Fundamentals of Mathematics

Course Code: 23B31MA111

Lecture-2,3

CO1

Module: Sets, Relations and Functions

Contents to be Covered

Contents

1. Subsets
2. Power Set
3. Universal Set
4. Venn Diagrams
5. Operations on Sets
6. Numerical and examples

Reference

1. Mathematics Textbook for Class XI, NCERT, 2013.

SUBSETS

- ***A set A is said to be a subset of a set B if every element of A is also an element of B.***

$$A \subset B \text{ if } a \in A \Rightarrow a \in B$$

We read the above statement as "*A is a subset of B if a is an element of A implies that a is also an element of B*". If A is not a subset of B, we write $A \not\subset B$.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have $B \subset A$. In this case, A and B are the same sets so that we have

$$A \subset B \text{ and } B \subset A \Leftrightarrow A = B$$

It follows from the above definition that every set A *is a subset of itself*, i.e., $A \subset A$. Since the empty set ϕ has no elements, we agree to say that ϕ *is a subset of every set*. We now consider some examples :

(i) The set **Q** of rational numbers is a subset of the set **R** of real numbes, and we write $\mathbf{Q} \subset R$.

(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write $B \subset A$.

(iii) Let $A = \{1, 3, 5\}$ and $B = \{x : x$ is an odd natural number less than $6\}$. Then $A \subset B$ and $B \subset A$ and hence $A = B$.

(iv) Let $A = \{ a, e, i, o, u\}$ and $B = \{ a, b, c, d\}$. Then A is not a subset of B, also B is not a subset of A.

Let A and B be two sets. If $A \subset B$ and $A \neq B$, then A is called a *proper subset of* B and B is called *superset* of A. For example,

$A = \{1, 2, 3\}$ is a proper subset of $B = \{1, 2, 3, 4\}$.

If a set A has only one element, we call it a *singleton set*. Thus,$\{ a \}$ is a singleton set.

Subsets of R

Let $a, b \in \mathbf{R}$ and $a < b$. Then the set of real numbers

$\{ y : a < y < b\}$ is called an *open interval* and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called *closed interval* and is denoted by $[a, b]$. Thus

$$[a, b] = \{x : a \leq x \leq b\}$$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \leq x < b\}$ is an *open interval* from a to b, including a but excluding b.

$(a, b] = \{x : a < x \leq b\}$ is an *open interval* from a to b including b but excluding a.

Power Set

- ***The collection of all subsets of a set A is called the power set of A.***
- It is denoted by P(A).

if A = { 1, 2 }, then

$$P(A) = \{ \phi, \{1\}, \{2\}, \{1,2\} \}$$

Also, note that $n[P(A)] = 4 = 2^2$

In general, if A is a set with $n(A) = m$, then it can be shown that $n[P(A)] = 2^m$.

Universal Set

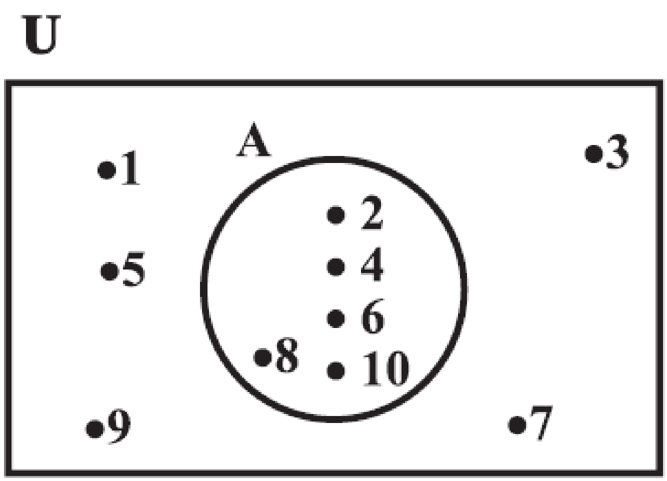
- Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. For example, while studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the "Universal Set". The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.
- For example, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set R of real numbers. For another example, in human population studies, the universal set consists of all the people in the world.

