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Optimal designs for conjoint experiments☆

Roselinde Kessels[a,*], Peter Goos[b], Martina Vandebroek[c,d]

[a]North Carolina State University, College of Textiles, Campus Box 8301, Raleigh, NC 27695, USA
[b]Department of Mathematics, Statistics and Actuarial Sciences, Faculty of Applied Economics, Universiteit Antwerpen, Prinsstraat 13, 2000 Antwerpen, Belgium
[c]Faculty of Business and Economics, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium
[d]Leuven Statistics Research Centre, W. de Croylaan 54, 3001 Leuven-Heverlee, Belgium

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Abstract

In conjoint experiments, each respondent receives a set of profiles to rate. Sometimes, the profiles are expensive prototypes that respondents have to test before rating them. Designing these experiments involves determining how many and which profiles each respondent has to rate and how many respondents are needed. To that end, the set of profiles offered to a respondent is treated as a separate block in the design and a random respondent effect is used in the model because profile ratings from the same respondent are correlated. Optimal conjoint designs are then obtained by means of an adapted version of an algorithm for finding $\mathscr{D}$-optimal split-plot designs. A key feature of the design construction algorithm is that it returns the optimal number of respondents and the optimal number of profiles each respondent has to evaluate for a given number of profiles. The properties of the optimal designs are described in detail and some practical recommendations are given.

Keywords: Conjoint experiments; $\mathscr{D}$-optimality; Optimal block design; Optimal block sizes; Prototype testing

1. Introduction

In marketing, conjoint experiments have frequently been carried out to measure consumer preferences for the attributes of various products or services, jointly referred to as goods (Green et al., 2001). They have been conducted for new product development, pricing, advertising, and various other things, in many different businesses around the world (Wittink and Cattin, 1989; Wittink et al., 1994; Gustafsson et al., 2003). In a conjoint experiment, respondents are usually asked to rate a set of goods on a scale. The goods that have to be rated are combinations of attribute levels, and they are referred to as profiles in the conjoint literature. Sometimes, the goods need to be rated on a monetary scale. In that case, the rating is sometimes called a reservation price.

In conjoint experiments where the profiles are physical prototypes, budgetary constraints usually force the researcher to use only a small number of profiles and a restricted number of respondents. Also, the number of prototypes that can be assigned to each respondent is limited because he or she must test these prototypes in advance. Given this

☆ The computer program used to generate the conjoint designs discussed in this paper, along with sample input and output files, can be downloaded from www.ua.ac.be/peter.goos.

* Corresponding author. Tel.: +1 919 513 1089; fax: +1 919 515 6621.
E-mail addresses: rkessel@ncsu.edu (R. Kessels), peter.goos@ua.ac.be (P. Goos), martina.vandebroek@econ.kuleuven.be (M. Vandebroek).

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experimental situation the researcher wants to elicit as much information as possible on the utilities the respondents derive from the attribute levels of the good. These utilities are called part-worths and correspond to the parameters of a statistical model. An accurate estimation of these parameters allows the researcher to learn about consumers' trade-offs as well as to make precise predictions about their future purchasing behavior.

Now, the quality of these inferences highly depends on the profiles and the number of test persons used in the conjoint study. This is especially so if only a small number of profiles can be used. Also, the assignment of the profiles to the subjects plays a key role. It is therefore important to plan the conjoint study carefully and to use experimental design principles. The experimental design literature on conjoint experiments is, however, silent about how to carefully select sets of alternative prototypes to be evaluated by the respondents. For example, if 30 prototypes can be developed from a set of many possible ones, then the literature does neither provide a tailor-made answer about how to select those 30 prototypes, nor about the ideal number of test persons or the assignment of the 30 selected alternatives to these respondents. To provide answers to these questions is the goal of this paper.

The method we adopt to solve the conjoint design problem is based on the optimal design approach for blocked and split-plot experiments advocated by Goos and Vandebroek (2001a,b, 2003, 2004) and Goos (2002, 2006). Block designs are heavily used in industry and agriculture when not all the observations can be carried out under homogeneous circumstances, for example, when more than one batch of material is required or when the experiment takes up more than one day. Split-plot designs are special cases of block designs where some of the experimental factors stay constant within each block. In all of the work on block and split-plot designs, the assumed model is the linear random block effects model. Like Brazier et al. (2002) we adopt this model in the conjoint setting and refer to it as the linear random respondent effects model.

The motivation for this model is as follows. It is reasonable to assume that respondents are randomly selected from a population and that they are heterogeneous. Respondent heterogeneity is due to variations in terms of age, experience with the good under study, physical characteristics, cognitive abilities, and so forth. The consequence of this heterogeneity is that profile ratings from different respondents are more dissimilar than profile ratings from the same respondent. The likeness of the ratings from a single respondent is nothing but a positive correlation. To capture the heterogeneity between respondents, or the correlation within respondents, a random effect is included in the model.

Drawing on the random respondent effects model, a conjoint design consists of blocks or sets of profiles that are each offered to a different respondent and the number of respondents is equal to the number of blocks in the design. Note that we focus on main-effects conjoint designs only. To evaluate different conjoint designs, we use the $\mathscr{D}$-optimality criterion (see, for example, Atkinson and Donev, 1992) that seeks designs that minimize the determinant of the variance–covariance matrix of the parameter estimators. To search for $\mathscr{D}$-optimal conjoint designs, we use an adapted version of the split-plot design construction algorithm of Goos and Vandebroek (2004). The adapted algorithm finds the $\mathscr{D}$-optimal number of respondents, the $\mathscr{D}$-optimal number of profiles for each respondent, and the $\mathscr{D}$-optimal design profiles for a given number of profiles. The algorithm imposes no restrictions on the number of profiles each respondent has to rate. As a result, it is possible that the algorithm returns designs in which not all respondents have to rate the same number of profiles. Such a result would not be unexpected because Atkinson and Donev (1992) as well as Goos (2002, 2006) already described situations where $\mathscr{D}$-optimal block and split-plot designs have unequal block sizes.

Until now, researchers have often used balanced incomplete block (BIB) designs to cope with respondent heterogeneity in preference rating. Cochran and Cox (1957) recommended BIB designs because most of them contain blocks with six or fewer units. Also, BIB designs are optimal for parameter estimation. Regrettably, BIB designs only exist for specific numbers of observations, attribute levels and blocks. Consequently, there are many experimental situations for which no BIB design is available, and where an algorithmic design construction is required.

This paper is organized as follows. First, the random respondent effects model is introduced in Section 2. Section 3 explains how to analyze a conjoint experiment assuming this model and Section 4 discusses the design criterion. Next, Section 5 presents the design construction algorithm and Section 6 describes the computational results. Finally, Section 7 summarizes the paper and proposes future research directions.

2. Model

The model used to design and analyze rating-based conjoint experiments is a random respondent effects model. Suppose a conjoint experiment is set up for n profiles. In total, b respondents are appointed who each rate a different

set or block of profiles so as to be able to estimate all parameters. The sizes of these profile sets, which we denote by m_i $(i = 1, \ldots, b)$, may be unequal. The n profiles used in the experiment are thus arranged in b sets of sizes $m_1, \ldots, m_b$, where $n = \sum_{i=1}^{b} m_i$. If we assume that the respondents are heterogeneous and randomly selected from a prespecified population, then the rating U_{ij} provided by respondent i for profile j is modelled as

$$U_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta} + \gamma_i + \varepsilon_{ij}. \tag{1}$$

In this model, $\mathbf{x}_{ij}$ is a $(k+1) \times 1$ vector having a one as its first element and codings for the attribute levels of the jth profile rated by respondent i as its remaining k elements. We thus assume that the model contains main effects only. The vector $\boldsymbol{\beta} = [\beta_0, \ldots, \beta_k]'$ is the $(k+1) \times 1$ unknown fixed parameter vector with β_0 the intercept and $\beta_1, \ldots, \beta_k$ the part-worths or weights associated with the attribute levels. Each part-worth reflects the importance of an attribute level as perceived by the average respondent. The term γ_i represents the random effect of respondent i and ε_{ij} is a random error term.

The attributes are treated as categorical factors and their levels are coded by means of effects-type coding. For a two-level attribute, this means that one level is coded as 1 and the other level as -1. For a three-level attribute the codings are [1 0], [0 1] and [-1 -1]. For a four-level attribute they are [1 0 0], [0 1 0], [0 0 1] and [-1 -1 -1], and so forth for higher-level attributes. Other types of coding may be used too because the coding has no effect on the $\mathscr{D}$-optimal designs found (see Goos, 2002, pp. 37–40).

