Automatic Control

Steady state properties of stable LTI systems

Steady state analysis of stable LTI systems

Consider a strictly proper, asymptotically stable and minimal (=> BIBO stable) LTI SISO dynamical system described by the transfer function:

$$H(s) = \frac{N_H(s)}{D_H(s)} = \frac{N_H(s)}{(s-p_1)^{\mu_1}(s-p_2)^{\mu_2}\cdots(s-p_h)^{\mu_h}} = \frac{N_H(s)}{\prod_{i=1}^{n_h}(s-p_i)^{\mu_i}}$$

with n_h distinct poles p_i , $i = 1$, ... , n_h of multiplicity μ_i and such that $\mathbb{R}\mathrm{e}(p_i) < 0, \ \forall i$

For such a system, let's study the zero state output response $y(t)$ in the presence of an input $u(t)$ with Laplace transform given by:

$$U(s) = \frac{N_U(s)}{D_U(s)} = \frac{N_U(s)}{(s-q_1)^{m_1}(s-q_2)^{m_2}\ldots(s-q_{n_u})^{m_u}} = \frac{N_U(s)}{\prod_{j=1}^{n_u}(s-q_j)^{m_j}}$$

Steady state response

$$H(s)=\frac{N_H(s)}{D_H(s)}=\frac{N_H(s)}{(s-p_1)^{\mu_1}(s-p_2)^{\mu_2}\cdots(s-p_h)^{\mu_h}}=\frac{N_H(s)}{\prod_{i=1}^{n_h}(s-p_i)^{\mu_i}}$$

$$U(s)=\frac{N_U(s)}{D_U(s)}=\frac{N_U(s)}{(s-q_1)^{m_1}(s-q_2)^{m_2}\ldots(s-q_{n_u})^{m_u}}=\frac{N_U(s)}{\prod_{j=1}^{n_u}(s-q_j)^{m_j}}$$

For simplicity, suppose that the polynomials $D_H(s)$ and $D_U(s)$ have not common roots

The output can be computed as

$$Y(s)=H(s)U(s)=\frac{N_H(s)}{\prod_{i=1}^{n_h}(s-p_i)^{\mu_i}}\cdot\frac{N_U(s)}{\prod_{j=1}^{n_u}(s-q_j)^{m_j}}$$

Steady state response

$$Y(s) = H(s)U(s) = \frac{N_H(s)}{\prod_{i=1}^{n_h}(s-p_i)^{\mu_i}} \cdot \frac{N_U(s)}{\prod_{j=1}^{n_u}(s-q_j)^{m_j}} =$$

$$= \frac{R_{1,1}}{s-p_1} + \frac{R_{1,2}}{(s-p_1)^2} + \ldots + \frac{R_{1,n_1}}{(s-p_1)^{n_1}} + \frac{R_{2,1}}{s-p_2} + \frac{R_{2,2}}{(s-p_2)^2} + \ldots + \frac{R_{2,n_2}}{(s-p_2)^{n_2}} +$$

$$+ \ldots + \frac{R_{h,1}}{s-p_h} + \frac{R_{h,2}}{(s-p_h)^2} + \ldots + \frac{R_{h,n_h}}{(s-p_h)^{n_h}} +$$

$$+ \frac{Q_{1,1}}{s-q_1} + \frac{Q_{1,2}}{(s-q_1)^2} + \ldots + \frac{Q_{1,m_1}}{(s-q_1)^{m_1}} + \frac{Q_{2,1}}{s-q_2} + \frac{Q_{2,2}}{(s-q_2)^2} + \ldots + \frac{Q_{2,m_2}}{(s-q_2)^{m_2}} +$$

$$+ \ldots + \frac{Q_{n_u,1}}{s-q_{n_u}} + \frac{Q_{n_u,2}}{(s-q_{n_u})^2} + \ldots + \frac{Q_{n_u,m_u}}{(s-q_{n_u})^{m_u}} =$$

