**Appendix A: Alternative RDE combustor model**

An alternative model for the RDE combustor was considered. Although it was not used to solve the flow across the combustor in our final analysis, because it has not been used before by any research group, we think it is worth mentioning it.

The purpose of this method is to simplify the analysis of the RDE by looking at it from the reference frame of the detonation instead of looking at it from a fixed reference frame. By doing so the problem stops being a time dependent problem and an analytical solution can be found. However, some of the assumptions made in order to simplify the problem have not been proven to be correct. The main assumptions made are:

- The effect of the inertial forces can be neglected because they are small compared with other forces.
- The velocity in the azimuthal direction $(v_\theta)$ can be neglected in this analysis because it will not contribute to the thrust generation.
- Isentropic flow after the detonation.

In the reference frame of the detonation, all the physical properties of the flow (i.e. pressure, density, temperature,… ) depend on the angular position in the combustor $(\theta)$ but not in time. The problem becomes steady. The change of reference frame has been made by means of the velocity of the detonation $v_{CJ}$

$$v_{CJ} = R_m \frac{d\theta}{dt} \quad where\ R_m = \frac{1}{2}(R_{out} + R_{in})$$

Integrating this equation and imposing the boundary condition that at $t = 0,\ \theta = 0$ the following relation between $t$ and $\theta$ is obtained.

$$t = \frac{R_m}{v_{CJ}}\theta \rightarrow t_c = \frac{2\pi R_m}{v_{CJ}}$$

From experimental measurement it is known that the pressure decays exponentially with time.

$$p_c(t) = p_{min} PR_{CJ} \exp(-\lambda t) \quad where\ \lambda = \frac{\ln(PR_{CJ})}{tc}$$

Converting this equation into a function dependent on $\theta$ the following equation is obtained:

$$p_c(\theta) = p_{min} PR_{CJ} \exp\left(-\ln(PR_{CJ})\frac{\theta}{2\pi}\right) \quad where\ \theta \sim rad \quad \left| \begin{array}{l} p_c(\theta = 0) = p_{min}PR_{CJ} = p_{max} \\ \\ p_c(\theta = 2\pi\ rad) = p_{min} \end{array} \right.$$

Recalling the assumption of isentropic flow after the detonation the temperature evolution can be modeled as:

$$\frac{T_c(\theta)}{T_{max}} = \left(\frac{p_c(\theta)}{p_{min}PR_{CJ}}\right)^{\frac{\gamma-1}{\gamma}} \rightarrow T_c(\theta) = T_{max}\left(\frac{p_c(\theta)}{p_{min}PR_{CJ}}\right)^{\frac{\gamma-1}{\gamma}}$$

The mass flow at the exit of the combustor must be the airflow $(\dot{m}_0)$ plus the fuel mass $(\dot{m}_f)$. Applying mass continuity to a control volume that encloses the combustor volume the following relation arises.

$$\dot{m} = \dot{m}_0 + \dot{m}_f = \rho_c(\theta)u_c(\theta)A_{combustor} = \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \rho_c(\theta)u_c(\theta) r dr d\theta$$

This equation cannot be solved because the shape of $u_c(\theta)$ is not known a priori. In order to obtain a solution the average value of $u_c(\theta)$ is used, $\bar{u}_c$ is an average value of $u_c(\theta)$.

Solving the equation for $\bar{u}_c$ the following analytical solution is obtained.

$$\bar{u}_c = \frac{\dot{m}}{\frac{1}{2}(R_{out}^2 - R_{in}^2)\, C} \quad where\ \ C = \frac{p_{max}\, 2\pi\gamma}{RT_{max}\ln(PR_{CJ})}\left[\exp\left(\frac{ln(PR_{CJ})}{\gamma}\right) - 1\right]$$

The stagnation properties can be calculated using $\bar{u}_c$ and assuming that $v_\theta$ is zero. Once these values are obtained the problem is closed and all the flow properties are known.

$$M_c(\theta) = \frac{\bar{u}_c}{\sqrt{\gamma RT(\theta)}} \qquad T_{tc}(\theta) = T_c(\theta) + \frac{\bar{u}_c^2}{2Cp} \qquad p_{tc}(\theta) = p_c(\theta)\left(1 + \frac{\gamma-1}{2}M_c^2(\theta)\right)^{\frac{\gamma-1}{\gamma}}$$

**Appendix B: Chapman-Jouguet detonation model**

Before having available the Matlab version of the NASA CEA computer program, the properties of the flow after a Chapman-Jouguet detonation were obtained using a Matlab code based on this simple model.