In matrix notation, model (1) can be written as

$$\boldsymbol{U} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon}, \tag{2}$$

where $\boldsymbol{U}$ is a vector of n profile ratings, the vector $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_b]'$ contains the b random respondent effects and $\boldsymbol{\epsilon}$ is a random error vector. The matrices $\mathbf{X}$ and $\mathbf{Z}$ have dimensions $n \times (k+1)$ and $n \times b$, respectively. $\mathbf{X}$ is given by

$$\mathbf{X} = [\mathbf{X}_1', \ldots, \mathbf{X}_b']', \tag{3}$$

where $\mathbf{X}_i = [\mathbf{x}_{i1}, \ldots, \mathbf{x}_{im_i}]'$ collects the profiles rated by respondent i and $\mathbf{Z}$ is defined as

$$\mathbf{Z} = \mathrm{diag}[\mathbf{1}_{m_1}, \ldots, \mathbf{1}_{m_b}], \tag{4}$$

where $\mathbf{1}_{m_i}$ is a $m_i \times 1$ vector of ones. It is assumed that

$$\mathrm{E}(\boldsymbol{\epsilon}) = \mathbf{0}_n \quad \text{and} \quad \mathrm{Cov}(\boldsymbol{\epsilon}) = \sigma_{\boldsymbol{\epsilon}}^2\mathbf{I}_n, \tag{5}$$

$$\mathrm{E}(\boldsymbol{\gamma}) = \mathbf{0}_b \quad \text{and} \quad \mathrm{Cov}(\boldsymbol{\gamma}) = \sigma_{\boldsymbol{\gamma}}^2\mathbf{I}_b, \tag{6}$$

and

$$\mathrm{Cov}(\boldsymbol{\gamma}, \boldsymbol{\epsilon}) = \mathbf{0}_{b\times n}, \tag{7}$$

where $\sigma_{\boldsymbol{\epsilon}}^2$ is the within-respondents variance and $\sigma_{\boldsymbol{\gamma}}^2$ is the between-respondents variance. Under these assumptions, the variance–covariance matrix $\mathbf{V}$ of the profile ratings $\boldsymbol{U}$ can be written as

$$\begin{aligned}
\mathbf{V} &= \mathrm{Cov}(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon}) \\
&= \mathrm{Cov}(\mathbf{Z}\boldsymbol{\gamma}) + \mathrm{Cov}(\boldsymbol{\epsilon}) \\
&= \sigma_{\boldsymbol{\gamma}}^2\mathbf{Z}\mathbf{Z}' + \sigma_{\boldsymbol{\epsilon}}^2\mathbf{I}_n \\
&= \sigma_{\boldsymbol{\gamma}}^2\mathrm{diag}[\mathbf{1}_{m_1}\mathbf{1}_{m_1}', \ldots, \mathbf{1}_{m_b}\mathbf{1}_{m_b}'] + \sigma_{\boldsymbol{\epsilon}}^2\mathrm{diag}[\mathbf{I}_{m_1}, \ldots, \mathbf{I}_{m_b}] \\
&= \mathrm{diag}[\sigma_{\boldsymbol{\epsilon}}^2\mathbf{I}_{m_1} + \sigma_{\boldsymbol{\gamma}}^2\mathbf{1}_{m_1}\mathbf{1}_{m_1}', \ldots, \sigma_{\boldsymbol{\epsilon}}^2\mathbf{I}_{m_b} + \sigma_{\boldsymbol{\gamma}}^2\mathbf{1}_{m_b}\mathbf{1}_{m_b}'].
\end{aligned} \tag{8}$$

Substituting

$$\mathbf{V}_i = \sigma_{\boldsymbol{\epsilon}}^2\mathbf{I}_{m_i} + \sigma_{\boldsymbol{\gamma}}^2\mathbf{1}_{m_i}\mathbf{1}_{m_i}' \tag{9}$$

leads to the variance–covariance matrix

$$\mathbf{V} = \mathrm{diag}[\mathbf{V}_1, \ldots, \mathbf{V}_b]. \tag{10}$$

Note that the matrices $\mathbf{V}_i$ are compound symmetric: the main diagonals of these matrices contain the constant variances of the profile ratings, while the off-diagonal elements are constant covariances. The coefficient of correlation between two profile ratings from the same respondent is equal to

$$\rho = \frac{\sigma_\gamma^2}{\sigma_\epsilon^2 + \sigma_\gamma^2}. \tag{11}$$

This ratio, which lies between 0 and 1, measures the proportion of the total variability that is accounted for by the differences between respondents.

A special case of the random respondent effects model arises when each of the respondents in the conjoint experiment rates only one profile. In that case, $b = n$ and $m_1 = \cdots = m_b = 1$ so that $\mathbf{V}$ is a diagonal matrix with $\sigma_\gamma^2 + \sigma_\epsilon^2$ as its diagonal elements. The ratings are then uncorrelated and σ_γ^2 and σ_ϵ^2 cannot be estimated separately. The model can then be written as

$$\boldsymbol{U} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{v}, \tag{12}$$

where $\boldsymbol{v}$, which has a zero mean and covariance matrix $(\sigma_\gamma^2 + \sigma_\epsilon^2)\mathbf{I}_n$, captures both the random respondent effects contained in $\boldsymbol{\gamma}$ and the random errors in $\boldsymbol{\epsilon}$. As the errors in this simplified model are uncorrelated, it should be clear that this is the model used for analyzing data from a completely randomized design.

3. Estimation

If the error terms and the respondent effects are normally distributed, the maximum likelihood estimator of the unknown parameter vector $\boldsymbol{\beta}$ in (1) and (2) is the generalized least squares (GLS) estimator (see, for example, Gilmour and Trinca, 2000; Goos, 2002)

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\boldsymbol{U}, \tag{13}$$

with variance–covariance matrix

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}. \tag{14}$$

Sometimes, the variance components σ_γ^2 and σ_ϵ^2 are known from previous experimentation so that the estimator $\hat{\boldsymbol{\beta}}$ and its variance–covariance matrix can be immediately obtained. Most often, however, the variances σ_γ^2 and σ_ϵ^2 are unknown and, therefore, (13) and (14) cannot be applied directly. Instead, the variance components σ_γ^2 and σ_ϵ^2 have to be estimated, for example via restricted maximum likelihood (REML) (Gilmour and Trinca, 2000). The estimates $\hat{\sigma}_\gamma^2$ and $\hat{\sigma}_\epsilon^2$ are then substituted in the GLS estimator (13), yielding the feasible GLS estimator

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\boldsymbol{U}, \tag{15}$$

where

$$\hat{\mathbf{V}} = \hat{\sigma}_\epsilon^2\mathbf{I}_n + \hat{\sigma}_\gamma^2\mathbf{Z}\mathbf{Z}'.$$

In that case, the variance–covariance matrix (14) can be approximated by

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1}. \tag{16}$$

4. Design criterion

We evaluate alternative conjoint design options by means of the $\mathscr{D}$-optimality criterion. The $\mathscr{D}$-optimality criterion seeks designs that minimize the determinant of the variance–covariance matrix (14), or equivalently, that maximize the determinant of the information matrix $\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$. $\mathscr{D}$-optimal conjoint designs therefore minimize the generalized

variance of the parameter estimators (Atkinson and Donev, 1992). Goos and Vandebroek (2001b) showed that, because of the compound symmetric error structure of model (2), the information matrix can be written as

$$\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} = \frac{1}{\sigma_\epsilon^2}\left\{\mathbf{X}'\mathbf{X} - \sum_{i=1}^{b}\frac{\sigma_\gamma^2/\sigma_\epsilon^2}{1+m_i(\sigma_\gamma^2/\sigma_\epsilon^2)}(\mathbf{X}_i'\mathbf{1}_{m_i})(\mathbf{X}_i'\mathbf{1}_{m_i})'\right\}. \tag{17}$$

In terms of the coefficient of correlation ρ, (17) can be rewritten as

$$\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} = \frac{1}{\sigma_\epsilon^2}\left\{\mathbf{X}'\mathbf{X} - \sum_{i=1}^{b}\frac{\rho}{1+\rho(m_i-1)}(\mathbf{X}_i'\mathbf{1}_{m_i})(\mathbf{X}_i'\mathbf{1}_{m_i})'\right\}. \tag{18}$$

We call a design that maximizes the determinant of this matrix a $\mathscr{D}$-optimal conjoint design. The $\mathscr{D}$-optimal conjoint design depends on the relative magnitude of σ_γ^2 and σ_ϵ^2 through ρ, but not on their absolute magnitudes.

A special case of a conjoint design is one where each respondent rates a single profile. This design is nothing but a completely randomized design for which $\mathbf{V}$ is diagonal and the information matrix reduces to

$$\frac{1}{\sigma_\epsilon^2+\sigma_\gamma^2}\mathbf{X}'\mathbf{X}. \tag{19}$$

A design that maximizes the determinant of this matrix is therefore referred to as a $\mathscr{D}$-optimal completely randomized design in the rest of this article.

