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# Supporting Web Material: Adaptively-Tuned Particle Swarm Optimization with Application to Spatial Design

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# S1. PSO and BBPSO details

In Section 2 we introduced PSO, BBPSO, and our adaptively tuned variants of both. Here we describe in detail several modifications to both PSO and BBPSO. Most of these are so-called standard modifications from Clerc (2011).

## S1.1. PSO

The standard PSO algorithm is given by equation (1). We consider several additions and modifications to this algorithm below. Each of these modifications is combined with adaptively tuned inertia as in equation (4) to create the AT-PSO algorithms we employ.

*S1.1.1. Initialization:* In order to initialize the swarm, the number of particles, their initial locations, and their initial velocities have to be chosen. Clerc (2011) suggests making the swarm size a function of the dimension of the search space, but notes that this is known to be suboptimal. We use their alternative suggest to use a default swarm size of 40. We assume the search space is a $D$-dimensional hypercube given by $\bigtimes_{j=1}^{D}[min_j, max_j]$. Then each particle's initial location is randomly generated uniformly on the cube, i.e. $\theta_{ij}(0) \overset{ind}{\sim} U(min_j, max_j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, D$. Each particle's velocity is initialized based on its location via $v_{ij}(0) \overset{ind}{\sim} U(min_j - x_{ij}(0), max_j - x_{ij}(0))$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, D$.

In Section 4.3 each design point is constrained to be in Harris County, TX, which is not a rectangle. We initialize the swarm by placing initial design points on the smallest rectangle containing Harris County. Design points outside of Harris County will quickly move back into the county due to the confinement strategy discussed next.

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*S1.1.2. Confinement:* Even though the initialization of the swarm is confined to a hypercube, nothing prevents any given particle from leaving the search space. There are a number of things that can be done in order to solve this problem and each has advantages and disadvantages depending on the situation. We focus on two approaches. One is to move any particle that leaves the search space to the nearest point still inside the search space and then adjust its velocity, e.g. Clerc (2011) suggests that whenever $x_{ij}(k) > max_j$, it should be set to $x_{ij}(k) = max_j$ and similarly when $x_{ij}(k) < min_j$ it should be set to $x_{ij}(k) = min_j$, and in both cases the velocity along that dimension should be reversed and halved, i.e. $v_{ij}(k) = -0.5v_{ij}(k)$. This causes particles to bounce off the boundary and move back toward the middle of the search space. This is the default strategy we use. This strategy can also be used for more complex search spaces, and in Section 4.3 when a particle leaves Harris County, we move it back to the nearest point in the county and reset the particle's velocity as in the previous paragraph. In order to move the particle we use the `gNearestPoint` function from the `rgeos` R package; Bivand & Rundel (2016); R Core Team (2016).

*S1.1.3. Redraw Neighborhoods:* In Section 2 we briefly described a variant of the stochastic star neighborhood which we use. This neighborhood is stochastic, meaning that each particle's neighbors are randomly drawn when the algorithm is initialized. After an iteration in which the best known value of the objective function is unchanged, each particle's neighbors are randomly redrawn according to the same distribution.

*S1.1.4. Asynchronous Updates:* The way we defined PSO in equation (1) each particle can update simultaneously. This means that the algorithm is parallelizable, which is a major advantage for implementation on modern GPUs. However, asynchronously updating the particles typically results in faster converging algorithms when it is computationally feasible. In an asynchronous update from period $k$ to $k + 1$, particle $i$ recognizes that particle $i - 1$ has already updated its personal best location to $\boldsymbol{p}_{i-1}(k + 1)$ by the time it is $i$'s turn to update. So $i$ computes its group best location taking this into account. This causes particle $i = 1$ to behave differently from particle $i = n$ since particle $n$ always has better information in order to perform its update, so every iteration the particles are randomly reordered. More formally, before every iteration sample $o_1(k + 1), o_2(k + 1), \ldots, o_n(k + 1)$ from $\{1, 2, \ldots, n\}$ without replacement. Then the particles update starting with $o_1$, $o_2$, etc., where the group best update becomes $\boldsymbol{g}_{o_i}(k + 1) = \arg\min_{\{\boldsymbol{p}_{o_j}(k+1)|j \in \mathcal{N}_i\}} Q(\boldsymbol{p}_j(k_j^*))$ where $k_j^* = k + 1$ if $j < i$ and $k$ otherwise. We asynchronously update in all of our algorithms.

