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Supplemental Web Material: Adaptively-Tuned Particle Swarm Optimization with Application to Spatial Design

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A. PSO and BBPSO details

In Section 2 we introduced PSO, BBPSO, and our adaptively tuned variants of both. Here we describe in detail several modifcations to both PSO and BBPSO.

A.1. PSO

The standard PSO algorithm is given by equation (1). We consider several additions and modifications to this algorithm below. Each of these modifications is combined with adaptively tuned inertia as in equation (4) to create the AT-PSO algorithms we employ.

A.1.1. Initialization: In order to initialize the swarm, the number of particles, their initial locations, and their initial velocities have to be chosen. Clerc (2011) suggests making the swarm size a function of the dimension of the search space, but notes that this is known to be suboptimal. We use their alternative suggest to use a default swarm size of 40. We assume the search space is a D-dimensional hypercube given by $\bigtimes_{j=1}^{D}[min_j, max_j]$. Then each particle's initial location is randomly generated uniformly on the cube, i.e. $\theta_{ij}(0) \overset{ind}{\sim} U(min_j, max_j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, D$. Each particle's velocity is initalized based on its location via $v_{ij}(0) \overset{ind}{\sim} U(min_j - x_{ij}(0), max_j - x_{ij}(0))$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, D$.

In Section 3.3 each design point is constrained to be in Harris County, TX, wich is not a rectangle. We intialize the swarm by placing intial design points on the smallest rectangle containing Harris County. Design points outside of Harris County will quickly move back into the county due to the confinement strategy.

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A.1.2. Confinement: Even though the initialization of the swarm is confined to a hypercube, nothing prevents any given particle from leaving the search space. There are a number of things that can be done in order to solve this problem and each has advantages and disadvantages depending on the situation (Helwig & Wanka, 2007). We focus on two approaches. One is to move any particle that leaves the search space to the nearest point still inside the search space and then adjust its velocity, e.g. Clerc (2011) suggests that whenever $x_{ij}(k) > max_j$, it should be set to $x_{ij}(k) = max_j$ and similarly when $x_{ij}(k) < min_j$ it should be set to $x_{ij}(k) = min_d$, and in both cases the velocity along that dimension should be reversed and halved, i.e. $v_{ij}(k) = -0.5v_{ij}(k)$. This causes particles to bounce off the boundary and move back towards the middle of the search space. This is the default strategy we use.

Another strategy is to simply define the objective function to be $\pm\infty$ outside of the search space ($+$ when minimizing, $-$ when maximizing) so that the swarm tries to stay in bounds naturally. In Section 3.3 we employ this method in combination with the other strategy. When a design point is proposed outside of the smallest rectangle containing Harris County, we move it back to the edge of the rectangle using the method described in the previous paragraph. All points inside the rectangle but outside Harris County are defined to have infinite MSPE variance in order to further restrict the swarm to the desired area.

A.1.3. Redraw Neighborhoods: In Section 2 we briefly described a variant of the stochastic star neighborhood which we use. This neighborhood is stochastic, meaning that each particle's neighbors are randomly drawn when the algorithm is initialized. After an iteration in which the best known value of the objective function is unchanged, each particle's neighbors are randomly redrawn according to the same distribution.

A.1.4. Asynchronous Updates: The way we defined PSO in equation (1) each particle can update simultaneously. This means that the algorithm is parallelizable, which is a major advantage for implementation on modern GPUs. However, asynchronously updating the particles typically results in faster converging algorithms when it is computationally feasible. In an asynchronous update from period k to $k+1$, particle i recognizes that particle $i-1$ has already updated its personal best location to $\boldsymbol{p}_{i-1}(k+1)$ by the time it is i's turn to update. So i computes its group best location taking this into account. This causes particle $i = 1$ to behave differently from particle $i = n$ since particle n always has better information in order to perform its update, so every iteration the particles are randomly reordered. More formally, before every iteration sample $o_1(k+1), o_2(k+1), \ldots, o_n(k+1)$ from $\{1, 2, \ldots, n\}$ without replacement. Then the particles update starting with o_1, o_2, etc., where the group best update becomes $\boldsymbol{g}_{o_i}(k+1) = \arg\min_{\{\boldsymbol{p}_{o_j}(k+1)|j\in\mathcal{N}_i\}} Q(\boldsymbol{p}_j(k_j^*))$ where $k_j^* = k+1$ if $j < i$ and k otherwise. We asynchronously update in all of our algorithms.

A.1.5. Coordinate Free Velocity Updates: The standard velocity update in equation (1) is well known to bias the algorithm towards locations near the coordinate axes and especially the origin (Monson & Seppi, 2005; Spears et al., 2010). In general we may not know if the true optimum is near an axis, so this behavior is undesirable. There are several alternative velocity updates available, e.g. in Monson & Seppi (2005). We use the coordinate free (CF) update suggested by Clerc (2011). First define the center of gravity for particle i to be $\boldsymbol{C}_i(k) = \boldsymbol{\theta}_i(k) + \phi_1\{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}/3 + \phi_2\{\boldsymbol{g}_i(k) - \boldsymbol{\theta}_i(k)\}/3$. Let $\mathcal{H}_i(k)$ denote the hypersphere centered at $\boldsymbol{C}_i(k)$ with radius $||\boldsymbol{C}_i(k) - \boldsymbol{\theta}_i(k)||$ where $||.||$ denotes Euclidean distance. Then a new point $\boldsymbol{\theta}_i'(k)$ is drawn randomly from $\mathcal{H}_i(k)$ by sampling a direction and a radius, each uniformly. This is *not* the same as drawing uniformly over $\mathcal{H}_i(k)$ and in fact favors points near the center. Then the CF velocity update is given by $\boldsymbol{v}_i(k+1) = \omega\boldsymbol{v}_i(k) + \boldsymbol{x}_i'(k)$. We use both the standard and CF velocity updates in our algorithms. The standard PSO algorithm with each feature in this subsection including the CF velocity update is what Clerc (2011) calls SPSO 2011.

A.1.6. When Personal Best = Group Best: When a particle's personal best and group best locations coincide, it is often advantageous to allow the particle to explore more than usual. In the standard velocity update we do this by removing the social term so that $\boldsymbol{v}_i(k+1) = \omega \boldsymbol{v}_i(k) + \phi_1 \boldsymbol{r}_{1i}(k) \circ \{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}$. In the CF velocity update we change the center of gravity to ignore the social term so that $\boldsymbol{C}_i(k) = \boldsymbol{\theta}_i(k) + \phi_2\{\boldsymbol{p}_i(k) - \boldsymbol{\theta}_i(k)\}/2$.

A.2. BBPSO

The standard BBPSO algorithm was introduced by Kennedy (2003) and updates from t to $t+1$ via equation (2). We use each of the features in Section A.1 in our BBPSO algorithms, though some of them need to be modified for the BBPSO setting. We list them below along with another modification of BBPSO which we employ. Each of these modications are combined with adaptively tuning a scale parameter as in equation (3) to create our AT-BBPSO algorithms.

A.2.1. BBPSOxp: A commonly used variant of BBPSO also introduced by Kennedy (2003) is called BBPSOxp. In this variant, each coordinate of each particle has a 50% chance of updating according to (2) and a 50% chance of moving directly to that particle's personal best location on that coordinate. In other words

$$\theta_{ij}(k+1) = \begin{cases} N\left(\frac{p_{ij}(k)+g_{ij}(k)}{2}, h_{ij}^2(k)\right) & \text{with probability } 0.5 \\ p_{ij}(k) & \text{otherwise,} \end{cases} \tag{1}$$

where $h_{ij}(k) = |p_{ij}(k) - g_{ij}(k)|$. We use both xp and non-xp versions of our BBPSO algorithms.

A.2.2. CF BBPSO: BBPSO's update also depends on the coordinate system since each coordinate of $\boldsymbol{\theta}$ gets a different standard deviation. We employ BBPSO algorithms using the default standard deviation, but also using a coordinate free standard deviation given by $h_{ij}(k) = ||\boldsymbol{p}_i(k) - \boldsymbol{g}_i(k)||$.