Venn Diagrams

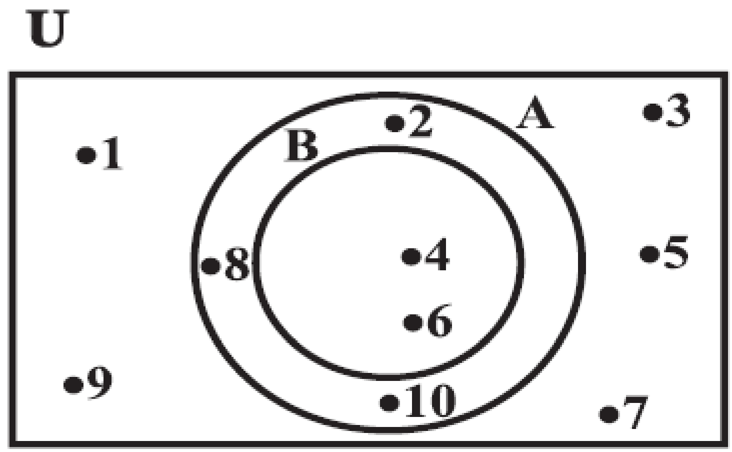
- ***Most of the relationships between sets can be represented by means of diagrams which are known as Venn diagrams.***

These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

Ex: In diagram below, U = {1,2,3, …, 10} is the universal set of which A = {2,4,6,8,10} is a subset.



Ex: In diagram below, $U = \{1,2,3, \ldots, 10\}$ is the universal set of which $A = \{2,4,6,8,10\}$ and $B = \{4, 6\}$ are subsets, and also B is a subset of A.



Operations on Sets

Operations when performed on two sets give rise to another set. (like numbers)

Union of sets

The union of two sets A and B is the set C which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write

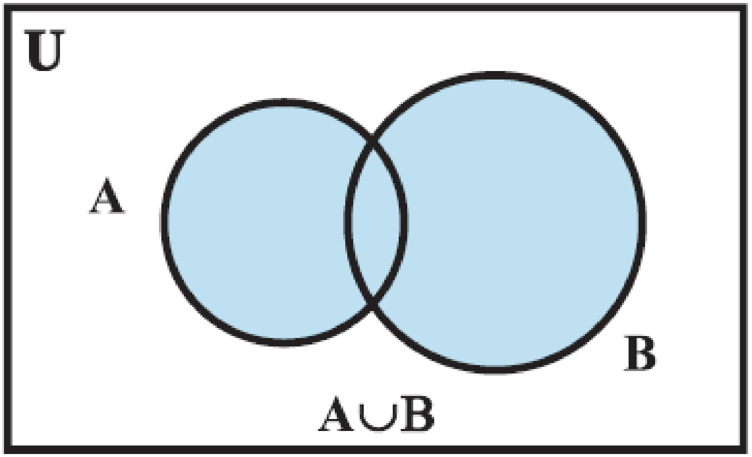
$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

Example:

Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find A U B.

Solution We have A U B = { 2, 4, 6, 8, 10, 12}

The union of two sets can be represented by a Venn diagram as shown in Figure below



Example:

Let $A = \{ a, e, i, o, u \}$ and $B = \{ a, i, u \}$. Show that $A \cup B = A$

Solution We have, $A \cup B = \{ a, e, i, o, u \} = A$.

This example illustrates that union of sets A and its subset B is the set A itself, i.e., if $B \subset A$, then $A \cup B = A$.