*S1.1.5. Coordinate Free Velocity Updates:* The standard velocity update in equation (1) is well known to bias the algorithm toward locations near the coordinate axes and especially the origin; Monson & Seppi (2005); Spears et al. (2010). In general we may not know if the true optimum is near an axis, so this behavior is undesirable. There are several alternative velocity updates available, e.g. in Monson & Seppi (2005). We use the coordinate free (CF) update suggested by Clerc (2011). First define the center of gravity for particle $i$ to be $\boldsymbol{C}_i(k) = \boldsymbol{\theta}_i(k) + \phi_1\{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}/3 + \phi_2\{\boldsymbol{g}_i(k) - \boldsymbol{\theta}_i(k)\}/3$. Let $\mathcal{H}_i(k)$ denote the hypersphere centered at $\boldsymbol{C}_i(k)$ with radius $||\boldsymbol{C}_i(k) - \boldsymbol{\theta}_i(k)||$ where $||.||$ denotes Euclidean distance. Then a new point $\boldsymbol{\theta}_i'(k)$ is drawn randomly from $\mathcal{H}_i(k)$ by sampling a direction and a radius, each uniformly. This is *not* the same as drawing uniformly over $\mathcal{H}_i(k)$ and in fact favors points near the center. Then the CF velocity update is given by $\boldsymbol{v}_i(k + 1) = \omega\boldsymbol{v}_i(k) + \boldsymbol{x}_i'(k)$. We use both the standard and CF velocity updates in our algorithms. The standard PSO algorithm with each feature in this subsection including the CF velocity update is what Clerc (2011) calls SPSO 2011.

*S1.1.6. When Personal Best = Group Best:* When a particle's personal best and group best locations coincide, it is often advantageous to allow the particle to explore more than usual. In the standard velocity update we do this

by removing the social term so that $\boldsymbol{v}_i(k+1) = \omega\boldsymbol{v}_i(k) + \phi_1\boldsymbol{r}_{1i}(k) \circ \{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}$. In the CF velocity update we change the center of gravity to ignore the social term so that $\boldsymbol{C}_i(k) = \boldsymbol{\theta}_i(k) + \phi_2\{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}/2$.

## S1.2. BBPSO

The standard BBPSO algorithm was introduced by Kennedy (2003) and updates from $t$ to $t+1$ via equation (2). We use each of the features in Section S1.1 in our BBPSO algorithms, though some of them need to be modified for the BBPSO setting. We list them below along with another modification of BBPSO which we employ. Each of these modifications are combined with adaptively tuning a scale parameter as in equation (3) to create our AT-BBPSO algorithms.

*S1.2.1. BBPSOxp:* A commonly used variant of BBPSO also introduced by Kennedy (2003) is called BBPSOxp. In this variant, each coordinate of each particle has a 50% chance of updating according to (2) and a 50% chance of moving directly to that particle's personal best location on that coordinate. In other words

$$\theta_{ij}(k+1) = \begin{cases} N\left(\frac{p_{ij}(k)+g_{ij}(k)}{2}, h_{ij}^2(k)\right) & \text{with probability } 0.5 \\ p_{ij}(k) & \text{otherwise,} \end{cases} \tag{1}$$

where $h_{ij}(k) = |p_{ij}(k) - g_{ij}(k)|$. We use both xp and non-xp versions of our BBPSO algorithms.

*S1.2.2. CF BBPSO:* BBPSO's update also depends on the coordinate system since each coordinate of $\boldsymbol{\theta}$ gets a different standard deviation. We employ BBPSO algorithms using the default standard deviation, but also using a coordinate free standard deviation given by $h_{ij}(k) = ||\boldsymbol{p}_i(k) - \boldsymbol{g}_i(k)||$.

*S1.2.3. When Personal Best = Group Best in BBPSO:* A downside of both BBPSO and BBPSOxp is that any particle whose personal best is currently its group best location does not move due to the definition of the standard deviation term. Several methods have been proposed to overcome this; e.g. Hsieh & Lee (2010) and Zhang et al. (2011). Zhang et al. (2011) propose using mutation and crossover operations for the group best particle. To do this, each group best particle randomly selects three other distinct particles from the entire swarm, $i_1$, $i_2$, and $i_3$, and updates according to

$$\theta_{ij}(k+1) = p_{i_1j}(k) + 0.5\{p_{i_2j}(k) - p_{i_3j}(k)\}. \tag{2}$$

This combines easily with BBPSOxp to update each coordinate of each particle with $h_{ij}(k) = 0$ according to (2) and the rest according to (1).

# S2. Tables for Comparing PSO and BBPSO algorithms

Tables S2.1-S2.6 contain the simulation results for objective functions 1-6 respectively (OF1, OF2, etc.). We use several measures to quantify how well each algorithm finds the global minimum. First, each table includes the mean and standard deviation of the absolute difference between the true global minimum and the algorithm's estimated global minimum across all 40 replications, denoted by Mean and SD. Second, each table includes a convergence criterion — the proportion of the replications that came within 0.01 of the true global minimum, denoted by $\widehat{P}$. Finally, $\widehat{K}$ denotes the median number of iterations until the algorithm reaches the convergence criterion; $\widehat{K} > 1000$ indicates that the algorithm did not converge in the allotted 1,000 iterations in at least of 50% of replications. Mean, $\widehat{P}$, and