$$= \sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j} \frac{R_{j,k}}{(s-p_j)^k} + \sum_{j=1}^{n_q}\sum_{k=1}^{m_j} \frac{Q_{j,k}}{(s-q_j)^k}$$

$$Y(s)=H(s)U(s)=\sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j}\frac{R_{j,k}}{(s-p_j)^k}+\sum_{j=1}^{n_q}\sum_{k=1}^{m_j}\frac{Q_{j,k}}{(s-q_j)^k}$$

$$y(t)=$$

$$\left.\begin{aligned}
&R_{1,1}e^{p_1t}+R_{1,2}te^{p_1t}\ldots+R_{1,n_1}\frac{t^{n_1-1}}{(n_1-1)!}e^{p_1t}+\\
&+R_{2,1}e^{p_2t}+R_{2,2}te^{p_2t}\ldots+R_{2,n_2}\frac{t^{n_2-1}}{(n_2-1)!}e^{p_2t}+\ldots\\
&+R_{h,1}e^{p_ht}+R_{h,2}te^{p_ht}\ldots+R_{h,n_h}\frac{t^{n_h-1}}{(n_h-1)!}e^{p_ht}
\end{aligned}\right\}\text{system natural modes}$$

$$\left.\begin{aligned}
&+Q_{1,1}e^{q_1t}+Q_{1,2}te^{q_1t}+\ldots+Q_{1,m_1}\frac{t^{m_1-1}}{(m_1-1)!}e^{q_1t}+\\
&+Q_{2,1}e^{q_2t}+Q_{2,2}te^{q_2t}+\ldots+Q_{2,m_2}\frac{t^{m_2-1}}{(m_2-1)!}e^{q_2t}+\\
&+\ldots+Q_{n_q,1}e^{q_{n_u}t}+Q_{n_q,2}te^{q_{n_u}t}+\ldots+Q_{n_u,m_q}\frac{t^{m_q-1}}{(m_q-1)!}e^{q_{n_u}t}=
\end{aligned}\right\}\text{"input modes"}$$

$$=\sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j}R_{j,k}\frac{t^{k-1}}{(k-1)!}e^{p_jt}+\sum_{j=1}^{n_u}\sum_{k=1}^{m_j}Q_{j,k}\frac{t^{k-1}}{(k-1)!}e^{q_jt}$$

$$Y(s) = H(s)U(s) = \frac{N_H(s)}{\prod_{i=1}^{n_h}(s-p_i)^{\mu_i}} \cdot \frac{N_U(s)}{\prod_{j=1}^{n_u}(s-q_j)^{m_j}}$$

$$Y(s) = \sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j}\frac{R_{j,k}}{(s-p_j)^k} + \sum_{j=1}^{n_q}\sum_{k=1}^{m_j}\frac{Q_{j,k}}{(s-q_j)^k}$$

$$y(t) = \sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j} R_{j,k}\frac{t^{k-1}}{(k-1)!}e^{p_j t} + \sum_{j=1}^{n_q}\sum_{k=1}^{m_j} Q_{j,k}\frac{t^{k-1}}{(k-1)!}e^{q_j t}$$

i.e. the output response y(t) is a linear composition of the:

- system natural modes
- input modes

$$y(t) = \underbrace{\sum_{j=1}^{n_h}\sum_{k=1}^{\mu_j} R_{j,k}\frac{t^{k-1}}{(k-1)!}e^{p_j t}}_{y_{tr}(t)} + \underbrace{\sum_{j=1}^{n_q}\sum_{k=1}^{m_j} Q_{j,k}\frac{t^{k-1}}{(k-1)!}e^{q_j t}}_{y_{ss}(t)}$$

The output response is the sum of two contributions.