Example:

construct your own 2 examples

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$
(Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $A \cup A = A$ (Idempotent law)

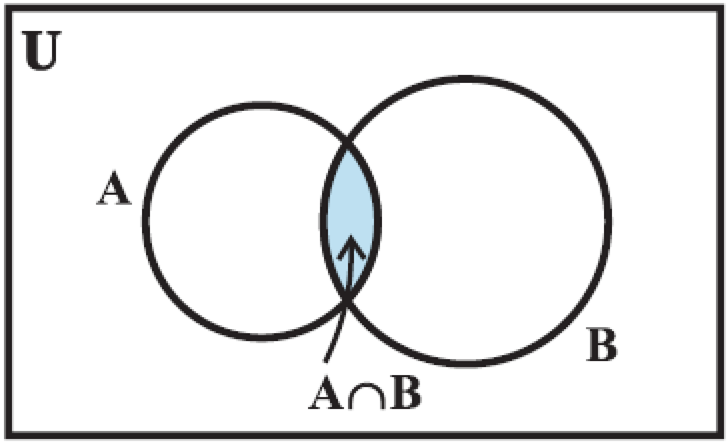
(v) $U \cup A = U$ (Law of U)

Intersection of sets

The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The union of two sets can be represented by a Venn diagram as shown in Figure below



Example:

Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$. Find $A \cap B$

Solution We have $A \cap B = \{6, 8\}$

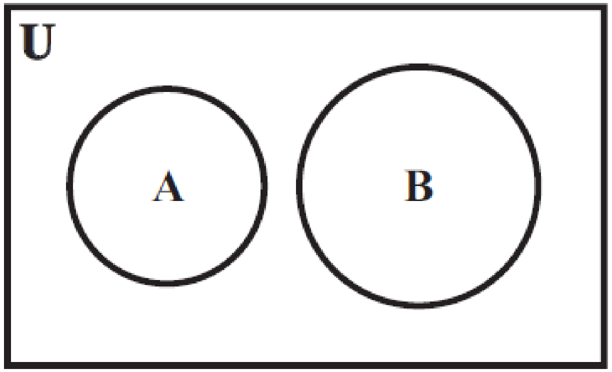
Example:

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$ and hence show that $A \cap B = B$.

Solution We have $A \cap B = \{2, 3, 5, 7\} = B$. We note that $B \subset A$ and that $A \cap B = B$.

If A and B are two sets such that $A \cap B = \phi$, then A and B are called *disjoint sets.*

For example, let $A = \{ 2, 4, 6, 8 \}$ and $B = \{ 1, 3, 5, 7 \}$. Then A and B are disjoint sets, because there are no elements which are common to A and B. The disjoint sets can be represented by



Some Properties of Operation of Intersection

(i) $A \cap B = B \cap A$ (Commutative law).

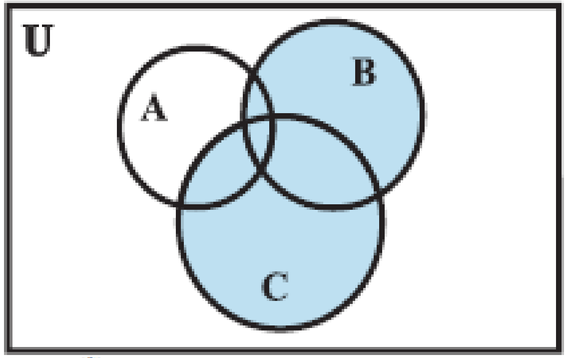
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).

(iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U).

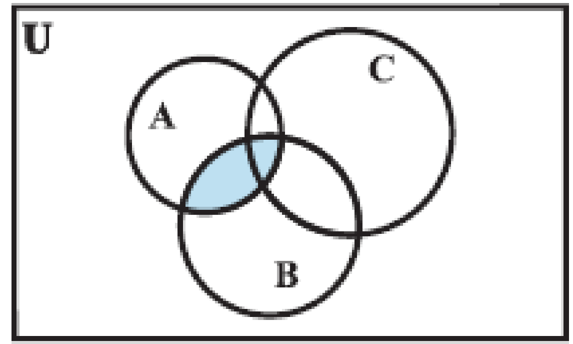
(iv) $A \cap A = A$ (Idempotent law)

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law) i. e., $\cap$ distributes over $\cup$