$\widehat{K}$ can be thought of as how close on average the algorithm gets to the global minimum, what proportion of the time it converges, and long it takes to converge respectively. Values for the Mean and SD that are greater than 10,000 are omitted to keep the size of the tables manageable.

| OF1;  Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 0.00 | 0.00 | 1.00 | 205.5 | 0.00 | 0.00 | 1.00 | 358.5 | 384.26 | 891.68 | 0.32 | > 1000 |
| PSO2 | 0.00 | 0.00 | 1.00 | 113 | 0.00 | 0.00 | 1.00 | 200.5 | 724.66 | 1160.10 | 0.20 | > 1000 |
| PSO1-CF | 173.52 | 201.37 | 0.00 | > 1000 | 174.60 | 139.70 | 0.00 | > 1000 | 1633.30 | 975.59 | 0.00 | > 1000 |
| PSO2-CF | 164.60 | 117.21 | 0.00 | > 1000 | 122.56 | 97.48 | 0.00 | > 1000 | 1956.30 | 1074.90 | 0.00 | > 1000 |
| DI-PSO1 | 0.00 | 0.00 | 1.00 | 214 | 0.00 | 0.00 | 1.00 | 265.5 | 1467.50 | 1543.60 | 0.00 | > 1000 |
| DI-PSO2 | 0.00 | 0.00 | 1.00 | 187 | 0.00 | 0.00 | 1.00 | 233.5 | 1641.30 | 1566.30 | 0.00 | > 1000 |
| DI-PSO1-CF | 1709.90 | 897.14 | 0.00 | > 1000 | 694.98 | 346.65 | 0.00 | > 1000 | 2418.30 | 1140.10 | 0.00 | > 1000 |
| DI-PSO2-CF | 1535.30 | 848.11 | 0.00 | > 1000 | 790.97 | 429.16 | 0.00 | > 1000 | 2435.10 | 1173.80 | 0.00 | > 1000 |
| AT1-PSO1 | 0.00 | 0.00 | 1.00 | 186 | 0.00 | 0.00 | 1.00 | 263 | 1962.50 | 1777.20 | 0.00 | > 1000 |
| AT1-PSO2 | 0.00 | 0.00 | 1.00 | 183 | 0.00 | 0.00 | 1.00 | 247 | 932.07 | 1118.20 | 0.00 | > 1000 |
| AT1-PSO1-CF | 329.08 | 260.13 | 0.00 | > 1000 | 294.80 | 166.07 | 0.00 | > 1000 | 2489.30 | 1040.10 | 0.00 | > 1000 |
| AT1-PSO2-CF | 342.09 | 293.48 | 0.00 | > 1000 | 268.43 | 181.01 | 0.00 | > 1000 | 2096.10 | 1131.30 | 0.00 | > 1000 |
| AT2-PSO1 | 0.00 | 0.00 | 1.00 | 112 | 0.00 | 0.00 | 1.00 | 154 | 232.14 | 205.78 | 0.00 | > 1000 |
| AT2-PSO2 | 0.00 | 0.00 | 1.00 | 117 | 0.00 | 0.00 | 1.00 | 150.5 | 127.58 | 164.99 | 0.00 | > 1000 |
| AT2-PSO1-CF | 165.98 | 133.79 | 0.00 | > 1000 | 196.39 | 130.15 | 0.00 | > 1000 | 1550.60 | 696.54 | 0.00 | > 1000 |
| AT2-PSO2-CF | 118.22 | 77.93 | 0.00 | > 1000 | 160.47 | 118.92 | 0.00 | > 1000 | 1453.20 | 922.27 | 0.00 | > 1000 |
| AT1-BBPSO | 0.00 | 0.00 | 1.00 | 740 | 0.00 | 0.00 | 1.00 | 756 | 0.00 | 0.00 | 1.00 | 679.5 |
| AT1-BBPSOxp | 0.00 | 0.00 | 1.00 | 822.5 | 0.00 | 0.00 | 1.00 | 831 | 0.00 | 0.00 | 1.00 | 694 |
| AT1-BBPSO-CF | 0.00 | 0.00 | 1.00 | 637 | 0.00 | 0.00 | 1.00 | 642 | 0.00 | 0.00 | 1.00 | 572.5 |
| AT1-BBPSOxp-CF | 0.00 | 0.00 | 1.00 | 724.5 | 0.00 | 0.00 | 1.00 | 717 | 0.00 | 0.00 | 1.00 | 607.5 |
| AT2-BBPSO | 0.00 | 0.00 | 1.00 | 445.5 | 0.00 | 0.00 | 1.00 | 465 | 0.00 | 0.00 | 1.00 | 451 |
| AT2-BBPSOxp | 0.00 | 0.00 | 1.00 | 501 | 0.00 | 0.00 | 1.00 | 511 | 0.00 | 0.00 | 1.00 | 479 |
| AT2-BBPSO-CF | 0.00 | 0.00 | 1.00 | 386.5 | 0.00 | 0.00 | 1.00 | 404.5 | 0.00 | 0.00 | 1.00 | 392 |
| AT2-BBPSOxp-CF | 0.00 | 0.00 | 1.00 | 436.5 | 0.00 | 0.00 | 1.00 | 458 | 0.00 | 0.00 | 1.00 | 420 |

