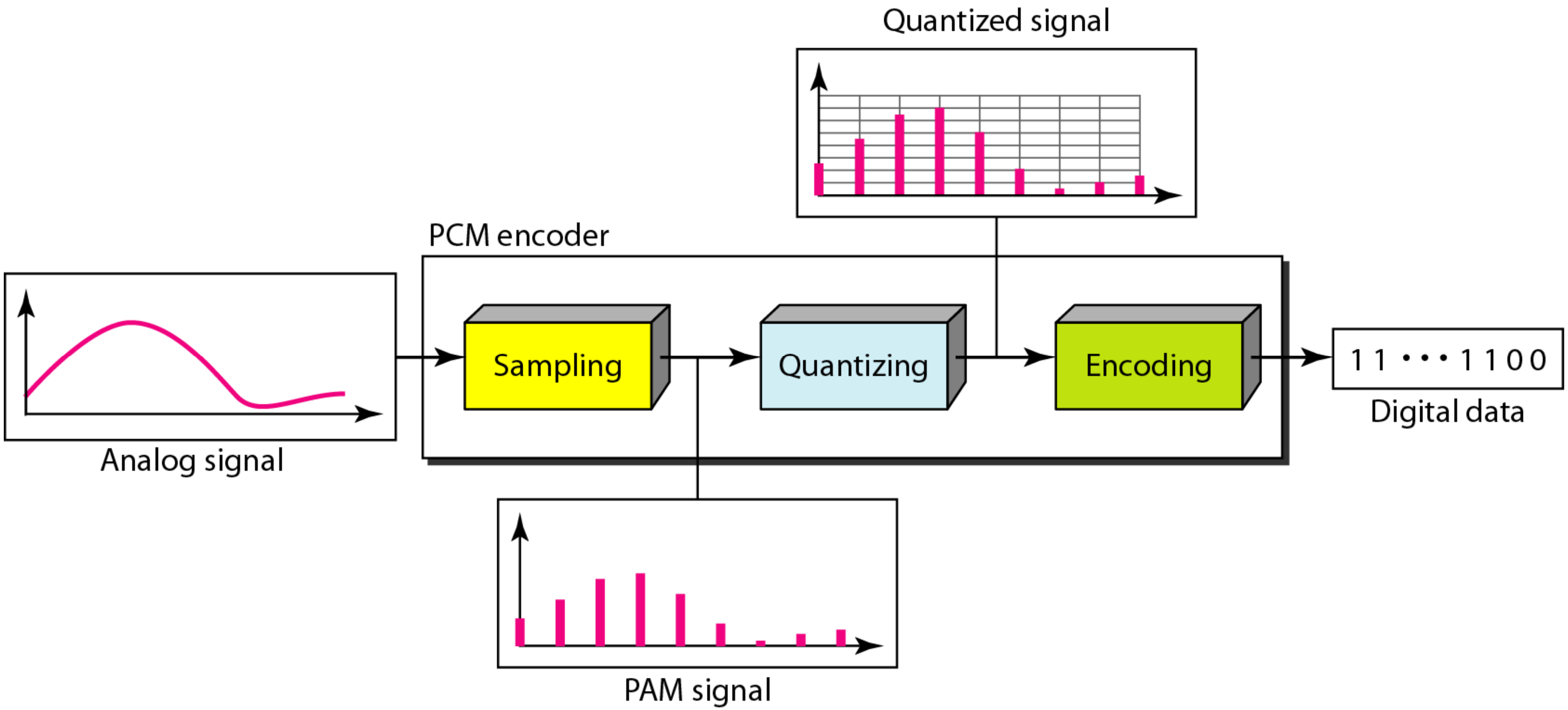
Analog to digital conversion consists of three steps :

- Sampling
- Quantization
- Binary encoding

Before we sample, we have to filter the signal to limit the maximum frequency of the signal as it affects the sampling rate.

Filtering should ensure that we do not distort the signal, i.e. remove high frequency components that affect the signal shape.

Analog to digital conversion



Sampling

Analog signal is sampled every T_S secs.

T_s is referred to as the sampling interval.

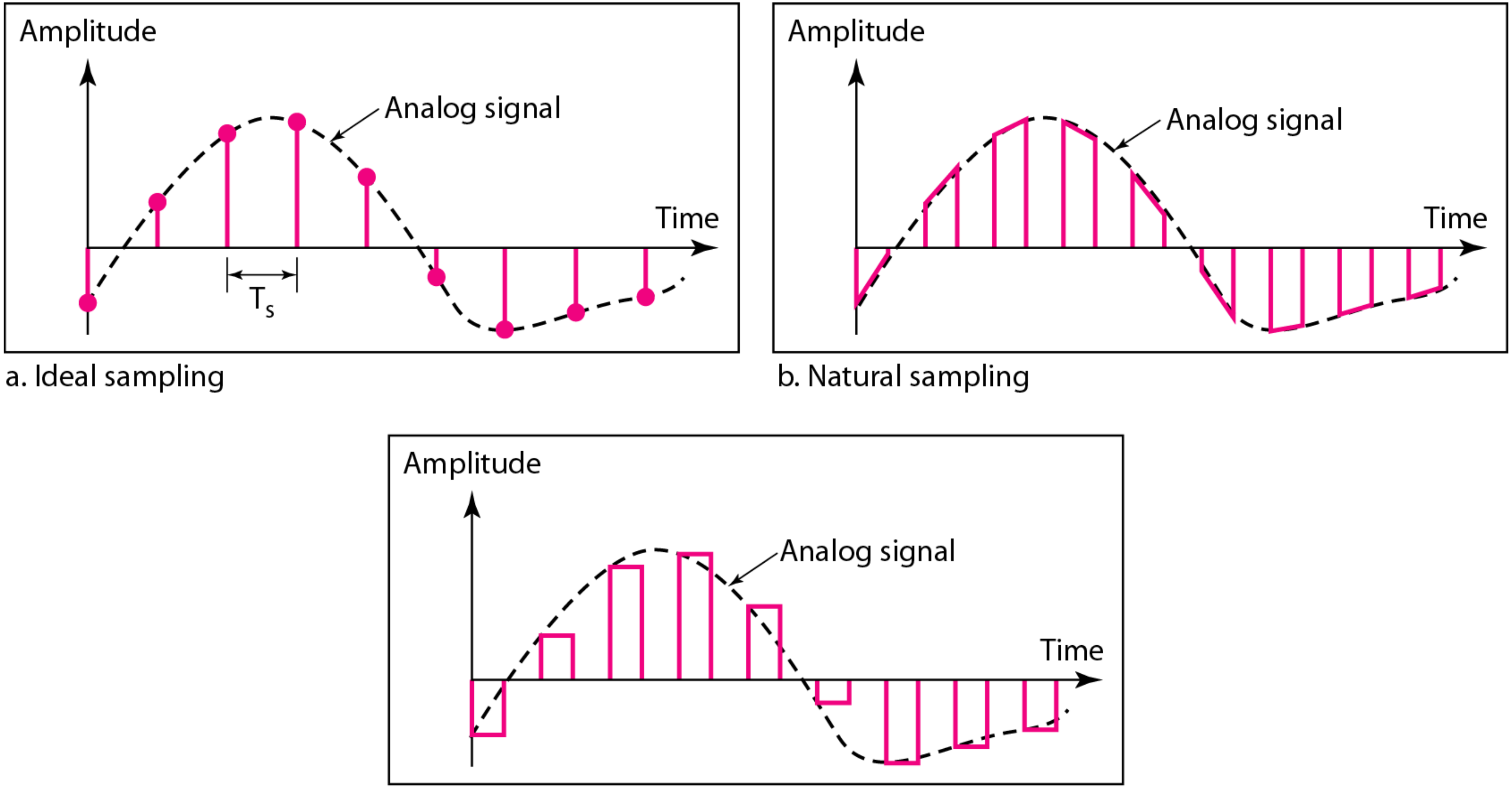
$f_s = 1/T_s$ is called the sampling rate or sampling frequency.

There are 3 sampling methods:

- Ideal - an impulse at each sampling instant
- Natural - a pulse of short width with varying amplitude
- Flattop - sample and hold, like natural but with single amplitude value

The process is referred to as pulse amplitude modulation PAM and the outcome is a signal with analog (non integer) values

Sampling



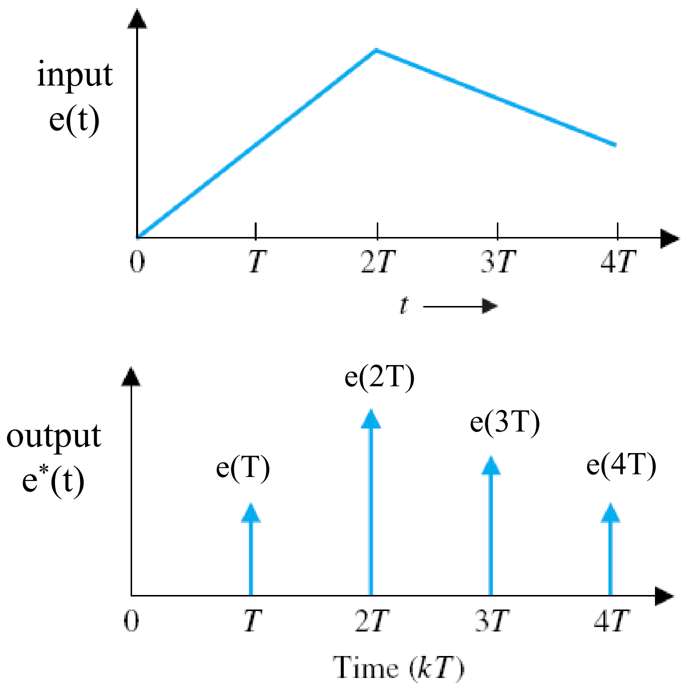
a. Ideal sampling

b. Natural sampling

c. Flat-top sampling

Sampling characteristics

As a first step toward the analysis and design of digital control systems, it is necessary to understand the effects of sampling. A sampler is basically a switch that closes every T seconds for one instant of time, say τ ($\tau << T$ of course).

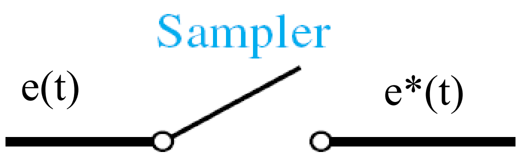


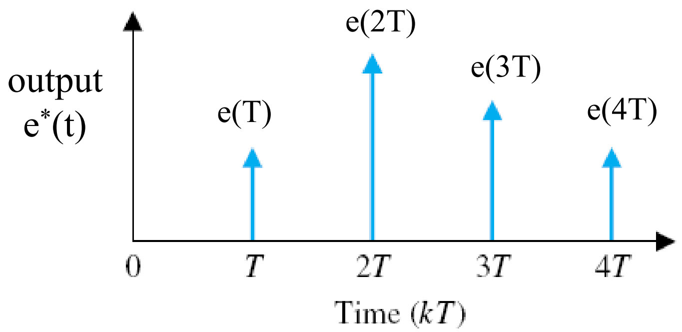
The sampler output is e*(t). e*(t) equals e(t) over τ during each interval when the switch is closed and e*(t) is zero between samples.

