

An Interactive Shader for Natural Diffraction Gratings

Bachelorarbeit

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Michael Single

2014

Leiter der Arbeit:
Prof. Dr. Matthias Zwicker
Institut für Informatik und angewandte Mathematik

Abstract

In nature, animals exhibit structural colors because of the physical interaction of light with the surface nanostructure of their exterior skin. In his pioneering work, J.Stam developed a reflectance model based on wave optics capturing the effect of diffraction from surface nanostructures. His model is limited by an accurate estimate of the correlation function using statistical properties of the surface's height field. We propose an adaption of his BRDF model that can handle complex natural gratings. Furthermore, we describe a method for interactively rendering of diffraction effects due to interaction of light with biological nano-structures such as snake skin. As input data, our method uses discrete height fields of natural gratings acquired by using atomic force microscopy (AFM) and employ Fourier Optics. Based on Taylor Series approximation for the phase shifts at the nanoscale surface, we leverage the precomputation of the discrete Fourier Transformations, involved in our model, to achieve interactive rendering speed (about 5-15 fps). We demonstrate results of our approach using surface nano-structures of two snake species, namely the *Elaphe* and *Xenopeltis* species, when applied to a measured snake geometry. Lastly, we evaluate the quality of our method by comparing its (peak) viewing angles with maximum reflectance for a fixed incident beam with those resulting from the grating equation at different wavelengths. We conclude that our method produces accurate results for complex, natural gratings at interactive speed.

Contents

| | | |
|----------|--|-----------|
| A | Signal Processing Basics | 1 |
| A.1 | Fourier Transformation | 1 |
| A.2 | Convolution | 3 |
| A.3 | Taylor Series | 3 |
| B | Summary of Stam's Derivations | 4 |
| C | Derivation Steps in Detail | 7 |
| C.1 | Taylor Series Approximation | 7 |
| C.1.1 | Proof Sketch of 1. | 7 |
| C.1.2 | Part 2: Find such an N | 7 |
| C.2 | PQ approach | 9 |
| C.2.1 | One dimensional case | 9 |
| C.2.2 | Two dimensional case | 10 |
| D | Miscellaneous Transformations | 12 |
| D.1 | Fresnel Term - Schlick's approximation | 12 |
| D.2 | Spherical Coordinates and Space Transformation | 13 |
| D.3 | Tangent Space | 13 |
| | List of Tables | 15 |
| | List of Figures | 15 |
| | List of Algorithms | 16 |
| | Bibliography | 17 |

Appendix A

Signal Processing Basics

A signal is a function that conveys information about the behavior or attributes of some phenomenon. In the physical world, any quantity exhibiting variation in time or variation in space (such as an image) is potentially a signal that might provide information on the status of a physical system, or convey a message between observers.

The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components. The output of the transformation represents the image in the Fourier or frequency domain, while the input image is the spatial domain equivalent. In the Fourier domain image, each point represents a particular frequency contained in the spatial domain image.

A.1 Fourier Transformation

The Fourier-Transform is a mathematical tool which allows to transform a given function or rather a given signal from defined over a time- (or spatial-) domain into its corresponding frequency-domain.

Let f an measurable function over \mathbb{R}^n . Then, the continuous Fourier Transformation(**FT**), denoted as $\mathcal{F}\{f\}$ of f , ignoring all constant factors in the formula, is defined as:

$$\mathcal{F}_{FT}\{f\}(w) = \int_{\mathbb{R}^n} f(x)e^{-iwt}dt \quad (\text{A.1})$$

whereas its inverse transform is defined like the following which allows us to obtain back the original signal:

$$\mathcal{F}_{FT}^{-1}\{f\}(w) = \int_{\mathbb{R}} \mathcal{F}\{w\}e^{iwt}dt \quad (\text{A.2})$$

Usual w is identified by the angular frequency which is equal $w = \frac{2\pi}{T} = 2\pi v_f$. In this connection, T is the period of the resulting spectrum and v_f is its corresponding frequency.

