

Diffraction Shader

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1 Introduction

1.1 Motivation

In Nature, coloring mostly comes from the inherent colors of materials but sometimes colorization has a pure physical origin such as the effect diffraction or interference of light. Both phenomenon are causing the so called structural colorization, which is the production of color through the interaction of visible light with microscopically structured surfaces. Therefore we perceive color when the different wavelengths composing white light are selectively interfered with by matter (absorbed, reflected, refracted, scattered, or diffracted) on their way to our eyes, or when a non-white distribution of light has been emitted. In animals, such as feathers of birds and the scales of butterflies, interference is created by a range of photonic mechanisms, including diffraction grating, selective mirrors, photonic crystals. The connection between microscopic structures and coloration has been observed by Robert Hooke in the early seventeenth century. The discovery of the wave nature of light led to the conclusion that the cause for the coloration lies in wave interference.

In the field of computer graphics, many researchers have been attempted rendering of structural colors but most of the techniques so far, however, are either too slow for interactive rendering or rely on simplifying assumption, like modeling light as rays, to achieve real-time performance.

1.2 Goal

The purpose of this thesis is to render realtime the effect of diffraction on different snakes skins in a photorealistic manor. In oder to achieve this purpose we will rely J. Stam's formulation of a BRDF which basically describes the effect of diffraction on a given surface assuming one knows the hightfield on this surface. In our case, those heightfields are small patches of the nanostructure of the snake skin provided by GENEVA taken by MIKROSKOP. In his Paper, J. Stam assuming distribution on his heightfields whereas we require a an explicit provided hightfield of the surface or at least a small patch. Therefore, this work can be considered as an extension of J. Stam's derivations for the case one is provided by a explicit height field on a quasi-periodic structure. Since one goal of this work is to render in realtime, we have to perform also precomputations which will require us to slightly modify

Stam's main derivation.
 what we want to achieve

1.3 Basis: J. Stam

——— stam employee scalar wave theory of diffraction - approximation the light wave is assumed to be a complex valued scalar disturbance Φ Theory completely ignores the polarization of light, so its results are therefore restricted to unpolarized light but common light sources such as the sun and light bulbs are totally unpolarized

also assume that the waves emanating from the source are stationary. means that the wave is a superposition of independent monochromatic waves and consequently we can restrict our analysis to a wave having a definite wavelength λ associated with i

GOAL main task in the theory of diffraction is to solve this wave equation for different geometries. we are interested in computing the reflected waves from various types of surfaces

More precisely, we want to compute the wave Φ_2 equal to the reflection of an incoming planar monochromatic wave $\phi_2 = e^{ik_1 \cdot x}$ traveling in the direction k_1 from a surface S .

The equation relating the reflected field to the incoming field is known as the Kirchhoff integral. This equation is a formalization of Huygen's well-known principle that states that if one knows the wavefront at a given moment, the wave at a later time can be deduced by considering each point on the first wave as the source of a new disturbance, i.e. once the field on the surface is known, the field everywhere else away from the surface can be computed.

which will result in $\phi_2 = \frac{ike^{ikR}}{Fv-p} * \int_S n * e^{ikv \cdot s} ds$ where x is the observing point, k_2 is equal to the unit vector pointing from the origin of the surface towards the point x .

In this paper we will provide in every case the BRDF corresponding to our reflection model. In the applied optics literature, when dealing with scattered waves from a surface, one does not usually define the BRDF but rather the differential scattering cross-section $\sigma^0 = 4\pi \lim_{R \rightarrow \infty} R^2 \frac{abs(\phi_2)^2}{abs(\phi_1)^2}$

The relationship between the BRDF and the scattering cross section can be shown to be equal to $BRDF = \frac{1}{4\pi A} * \frac{\sigma^0}{\cos(\theta_1)\cos(\theta_2)}$

His derivation he shows that In this section we demonstrate that the Kirchhoff integral of Equation 2 can be computed analytically. we restrict ourselves to the reflection of waves from height fields. We assume that the surface is defined as an elevation over the xy-plane using fourier analysis he derives his equation 9 $BRDF_{\lambda}(w_i, w_r) = \frac{F^2 G}{\lambda^2 A w^2} abs(P(\frac{u}{\lambda}, \frac{v}{\lambda}))^2$ which is our starting point.

1.4 Stams derivation

In his Paper Diffraction Shader, Jos Stam derives a BRDF which modeling the effect of diffraction for various analytical anisotropic reflection models using the scalar Kirchhof theory and the theory of random processes. By employing the so called wave theory of diffraction [source 5 in stams paper] in which a wave is assumed to be a complex valued scalar. It's noteworthy, that stam's BRDF formulation does not take into account the polarization of the light. Nevertheless, light sources like sunlight and light bulbs are unpolarized. In our simulations we will always assume we have given a directional light source, i.e. sunlight. Hence, we can use stam's model for our derivations

A further assumption in Stam's Paper is, the emanated waves from the source are stationary - sunlight once again. Which implies the wave is a superposition of independent monochromatic waves. This implies that each wave is associated to a definite wavelength λ .

Mention Helmholtz equation, which has the solution $k = \frac{2\pi}{\lambda}$ which is the wavenumber

Stam starts his derivations by above's assumptions and by applying the Kirchhoff integral, which describes the reflected field and the Huygen's principle, which states, when somebody knows the wavefront at a given moment, the wave at a later time can be deduced by considering each point on the first wave as the source of a new disturbance.

$$\psi_2 = \frac{ike^{iKR}}{4\pi R} (F\mathbf{v} - \mathbf{p}) \cdot \int_S \hat{\mathbf{n}} e^{ik\mathbf{v} \cdot \mathbf{s}} d\mathbf{s} \quad (1)$$

In optics, when dealing with scattered waves, one does use differential scattering cross-section rather than a BRDF which has the following identity:

$$\sigma^0 = 4\pi \lim_{R \rightarrow \infty} R^2 \frac{\langle |\psi_2|^2 \rangle}{\langle |\psi_1|^2 \rangle} \quad (2)$$

Relationship between the BRDF and the scattering cross section is the following:

$$BRDF = \frac{1}{4\pi} \frac{1}{A} \frac{\sigma^0}{\cos(\theta_1)\cos(\theta_2)} \quad (3)$$

Whereas θ_1 and θ_2 are the angles that the vectors \hat{k}_1 and \hat{k}_2 make with the vertical direction.

ADD FIGURE for k_1, k_2

where R is the distance from the center of the patch to the receiving point x_p , \hat{n} is the normal of the surface at s and the vectors:

$$\mathbf{v} = \hat{k}_1 - \mathbf{k}_1 = (u, v, w)$$

$$\mathbf{p} = \hat{k}_1 + \mathbf{k}_1$$

During his derivations, Stam provides a analytical representation for the Kirchhoff integral by using his assumptions. He restricts himself to the reflection of waves from height fields $h(x, y)$ with the assumption that the surface is defined as an elevation over the (x, y) plane using the surface plane approximation. Which will lead him to the following identity for the Kirchhoff integral

$$\mathbf{I}(ku, kv) = \int \int \frac{1}{ikw} (-p_x, -p_y, ikwp) \quad (4)$$

whereas

$$p(x, y) = e^{ikwh(x, y)} \quad (5)$$

We the observation that the integral is a Fourier transform by $-iku$ and $-ikv$ which will lead us to his final derivation, using the identity of BRDF, and computing the limits:

$$BRDF = \frac{k^2 F^2 G}{4\pi^2 A w^2} \langle |P(ku, kv)|^2 \rangle \quad (6)$$

Where

$$G = \frac{(1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{\cos(\theta_1)\cos(\theta_2)} \quad (7)$$

and $P(x,y)$ is the Fourier transform of the function $p(x,y)$ from above.
This identity for the BRDF is the starting point for our derivations.