The first one (i.e. $y_{tr}(t)$) depends on the system natural modes only, while the second (i.e. $y_{ss}(t)$) depends on the input modes only

Recalling that the system has been assumed asymptotically stable

(i.e. $\mathbb{R}\mathrm{e}(p_j) < 0$, $j = 1, \ldots, n_h$), then, when $t \to \infty$:

1. $y_{tr}(t) \to 0$ (all the system natural modes are convergent)
   $y_{tr}(t) \to$ **transient response**
2. $y(t) \to y_{ss}(t)$ (i.e. for sufficiently high time instant, the system output coincides with $y_{ss}(t)$)
   $y_{ss}(t) \to$ **steady-state response**

Remark

In the presence of common roots in the polynomials $D_H(s)$ and $D_U(s)$, the output response is the sum of three terms. In particular:

1. a term that includes all the poles of $H(s)$ that are not poles of $U(s)$ → **transient response**
2. a "mixed term" that includes the common poles of $H(s)$ and $U(s)$ → **transient response** (since such poles have strictly negative real part)
3. a term that includes all the poles of $U(s)$ that are not poles of $H(s)$ → **steady-state response**

Thus, the output response can be still represented as the sum of a transient response $y_{tr}(t)$ such that $y_{tr}(t) \to 0$ and a steady state response $y_{ss}(t)$ such that $y(t) \to y_{ss}(t)$

Example 1

$$H(s) = \frac{1}{s^2 + s + 1} = \frac{1}{(s + 0.5 - j0.866)(s + 0.5 + j0.866)}$$

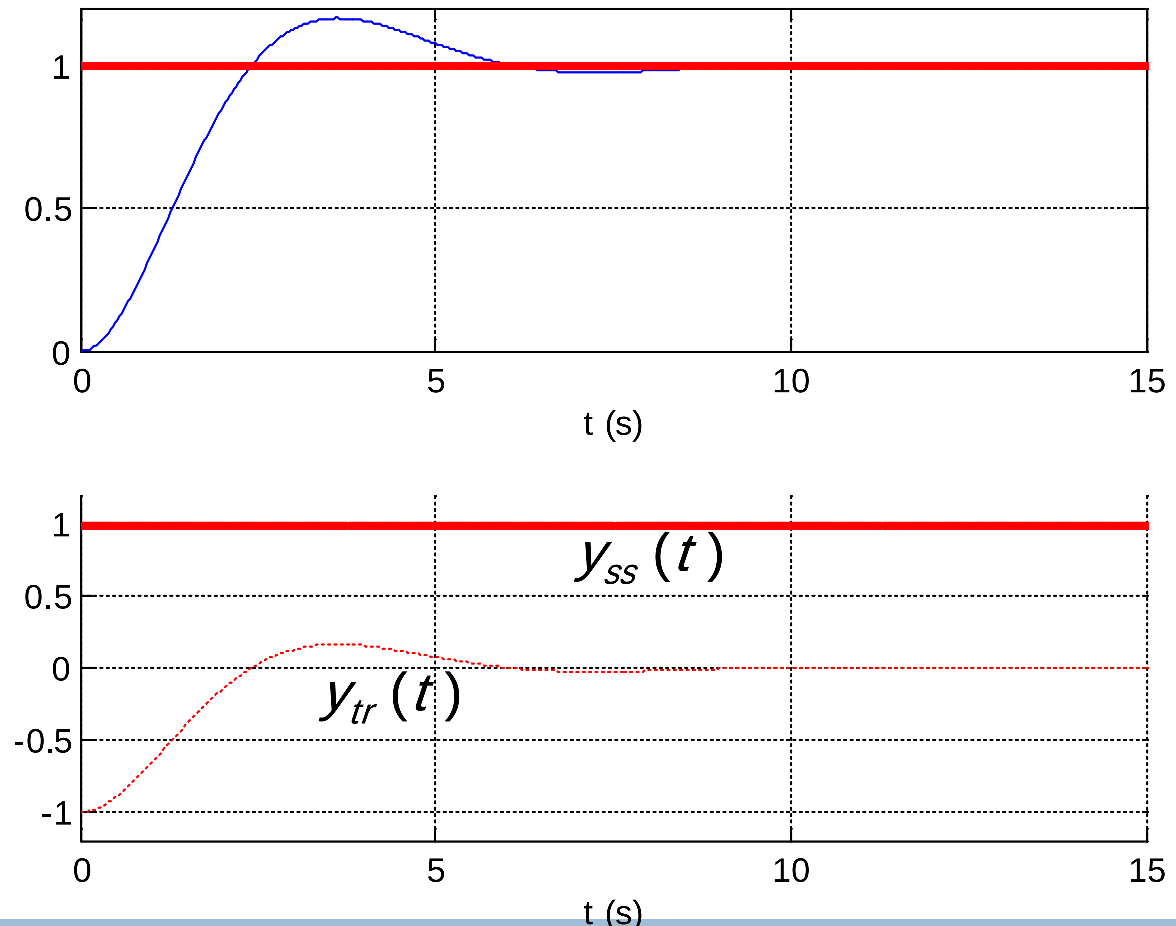
$$U(s) = \frac{1}{s}, (u(t) = \varepsilon(t))$$

