
In-Sample and Out-of-Sample Sharpe Ratios for Linear Predictive Models

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Abstract

We study how much the in-sample performance of trading strategies based on linear predictive models is reduced out-of-sample due to overfitting. More specifically, we compute the in- and out-of-sample means and variances of the corresponding PnLs and use these to derive a closed-form approximation for the corresponding Sharpe ratios. We find that the out-of-sample “replication ratio” diminishes for complex strategies with many assets and based on many weak rather than a few strong trading signals, and increases when more training data is used. The substantial quantitative importance of these effects is illustrated with an empirical case study for commodity futures following the methodology of Gârleanu and Pedersen [GP13].

1 Introduction

A standard approach to build systematic trading strategies is to first fit some predictive model to historical training data. For example, Gârleanu and Pedersen [GP13] regress price changes of commodity futures against past returns with various lookback periods. Using the forecasting model, one can then form a trading strategy, e.g., by combining forecasts for several assets using mean-variance analysis, and “backtest” its performance on historical data. The strategy’s “out-of-sample” performance on new testing data or in live trading will typically be worse than its “in-sample” performance on the training data for a number of reasons. But how much worse?

A number of empirical studies finds that the out-of-sample performance of ETFs, anomalies and various other strategies tends to be significantly lower than the corresponding in-sample performance, with estimates ranging widely depending on the dataset [BL15; Wie+16; MP16; SLP17; FRT22].¹ One commonly used rule of thumb is to haircut the in-sample performance by 50% [HL15]. While such rules of thumb are useful, we seek a more systematic approach, enabling us to estimate the out-of-sample performance of different trading strategies based on their characteristics, and without having to resort to a new large-scale simulation study each time the strategy is modified.

To this end, we focus on one crucial issue in this context: *overfitting*.² Kan, Wang, and Zheng [KWZ24] obtain analytical results in this spirit for the simplest setting where investment opportunities are constant, in that expected asset returns and covariances do not change over time. When these model parameters are estimated from a finite training dataset, Kan, Wang, and Zheng

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¹ More generally, as in many other quantitative disciplines [Ioa05], there is a controversial ongoing debate whether there is a “replication crisis” in financial economics [HLZ16; JKP23].

² Another key driver of lower out-of-sample performance is *multiple testing* studied by [BL14; HL15]. While multiple testing may lead to overfitting, we focus on overfitting from a single test perspective.

derive explicit formulas for the “replication ratio”, that is, the fraction of the in-sample Sharpe ratio that can be recovered out-of-sample.

In the present paper, we extend this analysis to a setting closer to that of Gârleanu and Pedersen and many related papers, where trading strategies exploit several trading signals per asset and also take into account that these signals are not fully persistent but fluctuate over time. More specifically, we assume that expected returns are linear functions of iid trading signals drawn from a multivariate normal distribution. When the variance of this distribution tends to zero, the constant-parameter model of Kan, Wang, and Zheng is recovered as a special case. Conversely, our iid expected returns correspond to the limiting case of short-lived signals in the setting of Gârleanu and Pedersen.

The relationship between signals and expected returns can be estimated empirically by linear regression. For portfolios corresponding to this plug-in estimate, we compute the (unconditional) in- and out-of-sample means and variances of the P&L and derive closed-form approximations for the replication ratio, i.e., the fraction of the in-sample Sharpe ratio that can be achieved out-of-sample.

We find that as in Kan, Wang, and Zheng [KWZ24] the replication ratio is increasing in the size of the training dataset and the magnitude of the “true” Sharpe ratio of the strategy (i.e., the exact unconditional Sharpe ratio computed in the model rather than based on estimated parameters). Indeed, when the true Sharpe ratio is low the strategies are more prone to overfitting, and small increases in the true Sharpe ratio yield significant improvements in the replication ratio. Once the true Sharpe ratio is sufficiently large, the model is more robust to overfitting and the replication ratio slowly converges towards 100%.

Conversely, the replication ratio is decreasing in the number of model parameters. In Kan, Wang, and Zheng [KWZ24] this quantity only depends on the number of assets; in our setting, it is determined both by the number of assets and by the trading signals per asset. The takeaway message is that complex strategies are more prone to overfitting, which needs to be balanced with sufficiently long backtesting periods.

The practical relevance of these theoretical results is in turn explored in an empirical case study that follows the implementation in [GP13]. To wit, we fit a linear predictive model to a set of commodity futures forecast using momentum-style signals, and in turn use this as a simulation engine. We test our approach on simulations with both Gaussian and fat-tailed returns, and for both iid and autoregressive signals. Our analytical approximations provide very good predictions in the Gaussian iid setting where they were derived, but also perform encouragingly well with autoregressive signals driven by heavy-tailed noise, confirming the practicality of our results in realistic settings.

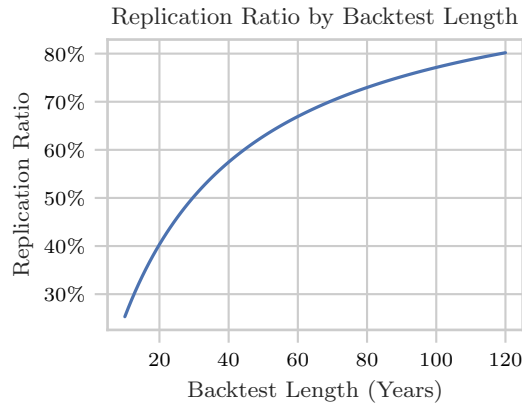


Figure 1: The replication ratio for the case study in Section 5 with varying backtest length.

We observe that a ten-year backtest yields an expected replication ratio of just 30% (Figure 1). This steep haircut arises from the large number of parameters in the model; it can be reduced by

imposing constraints on the model, such as regularisation or a panel regression as in the empirical section of [GP13].

In summary, our results suggest the following broad practical conclusions: be wary of low Sharpe ratio strategies, keep your models as simple as possible, and use the longest sensible backtest period available. While much of this advice aligns with practitioners' intuition, our results allow one to pinpoint how much of a haircut one should expect out-of-sample depending on the characteristics of the backtest, rather than broad rules of thumb. Additionally, estimating the out-of-sample haircut by simulation is often numerically slow. In contrast, our closed-form expressions permit to accelerate this time-sensitive process by instantly computing the expected out-of-sample haircuts.

The remainder of this paper is organized as follows. Section 2 sets up the model and differentiates between the true and the sample Sharpe ratios. Section 3 presents the main results and their interpretation. Section 4 compares our results to the existing literature on overfitting. Section 5 applies our results in an empirical case study using commodity futures, and Section 6 concludes. Appendix A and C provide derivations of results used in the main text, Appendix B provides the general case of the main result, and Appendix D provides details of the Monte-Carlo analysis we use to check the approximations used.

2 Model Setup

We consider a researcher who wants to fit a linear model to forecast asset returns and to trade based on these predictions. The researcher has access to some historical data observed at discrete times $\mathcal{T}_1 = \{0, \dots, T_1\}$. We denote the returns per time step by $\mathbf{r}_{t+1} \in \mathbb{R}^m$ and the predictive signals by $\mathbf{s}_t \in \mathbb{R}^p$ for each $t \in \mathcal{T}_1$. The researcher assumes that the relationship between the signals and the returns follows a linear model with parameter $\beta \in \mathbb{R}^{m \times p}$,

$$\mathbf{r}_{t+1} = \beta \mathbf{s}_t + \epsilon_{t+1},$$

where $\epsilon_{t+1} \in \mathbb{R}^m$ denotes the unexplained residual returns. We assume that the sequences $\{\mathbf{s}_t\}$ and $\{\epsilon_{t+1}\}$ are independent of each other, and iid across time with distributions \mathbf{s} and ϵ respectively, which are assumed to be multivariate Gaussian:³

$$\mathbf{s} \sim \mathcal{N}(\mu_s, \Sigma_s) \quad \text{and} \quad \epsilon \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon).$$

We note here that trading signals are typically assumed to be autoregressive as in [GP13]. However, imposing the iid assumption as in [DL18] makes the calculations below substantially more tractable, and we demonstrate using simulation experiments in Section 5 that the resulting formulas also provide very good approximations for autoregressive signals fitted using the methodology from the empirical part of [GP13].

We impose without loss of generality that the signals are centred, except for a possible intercept term, i.e. $\mu_s = (\mu_{s,1}, 0, \dots)$ where $\mu_{s,1} = 1$ if the model includes a constant (intercept) term, and $\mu_{s,1} = 0$ otherwise; removing the intercept simplifies many of the expressions, but we leave the expressions general as including a drift term may be desirable for some applications such as long-only equity portfolios.

Assuming β is known, the researcher can use the signal \mathbf{s}_t at time t to accurately predict the expected returns $\mathbb{E}_t[\mathbf{r}_{t+1}] = \beta \mathbf{s}_t$ conditional on the information at time t , and in turn trade a portfolio of the form $\mathbf{w}_t = \mathbf{Z} \beta \mathbf{s}_t$ where $\mathbf{Z} \in \mathbb{R}^{m \times m}$ is some symmetric weight matrix. For example, the standard (one-period) Markowitz portfolio is given by $\mathbf{w}_t = \Sigma_t^{-1} \mu_t$ where Σ_t and μ_t denote the covariance matrix and expected returns of the traded assets conditional on the information available at time t . In the present context, these are given by

$$\mathbb{E}_t[\beta \mathbf{s}_t + \epsilon_{t+1}] = \beta \mathbf{s}_t, \quad \text{Cov}_t[\beta \mathbf{s}_t + \epsilon_{t+1}] = \Sigma_\epsilon,$$

and thus the one-step Markowitz optimal portfolio is of the form $\mathbf{w}_t = \mathbf{Z} \beta \mathbf{s}_t$ with $\mathbf{Z} = \Sigma_\epsilon^{-1}$. Somewhat less common but still relevant choices of \mathbf{Z} include $\text{diag}(\Sigma_\epsilon)^{-1}$, corresponding to an

³ We focus on the Gaussian case as it simplifies the exposition in many places. One can compute the true Sharpe ratio when the signals/noise are non-Gaussian, but we omit it here as the expression is verbose and broadly similar.

inverse variance allocation by asset, or \mathbf{I}_m , corresponding to a risk indifferent portfolio. The researcher can then test this portfolio against the returns to compute the Profit & Loss per time step,

$$\text{PnL}_t = \mathbf{w}_t^\top \mathbf{r}_{t+1}.$$

2.1 The True Sharpe Ratio

The Sharpe ratio typically refers to a summary statistic that compares the sample mean of a time series of returns to their sample standard deviation. In the model from the previous section, we can compare this to its “true” equivalent, defined as the ratio of the expected return and standard deviation of the daily P&L, averaged over the distributions of the signal and noise terms:

$$\text{SR} := \frac{\mathbb{E}[\text{PnL}_t]}{\sqrt{\mathbb{V}[\text{PnL}_t]}} = \frac{\mathbb{E}[\mathbf{w}_t^\top \mathbf{r}_{t+1}]}{\sqrt{\mathbb{V}[\mathbf{w}_t^\top \mathbf{r}_{t+1}]}} = \frac{\mathbb{E}[(\mathbf{Z}\boldsymbol{\beta}\mathbf{s}_t)^\top (\boldsymbol{\beta}\mathbf{s}_t + \boldsymbol{\epsilon}_{t+1})]}{\sqrt{\mathbb{V}[(\mathbf{Z}\boldsymbol{\beta}\mathbf{s}_t)^\top (\boldsymbol{\beta}\mathbf{s}_t + \boldsymbol{\epsilon}_{t+1})]}}. \quad (2.1)$$

Using identities for quadratic expectations of Gaussian vectors, this can be computed as

$$\text{SR} = \frac{\text{tr}(\mathbf{G}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\mu}_s}{\sqrt{2 \text{tr}((\mathbf{G}\boldsymbol{\Sigma}_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\Sigma}_s \mathbf{G}\boldsymbol{\mu}_s + \text{tr}(\mathbf{F}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{F}\boldsymbol{\mu}_s}}, \quad (2.2)$$

where

$$\mathbf{F} := \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta} \quad \text{and} \quad \mathbf{G} := \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\beta}.$$

See Appendix A for the derivation of this expression. To provide some intuition for this result, suppose for simplicity that the signals include no intercept ($\boldsymbol{\mu}_s = 0$). Then, (2.2) simplifies to

$$\text{SR} = \frac{\text{tr}(\boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\beta}\boldsymbol{\Sigma}_s)}{\sqrt{2 \text{tr}((\boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\beta}\boldsymbol{\Sigma}_s)^2) + \text{tr}(\boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}\boldsymbol{\Sigma}_s)}}.$$

In this form, clearly the expected return (numerator) increases with the Frobenius norm $\|\boldsymbol{\beta}\|_F$ (which measures the “predictability” of future asset returns), however the variance (denominator) also increases. For a single asset or signal, the Sharpe ratio is monotonically increasing in $\|\boldsymbol{\beta}\|_F$, but with multiple assets or signals the relationship is no longer obvious. We also see that the Sharpe ratio increases as the signal-to-noise ratio rises with smaller $\|\boldsymbol{\Sigma}_\epsilon\|_F$. Similarly to $\boldsymbol{\beta}$, the relationship with the signal variance $\boldsymbol{\Sigma}_s$ is generally not monotone, but for parameters similar to our empirical case study in Section 5 the Sharpe ratio typically increases when $\|\boldsymbol{\Sigma}_s\|_F$ increases relative to $\|\boldsymbol{\Sigma}_\epsilon\|_F$, again indicating an improvement in the signal-to-noise ratio of the model.