In our study in Section 6, we show that $\mathscr{D}$-optimal conjoint designs have larger determinants than $\mathscr{D}$-optimal completely randomized designs and thus that $\mathscr{D}$-optimal conjoint designs, in general, are statistically more efficient than completely randomized designs. While comparing the determinants, we assume without loss of generality that the total variance, $\sigma_\epsilon^2+\sigma_\gamma^2$, is 1. The determinant of the information matrix of a completely randomized design then becomes

$$|\mathbf{X}'\mathbf{X}|, \tag{20}$$

while that for a conjoint design can be computed as

$$(1-\rho)^{-(k+1)}\left|\mathbf{X}'\mathbf{X} - \sum_{i=1}^{b}\frac{\rho}{1+\rho(m_i-1)}(\mathbf{X}_i'\mathbf{1}_{m_i})(\mathbf{X}_i'\mathbf{1}_{m_i})'\right|, \tag{21}$$

with $k+1$ the number of parameters in $\boldsymbol{\beta}$. As in most of the optimal design literature, we express these determinants on a per parameter basis, and this leads us to defining

$$|\mathbf{X}'\mathbf{X}|^{1/(k+1)} \tag{22}$$

and

$$(1-\rho)^{-1}\left|\mathbf{X}'\mathbf{X} - \sum_{i=1}^{b}\frac{\rho}{1+\rho(m_i-1)}(\mathbf{X}_i'\mathbf{1}_{m_i})(\mathbf{X}_i'\mathbf{1}_{m_i})'\right|^{1/(k+1)} \tag{23}$$

as the $\mathscr{D}$-criterion values of a completely randomized design and a conjoint design, respectively.

5. Conjoint design algorithm

The algorithm we used to construct $\mathscr{D}$-optimal designs for conjoint experiments is an adaptation of that of Goos and Vandebroek (2004) for the production of $\mathscr{D}$-optimal split-plot designs allowing for variable block sizes. For given sample size n and degree of correlation ρ, the conjoint design algorithm seeks the $\mathscr{D}$-optimal number of respondents (that is, the $\mathscr{D}$-optimal number of blocks), the $\mathscr{D}$-optimal number of profiles for each respondent (in other words, the optimal block sizes) and the $\mathscr{D}$-optimal design profiles.

The algorithm begins by randomly generating a design. The starting design is then improved iteratively by swapping profiles from different blocks and exchanging profiles in the design with profiles from a candidate set. To avoid being

stuck in a locally optimal design, more than one starting design is generated and the design search is repeated. Each repetition is called a try and the conjoint design with the best $\mathscr{D}$-criterion value found over all tries is referred to as the $\mathscr{D}$-optimal conjoint design. Obviously, the more observations, attributes and attribute levels are involved, the more designs are possible and the higher the chance the algorithm yields local optima that are far from the global optimum. Therefore, for large problems, we used at least 1000 tries.

To benchmark the results of our algorithm, we used the algorithm of Goos and Vandebroek (2001a) to which we refer as the fixed block size algorithm because it restricts its searches to conjoint designs with predetermined block sizes. Whereas the conjoint design algorithm generates the $\mathscr{D}$-optimal number of respondents, b, and the $\mathscr{D}$-optimal number of profiles for each respondent, $m_1, \ldots, m_b$, the fixed block size algorithm requires the values b and $m_1, \ldots, m_b$ as an input. The fact that the fixed block size algorithm constrains the design structure by imposing the specification of b and $m_1, \ldots, m_b$ means that the resulting designs do not necessarily have the optimal blocking structure and therefore may not have the highest $\mathscr{D}$-criterion value possible. The conjoint design algorithm relaxes this restriction so that it does find the optimal design structure. For a given number of profiles n, it requires, however, more tries and computing time compared with the fixed block size algorithm.

In the next section, we show that the $\mathscr{D}$-optimal conjoint designs are not very sensitive to the specified value for ρ so that a rough estimate for ρ usually suffices as an input to the design construction algorithm. In practice, a reasonable value for ρ is often 0.5. For example, a dataset collected in a health economics study by Brazier et al. (2002) yielded an estimate for ρ of 0.62. Also, four datasets from sensory experiments conducted at the laboratories of the multinational brewer InBev yielded estimates for ρ of 0.48, 0.46, 0.36 and 0.41.

6. Results

We now present a selection of computational results from which we derive recommendations to produce $\mathscr{D}$-optimal conjoint designs when the profile construction is expensive. We first show that it is statistically justified to apply these designs instead of $\mathscr{D}$-optimal completely randomized designs. We then proceed with a discussion of the $\mathscr{D}$-optimal blocking structures and the computing times needed. Also, we deal with some practical issues and seek ways to save computing time.

6.1. Designs under investigation

For our study we computed $\mathscr{D}$-optimal conjoint designs for five scenarios. The first scenario involved six two-level attributes and is denoted by (2, 2, 2, 2, 2, 2). All other scenarios involved only four attributes. The second scenario has all four attributes at three levels. We denote this scenario by (3, 3, 3, 3). The next three scenarios possess increasing amounts of heterogeneity in the numbers of attribute levels. The third scenario has the first attribute at two levels, the next two attributes at three levels and the fourth attribute at four levels. We refer to it as (2, 3, 3, 4). The fourth scenario is similar to the third one except for the fourth attribute which has five levels: (2, 3, 3, 5). Finally, the fifth scenario has a different number of levels for each of its attributes: (2, 3, 4, 5).

Table 1 contains further information about the setup of our conjoint design study. For each scenario we indicated the number of candidate profiles and the number of model parameters, $k + 1$, in $\boldsymbol{\beta}$ after coding the levels. The table also contains the sample sizes, n, of the conjoint designs we generated. We considered more sample sizes in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios than in the other scenarios to perform some additional studies on these cases. These are described in Sections 6.5–6.7. For each of the scenarios and sample sizes in Table 1, we computed $\mathscr{D}$-optimal designs

Table 1
Setup of the conjoint design study

Scenario	# cand.	$k + 1$	n
(2, 2, 2, 2, 2, 2)	64	7	20, 30, 40, 50, 60, 70
(3, 3, 3, 3)	81	9	20, 24, 30, 36, 40, 50, 60, 70, 72, 81
(2, 3, 3, 4)	72	9	20, 24, 30, 36, 40, 50, 60, 70, 72, 81
(2, 3, 3, 5)	90	10	20, 30, 40, 50, 60, 70
(2, 3, 4, 5)	120	11	20, 30, 40, 50, 60, 70

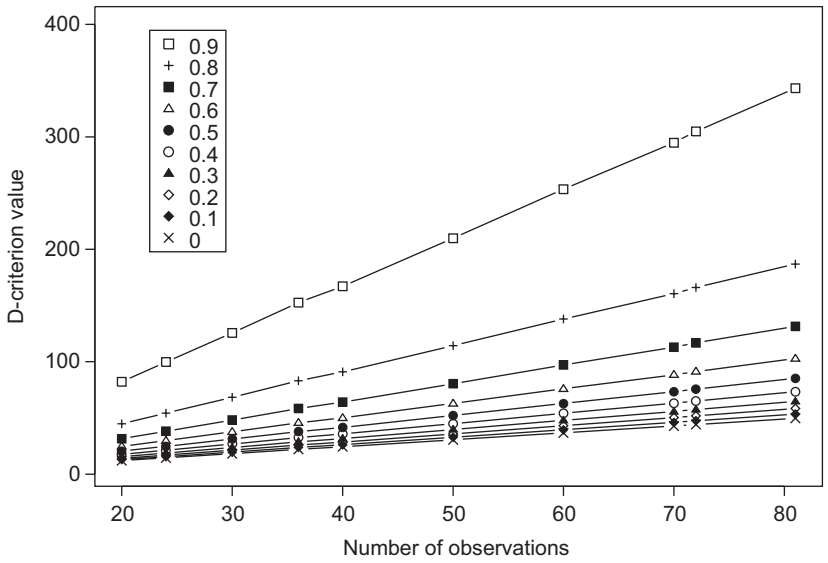

Fig. 1. $\mathcal{D}$-criterion values of the $\mathcal{D}$-optimal conjoint designs ($\rho > 0$) and the $\mathcal{D}$-optimal completely randomized designs ($\rho = 0$) in scenario (3, 3, 3, 3).

and their $\mathcal{D}$-criterion values for ρ values ranging from 0 to 0.9. Note that the designs obtained for $\rho = 0$ (that is, in the absence of correlation) can be viewed as $\mathcal{D}$-optimal completely randomized designs.

6.2. $\mathcal{D}$-optimal conjoint designs versus $\mathcal{D}$-optimal completely randomized designs

The $\mathcal{D}$-criterion values of the $\mathcal{D}$-optimal conjoint designs and $\mathcal{D}$-optimal completely randomized designs for the five scenarios described in Table 1 are given in Table A.1 of Appendix A. We calculated these values using (22) and (23) to compare the $\mathcal{D}$-optimal conjoint designs with the $\mathcal{D}$-optimal completely randomized designs for given sample sizes and scenarios. To gain better insight in the $\mathcal{D}$-criterion values associated with each scenario, we plotted them against the sample size for the different degrees of correlation. Fig. 1 contains the graph for the (3, 3, 3, 3) scenario. The plots for the other scenarios show similar trends and have therefore been left out.