By using Fourier Analysis, which is the approach to approximate any function by sums of simpler trigonometric functions, we gain the so called Discrete Time Fourier Transform (in short **DTFT**). The DTFT operates on a discrete function. Usually, such an input function is often created by digitally sampling a continuous function. The DTFT itself is operation on a discretized signal on a continuous, periodic frequency domain and looks like the following:

$$\mathcal{F}_{DTFT}\{f\}(w) = \sum_{-\infty}^{\infty} f(x)e^{-iwx} \quad (\text{A.3})$$

Note that the DTFT is not practically suitable for digital signal processing since there a signal can be measured only in a finite number of points. Thus, we can further discretize the frequency domain and will get then the Discrete Fourier Transformation (in short **DFT**) of the input signal:

$$\mathcal{F}_{DFT}\{f\}(w) = \sum_{n=0}^{N-1} f(x)e^{-iwn} \quad (\text{A.4})$$

Where the angular frequency w_n is defined like the following $w_n = \frac{2\pi n}{N}$ and N is the number of samples within an equidistant period sampling.

Any continuous function $f(t)$ can be expressed as a series of sines and cosines. This representation is called the Fourier Series (denoted by *FS*) of $f(t)$.

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \quad (\text{A.5})$$

where

$$\begin{aligned} a_0 &= \int_{-\pi}^{\pi} f(t)dt \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)dt \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)dt \end{aligned} \quad (\text{A.6})$$

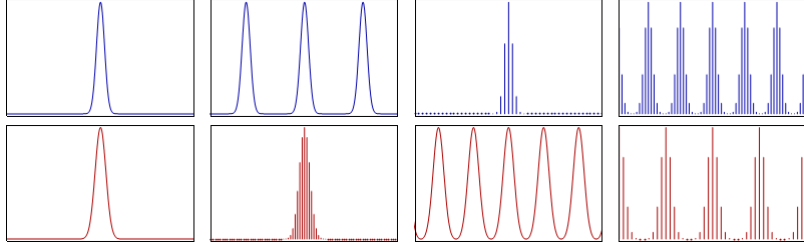


Figure A.1: Relationship¹ between the continuous Fourier transform and the discrete Fourier transform: Left column: A continuous function (top) and its Fourier transform A.1 (bottom). Center-left column: Periodic summation of the original function (top). Fourier transform (bottom) is zero except at discrete points. The inverse transform is a sum of sinusoids called Fourier series A.5. Center-right column: Original function is discretized (multiplied by a Dirac comb) (top). Its Fourier transform (bottom) is a periodic summation (DTFT) of the original transform. Right column: The DFT A.4 (bottom) computes discrete samples of the continuous DTFT A.3. The inverse DFT (top) is a periodic summation of the original samples.

¹image of illustration has been taken from wikipedia

| Spetail signal $f(t)$ is | Operator | Transformed frequency signal $\hat{f}(\omega)$ is |
|--------------------------------|----------|---|
| continuous and periodic in t | FS A.5 | only discrete in ω |
| only continuous in t | FT A.1 | only continuous in ω |
| only discrete in t | DTFT A.3 | continuous and periodic in ω |
| discrete and periodic in t | DFT A.4 | discrete and periodic in ω |

Table A.1: Fourier operator to apply for a given spatial input signal and the properties of its resulting output signal in frequency space

A.2 Convolution

The convolution $f * g$ of two functions $f, g: \mathbb{R}^n \rightarrow \mathbb{C}$ is defined as:

$$(f * g)(t) = \int_{\mathbb{R}^n} f(t)g(t - x)dx \quad (\text{A.7})$$

Note that the Fourier transform of the convolution of two functions is the product of their Fourier transforms. This is equivalent to the fact that Convolution in spatial domain is equivalent to multiplication in frequency domain. Therefore, the inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms. Last an illustration of the relationships between the previous presented Fourier transformations and different given input signals. First an concrete example shown in Figure A.1. Table A.1 tells what Fourier transformation operator has to be applied to which kind of input signal and what properties its resulting Fourier transform will have.