2.2 The In-Sample Sharpe Ratio

When a given predictive model is deployed in practice, its parameters $\boldsymbol{\beta}$ must be estimated. In the linear model we consider here, this can be done using ordinary least-squares (OLS) by stacking the returns across all time steps into the matrix $\mathbf{R} \in \mathbb{R}^{m \times T_1}$ and the signals across all time steps into the matrix $\mathbf{S} \in \mathbb{R}^{p \times T_1}$ and computing the OLS estimator

$$\hat{\boldsymbol{\beta}} = \mathbf{R}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}. \quad (2.3)$$

Although this is the best estimator for $\boldsymbol{\beta}$ in many senses (unbiased, L^2 optimal, consistent, etc.), it is also naturally overfit to the particular realizations of the signal and returns in the in-sample period, which will clearly differ in the out-of-sample period.

Moreover, using this estimate for $\boldsymbol{\beta}$ for backtesting on the same in-sample dataset naturally induces “look-ahead bias”, as data from the full sample period has been used to estimate the model parameters which are then used to test the trading strategy on the same period.⁴

⁴ The look-ahead bias could be avoided using a walk-forward regression as in [Jou+24], but this does not avoid the issue of overfitting.

To determine the magnitude of these effects, we can also stack the residuals ϵ_{t+1} into the matrix $\mathbf{E} \in \mathbb{R}^{m \times T_1}$ and with a substitution of the model for the returns into (2.3) we obtain

$$\hat{\beta} = \beta + \mathbf{E}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}.$$

This expression enables us to decompose the estimated regression parameter into the true regression parameter plus some sampling error. If the researcher trades the portfolio $\hat{\mathbf{w}}_t = \mathbf{Z}\hat{\beta}_t$ using this estimate, the corresponding P&L per time step is $\widehat{\text{PnL}}_t = \hat{\mathbf{w}}_t^\top \mathbf{r}_{t+1}$. The sample average and variance in the training data set in turn are

$$\hat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}] := \frac{1}{T_1} \sum_{t=0}^{T_1-1} \widehat{\text{PnL}}_t, \quad (2.4)$$

$$\hat{\mathbb{V}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}] := \frac{1}{T_1 - 1} \sum_{t=0}^{T_1-1} \left(\widehat{\text{PnL}}_t - \hat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}] \right)^2. \quad (2.5)$$

Together, these lead to the in-sample Sharpe ratio

$$\text{SR}_{\text{IS}} := \frac{\hat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}]}{\sqrt{\hat{\mathbb{V}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}]}.$$

The issue here is that $\hat{\beta}$ has been estimated using the same returns observations \mathbf{r}_{t+1} which the model is then tested against, i.e. the model has been fit to the specific realization of the noise in the historical period. Expanding out the sample P&L,

$$\begin{aligned} \widehat{\text{PnL}}_t &= \hat{\mathbf{w}}_t^\top \mathbf{r}_{t+1} = (\Sigma_\epsilon^{-1} \hat{\beta}_t \mathbf{s}_t)^\top (\beta \mathbf{s}_t + \epsilon_{t+1}) \\ &= (\Sigma_\epsilon^{-1} \beta \mathbf{s}_t)^\top \beta \mathbf{s}_t + (\Sigma_\epsilon^{-1} \beta \mathbf{s}_t)^\top \epsilon_{t+1} \leftarrow \text{Truth} \\ &+ \left(\Sigma_\epsilon^{-1} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \right)^\top \beta \mathbf{s}_t \leftarrow \text{Misestimation} \\ &+ \left(\Sigma_\epsilon^{-1} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \right)^\top \epsilon_{t+1} \leftarrow \text{Overfitting} \end{aligned}$$

We observe that by using $\hat{\beta}$ instead of β we obtain a combination of the true P&L and some additional P&L as a result of the estimation error in β . If \mathbf{E} (the stacked matrix of residuals) is independent of the other terms then as $\mathbb{E}[\mathbf{E}] = \mathbf{0}$ the “Misestimation” and “Overfitting” terms will have zero expectation. If we use $\hat{\beta}$ to test the strategy in the in-sample period, then ϵ_{t+1} is a column of \mathbf{E} , and thus they are not independent, leading to overfitting, this is how the look ahead bias discussed earlier inflates the in-sample performance. In the out-of-sample period the “overfitting term vanishes” (because the new independent realization of the noise are independent) but the misestimation term is still present there as the estimation error has a non-trivial variance.

2.3 The Out-of-Sample Sharpe Ratio

After estimating the predictive model on the training dataset, the researcher deploys the model out-of-sample at discrete trading times $\mathcal{T}_2 = \{T_1 + 1, \dots, T_1 + T_2 + 1\}$. The corresponding sample mean and variance then lead to the out-of-sample Sharpe ratio

$$\text{SR}_{\text{OOS}} := \frac{\hat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2}]}{\sqrt{\hat{\mathbb{V}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2}]}.$$

The difference, of course, is that in the future period $\hat{\beta}$ is not fit to the particular observations of the returns and signals, and thus the out-of-sample Sharpe ratio will typically be lower than the

in-sample Sharpe ratio previously calculated. We also see from the expressions derived below that the out-of-sample Sharpe ratio will be lower than the true Sharpe ratio due to misestimation of the regression parameter causing increased volatility over trading with the true β .

The questions therefore are: How much lower? And what can be done to mitigate this bias?

3 Main Results

The unconditional means of the in-sample and out-of-sample Sharpe ratio, $\mathbb{E}[\text{SR}_{\text{IS}}]$ and $\mathbb{E}[\text{SR}_{\text{OOS}}]$, cannot be computed in closed form. We can however compute the expected in-sample and out-of-sample expected return and variance. For clarity of exposition, we present here our main result for a special (but extremely common) case, taking $\mathbf{Z} = \boldsymbol{\Sigma}_\epsilon^{-1}$, $\mu_s = \mathbf{0}$ and $\boldsymbol{\Sigma}_s = \mathbf{I}_p$, i.e. the Markowitz portfolio for a set of centred standardised signals. The general version of this result is reported in Appendix B.

Proposition 3.1. *The expected average in-sample P&L and out-of-sample P&L is given by*

$$\begin{aligned}\mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\widehat{\text{PnL}}_t \right)_{t \in \mathcal{T}_1} \right] \right] &= \text{tr}(\mathbf{\Gamma}) + \frac{pm}{T_1}, \\ \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\widehat{\text{PnL}}_u \right)_{u \in \mathcal{T}_2} \right] \right] &= \text{tr}(\mathbf{\Gamma}).\end{aligned}$$

The expected variance of the in-sample and out-of-sample P&L is given by

$$\begin{aligned}\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left(\widehat{\text{PnL}}_t \right)_{t \in \mathcal{T}_1} \right] \right] &= 2 \text{tr}(\mathbf{\Gamma}^2) + (c_1 + \tilde{c}_1) \text{tr}(\mathbf{\Gamma}) + c_2 + \tilde{c}_2 + \varepsilon, \\ \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left(\widehat{\text{PnL}}_u \right)_{u \in \mathcal{T}_2} \right] \right] &= 2 \text{tr}(\mathbf{\Gamma}^2) + c_1 \text{tr}(\mathbf{\Gamma}) + c_2,\end{aligned}$$

where

$$\begin{aligned}\mathbf{\Gamma} &= \boldsymbol{\beta}^\top \boldsymbol{\Sigma}_\epsilon^{-1} \boldsymbol{\beta} \boldsymbol{\Sigma}_s, \\ c_1 &= 1 + \frac{p+1}{T_1 - p - 1}, \quad \tilde{c}_1 = \frac{2p+5}{T_1 - p - 1} + \frac{2m(p^2 + p + 2T_1)}{T_1(T_1 - p - 1)}, \\ c_2 &= \frac{mp}{T_1 - p - 1}, \quad \tilde{c}_2 = \frac{mp(2m + p + T_1 + 4)}{T_1(T_1 + 2)} - \frac{2m^2 p^2}{T_1^2(T_1 + 2)} - \frac{mp}{T_1 - p - 1},\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\varepsilon &= \text{tr} \left(\text{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right) + \text{tr} \left(\text{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right) \\ &\quad + \text{tr} \left(\text{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right).\end{aligned}$$

The proof relies on a combination of results on the expectations of Gaussian vectors or matrices, properties of projection matrices and matrix statistics tricks. For better readability, these lengthy computations are deferred to Appendix C. The in-sample variance is almost fully explicit bar the ε term which is not possible to derive exactly. In practice, we approximate the in-sample variance by simply dropping this term as it is typically very small compared to the other terms in our experience. The accuracy of this approximation is checked in section 3.1. Here, let us mention a few implications.

Remark 1 (Expected Mean Return). Examining the difference between the expected in-sample and out-of-sample returns, we see that the in-sample return is inflated by $\frac{pm}{T_1}$. Thus, as the number of in-sample observations T_1 increases, our in-sample P&L will converge to our out-of-sample P&L. Conversely, as the number of signals p increases the overfitting bias becomes arbitrarily large. Increasing the number of assets m will also lead to further inflation of the in-sample P&L (see Figure 6).

Remark 2 (Expected Variance). Inspecting the expected variance, we see that $\tilde{c}_1 > 0$ provided $T_1 > p + 1$, which we would always expect to be the case, but whether $\tilde{c}_2 > 0$ depends on the interplay between the length T_1 of the backtest, as well as the number m and p of assets and signals. For a range of backtest lengths, Figure 2 displays the combinations of m, p for which $\tilde{c}_2 > 0$. Generally, when $p < m$ (with some sublinearity for large m) then $\tilde{c}_2 > 0$, and for large enough T_1 then p must be much larger than m for $\tilde{c}_2 < 0$. Overfitting β thus causes the in-sample variance to be marginally larger than the out-of-sample variance for small m , but for large m it can be significantly higher. Further, the out-of-sample variance can be much larger than the true variance, which causes the out-of-sample Sharpe to be lower than the true theoretical Sharpe ratio if β was known.

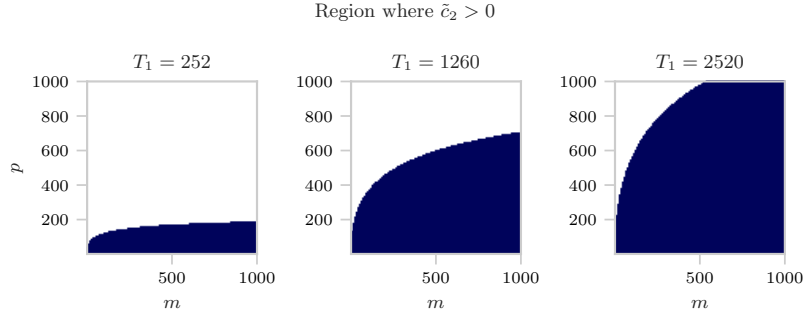


Figure 2: Values of m, p, T_1 where $\tilde{c}_2 > 0$.