$$Y(s) = H(s)U(s) = \frac{1}{s(s + 0.5 - j0.866)(s + 0.5 + j0.866)} =$$

$$= \underbrace{\frac{-0.5 + j0.2887}{(s + 0.5 - j0.866)} + \frac{-0.5 - j0.2887}{(s + 0.5 + j0.866)}}_{Y_{tr}(s)} + \underbrace{\frac{1}{s}}_{Y_{ss}(s)}$$

$$y(t) = \underbrace{1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\varepsilon(t)}_{y_{tr}(t)} + \underbrace{\varepsilon(t)}_{y_{ss}(t)}$$

$$y(t) = \boxed{1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\varepsilon(t)} + \boxed{\varepsilon(t)}$$



Example 2

$$H(s)=\frac{1}{s^2+s+1}=\frac{1}{(s+0.5-j0.866)(s+0.5+j0.866)}$$

$$U(s)=\frac{1.5}{s^2+1.5^2},(u(t)=\sin(1.5t))$$

$$Y(s)=H(s)U(s)=\frac{1}{(s+0.5-j0.866)(s+0.5+j0.866)(s-j1.5)(s+j1.5)}=$$

$$=\underbrace{\frac{-0.1967+j0.1639}{(s+0.5-j0.866)}+\frac{-0.1967-j0.1639}{(s+0.5+j0.866)}}_{Y_{tr}(s)}+\underbrace{\frac{0.1967-j0.3975}{(s-j1.5)}+\frac{0.1967+j0.0.3975}{(s+j1.5)}}_{Y_{ss}(s)}$$

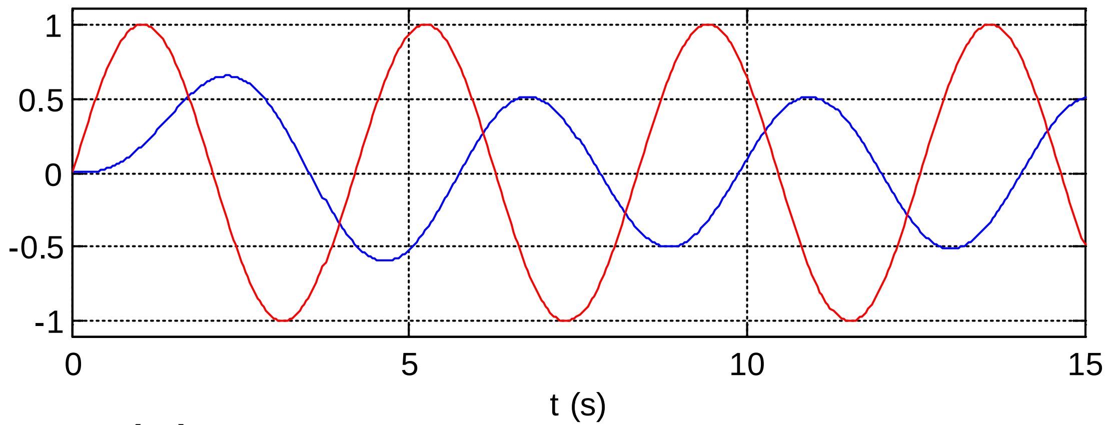
$$y(t)=\underbrace{0.8871e^{-0.5t}\cos\left(0.866t-1.1113\right)\varepsilon(t)}_{y_{tr}(t)}+$$