The CJ detonation was modeled as a shock wave followed by a heat addition region. A sketch of this detonation model can be seen in Figure []

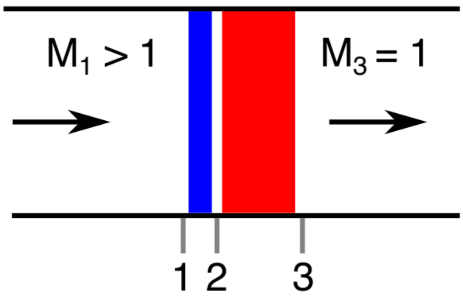


Figure [2], the shock wave is blue and the heat addition zone is red (from [3])

The problem is solved from the shock wave reference frame. However, the inertial forces that may appear due this change to a relative reference frame are considered to be negligible small. In this analysis the properties of the air are allowed to change depending on the static temperature. The fuel to air ratio is expected to be very small. Due to this fact, the properties before and after the detonation are estimated assuming that the contribution of the burned fuel to $Cp$, $\gamma$ and $R$ is negligible and that the fluid before and after the detonation is only air.

$$Cp = Cp_{air}(T) \qquad R = 287\frac{J}{kg\ K} \qquad \gamma = \frac{Cp_{air}(T)}{Cp_{air}(T) - R}$$

This problem has been solved using the 1-D form of the equations of continuity, momentum and energy. These equations expressed in terms of the stagnation properties of the flow and the Mach number can be written as:

Continuity equation
$$\left(\frac{p_{t1}A_1}{\sqrt{T_{t1}}}\right)\sqrt{\frac{\gamma_1}{R}}D_1 = \left(\frac{p_{t2}A_2}{\sqrt{T_{t2}}}\right)\sqrt{\frac{\gamma_2}{R}}D_2$$

Momentum equation
$$p_{t1}A_1G_1 + F_x = p_{t2}A_2G_2$$

Conservation of energy
$$f\Delta\text{HV} = Cp_2T_{t2} - Cp_1T_{t1}$$

Where D, G and N are functions of $M$ and $\gamma$

$$D(M,\gamma) = \frac{M}{\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$G(M,\gamma) = \frac{1+\gamma M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}}$$

$$N(M,\gamma) = \frac{D(M,\gamma)}{G(M,\gamma)} = \frac{M\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}}}{(1+\gamma M^2)}$$

Shock wave

Shock waves are adiabatic by definition, so the stagnation temperature before and after the shock are equal. In addition, the shock does not add any external force into the flow. Dividing the continuity by the momentum equation the following relation arises

$$\sqrt{\gamma_1}N(M_1,\gamma_1) = \sqrt{\gamma_2}N(M_2,\gamma_2)$$

Heat addition region

As in the shock wave during the heat addition there are not external forces acting on the flow but, in this region, the stagnation temperature and pressure change. Dividing continuity by momentum and simplifying the energy equation the following equations are obtained:

$$\sqrt{\frac{\gamma_2}{T_{t2}}}N(M_2,\gamma_2) = \sqrt{\frac{\gamma_1}{T_{t1}}}N(M_1,\gamma_1) = \sqrt{\frac{\gamma_3}{T_{t3}}}N(M_3,\gamma_3)$$

$$f\Delta HV = Cp_3T_{t3} - Cp_2T_{t2} = Cp_3T_{t3} - Cp_1T_{t1}$$

Imposing the Chapman-Jouguet condition of choked flow after the detonation and solving this system of equations for $M_1$ it is possible to obtain the Mach number of the detonation wave $\left(M_1 = M_{CJ}\right)$

$$\sqrt{\frac{\gamma_1}{\gamma_3}\frac{\frac{Cp_1}{Cp_3}T_{t1}+\frac{f\Delta HV}{Cp_3}}{Tt1}}\,N(M_1,\gamma_1) - N(M_3=1,\gamma_3) = 0 \quad where\ T_{t1} = T_1\left(1+\frac{\gamma_1-1}{2}M_1^2\right)$$

Once the Mach number of the detonation is known it is possible to determine all the other properties of the detonation.