A.2.3. When Personal Best = Group Best in BBPSO: A downside of both BBPSO and BBPSOxp is that any particle whose personal best is currently its group best location does not move due to the definition of the standard deviation term. Several methods have been proposed to overcome this; e.g. Hsieh & Lee (2010) and Zhang et al. (2011). Zhang et al. (2011) propose using mutation and crossover operations for the group best particle. To do this, each group best particle randomly selects three other distinct particles from the entire swarm, i_1, i_2, and i_3, and updates according to

$$\theta_{ij}(k+1) = p_{i_1 j}(k) + 0.5\{p_{i_2 j}(k) - p_{i_3 j}(k)\}. \tag{2}$$

This combines easily with BBPSOxp to update each coordinate of each particle with $h_{ij}(k) = 0$ according to (2) and the rest according to (1).

B. Comparing PSO and BBPSO algorithms

[THIS SECTION IS OUT OF DATE AND WILL BE UPDATED ONCE THE NEW SIMULATIONS COME OFF]

In order to compare AT-BBPSO to other PSO variants, we employ a subset of test functions used in Hsieh & Lee (2010). Each function is listed in Table 1 along with the global maximum and argmax, and the initialization range for the simulations. Further description of many of these functions can be found in Clerc (2010). For each function, we

Equation	ArgMax	Maximum	Initialization
$Q_1(\boldsymbol{\theta}) = -\sum_{i=1}^{D} \theta_i^2$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_1(\boldsymbol{\theta}^*) = 0$	$(50, 100)^D$
$Q_2(\boldsymbol{\theta}) = -\sum_{i=1}^{D} \left(\sum_{j=1}^{i} \theta_j\right)^2$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_2(\boldsymbol{\theta}^*) = 0$	$(50, 100)^D$
$Q_3(\boldsymbol{\theta}) = -\sum_{i=1}^{D-1} \left[100\{\theta_{i+1} + 1 - (\theta_i + 1)^2\} + \theta_i^2\right]$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_3(\boldsymbol{\theta}^*) = 0$	$(15, 30)^D$
$Q_4(\boldsymbol{\theta}) = 9D - \sum_{i=1}^{D}\{\theta_i^2 - \cos(2\pi\theta_i) + 10\}$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_4(\boldsymbol{\theta}^*) = 0$	$(2.56, 5.12)^D$
$Q_5(\boldsymbol{\theta}) = -\frac{1}{4000}\vert\vert\boldsymbol{\theta}\vert\vert^2 + \prod_{i=1}^{D} \cos\left(\frac{\theta_i}{\sqrt{i}}\right) - 1$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_5(\boldsymbol{\theta}^*) = 0$	$(300, 600)^D$
$Q_6(\boldsymbol{\theta}) = 20\exp\left(-0.2\sqrt{\frac{1}{D}\vert\vert\boldsymbol{\theta}\vert\vert}\right)$			
$+ \exp\left\{\frac{1}{D}\sum_{i=1}^{D} \cos(2\pi\theta_i)\right\} - 20 - \exp(1)$	$\boldsymbol{\theta}^* = \mathbf{0}$	$Q_6(\boldsymbol{\theta}^*) = 0$	$(16, 32)^D$

Table 1. Test functions for evaluating PSO algorithms. The dimension of $\boldsymbol{\theta}$ is D and $||.||$ is the Euclidean norm: $||\boldsymbol{\theta}|| = \sqrt{\sum_{i=1}^{D} \theta_i^2}$.

set $D = 20$ so the domain of each function is $\Re^{20}$. For each function, the PSO algorithms are initialized in a range that does not contain the true maximum.

We use several PSO algorithms in the simulation study. The standard PSO algorithm uses the parameter values suggested by Blum & Li (2008) and Clerc & Kennedy (2002). The AT-BBPSO variants are implemented a wide variety of parameter values, but all have the scale parameter initialized at $\sigma(0) = 1$, and both set $c = 0.1$. The AT-PSO variants are initialized at $\omega(0) = 1$ and also set $c = 0.1$. In addition, each algorithm is implemented using each of three neighborhood structures — the global, ring-3, and ring-1 neighborhoods. Each algorithm was used to optimize each objective function for 500 iterations over 50 replications using 20 particles. Initializations were changed across replications but held constant across algorithms. The standard PSO, DI-PSO, and AT-PSO algorithms initialized their velocity terms using the same method as a function of the initial locations of the particles, which we will denote by $\boldsymbol{x}_i$ for particle i. Let $x_{max,j}$ be the maximum initial value of coordinate j of $\boldsymbol{x}_i$ for each particle i, and let $x_{min,j}$ be the corresponding minimum. Then let $d_{max} = max_j x_{max,j} - x_{min,j}$. Then we initialize the velocities with $v_{ij}(0) \overset{iid}{\sim} U(-d_{max}/2, d_{max}/2)$. Tables 2-7 contain the simulation results for objective functions 1-6 respectively (OF1, OF2, etc.). We use several measures to quantify how well each algorithm finds the global maximum. First, each table includes the mean and standard deviation of the absolute difference between the true global maximum and the algorithm's estimated global maximum across all 50 replications, denoted by Mean and SD. Second, each table includes a convergence criterion — the proportion of the replications that came within 0.01 of the true global maximum, denoted by $\widehat{p}$. Finally, $\widehat{k}$ denotes the median number of iterations until the algorithm reaches the convergence criterion. When $\widehat{p} < 0.5$ then $\widehat{k} = \infty$ since greater than 50% of the replications did not converge in the maximum number of iterations allowed. Then Mean, $\widehat{p}$, and $\widehat{k}$ can be thought of how close the algorithm gets to the global maximum on average, what proportion of the time it converges, and long it takes to converge respectively.

We highlight only some of the features of these tables. First and foremost, PSO, BBPSO-MC, and BBPSOxp-MC almost always do worse than their AT cousins. Adaptively tuning either the scale or inertia parameter leads to gains in all three of our measures, sometimes large. The comparison is starkest between BBPSO variants and AT-BBPSO variants, partially because BBPSO tends to be pretty bad but also because AT-BBPSO does very well. Second, for most non-AT algorithms the more restrictive neighborhoods appear to yield algorithms which do a better job of finding the global max. For the AT-PSO algorithms, it appears that the target improvement rate (R^*) and the neighborhood interact. When the rate is high a more restrictive neighborhood is preferable, while when the rate is low a less restrictive neighborhood is preferable. On the other hand, for the AT-BBPSO algorithms, the opposite appears to be true — when the target improvement rate is high, a less restrictive neighborhood is desirable and vice versa. In addition, the best AT-BBPSO algorithms often use the global neighborhood while for the other classes of algorithms their best versions typically use the ring-1 or ring-3 neighborhood.

For the DI-PSO algorithms often there is a parameter-neighborhood combination that does well, typically from setting $\alpha = 200$ (20% of the 500 iterations) and $\beta = 1$ and using either the ring-1 or ring-3 neighborhood. One of these combinations typically does the best of all the DI-PSO algorithms but they can sometimes still do much worse than the best alternatives (e.g., for OF2). In the AT-BBPSO algorithms, it is not always clear what the best parameter settings are, but good default values appear to be $df = 3$ or 5 and $R^* = 0.3$ or 0.5. When these values are good, they often lead to the best performing PSO algorithms we consider. However, it is not always clear whether to use AT-BBPSO-MC or AT-BBPSOxp-MC. For some objective functions, e.g. OF4, the xp version is consistently better than the non-xp alternative, but the opposite is true for others, e.g. OF2.. AT-PSO is also often very competitive with $R^* = 0.3$ or $R^* = 0.5$, though again sometimes different parameter settings also appear to work well.

The DI-PSO and AT-PSO algorithms are similar conceptually, but often yield very different results. DI-PSO deterministically reduces the inertia parameter over time in the same manner given a fixed set of parameter values (α and β), while AT-PSO dynamically adjusts the inertia parameter to hit a target improvement rate. Figure 1 plots the inertia over time for the DI-PSO algorithm with $\alpha = 200$ and $\beta = 1$, and observed inertia over time for one replication of the AT-PSO algorithm with target rate $R^* = 0.5$ and ring-1 neighborhood for OF1 and one replication for OF6. All three algorithms have an initial inertia of $\omega(0) = 1$. While DI-PSO smoothly decreases its inertia with a slowly decreasing rate, AT-PSO very quickly drops its inertia for OF1 to about 0.55 then bounces around around near that point. It also jumps up above 1 initially, imploring the particles to cast a wider net in search of higher value areas of the search space. This is pretty typical behavior for the inertia parameter of AT-PSO — it tends to bounce around a level which is approximately the average over time of the DI-PSO's inertia, though lower values of R^* will result in higher inertias. In this way, AT-PSO alternates periods of exploration (relatively high inertia) and periods of exploitation (relatively low inertia). The main exception to this pattern is when AT-PSO converges around a local maximum. In this case, inertia plummets to zero as the particles settle down. This is precisely what happens for OF6 in Figure 1, though in this case the maximum is not global — Table 7 indicates that ring-1 AT-PSO with $R^* = 0.5$ never converged to the global max. In optimization problems with multiple local optima, both AT-PSO and AT-BBPSO variants can exhibit this behavior and prematurely converge to a local optima, so they may not be advantageous for those problems.