A.3 Taylor Series

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

The Taylor series \mathcal{T} of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series:

$$\mathcal{T}(f; a)(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n \quad (\text{A.8})$$

Appendix B

Summary of Stam's Derivations

In his paper about Diffraction Shader, J. Stam derives a BRDF which is modeling the effect of diffraction for various analytical anisotropic reflexion models relying on the so called scalar wave theory of diffraction for which a wave is assumed to be a complex valued scalar. It's noteworthy, that Stam's BRDF formulation does not take into account the polarization of the light. Fortunately, light sources like sunlight and light bulbs are unpolarized.

A further assumption in Stam's Paper is, the emanated waves from the source are stationary, which implies the wave is a superposition of independent monochromatic waves. This implies that each wave is associated to a definite wavelength λ . However, sunlight once again fulfills this fact.

In our simulations we will always assume we have given a directional light source, i.e. sunlight. Hence, Stam's model can be used for our derivations.

For his derivations Stam uses the Kirchhoff integral¹, which is relating the reflected field to the incoming field. This equation is a formalization of Huygen's well-known principle that states that if one knows the wavefront at a given moment, the wave at a later time can be deduced by considering each point on the first wave as the source of a new disturbance. Mathematically speaking, once the field $\psi_1 = e^{ik\mathbf{x}\cdot\mathbf{s}}$ on the surface is known, the field ψ_2 everywhere else away from the surface can be computed. More precisely, we want to compute the wave ψ_2 equal to the reflection of an incoming planar monochromatic wave $\psi_1 = e^{ik\omega_i\cdot\mathbf{x}}$ traveling in the direction ω_i from a surface S to the light source. Formally, this can be written as:

$$\psi_2(\omega_i, \omega_r) = \frac{ike^{iKR}}{4\pi R} (F(-\omega_i - \omega_r) - (-\omega_i + \omega_r)) \cdot I_1(\omega_i, \omega_r) \quad (\text{B.1})$$

with

$$I_1(\omega_i, \omega_r) = \int_S \hat{\mathbf{n}} e^{ik(-\omega_i - \omega_r)\cdot\mathbf{s}} d\mathbf{s} \quad (\text{B.2})$$

In applied optics, when dealing with scattered waves, one does use differential scattering cross-section rather than defining a BRDF which has the following identity:

$$\sigma^0 = 4\pi \lim_{R \rightarrow \infty} R^2 \frac{\langle |\psi_2|^2 \rangle}{\langle |\psi_1|^2 \rangle} \quad (\text{B.3})$$

where R is the distance from the center of the patch to the receiving point x_p , $\hat{\mathbf{n}}$ is the normal of the surface at \mathbf{s} and the vectors:

¹See http://en.wikipedia.org/wiki/Kirchhoff_integral_theorem for further information.

The relationship between the BRDF and the scattering cross section can be shown to be equal to

$$BRDF = \frac{1}{4\pi} \frac{1}{A} \frac{\sigma^0}{\cos(\theta_i)\cos(\theta_r)} \quad (B.4)$$

where θ_i and θ_r are the angles of incident and reflected directions on the surface with the surface normal n . See ??.

The components of vector resulting by the difference between these direction vectors: In order to simplify the calculations involved in his vectorized integral equations, Stam considers the components of vector

$$(u, v, w) = -\omega_i - \omega_r \quad (B.5)$$

explicitly and introduces the equation:

$$I(ku, kv) = \int_S \hat{\mathbf{n}} e^{ik(u,v,w) \cdot \mathbf{s}} d\mathbf{s} \quad (B.6)$$

which is a first simplification of B.2. Note that the scalar w is the third component of ?? and can be written as $w = -(\cos(\theta_i) + \cos(\theta_r))$ using spherical coordinates. The scalar $k = \frac{2\pi}{\lambda}$ represent the wavenumber.