1.5 Previous work

1.6 Overview

Robert Hooke schlangenbsp Stam

ABSTRACT:content

interactive rendering of diffraction effect produced by biological nano-structures such as snake skin surface gratings. Images AFM captures nano-structure, responsible for color, colorization due to wave interference. rendering technique, construction of BRDF. reflection light on surface in computer graphics

START STAM SUMMARY —————

abstract: before: most reflection models empirically or based on ray-theory of light. now: new reflection model based on wave theory modeling the effect of diffraction

Introduction =====

Diffraction occurs when the surface detail is comparable to the wavelength of light Fourier analysis has enabled us to tackle the difficult task of computing the reflected waves off of these surfaces

Diffraction is a purely wave-like phenomenon which cannot be modeled using the standard ray theory of light. Diffraction should not be confused with the related phenomenon of interference Interesting diffraction phenomena, however, occur mostly when the surface detail is highly anisotropic, viz. non-isotropic Interference produces colorful effects due to the phase differences caused by a wave traversing thin media of different indices of refraction Interference, unlike diffraction, can be modeled using the ray theory of light alone

well known from physical optics that ray theory is only an approximation of the more fundamental wave theory + ray theory is sufficient to visually capture the reflected field from many commonly occurring surfaces + common belief that models based on wave theory are computationally too expensive to be of any use in computer graphics

derive analytical reflection models based on wave theory that capture the effects of diffraction

Wave Theory and Computer Graphics =====

END STAM SUMMARY

structural coloration is production of color through the interaction of light with nanoscale surface structure. Color production is due to wave interference with quasiperiodic structures whose periodicity leads to interaction with visible light.

the interference itself may be caused by a variety of photonic mechanisms such as diffraction gratings, 2d and 3d photonic crystals or crystal fibres.

Robert Hooke = observed connection between microscopic structures and colorisation wave nature of light led to conclusion that the cause for the coloration lies in wave interference.

derive reflectance models suitable for interactive rendering directly from physical measurements of biological structures, focused on structural colors generated by diffraction gratings. in particular our approach applies to surfaces with quasiperiodic structures at nanometer scale that can be represented as heightfields of different snakes sheds using AFM.

goal: interactive rendering of diffraction effects from complex physical nanostructures acquired using electron or atomic force microscopy (AFM). We obtained snake skin surface gratings geometry using afm, which delivers height field data representing surface structures at the nanometer scale. such height fields are denoted as $h(x,y)$.

the nanostructures we are interested in exhibit periodicity on the scale of wavelength of visible light. Hence rendering requires taking into account the wave nature of light to reproduce these effects.

=> stam paper: under certain conditions formulate conventional BRDFs for such nanostructures. assumes surface is observed from a large enough distance and then uses the kirchoff integral

stam formulates for a heightfield auxiliary function $p(x,y) = e^{i\mathbf{w}k\mathbf{h}(x,y)}$ where $\mathbf{w} = -(\cos(\theta_i) + \cos(\theta_r))$ and θ_i and θ_r are the angles of incident and reflected directions with the surface normal (ADD picture) and the wavenumber $k = \frac{2\pi}{\lambda}$

$BRDF_\lambda(w_i, w_r)$ is BRDF where wavelength λ w_i and w_r are incident and reflected normalized directions vectors, pointing away from the given surface. Which can be written, using the fourier transform (FT) $P(u,v) = F(p)(u,v)$, as: $BRDF_\lambda(w_i, w_r) = \frac{F^2 G}{\lambda^2 A w^2} \text{abs}(P(\frac{u}{\lambda}, \frac{v}{\lambda}))^2$ where F represents the

Fresnel term, u, v, w are derived from the incident and reflected directions as $(u, v, w) = -\omega_i - \omega_r$, $\text{abs}(P)$ represents the expected value of a random variable X and A is an area of integration on the surface that is considered to contribute to diffraction, G is the geometry term which is $G = \frac{(1+w_r*w_i)^2}{\cos(\theta_i)\cos(\theta_r)}$

reviewer summaries:

rendering diffraction colors caused by coherent (as opposed to statistical) surface micro-geometry. tables containing the result of the spectral integration predicted by the model of Stam. The theory is applied to render diffraction caused by acquired geometric models of the microstructure of snake skin, so as to reproduce the iridescent effects displayed by these animals.

PEWPEW

Summary: The paper suggests an interactive rendering technique for diffraction effects of (biological) nano-structures. The results presented are for biological structures very particular snake species TODO: add real snake images in final results

PEWPEW

Summary: An algorithm and a representation for rendering diffraction BRDFs from measured nano-scale patterns are presented. The paper is based on the theoretical framework presented by Stam in Diffraction Shaders, SIGGRAPH99. The author solves Stam's equation with the same Taylor expansion trick. Since they use measured surfaces instead of statistical surface profiles, they precompute and store in lookup tables what is no more analytically available compared to Stam's method.

PEWPEW Summary: Pictures are produced from a previously published formula for the BRDF of a quasiperiodic nanostructure that can be represented as a heightfield. Whereas the previous work models the heightfield as a probabilistic superposition of bumps and proceeds to derive an analytical expression for the BRDF, this work actually takes the heightfield from measurement. Real time performance is achieved with a representation of the formula as a power series over a variable related to the viewing and lighting directions. Values closely related to the coefficients in that power series are precomputed.

2 Theoretical Background

2.1 Definitions

2.1.1 Electromagnetic radiation

WIKI : http://en.wikipedia.org/wiki/Electromagnetic_radiation

Electromagnetic radiation (EM radiation or EMR) is one of the fundamental phenomena of electromagnetism, behaving as waves propagating through space, and also as photon particles traveling through space, carrying radiant energy.

EMR is characterized by the frequency or wavelength of its wave. The electromagnetic spectrum, in order of increasing frequency and decreasing wavelength, consists of radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays and gamma rays.

The eyes of various organisms sense a somewhat variable but relatively small range of frequencies of EMR called the visible spectrum or light. Higher frequencies correspond to proportionately more energy carried by each photon; for instance, a single gamma ray photon carries far more energy than a single photon of visible light.

EMR carries energy—sometimes called radiant energy—through space continuously away from the source

In general, EM radiation (the designation '*radiation*' excludes static electric and magnetic and near fields) is classified by wavelength into radio, microwave, infrared, the visible spectrum we perceive as visible light, ultraviolet, X-rays, and gamma rays. Arbitrary electromagnetic waves can always be expressed by Fourier analysis in terms of sinusoidal monochromatic waves, which in turn can each be classified into these regions of the EMR spectrum.