Based on these simulations, our default recommendation is to use AT-BBPSO-MC or AT-BBPSOxp-MC with $R^* = 0.3$ or 0.5 and $df = 3$ or $df = 5$ along with the global neighborhood. These algorithms will not always be the best of the PSO algorithms, but they will often be very good. AT-PSO with $R^* = 0.3$ or $R^* = 0.5$ with a restrictive neighborhood topology such as ring-3 also tends to be a very good choice, though perhaps less consistent than the AT-BBPSO variants. Default PSO also performs rather well and is a good baseline algorithm to use for comparisons.

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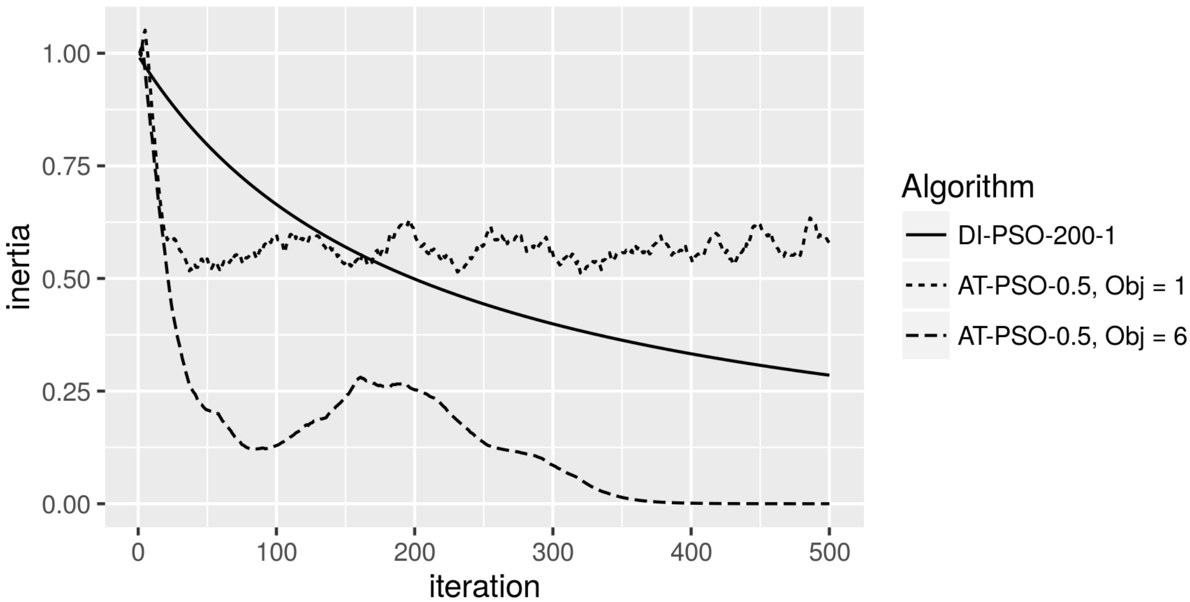

Figure 1. Inertia over time for the DI-PSO algorithm with $\alpha = 200$ and $\beta = 1$, and for one replication of the AT-PSO-0.5 algorithm for each of OFs 1 and 6.

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OF1	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	4015.81	3124.33	0.00	∞	0.00	0.01	0.98	362.00	0.01	0.01	0.78	488.50
BBPSO-MC	84517.26	8944.33	0.00	∞	83674.80	9956.73	0.00	∞	84784.51	9451.47	0.00	∞
BBPSOxp-MC	84104.44	8192.44	0.00	∞	82952.29	9655.82	0.00	∞	85886.72	7182.36	0.00	∞
AT-BBPSO-MC												
$df = 1$, $R^* = 0.1$	1.59	0.47	0.00	∞	2.08	0.73	0.00	∞	2.10	0.89	0.00	∞
$df = 1$, $R^* = 0.3$	0.00	0.00	1.00	445.00	0.01	0.00	0.96	467.00	0.01	0.01	0.54	499.50
$df = 1$, $R^* = 0.5$	0.00	0.00	1.00	289.00	0.00	0.00	1.00	309.50	0.00	0.00	1.00	348.00
$df = 1$, $R^* = 0.7$	0.00	0.00	1.00	223.00	0.00	0.00	1.00	256.00	0.00	0.00	1.00	329.00
$df = 3$, $R^* = 0.1$	6.92	2.41	0.00	∞	6.92	1.79	0.00	∞	5.10	2.02	0.00	∞
$df = 3$, $R^* = 0.3$	0.01	0.00	0.22	∞	0.02	0.01	0.06	∞	0.03	0.01	0.00	∞
$df = 3$, $R^* = 0.5$	0.00	0.00	1.00	331.50	0.00	0.00	1.00	347.50	0.00	0.00	1.00	387.50
$df = 3$, $R^* = 0.7$	0.00	0.00	1.00	259.50	0.00	0.00	1.00	288.50	0.00	0.00	1.00	400.50
$df = 5$, $R^* = 0.1$	11.50	2.93	0.00	∞	11.72	2.53	0.00	∞	8.14	2.55	0.00	∞
$df = 5$, $R^* = 0.3$	0.03	0.01	0.00	∞	0.04	0.01	0.00	∞	0.06	0.02	0.00	∞
$df = 5$, $R^* = 0.5$	0.00	0.00	1.00	356.50	0.00	0.00	1.00	368.50	0.00	0.00	1.00	415.50
$df = 5$, $R^* = 0.7$	0.00	0.00	1.00	288.50	0.00	0.00	1.00	326.50	222.69	288.53	0.02	∞
$df = \infty$, $R^* = 0.1$	42.86	10.31	0.00	∞	38.44	8.12	0.00	∞	20.28	8.59	0.00	∞
$df = \infty$, $R^* = 0.3$	0.14	0.05	0.00	∞	0.15	0.03	0.00	∞	0.25	0.08	0.00	∞
$df = \infty$, $R^* = 0.5$	0.00	0.00	1.00	415.00	0.00	0.00	1.00	433.00	0.01	0.00	0.94	489.00
$df = \infty$, $R^* = 0.7$	0.00	0.00	1.00	352.00	0.00	0.00	0.98	441.50	16949.57	3141.36	0.00	∞
AT-BBPSOxp-MC												
$df = 1$, $R^* = 0.1$	5.71	1.58	0.00	∞	6.00	1.76	0.00	∞	3.91	1.75	0.00	∞
$df = 1$, $R^* = 0.3$	0.02	0.01	0.00	∞	0.04	0.01	0.00	∞	0.05	0.02	0.00	∞
$df = 1$, $R^* = 0.5$	0.00	0.00	1.00	358.00	0.00	0.00	1.00	381.00	0.00	0.00	1.00	419.50
$df = 1$, $R^* = 0.7$	0.00	0.00	1.00	291.00	0.00	0.00	1.00	326.00	0.00	0.01	0.98	432.50
$df = 3$, $R^* = 0.1$	17.42	4.87	0.00	∞	16.91	4.41	0.00	∞	8.64	3.12	0.00	∞
$df = 3$, $R^* = 0.3$	0.10	0.03	0.00	∞	0.12	0.04	0.00	∞	0.15	0.05	0.00	∞
$df = 3$, $R^* = 0.5$	0.00	0.00	1.00	412.00	0.00	0.00	1.00	431.50	0.01	0.00	0.92	470.50
$df = 3$, $R^* = 0.7$	0.00	0.00	1.00	350.50	0.00	0.01	0.98	438.00	4193.16	2013.59	0.00	∞
$df = 5$, $R^* = 0.1$	28.39	6.37	0.00	∞	28.42	7.58	0.00	∞	13.22	5.27	0.00	∞
$df = 5$, $R^* = 0.3$	0.19	0.05	0.00	∞	0.22	0.07	0.00	∞	0.26	0.10	0.00	∞
$df = 5$, $R^* = 0.5$	0.00	0.00	1.00	440.00	0.00	0.00	1.00	466.00	0.03	0.02	0.04	∞
$df = 5$, $R^* = 0.7$	0.00	0.00	1.00	405.50	505.43	755.11	0.02	∞	17674.52	4505.53	0.00	∞
$df = \infty$, $R^* = 0.1$	65.18	14.92	0.00	∞	60.90	13.82	0.00	∞	26.77	10.11	0.00	∞
$df = \infty$, $R^* = 0.3$	0.50	0.11	0.00	∞	0.66	0.14	0.00	∞	0.82	0.25	0.00	∞
$df = \infty$, $R^* = 0.5$	0.01	0.00	0.82	492.00	0.03	0.01	0.02	∞	0.85	0.75	0.00	∞
$df = \infty$, $R^* = 0.7$	0.19	0.33	0.04	∞	17369.34	3631.71	0.00	∞	35172.80	4377.92	0.00	∞
DI-PSO												
$\alpha = 50$, $\beta = 1$	6703.90	3993.99	0.00	∞	543.02	683.48	0.00	∞	51.63	95.71	0.00	∞
$\alpha = 50$, $\beta = 2$	7823.89	4787.27	0.00	∞	1821.42	1508.50	0.00	∞	499.64	441.33	0.00	∞
$\alpha = 50$, $\beta = 4$	15373.54	6906.92	0.00	∞	5572.40	3137.25	0.00	∞	3705.93	2324.41	0.00	∞
$\alpha = 100$, $\beta = 1$	3928.51	3855.22	0.00	∞	236.79	733.83	0.00	∞	0.99	4.03	0.20	∞
$\alpha = 100$, $\beta = 2$	7618.89	4250.92	0.00	∞	1206.09	1393.42	0.00	∞	84.99	137.32	0.00	∞
$\alpha = 100$, $\beta = 4$	13044.34	6195.43	0.00	∞	4218.13	2881.34	0.00	∞	1745.64	1532.00	0.00	∞
$\alpha = 200$, $\beta = 1$	2321.68	2088.02	0.00	∞	178.09	618.06	0.02	∞	0.00	0.00	1.00	383.50
$\alpha = 200$, $\beta = 2$	6427.62	3741.12	0.00	∞	305.62	523.71	0.00	∞	5.45	12.55	0.00	∞
$\alpha = 200$, $\beta = 4$	12004.75	5708.12	0.00	∞	2236.94	1729.58	0.00	∞	475.93	503.87	0.00	∞
AT-PSO												
$R^* = 0.1$	111.02	237.73	0.00	∞	211.79	889.18	0.00	∞	12223.23	9590.30	0.00	∞
$R^* = 0.3$	54.27	252.08	0.04	∞	0.00	0.00	1.00	347.00	0.00	0.01	0.88	457.00
$R^* = 0.5$	329.77	586.04	0.00	∞	0.00	0.00	1.00	274.50	0.00	0.00	1.00	303.50
$R^* = 0.7$	2131.94	2209.74	0.00	∞	0.04	0.21	0.92	381.50	0.00	0.00	1.00	341.50