The temperature after the detonation ($T_3\ or\ T_{max}$) is obtained from the energy equation.

$$T_3 = T_{max} = \frac{f\Delta HV + Cp_1T_{t1}}{Cp_3\left(\frac{\gamma_3+1}{2}\right)}$$

The pressure ratio of the detonation is calculated using the following relations:

$$\frac{p_{t1}}{\sqrt{T_{t1}}}\sqrt{\gamma_1}D(M_1,\gamma_1) = \frac{p_{t3}}{\sqrt{T_{t3}}}\sqrt{\gamma_1}D(M_3=1,\gamma_3) \rightarrow \frac{p_{t3}}{p_{t1}} = \sqrt{\frac{T_{t3}}{T_{t1}}\frac{\gamma_1}{\gamma_3}}\frac{D(M_1,\gamma_1)}{D(M_3=1,\gamma_3)}$$

$$Pressure\ ratio = \frac{p_3}{p_1} = \frac{p_{t3}}{p_{t1}}\frac{\left(1+\frac{\gamma_1-1}{2}M_1^2\right)^{\frac{\gamma_1}{\gamma_1-1}}}{\left(\frac{\gamma_3+1}{2}\right)^{\frac{\gamma_3}{\gamma_3-1}}}$$

In Table [] the results obtained from this code and CEA are compared for the case of burning hydrogen in air at stoichiometric fuel to air ratio ($T_1 = 750\ K,\ p_1 = 2\ atm$)

| | Matlab code | CEA | Error [%] |
|---|---|---|---|
| Detonation Mach number | 2.868 | 3.029 | 5.32 |

| | | | |
|---|---|---|---|
| $[-]$ | | | |
| Temperature after the detonation $[K]$ | 2474 | 3076 | 19.57 |
| Detonation pressure ratio $[-]$ | 5.633 | 6.289 | 10.43 |

Table [4], detonation properties

Although there are discrepancies between both solutions the results are similar for most of the values. The main source of error comes from the fact that CEA solves the problem using chemical equilibrium and for the Matlab code we developed the flow is assumed to be composed only of air, neglecting the contribution of other species that will be present in the mixture.

**Appendix C: Air properties**

Air properties such as viscosity, specific heat and conductivity are obtained from different sources.

Viscosity [4]