The behavior of EM radiation depends on its frequency. Lower frequencies have longer wavelengths, and higher frequencies have shorter wavelengths, and are associated with photons of higher energy. There is no fundamental limit known to these wavelengths or energies, at either end of the spectrum, although photons with energies near the Planck energy or exceeding it (far too high to have ever been observed) will require new physical theories to describe.

2.1.2 Radiant energy

WIKI : http://en.wikipedia.org/wiki/Radiant_energy Radiant energy is the energy of electromagnetic waves.[1] The quantity of radiant energy may be calculated by integrating radiant flux (or power) with respect to time and, like all forms of energy, its SI unit is the joule. The term is used particularly when radiation is emitted by a source into the surrounding environment. Radiant energy may be visible or invisible to the human eye.

The term "*radiantenergy*" is most commonly used in the fields of radiometry, solar energy, heating and lighting,

2.1.3 Radiance

WIKI : <http://en.wikipedia.org/wiki/Radiance> Radiance and spectral radiance are measures of the quantity of radiation that passes through or is emitted from a surface and falls within a given solid angle in a specified direction. They are used in radiometry to characterize diffuse emission and reflection of electromagnetic radiation. In astrophysics, radiance is also used to quantify emission of neutrinos and other particles. The SI unit of radiance is watts per steradian per square metre ($\text{W} \cdot \text{sr}^{-1} \cdot \text{m}^{-2}$), while that of spectral radiance is $\text{W} \cdot \text{sr}^{-1} \cdot \text{m}^{-2} \cdot \text{Hz}^{-1}$ or $\text{W} \cdot \text{sr}^{-1} \cdot \text{m}^{-3}$ depending on whether the spectrum is a function of frequency or of wavelength.

Radiance characterizes total emission or reflection. Radiance is useful because it indicates how much of the power emitted by an emitting or reflecting surface will be received by an optical system looking at the surface from some angle of view. In this case, the solid angle of interest is the solid angle subtended by the optical system's entrance pupil. Since the eye is an optical system, radiance and its cousin luminance are good indicators of how bright an object will appear. For this reason, radiance and luminance are both sometimes called "*brightness*". This usage is now discouraged – see Brightness for a discussion. The nonstandard usage of "*brightness*" for "*radiance*" persists in some fields, notably laser physics

$$\text{Def } L = \frac{d^2\Phi}{dA d\Omega \cos(\theta)} \approx \frac{\Phi}{\Omega A \cos(\theta)}$$

2.1.4 Signal

A signal is a function that conveys information about the behavior or attributes of some phenomenon. In the physical world, any quantity exhibiting

variation in time or variation in space (such as an image) is potentially a signal that might provide information on the status of a physical system, or convey a message between observers

2.1.5 Fourier Transformation

The Fourier-Transform is a mathematical tool which allows to transform a given function or rather a given signal from defined over a time- (or spatial-) domain into its corresponding frequency-domain.

Let f an measurable function over \mathcal{R}^n . Then, the coninious Fourier Transformation, denoted as FT, $\mathcal{F}\{t\}$ of f is defined as, ignoring all constant factors in the formula:

$$\mathcal{F}\{w\}_{FT} = \int_{\mathcal{R}^n} f(x) e^{-iwt} dt \quad (8)$$

whereas its inverse transform is defined like the following which allows us to obtain back the original signal:

$$\mathcal{F}\{w\}_{FT}^{-1} = \int_{\mathcal{R}^n} \mathcal{F}\{w\} e^{iwt} dt \quad (9)$$

By using fourier analysis, which is the approach to approximate any function by sums of simpler trigonometric functions, we gain the so called discrete time fourier transform (in short DTFT). The DTFT operates on a discrete function. Usually, such an input function is often created by diitally sampling a continius function. The DTFT itself is operation on a discretized signal on a continious, periodic frequency domain and looks like the following:

$$\mathcal{F}\{w\}_{DFT} = \sum_{-\infty}^{\infty} f(x) e^{(-iwk)} \quad (10)$$

we can further discretize the frequency domain and will get then the discrete fourier transformation (in short DFT) of the input signal:

$$\mathcal{F}\{w\}_{DFT} = \sum_{n=0}^{N-1} f(x) e^{(-i w_n k)} \quad (11)$$

Where the angular frequency w_n is defined like the following $w_n = \frac{2\pi n}{N}$ and N is the number of samples within an equidistant periode sampling.

2.1.6 Convolution

$$(f * g)(t) = \int_{\mathcal{R}^n} f(t)g(t - x)dx \quad (12)$$

Note that the Fourier transform of the convolution of two functions is the product of their Fourier transforms. This is equivalent to the fact that Convolution in spatial domain is equivalent to multiplication in frequency domain. Therefore, the inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

2.1.7 BRDF

The bidirectional reflectance distribution function, in short BRDF, denoted as $f_r(w_i, w_r)$ is a four dimensional function that defines how light is reflected at an opaque surface. The function takes a negative incoming light direction, ω_i , and outgoing direction, ω_r , both defined with respect to the surface normal \mathbf{n} and returns the ratio of reflected radiance exiting along ω_r to the irradiance incident on the surface from direction ω_i

$$f_r(w_i, w_r) = \frac{dL_r(w_r)}{dE_i(w_i)} \quad (13)$$

$$= \frac{dL_r(w_r)}{L_i(w_i)\cos(\theta_i)dw_i} \quad (14)$$

where L is radiance, or power per unit solid-angle-in-the-direction-of-a-ray per unit projected-area-perpendicular-to-the-ray, E is irradiance, or power per unit surface area, and θ_i is the angle between ω_i and the surface normal, \mathbf{n} . The index i indicates incident light, whereas the index r indicates reflected light.

The reason the function is defined as a quotient of two differentials and not directly as a quotient between the undifferentiated quantities, is because other irradiating light than $dE_i(\omega_i)$, which are of no interest for $f_r(\omega_i, \omega_r)$, might illuminate the surface which would unintentionally affect $L_r(\omega_r)$, whereas $dL_r(\omega_r)$ is only affected by $dE_i(\omega_i)$.

2.1.8 Taylor Series

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series:

$$\mathcal{T}(f; a)(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n \quad (15)$$

2.1.9 The Effect Of Diffraction

+ mention fraunhofer diffraction, fresnel diffraction?

2.1.10 BRDF - Spectral Rendering

In Computer Graphics, spectral rendering is where a scene's light transport is modeled considering the whole span of wavelengths instead of R,G,B values (still relating on geometric optic, which ignore wave phase). The motivation is that real colors of the physical world are spectrum; trichromatic colors are only inherent to Human Visual System.

2.2 Derivations

2.2.1 Taylor Series Approximation

Based on J. Stam's Paper about Diffraction shaders we will show that there is an approximation of his equation (5), $\mathbf{p}(\mathbf{x}, \mathbf{y})$, for a explicitly given heightfield $\mathbf{h}(\mathbf{x}, \mathbf{y})$. This approximation is achieved by using Taylor-Series and using this identity we will further be able to approximate the Fourier-Transformation of $\mathbf{p}(\mathbf{x}, \mathbf{y})$, denoted as $\mathbf{P}(\mathbf{u}, \mathbf{v})$. Finally we will give an error bound for this approximation.