3.1 Accuracy of the Approximation of the In-Sample Variance

In our empirical study for commodity futures in Section 5, we find that ε from Proposition 3.1 are negative but small relative to the other terms, with the estimated variance being 3% smaller than the analytical result with $\varepsilon = 0$. To support the robustness of this observation, we also run separate simulations where the model parameters are selected randomly from distributions centred around the empirical point estimates (full details can be found in Appendix D), and then used to sample signals and returns to check how large the ε terms are for a wide range of parameter values. We find that for T_1 sufficiently larger than p , ε are small, less than 3% for $p/T_1 < 0.1$ as can be seen in Figure 3. This is equivalent to having at least ten data points per signal which in fact is a common rule of thumb [Har06, Sec. 4.4]. Based on these observations, we henceforth disregard ε in our analysis.

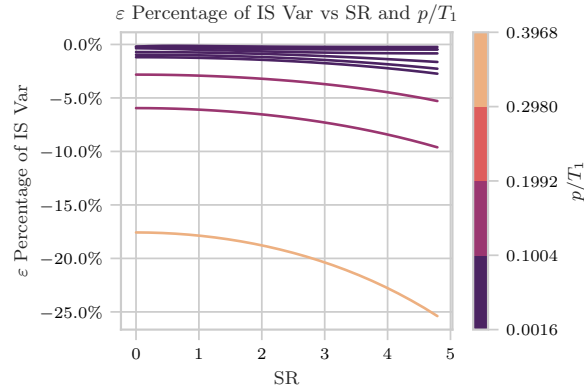


Figure 3: Magnitude of $\sum_i \varepsilon_i$ as a percentage of the in-sample variance $\mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right]$.

3.2 The Replication Ratio

In view of Proposition B.1 and the discussion in Section 3.1 we propose the following approximations to the expected in- and out-of-sample Sharpe ratios:

$$\begin{aligned} \mathbb{E}[\text{SR}_{\text{IS}}] &\approx \text{SR}_{\text{EIS}} := \frac{\mathbb{E} \left[\widehat{\mathbb{E}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right]}{\sqrt{\mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right]}}, \\ \mathbb{E}[\text{SR}_{\text{OOS}}] &\approx \text{SR}_{\text{EOOS}} := \frac{\mathbb{E} \left[\widehat{\mathbb{E}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2} \right] \right]}{\sqrt{\mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2} \right] \right]}. \end{aligned} \quad (3.2)$$

Here we take $\varepsilon = 0$ in $\mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right]$ as discussed in Section 3.1. Moreover, we assume that the difference between taking expectations of the numerator and denominator individually, rather than jointly, is small. We again check this by the same simulation process as Section 3.1 and find that indeed the convexity adjustment is generally very small, with maximum errors of 0.01 for the in-sample Sharpe ratio and 0.005 for the out-of-sample Sharpe ratio, as can be seen in Figure 4. Compared to the typical Sharpe ratios that are required to make a strategy potentially appealing in the first place, these are negligible errors.

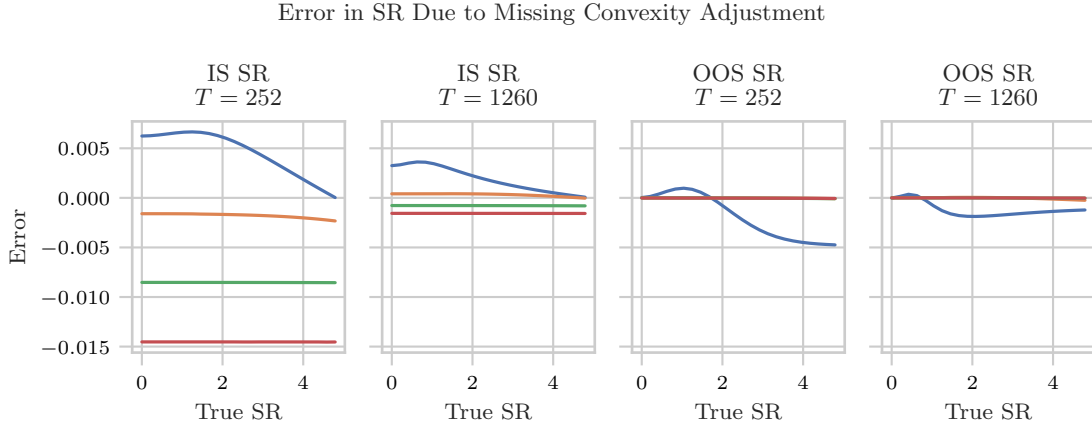


Figure 4: Difference between the expected Sharpe Ratio $\mathbb{E}[\text{SR}]$ and the expressions in (3.2), i.e. the error due to the missing convexity adjustment for both in-sample and out-of-sample Sharpe ratios.

Armed with these analytical approximations, we can study the “replication ratio”, i.e., the fraction $\text{SR}_{\text{EOOS}}/\text{SR}_{\text{IS}}$ of the in-sample Sharpe ratio that is recovered out of sample. A replication ratio of 100% means that the strategy completely replicates out of sample. Conversely, a replication ratio of 0% means that it returns zero, e.g., because the signals are completely uninformative (i.e., the true parameter β is $\mathbf{0}$) and thus the expected out-of-sample Sharpe ratio is zero.

Univariate Case To gain some intuition we first evaluate the replication ratio in the simplest case with one asset ($m = 1$) and one dynamic signal with no intercept ($p = 1$) and where we scale σ_s to be equal to σ_ϵ , which is always possible as one can always scale the original signal to

achieve this. In this case the formulas derived above simplify as

$$\begin{aligned} \text{SR} &= \frac{\beta^2}{\sqrt{2\beta^4 + \beta^2}}, \\ \text{SR}_{\text{EIS}} &= \frac{\beta^2 + \frac{1}{T_1}}{\sqrt{2\beta^4 + \left(1 + \frac{15}{T_1-2} - \frac{2}{T_1}\right)\beta^2 + \frac{4}{T_1} - \frac{3}{T_1+2} - \frac{1}{T_1^2}}}, \\ \text{SR}_{\text{EOOS}} &= \frac{\beta^2}{\sqrt{2\beta^4 + \left(1 + \frac{2}{T_1-2}\right)\beta^2 + \frac{1}{T_1-2}}}. \end{aligned}$$

Whence, the replication ratio only depends on the size T_1 of the training dataset and the parameter β (or, equivalently, the true Sharpe ratio) in this case:

$$\frac{\text{SR}_{\text{EOOS}}}{\text{SR}_{\text{EIS}}} = \frac{\beta^2 \sqrt{2\beta^4 + \left(1 + \frac{15}{T_1-2} - \frac{2}{T_1}\right)\beta^2 + \frac{4}{T_1} - \frac{3}{T_1+2} - \frac{1}{T_1^2}}}{\left(\beta^2 + \frac{1}{T_1}\right) \sqrt{2\beta^4 + \left(1 + \frac{2}{T_1-2}\right)\beta^2 + \frac{1}{T_1-2}}}. \quad (3.3)$$

We plot a heat map of this relationship in Figure 5. As the Sharpe ratio (equivalently β) increases, overfitting becomes less of an issue. Indeed, as the signal-to-noise ratio becomes arbitrarily large for $\beta \rightarrow \infty$, the replication ratio approaches 100% ($\text{SR}_{\text{EOOS}}/\text{SR}_{\text{EIS}} \rightarrow 1$). Similarly, the difference between in- and out-of-sample performance also vanishes as the training data set becomes arbitrarily large ($T_1 \rightarrow \infty$). Both these relationships are nonlinear, with large initial improvements that eventually saturate.

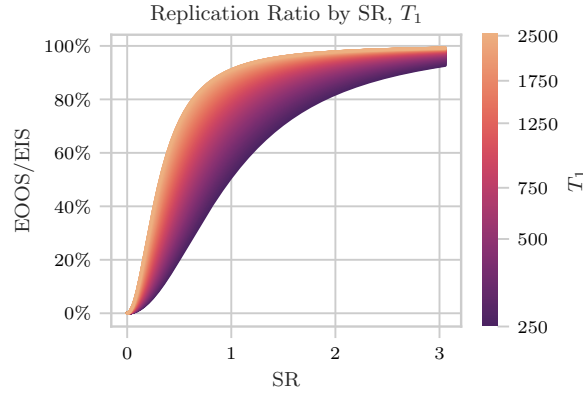


Figure 5: Replication ratio $\frac{\text{SR}_{\text{EOOS}}}{\text{SR}_{\text{EIS}}}$ in the 1-asset, 1-signal case as in (3.3), with varying SR, T_1 .

Multivariate Case We now turn to the case of multiple assets and signals ($m, p > 1$). To illustrate some of the basic mechanisms, we look at a representative example. To wit, we consider an in-sample Sharpe ratio of 2 with a ten-year backtest of daily returns ($T_1 = 2520$), and assume that assets and signals are uncorrelated, $\Sigma_s = \mathbf{I}_p$, $\Sigma_\epsilon = \mathbf{I}_m$. For simplicity, we also assume that the signals include no intercept ($\mu_s = \mathbf{0}$) and that investments are directly proportional to the signals without reweighing ($\mathbf{Z} = \mathbf{I}_m$). Finally, to achieve the desired in-sample Sharpe ratio of 2, we let $\beta = k\mathbf{1}_{m,p}$ for a suitable constant k and the matrix $\mathbf{1}_{m,p} \in \mathbb{R}^{m \times p}$ of all ones.

This is a kind of worst-case scenario for overfitting, because independent signals and returns introduce a maximal amount of freedom into the model. In this case the replication ratio reads

$$\frac{\text{SR}_{\text{EOOS}}}{\text{SR}_{\text{EIS}}} = \frac{k^2 pm}{k^2 pm + \frac{pm}{T_1}} \frac{\sqrt{2(k^2 pm)^2 + (c_1 + \tilde{c}_1)k^2 pm + c_2 + \tilde{c}_2}}{\sqrt{2(k^2 pm)^2 + c_1 k^2 pm + c_2}}, \quad (3.4)$$

for the constants c_i, \tilde{c}_i defined in (3.1). Whence, for independent signals and residuals, we obtain a similar expression as in the univariate case (3.3).

Figure 6 shows that as the number of assets increases, the replication ratio (3.4) rapidly decreases. Why does this happen? For each additional asset or signal we increase the freedom in the model to overfit to the historical period, therefore reducing the replication ratio.

One way to address this is to reduce the number of model parameters, e.g., by running a pooled regression that assumes that – up to some suitable normalizations – a family of common signals affects all assets in the same manner. For example, [GP13] normalize their momentum signals for each asset to make them comparable in the cross-section and then only fit one parameter across all stocks for each of these signals, rather than many stock-specific coefficients.⁵

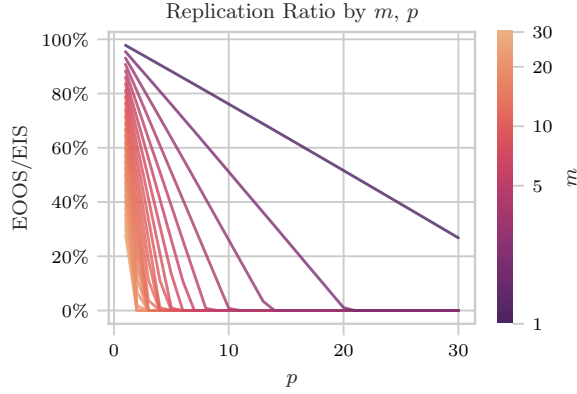


Figure 6: Replication ratio $\frac{SR_{EOOS}}{SR_{EIS}}$ for increasing assets m and signals p , and fixed in-sample Sharpe ratio $SR_{IS} = 2$ with $T_1 = 2520$.

3.3 Autoregressive Signals

So far, we assumed for analytical tractability that signals and residuals are iid. A typical assumption (in [GP13] for example) is that instead the signal follows an AR(1) process, so that signals are persistent and investment opportunities in turn autocorrelated across time. Such persistent signals are not analytically tractable in our framework, but we can assess the corresponding replication ratios via simulations. We examine the simplest case with one asset and one predictive signal:

$$r_{t+1} = \beta s_t + \epsilon_{t+1},$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$ and (s_t) is an AR(1) process with persistence parameter ϕ ,⁶

$$s_t = \phi s_{t-1} + u_t,$$

for a centred white noise process (u_t) with unit variance. In this case, the true Sharpe ratio is

$$SR = \frac{\mathbb{E}[w_t r_{t+1}]}{\sqrt{\mathbb{V}[w_t r_{t+1}]}} = \frac{\beta^2 \sigma_s^2}{\sqrt{2\beta^4 \sigma_s^4 + \beta^2 \sigma_s^2}} = \frac{\beta}{\sqrt{1 + 2\beta^2 - \phi^2}}, \quad (3.5)$$

where we have taken $w_t = \beta s_t$, κ_s is the kurtosis (not excess kurtosis) of the signal's noise, and assumed $\beta > 0$ without loss of generality. For the second equality in (3.5), we have used that $\kappa_s = 3$ for u_t normally distributed, and $\sigma_s^2 = \frac{1}{1-\phi^2}$.