It turns out that, for each number of profiles n, the $\mathcal{D}$-optimal completely randomized design is outperformed by each of the corresponding $\mathcal{D}$-optimal conjoint designs. This is in line with the results of Goos (2002, p. 133), who studied a second-order response surface model in four (quantitative) variables. The results imply that it is not a good idea to have every profile rated by a different respondent. On the contrary, it is better to have fewer respondents than profiles and have each respondent rate several profiles. This is exactly what is done in a conjoint design. Moreover, Fig. 1 clearly shows that the higher the correlation, the larger the efficiency gain of using a $\mathcal{D}$-optimal conjoint design instead of a $\mathcal{D}$-optimal completely randomized design. Fig. 1 further reveals that the $\mathcal{D}$-criterion values increase linearly with the sample size and do not saturate after a certain number of observations.

6.3. $\mathcal{D}$-optimal blocking structures

Because the $\mathcal{D}$-optimal conjoint designs are statistically more efficient than the $\mathcal{D}$-optimal completely randomized designs, it is better to have a select number of respondents evaluate several profiles than to have n respondents evaluate one profile each. We now examine the $\mathcal{D}$-optimal conjoint designs to derive more precisely what the optimal number of respondents is for a specific conjoint setting, what the optimal number of profiles is for each of them to rate, and what the optimal profiles are. Table 2 shows the optimal blocking structures for scenario (2, 2, 2, 2, 2, 2), whereas Table 3 contains the blocking structures of the $\mathcal{D}$-optimal conjoint designs for the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios. The blocking structures pertaining to the (2, 3, 3, 5) and (2, 3, 4, 5) scenarios appear in Tables 4 and 5, respectively. The designs themselves are not shown, but can be obtained from the authors.

Each row in Tables 2–5 corresponds to a different design. This means that most of the designs for a particular sample size and scenario are optimal for various degrees of correlation. Consequently, the $\mathcal{D}$-optimal conjoint designs are

Table 2
Blocking structures of the $\mathscr{D}$-optimal conjoint designs for scenario (2, 2, 2, 2, 2, 2)

Design	n	ρ	Blocking structure	b
1	20	{0.1, 0.2}	$m_1, m_2, m_3 = 4 \mid m_4, \ldots, m_7 = 2$	7
2	20	{0.3, ..., 0.7}	$m_1 = 4 \mid m_2, \ldots, m_9 = 2$	9
3	20	{0.8, 0.9}	$m_1, \ldots, m_{10} = 2$	10
4	30	{0.1, ..., 0.9}	$m_1, \ldots, m_{15} = 2$	15
5	40	{0.1, ..., 0.9}	$m_1, \ldots, m_{20} = 2$	20
6	50	{0.1, ..., 0.9}	$m_1, \ldots, m_{25} = 2$	25
7	60	{0.1, ..., 0.9}	$m_1, \ldots, m_{30} = 2$	30
8	70	{0.1, ..., 0.9}	$m_1, \ldots, m_{35} = 2$	35

The optimal number of blocks is denoted by b, and the optimal block sizes are denoted by $m_1, \ldots, m_b$.

Table 3
Blocking structures of the $\mathscr{D}$-optimal conjoint designs for scenarios (3, 3, 3, 3) and (2, 3, 3, 4)

Design	n	Scenario	ρ	Blocking structure	b
1	20	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_6 = 3 \mid m_7 = 2$	7
2	20	(2, 3, 3, 4)		$m_1, \ldots, m_4 = 3 \mid m_5, m_6 = 4$	6
3	24	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_8 = 3$	8
4	24	(2, 3, 3, 4)			
5	30	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{10} = 3$	10
6	30	(2, 3, 3, 4)	{0.1, ..., 0.6}	$m_1, \ldots, m_{10} = 3$	10
7	30		{0.7, 0.8, 0.9}	$m_1, \ldots, m_6 = 3 \mid m_7, m_8, m_9 = 4$	9
8	36	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{12} = 3$	12
9	36	(2, 3, 3, 4)			
10	40	(3, 3, 3, 3)	{0.1, 0.2}	$m_1, \ldots, m_{13} = 3 \mid m_{14} = 1$	14
11	40		{0.3, ..., 0.9}	$m_1, \ldots, m_{12} = 3 \mid m_{13} = 4$	13
12	40	(2, 3, 3, 4)	{0.1, ..., 0.4}	$m_1, \ldots, m_{12} = 3 \mid m_{13} = 4$	13
13	40		{0.5, ..., 0.9}	$m_1, \ldots, m_8 = 3 \mid m_9, \ldots, m_{12} = 4$	12
14	50	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{16} = 3 \mid m_{17} = 2$	17
15	50	(2, 3, 3, 4)		$m_1, \ldots, m_{14} = 3 \mid m_{15}, m_{16} = 4$	16
16	60	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{20} = 3$	20
17	60	(2, 3, 3, 4)			
18	70	(3, 3, 3, 3)	{0.1, 0.2}	$m_1, \ldots, m_{23} = 3 \mid m_{24} = 1$	24
19	70		{0.3, ..., 0.9}	$m_1, \ldots, m_{22} = 3 \mid m_{23} = 4$	23
20	70	(2, 3, 3, 4)	{0.1, ..., 0.9}	$m_1, \ldots, m_{22} = 3 \mid m_{23} = 4$	23
21	72	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{24} = 3$	24
22	72	(2, 3, 3, 4)			
23	81	(3, 3, 3, 3)	{0.1, ..., 0.9}	$m_1, \ldots, m_{27} = 3$	27
24	81	(2, 3, 3, 4)			

The optimal number of blocks is denoted by b, and the optimal block sizes are denoted by $m_1, \ldots, m_b$.

fairly robust against misspecifications of the degree of correlation. Goos (2002, p. 122) observed a similar result while computing $\mathscr{D}$-optimal blocked response surface designs for several degrees of correlation.

A number of interesting observations can be made based on Tables 2–5. Table 2 shows that, in many cases, having two profiles evaluated by each respondent is optimal for scenario (2, 2, 2, 2, 2, 2), while Table 3 reveals that assigning three profiles to each of the respondents is often optimal in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios. However, in the (2, 3, 3, 4) scenario it is sometimes better to administer four profiles to one or more respondents, especially when the

Table 4
Blocking structures of the $\mathcal{D}$-optimal conjoint designs for scenario (2, 3, 3, 5)

Design	n	ρ	Blocking structure	b
1	20	{0.1, 0.2}	$m_1, \ldots, m_4 = 3 \mid m_5, m_6 = 4$	6
2	20	{0.3, 0.4, 0.5}	$m_1, m_2 = 3 \mid m_3, m_4 = 5 \mid m_5 = 4$	5
3	20	{0.6, ..., 0.9}	$m_1, \ldots, m_4 = 5$	4
4	30	{0.1}	$m_1, \ldots, m_{10} = 3$	10
5	30	{0.2, 0.3, 0.4}	$m_1, \ldots, m_5 = 3 \mid m_6, m_7, m_8 = 5$	8
6	30	{0.5, ..., 0.9}	$m_1, \ldots, m_6 = 5$	6
7	40	{0.1, 0.2}	$m_1, \ldots, m_{10} = 3 \mid m_{11}, m_{12} = 5$	12
8	40	{0.3, ..., 0.8}	$m_1, \ldots, m_5 = 3 \mid m_6, \ldots, m_{10} = 5$	10
9	40	{0.9}	$m_1, \ldots, m_8 = 5$	8
10	50	{0.1, 0.2, 0.3}	$m_1, \ldots, m_{10} = 3 \mid m_{11}, \ldots, m_{14} = 5$	14
11	50	{0.4, ..., 0.7}	$m_1, \ldots, m_5 = 3 \mid m_6, \ldots, m_{12} = 5$	12
12	50	{0.8, 0.9}	$m_1, \ldots, m_{10} = 5$	10
13	60	{0.1}	$m_1, \ldots, m_{20} = 3$	20
14	60	{0.2}	$m_1, \ldots, m_{15} = 3 \mid m_{16}, m_{17}, m_{18} = 5$	18
15	60	{0.3, 0.4}	$m_1, \ldots, m_{10} = 3 \mid m_{11}, \ldots, m_{16} = 5$	16
16	60	{0.5, ..., 0.9}	$m_1, \ldots, m_5 = 3 \mid m_6, \ldots, m_{14} = 5$	14
17	70	{0.1}	$m_1, \ldots, m_{20} = 3 \mid m_{21}, m_{22} = 5$	22
18	70	{0.2}	$m_1, \ldots, m_{15} = 3 \mid m_{16}, \ldots, m_{20} = 5$	20
19	70	{0.3, 0.4}	$m_1, \ldots, m_{10} = 3 \mid m_{11}, \ldots, m_{18} = 5$	18
20	70	{0.5, ..., 0.9}	$m_1, \ldots, m_5 = 3 \mid m_6, \ldots, m_{16} = 5$	16

The optimal number of blocks is denoted by b, and the optimal block sizes are denoted by $m_1, \ldots, m_b$.