2.2.2 Taylor Series of \mathbf{p}

Given $p(x, y) = e^{ikwh(x, y)}$ from Stam's Paper where $h(x, y)$ is here a given heightfield. Also given the definition $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{y^n}{n!}$ where y can be real or even complex valued - note this identity can either be derived by power series or by Taylor-Series (using the derivatives of the exp-function and developing the Taylor-Series around the point $a=0$). Let us now set $y = ikwh(x, y)$ where i is the imaginary number. For simplification, let us denote $h(x, y)$ as h . It follows by our previous stated identities:

$e^y = 1 + (ikwh) + \frac{1}{2!}(ikwh)^2 + \frac{1}{3!}(ikwh)^3 + \dots = \sum_{n=0}^{\infty} \frac{(ikwh)^n}{n!}$. Hence it holds $p(x, y) = \sum_{n=0}^{\infty} \frac{(ikwh(x, y))^n}{n!}$.

2.2.3 Fourier Transformation of function p

Let us now compute the Fourier Transformation of $p(x, y)$ from above: $\mathcal{F}\{p\}(u, v) = \mathcal{F}\left\{\sum_{n=0}^{\infty} \frac{(ikwh(x, y))^n}{n!}\right\} = \mathcal{F} \text{ lin Operator } \sum_{n=0}^{\infty} \mathcal{F}\left\{\frac{(ikwh(x, y))^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{(ikwh)^n}{n!} \mathcal{F}\{h(x, y)^n\}$.

Hence it follows: $P(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta)$.

NB: $\mathcal{F}\{h^n\}(u, v)$ denotes the two dimensional Fourier Transformation of $p(x, y)$ and can be numerically computed by the two dimensional **DFT** or rather by the two dimensional **FFT** over $h(x, y)$.

2.2.4 Approximation of function P

Next we are going to look for an $N \in \mathbb{N}$ s.t. $\sum_{n=0}^N \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta) \approx P(\alpha, \beta)$. is a good approximation. We have to prove two things:

1. Show that there exist such an $N \in \mathbb{N}$ s.t the approximation holds true.
2. Find a value for B s.t. this approximation is below a certain error bound, for example machine precision ϵ .

2.2.5 Proof Sketch of 1.

By the **ratio test** (see [1]) we can show that the series $\sum_{n=0}^{\infty} \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta)$ converges absolutely:

Proof: Consider $\sum_{k=0}^{\infty} \frac{y^k}{k!}$ where $a_k = \frac{y^k}{k!}$. By the definition of the ratio test for series it follows: $\forall y : \limsup_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \limsup_{k \rightarrow \infty} \frac{y}{k+1} = 0$

Thus this series converges absolutely, no matter what value we will pick for y .

2.2.6 Part 2: Find such an N

Let $f(x) = e^x$. We can formulate its Taylor-Series, stated above. Let $P_n(x)$ denote the n -th Taylor-Polynomial, $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$, where a is our developing point (here, in this case $a=0$). We can define the error of the n -th Taylor-Polynomial to be $E_n(x) = f(x) - P_n(x)$. That error is the actual value minus the Taylor polynomial. It holds true: $|E_n(x)| = |f(x) - P_n(x)|$.

By using the Lagrangien Error Bound - (see source [2]) it follows: $|E_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$ with $a=0$, where M is some value satisfying $|f^{(n+1)}(x)| \leq M$ on the interval $I = [a, x]$. Since we are interested in an upper bound of the error and since a is known, we can reformulate the interval as $I = [0, x_{max}]$, where $x_{max} = |i| * k_{max} * w_{max} * h_{max}$, since we are interested in computing an error bound for $e^{ikwh(x,y)}$. From Stam's Paper about diffraction shader we know some paramters for the length, width and height for a given sample patch, i.e. heightfield $h(x,y)$ and when using those parameters are able to find a explicit number for x_{max} .

Facts we are using from Stam's Paper:

- Height of bump: 0.15micro meters
- Width of a bump: 0.5micro meters
- Length of a bump: 1micro meters
- $k = \frac{2\pi}{\lambda}$ is the wavenumber and $\lambda \in [\lambda_{min}, \lambda_{max}]$ its wavelength hence $k_{max} = \frac{2\pi}{\lambda_{min}}$
- w is a component of the vector $\vec{v} = \vec{k}_1 - \vec{k}_2 = (u, v, w)$, where \vec{k}_1 and \vec{k}_2 are **normalized** direction vectors and this each component can have a value in range $[-2, 2]$.
- for simplification, assume $[\lambda_{min}, \lambda_{max}] = [400nm, 700nm]$.

Hence $x_{max} = |i| * k_{max} * w_{max} * h_{max} = k_{max} * w_{max} * h_{max} = 2 * (\frac{2\pi}{4*10^{-7}m}) * 1.5 * 10^{-7} = 1.5\pi$ and it follows for our intervall $I = [0, 1.5\pi]$. Next we are going to find the value for M . Since the exponential function is monoton growing (on the interval I) and the derivative of the **exp** function is the exp function itself, we can find such an M : $M = e^{x_{max}} = \exp(1.5\pi)$ and $|f^{(n+1)}(x)| \leq M$ holds. With $|E_n(x_{max})| \leq \frac{M}{(n+1)!} |x_{max} - a|^{n+1} = \frac{\exp(1.5\pi) * (1.5\pi)^{n+1}}{(n+1)!}$ we now can find a value of n for a given bound, i.e. we can find an value of $N \in \mathbb{N}$ s.t. $\frac{\exp(1.5\pi) * (1.5\pi)^{N+1}}{(N+1)!} \leq \epsilon$. With Octave/Matlab we can see:

- if $N=20$ then $\epsilon \approx 2.9950 * 10^{-4}$
- if $N=25$ then $\epsilon \approx 8.8150 * 10^{-8}$
- if $N=30$ then $\epsilon \approx 1.0050 * 10^{-11}$

2.2.7 Conclusion

With this approach we have that $\sum_{n=0}^{25} \frac{(ikwh)^n}{n!} \mathcal{F}\{h^n\}(\alpha, \beta)$ is an approximation of $P(u, v)$ with error $\epsilon \approx 8.8150 * 10^{-8}$. This means we can precompute 25 Fourier Transformations (for example via FFT2) and then sum them up in order to approximate $P(u, v)$ and $\epsilon \approx 8.8150 * 10^{-8}$. This approach will allow us to speed up our shader. Furthermore we see that when we just take 5 more iterations, we will reduce the error bound to the dimension of 10^{-11} .

NB: More explanation about how the shader itself works during our discussion.

2.2.8 Our derivations

EXPLAIN: Why do we want a formulation for $L_\lambda(w_r)$ in some words. what does it represent?