When nT is the current sample time, the current value of e*(t) is e(nT).

Ideal sampler

A symbolic representation of the ideal sampler is shown in the figure, where the input is e(t) and the output is e*(t). Note that e*(t) is not a physical signal. e*(t) may be represented mathematically by multiplying e(t) by a train of unit impulse functions:

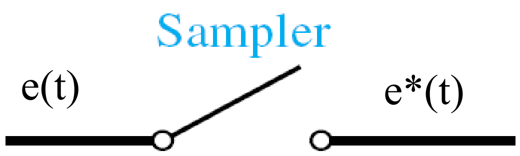


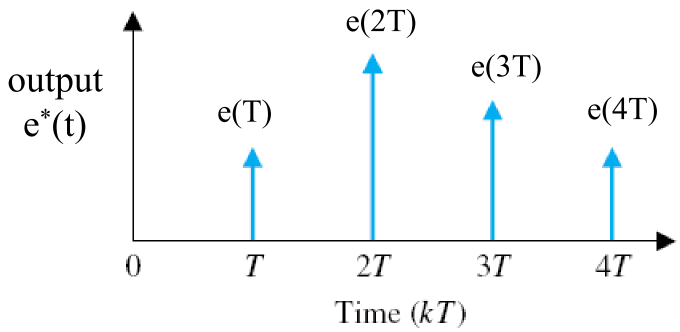


$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

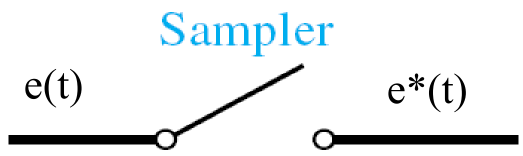
Ideal sampler (cont'd)

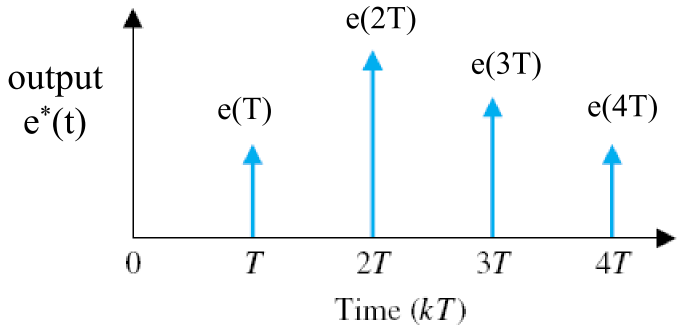
A symbolic representation of the ideal sampler is shown in the figure, where the input is e(t) and the output is e*(t). Note that e*(t) is not a physical signal. e*(t) may be represented mathematically by multiplying e(t) by a train of unit impulse functions:





$$
\begin{aligned}
e^*(t) &= e(t)\,\delta_T(t) \\
&= e(t)\delta(t) + e(t)\delta(t-T) + e(t)\delta(t-2T) + \ldots \\
&= e(0)\delta(t) + e(T)\delta(t-T) + e(2T)\delta(t-2T) + \ldots
\end{aligned}
$$





Or equivalently,

$$e^*(t) = \sum_{n=0}^{\infty} e(nT)\delta(t - nT)$$

Taking the Laplace transform of e*(t) yields E*(s), called the **starred transform**.

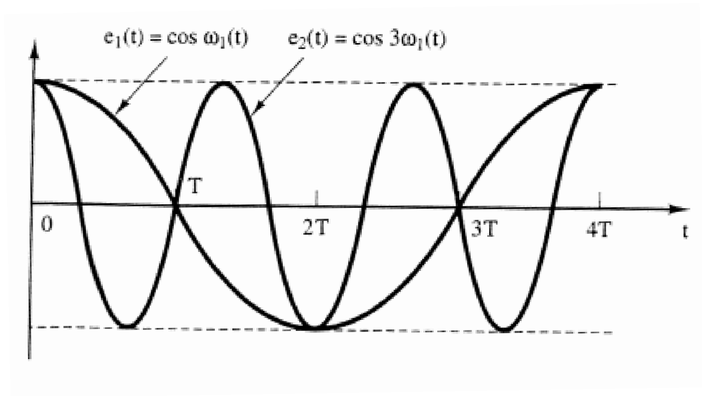
$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

Ideal sampler (cont'd)

The output signal of an ideal sampler is defined as the signal whose Laplace transform is

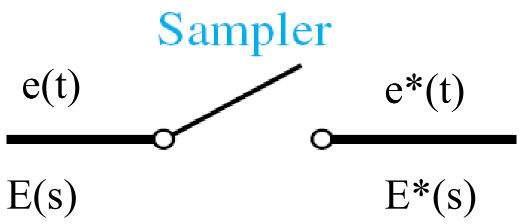
$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

where e(t) is the input signal to the sampler. If e(t) is discontinuous at t=nT, where n is an integer, then e(nT) is taken to be $e(nT^+)$, that is, the value of e(t) as t approaches nT from the right.



The output of the sampler is a function of e(t) only at t=nT, n=0,1,2... So many different input signals can result in the same output e*(t) or E*(s).

Ideal sampler (cont'd)



$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

No transfer function exists for the ideal sampler, and this property of the sampler complicates the analysis of sampled-data systems.

Example 1. Determine E*(s) for the unit step.

By definition of the (ideal) sampler,

$$E^*(s) = \sum_{n=0}^{\infty} e(nT) e^{-nTs}$$

for the unit step, e(nT)=1, n=0,1,2, ... , E*(s) can be expressed as

$$E^*(s) = 1 + e^{-Ts} + e^{-2Ts} + \dots$$

Using the relationship $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$

E*(s) can be written in closed form: $E^*(s) = \frac{1}{1-e^{-Ts}}, \qquad \left|e^{-Ts}\right| < 1$

Example 2. Determine E*(s) for $e(t)=e^{-t}$.

Again we use the definition of the (ideal) sampler,

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$
$$= 1 + e^{-T}e^{-Ts} + e^{-2T}e^{-2Ts} + \cdots$$
$$= 1 + e^{-(1+s)T} + \left(e^{-(1+s)T}\right)^2 + \cdots$$

According to $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$

E*(s) can be written in closed form: $E^*(s) = \dfrac{1}{1-e^{-(1+s)T}}, \qquad \left|e^{-(1+s)T}\right| < 1$

Evaluation of E*(s)

E*(s) has limited usefulness in system analysis because it is defined as an infinite series:

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

However, for many useful time functions, E*(s) can be expressed in closed form, as we have seen in the examples.

Introduction to digital control

Spectrum of sampled signals - Aliasing

In order to find the spectrum of the sampled signal, we introduce an alternative form of the pulse train which, we recall, is a periodical function.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} \quad \text{where} \quad \omega_s = 2\pi / T$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \left[e^{-jk\omega_s t} \right]_{t=0} = \frac{1}{T}$$

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_s t}$$

In view of this form of the pulse train, the sampled signal can be written as

$$e^*(t) = e(t)\,\delta_T(t) = e(t)\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_s t} = \frac{1}{T}\sum_{k=-\infty}^{\infty}e(t)e^{jk\omega_s t}$$

Which Laplace transform is

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty}E(s - jk\omega_s)$$

An important s-plane property of E*(s) is now given.

Property - $E^*(s)$ is periodic in s with period $j\omega_s$, that is

$$E^*(s) = E^*(s + jk\omega_s)$$

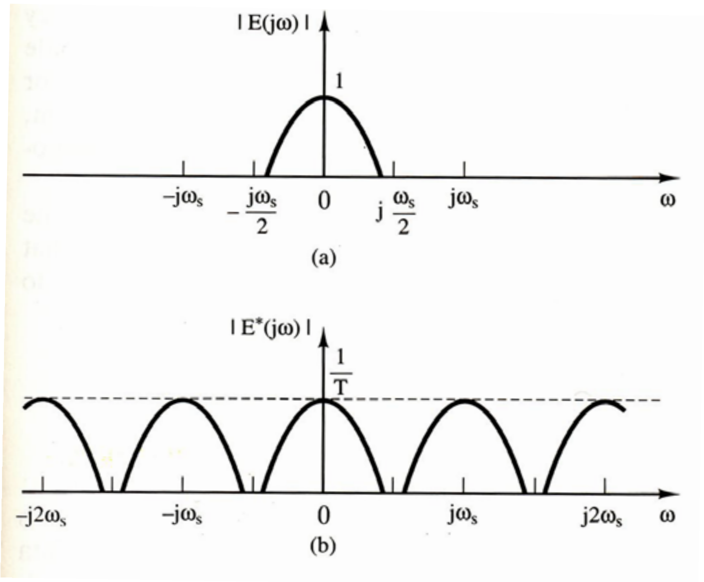
where k is an integer, and ω_s is the radian sampling frequency, $\omega_s = 2\pi/T$.

Effects of sampling

Suppose the sampler input signal e(t) has the amplitude spectrum $|E(j\omega)|$, as shown in the figure. The sampling operation

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s - jk\omega_s)$$

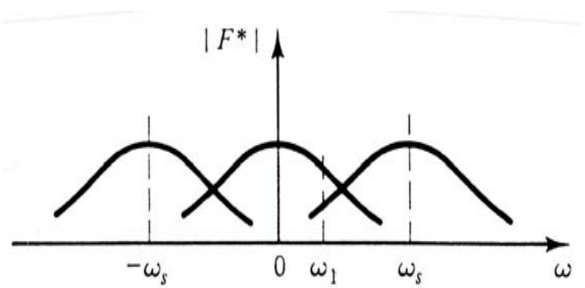
will lead to the amplitude spectrum $|E^*(j\omega)|$ of the sampler output e*(t), which is also shown in the figure.



Hence the effect of ideal sampling is to replicate the original spectrum centered at ω_s, at $2\omega_s$, at $-\omega_s$, at $-2\omega_s$, and so on.

To reconstruct the original signal successfully, it is well known that the sampling theorem must be satisfied.

Sampling theorem. The sampling frequency ω_s must at least equal twice the value of the highest significant frequency in the signal.



Frequency aliasing is the typical consequence of violating the sampling theorem. In the figure, at frequency ω_1 the central band contributes $|E^*(j\omega_1)|$, and the side band

$|E^*(j(\omega_s-\omega_1))|$.

The latter is an *alias* of ω_1.