Table 2. Simulation results for OF1. See text for description.

OF2	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	10915.05	17201.47	0.00	∞	1246.38	1002.75	0.00	∞	5738.34	3835.37	0.00	∞
BBPSO-MC	9081909.83	1958288.73	0.00	∞	9194595.70	2157840.19	0.00	∞	8830368.63	2234188.42	0.00	∞
BBPSOxp-MC	9138885.89	2265920.11	0.00	∞	8889637.56	2035964.50	0.00	∞	9038770.99	1858659.66	0.00	∞
AT-BBPSO-MC												
$df=1,\ R^*=0.1$	950.30	809.12	0.00	∞	690.95	535.97	0.00	∞	743.12	728.28	0.00	∞
$df=1,\ R^*=0.3$	300.03	258.36	0.00	∞	685.68	458.39	0.00	∞	1091.45	1375.54	0.00	∞
$df=1,\ R^*=0.5$	440.38	436.08	0.00	∞	1419.88	1239.43	0.00	∞	3652.45	4072.28	0.00	∞
$df=1,\ R^*=0.7$	1615.81	1539.22	0.00	∞	4330.31	4863.92	0.00	∞	8858.46	4578.27	0.00	∞
$df=3,\ R^*=0.1$	196.07	88.33	0.00	∞	226.17	163.16	0.00	∞	252.63	187.08	0.00	∞
$df=3,\ R^*=0.3$	23.27	17.11	0.00	∞	54.95	48.43	0.00	∞	189.83	399.03	0.00	∞
$df=3,\ R^*=0.5$	30.94	29.21	0.00	∞	185.87	161.77	0.00	∞	1073.22	1233.74	0.00	∞
$df=3,\ R^*=0.7$	222.12	524.53	0.00	∞	939.33	789.54	0.00	∞	5897.58	4513.86	0.00	∞
$df=5,\ R^*=0.1$	198.27	75.35	0.00	∞	175.31	80.21	0.00	∞	205.86	117.06	0.00	∞
$df=5,\ R^*=0.3$	12.25	9.35	0.00	∞	24.76	18.02	0.00	∞	60.79	63.04	0.00	∞
$df=5,\ R^*=0.5$	11.51	7.41	0.00	∞	63.91	51.53	0.00	∞	458.47	317.21	0.00	∞
$df=5,\ R^*=0.7$	61.67	36.27	0.00	∞	378.27	218.82	0.00	∞	7696.71	5485.99	0.00	∞
$df=\infty,\ R^*=0.1$	280.79	97.77	0.00	∞	267.44	87.93	0.00	∞	287.37	128.28	0.00	∞
$df=\infty,\ R^*=0.3$	6.75	3.83	0.00	∞	13.91	9.02	0.00	∞	39.37	32.01	0.00	∞
$df=\infty,\ R^*=0.5$	4.96	3.73	0.00	∞	30.82	22.83	0.00	∞	269.58	161.10	0.00	∞
$df=\infty,\ R^*=0.7$	51.31	38.92	0.00	∞	550.48	301.85	0.00	∞	1100097.72	493290.33	0.00	∞
AT-BBPSOxp-MC												
$df=1,\ R^*=0.1$	1620.04	724.59	0.00	∞	1434.81	637.94	0.00	∞	1004.77	608.51	0.00	∞
$df=1,\ R^*=0.3$	1117.65	614.18	0.00	∞	1541.55	913.10	0.00	∞	1372.47	1108.92	0.00	∞
$df=1,\ R^*=0.5$	2424.76	1458.65	0.00	∞	3639.53	2048.74	0.00	∞	5212.21	3221.11	0.00	∞
$df=1,\ R^*=0.7$	5685.79	3365.32	0.00	∞	6746.82	4254.76	0.00	∞	13170.58	7291.81	0.00	∞
$df=3,\ R^*=0.1$	609.04	289.92	0.00	∞	568.65	326.15	0.00	∞	562.17	301.77	0.00	∞
$df=3,\ R^*=0.3$	231.40	167.47	0.00	∞	320.46	308.01	0.00	∞	366.68	274.58	0.00	∞
$df=3,\ R^*=0.5$	360.34	224.38	0.00	∞	1026.42	709.48	0.00	∞	5089.66	5857.62	0.00	∞
$df=3,\ R^*=0.7$	1426.88	1209.92	0.00	∞	5568.83	7882.29	0.00	∞	38081.89	31747.85	0.00	∞
$df=5,\ R^*=0.1$	476.81	215.75	0.00	∞	497.74	194.10	0.00	∞	540.99	263.49	0.00	∞
$df=5,\ R^*=0.3$	109.01	45.98	0.00	∞	170.67	120.10	0.00	∞	272.26	192.17	0.00	∞
$df=5,\ R^*=0.5$	243.73	154.52	0.00	∞	448.77	241.51	0.00	∞	2147.17	1308.53	0.00	∞
$df=5,\ R^*=0.7$	794.68	388.65	0.00	∞	17971.50	16875.10	0.00	∞	507347.01	409181.08	0.00	∞
$df=\infty,\ R^*=0.1$	609.19	197.59	0.00	∞	585.88	163.05	0.00	∞	536.88	263.19	0.00	∞
$df=\infty,\ R^*=0.3$	70.82	44.69	0.00	∞	111.39	79.50	0.00	∞	220.30	168.06	0.00	∞
$df=\infty,\ R^*=0.5$	114.86	88.86	0.00	∞	329.69	169.04	0.00	∞	2889.16	2088.86	0.00	∞
$df=\infty,\ R^*=0.7$	2469.49	2799.80	0.00	∞	1483405.94	407996.51	0.00	∞	2535125.64	778910.46	0.00	∞
DI-PSO												
$\alpha=50,\ \beta=1$	40714.56	24208.86	0.00	∞	16548.67	9312.34	0.00	∞	7839.43	5031.49	0.00	∞
$\alpha=50,\ \beta=2$	47763.21	31361.92	0.00	∞	24627.59	11610.10	0.00	∞	11660.25	6362.56	0.00	∞
$\alpha=50,\ \beta=4$	62225.74	28459.35	0.00	∞	31511.13	15145.07	0.00	∞	16416.52	7831.71	0.00	∞
$\alpha=100,\ \beta=1$	26613.11	14611.63	0.00	∞	10168.37	6029.10	0.00	∞	5378.87	3302.21	0.00	∞
$\alpha=100,\ \beta=2$	46177.71	37974.29	0.00	∞	17795.63	8897.56	0.00	∞	9818.51	4708.72	0.00	∞
$\alpha=100,\ \beta=4$	64833.73	36534.52	0.00	∞	28627.71	14257.04	0.00	∞	14536.17	7921.94	0.00	∞
$\alpha=200,\ \beta=1$	21611.55	12914.90	0.00	∞	5699.97	3916.44	0.00	∞	3499.49	1865.32	0.00	∞
$\alpha=200,\ \beta=2$	36405.44	27997.69	0.00	∞	12492.90	6829.93	0.00	∞	6944.87	4559.84	0.00	∞
$\alpha=200,\ \beta=4$	57723.16	30860.32	0.00	∞	21807.81	11567.18	0.00	∞	12201.41	5365.69	0.00	∞
AT-PSO												
$R^*=0.1$	8823.66	9727.43	0.00	∞	8852.96	6912.25	0.00	∞	58369.56	38288.17	0.00	∞
$R^*=0.3$	6658.84	6659.87	0.00	∞	769.74	850.01	0.00	∞	5183.03	4046.47	0.00	∞
$R^*=0.5$	20994.61	16398.66	0.00	∞	1713.89	2589.20	0.00	∞	1877.08	1465.51	0.00	∞
$R^*=0.7$	35128.87	17886.90	0.00	∞	7126.76	5756.50	0.00	∞	5771.37	3581.95	0.00	∞