Definition of $BRDF(w_i, w_r) := f_r(w_i, w_r) = \frac{dL_r(w_r)}{dE_i(w_i)} = \frac{dL_r(w_r)}{L_i(w_i)\cos(\theta_i)dw_i}$

Hence, we can derive the following expression:

$$\begin{aligned} f_r(w_i, w_r) &= \frac{dL_r(w_r)}{L_i(w_i)\cos(\theta_i)dw_i} \\ \Rightarrow f_r(w_i, w_r)L_i(w_i)\cos(\theta_i)dw_i &= dL_r(w_r) \\ \Rightarrow \int_{\Omega} f_r(w_i, w_r)L_i(w_i)\cos(\theta_i)dw_i &= \int_{\Omega} dL_r(w_r) \\ \Rightarrow L_r(w_r) &= \int_{\Omega} f_r(w_i, w_r)L_i(w_i)\cos(\theta_i)dw_i \end{aligned}$$

We assume, that our incident light is a directional light source like sunlight and therefore its radiance is given as $L_\lambda(w) = I(\lambda)\delta(w - w_i)$ where $I(\lambda)$ is the intensity of the relative spectral power for the wavelength λ . Thus we get for our the brdf formulation:

$$L_\lambda(w_r) = \int_{\Omega} BRDF_\lambda(w_i, w_r)L_\lambda(w_i)\cos(\theta_i)dw_i \quad (16)$$

$$= BRDF_\lambda(w_i, w_r)I(\lambda)\cos(\theta_i) \quad (17)$$

where w_i is the solid angle for the incoming light, θ_i is the angle of incidence, w_r is the solid angle for the reflected light, λ wavelength, Ω is the hemisphere we of integration for the incoming light. Radiance reflected by

given surface in given direction: $L_\lambda(w_i)$ is the incomming radiance, $L_\lambda(w_r)$ is the reflected radiance

For the $BRDF(w_i, w_r)$ we are going to use the formulation dervied by Stam described above which looks like this using the fact that wavenumber $k = \frac{2\pi}{\lambda}$:

$$\begin{aligned} BRDF(w_i, w_r) &= \frac{k^2 F^2 G}{4\pi^2 A w^2} \langle |P(ku, kv)|^2 \rangle \\ &= \frac{k^2 F^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{\cos(\theta_1) \cos(\theta_2) 4\pi^2 A w^2} \langle |P(ku, kv)|^2 \rangle \\ &= \frac{4\pi^2 F^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{\cos(\theta_1) \cos(\theta_2) 4\pi^2 A \lambda^2 w^2} \langle |P(ku, kv)|^2 \rangle \\ &= \frac{F(w_i, w_r)^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{\cos(\theta_1) \cos(\theta_2) A \lambda^2 w^2} \langle |P(ku, kv)|^2 \rangle \end{aligned}$$

where $\hat{\mathbf{k}}_t$ represents a unit vector whose spherical coordinates are given by the solid angle t . Since we are going to integrate over a sphere Ω we can write the component $w = (\cos(\theta_i) + \cos(\theta_r))$ SHOW WHY WE ARE ALLOWED TO WRITE IT LIKE THIS => SPHERICAL COORDINATES DIFFERENCE $(k_1 - k_2) = (u, v, w)$ and so on. this our identity for $L_r(w_r)$ will lead us to the following identiy using our identity :

$$\begin{aligned} L_\lambda(w_r) &= \frac{F(w_i, w_r)^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{A \lambda^2 \cos(\theta_i) \cos(\theta_r) (\cos(\theta_i) + \cos(\theta_r))^2} \left\langle \left| P_{cont}\left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda}\right) \right|^2 \right\rangle \cos(\theta_i) I(\lambda) \\ &= I(\lambda) \frac{F(w_i, w_r)^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{\lambda^2 A (\cos(\theta_i) + \cos(\theta_r))^2 \cos(\theta_r)} \left\langle \left| P_{cont}\left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda}\right) \right|^2 \right\rangle \\ &= I(\lambda) \frac{F(w_i, w_r)^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{\lambda^2 A (\cos(\theta_i) + \cos(\theta_r))^2 \cos(\theta_r)} \left\langle \left| T_0^2 P_{dft}\left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda}\right) \right|^2 \right\rangle \end{aligned}$$

P_{cont} is the continious inverse Fourier transform for the taylor seies of our hight-field representing the nano structure, i.e. $P(k, l) = \mathcal{F}^{-1}\{p\}(k, l)$ and P_{dft} is the dicrete-time inverse Fourier Transform for the same problem domain and T_0 the sampling distance for the discretization pf $p(x, y)$ assuming equal and uniform sampling in both dimensions x, y .

2.3 Relative BRDF

reason why relative brdf: In order to scale the reflectance such that we are able to texture. convex combination reflectance with texture. Scale illumination.

Let us examine what $L_\lambda(w_r)$ will be for $w_r = w_0 := (0, 0, *)$ i.e. specular reflection case, denoted as $L_\lambda^{spec}(w_0)$. When we know the expression for $L_\lambda^{spec}(w_0)$ we would be able to compute the relative reflected radiance for our problem by simply dividing $L_\lambda(w_r)$ by $L_\lambda^{spec}(w_0)$, denoted as

$$\rho_\lambda(w_i, w_r) = \frac{L_\lambda(w_r)}{L_\lambda^{spec}(w_0)} \quad (18)$$

But first, let us derive the following expression:

$$\begin{aligned} L_\lambda^{spec}(w_0) &= I(\lambda) \frac{F(w_0, w_0)^2 (1 - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})^2}{\lambda^2 A (\cos(0) + \cos(0))^2 \cos(0)} \langle |T_0^2 P_{dtft}(0, 0)|^2 \rangle \\ &= I(\lambda) \frac{F(w_0, w_0)^2 (1 + 1)^2}{\lambda^2 A (1 + 1)^2 1} |T_0^2 N_{sample}|^2 \\ &= I(\lambda) \frac{F(w_0, w_0)^2}{\lambda^2 A} |T_0^2 N_{sample}|^2 \end{aligned}$$

Where $N_{samples}$ is the number of samples of the dtft.

Thus, we can plug our last derived expression into the definition for the relative reflectance radiance in the direction w_r and will get:

$$\begin{aligned} \rho_\lambda(w_i, w_r) &= \frac{L_\lambda(w_r)}{L_\lambda^{spec}(w_0)} \\ &= \frac{I(\lambda) \frac{F(w_i, w_r)^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{\lambda^2 A (\cos(\theta_i) + \cos(\theta_r))^2 \cos(\theta_r)} \langle |T_0^2 P_{dtft}(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda})|^2 \rangle}{I(\lambda) \frac{F(w_0, w_0)^2}{\lambda^2 A} |T_0^2 N_{sample}|^2} \\ &= \frac{F^2(w_i, w_r) (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{F^2(w_0, w_0) (\cos(\theta_i) + \cos(\theta_r))^2 \cos(\theta_r)} \langle \left| \frac{P_{dtft}(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda})}{N_{samples}} \right|^2 \rangle \end{aligned}$$

for simplification and a better overview, let us introduce the following expression, the so called gain factor

$$C(w_i, w_r) = \frac{F^2(w_i, w_r)(1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{F^2(w_0, w_0)(\cos(\theta_i) + \cos(\theta_r))^2 \cos(\theta_r) N_{samples}^2} \quad (19)$$