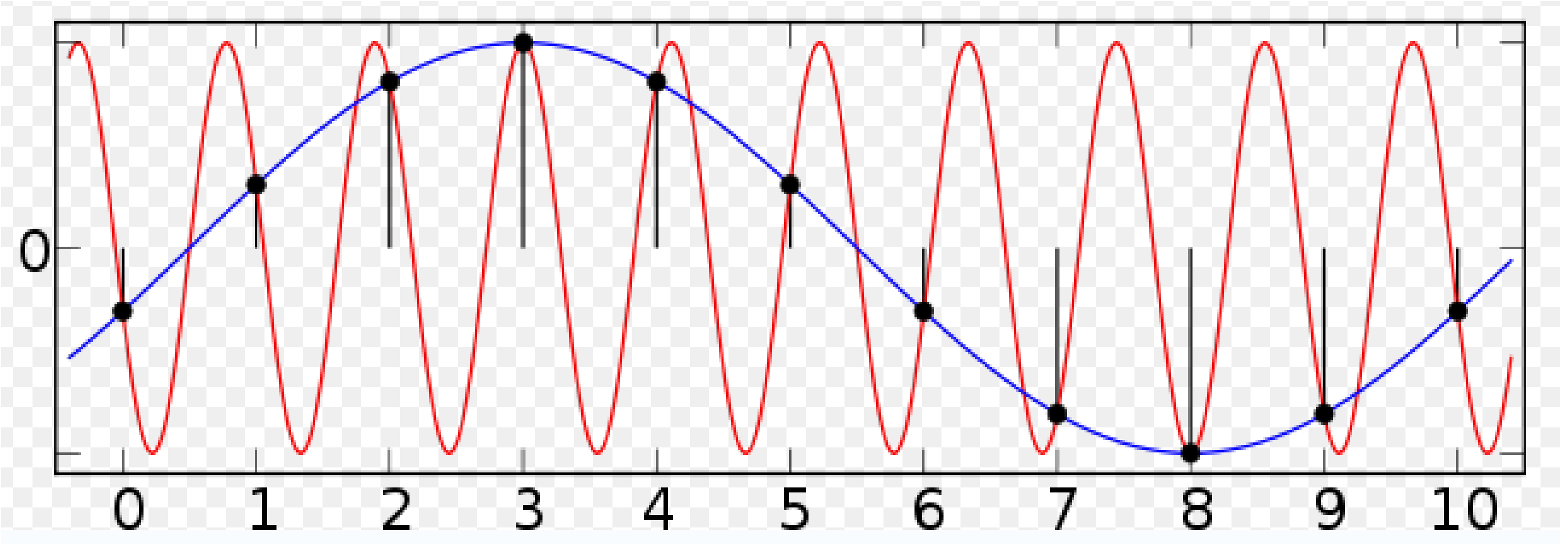
Spectrum of a Sampled Signal - Aliasing

High and low frequency samples are indistinguishable

Results in improper conversion of the input signal

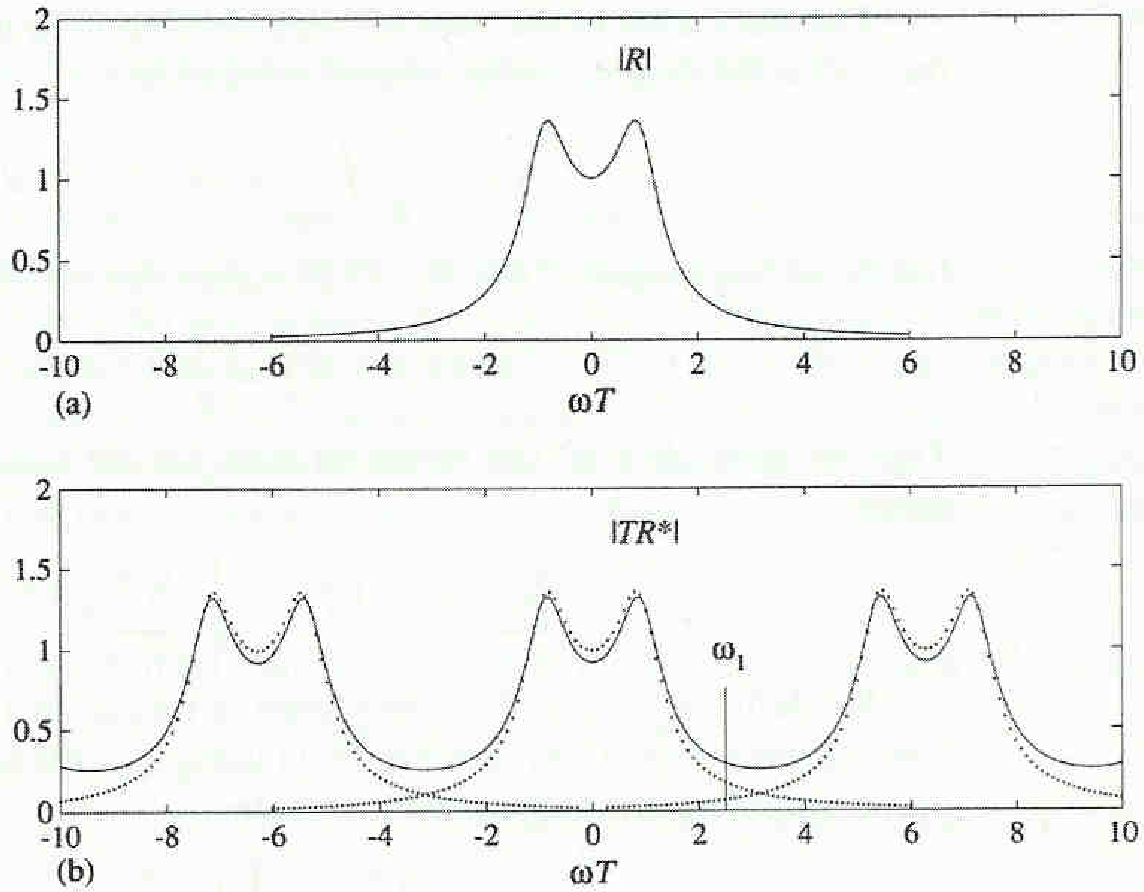
Usually exists when Nyquist Criterion is violated

Prevented through the use of Low-Pass (Anti-aliasing) Filters



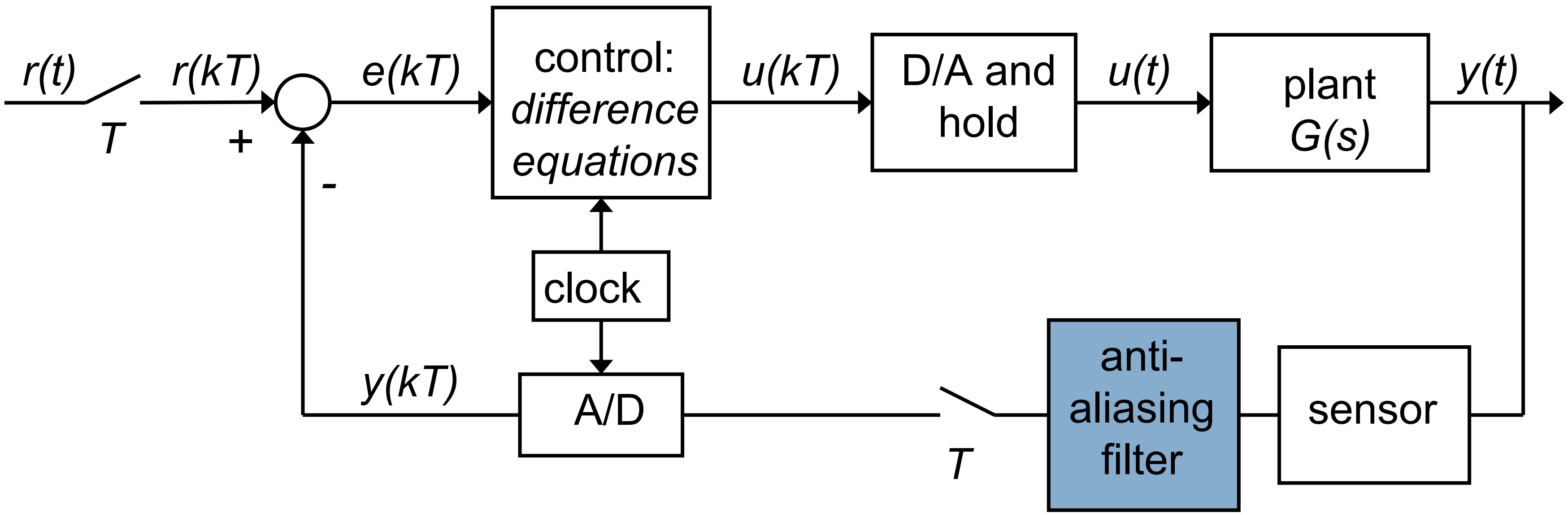
Spectrum of a Sampled Signal - Aliasing

Figure 5.4
(a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing aliasing



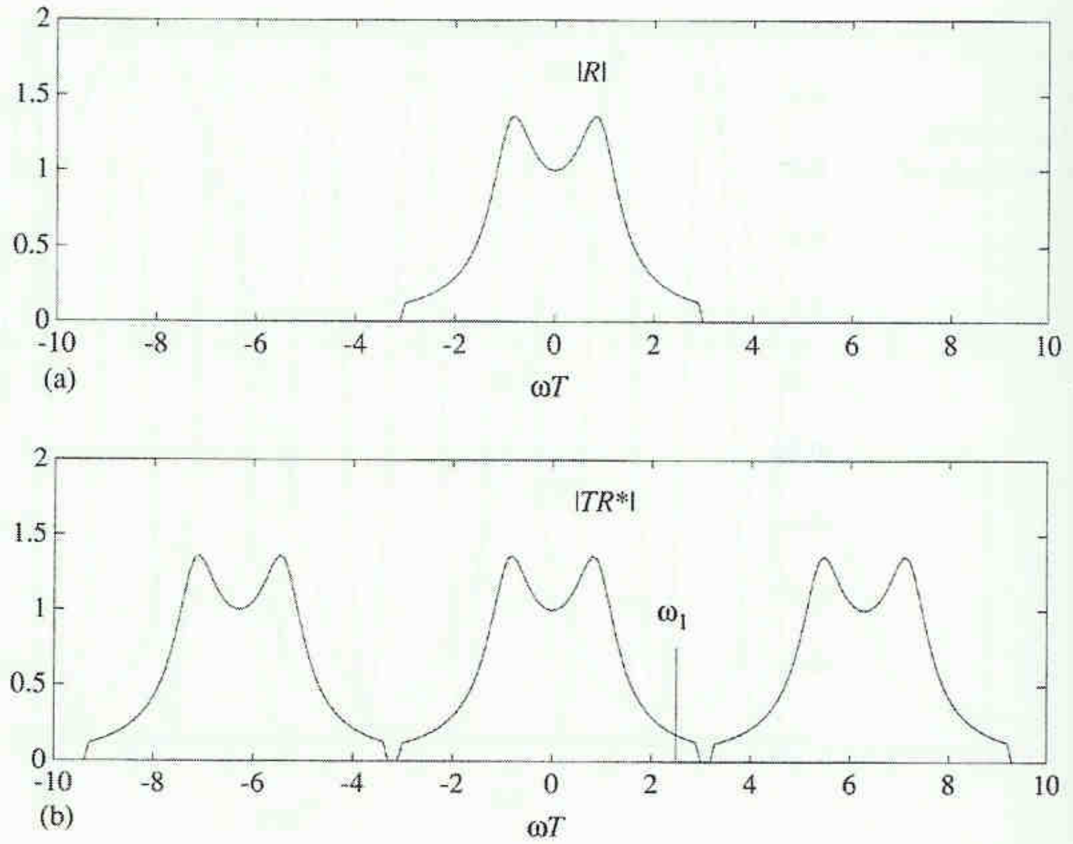
Anti-aliasing filter

Removing (unnecessary) high frequencies through an anti-aliasing filter



Spectrum of a Sampled Signal - Aliasing

Figure 5.6
(a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing removal of aliasing with an antialiasing filter