⁵ This is reminiscent to the fitting of price impact models where, after normalizing for each asset's volatility and trading volume, universal models also often outperform stock-specific models out of sample, as in [MWV24] and references therein.

⁶ We assume that $|\phi| < 1$ so that (s_t) is weakly stationary thus $\mathbb{E}[s_t] = 0$ and $\mathbb{V}[s_t] = \frac{1}{1-\phi^2}$.

To understand the impact of the persistence of AR(1) signal compared to their iid counterparts, we proceed by Monte-Carlo simulations. We let $T_2 = 1260$ and vary $T_1 \in \{252, 1260, 2520\}$, $\phi \in \{0, 0.5, 0.9, 0.99, 0.999\}$ and $SR \in [0, 2]$. Note that $\phi = 0.9$ corresponds to a signal half-life of 6.5 (daily) time-steps, whereas $\phi = 0.999$ implies a half-life of 690 time-steps. For a given signal persistence ϕ we vary the parameter β to achieve the desired true Sharpe ratio in each case using (3.5).

In Figure 7 we compare the replication ratios (and their difference to their counterparts computed in our model with iid signals) for different signal persistences parameterized by ϕ . Unless ϕ is very close to 1 (> 0.99), the difference compared to the iid case $\phi = 0$ is surprisingly small. Thus, although our assumption of iid signals may initially seem like a strong one, it in fact has little impact on the replication ratio in practice.

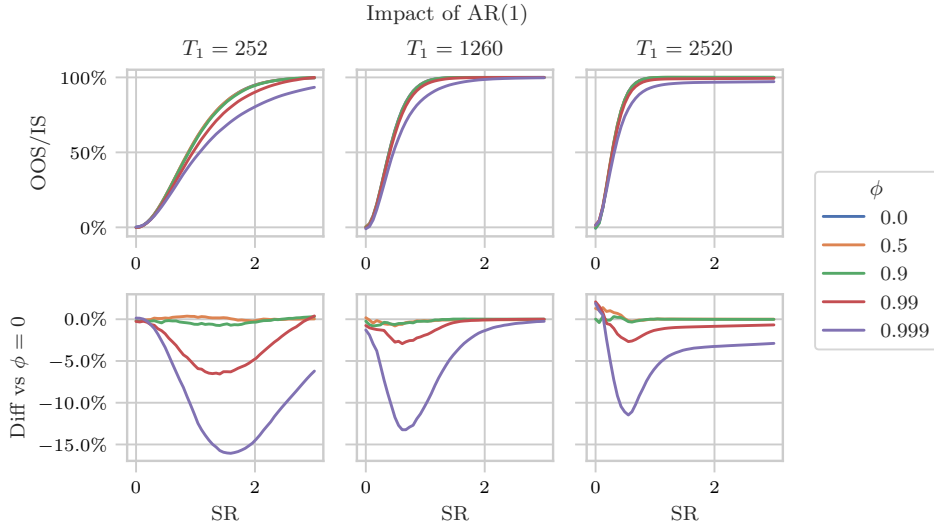


Figure 7: Impact of s_t following an AR(1) process. The top row shows the replication ratio, and the bottom row shows the simple difference of this ratio vs $\phi = 0$, corresponding to the iid case.

4 Comparison with the Literature

Most closely related to the present study is the work of Kan, Wang, and Zheng [KWZ24]. They compare the expected in- and out-of-sample Sharpe ratios of the Markowitz portfolio for m assets whose expected returns are estimated on a training dataset of size T . They find that the expected in-sample Sharpe ratio is

$$\mathbb{E}[\hat{\theta}] = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{T-m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{T-m}{2}\right)} {}_1F_1\left(-\frac{1}{2}; \frac{m}{2}; -\frac{m\theta^2}{2}\right),$$

where $\Gamma(\cdot)$ is the Gamma function, ${}_1F_1(\cdot; \cdot; \cdot)$ the confluent hypergeometric function, and $\theta = \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}$ the true Sharpe ratio of the Markowitz portfolio. Likewise, the expected out-of-sample Sharpe ratio can also be computed in closed form as

$$\mathbb{E}[\tilde{\theta}] = \frac{\theta^2 \sqrt{T} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{T-m+2}{2}\right) \Gamma\left(\frac{T}{2}\right)}{\sqrt{2} \Gamma\left(\frac{m+2}{2}\right) \Gamma\left(\frac{T-m+1}{2}\right) \Gamma\left(\frac{T+1}{2}\right)} {}_1F_1\left(\frac{1}{2}; \frac{m+2}{2}; -\frac{T\theta^2}{2}\right).$$

In the framework of [GP13], the setting of [KWZ24] corresponds to using a single, infinitely persistent trading signal ($\phi \rightarrow 1$) for each asset. In contrast, we consider multiple signals per

asset that decay quickly ($\phi \rightarrow 0$). Incorporating multiple signals per asset allows us to cover many common trading strategies, such as momentum strategies based on several different moving averages of past returns. Figure 8 illustrates how overfitting becomes an increasingly important concern when allowing for additional freedom in the model by considering multiple signals per asset. We fix $T = T_1 = T_2 = 2520$, $SR = 1.5$, vary the number of assets m in both models and set either $p = 1$ or $p = 10$ in our model. We observe that for a single signal ($p = 1$) our haircuts are similar to the ones of [KWZ24], but for $p = 10$ signals our haircut is much more severe.

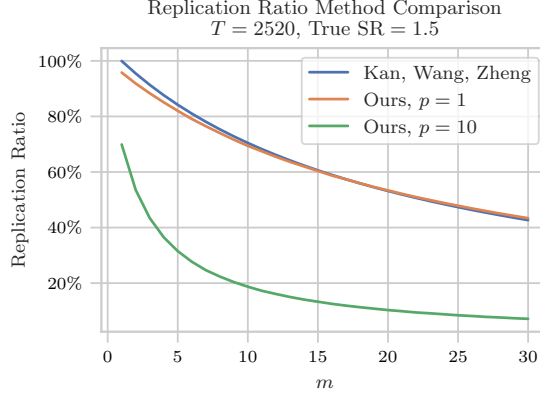


Figure 8: Comparison of the replication ratio for Kan, Wang, and Zheng and ours with $T = T_1 = T_2 = 2520$ and $SR = 1.5$ for varying m . For ours we set $p = 10$ or $p = 1$.

An important common conclusion of both [KWZ24] and the present study is the non-linear dependence of the replication ratio on the true Sharpe ratio. To wit, for small values of the true Sharpe ratio, the expected out-of-sample Sharpe ratio is much lower than the expected in-sample Sharpe ratio. A takeaway from this is that when developing investment strategies, it is preferable to develop a single high Sharpe strategy than many low Sharpe strategies. This aligns with the accepted wisdom of successful practitioners, for example Peter Muller (Founder, PDT Partners) who quipped [Mul01]:

In my opinion it is far better to refine an individual strategy...than to attempt to put together lots of weaker strategies. Depth is more important than breadth for investment strategies... I would much rather have a single strategy with an expected Sharpe ratio of 2 than a strategy that has an expected Sharpe ratio of 2.5 formed by putting together five supposedly uncorrelated strategies each with an expected Sharpe ratio of 1.

5 Predicting Commodity Futures

To test our method on real data, we follow the approach of Gârleanu and Pedersen [GP13] in the empirical application section of their seminal paper on dynamic trading. We select the commodity futures in Table 1 and collect close prices for the closest-to-expiry contracts from May 1998 to December 2023. The daily returns are computed as close-to-close percent changes in prices.

As in [GP13], we compute the 5-day, 1-year and 5-year Sharpe ratios for each asset and use these as signals. In effect these are normalized versions of the moving averages typically used in momentum-style strategies. We additionally transform the signals to be mean zero and unit variance. Normalizing the variances enables us to easily apply L^2 -regularization when fitting β to calibrate the model we later use as a simulation engine; the centring is required per our conditions on the signals in the theoretical model. This yields a total of 36 dynamic (cross-sectional) signals, which with the addition of the asset-specific intercept yields 37 regressors per asset.

To assess the validity of our results we need to compare our analytical approximations for the expected in-sample and out-of-sample Sharpe ratios, SR_{EIS} and SR_{EOOS} , to empirically observed

Commodity	Venue	Ticker
Aluminium	LME	MAL3=LX
Copper	LME	MCU3=LX
Nickel	LME	MNI3=LX
Zinc	LME	MZN3=LX
Lead	LME	MPB3=LX
Gasoil	ICE	LGOc1
WTI	NYMEX	CLc1
Gold	COMEX	GCc1
Silver	COMEX	SIc1
Coffee	NYBOT	KCc1
Cocoa	NYBOT	CCc1
Sugar	NYBOT	SBc1

Table 1: Commodities selected for testing from the LME (London Metal Exchange), ICE (Intercontinental Exchange), NYMEX (New York Mercantile Exchange), COMEX (Commodity Exchange) and NYBOT (New York Board of Trade).

equivalents. To do this we clearly require more than the one realization of the returns available historically. To overcome this, we resort to simulating many realizations of a fitted model and use these to check our results. The first step in enabling this simulation is to estimate a model for the returns, which naturally we take as

$$\mathbf{r}_{t+1} = \boldsymbol{\beta}\mathbf{s}_t + \boldsymbol{\epsilon}_{t+1}.$$

We estimate $\boldsymbol{\beta} \in \mathbb{R}^{37 \times 12}$ by ridge regression. All fitted parameter values from this section can be found on the authors' webpages.

As mentioned in the introduction, the haircuts we observe for these trials are quite steep, and they could be improved by reducing the amount of freedom in the model, for example by panel regression. Indeed, making the structural assumption that the (suitably normalized) momentum signals have universal weights across assets reduces the number of model parameters for $m = 12$ assets and $p = 3 \times 12 + 1 = 37$ signals each from 444 to 3. This increases the replication ratio considerably from 30% to 71% for a ten-year backtest.⁷

5.1 Gaussian and iid Signals and Residuals

We first test our approach in the case where the data-generating model satisfies our key assumptions, namely that the signals and residuals are Gaussian and iid. To this end, we fit a multivariate Normal to the observed signals and residuals from the estimated model above.⁸ We then use these new samples to construct new realizations of the returns using the $\boldsymbol{\beta}$ parameter we estimated by ridge regression above.

We simulate 13 years, with 252 trading days per year, of new signals and returns, and use 10 years as in-sample and 3 years as out-of-sample data. For each in-sample simulation we compute the OLS estimator for $\hat{\boldsymbol{\beta}} = \mathbf{R}\mathbf{S}^T(\mathbf{S}\mathbf{S}^T)^{-1}$, where $\mathbf{R} \in \mathbb{R}^{12 \times 252 \cdot 10}$ is the stacked matrix of returns and $\mathbf{S} \in \mathbb{R}^{37 \times 252 \cdot 10}$ is the stacked matrix of signals. We then form the portfolio $\hat{\mathbf{w}}_t = \boldsymbol{\Sigma}_\epsilon^{-1}\hat{\boldsymbol{\beta}}\mathbf{s}_t$ as described in Section 2. This portfolio is then tested on the in-sample and out-of-sample periods to compute the observed in-sample and out-of-sample Sharpe ratios for each simulation.

In Figure 9 we see that the simulated expected in-sample and out-of-sample Sharpe ratios almost perfectly align with our analytical results, and so does the replication ratio. This shows that with real-world parameter values our analytical approximations are highly accurate. In the next section we additionally check the robustness of our results when our assumptions are violated.

⁷ A similar replication ratio is obtained when the out-of-sample values are estimated from the real historical data set rather than the fitted full model. On real data, the panel model in fact dominates the full model also in absolute terms, a sharp reversal given the latter's substantially better in-sample performance.

⁸ Note that both $\boldsymbol{\mu}_s$ and $\boldsymbol{\mu}_\epsilon$ will be zero by construction, so the covariance of the signals $\boldsymbol{\Sigma}_s$ and residuals $\boldsymbol{\Sigma}_\epsilon$ is all we need to generate new samples of each.