Table 3. Simulation results for OF2. See text for description.

OF3	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	1242764.67	2075592.00	0.00	∞	121.33	104.18	0.00	∞	241.45	255.69	0.00	∞
BBPSO-MC	232830400.95	37473167.09	0.00	∞	247662755.14	43820300.43	0.00	∞	247321508.63	39096259.33	0.00	∞
BBPSOxp-MC	235491326.54	42124202.61	0.00	∞	242679907.63	42872974.55	0.00	∞	253599464.23	43615037.04	0.00	∞
AT-BBPSO-MC												
$df = 1,\ R^* = 0.1$	382.54	325.35	0.00	∞	486.70	533.46	0.00	∞	669.09	932.66	0.00	∞
$df = 1,\ R^* = 0.3$	233.17	429.80	0.00	∞	182.89	272.06	0.00	∞	352.64	465.63	0.00	∞
$df = 1,\ R^* = 0.5$	315.43	497.55	0.00	∞	159.55	216.03	0.00	∞	254.78	669.72	0.00	∞
$df = 1,\ R^* = 0.7$	235.08	455.38	0.00	∞	96.32	106.25	0.00	∞	189.20	288.45	0.00	∞
$df = 3,\ R^* = 0.1$	545.05	481.33	0.00	∞	629.79	657.37	0.00	∞	567.75	585.07	0.00	∞
$df = 3,\ R^* = 0.3$	176.81	186.22	0.00	∞	172.84	204.18	0.00	∞	256.32	426.46	0.00	∞
$df = 3,\ R^* = 0.5$	159.09	170.77	0.00	∞	128.99	127.25	0.00	∞	137.56	202.68	0.00	∞
$df = 3,\ R^* = 0.7$	155.74	399.33	0.00	∞	132.16	339.64	0.00	∞	82.12	170.21	0.00	∞
$df = 5,\ R^* = 0.1$	833.71	627.25	0.00	∞	928.42	982.63	0.00	∞	450.93	444.86	0.00	∞
$df = 5,\ R^* = 0.3$	283.78	431.37	0.00	∞	135.95	98.01	0.00	∞	229.39	314.25	0.00	∞
$df = 5,\ R^* = 0.5$	198.53	185.97	0.00	∞	173.66	278.73	0.00	∞	118.79	114.49	0.00	∞
$df = 5,\ R^* = 0.7$	142.06	190.31	0.00	∞	64.57	98.26	0.00	∞	108.71	538.33	0.00	∞
$df = \infty,\ R^* = 0.1$	1343.31	757.87	0.00	∞	1056.96	522.89	0.00	∞	958.18	773.33	0.00	∞
$df = \infty,\ R^* = 0.3$	230.32	320.08	0.00	∞	138.02	120.68	0.00	∞	212.78	331.90	0.00	∞
$df = \infty,\ R^* = 0.5$	225.28	219.75	0.00	∞	156.78	132.24	0.00	∞	120.23	106.23	0.00	∞
$df = \infty,\ R^* = 0.7$	131.34	91.53	0.00	∞	56.10	76.80	0.00	∞	179.69	683.04	0.00	∞
AT-BBPSOxp-MC												
$df = 1,\ R^* = 0.1$	616.80	278.83	0.00	∞	633.00	276.25	0.00	∞	490.75	350.79	0.00	∞
$df = 1,\ R^* = 0.3$	152.80	179.75	0.00	∞	163.64	131.42	0.00	∞	140.13	158.06	0.00	∞
$df = 1,\ R^* = 0.5$	139.49	269.65	0.00	∞	99.39	225.41	0.00	∞	202.26	429.55	0.00	∞
$df = 1,\ R^* = 0.7$	149.01	459.29	0.00	∞	73.54	126.07	0.00	∞	188.28	370.67	0.00	∞
$df = 3,\ R^* = 0.1$	961.94	370.97	0.00	∞	931.38	294.49	0.00	∞	709.41	452.45	0.00	∞
$df = 3,\ R^* = 0.3$	168.44	279.24	0.00	∞	144.15	200.78	0.00	∞	120.33	48.68	0.00	∞
$df = 3,\ R^* = 0.5$	123.39	102.98	0.00	∞	97.56	50.88	0.00	∞	77.47	81.83	0.00	∞
$df = 3,\ R^* = 0.7$	49.75	70.72	0.00	∞	33.96	52.12	0.00	∞	3586.02	9973.24	0.00	∞
$df = 5,\ R^* = 0.1$	1229.87	526.97	0.00	∞	1171.93	406.81	0.00	∞	727.95	436.07	0.00	∞
$df = 5,\ R^* = 0.3$	267.86	435.67	0.00	∞	124.80	75.86	0.00	∞	157.72	217.77	0.00	∞
$df = 5,\ R^* = 0.5$	137.08	104.13	0.00	∞	91.44	86.38	0.00	∞	75.74	95.04	0.00	∞
$df = 5,\ R^* = 0.7$	66.87	126.23	0.00	∞	19.28	17.08	0.00	∞	82498.29	102684.12	0.00	∞
$df = \infty,\ R^* = 0.1$	2002.70	822.46	0.00	∞	1718.67	698.34	0.00	∞	1158.47	903.16	0.00	∞
$df = \infty,\ R^* = 0.3$	137.09	84.51	0.00	∞	116.46	66.60	0.00	∞	114.47	57.68	0.00	∞
$df = \infty,\ R^* = 0.5$	121.71	103.55	0.00	∞	104.78	67.31	0.00	∞	41.15	33.00	0.00	∞
$df = \infty,\ R^* = 0.7$	24.31	24.58	0.00	∞	24.31	20.57	0.00	∞	2088154.00	1258187.91	0.00	∞
DI-PSO												
$\alpha = 50,\ \beta = 1$	4180196.82	4677639.48	0.00	∞	354071.40	579847.34	0.00	∞	70128.62	165037.08	0.00	∞
$\alpha = 50,\ \beta = 2$	6964209.15	8322398.93	0.00	∞	2341768.78	4277548.98	0.00	∞	1232022.80	1464906.82	0.00	∞
$\alpha = 50,\ \beta = 4$	20143670.82	15026478.57	0.00	∞	11740754.91	8374232.25	0.00	∞	6766696.51	4472751.26	0.00	∞
$\alpha = 100,\ \beta = 1$	1610827.18	3342137.70	0.00	∞	55104.82	171942.01	0.00	∞	1691.16	3160.57	0.00	∞
$\alpha = 100,\ \beta = 2$	6945649.64	6844592.19	0.00	∞	1001725.17	1774982.73	0.00	∞	150478.59	213009.16	0.00	∞
$\alpha = 100,\ \beta = 4$	21536033.79	15337048.80	0.00	∞	6691579.83	5932202.88	0.00	∞	4328277.03	4060546.38	0.00	∞
$\alpha = 200,\ \beta = 1$	1543367.07	2077471.63	0.00	∞	4350.55	16257.10	0.00	∞	266.44	453.75	0.00	∞
$\alpha = 200,\ \beta = 2$	6626917.67	9093854.31	0.00	∞	299834.83	1040979.02	0.00	∞	5848.88	14329.18	0.00	∞
$\alpha = 200,\ \beta = 4$	17830796.63	14245007.26	0.00	∞	3367710.63	3194612.95	0.00	∞	776424.69	865443.75	0.00	∞
AT-PSO												
$R^* = 0.1$	5779.95	14928.86	0.00	∞	297272.41	1261108.56	0.00	∞	33785179.04	23020834.78	0.00	∞
$R^* = 0.3$	869.69	2100.98	0.00	∞	131.15	175.33	0.00	∞	184.15	186.17	0.00	∞
$R^* = 0.5$	71363.37	218211.42	0.00	∞	190.13	271.20	0.00	∞	134.32	200.92	0.00	∞
$R^* = 0.7$	1572669.78	3884482.27	0.00	∞	391.10	857.02	0.00	∞	535.09	1277.83	0.00	∞