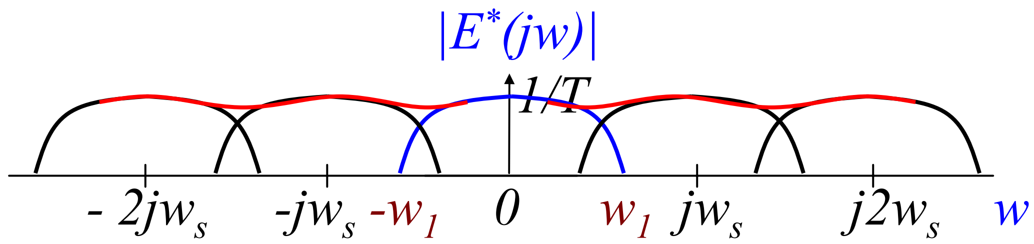
Introduction to digital control

Signal reconstruction through ideal low pass filter

Signal reconstruction through ideal low pass filter

According to the Nyquist theorem, the sampling rate must be at least twice the highest frequency contained in the signal**.**

In theory, an ideal low-pass filter could reconstruct exactly the original signal e(t) from the sampled signal e*(t) if the bandwidth of the filter were ωs/2 and the highest frequency component of e(t) is less than ωs/2. However, since ideal filters do not exist in physically realizable systems, we must employ approximations.



Signal reconstruction through ideal low pass filter

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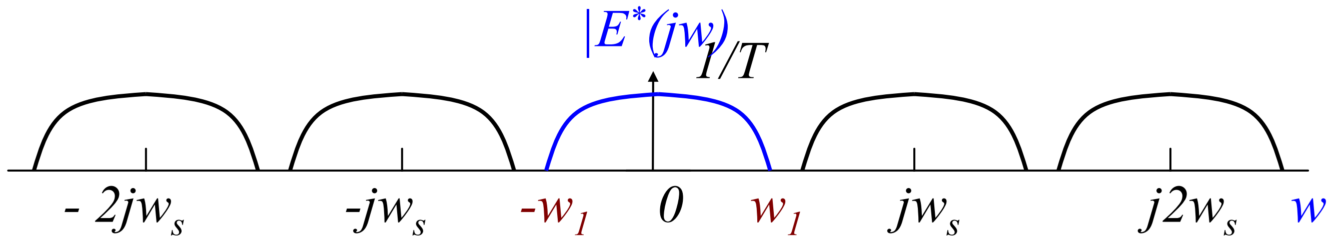
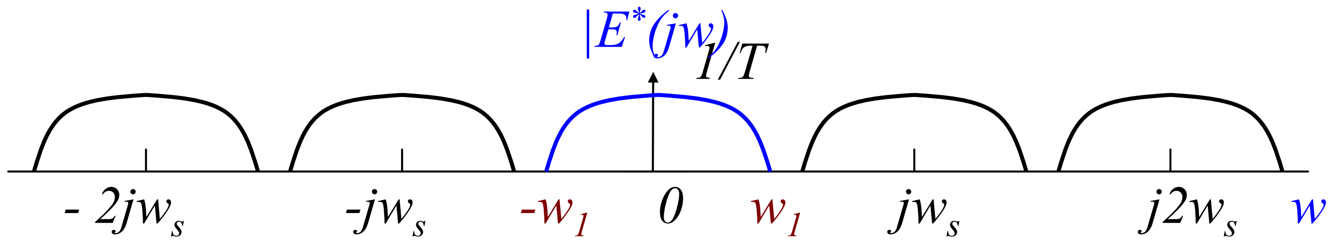
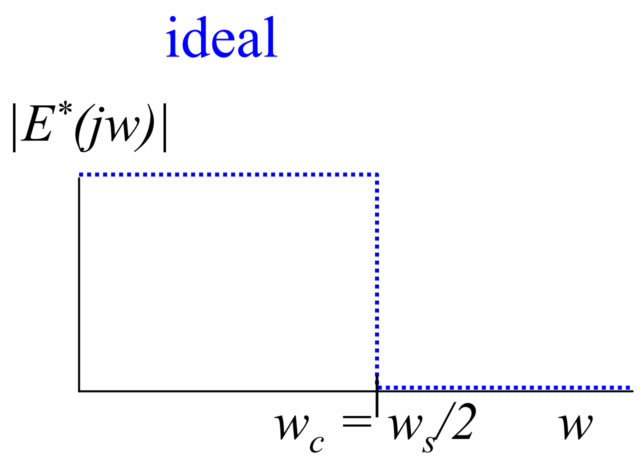
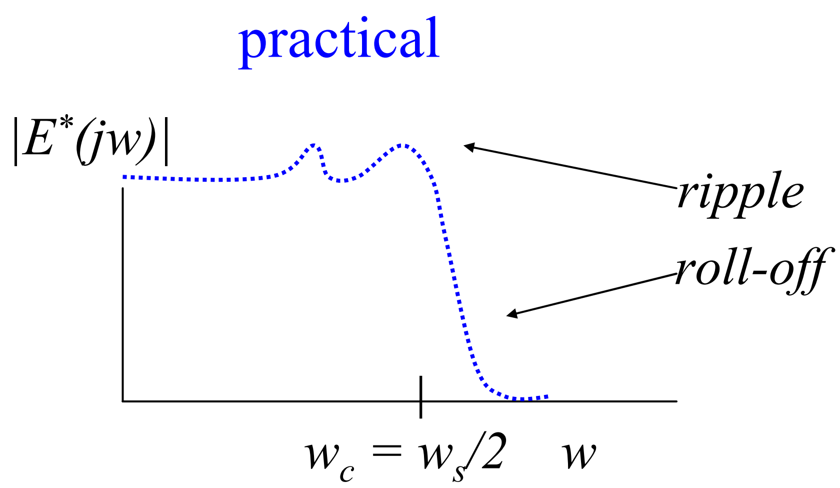
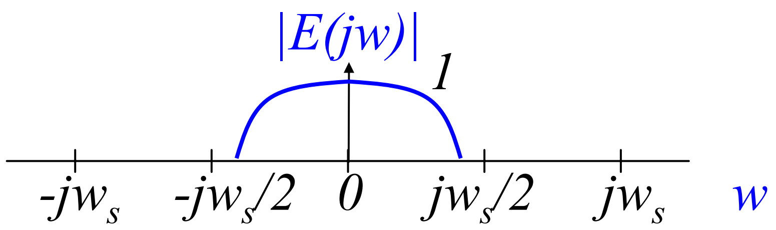


Signal reconstruction through ideal low pass filter









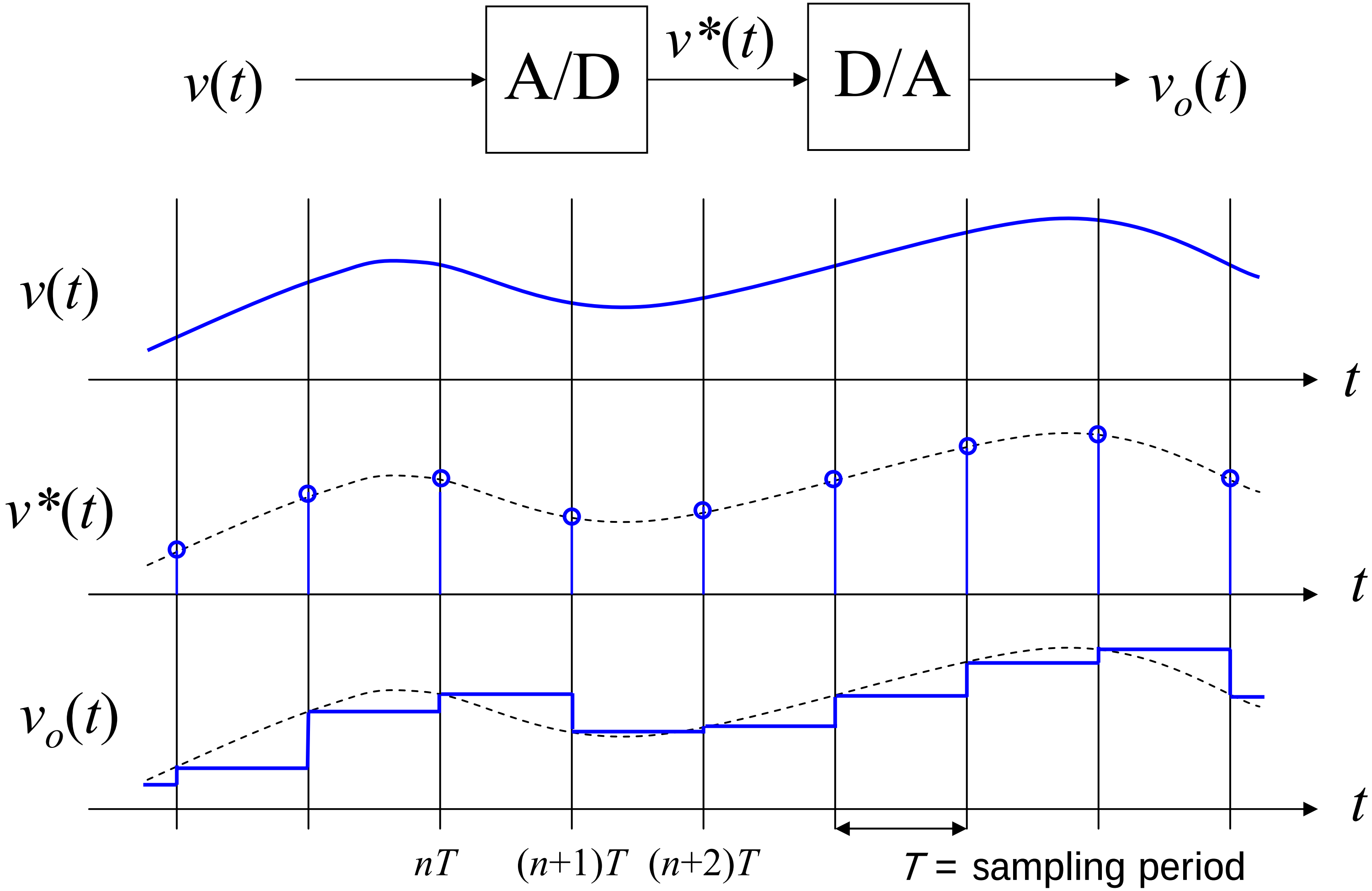
Introduction to digital control

Signal reconstruction through hold devices

Signal reconstruction through hold devices

- digital computer operates on discrete components (not continuous)
- continuous signal *e(t)* must be sampled
- reconstruct discrete approximation of *e(t)* using sampled components
- if the sampling theorem is fulfilled, then e(t) can be recovered with an ideal (LPF)
- since ideal LPF's don't exist, they are approximated using data hold devices
- Data holds are data reconstruction devices that approximate, in some sense, an ideal low-pass filter