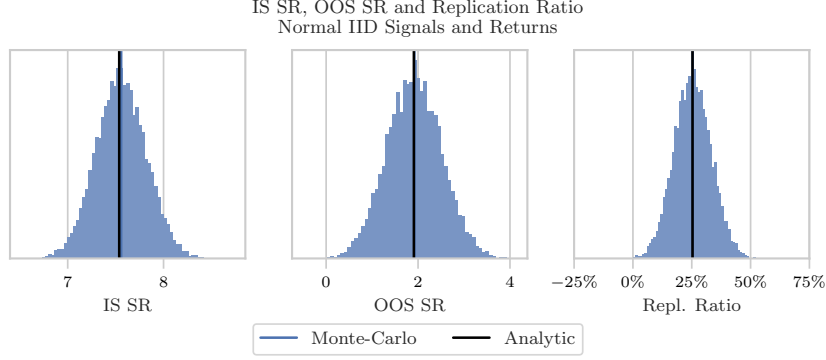


Figure 9: The expected in-sample Sharpe ratio, expected out-of-sample Sharpe ratio and replication ratio for the experiments in Section 5.1.

5.2 AR(1) Signals and t -distributed Residuals and Shocks

To simulate non-iid and non-Gaussian signals and residuals we now allow the signals (excluding the intercept) to follow an AR(1) process,

$$\mathbf{s}'_t = \Phi \mathbf{s}'_{t-1} + \mathbf{u}_t,$$

where $\mathbf{s}'_t \in \mathbb{R}^{36}$ denotes the non-intercept signals, $\mathbf{u}_t \in \mathbb{R}^{36}$ denotes the shocks to the AR(1) process, and $\Phi \in \mathbb{R}^{36 \times 36}$ denotes the AR(1) parameter. Additionally, rather than having ϵ_{t+1} and \mathbf{u}_t be multivariate Gaussian, we fit a t -distribution to each dimension of the residuals and shocks and use these to simulate residuals and shocks with fat tails. To induce the original covariance structure of the residuals and shocks we scale the independent samples by the Cholesky transformation of the relevant covariance matrix, Σ_ϵ or Σ_u .

In Figure 10 we see that even when our assumptions are violated, our analytical replication ratio is still remarkably close to its empirically observed counterpart. Relaxing the Gaussian assumption to allow for fat tails causes the in-sample Sharpe ratio to decrease, but also causes the out-of-sample Sharpe ratio to decrease. Additionally, allowing the signals to be AR(1) rather than iid causes the out-of-sample Sharpe ratio to decrease, however this effect is rather small. Finally, as the out-of-sample and in-sample Sharpe ratio are affected by the non-Gaussian returns, and the AR(1) signals have a relatively small effect, the replication ratio with both is almost identical to our analytical value under the Gaussian and iid assumptions. This confirms the practical relevance of our analytical results.

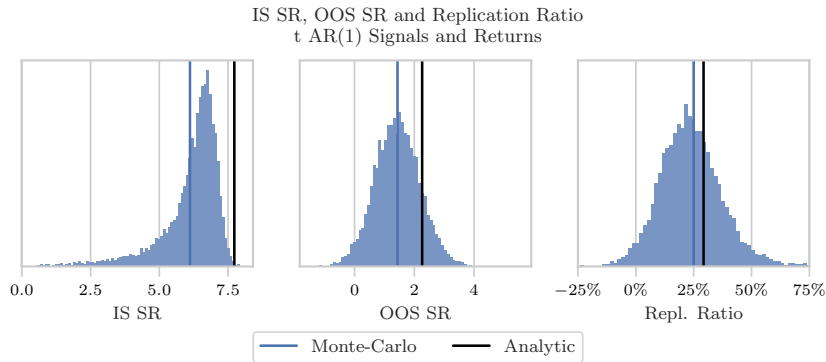


Figure 10: The expected in-sample Sharpe ratio, expected out-of-sample Sharpe ratio and replication ratio for the experiments in Section 5.2.

6 Conclusion

This paper derives analytical approximations for the in-sample and out-of-sample Sharpe ratios of portfolios constructed using linear prediction models. We show that increasing either the number of signals or assets too much makes this procedure susceptible to overfitting and thereby yields wildly overestimated in-sample Sharpe ratios.

We show that low true Sharpe ratio signals are particularly vulnerable to overfitting. Conversely, by extending the length of the in-sample period one can reduce the overfitting risk, and can produce a higher replication ratio out of sample.

We test our results on commodity futures using momentum-style signals and find that allowing AR(1) signals and non-Normal signals/residuals does not significantly impact the validity of our results. In particular, once we match the theoretical out-of-sample Sharpe ratio to the observed value, we see that the replication ratio is primarily a function of the out-of-sample Sharpe ratio and the curves for the AR(1) signals closely matches those of iid signals.

From this analysis, it seems that the best way to minimize the potential for overfitting is to minimize the number of signals and assets that are being used for any predictive model used to trade, and utilize the largest amount of data possible. To conclude, we quote Nick Patterson [Pat16] on his experience at Renaissance Technologies, which is very much in line with these results:

It's funny that I think the most important thing to do on data analysis is to do the simple things right. So, here's a kind of non-secret about what we did at Renaissance: in my opinion, our most important statistical tool was simple regression with one target and one independent variable.

They might be on to something.

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A Derivation of Equation (2.2)

To derive eq. (2.2) we need to derive two expressions,

$$\mathbb{E}[(\mathbf{Z}\beta\mathbf{s}_t)^\top(\beta\mathbf{s}_t + \epsilon_{t+1})], \quad \text{and,} \quad \mathbb{V}[(\mathbf{Z}\beta\mathbf{s}_t)^\top(\beta\mathbf{s}_t + \epsilon_{t+1})].$$

We start with the former,

$$\begin{aligned} \mathbb{E}[(\mathbf{Z}\beta\mathbf{s}_t)^\top(\beta\mathbf{s}_t + \epsilon_{t+1})] &= \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t + \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t], \\ &= \text{tr}(\mathbf{G}\Sigma_s) + \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\mu}_s, \end{aligned}$$

where $\mathbf{G} := \beta^\top \mathbf{Z} \beta$. We have used $\mathbb{E}[\epsilon_{t+1}] = 0$ in the second equality, and identity 5.24 from [Bro20] in the final equality.

Next we tackle the variance,

$$\begin{aligned} \mathbb{V}[(\mathbf{Z}\beta\mathbf{s}_t)^\top(\beta\mathbf{s}_t + \epsilon_{t+1})] &= \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t + \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t] + \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}] + \text{Cov}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t, \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t] + \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= 2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= 2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1} \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}] - \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}]^2, \\ &= 2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1} \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}], \\ &= 2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \mathbb{E}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \Sigma_\epsilon \mathbf{Z} \beta \mathbf{s}_t], \\ &= 2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \text{tr}(\mathbf{F}\Sigma_s) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s, \end{aligned}$$

where $\mathbf{F} := \beta^\top \mathbf{Z} \Sigma_\epsilon \mathbf{Z} \beta$. We have used $\mathbb{E}[\epsilon_{t+1}] = 0$ to deduce $\text{Cov}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t, \mathbf{s}_t^\top \beta^\top \mathbf{Z} \epsilon_{t+1}] = 0$, identity 379 from [PP12, sec. 8.2.2] to compute $\mathbb{V}[\mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t]$, identity 322 from [PP12, sec. 6.2.2] to compute the penultimate line and identity 5.24 from [Bro20] in the final equality.

Thus,

$$\text{SR} = \frac{\text{tr}(\mathbf{G}\Sigma_s) + \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\mu}_s}{\sqrt{2 \text{tr}((\mathbf{G}\Sigma_s)^2) + 4\boldsymbol{\mu}_s^\top \mathbf{G} \Sigma_s \mathbf{G} \boldsymbol{\mu}_s + \text{tr}(\mathbf{F}\Sigma_s) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s}},$$

where

$$\mathbf{F} := \beta^\top \mathbf{Z} \Sigma_\epsilon \mathbf{Z} \beta \quad \text{and} \quad \mathbf{G} := \beta^\top \mathbf{Z} \beta.$$

B General Case Result

Proposition B.1. *The expected average in-sample PEL is*

$$\mathbb{E} \left[\widehat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}] \right] = \text{tr}(\mathbf{G}\Sigma_s) + \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\mu}_s + \frac{p}{T_1} \text{tr}(\mathbf{Z}\Sigma_\epsilon).$$

The expected variance of the in-sample $P\mathcal{E}L$ is

$$\begin{aligned}
& \mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right] \\
&= 2 \operatorname{tr} \left((\mathbf{G}\boldsymbol{\Sigma}_s)^2 \right) + 4\boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\Sigma}_s \mathbf{G}\boldsymbol{\mu}_s + \operatorname{tr}(\mathbf{F}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{F}\boldsymbol{\mu}_s \\
&\quad + \frac{3}{T_1 - p - 1} \operatorname{tr} \left(2\mathbf{F}(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + \boldsymbol{\mu}_s^\top \mathbf{F}\boldsymbol{\mu}_s \cdot (\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1} (\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + p\mathbf{F}\boldsymbol{\Sigma}_s \right) \\
&\quad + \operatorname{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \left(\frac{p(p + T_1 + 4)}{T_1(T_1 + 2)} - \frac{2\mathbf{1}_p^\top \boldsymbol{\mu}_s (p - T_1 - 2)(p - T_1)}{T_1^2(T_1 + 1)(T_1 + 2)} \right) \\
&\quad - \operatorname{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \frac{2(p - T_1) \left((p - 2)\mathbf{1}_p^\top \boldsymbol{\mu}_s + T_1(p - \mathbf{1}_p^\top \boldsymbol{\mu}_s) + p \right)}{T_1^2(T_1 + 1)(T_1 + 2)} \\
&\quad + \frac{2}{T_1 - p - 1} \operatorname{tr}(\mathbf{Z}\boldsymbol{\Sigma}_\epsilon) \operatorname{tr} \left(2\mathbf{G}(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + \boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\mu}_s \cdot (\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1} (\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + p\mathbf{G}\boldsymbol{\Sigma}_s \right) \\
&\quad - \frac{2p}{T_1} \operatorname{tr}(\mathbf{Z}\boldsymbol{\Sigma}_\epsilon) \left(\operatorname{tr}(\mathbf{G}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\mu}_s \right) \\
&\quad + \varepsilon_1 + \varepsilon_2 + \varepsilon_3,
\end{aligned}$$

where $\mathbf{F} = \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta}$, $\mathbf{G} = \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta}$ and

$$\begin{aligned}
\varepsilon_1 &= \operatorname{tr} \left(\operatorname{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right), \\
\varepsilon_2 &= \operatorname{tr} \left(\operatorname{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right), \\
\varepsilon_3 &= \operatorname{tr} \left(\operatorname{Cov} \left[(\mathbf{S}\mathbf{S}^\top)^{-1}, \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right),
\end{aligned}$$

for any $t \in \mathcal{T}_1$ by the iid property of $(\mathbf{s}_t)_{t \in \mathcal{T}_1}$.

The expected average out-of-sample $P\mathcal{E}L$ is

$$\mathbb{E} \left[\widehat{\mathbb{E}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2} \right] \right] = \operatorname{tr}(\mathbf{G}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\mu}_s,$$

The expected variance of the out-of-sample $P\mathcal{E}L$ is

$$\begin{aligned}
& \mathbb{E} \left[\widehat{\mathbb{V}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_2} \right] \right] \\
&= 2 \operatorname{tr} \left((\mathbf{G}\boldsymbol{\Sigma}_s)^2 \right) + 4\boldsymbol{\mu}_s^\top \mathbf{G}\boldsymbol{\Sigma}_s \mathbf{G}\boldsymbol{\mu}_s + \operatorname{tr}(\mathbf{F}\boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{F}\boldsymbol{\mu}_s \\
&\quad + \boldsymbol{\mu}_s^\top \left(2\mathbf{D}\boldsymbol{\Sigma}_s^\top \mathbf{F} + \operatorname{tr}(\mathbf{F}\boldsymbol{\Sigma}_s^\top) \mathbf{D} + \operatorname{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) \mathbf{F} \right) \boldsymbol{\mu}_s + \operatorname{tr}(\mathbf{D}\boldsymbol{\Sigma}_s^\top \mathbf{F}\boldsymbol{\Sigma}_s) + \operatorname{tr}(\mathbf{D}\boldsymbol{\Sigma}_s) \operatorname{tr}(\mathbf{F}\boldsymbol{\Sigma}_s^\top) \\
&\quad + \operatorname{tr}((\mathbf{Z}\boldsymbol{\Sigma}_\epsilon)^2) \left(\operatorname{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) + \boldsymbol{\mu}_s^\top \mathbf{D}\boldsymbol{\mu}_s \right),
\end{aligned}$$

where $\mathbf{D} := \frac{(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1}}{T_1 - p - 1}$.