Table 5
Blocking structures of the $\mathcal{D}$-optimal conjoint designs for scenario (2, 3, 4, 5)

Design	n	ρ	Blocking structure	b
1	20	{0.1, ..., 0.9}	$m_1, \ldots, m_5 = 4$	5
2	30	{0.1}	$m_1, \ldots, m_6 = 4 \mid m_7, m_8 = 3$	8
3	30	{0.2, ..., 0.9}	$m_1, \ldots, m_5 = 4 \mid m_6, m_7 = 5$	7
4	40	{0.1, ..., 0.9}	$m_1, \ldots, m_{10} = 4$	10
5	50	{0.1, ..., 0.9}	$m_1, \ldots, m_{10} = 4 \mid m_{11}, m_{12} = 5$	12
6	60	{0.1, ..., 0.9}	$m_1, \ldots, m_{15} = 4$	15
7	70	{0.1}	$m_1, \ldots, m_{16} = 4 \mid m_{17}, m_{18} = 3$	18
8	70	{0.2, ..., 0.9}	$m_1, \ldots, m_{15} = 4 \mid m_{16}, m_{17} = 5$	17

The optimal number of blocks is denoted by b, and the optimal block sizes are denoted by $m_1, \ldots, m_b$.

correlation, ρ, is large. Also in the (3, 3, 3, 3) scenario blocks of size 4 appear for higher correlations. In these cases, the optimal number of respondents decreases with ρ. The result that given numbers of profiles are grouped in fewer (and thus larger) blocks when the correlation is increased was also found by Goos and Vandebroek (2004), who studied $\mathcal{D}$-optimal split-plot designs for response surface experiments in industry.

While the optimal blocking structure of the (2, 3, 3, 4) scenario corresponds to that of the (3, 3, 3, 3) scenario, the optimal blocking structure of the (2, 3, 3, 5) scenario largely differs from it. Table 4 shows that, in general, for the (2, 3, 3, 5) scenario it is most efficient to assign sets of three and/or five profiles to the respondents. The lower the correlation, the more sets of size 3 appear in the optimal design. The higher the correlation, the more sets of size 5 are included. As a result, there is again a stronger grouping tendency at higher correlations, but it is more pronounced here than in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios. As can be seen, three or four design structures are possible for almost each sample size in the (2, 3, 3, 5) scenario. This is more than the one or two design structures for each sample size in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios.

In the optimal blocking structures of the most heterogeneous level setting, the (2, 3, 4, 5) scenario, contained in Table 5, the blocks of size 3 from in the previous scenarios mostly disappeared and were replaced by blocks of size 4. It is thus generally most efficient to present four profiles to each of the respondents. Here, a slightly larger grouping tendency can be observed for large values of ρ when $n = 30$ and 70.

Note that the $\mathscr{D}$-optimal conjoint design for a given scenario, sample size and degree of correlation is not unique. For each design problem a number of equivalent designs (with the same blocking structures as in Tables 2–5) exist. This is because the $\mathscr{D}$-optimal conjoint designs are constructed for a random respondent effects model with main effects only. Therefore, the $\mathscr{D}$-optimal conjoint design for 81 observations in the (3, 3, 3, 3) scenario is equivalent to a 81-run $\mathscr{D}$-optimal 3^4 factorial design arranged in 27 blocks of size 3. This design includes all 81 candidate profiles and appears in Table B.1 of Appendix B. It was created using Appendix 5A of Wu and Hamada (2000). A design that is equivalent to it can be obtained by applying the swap procedure in the design construction algorithm to the 81 candidate profiles in order to create a design with block sizes of 3. Also the $\mathscr{D}$-optimal conjoint design for 72 observations in the (2, 3, 3, 4) scenario can be shown to be equivalent to a design involving all candidate profiles arranged in blocks of size 3. Applying the swap procedure to the 72 candidate profiles for that scenario yielded the $\mathscr{D}$-optimal arrangement shown in Table B.2 of Appendix B.

6.4. Compromising between practical and optimal blocking structures

In this section, we investigate whether some of the optimal blocking structures can be slightly adapted to result in more practical structures that still possess high $\mathscr{D}$-criterion values. These more practical conjoint designs are computed using the fixed block size algorithm, with the practical blocking structures as inputs. These designs are $\mathscr{D}$-optimal for a given number of blocks b and given block sizes $m_1, \ldots, m_b$, but they are suboptimal within the class of all possible conjoint designs because fixing the block sizes in advance restricts the realm of possible conjoint designs.

We consider the following five cases each involving a different scenario. The first case is concerned with scenario (2, 2, 2, 2, 2, 2), where some of the respondents have to rate four profiles when $n = 20$ even though the dominant block size is 2. For logistic reasons, it may be more convenient to have 10 respondents evaluate two profiles each. The second case concerns the conjoint designs in the (3, 3, 3, 3) scenario in which one or two profiles are administered to one of the respondents. It may be more sensible, however, to assign four profiles instead of three to one or two respondents so that fewer respondents are needed. The third case covers the designs in the (2, 3, 3, 4) scenario in which four profiles are presented to more than two respondents. It would be very convenient if the blocking structure of three profiles per respondent could be extended to these instances. The fourth case involves all the designs in the (2, 3, 3, 5) scenario as it would be convenient to have one fixed block size of either 3 or 5. The fifth case is concerned with the designs in the (2, 3, 4, 5) scenario with blocks of size 3. In that case, it would be very useful if the blocks of size 3 could be replaced with blocks of size 5.

We discuss these five cases more at length and investigate how much one loses in terms of the $\mathscr{D}$-criterion value by using the best possible designs with more practical blocking structures instead of the optimal ones. To determine the efficiency losses of using a conjoint design with a suboptimal blocking structure, we calculate how many observations would be saved if a $\mathscr{D}$-optimal conjoint design with an optimal blocking structure were applied whose $\mathscr{D}$-criterion value equals that of the suboptimal conjoint design. In other words, we express the losses in $\mathscr{D}$-criterion value in terms of the number of redundant observations of the design with the suboptimal blocking structure. In Appendix C we describe in detail how to compute the number of redundant observations.

Case 1: *The* (2, 2, 2, 2, 2, 2) *scenario.* In the (2, 2, 2, 2, 2, 2) scenario it is clear that the dominant block size of the $\mathscr{D}$-optimal conjoint designs is 2. For $n = 20$ and $\rho \leqslant 0.7$, it is, however, optimal to have one or three respondents evaluate four profiles each. As it is much simpler for a researcher when every respondent is subject to the same regime of having to test and rate two profiles, it is worth investigating what loss in $\mathscr{D}$-criterion value is incurred if a homogeneous block size of 2 is used. It turns out that this does not lead to redundant observations, so that the loss is negligible. As a result, it is efficient to use blocks of size 2 in this scenario where all attributes have two levels.

Cases 2 *and* 3: *The* (3, 3, 3, 3) *and* (2, 3, 3, 4) *scenarios.* In the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios it is often most efficient to assign three profiles to the respondents. In some cases in the (3, 3, 3, 3) scenario where the sample size is not a multiple of 3, it is optimal to administer one or two profiles to one of the respondents. Table 3 shows this is the case for all designs with a sample size of 20 and 50, and for the designs with a sample size of 40 and 70 at lower correlations. The assignment of one or two profiles to one respondent is not very attractive from a practical standpoint,

especially when costs are associated with each respondent. Therefore, we calculated the losses in the $\mathscr{D}$-criterion value when assigning four profiles instead of three to one or two respondents so as to save on one respondent. These losses turn out to be so small that there are no redundant observations. As a result, the new blocking structures constitute a good compromise between practical and statistical efficiency.

Note that the assignment of four profiles to one or two respondents is most efficient for the remainder of the cases in the (3, 3, 3, 3) scenario and for most of the cases in the (2, 3, 3, 4) scenario where the sample size is not a multiple of 3. An exception to the rule of administering three profiles to the respondents and four profiles to one or two respondents in case the sample size is not a multiple of 3 are the designs in the (2, 3, 3, 4) scenario with a sample size of 30 and 40 at higher correlations. For these instances, it is optimal to present four profiles to more than two respondents. We examined whether it is possible to spread the profiles more equally over respondents in blocks of 3. In the case of 40 observations this means that one respondent receives four profiles. Also here, the resulting losses in the $\mathscr{D}$-criterion value turn out to be negligible so that there are no redundant observations.