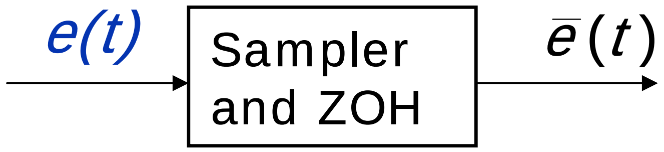
Example: A/D and D/A conversion



T = sampling period

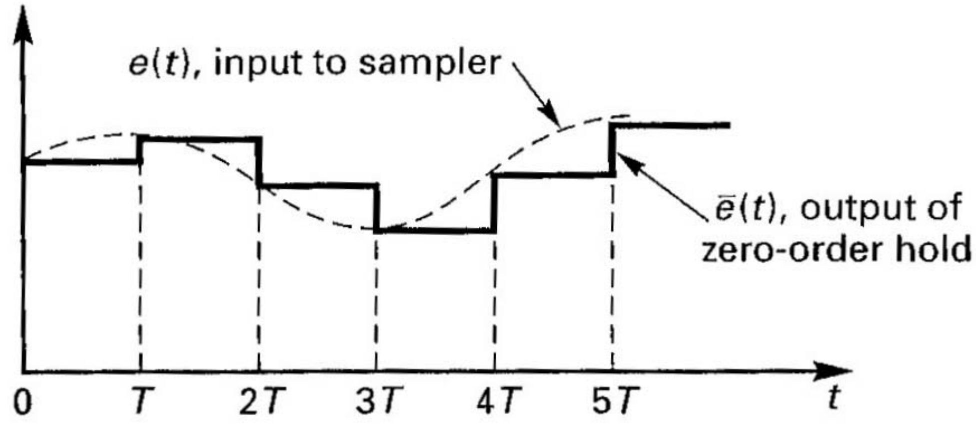
$1/T$ = sampling frequency

The most common and simplest data reconstruction device is the zero-order hold (ZOH). The ZOH clamps the output to a value equal to the input at the sampling instant, i.e.,

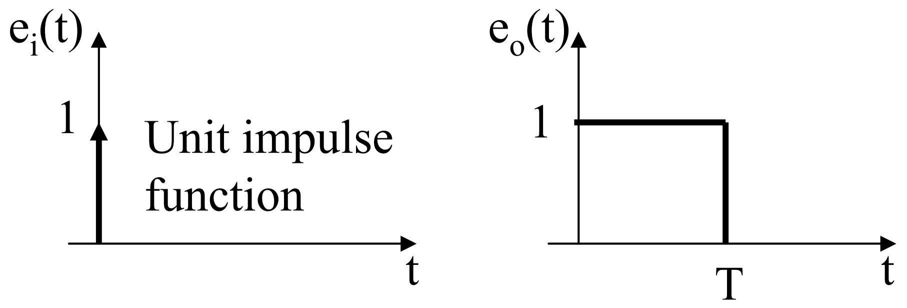


$$\bar{e}(t) = e(nT), \quad nT \le t < (n+1)T$$

The operation of a sampler/ZOH combination is described by the signals shown in the figure.



Let us derive the transfer function of the ZOH. Since the ZOH is preceded by an ideal sampler, the input to the ZOH are only impulse functions. Suppose the input $e_i(t)$ to the ZOH is a unit impulse, the output $e_o(t)$ of the ZOH is therefore



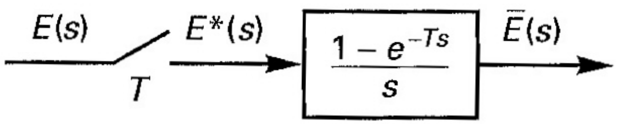
$$e_o(t) = u(t) - u(t-T)$$

Zero-order hold (cont'd)

where u(t) is the unit step function. Taking the Laplace transform of the above expression yields the transfer function of the ZOH, denoted by $G_{h0}(s)$.

$$G_{h0}(s) = \frac{1 - e^{-Ts}}{s}$$

Having derived the TF of the ZOH, we can obtain a mathematical model of the overall sample-hold operation.



where $$E^{*}(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

Frequency response of the ZOH

The frequency response of the ZOH can be derived by replacing s with jω in the TF of the ZOH.

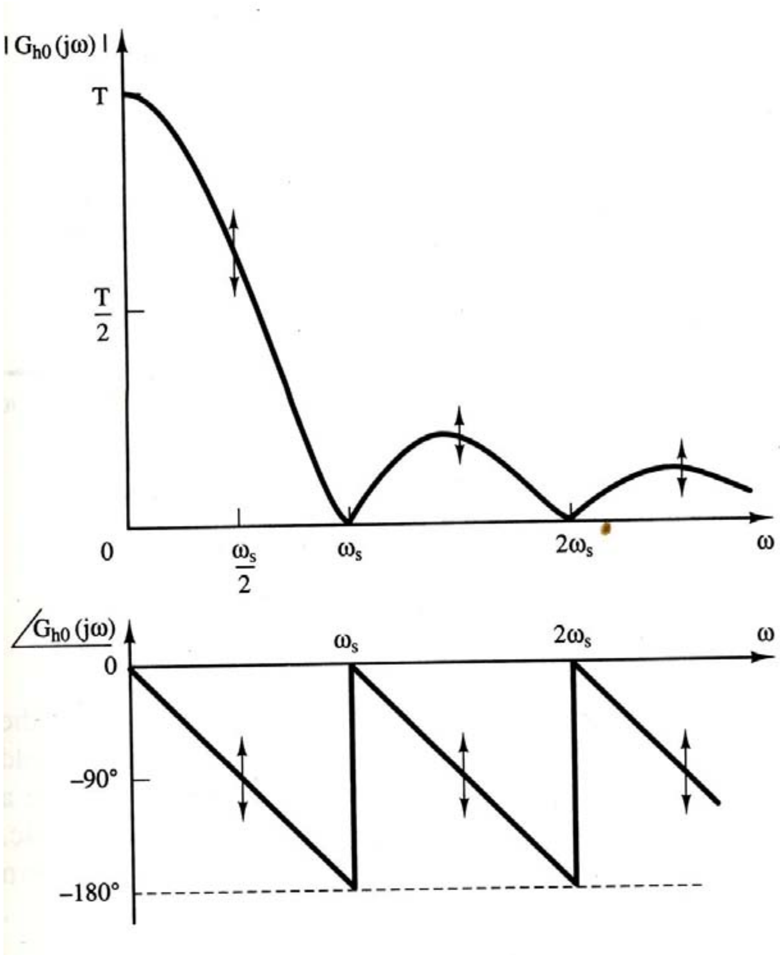
$$G_{h0}(j\omega) = \left.\frac{1-e^{-Ts}}{s}\right|_{s=j\omega} = \frac{1-e^{-j\omega T}}{j\omega} e^{j\omega T/2} e^{-j\omega T/2} =$$

$$= \frac{2e^{-j\omega T/2}}{\omega} \cdot \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} =$$

$$= T\,\frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega T/2} = T\,\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} e^{-j\pi\omega/\omega_s} =$$

$$= T\ \mathrm{sinc}(\omega/\omega_s) e^{-j\pi\omega/\omega_s}$$

Frequency response of the ZOH



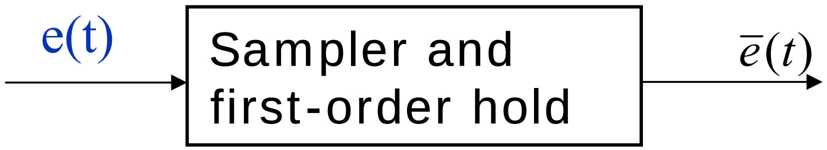
Frequency spectrum of G_{h0} $(j\omega)$.
Note the phase lag introduced by ZOH.
At ω=ωs, the phase angle is -180°, which is an appreciable amount of phase lag.