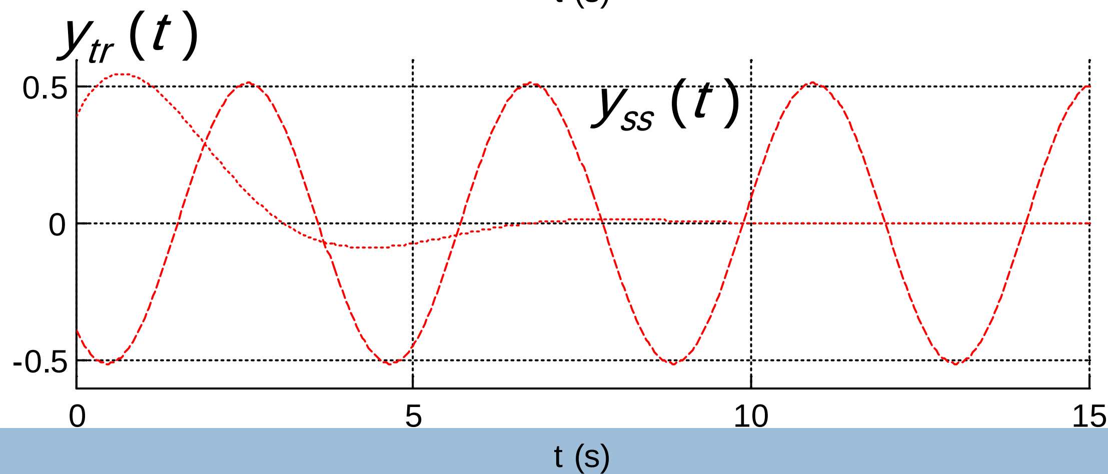
$$+\underbrace{0.5121\cos\left(1.5t+2.4469\right)\varepsilon(t)}_{y_{ss}(t)}$$

Example 2

$u(t) = \sin(1.5t)$

$$y(t) = \boxed{0.8871 e^{-0.5t} \cos\left(0.866t - 1.1113\right) \varepsilon(t)} + \boxed{0.5121 \cos\left(1.5t + 2.4469\right) \varepsilon(t)}$$





Steady state properties of LTI stable systems

Steady-state properties of an LTI system concern the time behavior of the output (and state) for sufficiently high time values

LTI asymptotically stable system have the property that, for the class of input signals with real rational Laplace transform such as exponential functions, the output (and state) response at steady state (i.e. for sufficiently high time values) has the same shape as the input (except for scaling factors and time shifts)

Special cases of input signals belonging to the considered class, such as:

- step signals
- harmonic signals (sinusoid, ...)

and their linear combinations are of practical interest for the analysis and design of control systems

Steady-state response: step input

Let

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

be the transfer function (tf) of a stable LTI system.