C Proof of Proposition B.1

Recall that the classical Ordinary Least Square method yields the estimator

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \mathbf{E}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1},$$

where $\mathbf{E} \in \mathbb{R}^{m \times T_1}$ is the matrix of stacked residuals. This is a more convenient expression to work with than (2.3), and we will use it to evaluate the expectations of the sample means, variances and covariances. Since $(\mathbf{s}_t)_t$ and $(\boldsymbol{\epsilon}_{t+1})_t$ are iid sequences, the expectation of the sample expectation, variance and covariance are simply the true expectation, variance and covariance respectively.

C.1 In-sample expected average

From the definition (2.4), the in-sample expected P&L $\mathbb{E}[\widehat{\mathbb{E}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}]]$ can be evaluated as

$$\begin{aligned} & \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}}_{\mathbf{s}_t})^\top (\boldsymbol{\beta}_{\mathbf{s}_t} + \boldsymbol{\epsilon}_{t+1}) \right)_{t \in \mathcal{T}_1} \right] \right] \\ &= \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\{\mathbf{Z} \widehat{\boldsymbol{\beta}}_{\mathbf{s}_t}\}^\top \boldsymbol{\beta}_{\mathbf{s}_t} \right)_{t \in \mathcal{T}_1} \right] \right] + \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\{\mathbf{Z} \widehat{\boldsymbol{\beta}}_{\mathbf{s}_t}\}^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right], \\ &= \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left\{ \mathbf{Z} \left(\boldsymbol{\beta} + \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \right) \mathbf{s}_t \right\}^\top \boldsymbol{\beta}_{\mathbf{s}_t} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left\{ \mathbf{Z} \left(\boldsymbol{\beta} + \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \right) \mathbf{s}_t \right\}^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right], \\ &= \mathbb{E} \left[(\mathbf{Z} \boldsymbol{\beta}_{\mathbf{s}_t})^\top \boldsymbol{\beta}_{\mathbf{s}_t} \right] + \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left\{ (\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta}_{\mathbf{s}_t} \right\} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left\{ \mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \right\}^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right], \end{aligned}$$

since the sample mean is an unbiased estimator, and since $(\mathbf{s}_t)_t$ and $(\boldsymbol{\epsilon}_{t+1})_t$ are respectively iid, and $\mathbb{E}[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$.

Since we can “extract” the t -th column from a matrix \mathbf{X} by multiplying it by a vector \mathbf{e}_t which has 1 in the t -th position and zero everywhere else, then

$$\mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left(\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \right)^\top \boldsymbol{\beta}_{\mathbf{s}_t} \right)_{t \in \mathcal{T}_1} \right] \right] = \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S} \mathbf{E}^\top \mathbf{Z} \boldsymbol{\beta}_{\mathbf{s}_t} \right] = 0, \quad \text{for all } t \in \mathcal{T}_1,$$

since \mathbf{E} has a Matrix Normal distribution $\mathcal{MN}(\mathbf{0}, \boldsymbol{\Sigma}_\epsilon, \mathbf{I}_{T_1})$ by assumption and since \mathbf{Z} is assumed symmetric. As previously mentioned, the first equality is true for all $t \in \mathcal{T}_1$ by the iid property of $(\mathbf{s}_t)_t$, and the fact the sample mean is an unbiased estimator. The next term reads

$$\begin{aligned} & \mathbb{E} \left[\widehat{\mathbb{E}} \left[\left(\left(\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \right)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] = \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S} \mathbf{E}^\top \mathbf{Z} \mathbf{E} \mathbf{e}_t \right], \quad \text{for any } t \in \mathcal{T}_1, \\ &= \mathbf{e}_t^\top \mathbb{E}[\mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S}] \mathbb{E}[\mathbf{E}^\top \mathbf{Z} \mathbf{E}] \mathbf{e}_t, \\ &= \text{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon) \mathbf{e}_t^\top \mathbb{E}[\mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S}] \mathbf{e}_t = \frac{p}{T_1} \text{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon), \end{aligned}$$

where we have used the Matrix Normal identity

$$\mathbb{E}[\mathbf{X} \mathbf{A} \mathbf{X}^\top] = \mathbf{U} \text{tr}(\mathbf{A}^\top \mathbf{V}) + \mathbf{M} \mathbf{A} \mathbf{M}^\top,$$

with $\mathbf{X} \sim \mathcal{MN}(\mathbf{M}, \mathbf{U}, \mathbf{V})$ [Gup00, Theorem 2.3.5]. We also used that $\text{tr}(\mathbb{E}[\mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S}]) = p$ since the projection matrix $\mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S}$ is idempotent and thus its trace equals its rank, and

since $(\mathbf{s}_t)_t$ are iid, then the expectation of each element of the diagonal of $\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}\mathbf{S}$ are equal, thus $\mathbf{e}_t^\top \mathbb{E}[\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}\mathbf{S}]\mathbf{e}_t = \frac{p}{T_1}$. Therefore, the in-sample expected P&L is given by

$$\mathbb{E} \left[\widehat{\mathbb{E}} \left[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1} \right] \right] = \text{tr} \left(\boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \right) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\mu}_s + \frac{p}{T_1} \text{tr} (\mathbf{Z} \boldsymbol{\Sigma}_\epsilon).$$

C.2 In-sample expected variance

We turn next to the expected variance $\mathbb{E}[\widehat{\mathbb{V}}[(\widehat{\text{PnL}}_t)_{t \in \mathcal{T}_1}]]$:

$$\begin{aligned} & \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}}_{\mathbf{s}_t})^\top (\boldsymbol{\beta} \mathbf{s}_t + \boldsymbol{\epsilon}_{t+1}) \right)_{t \in \mathcal{T}_1} \right] \right] \\ &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left(\left(\mathbf{Z} (\boldsymbol{\beta} + \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1}) \mathbf{s}_t \right)^\top \boldsymbol{\beta} \mathbf{s}_t + \left(\mathbf{Z} (\boldsymbol{\beta} + \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1}) \mathbf{s}_t \right)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right], \\ &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t + (\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t + (\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} + (\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right], \\ &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] + \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] + \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] + \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &+ \mathbb{E} \left[2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right]. \end{aligned}$$

Since the signals and residuals are iid, and the sample variance is an unbiased estimator for the population variance, we can replace all the expected sample variance terms with the true variance. We are thus left with computing these true variances. The first two terms are

$$\begin{aligned} & \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] = \mathbb{V} \left[(\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right] = 2 \text{tr} \left(\mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \right) + 4 \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\mu}_s, \\ & \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] = \mathbb{V} \left[(\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right] = \text{tr} \left(\mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \right) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\mu}_s, \end{aligned}$$

where the first equality in both expressions is for all $t \in \mathcal{T}_1$, and the first equality comes from the iid assumption and the fact the sample variance is an unbiased estimator.

The next term is

$$\begin{aligned}
 \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] &= \mathbb{V} \left[(\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right], \quad \text{for all } t \in \mathcal{T}_1 \\
 &= \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)^2 \right] - \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right) \right]^2 \\
 &= \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)^2 \right],
 \end{aligned}$$

since $\mathbb{E}[\mathbf{E}] = 0$ and \mathbf{E} and \mathbf{S} are independent. To evaluate this further we again use $\mathbf{s}_t = \mathbf{S}\mathbf{e}_t$ and let $\mathbf{H} := \mathbf{S}^\top (\mathbf{SS}^\top)^{-1} \mathbf{S}$ (the so called ‘hat’ matrix in linear regression), so that

$$\begin{aligned}
 \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)^2 \right] &= \mathbb{E} \left[\left(\mathbf{e}_t^\top \mathbf{S}^\top (\mathbf{SS}^\top)^{-1} \mathbf{S} \mathbf{E}^\top \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right)^2 \right] \\
 &= \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{E}^\top \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{H} \mathbf{E}^\top \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right] \\
 &= \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} (\mathbf{I}_{T_1} \mathbf{H}^\top \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t) \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right] \\
 &= \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right], \\
 &= \text{tr} \left(\mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right] \right) \\
 &= \text{tr} \left(\mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right] \right) \\
 &= \text{tr} \left(\mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right] + \mathcal{E}_1 \right),
 \end{aligned}$$

where

$$\mathcal{E}_1 := \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right] - \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right].$$

The \mathcal{E}_1 term results from the fact that $(\mathbf{SS}^\top)^{-1}$ is of course not independent of the other terms, however experimentally we find that $\text{tr}(\mathcal{E}_1)$ is negligible compared to the trace of the expectation product and decreases rapidly in the ratio p/T_1 . Provided we are in the regime where $T_1 \gg p$, then this approximation is reasonable as can be seen in Figure 3.

With this, we can finish the evaluation. To simplify the expression it is convenient to swap $\mathbf{S}\mathbf{e}_t$ with \mathbf{s}_t again, and define the matrix $\mathbf{F} := \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta$, so that

$$\begin{aligned}
 &\mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)^2 \right] \\
 &= \text{tr} \left(\mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \mathbf{Z}^\top \mathbf{S} \mathbf{e}_t \mathbf{Z} \beta \mathbf{s}_t \mathbf{s}_t^\top \right] + \mathcal{E}_1 \right) \\
 &= \text{tr} \left[\frac{(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1}}{T_1 - p - 1} \left(2 \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) \mathbf{F} \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s \left(\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + \text{tr}(\mathbf{F} \boldsymbol{\Sigma}_s) \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) \right) \right] + \varepsilon_1 \\
 &= \frac{1}{T_1 - p - 1} \text{tr} \left(2 \mathbf{F} \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s \cdot \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right)^{-1} \left(\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + p \mathbf{F} \boldsymbol{\Sigma}_s \right) + \varepsilon_1,
 \end{aligned}$$

with $\varepsilon_1 := \text{tr}(\mathcal{E}_1)$. Thus, the expected variance term is given by

$$\begin{aligned}
 &\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \beta \mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= \frac{1}{T_1 - p - 1} \text{tr} \left(2 \mathbf{F} \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s \cdot \left(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right)^{-1} \left(\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top \right) + p \mathbf{F} \boldsymbol{\Sigma}_s \right) + \varepsilon_1.
 \end{aligned}$$

Moving on, we tackle the next term in the expected variance similarly:

$$\begin{aligned}
 \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] &= \mathbb{V} \left[(\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right], \quad \text{for all } t \in \mathcal{T}_1, \\
 &= \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)^2 \right] - \mathbb{E} \left[(\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right]^2 \\
 &= \mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)^2 \right] - \left(\text{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon) \frac{p}{T_1} \right)^2.
 \end{aligned}$$