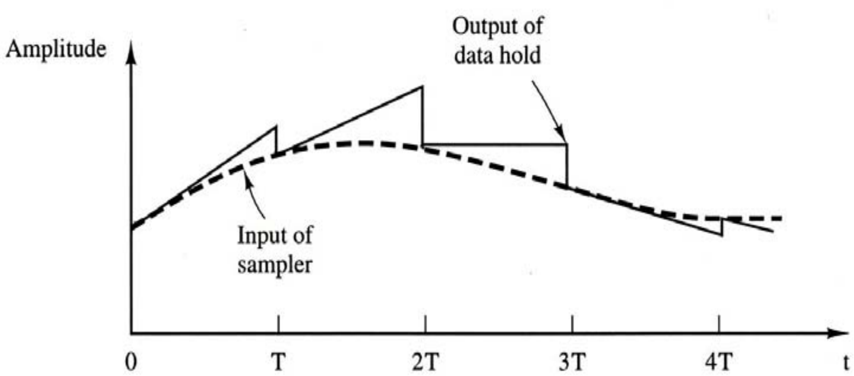
First-order hold

The first-order hold is also a type of date reconstruction device. The operation of the first-order hold can be described by

$$\bar{e}(t) = e(nT) + e'(nT)(t - nT), \quad nT \le t < (n+1)T$$

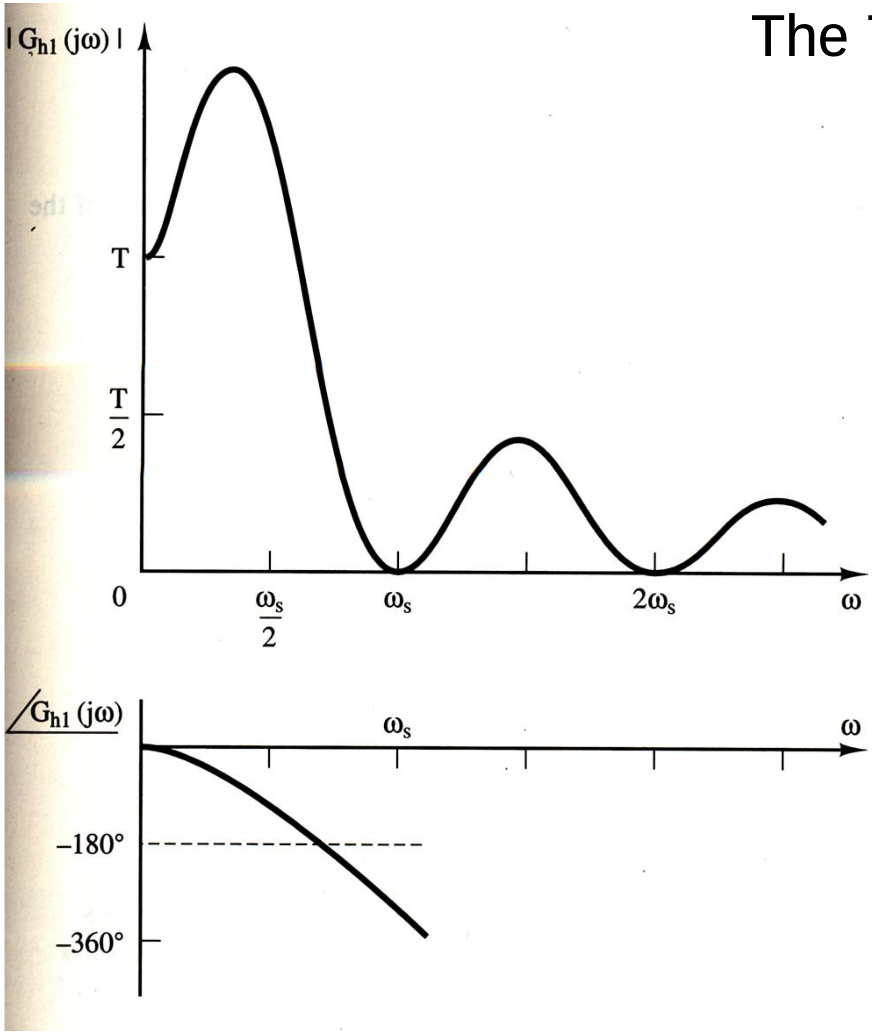
$$e'(nT) = \frac{e(nT) - e\big((n-1)T\big)}{T}$$





which indicates that the extrapolation is a straight line whose slope is determined by the values of the function at the sampling instants in the previous interval.

First-order hold (cont'd)



The TF of the first-order hold, $G_{h1}(s)$, is

$$G_{h1}(s) = \frac{1+Ts}{T}\left(\frac{1-e^{-Ts}}{s}\right)^2$$

The frequency response of the first-order hold is shown in the figure.

Introduction to digital control

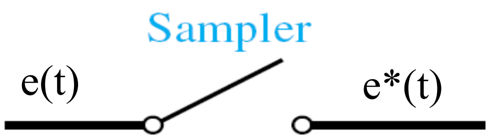
Relation between the Z-transform and the L-transform of the sampled signal

The relationship between E(z) and E*(s)

The z-transform of the number sequence {e(k)} is defined as

$$E(z) = Z\{e(k)\} = \sum_{k=0}^{\infty} e(k) z^{-k}$$

The starred transform for the time function e(t), where e(t) is the input to an ideal sampler, is written as



$$E^*(s) = \sum_{n=0}^{\infty} e(kT) e^{-kTs}$$

If we assume the number sequence {e(k)} is obtained from sampling a time function e(t), **and let $z=e^{Ts}$, then the starred transform becomes the z-transform**:

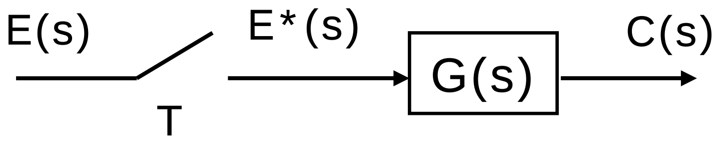
$$E(z) = E^*(s)\Big|_{e^{Ts}=z}$$

Introduction to digital control

The pulse transfer function

Open-loop systems containing samplers and digital filters

The sampled-data system in the figure shows that while the input is sampled, the output C(s) is still a continuous signal. However, since we are satisfied with finding the output **at the sampling instants**, then the **pulse transfer function** comes into play. Of course, we assume that the output is conceptually sampled in synchronisation with the input. Since



$$C(s) = G(s)E^*(s)$$

If we assume that *c(t)* is continuous at all sampling instants, taking the starred transform gives

$$C^*(s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} C(s + jm\omega_s)$$

Pulse transfer function

$$C^*(s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} G(s + jm\omega_s)E^*(s + jm\omega_s)$$

If we recall that $E^*(s)$ is periodic in s with period $j\omega s$, that is

$$E^*(s) = E^*(s + jm\omega_s)$$

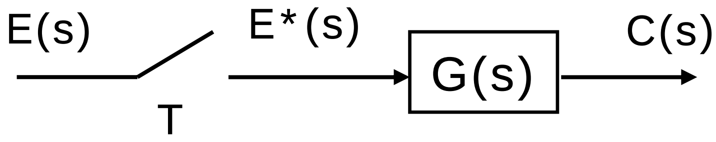
Then we get

$$C^*(s) = E^*(s)\frac{1}{T}\sum_{m=-\infty}^{\infty} G(s + jm\omega_s) = E^*(s)G^*(s)$$

That is

$$C^*(s) = \left(G(s)E^*(s)\right)^* = G^*(s)E^*(s)$$

Pulse transfer function (cont'd)

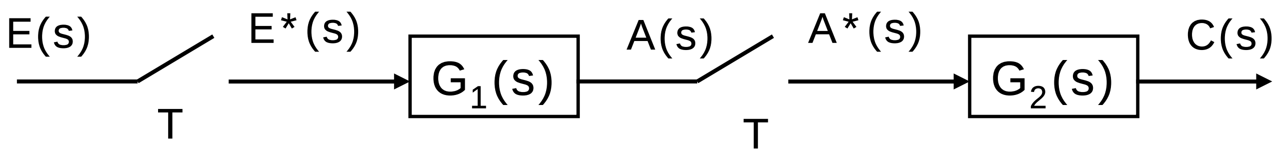


For the open-loop sampled-data system in the figure, based on the relationship between the starred transform and the Z-transform, we have obtained

$$C(z) = G(z)E(z)$$

where the **pulse transfer function G(z)** is the transfer function between the sampled input and the **output at the sampling instants**. Therefore, the pulse transfer function gives no information on the nature of the output c(t) between sampling instants.

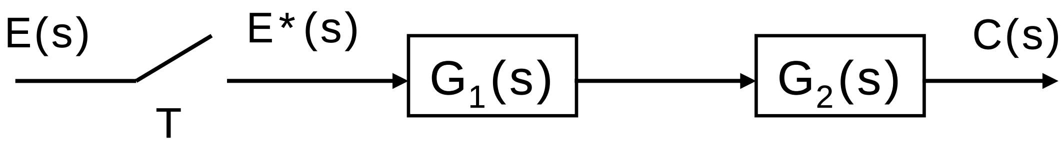
Example 1



$$C(s) = G_2(s)A^*(s) \quad \Rightarrow C^*(s) = G_2^*(s)A^*(s)$$
$$C(z) = G_2(z)A(z)$$
$$A(s) = G_1(s)E^*(s) \quad \Rightarrow A^*(s) = G_1^*(s)E^*(s)$$
$$A(z) = G_1(z)E(z)$$

$$C(z) = G_1(z)G_2(z)E(z)$$

Example 2



$$C(s) = G_1(s)G_2(s)E^*(s) \quad \Rightarrow C(z) = \overline{G_1G_2}(z)E(z)$$

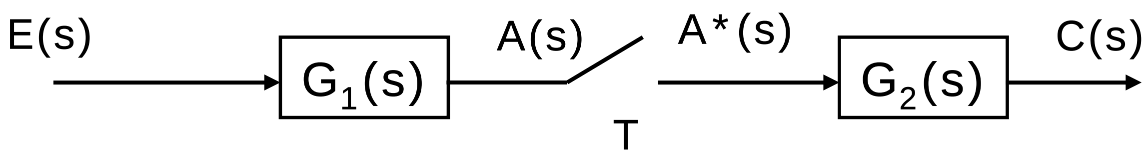
where

$$\overline{G_1G_2}(z) = Z\{G_1(s)G_2(s)\}$$

Note that $\overline{G_1G_2}(z) \neq G_1(z)G_2(z)$

Open-loop sampled-data systems - Examples

Example 3



$$C(s) = G_2(s)A^*(s) \quad \Rightarrow C(z) = G_2(z)A(z)$$

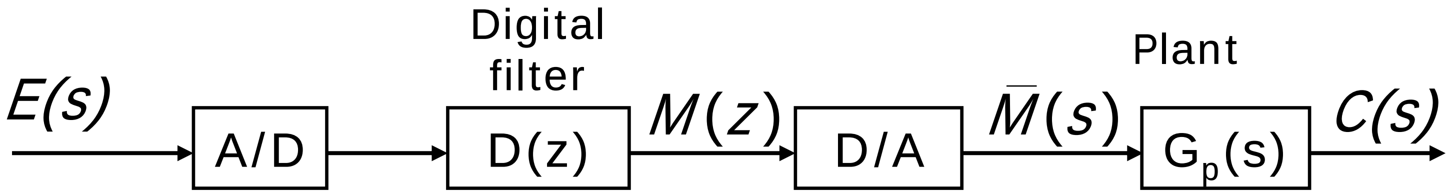
$$A(s) = G_1(s)E(s) \quad \Rightarrow A(z) = \overline{G_1E}(z)$$

$$C(z) = G_2(z)\overline{G_1E}(z)$$

For this system, a pulse transfer function cannot be written.

In general, if the input to a sampled-data system is applied directly to a continuous-time part of the system before being sampled, the z-transform of the system output cannot be expressed as a function of the z-transform of the input signal.