Using this substitute, we will end up with the following expression for the relative reflectance radiance

$$\rho_\lambda(w_i, w_r) = C(w_i, w_r) \left\langle \left| P_{dft} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 \right\rangle \quad (20)$$

using the previous definition for the relative reflectance radiance $\rho_\lambda(w_i, w_r) = \frac{L_\lambda(w_r)}{L_\lambda^{spec}(w_0)}$ which we can rearrange to the expression

$$L_\lambda(w_r) = \rho_\lambda(w_i, w_r) L_\lambda^{spec}(w_0) \quad (21)$$

Let us choose $L_\lambda^{spec}(w_0) = S(\lambda)$ such that it has the same profile as the relative spectral power distribution of CIE Standard Illuminant *D65*. Further, when integration over λ for a specular surface we should get *CIE_XYZ* values corresponding to the white point for *D65*

the corresponding tristimulus values using CIE colormatching functions for the *CIE_XYZ* values look like:

SEE HOW THIS DEFINITION DIFFERS FROM THE WIKIDEF AND HOW WE COULD END UP WITH A SIMILAR DEFINITION.

$$X = \int_\lambda L_\lambda(w_r) \bar{x}(\lambda) d\lambda \quad (22)$$

$$Y = \int_\lambda L_\lambda(w_r) \bar{y}(\lambda) d\lambda \quad (23)$$

$$Z = \int_\lambda L_\lambda(w_r) \bar{z}(\lambda) d\lambda \quad (24)$$

where \bar{x} , \bar{y} , \bar{z} are the color matching functions

Using our last finding for $L_\lambda(w_r)$ and the definition for the tristimulus values we can actually derive an expression for computing the colors for our brdf model. Since X, Y, Z are defined similarly, it satisfies to derive an explicit expression for just one tristimulus term, for example X. The other two will

look the same, except the we have to replace all X with Y or Z respectively. Therefore, we get:

$$\begin{aligned}
X &= \int_{\lambda} L_{\lambda}(w_r) \bar{x}(\lambda) d\lambda \\
&= \int_{\lambda} \rho_{\lambda}(w_i, w_r) L_{\lambda}^{spec}(w_0) \bar{x}(\lambda) d\lambda \\
&= \int_{\lambda} \rho_{\lambda}(w_i, w_r) S(\lambda) \bar{x}(\lambda) d\lambda \\
&= \int_{\lambda} C(w_i, w_r) \left\langle \left| P_{dft} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 \right\rangle S(\lambda) \bar{x}(\lambda) d\lambda \\
&= C(w_i, w_r) \int_{\lambda} \left\langle \left| P_{dft} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 \right\rangle S(\lambda) \bar{x}(\lambda) d\lambda \\
&= C(w_i, w_r) \int_{\lambda} \left\langle \left| P_{dft} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 \right\rangle S_x(\lambda) d\lambda
\end{aligned}$$

Where we used the definition $S_x(\lambda) \bar{x}(\lambda)$ in the last step.

2.3.1 Taylour approximation for BRDF

Using $P_{dft} = \mathcal{F}^{-1}\{p\}(u, v)$ defined in the section of the taylor approximation we get for the tristimulus value X, we will get:

$$\begin{aligned}
X &= C(w_i, w_r) \int_{\lambda} \left\langle \left| P_{dft} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 \right\rangle S_x(\lambda) d\lambda \\
&= C(w_i, w_r) \int_{\lambda} \left| \sum_{n=0}^N \frac{(wk)^n}{n!} \mathcal{F}^{-1}\{i^n h^n\} \left(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda} \right) \right|^2 S_x(\lambda) d\lambda
\end{aligned}$$

2.3.2 Sampling: Gaussian Window

why this identity works: The DFT of a discrete heightfield patch is equivalent to the DTFT of an infinitely periodic function consisting of replicas of the same discrete patch. By windowing with a window function that is zero outside the central replica, the convolution of either the DFT or the DTFT of heightfield with the fourier transform of the window becomes equivalent.

Let $window_g$ denote the gaussian window with $4\sigma_s \mu m$ where $\sigma_f = \frac{1}{2\pi\sigma_s}$ let us further substitute $\mathbf{t}(\mathbf{x}, \mathbf{y}) = i^n h(x, y)^n$

$$\mathcal{F}_{dtft}^{-1}\{\mathbf{t}\}(u, v) = \mathcal{F}_{fft}^{-1}\{\mathbf{t}\}(u, v) window_g(\sigma_f) \quad (25)$$

Therefore we can deduce the following expression from this:

$$\begin{aligned} \mathcal{F}_{dtft}^{-1}\{\mathbf{t}\}(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{fft}^{-1}\{\mathbf{t}\}(w_u, w_v) \phi(u - w_u, v - w_v) dw_u dw_v \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_i \sum_j F_{fft}^{-1}\{\mathbf{t}\}(w_u, w_v) \delta(w_u - w_i, w_v - w_j) \phi(u - w_u, v - w_v) dw_u dw_v \\ &= \sum_i \sum_j \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{fft}^{-1}\{\mathbf{t}\}(w_u, w_v) \delta(w_u - w_i, w_v - w_j) \phi(u - w_u, v - w_v) dw_u dw_v \\ &= \sum_i \sum_j F_{fft}^{-1}\{\mathbf{t}\}(w_u, w_v) \phi(u - w_u, v - w_v) \end{aligned}$$

where $\phi(x, y) = \pi e^{-\frac{x^2 + y^2}{2\sigma_f^2}}$

2.3.3 Aplitude smooting

Let us consider the so called 1-dimensional Box-function with length T which is defined as the following: ADD AN IMAGE OF BOXFUNCTION

$$Box(x) = \begin{cases} 1 & \text{if } x \leq T \\ 0 & \text{if } else \end{cases}$$

We assume, that our given heighfield can be represented as a 2-dimensional box-function. Note that we can use any explicit given constrained 2-dimensional function and will get some identities like we get from the box-function.

Further we are assuming that we can model the overall surface be assuming this heighfield being distributed in a periodic manor. Therefore, the whole surface can be represented like this $f(x) = \sum_{n=0}^N Box(x + nT_1, y + mT_2)$ assuming the given heighfield has the dimensions T_1 by T_2 . But let us first consider the 1-dimensional Box-function case before deriving an identity for the Fourier transform of our 2-dimensional Box-function, i.e. the fourier transform of our heighfield.