Table 4. Simulation results for OF3. See text for description.

OF4	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	29.94	15.21	0.00	∞	5.37	3.47	0.02	∞	3.24	1.44	0.00	∞
BBPSO-MC	237.31	18.67	0.00	∞	232.86	24.40	0.00	∞	232.69	21.86	0.00	∞
BBPSOxp-MC	229.80	22.24	0.00	∞	236.60	25.20	0.00	∞	230.84	21.05	0.00	∞
AT-BBPSO-MC												
$df = 1$, $R^* = 0.1$	4.96	1.58	0.00	∞	3.93	1.29	0.00	∞	4.07	1.88	0.00	∞
$df = 1$, $R^* = 0.3$	5.14	1.89	0.00	∞	2.76	1.45	0.06	∞	2.48	1.57	0.06	∞
$df = 1$, $R^* = 0.5$	6.32	2.16	0.00	∞	4.68	1.94	0.00	∞	4.24	2.36	0.02	∞
$df = 1$, $R^* = 0.7$	8.56	2.84	0.00	∞	5.78	1.67	0.00	∞	6.03	2.01	0.00	∞
$df = 3$, $R^* = 0.1$	5.44	1.30	0.00	∞	5.40	1.17	0.00	∞	5.39	1.54	0.00	∞
$df = 3$, $R^* = 0.3$	3.56	1.76	0.00	∞	1.59	1.24	0.20	∞	1.43	1.00	0.18	∞
$df = 3$, $R^* = 0.5$	4.07	1.85	0.00	∞	2.70	1.57	0.08	∞	2.57	1.44	0.02	∞
$df = 3$, $R^* = 0.7$	4.70	1.73	0.00	∞	3.83	1.72	0.00	∞	3.43	1.75	0.06	∞
$df = 5$, $R^* = 0.1$	6.63	1.34	0.00	∞	6.45	1.92	0.00	∞	6.72	1.86	0.00	∞
$df = 5$, $R^* = 0.3$	2.80	1.51	0.02	∞	1.24	1.12	0.24	∞	1.13	1.21	0.30	∞
$df = 5$, $R^* = 0.5$	4.11	2.16	0.02	∞	2.26	1.60	0.12	∞	1.90	1.43	0.16	∞
$df = 5$, $R^* = 0.7$	4.40	2.00	0.00	∞	2.91	1.48	0.00	∞	3.56	1.84	0.00	∞
$df = \infty$, $R^* = 0.1$	9.12	1.88	0.00	∞	9.17	1.58	0.00	∞	9.42	1.70	0.00	∞
$df = \infty$, $R^* = 0.3$	2.15	1.31	0.00	∞	0.72	0.92	0.00	∞	0.61	0.74	0.04	∞
$df = \infty$, $R^* = 0.5$	2.74	1.55	0.04	∞	1.62	1.38	0.18	∞	1.01	0.91	0.26	∞
$df = \infty$, $R^* = 0.7$	3.22	1.78	0.06	∞	2.11	1.56	0.18	∞	2.13	1.30	0.10	∞
AT-BBPSOxp-MC												
$df = 1$, $R^* = 0.1$	4.25	1.24	0.00	∞	4.70	1.25	0.00	∞	6.34	1.56	0.00	∞
$df = 1$, $R^* = 0.3$	0.74	0.87	0.04	∞	0.23	0.44	0.02	∞	0.51	0.74	0.00	∞
$df = 1$, $R^* = 0.5$	1.75	1.24	0.12	∞	0.95	1.00	0.38	∞	1.41	1.16	0.18	∞
$df = 1$, $R^* = 0.7$	3.07	1.65	0.02	∞	2.14	1.20	0.06	∞	3.10	1.62	0.02	∞
$df = 3$, $R^* = 0.1$	5.34	1.01	0.00	∞	6.29	1.31	0.00	∞	7.39	1.59	0.00	∞
$df = 3$, $R^* = 0.3$	0.25	0.45	0.00	∞	0.08	0.23	0.00	∞	0.19	0.41	0.02	∞
$df = 3$, $R^* = 0.5$	0.65	0.78	0.52	345.50	0.30	0.52	0.72	342.00	0.56	0.66	0.46	∞
$df = 3$, $R^* = 0.7$	1.43	1.21	0.22	∞	1.24	1.02	0.24	∞	1.85	1.12	0.08	∞
$df = 5$, $R^* = 0.1$	6.02	1.08	0.00	∞	6.50	1.08	0.00	∞	7.94	1.31	0.00	∞
$df = 5$, $R^* = 0.3$	0.10	0.26	0.00	∞	0.08	0.23	0.00	∞	0.10	0.33	0.02	∞
$df = 5$, $R^* = 0.5$	0.40	0.61	0.66	338.00	0.25	0.42	0.74	338.50	0.46	0.70	0.64	367.00
$df = 5$, $R^* = 0.7$	1.13	1.01	0.28	∞	0.75	0.85	0.44	∞	1.74	1.10	0.04	∞
$df = \infty$, $R^* = 0.1$	7.81	1.19	0.00	∞	8.10	1.36	0.00	∞	9.02	1.36	0.00	∞
$df = \infty$, $R^* = 0.3$	0.09	0.23	0.00	∞	0.03	0.01	0.00	∞	0.05	0.13	0.00	∞
$df = \infty$, $R^* = 0.5$	0.25	0.57	0.80	346.00	0.13	0.33	0.86	345.00	0.17	0.42	0.84	352.50
$df = \infty$, $R^* = 0.7$	0.90	0.89	0.36	∞	0.77	0.92	0.46	∞	2.66	1.39	0.00	∞
DI-PSO												
$\alpha = 50$, $\beta = 1$	32.08	11.51	0.00	∞	13.04	5.65	0.00	∞	6.33	3.37	0.00	∞
$\alpha = 50$, $\beta = 2$	38.18	17.28	0.00	∞	15.90	5.67	0.00	∞	10.13	4.19	0.00	∞
$\alpha = 50$, $\beta = 4$	56.49	22.89	0.00	∞	28.28	11.13	0.00	∞	15.83	7.08	0.00	∞
$\alpha = 100$, $\beta = 1$	28.51	12.71	0.00	∞	11.32	5.72	0.00	∞	4.66	2.48	0.00	∞
$\alpha = 100$, $\beta = 2$	32.85	13.96	0.00	∞	14.35	7.57	0.00	∞	6.86	2.78	0.00	∞
$\alpha = 100$, $\beta = 4$	49.67	16.91	0.00	∞	24.47	9.82	0.00	∞	13.74	6.18	0.00	∞
$\alpha = 200$, $\beta = 1$	22.39	12.14	0.00	∞	7.53	4.31	0.00	∞	3.92	2.11	0.02	∞
$\alpha = 200$, $\beta = 2$	36.56	14.29	0.00	∞	11.21	5.42	0.00	∞	5.28	2.77	0.00	∞
$\alpha = 200$, $\beta = 4$	55.67	19.91	0.00	∞	19.67	8.61	0.00	∞	11.18	5.41	0.00	∞
AT-PSO												
$R^* = 0.1$	7.73	5.50	0.00	∞	5.72	1.90	0.00	∞	23.02	10.87	0.00	∞
$R^* = 0.3$	14.10	6.74	0.00	∞	6.67	3.74	0.00	∞	3.77	2.04	0.02	∞
$R^* = 0.5$	23.89	8.43	0.00	∞	11.39	5.91	0.00	∞	6.06	2.92	0.00	∞
$R^* = 0.7$	34.91	12.20	0.00	∞	14.91	5.84	0.00	∞	9.80	3.78	0.00	∞