**Table S2.1.** Simulation results for OF1. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.

| OF2; Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 0.00 | 0.00 | 0.92 | 882.5 | 42.08 | 41.70 | 0.00 | $>1000$ | 3121.10 | 1900.60 | 0.00 | $>1000$ |
| PSO2 | 0.00 | 0.00 | 1.00 | 455 | 0.05 | 0.09 | 0.28 | $>1000$ | 2128.20 | 2264.80 | 0.00 | $>1000$ |
| PSO1-CF | 1198.70 | 727.12 | 0.00 | $>1000$ | 718.76 | 477.04 | 0.00 | $>1000$ | 3467.90 | 1973.20 | 0.00 | $>1000$ |
| PSO2-CF | 892.45 | 593.77 | 0.00 | $>1000$ | 639.33 | 342.00 | 0.00 | $>1000$ | 2966.50 | 1426.90 | 0.00 | $>1000$ |
| DI-PSO1 | 52.59 | 108.12 | 0.00 | $>1000$ | 225.76 | 210.47 | 0.00 | $>1000$ | 5494.70 | 2604.70 | 0.00 | $>1000$ |
| DI-PSO2 | 151.31 | 169.29 | 0.00 | $>1000$ | 516.50 | 282.07 | 0.00 | $>1000$ | 4323.90 | 1872.50 | 0.00 | $>1000$ |
| DI-PSO1-CF | 3793.40 | 1717.40 | 0.00 | $>1000$ | 1401.90 | 678.72 | 0.00 | $>1000$ | 4510.60 | 2058.90 | 0.00 | $>1000$ |
| DI-PSO2-CF | 3873.30 | 1935.70 | 0.00 | $>1000$ | 1566.30 | 733.85 | 0.00 | $>1000$ | 3637.10 | 1528.20 | 0.00 | $>1000$ |
| AT1-PSO1 | 0.00 | 0.00 | 1.00 | 478.5 | 0.08 | 0.18 | 0.25 | $>1000$ | 7808.60 | 2929.70 | 0.00 | $>1000$ |
| AT1-PSO2 | 0.00 | 0.00 | 1.00 | 481.5 | 0.01 | 0.01 | 0.85 | 924 | 4071.70 | 1647.50 | 0.00 | $>1000$ |
| AT1-PSO1-CF | 1424.90 | 620.02 | 0.00 | $>1000$ | 1021.10 | 509.66 | 0.00 | $>1000$ | 2910.80 | 1528.70 | 0.00 | $>1000$ |
| AT1-PSO2-CF | 1517.60 | 1070.00 | 0.00 | $>1000$ | 883.77 | 458.02 | 0.00 | $>1000$ | 2480.30 | 1419.50 | 0.00 | $>1000$ |
| AT2-PSO1 | 91.83 | 356.78 | 0.10 | $>1000$ | 0.40 | 0.85 | 0.12 | $>1000$ | 4323.20 | 1925.80 | 0.00 | $>1000$ |
| AT2-PSO2 | 5.98 | 20.86 | 0.52 | 979 | 0.23 | 1.24 | 0.60 | 965.5 | 2122.30 | 1402.00 | 0.00 | $>1000$ |
| AT2-PSO1-CF | 1358.10 | 766.94 | 0.00 | $>1000$ | 969.11 | 476.07 | 0.00 | $>1000$ | 2737.70 | 1082.90 | 0.00 | $>1000$ |
| AT2-PSO2-CF | 1078.80 | 640.14 | 0.00 | $>1000$ | 878.00 | 410.46 | 0.00 | $>1000$ | 2342.40 | 1427.20 | 0.00 | $>1000$ |
| AT1-BBPSO | 0.06 | 0.03 | 0.00 | $>1000$ | 0.01 | 0.01 | 0.62 | 989 | 0.00 | 0.00 | 0.90 | 916.5 |
| AT1-BBPSOxp | 0.67 | 0.41 | 0.00 | $>1000$ | 0.02 | 0.01 | 0.15 | $>1000$ | 0.01 | 0.00 | 0.85 | 944 |
| AT1-BBPSO-CF | 0.03 | 0.01 | 0.05 | $>1000$ | 0.00 | 0.00 | 0.97 | 916 | 0.00 | 0.00 | 1.00 | 819.5 |
| AT1-BBPSOxp-CF | 0.35 | 0.22 | 0.00 | $>1000$ | 0.01 | 0.01 | 0.72 | 972 | 0.00 | 0.00 | 1.00 | 873.5 |
| AT2-BBPSO | 0.00 | 0.00 | 1.00 | 852 | 0.00 | 0.00 | 0.92 | 846.5 | 0.00 | 0.00 | 1.00 | 660 |
| AT2-BBPSOxp | 0.45 | 0.63 | 0.00 | $>1000$ | 0.05 | 0.13 | 0.50 | 972 | 0.00 | 0.00 | 1.00 | 718 |
| AT2-BBPSO-CF | 0.00 | 0.00 | 1.00 | 821 | 0.01 | 0.02 | 0.92 | 825 | 0.00 | 0.00 | 1.00 | 636.5 |
| AT2-BBPSOxp-CF | 0.24 | 0.20 | 0.00 | $>1000$ | 0.04 | 0.08 | 0.52 | 962 | 0.00 | 0.00 | 1.00 | 682 |