During his derivations, Stam provides a analytical representation for the Kirchhoff integral assuming that each surface point $s(x, y)$ can be parameterized by $(x, y, h(x, y))$ where h is the height at the position (x, y) on the given (x, y) surface plane. Using the tangent plane approximation for the parameterized surface and plugging it into B.6 he will end up with:

$$\mathbf{I}(ku, kv) = \int \int (-h_x(x, y), -h_y(x, y), 1) e^{ikwh(x, y)} e^{ik(ux+vy)} dx dy \quad (B.7)$$

For further simplification Stam formulates auxillary function which depends on the provided height field:

$$p(x, y) = e^{ikwh(x, y)} \quad (B.8)$$

which will allow him to further simplify his equation B.7 to:

$$\mathbf{I}(ku, kv) = \int \int \frac{1}{ikw} (-p_x, -p_y, ikwp) dx dy \quad (B.9)$$

where he used that $(-h_x(x, y), -h_y(x, y), 1) e^{ikwh(x, y)}$ is equal to $\frac{(-p_x, -p_y, ikwp)}{ikw}$ using the definition of the partial derivatives applied to the function ??.

Let $P(x, y)$ denote the Fourier Transform (FT) of $p(x, y)$. Then, the differentiation with respect to x respectively to y in the Fourier domain is equivalent to a multiplication of the Fourier transform by $-iku$ or $-ikv$ respectively. This leads him to the following simplification for B.7:

$$\mathbf{I}(ku, kv) = \frac{1}{w} P(ku, kv) \cdot (u, v, w) \quad (B.10)$$

Let us consider the term $g = (F(-\omega_i - \omega_r) - (-\omega_i + \omega_r))$, which is a scalar factor of B.1. The dot product with g and $(-\omega_i - \omega_r)$ is equal $2F(1 + \omega_i \cdot \omega_r)$. Putting this finding and the identity B.10 into B.1 he will end up with:

$$\psi_2(\omega_i, \omega_r) = \frac{ike^{iKR}}{4\pi R} \frac{2F(1 + \omega_i \cdot \omega_r)}{w} P(ku, kv) \quad (B.11)$$

By using the identity *B.4*, this will lead us to his main finding:

$$BRDF_{\lambda}(\omega_i, \omega_r) = \frac{k^2 F^2 G}{4\pi^2 A w^2} \langle |P(ku, kv)|^2 \rangle \quad (\text{B.12})$$

where G is the so called geometry term which is equal:

$$G = \frac{(1 + \omega_i \cdot \omega_r)^2}{\cos(\theta_i) \cos(\theta_r)} \quad (\text{B.13})$$

Appendix C

Derivation Steps in Detail

C.1 Taylor Series Approximation

In order to prove equation ?? from section ?? we have to show the following: For any $N \in \mathbb{N}$ and

$$\sum_{n=0}^N \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta) \approx P(\alpha, \beta) \quad (\text{C.1})$$

we have to prove:

1. Show that there exist such an $N \in \mathbb{N}$ s.t the approximation holds true.
2. Find a value for B s.t. this approximation is below a certain error bound, for example machine precision ϵ .

C.1.1 Proof Sketch of 1.

By the **ratio test** relying on Taylor's Theorem¹ It is possible to show that the series $\sum_{n=0}^N \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta)$ converges absolutely:

Proof: Consider $\sum_{k=0}^{\infty} \frac{y^k}{k!}$ where $a_k = \frac{y^k}{k!}$. By applying the definition of the ratio test for this series it follows:

$$\forall y : \limsup_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \limsup_{k \rightarrow \infty} \frac{y}{k+1} = 0 \quad (\text{C.2})$$

Thus this series converges absolutely, no matter what value we will pick for y.

C.1.2 Part 2: Find such an N

Let $f(x) = e^x$. We can formulate its Taylor-Series, stated above. Let $P_n(x)$ denote the n-th Taylor polynomial,

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \quad (\text{C.3})$$

¹Please have a look at http://en.wikipedia.org/wiki/Taylor's_theorem in order to see a proper definition of the ratio test.

where a is our developing point (here a is equal zero).