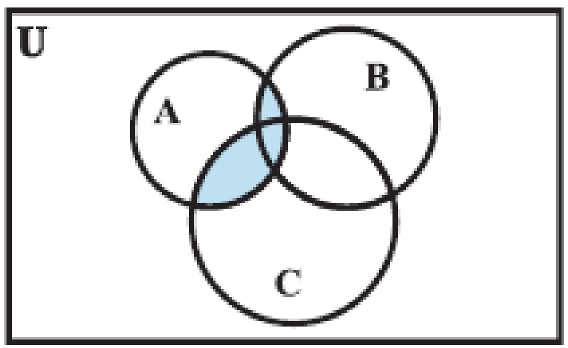
(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



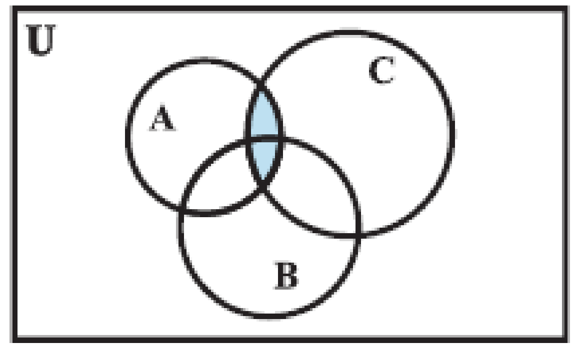
(i) $(B \cup C)$



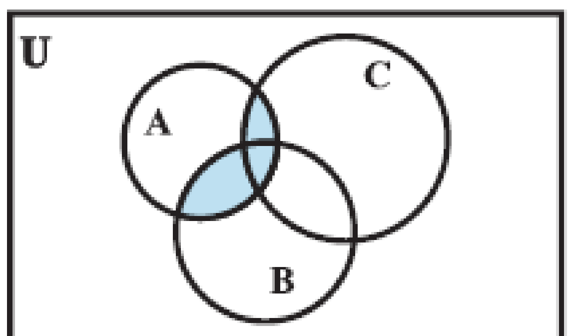
(iii) $(A \cap B)$



(ii) $A \cap (B \cup C)$



(iv) $(A \cap C)$



(v) $(A \cap B) \cup (A \cap C)$

Difference of sets

The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as " A minus B".

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

Example Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A.

Solution We have, A – B = { 1, 3, 5 }, since the elements 1, 3, 5 belong to A but not to B and B – A = { 8 }, since the element 8 belongs to B and not to A.

We note that A – B ≠ B – A.

Complement of a Set

Let U be the universal set and A be a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus,

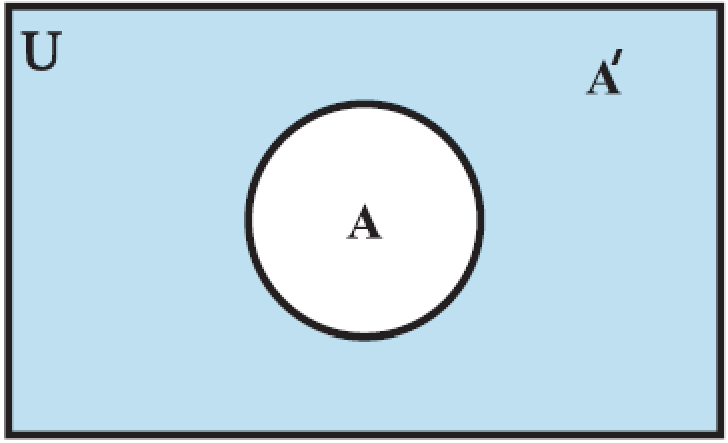
$$A' = \{x : x \in U \text{ and } x \notin A\}$$

Obviously $A' = U - A$

Example:

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$. Find A'.

Solution We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A. Hence $A' = \{2, 4, 6, 8, 10\}$.



Some Properties of Complement Sets

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation : $(A')' = A$

4. Laws of empty set and universal set $\phi' = U$ and $U' = \phi$.

Example Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$ and $B = \{3, 4, 5\}$.