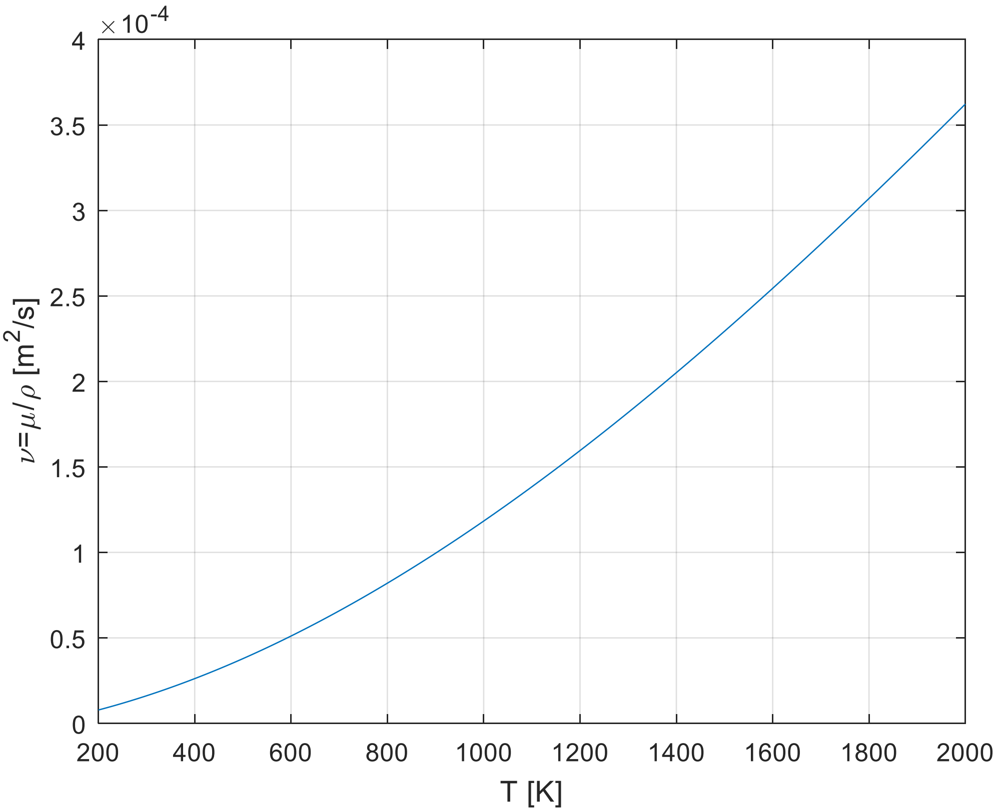


Figure [3] kinematic viscosity vs. static temperature

Specific heat [5]

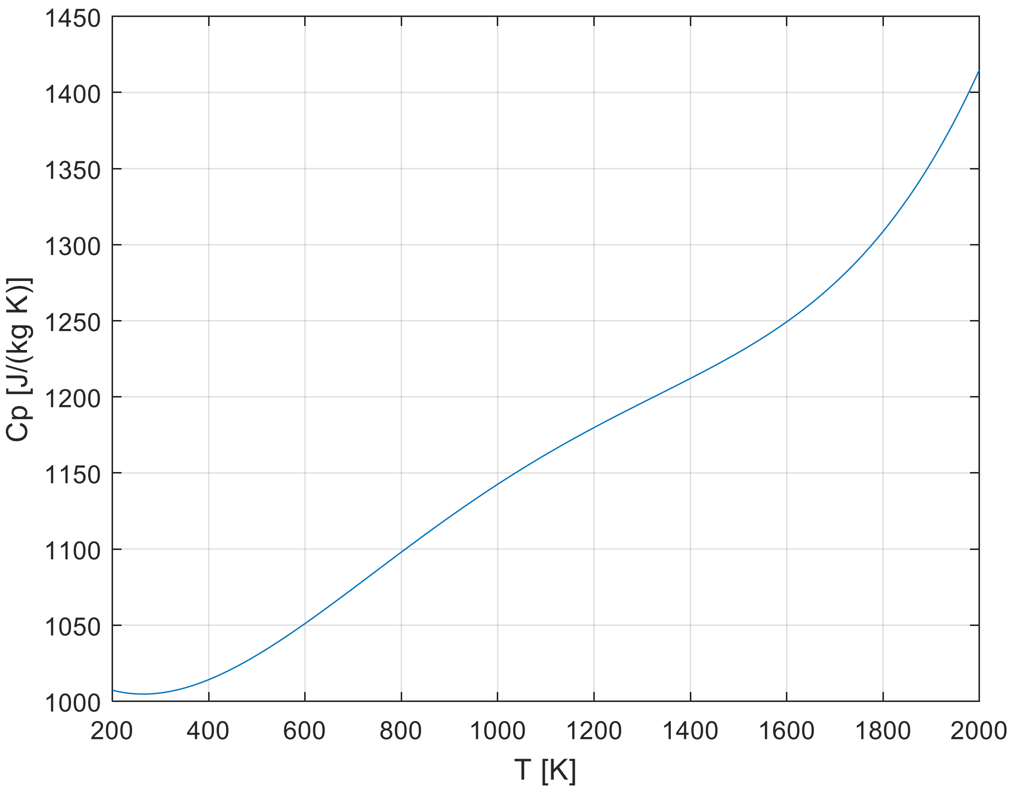


Figure [4] Specific heat vs. static temperature

<u>Conductivity</u> [6]