We can define the error of the n -th Taylor polynomial to be $E_n(x) = f(x) - P_n(x)$. the error of the n -th Taylor polynomial is difference between the value of the function and the Taylor polynomial. This directly implies $|E_n(x)| = |f(x) - P_n(x)|$. By using the Lagrangian Error Bound it follows:

$$|E_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad (C.4)$$

with $a = 0$, where M is some value satisfying $|f^{(n+1)}(x)| \leq M$ on the interval $I = [a, x]$. Since we are interested in an upper bound of the error and since a is known, we can reformulate the interval as $I = [0, x_{max}]$, where

$$x_{max} = \|i\| k_{max} w_{max} h_{max} \quad (C.5)$$

We are interested in computing an error bound for $e^{ikwh(x,y)}$. Assuming the following parameters and facts used within Stam's Paper:

- Height of bump: 0.15micro meters
- Width of a bump: 0.5micro meters
- Length of a bump: 1micro meters
- $k = \frac{2\pi}{\lambda}$ is the wavenumber, $\lambda \in [\lambda_{min}, \lambda_{max}]$ and thus $k_{max} = \frac{2\pi}{\lambda_{min}}$. Since $(u, v, w) = -\omega_i - \omega_r$ and both are unit direction vectors, each component can have a value in range $[-2, 2]$.
- for simplification, assume $[\lambda_{min}, \lambda_{max}] = [400nm, 700nm]$.

We get:

$$\begin{aligned} x_{max} &= \|i\| * k_{max} * w_{max} * h_{max} \\ &= k_{max} * w_{max} * h_{max} \\ &= 2 * \left(\frac{2\pi}{4 * 10^{-7}m} \right) * 1.5 * 10^{-7} \\ &= 1.5\pi \end{aligned} \quad (C.6)$$

and it follows for our interval $I = [0, 1.5\pi]$. Next we are going to find the value for M . Since the exponential function is monotonically growing (on the interval I) and the derivative of the **exp** function is the exponential function itself, we can find such an M :

$$\begin{aligned} M &= e^{x_{max}} \\ &= \exp(1.5\pi) \end{aligned}$$

and $|f^{(n+1)}(x)| \leq M$ holds. With

$$\begin{aligned} |E_n(x_{max})| &\leq \frac{M}{(n+1)!} |x_{max} - a|^{n+1} \\ &= \frac{\exp(1.5\pi) * (1.5\pi)^{n+1}}{(n+1)!} \end{aligned} \quad (C.7)$$

we now can find a value of n for a given bound, i.e. we can find an value of $N \in \mathbb{N}$ s.t. $\frac{\exp(1.5\pi) * (1.5\pi)^{N+1}}{(N+1)!} \leq \epsilon$. With Octave/Matlab we can see:

- if $N=20$ then $\epsilon \approx 2.9950 * 10^{-4}$
- if $N=25$ then $\epsilon \approx 8.8150 * 10^{-8}$
- if $N=30$ then $\epsilon \approx 1.0050 * 10^{-11}$

With this approach we have that $\sum_{n=0}^{25} \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta)$ is an approximation of $P(u, v)$ with error $\epsilon \approx 8.8150 * 10^{-8}$. This means we can precompute 25 Fourier Transformations in order to approximate $P(u, v)$ having an error $\epsilon \approx 8.8150 * 10^{-8}$.

C.2 PQ approach

C.2.1 One dimensional case

Since our series is bounded, we can simplify the right-hand-side of equation ???. Note that e^{-ix} is a complex number. Every complex number can be written in its polar form, i.e.

$$e^{-ix} = \cos(x) + i\sin(x) \quad (C.8)$$

Using the following trigonometric identities

$$\begin{aligned} \cos(-x) &= \cos(x) \\ \sin(-x) &= -\sin(x) \end{aligned} \quad (C.9)$$

combined with C.8 we can simplify the series ??? even further to:

$$\frac{1 - e^{iwT(N+1)}}{1 - e^{-iwT}} = \frac{1 - \cos(wT(N+1)) + i\sin(wT(N+1))}{1 - \cos(wT) + i\sin(wT)} \quad (C.10)$$

Equation C.10 is still a complex number, denoted as $(p + iq)$. Generally, every complex number can be written as a fraction of two complex numbers. This implies that the complex number $(p + iq)$ can be written as $(p + iq) = \frac{(a+ib)}{(c+id)}$ for any $(a+ib), (c+id) \neq 0$. Let us use the following substitutions:

$$\begin{aligned} a &:= 1 - \cos(wT(N+1)) & b &= \sin(wT(N+1)) \\ c &= 1 - \cos(wT) & d &= \sin(wT) \end{aligned} \quad (C.11)$$