Table 5. Simulation results for OF4. See text for description.

OF5	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	27.55	20.12	0.00	∞	0.03	0.03	0.28	∞	0.08	0.07	0.04	∞
BBPSO-MC	762.36	82.36	0.00	∞	785.33	69.84	0.00	∞	751.88	80.37	0.00	∞
BBPSOxp-MC	766.30	80.16	0.00	∞	785.43	64.92	0.00	∞	771.17	76.11	0.00	∞
AT-BBPSO-MC												
$df = 1,\ \ R^* = 0.1$	0.83	0.10	0.00	∞	0.88	0.09	0.00	∞	0.92	0.06	0.00	∞
$df = 1,\ \ R^* = 0.3$	0.02	0.01	0.24	∞	0.02	0.01	0.34	∞	0.05	0.03	0.02	∞
$df = 1,\ \ R^* = 0.5$	0.01	0.01	0.52	455.50	0.01	0.01	0.56	455.00	0.01	0.01	0.48	∞
$df = 1,\ \ R^* = 0.7$	0.02	0.01	0.36	∞	0.02	0.02	0.46	∞	0.04	0.04	0.16	∞
$df = 3,\ \ R^* = 0.1$	1.06	0.02	0.00	∞	1.05	0.02	0.00	∞	1.04	0.02	0.00	∞
$df = 3,\ \ R^* = 0.3$	0.08	0.03	0.00	∞	0.12	0.04	0.00	∞	0.32	0.15	0.00	∞
$df = 3,\ \ R^* = 0.5$	0.01	0.01	0.54	459.00	0.01	0.01	0.62	481.00	0.09	0.07	0.02	∞
$df = 3,\ \ R^* = 0.7$	0.01	0.01	0.46	∞	0.12	0.15	0.04	∞	205.76	57.74	0.00	∞
$df = 5,\ \ R^* = 0.1$	1.11	0.03	0.00	∞	1.11	0.03	0.00	∞	1.09	0.03	0.00	∞
$df = 5,\ \ R^* = 0.3$	0.18	0.05	0.00	∞	0.29	0.09	0.00	∞	0.66	0.15	0.00	∞
$df = 5,\ \ R^* = 0.5$	0.02	0.01	0.32	∞	0.02	0.01	0.26	∞	0.69	0.21	0.00	∞
$df = 5,\ \ R^* = 0.7$	0.01	0.01	0.32	∞	39.11	26.34	0.00	∞	480.85	48.89	0.00	∞
$df = \infty,\ R^* = 0.1$	1.47	0.11	0.00	∞	1.45	0.09	0.00	∞	1.25	0.08	0.00	∞
$df = \infty,\ R^* = 0.3$	0.69	0.11	0.00	∞	0.85	0.07	0.00	∞	0.97	0.05	0.00	∞
$df = \infty,\ R^* = 0.5$	0.04	0.02	0.00	∞	0.26	0.09	0.00	∞	9.61	10.88	0.00	∞
$df = \infty,\ R^* = 0.7$	0.82	0.19	0.00	∞	395.75	42.10	0.00	∞	638.33	59.56	0.00	∞
AT-BBPSOxp-MC												
$df = 1,\ \ R^* = 0.1$	1.01	0.02	0.00	∞	1.01	0.02	0.00	∞	1.00	0.03	0.00	∞
$df = 1,\ \ R^* = 0.3$	0.07	0.04	0.00	∞	0.13	0.06	0.00	∞	0.25	0.13	0.00	∞
$df = 1,\ \ R^* = 0.5$	0.01	0.01	0.54	465.50	0.01	0.01	0.44	∞	0.06	0.08	0.06	∞
$df = 1,\ \ R^* = 0.7$	0.02	0.02	0.46	∞	0.02	0.03	0.46	∞	1.20	0.54	0.00	∞
$df = 3,\ \ R^* = 0.1$	1.17	0.04	0.00	∞	1.16	0.04	0.00	∞	1.09	0.03	0.00	∞
$df = 3,\ \ R^* = 0.3$	0.56	0.12	0.00	∞	0.73	0.08	0.00	∞	0.87	0.09	0.00	∞
$df = 3,\ \ R^* = 0.5$	0.04	0.02	0.00	∞	0.33	0.14	0.00	∞	5.84	7.01	0.00	∞
$df = 3,\ \ R^* = 0.7$	7.20	9.68	0.00	∞	295.56	44.31	0.00	∞	439.15	53.40	0.00	∞
$df = 5,\ \ R^* = 0.1$	1.26	0.06	0.00	∞	1.30	0.06	0.00	∞	1.16	0.06	0.00	∞
$df = 5,\ \ R^* = 0.3$	0.80	0.08	0.00	∞	0.92	0.05	0.00	∞	1.00	0.02	0.00	∞
$df = 5,\ \ R^* = 0.5$	0.22	0.10	0.00	∞	0.93	0.09	0.00	∞	72.53	27.81	0.00	∞
$df = 5,\ \ R^* = 0.7$	172.86	53.14	0.00	∞	491.44	39.13	0.00	∞	527.93	56.57	0.00	∞
$df = \infty,\ R^* = 0.1$	1.71	0.13	0.00	∞	1.70	0.17	0.00	∞	1.34	0.14	0.00	∞
$df = \infty,\ R^* = 0.3$	0.99	0.04	0.00	∞	1.03	0.01	0.00	∞	1.06	0.03	0.00	∞
$df = \infty,\ R^* = 0.5$	0.82	0.12	0.00	∞	2.65	1.75	0.00	∞	255.87	50.83	0.00	∞
$df = \infty,\ R^* = 0.7$	451.87	38.28	0.00	∞	622.22	35.60	0.00	∞	586.09	47.80	0.00	∞
DI-PSO												
$\alpha = 50,\ \ \beta = 1$	55.37	25.87	0.00	∞	7.99	9.84	0.00	∞	1.07	0.35	0.00	∞
$\alpha = 50,\ \ \beta = 2$	73.04	43.14	0.00	∞	21.12	16.40	0.00	∞	6.20	4.72	0.00	∞
$\alpha = 50,\ \ \beta = 4$	142.00	66.28	0.00	∞	61.48	36.04	0.00	∞	27.71	18.29	0.00	∞
$\alpha = 100,\ \beta = 1$	32.00	20.76	0.00	∞	2.52	2.93	0.00	∞	0.22	0.19	0.00	∞
$\alpha = 100,\ \beta = 2$	78.45	51.91	0.00	∞	12.10	18.31	0.00	∞	1.76	1.50	0.00	∞
$\alpha = 100,\ \beta = 4$	139.13	62.71	0.00	∞	42.32	21.23	0.00	∞	15.40	12.26	0.00	∞
$\alpha = 200,\ \beta = 1$	28.74	20.48	0.00	∞	0.56	0.70	0.00	∞	0.06	0.07	0.18	∞
$\alpha = 200,\ \beta = 2$	76.70	56.65	0.00	∞	4.61	6.06	0.00	∞	0.52	0.34	0.00	∞
$\alpha = 200,\ \beta = 4$	130.62	67.27	0.00	∞	21.48	17.97	0.00	∞	5.24	5.82	0.00	∞
AT-PSO												
$R^* = 0.1$	3.28	6.03	0.00	∞	2.22	2.69	0.00	∞	111.55	65.99	0.00	∞
$R^* = 0.3$	0.74	1.00	0.02	∞	0.02	0.02	0.34	∞	0.06	0.06	0.16	∞
$R^* = 0.5$	4.06	8.44	0.00	∞	0.07	0.13	0.22	∞	0.01	0.02	0.56	414.50
$R^* = 0.7$	28.82	33.05	0.00	∞	0.43	1.25	0.04	∞	0.07	0.16	0.22	∞

Table 6. Simulation results for OF5. See text for description.