So in general, to construct conjoint designs for the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios, it is efficient to use blocks of three profiles and one or two blocks of four profiles in case the sample size is not a multiple of 3. This general blocking structure can be given as an input to the fixed block size algorithm to generate the $\mathscr{D}$-optimal conjoint designs within the class of more practical designs. We found that this strategy is appropriate for the designs with a sample size larger than 10. Because the blocking structure is provided as an input to the fixed block size algorithm, this algorithm takes much less computing time than the conjoint design algorithm which seeks the optimal blocking structure (see also Section 6.5).

Case 4: *The* (2, 3, 3, 5) *scenario.* In the (2, 3, 3, 5) scenario it is generally optimal to group profiles in blocks of 3 and/or 5. Instead of combining these two block sizes in a single design, it would be simpler to have a fixed block size of either 3 or 5. Therefore, we examined whether it is feasible to use blocks of size 3 for all or nearly all respondents and have only one or two blocks of size 5 in case the sample size is not divisible by 3. Also, we studied whether five profiles can be administered to all or nearly all respondents and three profiles to no more than four respondents in case the sample size is not divisible by 5. This blocking structure, where blocks of size 5 are used as often as possible, turns out to be the best as there are no redundant observations in this case. Assigning three profiles to as many respondents as possible results in redundant observations at higher correlations. We therefore recommend using blocks of size 5 for the construction of conjoint designs for the (2, 3, 3, 5) scenario. We found that this approach works well for the designs with any sample size larger than 11.

Case 5: *The* (2, 3, 4, 5) *scenario.* In the (2, 3, 4, 5) scenario it is optimal to spread the profiles over respondents in blocks of size 4. In most cases where the sample size is not a multiple of 4, five profiles are assigned to at most three respondents. We obtained this result by computing some additional $\mathscr{D}$-optimal conjoint designs for sample sizes other than the ones in Table 5. Sometimes, however, three profiles are administered to one or two respondents. This is the case for the designs with a sample size of 30 and 70 for a degree of correlation of 0.1. For these instances, we found that using blocks of size 4 and two blocks of size 5 instead of the optimal blocking structure yields no redundant observations. In general, assigning four profiles to the respondents is a good option. If the sample size n is not a multiple of 4, it is advisable to assign five profiles to at most three respondents. That blocking structure can thus be used as an input to the fixed block size algorithm for the (2, 3, 4, 5) scenario without too large a loss in the $\mathscr{D}$-criterion value. Note that only designs with a sample size larger than 12 can be constructed in this way.

To conclude, for each of the five scenarios examined we found a general blocking structure supported by one predominant block size. This suggests that the following two-stage strategy can be adopted for generating conjoint designs without too large a penalty in terms of the $\mathscr{D}$-criterion value. First, construct some relatively small $\mathscr{D}$-optimal conjoint designs with the conjoint design algorithm to identify the optimal blocking structure. Next, provide this structure as an input to the faster fixed block size algorithm to compute the larger designs. In the next section, we show that the two-stage approach is more attractive from a computational point of view than using the conjoint design algorithm.

6.5. Computing times of $\mathscr{D}$-optimal conjoint designs

We now illustrate the huge time savings that can be achieved by applying the fixed block size algorithm with a good blocking structure as an input to generate $\mathscr{D}$-optimal conjoint designs for fairly large sample sizes. To do so, we compare the computing times for the designs in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios using the conjoint design

Table 6
Computing times for 1000 tries of the conjoint design algorithm and the fixed block size algorithm, and numbers of tries used to generate the $\mathscr{D}$-optimal conjoint designs in the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios

n	Scenario	Conjoint design algorithm		Fixed block size algorithm	
		Time/1000 Tries	# Tries	Time/1000 Tries	# Tries
20	(3, 3, 3, 3)	01:17	2000	00:02	1000
20	(2, 3, 3, 4)	01:05	2000	00:01	1000
24	(3, 3, 3, 3)	02:04	3000	00:02	2000
24	(2, 3, 3, 4)	01:24	3000	00:02	2000
30	(3, 3, 3, 3)	02:54	3000	00:03	2000
30	(2, 3, 3, 4)	02:02	3000	00:03	2000
36	(3, 3, 3, 3)	03:58	4000	00:04	2000
36	(2, 3, 3, 4)	02:47	4000	00:03	2000
40	(3, 3, 3, 3)	04:52	4000	00:05	2000
40	(2, 3, 3, 4)	03:05	4000	00:04	2000
50	(3, 3, 3, 3)	08:31	5000	00:08	3000
50	(2, 3, 3, 4)	04:36	5000	00:07	3000
60	(3, 3, 3, 3)	11:05	6000	00:10	3000
60	(2, 3, 3, 4)	07:14	6000	00:09	3000
70	(3, 3, 3, 3)	15:42	7000	00:15	4000
70	(2, 3, 3, 4)	09:18	7000	00:15	4000
72	(3, 3, 3, 3)	16:44	7000	00:15	4000
72	(2, 3, 3, 4)	10:43	7000	00:15	4000
81	(3, 3, 3, 3)	21:19	8000	00:21	5000
81	(2, 3, 3, 4)	12:51	8000	00:18	5000

The times are expressed in hours:minutes.

algorithm and the fixed block size algorithm. In the input to the fixed block size algorithm, we specified that blocks of size 3 were required. We registered the computing times of Fortran 77 implementations of the conjoint design algorithm and the fixed block size algorithm using a Dell personal computer with a 1.80 GHz Intel Processor and 256 MB RAM. Computing times for 1000 tries of the two algorithms are displayed in Table 6 along with the numbers of tries used to create the $\mathscr{D}$-optimal conjoint designs discussed above. We believe these numbers are sufficient so as not to be stuck in locally optimal designs. The table clearly shows that many more tries are required when the conjoint design algorithm is used instead of the fixed block size algorithm.

The table also shows that the computing times for 1000 tries of the conjoint design algorithm are long and grow exponentially with the sample size. In contrast, the computing times for 1000 tries of the fixed block size algorithm are much shorter and increase much slower with the sample size. This, combined with the fact that the conjoint design algorithm requires many more tries, makes it take much more time to generate the $\mathscr{D}$-optimal conjoint designs with the conjoint design algorithm than with the fixed block size algorithm. Particularly for the designs with a sample size of 40 and more, the fixed block size algorithm saves a lot of time.

6.6. *Replicating $\mathscr{D}$-optimal conjoint designs*

As an alternative way to quickly generate relatively large $\mathscr{D}$-optimal conjoint designs, a researcher might consider replicating smaller ones. If this design approach resulted in minor losses in the $\mathscr{D}$-criterion value, it would be desirable from a practical standpoint. This is because replicating a small $\mathscr{D}$-optimal conjoint design is cheaper both computationally and financially. The financial benefit would result from having at least two respondents rate each set of profiles so that fewer profiles need to be manufactured. To evaluate the performance of such replicated design plans, we calculate

Table 7
Replication schemes for small $\mathscr{D}$-optimal conjoint designs in the (3, 3, 3, 3), (2, 3, 3, 4), (2, 3, 3, 5) and (2, 3, 4, 5) scenarios

Scenario	n		
	40	60	72
(3, 3, 3, 3) and (2, 3, 3, 4)	2×20	3×20 2×30	3×24 2×36
(2, 3, 3, 5) and (2, 3, 4, 5)	2×20	3×20 2×30	–

the number of redundant observations associated with them. So, we determine how many observations would be saved if a $\mathscr{D}$-optimal conjoint design were applied whose $\mathscr{D}$-criterion value equals that of the replicated design plan.

To derive the number of redundant observations of a replicated design plan, we need to compute its $\mathscr{D}$-criterion value. The $\mathscr{D}$-criterion value of a design plan with n observations consisting of c replicates of a $\mathscr{D}$-optimal conjoint design with a smaller sample size n_s^* is given by $c \times \mathscr{D}_{n_s^*}^{\text{opt}}$, where $\mathscr{D}_{n_s^*}^{\text{opt}}$ denotes the $\mathscr{D}$-criterion value of the small $\mathscr{D}$-optimal conjoint design.

Table 7 shows the replication schemes we set up for the (3, 3, 3, 3), (2, 3, 3, 4), (2, 3, 3, 5) and (2, 3, 4, 5) scenarios. For the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios we investigated whether we could replicate the $\mathscr{D}$-optimal conjoint designs with a sample size of 20, 24, 30 and 36 to carry out experiments with 40, 60 and 72 observations. For the (2, 3, 3, 5) and (2, 3, 4, 5) scenarios we studied replications of the $\mathscr{D}$-optimal conjoint designs with a sample size of 20 and 30 to conduct experiments with 40 and 60 observations.

For the (3, 3, 3, 3) and (2, 3, 3, 4) scenarios it turns out that there are almost no redundant observations when the $\mathscr{D}$-optimal conjoint designs with a sample size of 20, 24, 30 and 36 are replicated. However, caution should be exercised in replicating these designs more than thrice as we observed one redundant observation from replicating the designs with 20 and 24 observations thrice. The larger the number of replications used, the larger the loss in the $\mathscr{D}$-criterion value and the larger the number of redundant observations.