We can evaluate the first term as

$$\begin{aligned}
 &\mathbb{E} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)^2 \right] \\
 &= \mathbb{E} \left[\mathbf{e}_t \mathbf{H} \mathbf{E}^\top \mathbf{Z} \mathbf{E} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{E}^\top \mathbf{Z} \mathbf{E} \mathbf{H}^\top \mathbf{e}_t \right] \\
 &= \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \left(\mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{e}_t \right] + \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{H}^\top \mathbf{e}_t \right] \right) + \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{H} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{H}^\top \mathbf{e}_t \right] \\
 &= \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \left(\frac{p}{T_1} + \left(\frac{p(p+2)}{T_1(T_1+2)} - \mathbf{1}_p^\top \boldsymbol{\mu}_s \frac{2(T_1-p)(T_1-p+2)}{T^2(T+1)(T+2)} \right) \right) \\
 &\quad + \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \left(\frac{p(p+2)}{T_1(T_1+2)} - \mathbf{1}_p^\top \boldsymbol{\mu}_s \frac{2(T_1-p)(T_1-p+2)}{T^2(T+1)(T+2)} \right),
 \end{aligned}$$

where we have used the fact that when there is no intercept term the diagonal entries of the hat matrix \mathbf{H} are distributed $\beta \left(\frac{p}{2}, \frac{(T-p)}{2} \right)$ [Bel04, Appendix 2A], and for the case where there is an intercept term, Chatterjee and Hadi [CH88, Section 2.3.7] derive an approximate distribution for the diagonal elements of the hat matrix,

$$\frac{T_1 - p}{p - 1} \frac{h_{tt} - \frac{1}{T_1}}{1 - h_{tt}} \sim F(p - 1, T_1 - p),$$

where $F(p - 1, T_1 - p)$ denotes the F distribution with $p - 1$ and $T_1 - p$ degrees of freedom. However, per Rao [Rao09, p. 542] this is an exact distribution, rather than approximate, when $\boldsymbol{\mu}_s = \mathbf{0}$, which we have per our assumptions. With a little work this yields

$$\mathbb{E}[h_{tt}^2] = \frac{p(p+2)}{T_1(T_1+2)} - \frac{2(T_1-p)(T_1-p+2)}{T^2(T+1)(T+2)},$$

and noting that $\mathbf{1}_p^\top \boldsymbol{\mu}_s$ is zero with no intercept, and one with an intercept, this yields the final line of the derivation above. Putting it all together we obtain the expected variance term

$$\begin{aligned}
 &\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{ZES}^\top (\mathbf{SS}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \left(\frac{p}{T_1} + \left(\frac{p(p+2)}{T_1(T_1+2)} - \mathbf{1}_p^\top \boldsymbol{\mu}_s \frac{2(T_1-p)(T_1-p+2)}{T^2(T+1)(T+2)} \right) \right) \\
 &\quad + \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \left(\frac{p(p+2)}{T_1(T_1+2)} - \mathbf{1}_p^\top \boldsymbol{\mu}_s \frac{2(T_1-p)(T_1-p+2)}{T^2(T+1)(T+2)} \right) - \left(\text{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon) \frac{p}{T_1} \right)^2 \\
 &= \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \frac{p T_1 (T_1 + 1) (p + T_1 + 4) - 2 (p - T_1 - 2) (p - T_1) \mathbf{1}_p^\top \boldsymbol{\mu}_s}{T_1^2 (T_1 + 1) (T_1 + 2)} \\
 &\quad - \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \frac{2 (p - T_1) \left((p - 2) \mathbf{1}_p^\top \boldsymbol{\mu}_s + T_1 (p - \mathbf{1}_p^\top \boldsymbol{\mu}_s) + p \right)}{T_1^2 (T_1 + 1) (T_1 + 2)}.
 \end{aligned}$$

Next we begin to tackle the covariance terms, again we use the fact that the sample covariance is an unbiased estimator and the iid property. The first two covariance terms are zero:

$$\mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] = 2\text{Cov}[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t, (\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1}] = 0,$$

since $\mathbb{E}[\epsilon_{t+1}] = \mathbf{0}$, and

$$\mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] = 2\text{Cov} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t, (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right]$$

is null for all $t \in \mathcal{T}_1$ as well since $\mathbb{E}[\epsilon_{t+1}] = \mathbf{0}$. Regarding the next covariance term,

$$\begin{aligned} & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &= 2\text{Cov} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t, (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right], \quad \text{for all } t \in \mathcal{T}_1, \\ &= 2 \left(\mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] - \mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right] \mathbb{E} \left[(\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] \right). \end{aligned}$$

Starting with the first expectation,

$$\begin{aligned} & \mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] \\ &= \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{H} \mathbf{E}^\top \mathbf{Z} \mathbf{E} \mathbf{e}_t \right] \\ &= \text{tr}(\mathbf{Z}\Sigma_\epsilon) \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{H} \mathbf{E} \mathbf{e}_t \right] \\ &= \text{tr}(\mathbf{Z}\Sigma_\epsilon) \text{tr} \left(\mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t \mathbf{s}_t^\top \right] + \mathcal{E}_2 \right) \\ &= \frac{\text{tr}(\mathbf{Z}\Sigma_\epsilon) \text{tr} \left(2\beta^\top \mathbf{Z} \beta (\Sigma_s + \mu_s \mu_s^\top) + \mu_s^\top \beta^\top \mathbf{Z} \beta \mu_s (\Sigma_s + \mu_s \mu_s^\top)^{-1} (\Sigma_s - \mu_s \mu_s^\top) + p\beta^\top \mathbf{Z} \beta \Sigma_s \right)}{T_1 - p - 1} + \varepsilon_2, \end{aligned}$$

where we have proceeded as before with a mixed expectation of $(\mathbf{SS}^\top)^{-1}$ and \mathbf{S} , and

$$\mathcal{E}_2 = \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t \mathbf{s}_t^\top \right] - \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \beta \mathbf{s}_t \mathbf{s}_t^\top \right],$$

and $\varepsilon_2 := \text{tr}(\mathcal{E}_2)$. Next, the second expectation reads

$$\mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right] \mathbb{E} \left[(\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] = \left(\text{tr}(\beta\Sigma_s\beta^\top\mathbf{Z}) + \mu_s^\top \beta^\top \mathbf{Z} \beta \mu_s \right) \frac{p}{T_1} \text{tr}(\mathbf{Z}\Sigma_\epsilon).$$

Putting this together yields

$$\begin{aligned} & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\ &= \frac{2}{T_1 - p - 1} \text{tr}(\mathbf{Z}\Sigma_\epsilon) \text{tr} \left(2\beta^\top \mathbf{Z} \beta (\Sigma_s + \mu_s \mu_s^\top) + \mu_s^\top \beta^\top \mathbf{Z} \beta \mu_s \cdot (\Sigma_s + \mu_s \mu_s^\top)^{-1} (\Sigma_s - \mu_s \mu_s^\top) + p\beta^\top \mathbf{Z} \beta \Sigma_s \right) \\ &\quad - \frac{2p}{T_1} \text{tr}(\mathbf{Z}\Sigma_\epsilon) \left(\text{tr}(\beta\Sigma_s\beta^\top\mathbf{Z}) + \mu_s^\top \beta^\top \mathbf{Z} \beta \mu_s \right) + \varepsilon_2. \end{aligned}$$

The next covariance term is similar, namely

$$\begin{aligned}
 & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= 2\text{Cov} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1}, (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right], \quad \text{for all } t \in \mathcal{T}_1, \\
 &= 2 \left(\mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right] - \mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right] \mathbb{E} \left[(\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right] \right) \\
 &= 2\mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right],
 \end{aligned}$$

since $\mathbb{E}[\epsilon_{t+1}] = \mathbf{0}$. The first expectation can be evaluated as

$$\begin{aligned}
 & \mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right] \\
 &= \mathbb{E} \left[\mathbf{e}_t^\top \mathbf{S}^\top \beta^\top \mathbf{Z} \text{tr}(\mathbf{e}_t^\top \mathbf{S}^\top (\mathbf{SS}^\top)^{-1} \mathbf{S} \mathbf{e}_t) \Sigma_\epsilon \mathbf{Z} \beta \mathbf{S} \mathbf{e}_t \right] \\
 &= \text{tr} \left(\mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \mathbf{Z} \Sigma_\epsilon \mathbf{Z}^\top \beta \mathbf{s}_t \mathbf{s}_t^\top \right] + \mathcal{E}_3 \right) \\
 &= \frac{1}{T_1 - p - 1} \text{tr} \left(2\mathbf{F}(\Sigma_s + \mu_s \mu_s^\top) + \mu_s^\top \mathbf{F} \mu_s \cdot (\Sigma_s + \mu_s \mu_s^\top)^{-1} (\Sigma_s - \mu_s \mu_s^\top) + p\mathbf{F} \Sigma_s \right) + \varepsilon_3,
 \end{aligned}$$

where again we handle the $(\mathbf{SS}^\top)^{-1}$ terms as before, and

$$\mathcal{E}_3 = \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \mathbf{Z} \Sigma_\epsilon \mathbf{Z}^\top \beta \mathbf{s}_t \mathbf{s}_t^\top \right] - \mathbb{E} \left[(\mathbf{SS}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \beta^\top \mathbf{Z} \mathbf{Z} \Sigma_\epsilon \mathbf{Z}^\top \beta \mathbf{s}_t \mathbf{s}_t^\top \right],$$

and $\varepsilon_3 := \text{tr}(\mathcal{E}_3)$. Thus,

$$\begin{aligned}
 & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= \frac{2}{T_1 - p - 1} \text{tr} \left(2\mathbf{F}(\Sigma_s + \mu_s \mu_s^\top) + \mu_s^\top \mathbf{F} \mu_s \cdot (\Sigma_s + \mu_s \mu_s^\top)^{-1} (\Sigma_s - \mu_s \mu_s^\top) + p\mathbf{F} \Sigma_s \right) + \varepsilon_3,
 \end{aligned}$$

where again $\mathbf{F} = \beta^\top \mathbf{Z} \Sigma_\epsilon \mathbf{Z} \beta$. We turn now to the penultimate term:

$$\begin{aligned}
 & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= 2\text{Cov} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1}, (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right], \quad \text{for all } t \in \mathcal{T}_1, \\
 &= 2 \left(\mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] - \mathbb{E} \left[(\mathbf{Z}\beta\mathbf{s}_t)^\top \epsilon_{t+1} \right] \mathbb{E} \left[(\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right] \right) = 0,
 \end{aligned}$$

as the derivation yields a cubic in \mathbf{E} and $\mathbb{E}[\epsilon_{t+1}] = \mathbf{0}$.

For the final term, we can write

$$\begin{aligned}
 & \mathbb{E} \left[2\widehat{\text{Cov}} \left[\left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1}, \left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] \\
 &= 2\mathbb{E} \left[\widehat{\mathbb{E}} \left[\left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t (\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right. \\
 & \quad \left. - \widehat{\mathbb{E}} \left[\left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \beta\mathbf{s}_t \right)_{t \in \mathcal{T}_1} \right] \widehat{\mathbb{E}} \left[\left((\mathbf{ZES}^\top(\mathbf{SS}^\top)^{-1}\mathbf{s}_t)^\top \epsilon_{t+1} \right)_{t \in \mathcal{T}_1} \right] \right] = 0,
 \end{aligned}$$

which again yields terms with only \mathbf{E} or cubics in \mathbf{E} , and from the identity for cubic expectations we see that every term in the expectation includes $\boldsymbol{\mu}_\epsilon$ which is zero, and thus the full expectation is zero. We can now collate all the terms for the expected variance $\mathbb{E}[\widehat{\mathbb{V}}[(\widehat{\mathbf{PnL}}_t)_{t \in \mathcal{T}_1}]]$:

$$\begin{aligned}
 & \mathbb{E}[\widehat{\mathbb{V}}[(\widehat{\mathbf{PnL}}_t)_{t \in \mathcal{T}_1}]] \\
 &= 2 \operatorname{tr} \left((\mathbf{G} \boldsymbol{\Sigma}_s)^2 \right) + 4 \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\Sigma}_s \mathbf{G} \boldsymbol{\mu}_s + \operatorname{tr}(\mathbf{F} \boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s \\
 &+ \frac{3}{T_1 - p - 1} \operatorname{tr} \left(2 \mathbf{F}(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + \boldsymbol{\mu}_s^\top \mathbf{F} \boldsymbol{\mu}_s \cdot (\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1} (\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + p \mathbf{F} \boldsymbol{\Sigma}_s \right) \\
 &+ \operatorname{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}) \frac{p T_1 (T_1 + 1) (p + T_1 + 4) - 2 (p - T_1 - 2) (p - T_1) \mathbf{1}_p^\top \boldsymbol{\mu}_s}{T_1^2 (T_1 + 1) (T_1 + 2)} \\
 &- \operatorname{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z})^2 \frac{2 (p - T_1) \left((p - 2) \mathbf{1}_p^\top \boldsymbol{\mu}_s + T_1 (p - \mathbf{1}_p^\top \boldsymbol{\mu}_s) + p \right)}{T_1^2 (T_1 + 1) (T_1 + 2)} \\
 &+ \frac{2}{T_1 - p - 1} \operatorname{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon) \operatorname{tr} \left(2 \mathbf{G}(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\mu}_s \cdot (\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1} (\boldsymbol{\Sigma}_s - \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top) + p \mathbf{G} \boldsymbol{\Sigma}_s \right) \\
 &- \frac{2p}{T_1} \operatorname{tr}(\mathbf{Z} \boldsymbol{\Sigma}_\epsilon) \left(\operatorname{tr}(\mathbf{G} \boldsymbol{\Sigma}_s) + \boldsymbol{\mu}_s^\top \mathbf{G} \boldsymbol{\mu}_s \right) \\
 &+ \varepsilon_1 + \varepsilon_2 + \varepsilon_3,
 \end{aligned}$$

where

$$\mathbf{F} := \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \quad \text{and} \quad \mathbf{G} := \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta},$$

and

$$\begin{aligned}
 \varepsilon_1 &= \operatorname{tr} \left(\mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right] - \mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \mathbf{S} \mathbf{e}_t \mathbf{e}_t^\top \mathbf{S}^\top \right] \right), \\
 \varepsilon_2 &= \operatorname{tr} \left(\mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] - \mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right), \\
 \varepsilon_3 &= \operatorname{tr} \left(\mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \mathbf{T} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] - \mathbb{E} \left[(\mathbf{S} \mathbf{S}^\top)^{-1} \right] \mathbb{E} \left[\mathbf{s}_t \mathbf{s}_t^\top \boldsymbol{\beta}^\top \mathbf{Z} \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \mathbf{T} \boldsymbol{\beta} \mathbf{s}_t \mathbf{s}_t^\top \right] \right).
 \end{aligned}$$

C.3 Out-of-sample expected average

Next we derive the expected out-of-sample expected return and variance. We now have that $\widehat{\boldsymbol{\beta}}$ is independent of the realizations of the signal and the residuals, thus the derivation is simplified significantly. Firstly, the expected out-of-sample P&L is given by

$$\mathbb{E} \left[\widehat{\mathbb{E}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top (\boldsymbol{\beta} \mathbf{s}_t + \boldsymbol{\epsilon}_{t+1}) \right)_{t \in \mathcal{T}_2} \right] \right] = \mathbb{E}[(\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t] = \operatorname{tr}(\boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z}) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\mu}_s,$$

where we have no $\mathbb{E}[(\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1}]$ term since $\mathbb{E}[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$ and by independence.