**Table S2.2.** Simulation results for OF2. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.

| OF3;   Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 51.84 | 81.96 | 0.00 | > 1000 | 32.87 | 35.51 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| PSO2 | 24.09 | 40.64 | 0.00 | > 1000 | 34.97 | 56.51 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| PSO1-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| PSO2-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| DI-PSO1 | 73.91 | 118.06 | 0.00 | > 1000 | 138.79 | 237.03 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| DI-PSO2 | 73.15 | 101.70 | 0.00 | > 1000 | 285.84 | 515.43 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| DI-PSO1-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| DI-PSO2-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT1-PSO1 | 42.13 | 62.63 | 0.00 | > 1000 | 27.96 | 32.20 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT1-PSO2 | 29.68 | 49.48 | 0.02 | > 1000 | 31.32 | 39.42 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT1-PSO1-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT1-PSO2-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT2-PSO1 | 63.40 | 150.86 | 0.02 | > 1000 | 27.31 | 45.22 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT2-PSO2 | 45.50 | 127.02 | 0.00 | > 1000 | 21.57 | 33.17 | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT2-PSO1-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT2-PSO2-CF | | | 0.00 | > 1000 | | | 0.00 | > 1000 | | | 0.00 | > 1000 |
| AT1-BBPSO | 286.56 | 733.87 | 0.00 | > 1000 | 163.98 | 568.42 | 0.00 | > 1000 | 55.96 | 68.75 | 0.00 | > 1000 |
| AT1-BBPSOxp | 152.06 | 495.54 | 0.00 | > 1000 | 53.66 | 40.22 | 0.00 | > 1000 | 38.12 | 36.67 | 0.00 | > 1000 |
| AT1-BBPSO-CF | 121.97 | 358.24 | 0.00 | > 1000 | 199.26 | 573.65 | 0.00 | > 1000 | 36.63 | 52.87 | 0.00 | > 1000 |
| AT1-BBPSOxp-CF | 390.89 | 857.50 | 0.00 | > 1000 | 39.49 | 42.56 | 0.00 | > 1000 | 31.37 | 35.33 | 0.00 | > 1000 |
| AT2-BBPSO | 192.88 | 490.38 | 0.00 | > 1000 | 63.77 | 83.78 | 0.00 | > 1000 | 84.38 | 132.31 | 0.00 | > 1000 |
| AT2-BBPSOxp | 124.92 | 235.06 | 0.00 | > 1000 | 41.88 | 46.31 | 0.00 | > 1000 | 34.63 | 36.02 | 0.00 | > 1000 |
| AT2-BBPSO-CF | 325.19 | 767.79 | 0.00 | > 1000 | 115.76 | 398.59 | 0.00 | > 1000 | 71.89 | 121.28 | 0.00 | > 1000 |
| AT2-BBPSOxp-CF | 229.62 | 646.40 | 0.00 | > 1000 | 45.22 | 55.65 | 0.00 | > 1000 | 37.75 | 59.87 | 0.00 | > 1000 |