Hence, using C.11, it follows

$$\frac{1 - e^{iwT(N+1)}}{1 - e^{-iwT}} = \frac{(a+ib)}{(c+id)} \quad (C.12)$$

By rearranging the terms, it follows $(a+ib) = (c+id)(p+iq)$ and by multiplying its right hand-side out we get the following system of equations:

$$\begin{aligned} (cp - dq) &= a \\ (dp + cq) &= b \end{aligned} \quad (C.13)$$

After multiplying the first equation of C.13 by c and the second by d and then adding them together, we get using the law of distributivity new identities for p and q :

$$\begin{aligned} p &= \frac{(ac + bd)}{c^2 + d^2} \\ q &= \frac{(bc + ad)}{c^2 + d^2} \end{aligned} \quad (\text{C.14})$$

Using some trigonometric identities and putting our substitution from C.11 for a, b, c, d back into the current representation C.14 of p and q we will get:

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2} \left(\frac{\cos(wTN) - \cos(wT(N+1))}{1 - \cos(wT)} \right) \\ q &= \frac{\sin(wT(N+1)) - \sin(wTN) - \sin(wT)}{2(1 - \cos(wT))} \end{aligned} \quad (\text{C.15})$$

Since we have seen, that $\sum_{n=0}^N e^{-uwnT}$ is a complex number and can be written as $(p + iq)$, we now know an explicit expression for p and q . Therefore, the one dimensional inverse Fourier transform of S is equal:

$$\begin{aligned} \mathcal{F}^{-1}\{S\}(w) &= \mathcal{F}^{-1}\{f\}(w) \sum_{n=0}^N e^{-iwnT} \\ &= (p + iq)\mathcal{F}^{-1}\{f\}(w) \end{aligned} \quad (\text{C.16})$$

C.2.2 Two dimensional case

$$\begin{aligned} \mathcal{F}^{-1}\{S\}(w_1, w_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} h(x_1 + n_1 T_1, x_2 + n_2 T_2) e^{i w(x_1 + x_2)} dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} h(y_1, y_2) e^{i w((y_1 - n_1 T_1) + (y_2 + n_2 T_2))} dx_1 dx_2 \\ &= \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(y_1, y_2) e^{i w(y_1 + y_2)} e^{-i w(n_1 T_1 + n_2 T_2)} dy_1 dy_2 \\ &= \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} e^{-i w(n_1 T_1 + n_2 T_2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Box}(y_1, y_2) e^{i w(y_1 + y_2)} dy_1 dy_2 \\ &= \left(\sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} e^{-i w(n_1 T_1 + n_2 T_2)} \right) \mathcal{F}^{-1}\{h\}(w_1, w_2) \\ &= \left(\sum_{n_2=0}^{N_1} e^{-i w n_1 T_1} \right) \left(\sum_{n_2=0}^{N_2} e^{-i w n_2 T_2} \right) \mathcal{F}^{-1}\{h\}(w_1, w_2) \\ &= (p_1 + iq_1)(p_2 + iq_2) \mathcal{F}^{-1}\{h\}(w_1, w_2) \\ &= ((p_1 p_2 - q_1 q_2) + i(p_1 q_2 + q_1 p_2)) \mathcal{F}^{-1}\{h\}(w_1, w_2) \\ &= (p + iq) \mathcal{F}_{DFTT}\{h\}(w_1, w_2) \end{aligned} \quad (\text{C.17})$$

Where we have defined

$$\begin{aligned} p &:= (p_1 p_2 - q_1 q_2) \\ q &:= (p_1 p_2 + q_1 q_2) \end{aligned} \tag{C.18}$$

Appendix D

Miscellaneous Transformations

D.1 Fresnel Term - Schlick's approximation

The Fresnel's equations describe the reflection and transmission of electromagnetic waves at an interface. That is, they give the reflection and transmission coefficients for waves parallel and perpendicular to the plane of incidence. Schlick's approximation is a formula for approximating the contribution of the Fresnel term where the specular reflection coefficient R can be approximated by:

$$R(\theta) = R_0 + (1 - R_0)(1 - \cos \theta)^5 \quad (\text{D.1})$$

and

$$R_0 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

where θ is the angle between the viewing direction and the half-angle direction, which is halfway between the incident light direction and the viewing direction, hence $\cos \theta = (H \cdot V)$. And n_1, n_2 are the indices of refraction of the two medias at the interface and R_0 is the reflection coefficient for light incoming parallel to the normal (i.e., the value of the Fresnel term when $\theta = 0$ or minimal reflection). In computer graphics, one of the interfaces is usually air, meaning that n_1 very well can be approximated as 1.