We observed similar results for the replicated design plans in the (2, 3, 3, 5) and (2, 3, 4, 5) scenarios. For most design plans consisting of a $\mathscr{D}$-optimal conjoint design with a sample size of 20 or 30 there are no redundant observations. An exception, however, are the design plans obtained from the triple replication of the three $\mathscr{D}$-optimal conjoint designs with a sample size of 20 in the (2, 3, 3, 5) scenario (see Table 4). These plans have two redundant observations. This can be explained by the occurrence of blocks of size 4 in the optimal blocking structures of the $\mathscr{D}$-optimal conjoint designs at lower correlations. Also, these designs contain only as few as 20 different profiles.

We can conclude from these examples that it is efficient to replicate small $\mathscr{D}$-optimal conjoint designs for larger experiments if two conditions are met. First, the blocking structure of the small $\mathscr{D}$-optimal conjoint design has to match the predominant blocking structure of the given scenario. Second, the small conjoint design should contain an acceptable number of different profiles. The better this second condition is fulfilled, or the larger the sample size of the "small" design, the more replications that can be made.

6.7. Randomly distributing profiles from $\mathscr{D}$-optimal completely randomized designs

In practice, conjoint designs have often been constructed by generating a $\mathscr{D}$-optimal completely randomized design and assigning the profiles at random to the respondents. Although this approach is very fast, it is statistically inefficient. To illustrate this, we examine the performance of the $\mathscr{D}$-optimal completely randomized designs in the (3, 3, 3, 3) scenario when the profiles are randomly spread over respondents in blocks of size 3. If the sample size is not divisible by 3, four profiles are assigned to one or two respondents.

We randomly generated 1000 arrangements according to that blocking structure and computed their $\mathscr{D}$-criterion values for values of the correlation coefficient ranging from 0.1 to 0.9. We compared the average $\mathscr{D}$-criterion values with the $\mathscr{D}$-criterion values of the $\mathscr{D}$-optimal conjoint designs that are listed in Table A.1 of Appendix A. The average $\mathscr{D}$-efficiencies of the random arrangements decrease with the degree of correlation, from about 95% when $\rho = 0.1$ to

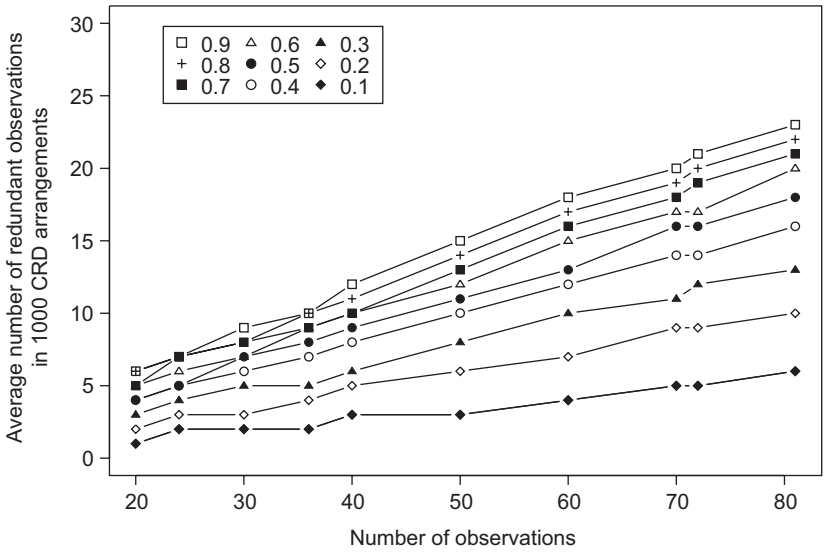

Fig. 2. Average numbers of redundant observations of 1000 random profile arrangements of the $\mathscr{D}$-optimal completely randomized designs (CRDs) in the (3, 3, 3, 3) scenario. Profiles are arranged according to a general blocking structure of three profiles per respondent.

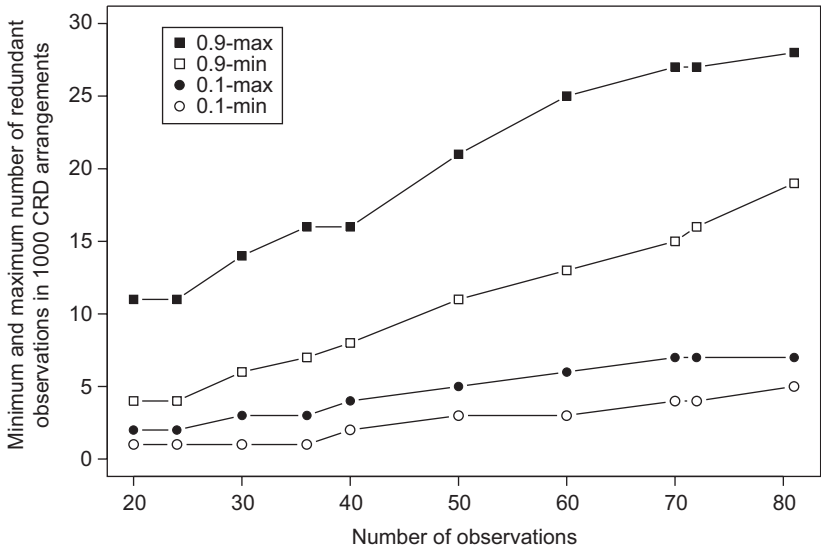

Fig. 3. Minimum and maximum numbers of redundant observations of 1000 random profile arrangements of the $\mathscr{D}$-optimal completely randomized designs (CRDs) in the (3, 3, 3, 3) scenario for $\rho = 0.1$ and 0.9. Profiles are arranged according to a general blocking structure of three profiles per respondent.

about 65% when $\rho = 0.9$. This suggests that randomly distributing the profiles of a completely randomized design over respondents is statistically not very efficient. To express the efficiency losses for the randomly generated designs, we computed the numbers of redundant observations corresponding to the average $\mathscr{D}$-criterion values. These numbers are shown in Fig. 2. As can be seen, the numbers of redundant observations are substantial and increase with the sample size and the degree of correlation.

Besides the average $\mathscr{D}$-criterion values, we also calculated the minimum and maximum $\mathscr{D}$-criterion values and the corresponding numbers of redundant observations. The minimum $\mathscr{D}$-criterion values result in the largest numbers of

redundant observations whereas the maximum $\mathscr{D}$-criterion values result in the smallest numbers. The maximum and minimum numbers of redundant observations for ρ values of 0.1 and 0.9 appear in Fig. 3. They serve as upper and lower bounds for the maximum and minimum numbers of redundant observations for the other degrees of correlation. We observe that the maximum values are more dispersed than the minimum ones, and that the spread between the maximum and minimum numbers increases with the degree of correlation.

7. Conclusion

In this paper, we constructed $\mathscr{D}$-optimal designs for conjoint experiments in which each respondent rates a small set of prototypes that must be tested in advance. We assumed that manufacturing these prototypes is expensive so that the number of design profiles, and thus also the number of observations n, is determined by the available budget. We used the linear model with a random respondent effect to take into account the fact that profile ratings from the same respondent are correlated. The resulting $\mathscr{D}$-optimal conjoint designs indicate how many respondents are required for a specific conjoint setting, and which and how many profiles each of them should evaluate.

We examined $\mathscr{D}$-optimal conjoint designs of various sample sizes in five scenarios. For each scenario, we were able to identify a dominant block size. We have also shown that searching for the $\mathscr{D}$-optimal number of respondents and the $\mathscr{D}$-optimal sets of profiles each of them has to rate is computationally very demanding, much more so than when the number of respondents and the number of profiles they have to rate are fixed in advance. Therefore, to generate large $\mathscr{D}$-optimal conjoint designs, we recommend to first determine the dominant blocking structure for the problem at hand by constructing some smaller $\mathscr{D}$-optimal conjoint designs. That blocking structure can then be given as an input to a faster design construction algorithm, that treats the blocking structure as fixed, to produce the large designs. We conjecture that good sample sizes for conjoint designs for a given scenario are multiples of the least common multiple of the numbers of attribute levels. This is because the blocking structures of such designs seem to correspond to the general blocking structure of the scenario which we exploit in the design construction algorithm. Verifying or disproving this conjecture by a rigorous study is a potential future research topic.

Another way to compute larger $\mathscr{D}$-optimal conjoint designs in a time-efficient manner is to replicate a smaller $\mathscr{D}$-optimal design that possesses the dominant blocking structure of the scenario under investigation and that involves a reasonable number of different profiles. This approach also requires fewer profiles to be manufactured because each distinct set of profiles can be rated by at least two respondents. This makes the experimental design cheaper financially.

Finally, we demonstrated that constructing $\mathscr{D}$-optimal completely randomized designs and arbitrarily distributing the profiles to the respondents is statistically inefficient on average.