**Table S2.3.** Simulation results for OF3. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.

*Prepared using staauth.cls*

| OF4;  Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 3.78 | 1.53 | 0.00 | > 1000 | 0.90 | 1.12 | 0.47 | > 1000 | 411.24 | 913.72 | 0.00 | > 1000 |
| PSO2 | 8.82 | 4.25 | 0.00 | > 1000 | 2.62 | 1.62 | 0.07 | > 1000 | 814.28 | 1218.90 | 0.00 | > 1000 |
| PSO1-CF | 504.88 | 318.43 | 0.00 | > 1000 | 337.63 | 182.82 | 0.00 | > 1000 | 1951.90 | 1010.30 | 0.00 | > 1000 |
| PSO2-CF | 337.14 | 160.49 | 0.00 | > 1000 | 253.11 | 127.14 | 0.00 | > 1000 | 1973.70 | 971.52 | 0.00 | > 1000 |
| DI-PSO1 | 7.68 | 2.34 | 0.00 | > 1000 | 3.02 | 1.76 | 0.07 | > 1000 | 1499.60 | 1649.80 | 0.00 | > 1000 |
| DI-PSO2 | 11.15 | 4.86 | 0.00 | > 1000 | 4.71 | 2.28 | 0.00 | > 1000 | 1497.30 | 1511.10 | 0.00 | > 1000 |
| DI-PSO1-CF | 1796.40 | 910.83 | 0.00 | > 1000 | 780.47 | 623.55 | 0.00 | > 1000 | 2276.80 | 1045.00 | 0.00 | > 1000 |
| DI-PSO2-CF | 1652.70 | 826.63 | 0.00 | > 1000 | 717.85 | 335.94 | 0.00 | > 1000 | 2295.10 | 941.03 | 0.00 | > 1000 |
| AT1-PSO1 | 4.76 | 2.15 | 0.00 | > 1000 | 2.16 | 1.76 | 0.17 | > 1000 | 1584.60 | 1101.90 | 0.00 | > 1000 |
| AT1-PSO2 | 6.85 | 2.12 | 0.00 | > 1000 | 2.43 | 1.49 | 0.05 | > 1000 | 772.81 | 909.62 | 0.00 | > 1000 |
| AT1-PSO1-CF | 963.98 | 737.17 | 0.00 | > 1000 | 470.14 | 242.39 | 0.00 | > 1000 | 2032.40 | 937.66 | 0.00 | > 1000 |
| AT1-PSO2-CF | 721.45 | 403.80 | 0.00 | > 1000 | 358.45 | 172.38 | 0.00 | > 1000 | 1780.40 | 650.79 | 0.00 | > 1000 |
| AT2-PSO1 | 7.44 | 3.39 | 0.00 | > 1000 | 3.50 | 1.66 | 0.00 | > 1000 | 108.91 | 112.21 | 0.00 | > 1000 |
| AT2-PSO2 | 10.01 | 4.71 | 0.00 | > 1000 | 5.02 | 2.70 | 0.00 | > 1000 | 122.32 | 191.30 | 0.00 | > 1000 |
| AT2-PSO1-CF | 495.75 | 240.51 | 0.00 | > 1000 | 454.87 | 259.16 | 0.00 | > 1000 | 1355.10 | 663.30 | 0.00 | > 1000 |
| AT2-PSO2-CF | 425.66 | 293.16 | 0.00 | > 1000 | 363.88 | 192.89 | 0.00 | > 1000 | 1209.60 | 810.10 | 0.00 | > 1000 |
| AT1-BBPSO | 3.22 | 1.39 | 0.05 | > 1000 | 0.46 | 0.53 | 0.30 | > 1000 | 1.15 | 1.23 | 0.35 | > 1000 |
| AT1-BBPSOxp | 0.05 | 0.15 | 0.00 | > 1000 | 0.03 | 0.01 | 0.00 | > 1000 | 0.14 | 0.35 | 0.65 | 974.5 |
| AT1-BBPSO-CF | 3.59 | 1.63 | 0.00 | > 1000 | 0.24 | 0.52 | 0.80 | 898.5 | 1.17 | 1.15 | 0.35 | > 1000 |
| AT1-BBPSOxp-CF | 0.15 | 0.34 | 0.75 | 985 | 0.01 | 0.00 | 0.85 | 973.5 | 0.12 | 0.32 | 0.88 | 865 |
| AT2-BBPSO | 4.59 | 1.93 | 0.00 | > 1000 | 0.62 | 0.79 | 0.52 | 633.5 | 1.14 | 1.18 | 0.40 | > 1000 |
| AT2-BBPSOxp | 0.17 | 0.43 | 0.85 | 656 | 0.00 | 0.00 | 1.00 | 672 | 0.12 | 0.32 | 0.88 | 633.5 |
| AT2-BBPSO-CF | 3.54 | 1.85 | 0.00 | > 1000 | 0.59 | 0.74 | 0.52 | 583.5 | 1.57 | 1.30 | 0.22 | > 1000 |
| AT2-BBPSOxp-CF | 0.09 | 0.36 | 0.92 | 591.5 | 0.00 | 0.00 | 1.00 | 614 | 0.10 | 0.29 | 0.90 | 598.5 |