Example 4



The open-loop sampled-data system shown in the above figure contains a digital filter represented by a transfer function D(z). The D/A converter has the characteristics of a ZOH, i.e.,

$$\bar{M}(s) = \frac{1 - e^{-Ts}}{s} M^*(s)$$

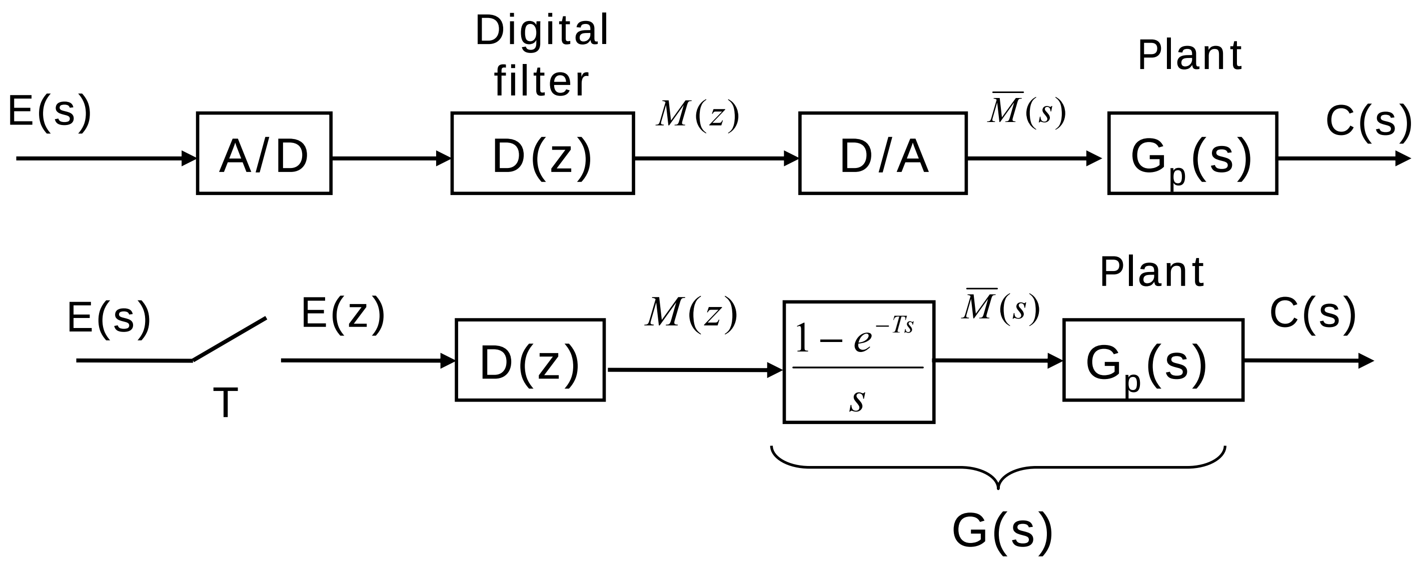
$$C(s) = G_p(s)\bar{M}(s) = G_p(s)\frac{1 - e^{-Ts}}{s} M^*(s)$$

$$C(s) = G_p(s)\frac{1 - e^{-Ts}}{s} D(z)\Big|_{z=e^{Ts}} E^*(s)$$

Open-loop systems containing digital filters-Examples

$$\Rightarrow \quad C(z) = Z\left\{G_p(s)\frac{1-e^{-Ts}}{s}\right\}D(z)E(z) = G(z)D(z)E(z)$$

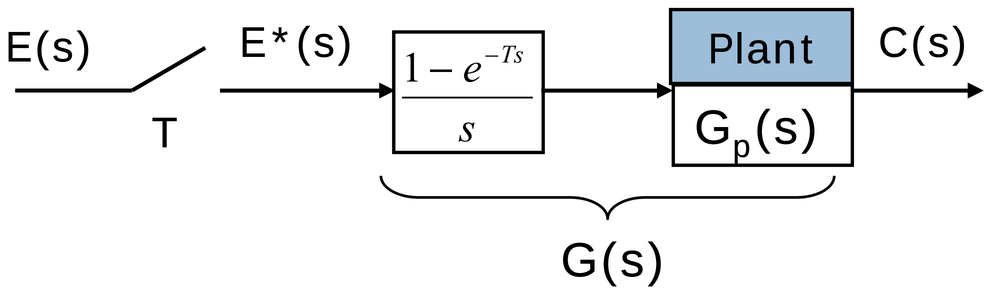
It is clear from the derivation in the previous slide that the combination of the A/D, digital filter, and D/A is accurately modeled by the combination of an ideal sampler, D(z), and ZOH.



Open-loop systems containing digital filters-Examples

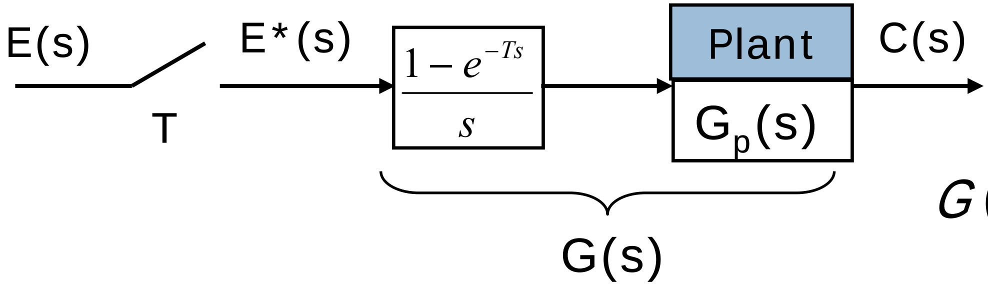
Example 5: Plant with ZOH.

Consider the open-loop sampled-data system in the figure. Find the pulse transfer function between the input E and the output C.



$$G(s) = G_{ho}(s)G_p(s) = \frac{1-e^{-Ts}}{s}G_p(s)$$

Open-loop systems containing digital filters-Examples



$$G(s) = G_{ho}(s)G_p(s) = \frac{1-e^{-Ts}}{s}G_p(s)$$

$$G(z) = Z\{G(s)\} = Z\left\{(1-e^{-sT})\frac{G_P(s)}{s}\right\} = Z\left\{\frac{G_P(s)}{s}\right\} - Z\left\{e^{-sT}\frac{G_P(s)}{s}\right\}$$

$$= Z\left\{\frac{G_P(s)}{s}\right\} - z^{-1}Z\left\{\frac{G_P(s)}{s}\right\} = (1-z^{-1})Z\left\{\frac{G_P(s)}{s}\right\}$$

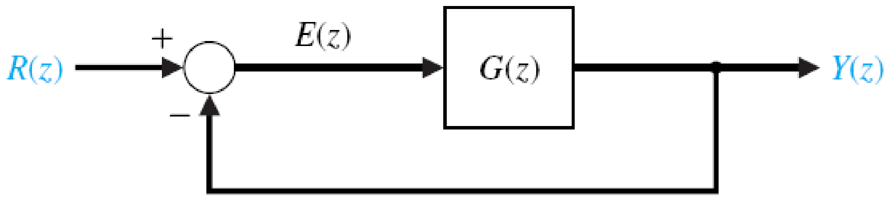
$$G(z) = (1-z^{-1})Z\left\{\frac{G_P(s)}{s}\right\}, \quad \mathrm{z} = \mathrm{e}^{\mathrm{sT}}$$

Introduction to digital control

Closed-loop systems containing samplers and digital filters

Closed-loop sampled-data systems

We have derived the output function C(z) of open-loop sampled-data systems of different configurations. We now derive the output function of closed-loop sampled-data systems. Consider the following simple closed-loop sampled-data system.



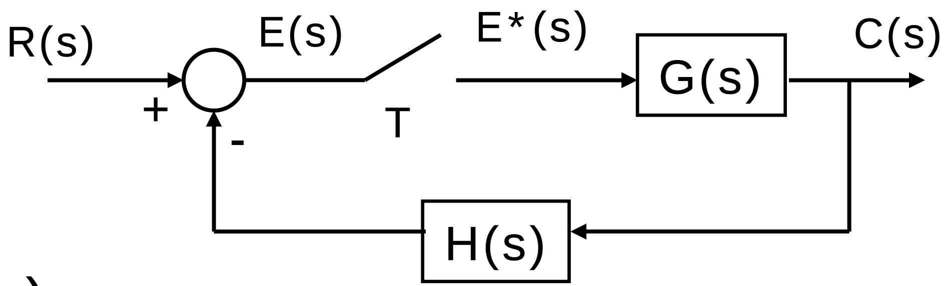
The closed-loop transfer function of the system is

$$G_{ry}(z) = \frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)}$$

$$Y(z) = \frac{G(z)}{1+G(z)} R(z)$$

Closed-loop sampled-data systems (cont'd)

Example 1. Consider the closed-loop sampled-data system:



$$C(s) = G(s)E^*(s)$$

$$E(s) = R(s) - H(s)C(s)$$

$$\Rightarrow E(s) = R(s) - G(s)H(s)E^*(s)$$

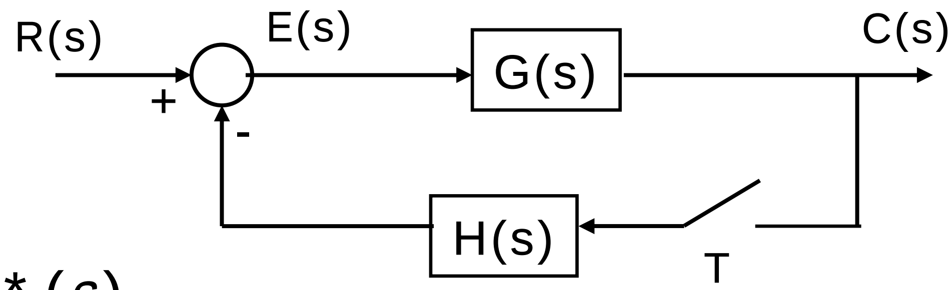
$$E^*(s) = R^*(s) - \overline{GH}^*(s)E^*(s) \quad \Rightarrow E^*(s) = \frac{R^*(s)}{1+\overline{GH}^*(s)}$$

$$C(s) = \frac{G(s)R^*(s)}{1+\overline{GH}^*(s)}$$

$$C^*(s) = \frac{G^*(s)R^*(s)}{1+\overline{GH}^*(s)} \quad \Rightarrow \quad \boxed{C(z) = \frac{G(z)R(z)}{1+\overline{GH}(z)}}$$

Closed-loop sampled-data systems (cont'd)

Example 2. Consider the closed-loop sampled-data system:



$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - H(s)C^*(s)$$

$$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$$

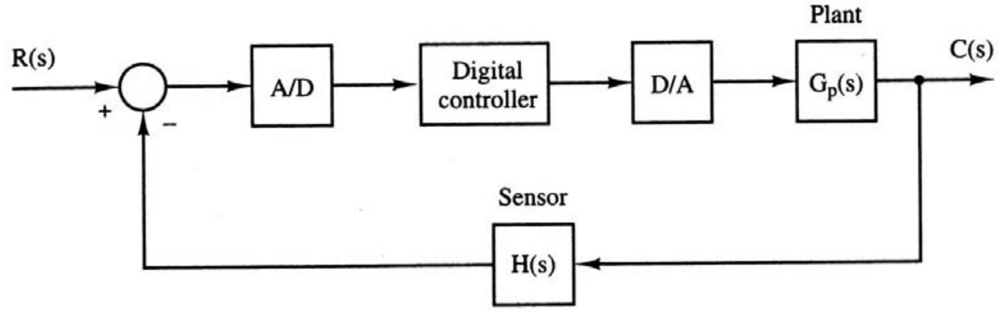
$$C^*(s) = \overline{GR}^*(s) - \overline{GH}^*(s)C^*(s)$$

$$C^*(s) = \frac{\overline{GR}^*(s)}{1+\overline{GH}^*(s)} \quad \Rightarrow \quad \boxed{C(z) = \frac{\overline{GR}(z)}{1+\overline{GH}(z)}}$$

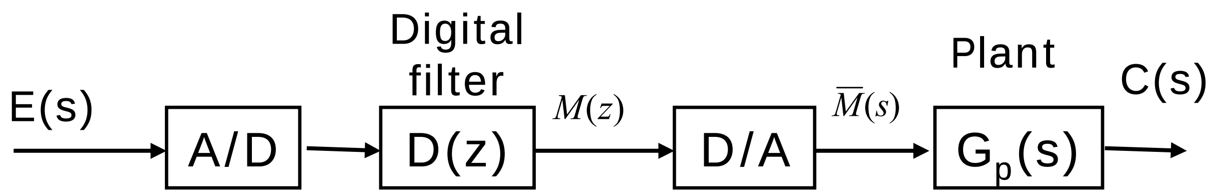
For this system, no transfer function can be derived. Why?

Closed-loop sampled-data systems (cont'd)

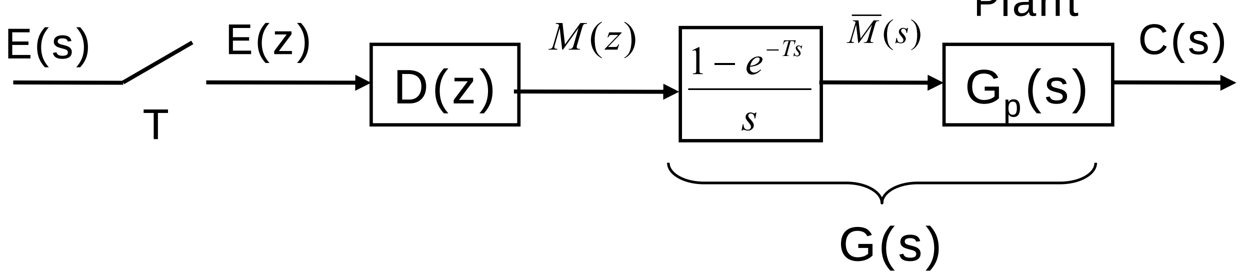
Example 3. Consider the closed-loop sampled-data system:



We have seen that

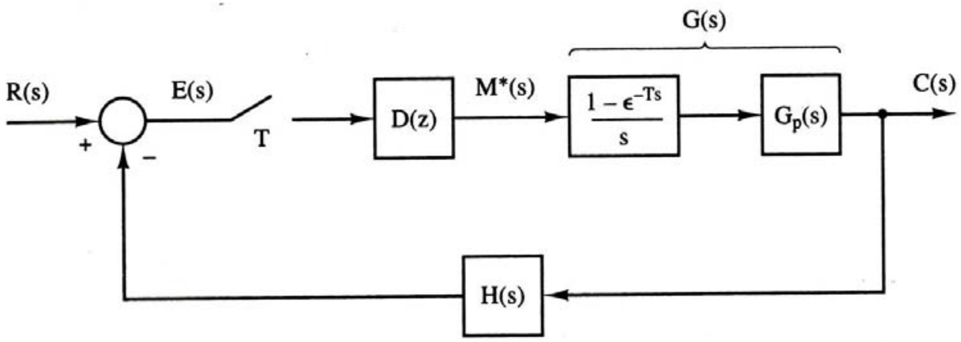


is equivalent to



Closed-loop sampled-data systems (cont'd)

The given system can be modeled by the following system:



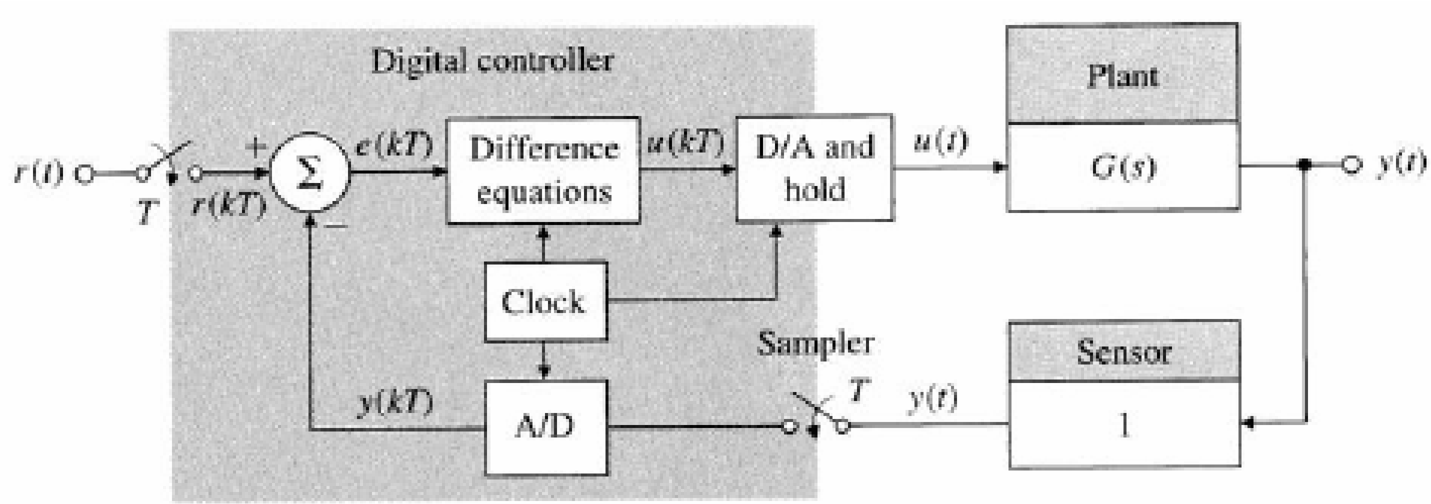
$$E = R - GHD^* E^*, \quad C = GD^* E^*$$

$$E^* = R^* - \overline{GH}^* D^* E^*, \quad C^* = G^* D^* E^*$$

$$C^* = \frac{G^* D^* R^*}{1 + \overline{GH}^* D^*} \quad \Rightarrow C(z) = \frac{D(z)G(z)}{1 + D(z)\overline{GH}(z)} R(z)$$

Closed-loop sampled-data systems (cont'd)

Example 4. Consider the closed-loop sampled-data system:



Derive a model of such a system based on a suitable combination of ideal samplers, D(z), ZOH, G(s), etc.

Introduction to digital control

Sampling rate selection

Sampling rate selection

Choice of a suitable sampling period T

Several aspects have to be taken into account:

- Sampling Theorem

  the sampling frequency must at least equal twice the value of the highest significant frequency in the signal

  → in a feedback system the highest significant frequency of all the signals in the loop is the system bandwidth ω_B , therefore a lower bound for the sampling frequency ω_s is:

$$\omega_s > 2\,\omega_B \rightarrow \omega_s > 4\,\omega_c$$

Choice of a suitable sampling period T

- ZOH filter: the D/A converter introduces a phase lag of

$$G_{ZOH}(s) = \frac{1-e^{-Ts}}{s} \approx \frac{T}{1+sT/2} \qquad \angle G_{ZOH}(j\omega) = -\frac{\omega T}{2}$$

In order to limit the phase lag at the cross-over frequency to small values (e.g. – 6° to – 3°) the following bound should be considered:

$$\angle G_{ZOH}(j\omega) = -\frac{\omega T}{2} = -\frac{\omega\pi}{\omega_s} \quad ; \qquad \omega_s = 2\pi / T$$

$$-6^\circ < -\frac{\omega\pi}{\omega_s} < -3^\circ \rightarrow 0.052 \text{ rad} < \frac{\omega\pi}{\omega_s} < 0.104 \text{ rad}$$

$$\underset{\omega=\omega_c}{\rightarrow} 30\omega_c < \omega_s < 60\omega_c \quad ; \qquad 0.1 < \omega_c T < 0.2$$

Choice of a suitable sampling period T

- Limitations → "small" T
  - improves conversion accuracy
  - decreases destabilizing effects
  - increases the cost of A/D and D/A devices

  a trade-off among such criteria is made for choosing T

  A final practical rule of thumb is to choose T within the interval

$$0.1/\omega_c < T < 0.2/\omega_c$$

provided that the chosen values satisfies HW and cost limitations

Introduction to digital control

Analog to digital conversion methods

Discretization through zero-pole-gain matching

Analog controller discretization can be obtained through the application of the sampling transformation $z = e^{sT}$ to zeros and poles of the analog controller $C_0(s)$

Given the zpk form

$$C_0(s) = K \frac{(s-q_1)(s-q_2)\ldots(s-q_n)}{s^r(s-p_1)(s-p_2)\ldots(s-p_{n-r})}$$

The **matched pole-zero (MPZ)** method: the discretized digital controller $C(z)$ is obtained as:

$$C(z) = K_d \frac{(z-e^{q_1T})(z-e^{q_2T})\ldots(z-e^{q_nT})}{(z-1)^r(z-e^{p_1T})(z-e^{p_2T})\ldots(z-e^{p_{n-r}T})}, K_d = K \frac{\prod_{i=1}^{n} q_i \prod_{i=1}^{n-r}(1-e^{p_iT})}{\prod_{i=1}^{n-r} p_i \prod_{i=1}^{n}(1-e^{q_iT})}$$

Remark: K_d is computed in order to guarantee that $C_0(s)$ and $C(z)$ have the same (generalized) DC gain

Discretization through zero-pole-gain matching

Example: (T = 1 s)

$$C_0(s) = 2\frac{1+\frac{s}{0.1}}{1+\frac{s}{10}} = 200\frac{s+10}{s+0.1}, K_c = \lim_{s\to 0} C_0(s) = 2$$

MPZ : $$C(z) = 21.0157\frac{z-e^{-0.1}}{z-e^{-10}} = 21.0157\frac{z-0.9048}{z-4.54\cdot 10^{-5}}, \lim_{z\to 1} C(z) = 2$$

Introduction to digital control

Digital controllers design through discretization of analog controllers