Note: A function f periodic with periode T means: $\forall x \in \mathcal{R} : Box(x) = Box(x + T)$

A so called bump can be represented by our 1-dimensional Box-function. We assume periodicity which is equivalent to: $f(x) = \sum_{n=0}^N \text{Box}(x + nT)$

We are interested in the 1-dimensional inverse Fourier transform of the 1-dimensional Box-function:

$$\begin{aligned}\mathcal{F}^{-1}\{f\}(w) &= \int f(x)e^{iwx}dx \\ &= \int_{-\infty}^{\infty} \sum_{n=0}^N \text{Box}(x + nT)e^{iwx}dx \\ &= \sum_{n=0}^N \int_{-\infty}^{\infty} \text{Box}(x + nT)e^{iwx}dx\end{aligned}$$

Next, apply the following substitution $x + nT = y$ which will lead us to:

$$\begin{aligned}x &= y - nT \\ dx &= dy\end{aligned}$$

Plugging this substitution back to the equation from above we will get

$$\begin{aligned}\mathcal{F}^{-1}\{f\}(w) &= \int f(x)e^{iwx}dx \\ &= \sum_{n=0}^N \int_{-\infty}^{\infty} \text{Box}(y)e^{iw(y-nT)}dy \\ &= \sum_{n=0}^N e^{-iwnT} \int_{-\infty}^{\infty} \text{Box}(y)e^{iwy}dy \\ &= \sum_{n=0}^N e^{-iwnT} \mathcal{F}\{\text{Box}\}(w) \\ &= \mathcal{F}^{-1}\{f\}(w) \sum_{n=0}^N e^{-iwnT}\end{aligned}$$

We used the fact that the term e^{-iwnT} is a constant when integrating along dy and the identity for the inverse Fourier transform of the Box function. Next, let us consider $\sum_{n=0}^N e^{-iwnT}$ further:

$$\begin{aligned}\sum_{n=0}^N e^{-uwnT} &= \sum_{n=0}^N (e^{-uwT})^n \\ &= \frac{1 - e^{iwT(N+1)}}{1 - e^{-iwT}}\end{aligned}$$

We recognize the geometric series identity for the left-handside of this equation. Since our series is bounded we can derive our right-handside.

Since e^{-ix} is a complex number and every complex number can be written in its polar form, i.e. $e^{-ix} = \cos(x) + i\sin(x)$ we can go even further, using the trigonometric identities that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$:

$$\frac{1 - e^{iwT(N+1)}}{1 - e^{-iwT}} = \frac{1 - \cos(wT(N+1)) + i\sin(wT(N+1))}{1 - \cos(wT) + i\sin(wT)}$$

Which is still a complex number ($p + iq$). Every complex number can be written as a fraction of two complex numbers. This means that the complex number ($p + iq$) can be written as $(p + iq) = \frac{(a+ib)}{(c+id)}$ for any $(a + ib), (c + id) \neq 0$. For our case, let us use the following substitutions:

$$a := 1 - \cos(wT(N+1)) \quad b = \sin(wT(N+1)) \quad (26)$$

$$c = 1 - \cos(wT) \quad d = \sin(wT) \quad (27)$$

hence it follows $\frac{1 - e^{iwT(N+1)}}{1 - e^{-iwT}} = \frac{(a+ib)}{(c+id)}$. By rearranging the terms it follows $(a + ib) = (c + id)(p + iq)$ and multiplying the right handside out we get the following system of equations:

$$(cp - dq) = a \quad (28)$$

$$(dp + cq) = b \quad (29)$$

Which gives lead us we some further math (trick: mult first eq. by c and 2nd by d , then adding them together. using distributivity and we have the identity for p for example, similar for q) to

$$p = \frac{(ac + bd)}{c^2 + d^2} \quad (30)$$

$$q = \frac{(bc + ad)}{c^2 + d^2} \quad (31)$$

Putting our substitution for a, b, c, d back into the current representation for p and q and using some trigonometric identities, this we then get:

$$p = \frac{1}{2} + \frac{1}{2} \left(\frac{\cos(wTN) - \cos(wT(N+1))}{1 - \cos(wT)} \right) \quad (32)$$

$$q = \frac{\sin(wT(N+1)) - \sin(wTN) - \sin(wT)}{2(1 - \cos(wT))} \quad (33)$$

Since we have seen, that $\sum_{n=0}^N e^{-iwnT}$ is a complex number and can be written as $(p + iq)$ and we know now the explicit identity for those p and q we get for the 1-dimensional Fourier transform of the 1-dimensional Box-function the following final identity:

$$\begin{aligned} \mathcal{F}^{-1}\{f\}(w) &= \mathcal{F}^{-1}\{f\}(w) \sum_{n=0}^N e^{-iwnT} \\ &= (p + iq) \mathcal{F}^{-1}\{Box\}(w) \end{aligned}$$

In order to derive next a identity for the Fourier transform for our 2-dim heighfield, we can proceed similarly, the only fact which changes is, that we are now in a 2-dimensional domain, i.e. we are about to compute a two-dimensional Fourier transform: Let us again use again a Box-function, this time a 2-dimensional Box-function $Box(x, y)$ just for the sake of convenience.

$$\begin{aligned}
\mathcal{F}^{-1}\{f\}(w_1, w_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} \text{Box}(x_1 + n_1 T_1, x_2 + n_2 T_2) e^{iw(x_1+x_2)} dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} \text{Box}(y_1, y_2) e^{iw((y_1-n_1 T_1)+(y_2+n_2 T_2))} dy_1 dy_2 \\
&= \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Box}(y_1, y_2) e^{iw(y_1+y_2)} e^{-iw(n_1 T_1+n_2 T_2)} dy_1 dy_2 \\
&= \sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} e^{-iw(n_1 T_1+n_2 T_2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Box}(y_1, y_2) e^{iw(y_1+y_2)} dy_1 dy_2 \\
&= \left(\sum_{n_2=0}^{N_1} \sum_{n_2=0}^{N_2} e^{-iw(n_1 T_1+n_2 T_2)} \right) \mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2) \\
&= \left(\sum_{n_2=0}^{N_1} e^{-iwn_1 T_1} \right) \left(\sum_{n_2=0}^{N_2} e^{-iwn_2 T_2} \right) \mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2) \\
&= (p_1 + iq_1)(p_2 + iq_2) \mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2) \\
&= ((p_1 p_2 - q_1 q_2) + i(p_1 p_2 + q_1 q_2)) \mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2) \\
&= (p + iq) \mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2)
\end{aligned}$$

Where we define $p := (p_1 p_2 - q_1 q_2)$ and $q := (p_1 p_2 + q_1 q_2)$. For this identity we used green's integration rule which allowed us to split the double integral to the product of two single integrations. Also, we used the definition of the 2-dimensional inverse Fourier transform of the Box-function. We applied the same substitution like we did in for the 1 dimensional case, but this time twice, once for each variable separately. The last step, substituting with p and q will be useful later in the implementation. The insight should be, that the product of two complex numbers is again a complex number. We will have to compute the absolute value of $\mathcal{F}^{-1}\{f\}(w_1, w_2)$ which will then be equal $(p^2 + q^2)^{\frac{1}{2}} |\mathcal{F}^{-1}\{\text{Box}\}(w_1, w_2)|$

2.3.4 Final Expression

As the last step of our series of derivations, we plug all our findings together to one big equation in order to compute the colors in the CIE_{XYZ} colorspace:

For a given heigh-field $h(x, y)$, representing a small patch of the nano-structure of our surface, the resulting CIE_XYZ caused by the effect of dif-fraction can be computed like the following:

Let $P(u, v, \lambda) = F_{fft}^{-1}\{i^n h^n\}(\frac{2\pi u}{\lambda}, \frac{2\pi v}{\lambda})$

$$\begin{pmatrix} X \\ X \\ Z \end{pmatrix} = C(w_i, w_r) \int_{\lambda} \sum_{n=0}^N \frac{(wk)^n}{n!} \sum_r \sum_s |P(u - w_r, v - w_s, \lambda)|^2 \phi(u - w_r, v - w_s) \begin{pmatrix} S_x(\lambda) \\ S_y(\lambda) \\ S_z(\lambda) \end{pmatrix} d\lambda \quad (34)$$

where $\phi(x, y) = \pi e^{-\frac{x^2+y^2}{2\sigma_f^2}}$ is the gaussian window. where w_s and w_r are ... explain them

3 Implementation

how to discretize from final derivation to computation? what do we have to precompute, what during runtime? how does the final algorithm look like explain shaders: vertex(geometry, precomp) - and fragment-shader(in local space-tspace) how from $cie_x yz$ to $cie_r gb$ how gamma correction how texturing can we do better?