**Table S2.4.** Simulation results for OF4. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.

| OF5; Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 0.03 | 0.03 | 0.30 | > 1000 | 0.01 | 0.01 | 0.57 | 441 | 0.52 | 0.65 | 0.07 | > 1000 |
| PSO2 | 0.02 | 0.02 | 0.38 | > 1000 | 0.01 | 0.01 | 0.60 | 205 | 0.56 | 0.60 | 0.07 | > 1000 |
| PSO1-CF | 0.89 | 0.20 | 0.00 | > 1000 | 0.96 | 0.14 | 0.00 | > 1000 | 1.45 | 0.28 | 0.00 | > 1000 |
| PSO2-CF | 0.94 | 0.15 | 0.00 | > 1000 | 0.99 | 0.09 | 0.00 | > 1000 | 1.48 | 0.31 | 0.00 | > 1000 |
| DI-PSO1 | 0.02 | 0.02 | 0.28 | > 1000 | 0.01 | 0.01 | 0.65 | 258.5 | 1.27 | 0.45 | 0.00 | > 1000 |
| DI-PSO2 | 0.02 | 0.03 | 0.38 | > 1000 | 0.01 | 0.02 | 0.65 | 239 | 1.36 | 0.42 | 0.00 | > 1000 |
| DI-PSO1-CF | 1.42 | 0.26 | 0.00 | > 1000 | 1.16 | 0.08 | 0.00 | > 1000 | 1.60 | 0.30 | 0.00 | > 1000 |
| DI-PSO2-CF | 1.33 | 0.18 | 0.00 | > 1000 | 1.15 | 0.09 | 0.00 | > 1000 | 1.58 | 0.28 | 0.00 | > 1000 |
| AT1-PSO1 | 0.02 | 0.02 | 0.42 | > 1000 | 0.01 | 0.01 | 0.78 | 253 | 1.37 | 0.28 | 0.00 | > 1000 |
| AT1-PSO2 | 0.02 | 0.02 | 0.32 | > 1000 | 0.01 | 0.01 | 0.82 | 239.5 | 1.44 | 0.56 | 0.00 | > 1000 |
| AT1-PSO1-CF | 1.09 | 0.22 | 0.00 | > 1000 | 1.04 | 0.10 | 0.00 | > 1000 | 1.64 | 0.26 | 0.00 | > 1000 |
| AT1-PSO2-CF | 1.03 | 0.21 | 0.00 | > 1000 | 1.04 | 0.11 | 0.00 | > 1000 | 1.61 | 0.26 | 0.00 | > 1000 |
| AT2-PSO1 | 0.03 | 0.03 | 0.38 | > 1000 | 0.01 | 0.01 | 0.52 | 167 | 0.93 | 0.28 | 0.00 | > 1000 |
| AT2-PSO2 | 0.03 | 0.03 | 0.35 | > 1000 | 0.01 | 0.01 | 0.65 | 132.5 | 0.85 | 0.27 | 0.00 | > 1000 |
| AT2-PSO1-CF | 0.97 | 0.25 | 0.00 | > 1000 | 1.02 | 0.10 | 0.00 | > 1000 | 1.42 | 0.21 | 0.00 | > 1000 |
| AT2-PSO2-CF | 0.91 | 0.19 | 0.00 | > 1000 | 0.97 | 0.10 | 0.00 | > 1000 | 1.38 | 0.21 | 0.00 | > 1000 |
| AT1-BBPSO | 0.01 | 0.01 | 0.55 | 875.5 | 0.00 | 0.01 | 0.85 | 555.5 | 0.00 | 0.01 | 0.82 | 535.5 |
| AT1-BBPSOxp | 0.01 | 0.01 | 0.72 | 631 | 0.00 | 0.00 | 1.00 | 623.5 | 0.00 | 0.00 | 0.95 | 566.5 |
| AT1-BBPSO-CF | 0.02 | 0.02 | 0.35 | > 1000 | 0.00 | 0.01 | 0.85 | 475 | 0.01 | 0.01 | 0.68 | 472 |
| AT1-BBPSOxp-CF | 0.01 | 0.01 | 0.78 | 518.5 | 0.00 | 0.00 | 1.00 | 549 | 0.00 | 0.00 | 0.97 | 469.5 |
| AT2-BBPSO | 0.02 | 0.02 | 0.40 | > 1000 | 0.00 | 0.01 | 0.92 | 366.5 | 0.01 | 0.01 | 0.72 | 406.5 |
| AT2-BBPSOxp | 0.01 | 0.01 | 0.78 | 387 | 0.00 | 0.00 | 0.97 | 400.5 | 0.00 | 0.01 | 0.90 | 380.5 |
| AT2-BBPSO-CF | 0.01 | 0.02 | 0.52 | 478.5 | 0.00 | 0.01 | 0.82 | 308.5 | 0.00 | 0.01 | 0.78 | 307 |
| AT2-BBPSOxp-CF | 0.01 | 0.01 | 0.72 | 370.5 | 0.00 | 0.00 | 1.00 | 342.5 | 0.00 | 0.01 | 0.80 | 349 |