D.2 Spherical Coordinates and Space Transformation

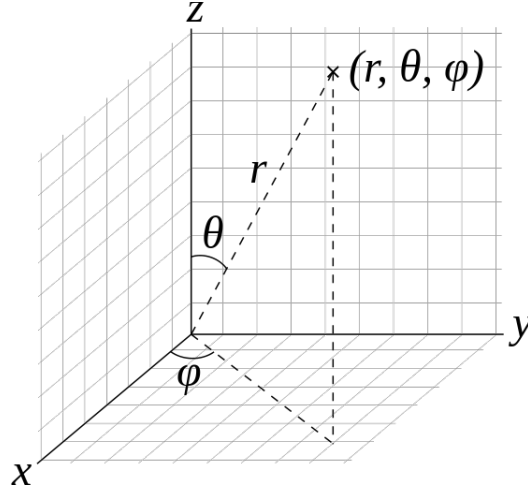


Figure D.1: Illustration¹ of Spherical coordinates (r, θ, ϕ) radius r , polar (inclination) angle θ , and azimuthal angle ϕ .

To define a spherical coordinate system like shown in figure D.1, two angles, the polar angle θ and the azimuthal ϕ plus a radius r are required. Then the Cartesian coordinates (x, y, z) may be retrieved from the spherical coordinates like the following: $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \exists r \in [0, \infty) \exists \phi \in [0, 2\pi] \exists \theta \in [0, \pi]$ s.t.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin(\theta) \cos(\phi) \\ r \sin(\theta) \sin(\phi) \\ r \cos(\theta) \end{pmatrix}$$

From the definition ?? of $(u, v, w) = -\omega_i - \omega_r$ and using spherical coordinates D.2, we get for w the following identity:

$$\begin{aligned} w &= -\omega_i - \omega_r \\ &= -(\omega_i + \omega_r) \\ &= -(\cos(\theta_i) + \cos(\theta_r)) \end{aligned} \tag{D.2}$$

and therefore w^2 is equal $(\cos(\theta_i) + \cos(\theta_r))^2$.

D.3 Tangent Space

The concept of tangentspace-transformation of tangent space is used in order to convert a point between world and tangent space. GLSL fragment shaders require normals and other vertex primitives declared at each pixel point, which mean that we have one normal vector at each texel and

¹image source of figure D.1 http://en.wikipedia.org/wiki/Spherical_coordinate_system

the normal vector axis will vary for every texel.

Think of it as a bumpy surface defined on a flat plane. If those normals were declared in the world space coordinate system, we would have to rotate these normals every time the model is rotated, even when just for a small amount. Since the lights, cameras and other objects are usually defined in world space coordinate system, and therefore, when they are involved in an calculation within the fragment shader, we would have to rotate them as well for every pixel. This would involve almost countless many object to world matrix transformations need to take place at the pixel level. Therefore, instead doing so, we transform all vertex primitives into tangent space within the vertex shader.

To make this point clear an example: Even we would rotate the cube in figure D.2, the tangent space axis will remain aligned with respect to the face. Which practically speaking, will save us from performing many space transformations applied pixel-wise within the fragment shader and instead allows us to perform the tangentspace transformation of every involved vertex primitive in the vertex-shader.