Find A', B', $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$.

Solution Clearly $A' = \{1, 4, 5, 6\}$, $B' = \{1, 2, 6\}$. Hence $A' \cap B' = \{1, 6\}$

Also $A \cup B = \{2, 3, 4, 5\}$, so that $(A \cup B)' = \{1, 6\}$

$$(A \cup B)' = \{1, 6\} = A' \cap B'$$

Exercise

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find (i) A' (ii) B' (iii) $(A \cup C)'$ (iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$

Some results using operations on sets

Let A and B be finite sets. If $A \cap B = \phi$, then

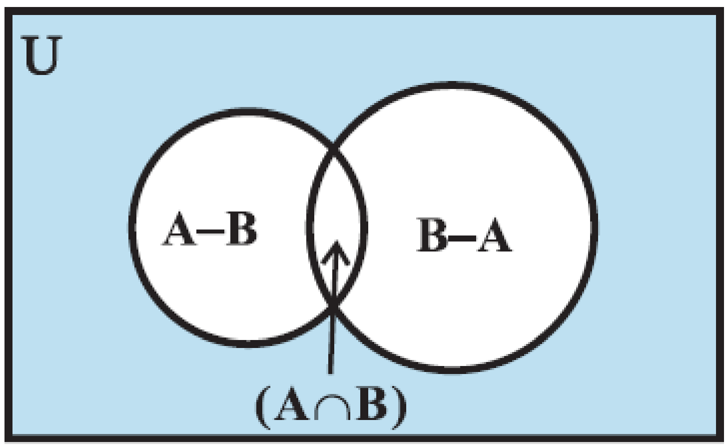
(i) $n(A \cup B) = n(A) + n(B)$ ... (1)

The elements in $A \cup B$ are either in A or in B but not in both as $A \cap B = \phi$. So, (1) follows immediately.

In general, if A and B are finite sets, then

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ... (2)

Note that the sets $A - B$, $A \cap B$ and $B - A$ are disjoint and their union is $A \cup B$



Therefore

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$
$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$
$$= n(A) + n(B) - n(A \cap B), \text{ which verifies (2)}$$

(iii) If A, B and C are finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$
$$- n(A \cap C) + n(A \cap B \cap C) \quad \ldots (3)$$

In fact, we have

$$n(A \cup B \cup C) = n(A) + n(B \cup C) - n[A \cap (B \cup C)] \quad [\text{by } (2)]$$
$$= n(A) + n(B) + n(C) - n(B \cap C) - n[A \cap (B \cup C)] \quad [\text{by } (2)]$$

Since $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we get

$$n[A \cap (B \cup C)] = n(A \cap B) + n(A \cap C) - n[(A \cap B) \cap (A \cap C)]$$
$$= n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)$$

Therefore

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$
$$- n(A \cap C) + n(A \cap B \cap C)$$

This proves (3).

Example

In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics ?

Solution Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach physics. In the statement of the problem, the word 'or' gives us a clue of union and the word 'and' gives us a clue of intersection. We, therefore, have

$$n(M \cup P) = 20,\ n(M) = 12 \text{ and } n(M \cap P) = 4$$

We wish to determine $n(P)$.

Using the result

$$n(M \cup P) = n(M) + n(P) - n(M \cap P),$$

we obtain

$$20 = 12 + n(P) - 4$$

Thus $\qquad n(P) = 12$

Hence 12 teachers teach physics.

Recall:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Example In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice.

Solution Let U denote the set of surveyed students and A denote the set of students taking apple juice and B denote the set of students taking orange juice. Then

$$n(U) = 400,\ n(A) = 100,\ n(B) = 150 \text{ and } n(A \cap B) = 75.$$

Now

$$\begin{aligned} n(A' \cap B') &= n(A \cup B)' \\ &= n(U) - n(A \cup B) \\ &= n(U) - n(A) - n(B) + n(A \cap B) \\ &= 400 - 100 - 150 + 75 = 225 \end{aligned}$$

Hence 225 students were taking neither apple juice nor orange juice.