The steady-state response $y_{ss}(t)$ due to a step input signal of amplitude $\bar{u}$:

$$u(t) = \bar{u}\,\varepsilon(t) \rightarrow U(s) = \frac{\bar{u}}{s}$$

is given by $y_{ss}(t) = \bar{y}\,\varepsilon(t) \rightarrow \bar{y} \underset{\substack{\uparrow \\ \text{final value theorem}}}{=} \bar{u} \cdot \lim_{s \to 0} sY(s) = \bar{u} \cdot \lim_{s \to 0} \cancel{s} H(s) \frac{1}{\cancel{s}} =$

$$= \bar{u} \cdot H(0) = \bar{u} \cdot \frac{b_0}{a_0}$$

Example

A minimal LTI system is described by the tf:

$$H(s) = \frac{1}{(s+2)(s+10)}$$

compute, if possible, the steady state output response in the presence of a step input with amplitude 2

The poles of $H(s)$ are -2 e -10
$\Rightarrow$ the system is BIBO stable
$\Rightarrow$ the system is asymptotically stable
$\Rightarrow$ the steady state response can be computed

Since $u(t) = \bar{u}\,\varepsilon(t) = 2\varepsilon(t) \Rightarrow$ the steady state response is:

$$y_{ss}(t) = \bar{y}\,\varepsilon(t) = H(0)\,\bar{u}\,\varepsilon(t) = \frac{1}{20} 2\,\varepsilon(t) = 0.1\,\varepsilon(t)$$

- Evaluation with Matlab of $\lim_{s \to 0} H(s)$ for a stable tf
- Statement **`dcgain`**:

```
>> s=tf('s');
>> H=1/((s+2)*(s+10));
>> K = dcgain(H)
K = 0.05
```

- The statement **`dcgain`** can also be used to compute the generalized steady-state gain (generalized dc-gain) $K = \lim_{s \to 0} s^r H(s)$
- Consider for example the tf $H(s) = \dfrac{1}{s(s+2)^2}, r = 1$

```
>> s=tf('s');
>> H=1/(s*(s+2)^2);
>> K = dcgain(s*H)
K = 0.25
```

Steady-state response: sinusoidal input

Let

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

be the tf of a stable LTI system.

The steady-state response $y_{ss}(t)$, $t \geq 0$, due to a sinusoidal input signal of amplitude $\bar{u}$ and frequency ω_0:

$$u(t) = \bar{u} \sin(\omega_0 t)$$

is given by(*)

$$y_{ss}(t) = \bar{y}(\omega_0) \sin(\omega_0 t + \varphi(\omega_0))$$

$$\rightarrow \bar{y}(\omega_0) = \bar{u} \cdot \left| H(j\omega_0) \right|, \varphi(\omega_0) = \angle(H(j\omega_0))$$

(*) for a complete proof see, e.g., *N. S. Nise, "Control System Engineering", 5th Edition, Wiley, 2008*

Example

A minimal LTI system is described by the tf:

$$H(s) = \frac{1}{(s+2)(s+10)}$$

compute, if possible, the steady state output response in the presence of a sinusoidal input $u(t) = 2\sin(0.5t)\,\varepsilon(t)$

The system is asymptotically stable (see Example 1)
$\Rightarrow$ the steady state response can be computed

Since $u(t) = \bar{u}\sin(\omega_0 t)\,\varepsilon(t) = 2\sin(0.5t)\,\varepsilon(t)$
$\Rightarrow$ The steady state response is:

$$y_{ss}(t) = \bar{y}\sin(\omega_0 t + \varphi)\,\varepsilon(t) = \bar{y}\sin(0.5t + \varphi)\,\varepsilon(t)$$

$$\bar{y} = \left|H(j\omega_0)\right| \cdot \bar{u} = 2\left|H(j0.5)\right|$$

$$\varphi = \arg H(j\omega_0) = \arg H(j0.5)$$

Example

$$y_{ss}(t) = \bar{y}\sin(\omega_0 t + \varphi)\,\varepsilon(t) = \bar{y}\sin(0.5t + \varphi)\,\varepsilon(t)$$

$$\bar{y} = \left|H(j\omega_0)\right| \cdot \bar{u} = 2\left|H(j0.5)\right|$$

$$\varphi = \arg H(j\omega_0) = \arg H(j0.5)$$

$$\left|H(j0.5)\right| = \left|(j0.5+2)(j0.5+10)\right|^{-1} = \left|j0.5+2\right|^{-1}\left|j0.5+10\right|^{-1} =$$