**Table S2.5.** Simulation results for OF5. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.

| OF6; Nbhd: | Global | | | | SS3 | | | | SS1 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Algorithm | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ | Mean | SD | $\widehat{P}$ | $\widehat{K}$ |
| PSO1 | 20.01 | 0.02 | 0.00 | > 1000 | 20.13 | 0.07 | 0.00 | > 1000 | 20.26 | 0.10 | 0.00 | > 1000 |
| PSO2 | 20.00 | 0.01 | 0.00 | > 1000 | 20.21 | 0.11 | 0.00 | > 1000 | 20.37 | 0.13 | 0.00 | > 1000 |
| PSO1-CF | 20.67 | 0.14 | 0.00 | > 1000 | 20.59 | 0.11 | 0.00 | > 1000 | 20.56 | 0.13 | 0.00 | > 1000 |
| PSO2-CF | 20.68 | 0.19 | 0.00 | > 1000 | 20.62 | 0.11 | 0.00 | > 1000 | 20.62 | 0.13 | 0.00 | > 1000 |
| DI-PSO1 | 20.02 | 0.03 | 0.00 | > 1000 | 20.06 | 0.04 | 0.00 | > 1000 | 20.11 | 0.05 | 0.00 | > 1000 |
| DI-PSO2 | 20.13 | 0.14 | 0.00 | > 1000 | 20.18 | 0.11 | 0.00 | > 1000 | 20.23 | 0.09 | 0.00 | > 1000 |
| DI-PSO1-CF | 20.43 | 0.11 | 0.00 | > 1000 | 20.33 | 0.10 | 0.00 | > 1000 | 20.29 | 0.11 | 0.00 | > 1000 |
| DI-PSO2-CF | 20.48 | 0.13 | 0.00 | > 1000 | 20.38 | 0.11 | 0.00 | > 1000 | 20.32 | 0.11 | 0.00 | > 1000 |
| AT1-PSO1 | 20.00 | 0.00 | 0.00 | > 1000 | 20.01 | 0.01 | 0.00 | > 1000 | 20.00 | 0.00 | 0.00 | > 1000 |
| AT1-PSO2 | 20.02 | 0.04 | 0.00 | > 1000 | 20.08 | 0.06 | 0.00 | > 1000 | 20.00 | 0.00 | 0.00 | > 1000 |
| AT1-PSO1-CF | 20.06 | 0.06 | 0.00 | > 1000 | 20.14 | 0.07 | 0.00 | > 1000 | 20.00 | 0.00 | 0.00 | > 1000 |
| AT1-PSO2-CF | 20.23 | 0.09 | 0.00 | > 1000 | 20.16 | 0.09 | 0.00 | > 1000 | 19.68 | 2.06 | 0.00 | > 1000 |
| AT2-PSO1 | 20.01 | 0.02 | 0.00 | > 1000 | 20.07 | 0.05 | 0.00 | > 1000 | 20.22 | 0.09 | 0.00 | > 1000 |
| AT2-PSO2 | 20.12 | 0.10 | 0.00 | > 1000 | 20.27 | 0.08 | 0.00 | > 1000 | 20.33 | 0.11 | 0.00 | > 1000 |
| AT2-PSO1-CF | 20.33 | 0.10 | 0.00 | > 1000 | 20.33 | 0.10 | 0.00 | > 1000 | 20.32 | 0.11 | 0.00 | > 1000 |
| AT2-PSO2-CF | 20.38 | 0.12 | 0.00 | > 1000 | 20.32 | 0.13 | 0.00 | > 1000 | 20.32 | 0.11 | 0.00 | > 1000 |
| AT1-BBPSO | 17.22 | 8.02 | 0.00 | > 1000 | 5.23 | 9.13 | 0.00 | > 1000 | 18.71 | 6.28 | 0.00 | > 1000 |
| AT1-BBPSOxp | 18.34 | 6.18 | 0.00 | > 1000 | 17.58 | 6.79 | 0.00 | > 1000 | 20.42 | 0.21 | 0.00 | > 1000 |
| AT1-BBPSO-CF | 15.63 | 9.14 | 0.07 | > 1000 | 0.53 | 3.30 | 0.42 | > 1000 | 13.68 | 9.62 | 0.22 | > 1000 |
| AT1-BBPSOxp-CF | 14.75 | 9.19 | 0.00 | > 1000 | 10.35 | 9.46 | 0.00 | > 1000 | 20.40 | 0.33 | 0.00 | > 1000 |
| AT2-BBPSO | 17.76 | 7.56 | 0.15 | > 1000 | 5.95 | 8.70 | 0.65 | 684.5 | 19.04 | 5.54 | 0.07 | > 1000 |
| AT2-BBPSOxp | 18.22 | 6.18 | 0.10 | > 1000 | 17.65 | 5.79 | 0.05 | > 1000 | 20.17 | 1.84 | 0.00 | > 1000 |
| AT2-BBPSO-CF | 18.33 | 6.76 | 0.07 | > 1000 | 2.06 | 6.25 | 0.90 | 628 | 14.47 | 9.01 | 0.22 | > 1000 |
| AT2-BBPSOxp-CF | 14.49 | 9.15 | 0.28 | > 1000 | 10.03 | 8.80 | 0.32 | > 1000 | 18.77 | 4.41 | 0.00 | > 1000 |

**Table S2.6.** Simulation results for OF6. Each algorithm was run for 1,000 iterations over 40 replications. Mean and SD denote the mean and standard deviation of minimum objective function values found over all replications, while $\widehat{P}$ is a denotes the proportion of all replicatons that came within 0.01 of the true global minimum (equal to zero), and $\widehat{K}$ denotes the median number of iterations until the algorithm came within 0.01 og the global minimum.