OF6	Global nbhd				Ring-3 nbhd				Ring-1 nbhd			
Algorithm	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$	Mean	SD	$\widehat{p}$	$\widehat{k}$
PSO	19.66	0.56	0.00	∞	14.77	8.35	0.02	∞	19.48	2.63	0.00	∞
BBPSO-MC	20.72	0.23	0.00	∞	20.74	0.22	0.00	∞	20.74	0.21	0.00	∞
BBPSOxp-MC	20.75	0.18	0.00	∞	20.73	0.18	0.00	∞	20.73	0.18	0.00	∞
AT-BBPSO-MC												
$df=1$, $R^*=0.1$	18.35	5.39	0.00	∞	9.73	9.19	0.00	∞	6.24	7.88	0.00	∞
$df=1$, $R^*=0.3$	18.65	4.66	0.00	∞	14.66	8.70	0.00	∞	11.40	9.71	0.00	∞
$df=1$, $R^*=0.5$	19.93	0.37	0.00	∞	18.90	4.71	0.04	∞	16.91	6.85	0.06	∞
$df=1$, $R^*=0.7$	19.69	2.00	0.00	∞	19.52	2.28	0.00	∞	18.05	5.13	0.00	∞
$df=3$, $R^*=0.1$	19.98	0.09	0.00	∞	19.48	3.55	0.00	∞	14.60	8.47	0.00	∞
$df=3$, $R^*=0.3$	19.82	0.05	0.00	∞	19.48	2.82	0.00	∞	19.94	0.54	0.00	∞
$df=3$, $R^*=0.5$	19.83	0.06	0.00	∞	19.95	0.30	0.00	∞	20.24	0.38	0.00	∞
$df=3$, $R^*=0.7$	19.85	0.10	0.00	∞	20.05	0.37	0.00	∞	20.34	0.39	0.00	∞
$df=5$, $R^*=0.1$	20.00	0.07	0.00	∞	20.12	0.25	0.00	∞	17.83	6.21	0.00	∞
$df=5$, $R^*=0.3$	19.83	0.04	0.00	∞	19.87	0.21	0.00	∞	20.12	0.45	0.00	∞
$df=5$, $R^*=0.5$	19.83	0.04	0.00	∞	19.90	0.26	0.00	∞	20.32	0.45	0.00	∞
$df=5$, $R^*=0.7$	19.85	0.12	0.00	∞	20.06	0.35	0.00	∞	20.42	0.39	0.00	∞
$df=\infty$, $R^*=0.1$	20.09	0.08	0.00	∞	20.16	0.14	0.00	∞	20.32	0.27	0.00	∞
$df=\infty$, $R^*=0.3$	19.83	0.03	0.00	∞	19.85	0.17	0.00	∞	19.98	0.33	0.00	∞
$df=\infty$, $R^*=0.5$	19.83	0.03	0.00	∞	19.89	0.20	0.00	∞	20.14	0.38	0.00	∞
$df=\infty$, $R^*=0.7$	19.83	0.06	0.00	∞	20.06	0.36	0.00	∞	20.36	0.37	0.00	∞
AT-BBPSOxp-MC												
$df=1$, $R^*=0.1$	15.26	7.90	0.00	∞	11.33	8.97	0.00	∞	9.05	8.54	0.00	∞
$df=1$, $R^*=0.3$	18.42	5.50	0.00	∞	17.06	6.87	0.00	∞	14.76	8.83	0.00	∞
$df=1$, $R^*=0.5$	20.19	0.20	0.00	∞	19.48	3.44	0.02	∞	17.05	6.20	0.00	∞
$df=1$, $R^*=0.7$	20.22	0.26	0.00	∞	20.08	0.67	0.00	∞	19.74	2.08	0.00	∞
$df=3$, $R^*=0.1$	20.19	0.16	0.00	∞	20.26	0.13	0.00	∞	19.58	3.36	0.00	∞
$df=3$, $R^*=0.3$	20.08	0.17	0.00	∞	20.23	0.18	0.00	∞	20.27	0.18	0.00	∞
$df=3$, $R^*=0.5$	20.19	0.13	0.00	∞	20.27	0.13	0.00	∞	20.33	0.25	0.00	∞
$df=3$, $R^*=0.7$	20.21	0.13	0.00	∞	20.32	0.11	0.00	∞	20.39	0.11	0.00	∞
$df=5$, $R^*=0.1$	20.25	0.08	0.00	∞	20.27	0.11	0.00	∞	20.32	0.12	0.00	∞
$df=5$, $R^*=0.3$	20.07	0.16	0.00	∞	20.20	0.17	0.00	∞	20.21	0.18	0.00	∞
$df=5$, $R^*=0.5$	20.14	0.15	0.00	∞	20.23	0.12	0.00	∞	20.33	0.14	0.00	∞
$df=5$, $R^*=0.7$	20.20	0.11	0.00	∞	20.31	0.13	0.00	∞	20.37	0.11	0.00	∞
$df=\infty$, $R^*=0.1$	20.18	0.08	0.00	∞	20.27	0.08	0.00	∞	20.29	0.12	0.00	∞
$df=\infty$, $R^*=0.3$	20.01	0.17	0.00	∞	20.14	0.18	0.00	∞	20.20	0.19	0.00	∞
$df=\infty$, $R^*=0.5$	20.06	0.15	0.00	∞	20.22	0.14	0.00	∞	20.27	0.15	0.00	∞
$df=\infty$, $R^*=0.7$	20.14	0.15	0.00	∞	20.30	0.13	0.00	∞	20.36	0.14	0.00	∞
DI-PSO												
$\alpha=50$, $\beta=1$	19.42	1.48	0.00	∞	18.52	3.08	0.00	∞	19.02	3.09	0.00	∞
$\alpha=50$, $\beta=2$	19.50	1.04	0.00	∞	17.94	3.01	0.00	∞	19.33	1.89	0.00	∞
$\alpha=50$, $\beta=4$	19.83	0.64	0.00	∞	19.66	1.00	0.00	∞	19.76	1.07	0.00	∞
$\alpha=100$, $\beta=1$	18.71	2.25	0.00	∞	16.04	5.48	0.00	∞	17.63	5.10	0.00	∞
$\alpha=100$, $\beta=2$	19.75	1.02	0.00	∞	19.27	1.95	0.00	∞	18.96	3.30	0.00	∞
$\alpha=100$, $\beta=4$	19.99	1.04	0.00	∞	19.80	1.11	0.00	∞	20.13	0.75	0.00	∞
$\alpha=200$, $\beta=1$	19.13	2.10	0.00	∞	14.83	6.47	0.00	∞	16.77	5.92	0.00	∞
$\alpha=200$, $\beta=2$	20.00	1.29	0.00	∞	19.63	2.37	0.00	∞	20.27	0.32	0.00	∞
$\alpha=200$, $\beta=4$	20.38	0.65	0.00	∞	20.19	0.65	0.00	∞	20.47	0.31	0.00	∞
AT-PSO												
$R^*=0.1$	18.67	4.43	0.00	∞	18.18	5.06	0.00	∞	20.18	0.66	0.00	∞
$R^*=0.3$	19.44	1.34	0.00	∞	16.84	6.30	0.04	∞	15.62	7.72	0.00	∞
$R^*=0.5$	19.69	0.92	0.00	∞	18.88	3.16	0.00	∞	19.66	2.29	0.00	∞
$R^*=0.7$	19.97	0.53	0.00	∞	19.76	0.69	0.00	∞	20.12	0.32	0.00	∞

Table 7. Simulation results for OF6. See text for description.