$$= \left(\sqrt{0.5^2+2^2}\sqrt{0.5^2+10^2}\right)^{-1} = \left(\sqrt{4.25}\sqrt{100.25}\right)^{-1} = 0.0484$$

$$\arg H(j0.5) = \arg\left(1 \big/ \left[(j0.5+2)(j0.5+10)\right]\right) =$$

$$= \arg(1) - \arg(j0.5+2) - \arg(j0.5+10) =$$

$$= 0 - \arctan(0.5/2) - \arctan(0.5/10) = -0.2949\,\text{rad}$$

- Evaluation of magnitude and phase of a tf with MatLab

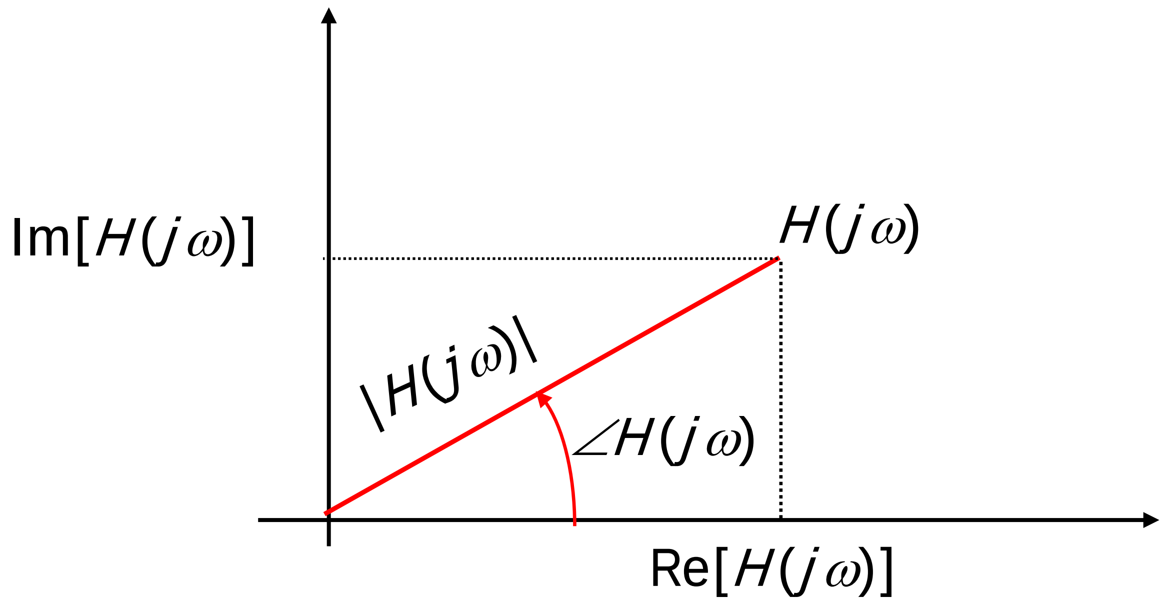
•Command **bode**

```
>> s=tf('s');
>> H=1/((s+2)*(s+10));
>> w0=0.5;
>> [m,f]=bode(H,w0)
m =
    0.0484
f =
  -16.8986
>> f_rad=f/180*pi
f_rad =
   -0.2949
```

The magnitude value is **not** expressed in dB, but in linear scale, the phase value is expressed in degrees

Frequency response function

The function $H(j\omega) : \mathbb{R}^+ \to \mathbb{C}$ of the variable $\omega \in \mathbb{R}^+$ is referred to as the **frequency response function** of the system:



$$H(j\omega) = \mathrm{Re}[H(j\omega)] + \mathrm{Im}[H(j\omega)] \to$$ **Cartesian representation**

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)} \to$$ **Polar representation**