Example There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to

(i) Chemical C_1 but not chemical C_2 (ii) Chemical C_2 but not chemical C_1

(iii) Chemical C_1 or chemical C_2

Solution Let U denote the universal set consisting of individuals suffering from the skin disorder, A denote the set of individuals exposed to the chemical C_1 and B denote the set of individuals exposed to the chemical C_2.

Here $n(U) = 200$, $n(A) = 120$, $n(B) = 50$ and $n(A \cap B) = 30$

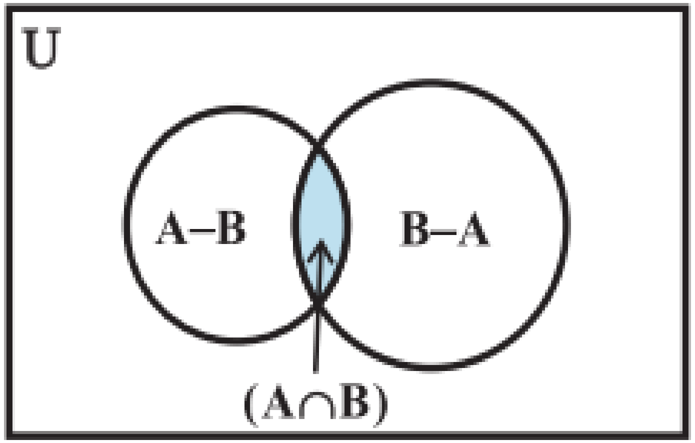
(i) From the Venn diagram given in Fig , we have

$$A = (A - B) \cup (A \cap B).$$

$$n(A) = n(A - B) + n(A \cap B) \quad \text{(Since A – B) and } A \cap B \text{ are disjoint.)}$$

or $n(A - B) = n(A) - n(A \cap B) = 120 - 30 = 90$

Hence, the number of individuals exposed to chemical C_1 but not to chemical C_2 is 90.



(ii) From the Fig 1.13, we have

$$B = (B - A) \cup (A \cap B).$$

and so, $n(B) = n(B - A) + n(A \cap B)$

(Since B – A and $A \cap B$ are disjoint.)

or $n(B - A) = n(B) - n(A \cap B)$

$= 50 - 30 = 20$

Thus, the number of individuals exposed to chemical C_2 and not to chemical C_1 is 20.

(iii) The number of individuals exposed either to chemical C_1 or to chemical C_2, i.e.,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 120 + 50 - 30 = 140.$$

Example Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

Solution Let U be the set of car owners investigated, M be the set of persons who owned car A and S be the set of persons who owned car B.

Given that $n(U) = 500$, $n(M) = 400$, $n(S) = 200$ and $n(S \cap M) = 50$.

Then $n(S \cup M) = n(S) + n(M) - n(S \cap M) = 200 + 400 - 50 = 550$

But $S \cup M \subset U$ implies $n(S \cup M) \le n(U)$.

This is a contradiction. So, the given data is incorrect.

Example 32 A market research group conducted a survey of 1000 consumers and reported tl ' 720 consumers like product A and 450 consumers like product B, what is the least number that must have liked both products?

Solution Let U be the set of consumers questioned, S be the set of consumers who liked the product A and T be the set of consumers who like the product B. Given that

$$n(U) = 1000,\ n(S) = 720,\ n(T) = 450$$

So $$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$

$$= 720 + 450 - n(S \cap T) = 1170 - n(S \cap T)$$

Therefore, $n(S \cup T)$ is maximum when $n(S \cap T)$ is least. But $S \cup T \subset U$ implies $n(S \cup T) \leq n(U) = 1000$. So, maximum values of $n(S \cup T)$ is 1000. Thus, the least value of $n(S \cap T)$ is 170. Hence, the least number of consumers who liked both products is 170.

You can discuss any doubts in Tutorial Sheets 1 and 2.

Thank You…