C.4 Out-of-sample expected variance

Next the variance is given by

$$\begin{aligned}
 & \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top (\boldsymbol{\beta} \mathbf{s}_t + \boldsymbol{\epsilon}_{t+1}) \right)_{t \in \mathcal{T}_2} \right] \right] \\
 &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] + \widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] + 2 \widehat{\operatorname{Cov}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2}, \left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right].
 \end{aligned}$$

The first term can be evaluated as

$$\begin{aligned} \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] + \widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \\ &\quad + 2 \widehat{\text{Cov}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2}, \left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right], \end{aligned}$$

where we must be careful to note that $\mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1}$, i.e. the error in $\widehat{\boldsymbol{\beta}}$, are the errors and signals from the first period, not the second. The first term of this is simply given by

$$\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] = \mathbb{V}[(\mathbf{Z} \boldsymbol{\beta} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t] = 2 \text{tr}(\mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top) + 4 \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\mu}_s,$$

where the first equality holds for all $t \in \mathcal{T}_2$ since \mathbf{s}_t are iid. The covariance term is zero since $\mathbb{E}[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$, so we are left with $\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right]$. With $\boldsymbol{\epsilon}_\beta := \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1}$,

$$\begin{aligned} \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] &= \mathbb{V} \left[(\mathbf{Z} \boldsymbol{\epsilon}_\beta \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right], \quad \text{for all } t \in \mathcal{T}_2, \\ &= \mathbb{E} \left[\text{tr} \left(\boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \left(\boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \boldsymbol{\beta} + \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\epsilon}_\beta \right) \boldsymbol{\Sigma}_s \right) + \boldsymbol{\mu}_s^\top \left(\boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \boldsymbol{\beta} + \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\epsilon}_\beta \right) \boldsymbol{\Sigma}_s \left(\boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \boldsymbol{\beta} + \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\epsilon}_\beta \right) \boldsymbol{\mu}_s \right]. \end{aligned}$$

To evaluate this we require some intermediate expectations,

$$\begin{aligned} \mathbb{E} \left[\boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\epsilon}_\beta^\top \mathbf{Z} \right] &= \mathbf{D} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z}, \\ \mathbb{E} \left[\boldsymbol{\epsilon}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\epsilon} \right] &= \mathbf{D} \text{tr} \left(\boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \right), \\ \mathbb{E} \left[\boldsymbol{\epsilon}_\beta \boldsymbol{\Sigma}_s \boldsymbol{\epsilon}_\beta^\top \right] &= \boldsymbol{\Sigma}_\epsilon \text{tr} \left(\boldsymbol{\Sigma}_s^\top \mathbf{D} \right), \end{aligned}$$

where

$$\mathbf{D} := \frac{(\boldsymbol{\Sigma}_s + \boldsymbol{\mu}_s \boldsymbol{\mu}_s^\top)^{-1}}{T_1 - p - 1}.$$

Putting this all together yields

$$\begin{aligned} &\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \mathbf{E} \mathbf{S}^\top (\mathbf{S} \mathbf{S}^\top)^{-1} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] \\ &= \text{tr} \left(\mathbf{D} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \right) + \text{tr}(\mathbf{D} \boldsymbol{\Sigma}_s) \text{tr} \left(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \right) \\ &\quad + \boldsymbol{\mu}_s^\top \left(2 \mathbf{D} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} + \text{tr} \left(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \right) \mathbf{D} + \text{tr} \left(\boldsymbol{\Sigma}_s^\top \mathbf{D} \right) \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \right) \boldsymbol{\mu}_s. \end{aligned}$$

Thus, the first term for the expected variance out-of-sample is

$$\begin{aligned} &\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z} \widehat{\boldsymbol{\beta}} \mathbf{s}_t)^\top \boldsymbol{\beta} \mathbf{s}_t \right)_{t \in \mathcal{T}_2} \right] \right] \\ &= 2 \text{tr}(\mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top) + 4 \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\mu}_s \\ &\quad + \text{tr} \left(\mathbf{D} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s \right) + \text{tr}(\mathbf{D} \boldsymbol{\Sigma}_s) \text{tr} \left(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \right) \\ &\quad + \boldsymbol{\mu}_s^\top \left(2 \mathbf{D} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} + \text{tr} \left(\boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z} \right) \mathbf{D} + \text{tr} \left(\boldsymbol{\Sigma}_s^\top \mathbf{D} \right) \boldsymbol{\beta}^\top \mathbf{Z} \boldsymbol{\Sigma}_\epsilon \mathbf{Z} \boldsymbol{\beta} \right) \boldsymbol{\mu}_s. \end{aligned}$$

We can proceed similarly for the second term,

$$\begin{aligned} \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\widehat{\boldsymbol{\beta}}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right] &= \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] + \widehat{\mathbb{V}} \left[\left((\mathbf{Z}\mathbf{E}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}\mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right. \\ &\quad \left. + 2\widehat{\text{Cov}} \left[\left((\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2}, \left((\mathbf{Z}\mathbf{E}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}\mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right]. \end{aligned}$$

Again the first term is simply

$$\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right] = \mathbb{V} \left[(\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right] = \text{tr}(\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t} \boldsymbol{\mu}_s,$$

and the final term is zero as \mathbf{E} is from the in-sample period, and therefore independent of $\boldsymbol{\epsilon}_{t+1}$ from the out-of-sample period, and $\mathbb{E}[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$. Thus, we need only to find the second term,

$$\begin{aligned} \mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\mathbf{E}\mathbf{S}^\top(\mathbf{S}\mathbf{S}^\top)^{-1}\mathbf{s}_t)^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right] &= \mathbb{E} \left[\text{tr}(\mathbf{Z}\boldsymbol{\epsilon}_\beta \boldsymbol{\Sigma}_s \boldsymbol{\epsilon}_\beta^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon) + \boldsymbol{\mu}_s^\top \boldsymbol{\epsilon}_\beta^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\epsilon}_\beta \boldsymbol{\mu}_s \right] \\ &= \text{tr} \left((\mathbf{Z}\boldsymbol{\Sigma}_\epsilon)^2 \right) \left(\text{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) + \boldsymbol{\mu}_s^\top \mathbf{D} \boldsymbol{\mu}_s \right). \end{aligned}$$

And therefore, the full second term is given by

$$\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\widehat{\boldsymbol{\beta}}_{\mathbf{s}_t})^\top \boldsymbol{\epsilon}_{t+1} \right)_{t \in \mathcal{T}_2} \right] \right] = \text{tr}(\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t} \boldsymbol{\mu}_s + \text{tr} \left((\mathbf{Z}\boldsymbol{\Sigma}_\epsilon)^2 \right) \left(\text{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) + \boldsymbol{\mu}_s^\top \mathbf{D} \boldsymbol{\mu}_s \right).$$

The third term, the covariance term, is necessarily zero by $\mathbb{E}[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$ and independence, thus the expected out-of-sample variance is given by

$$\begin{aligned} &\mathbb{E} \left[\widehat{\mathbb{V}} \left[\left((\mathbf{Z}\widehat{\boldsymbol{\beta}}_{\mathbf{s}_t})^\top (\boldsymbol{\beta}_{\mathbf{s}_t} + \boldsymbol{\epsilon}_{t+1}) \right)_{t \in \mathcal{T}_2} \right] \right] \\ &= 2 \text{tr}(\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top) + 4\boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t} \boldsymbol{\mu}_s \\ &\quad + \text{tr}(\mathbf{D}\boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}) + \text{tr}(\mathbf{D}\boldsymbol{\Sigma}_s) \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}) \\ &\quad + \boldsymbol{\mu}_s^\top \left(2\mathbf{D}\boldsymbol{\Sigma}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta} + \text{tr}(\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}) \mathbf{D} + \text{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta} \right) \boldsymbol{\mu}_s \\ &\quad + \text{tr}(\mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t}\boldsymbol{\beta}_{\mathbf{s}_t}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon) + \boldsymbol{\mu}_s^\top \boldsymbol{\beta}^\top \mathbf{Z}\boldsymbol{\Sigma}_\epsilon \mathbf{Z}\boldsymbol{\beta}_{\mathbf{s}_t} \boldsymbol{\mu}_s + \text{tr} \left((\mathbf{Z}\boldsymbol{\Sigma}_\epsilon)^2 \right) \left(\text{tr}(\boldsymbol{\Sigma}_s^\top \mathbf{D}) + \boldsymbol{\mu}_s^\top \mathbf{D} \boldsymbol{\mu}_s \right). \end{aligned}$$

D Monte-Carlo Details

As previously mentioned, we use Monte-Carlo simulations both to check that the approximations we have made in (3.2) are reasonable ($\varepsilon_i = 0$ and ignoring the convexity adjustment) and to support and provide additional analysis. To do this, we observe the point estimates of the parameters from the experiments using commodity futures in Section 5 and select distributions to draw new realizations within a cloud around these point estimates. Specifically,

- For the signals we let the standard deviation of the signals be 1, which is unrestrictive as we can always standardize any given signal.
- For the residuals we sample the variances from a χ^2 distribution, which is well known to be the sample distribution for variance.
- For both signals and residuals we draw correlation matrices from the LJK distribution [LKJ09], commonly used as a prior for correlation matrices in Bayesian modelling, and which can be considered the correlation equivalent of the Wishart distribution used for covariance matrices.

- We can then construct the covariance matrix $\Sigma = \text{diag}(\sigma)\mathbf{P}\text{diag}(\sigma)$, where σ is the vector of variances per signal/residual and \mathbf{P} is the sampled correlation matrix for the signal/residuals.
- For β we sample entries from a Laplace distribution, which provided the best fit when looking at the entries of real β matrices from Section 5.

With β and $\Sigma_\epsilon, \Sigma_s$ coming from real data we would expect certain structures such as clusters with the covariance matrix (e.g. similarity between stocks in the same sector), or similar parameters for certain stocks; this random sampling unfortunately misses out on this. However, from studying the expressions for the replication ratio it does not appear that these characteristics would impact our results. This is further confirmed by noting that when using real assets and signals in Section 5, the results remain valid. To achieve a desired true Sharpe ratio we simply scale the matrix β appropriately. Finally, we let $\mu_s = \mathbf{0}$ and do not include a drift term.

We note here that to simulate even 100 assets requires a significant amount of compute, particularly with large T , which realistically will almost always be the case for a strategy of this style. Thus, the analytical approximations are useful to enable quick comparative adjustments for strategies, without requiring lengthy simulations.

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Conflict of Interest Statement

The authors have no conflicts of interest to disclose.

Data Availability Statement

The data that support the findings of this study consists of publicly available commodity futures prices, which are accessible from various market data providers.

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