TODO: explain that there is the jrtr and the scene code - what are their responsibilities.

shader

In computergraphics, we are interested in rendering a given scene containing our 3d geometries by using so called shader programs. The purpose of such programs, which run directly on the gpu hardware device, is to compute the colorization and illumination of the objects living in our scene. This computation happens in several stages and depends on the provided input parameters like the camera, light sources, objects material constants and the desired rendering effect one is interested in to model. The shader stages are also modeled as small little programs, the so called vertex-, geometry- and fragment-shaders. Those stages are applied within the rendering pipeline sequentially.

Our shaders are written in GLSL, developed for OpenGL. The decision for using OpenGL has been made since the underlying framework which is responsible for the precomputation of all scene data is based on a framework written in Java using JOGL in order to communicate with the GPU and precompute all the relevant scene data. This framework, the so called jrtr framework has been developed as an exercise during the class computer graphics held by M. Zwicker which I attended in autumn 2012. The framework itself has been extended during this thesis quite a lot. Further, there are also some precomputations involved, performed in matlab. This is basically addressing all the required precomputations for the provided height-fields, referring to computation of the inverse two dimensional Fourier transformations which are further explained within this chapter.

It's noteworthy that all the vertices are processed within the vertex-shader, whereas the fragment shader's responsibility is to perform pixelwise rendering, using the input from the vertex shader. Just remember, fragments are determined by a triple of vertices. hence each pixel has assigned a trilinear interpolated value of all input parameters of its spanning vertices. Usually, all necessary transformations are applied vertex-wise, considering

the vertex-shader as the precomputation stage for the later rendering within the rendering pipeline, in the fragment-shader. In the geometry shader, new vertices around a considered vertex can be created. this is useful for debugging - displaying normals graphically for example.

In this section we are going to explain how to get a fragment-shader from our findings for our BRDF formulation from the last section. this fragment-shader will render the effect of diffraction on our given geometry pixelwise. Therefore, the quality of diffraction depends on the number of pixels we are going to use for the rendering process and this is directly determined by the resolution of the canvas in which the rendered images are being displayed. But, before we can start formulating our fragment-shader we first have to write our vertex shader which does all the precomputations.

By the end of the day we will end up with two different shaders, one which basically samples the whole lambda space using a gaussian window. This shader will be modeling the effect of diffraction completely but will also be rather slow. The other shader will use a gaussian window too but will just use a few wavenumber for the sampling process. Furthermore, this shader will thread specularly seperatly as a special case which will be more like an approximation.

tell how we are going to sample - uniformly along lambda - explain drawback of this approach - explain possible solutions for this issue. maybe refer to reference shader or leave this for the disscusion part.

3.1 Setup

explain geometry computation explain light(source) setup explain factories explain camera setup explain how materials are stored explain how assigned to jrtr explain how passed to glsl shader - see computer graphics slides maybe show schematically the architecture

3.2 Precomputations in Matlab

explain matlab code explain shifts explain what will be outputed

3.3 jrtr Framework

explain how this will work

3.4 GLSL Diffraction Shader

start using the final findings from chapter 2 and substitute explain how all the components are computed and why they are computed like this.

Algorithm 1 Vertex diffraction shader

foreach *Vertex* $v \in Shape$ **do**
end for

Algorithm 2 Fragment diffraction shader

```
foreach Pixel  $p \in \text{Fragment}$  do
   $BRDF_{XYZ}, BRDF_{RGB} = \text{vec4}(0.0)$ 
   $(u, v, w) = \hat{\mathbf{k}}_1 - \hat{\mathbf{k}}_2$ 
  for  $(\lambda = \lambda_{min}; \lambda \leq \lambda_{max}; \lambda = \lambda + \lambda_{step})$  do
     $k = \frac{2\pi}{\lambda}$ 
     $(w_u, w_v) = (ku, kv)$ 
     $w_{color} = (S_x(\lambda), S_y(\lambda), S_z(\lambda))$ 
    for  $(r)$  do
      for  $(s)$  do
         $coords = \text{getLookUpCoord}(r, s)$ 
         $P = \text{taylorApprox}(coords, k, w)$ 
         $w_{r,s} = \text{gaussianWeight}(dist)$ 
         $scale_{pq} = pqFactor(w_u, w_v)$ 
         $P* = scale_{pq}$ 
         $P_{abs} = |P|^2$ 
         $P_{abs}* = w_{r,s}$ 
         $BRDF_{XYZ} += \text{vec4}(P_{abs} * w_{color}, 0.0)$ 
      end for
    end for
  end for
   $BRDF_{XYZ} = BRDF_{XYZ} * C(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) * shadowF$ 
   $BRDF_{XYZ}.xyz = D_{65} * M_{XYZ-RGB} * BRDF_{XYZ}.xyz$ 
   $BRDF_{RGB}.xyz = D_{65} * M_{XYZ-RGB} * BRDF_{XYZ}.xyz$ 
   $BRDF_{RGB} = \text{gammaCorrect}(BRDF_{RGB})$ 
end for
```

4 Data Acquisition and Evaluation

what is this chapter about how is evaluation performed our shader

4.1 Diffraction Grating

list def from wiki

$$d\sin(\theta_m) = m\lambda$$

$$d(\sin(\theta_i) + \sin(\theta_m)) = m\lambda$$

$$\sin(\theta_m) = \left(\frac{m\lambda}{d} - \sin(\theta_i) \right)$$

$$\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \exists r \in [0, \infty) \exists \phi \in [0, 2\pi] \exists \theta \in [0, \pi] \text{ s.t.}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix}$$

4.2 Snake Skin Parameters

5 Results

differece of this shader compared to evaluation shader

6 Conclusion

explain why we did our derivations explain why our approach is a good idea explain how the straight foreward approach would behave compared to our approach, computing the fourier transformations straight away. explain what we achieved, summary say something about draw-backs and about limitations of current apporach say something about the ongoing paper

6.1 Further Work

6.1.1 Sources

- [1] http://en.wikipedia.org/wiki/Ratio_test
- [2] <http://math.jasonbhill.com/courses/fall-2010-math-2300-005/lectures/taylor-polynomial-error-bounds>