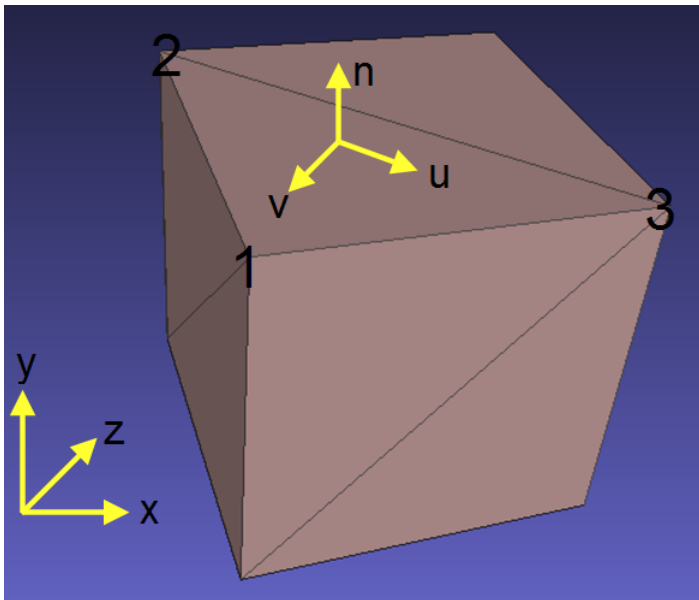


Figure D.2: Cube in world space (x, y, z) showing the tangen space (u, v, n) of its face $(2, 1, 3)$

List of Tables

| | |
|---|---|
| A.1 Fourier Transform Mapping | 3 |
|---|---|

List of Figures

| | | |
|-----|---|----|
| D.1 | Illustration of Spherical Coordinate System | 13 |
| D.2 | Illustration of a Tangent Space | 14 |

List of Algorithms

Bibliography

- [Bar07] BARTSCH, Hans-Jochen: *Taschenbuch Mathematischer Formeln*. 21th edition. HASNER, 2007. – ISBN 978–3–8348–1232–2
- [CT12] CUYPERS T., et a.: Reflectance Model for Diffraction. In: *ACM Trans. Graph.* 31, 5 (2012), September
- [DD14] D.S. DHILLON, et a.: Interactive Diffraction from Biological Nanostructures. In: *EUROGRAPHICS 2014/ M. Paulin and C. Dachsbacher* (2014), January
- [D.S14] D.S.DHILLON, M.Single I.Gaponenko M.C. Milinkovitch M. J.Teyssier: Interactive Diffraction from Biological Nanostructures. In: *Submitted at Computer Graphics Forum* (2014)
- [For11] FORSTER, Otto: *Analysis 3*. 6th edition. VIEWEG+TEUBNER, 2011. – ISBN 978–3–8348–1232–2
- [I.N14] I.NEWTON: *Opticks, reprinted*. CreateSpace Independent Publishing Platform, 2014. – ISBN 978–1499151312
- [JG04] JUAN GUARDADO, NVIDIA: Simulating Diffraction. In: *GPU Gems* (2004). <https://developer.nvidia.com/content/gpu-gems-chapter-8-simulating-diffraction>
- [LM95] LEONARD MANDEL, Emil W.: *Optical Coherence and Quantum Optics*. Cambridge University Press, 1995. – ISBN 978–0521417112
- [MT10] MATIN T.R., et a.: Correlating Nanostructures with Function: Structural Colors on the Wings of a Malaysian Bee. (2010), August
- [PAT09] PAUL A. TIPLER, Gene M.: *Physik für Wissenschaftler und Ingenieure*. 6th edition. Spektrum Verlag, 2009. – ISBN 978–3–8274–1945–3
- [PS09] P. SHIRLEY, S. M.: *Fundamentals of Computer Graphics*. 3rd edition. A K Peters, Ltd, 2009. – ISBN 978–1–56881–469–8
- [R.H12] R.HOOKE: *Micrographia, reprinted*. CreateSpace Independent Publishing Platform, 2012. – ISBN 978–1470079031
- [RW11] R. WRIGHT, et a.: *OpenGL SuperBible*. 5th edition. Addison-Wesley, 2011. – ISBN 978–0–32–171261–5
- [Sta99] STAM, J.: Diffraction Shaders. In: *SIGGRAPH 99 Conference Proceedings* (1999), August
- [T.Y07] T.YOUNG: *A course of lectures on natural philosophy and the mechanical arts Volume 1 and 2*. Johnson, 1807, 1807