A potentially interesting topic for further research, suggested by one of the reviewers, would be to check whether the blocking structures we found for the various scenarios assuming a main-effects model perform well for interaction-effects models too.

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Appendix A. $\mathscr{D}$-criterion values of the $\mathscr{D}$-optimal designs

The $\mathscr{D}$-criterion values of the $\mathscr{D}$-optimal conjoint designs and $\mathscr{D}$-optimal completely randomized designs for the five scenarios described in Table 1 are given in Table A.1.

Appendix B. $\mathscr{D}$-optimal conjoint designs for all candidate profiles

A $\mathscr{D}$-optimal conjoint design including all 81 candidate profiles in the (3, 3, 3, 3) scenario appears in Table B.1, while a $\mathscr{D}$-optimal conjoint design including all 72 candidate profiles in the (2, 3, 3, 4) scenario appears in Table B.2.

Table A.1
$\mathcal{D}$-criterion values of the $\mathcal{D}$-optimal conjoint designs ($\rho > 0$) and $\mathcal{D}$-optimal completely randomized designs ($\rho = 0$) for the five scenarios described in Section 6.1

n	ρ									
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
(a) (2, 2, 2, 2, 2, 2) scenario										
20	20.000	21.298	23.052	25.461	28.710	33.199	39.793	50.445	70.803	127.270
30	29.840	32.090	35.000	38.769	43.779	50.682	60.802	77.134	108.302	194.673
40	40.000	43.188	47.186	52.306	59.066	68.380	82.034	104.069	146.120	262.652
50	49.901	53.805	58.799	65.180	73.604	85.210	102.223	129.682	182.082	327.295
60	60.000	64.459	70.520	78.173	88.276	102.195	122.600	155.532	218.378	392.537
70	69.929	75.243	82.419	91.363	103.170	119.438	143.286	181.775	255.225	458.769
(b) (3, 3, 3, 3) scenario										
20	12.088	12.957	14.112	15.628	17.661	20.497	24.699	31.553	44.799	82.200
24	14.506	15.537	16.959	18.815	21.297	24.753	29.865	38.196	54.285	99.699
30	18.253	19.578	21.369	23.708	26.836	31.190	37.632	48.130	68.404	125.628
36	22.093	23.775	25.951	28.791	32.590	37.878	45.701	58.450	83.071	152.564
40	24.428	26.187	28.535	31.611	35.755	41.532	50.087	64.036	90.982	167.051
50	30.583	32.833	35.810	39.701	44.913	52.174	62.921	80.442	114.286	209.826
60	36.737	39.497	43.112	47.830	54.141	62.925	75.922	97.101	138.002	253.450
70	42.895	46.108	50.277	55.728	63.052	73.256	88.363	112.988	160.551	294.818
72	44.185	47.514	51.862	57.538	65.129	75.697	91.331	116.809	166.013	304.893
81	49.709	53.494	58.390	64.780	73.327	85.225	102.827	131.512	186.909	343.270
(c) (2, 3, 3, 4) scenario										
20	11.247	11.952	12.937	14.263	16.068	18.603	22.376	28.543	40.478	74.200
24	13.768	14.632	15.826	17.435	19.627	22.710	27.299	34.807	49.338	90.407
30	17.088	18.158	19.637	21.632	24.349	28.171	33.862	43.189	61.249	112.283
36	20.696	22.007	23.810	26.237	29.538	34.181	41.090	52.392	74.266	136.086
40	22.908	24.315	26.297	28.973	32.617	37.762	45.414	57.927	82.141	150.566
50	28.709	30.500	32.986	36.345	40.919	47.353	56.932	72.598	102.920	188.612
60	34.505	36.682	39.679	43.716	49.212	56.940	68.448	87.273	123.709	226.684
70	40.238	42.758	46.243	50.947	57.354	66.365	79.781	101.726	144.200	264.238
72	41.449	44.051	47.643	52.489	59.087	68.368	82.186	104.787	148.534	272.173
81	46.577	49.503	53.542	58.987	66.403	76.832	92.361	117.761	166.925	305.873
(d) (2, 3, 3, 5) scenario										
20	9.628	10.194	11.008	12.132	13.679	15.864	19.131	24.498	34.920	64.549
30	14.830	15.696	16.939	18.668	21.040	24.403	29.431	37.678	53.698	99.246
40	19.673	20.818	22.478	24.766	27.923	32.380	39.039	49.970	71.200	131.591
50	24.657	26.080	28.166	31.035	34.992	40.586	48.938	62.638	89.262	164.983
60	29.704	31.439	33.929	37.392	42.150	48.889	58.953	75.460	107.527	198.705
70	34.590	36.603	39.512	43.543	49.092	56.943	68.672	87.908	125.274	231.515
(e) (2, 3, 4, 5) scenario										
20	8.867	9.444	10.256	11.355	12.853	14.961	18.106	23.268	33.319	62.027
30	13.413	14.244	15.436	17.075	19.318	22.478	27.196	34.943	50.027	93.119
40	18.057	19.208	20.844	23.066	26.099	30.374	36.754	47.228	67.601	125.875
50	22.532	23.949	25.980	28.745	32.523	37.847	45.792	58.839	84.242	156.810
60	27.196	28.917	31.371	34.709	39.268	45.692	55.281	71.026	101.685	189.270
70	31.639	33.633	36.484	40.368	45.672	53.147	64.303	82.621	118.289	220.184

Appendix C. Number of redundant observations

The approach we propose to express losses in $\mathcal{D}$-efficiency when using suboptimal conjoint designs works as follows. For each setting of the attribute levels and degree of correlation, we regress the $\mathcal{D}$-criterion value of the $\mathcal{D}$-optimal conjoint designs in Table A.1 on the sample size. Denoting the sample size by n, the $\mathcal{D}$-criterion values by $\mathcal{D}_n^{\text{opt}}$, and

Table B.1
$\mathcal{D}$-optimal conjoint design involving all 81 candidate profiles in the (3, 3, 3, 3) scenario

Set	Attributes				Set	Attributes				Set	Attributes			
	1	2	3	4		1	2	3	4		1	2	3	4
1	1	1	1	1	10	1	1	2	1	19	1	1	3	1
1	3	2	2	2	10	3	2	3	2	19	3	2	1	2
1	2	3	3	3	10	2	3	1	3	19	2	3	2	3
2	2	1	1	1	11	2	1	2	1	20	2	1	3	1
2	1	2	2	2	11	1	2	3	2	20	1	2	1	2
2	3	3	3	3	11	3	3	1	3	20	3	3	2	3
3	3	1	1	1	12	3	1	2	1	21	3	1	3	1
3	2	2	2	2	12	2	2	3	2	21	2	2	1	2
3	1	3	3	3	12	1	3	1	3	21	1	3	2	3
4	1	2	1	1	13	1	2	2	1	22	1	2	3	1
4	3	3	2	2	13	3	3	3	2	22	3	3	1	2
4	2	1	3	3	13	2	1	1	3	22	2	1	2	3
5	2	2	1	1	14	2	2	2	1	23	2	2	3	1
5	1	3	2	2	14	1	3	3	2	23	3	1	2	3
5	3	1	3	3	14	3	1	1	3	23	1	3	1	2
6	3	2	1	1	15	3	2	2	1	24	3	2	3	1
6	2	3	2	2	15	2	3	3	2	24	2	3	1	2
6	1	1	3	3	15	1	1	1	3	24	1	1	2	3
7	1	3	1	1	16	1	3	2	1	25	1	3	3	1
7	3	1	2	2	16	3	1	3	2	25	3	1	1	2
7	2	2	3	3	16	2	2	1	3	25	2	2	2	3
8	2	3	1	1	17	2	3	2	1	26	2	3	3	1
8	1	1	2	2	17	1	1	3	2	26	1	1	1	2
8	3	2	3	3	17	3	2	1	3	26	3	2	2	3
9	3	3	1	1	18	3	3	2	1	27	3	3	3	1
9	2	1	2	2	18	1	2	1	3	27	2	1	1	2
9	1	2	3	3	18	2	1	3	2	27	1	2	2	3

the intercept and slope of the regression line by ψ and ω, respectively, the regression line can be written as

$$\mathcal{D}_n^{\text{opt}} = \psi + \omega n. \tag{C.1}$$

Consider now a suboptimal conjoint design with sample size n and $\mathcal{D}$-criterion value $\mathcal{D}_n^{\text{sub}}$ for a given scenario. Obviously, when plotted in a two-dimensional space formed by the sample size on the horizontal axis and the $\mathcal{D}$-criterion value on the vertical axis, the suboptimal design would fall below the regression line. This is shown in Fig. C.1. Now, on the regression line, there is one design which has the same $\mathcal{D}$-criterion value as the suboptimal design, that is $\mathcal{D}_n^{\text{sub}}$, but a smaller sample size. That design is equally good as the suboptimal design in terms of the $\mathcal{D}$-optimality criterion, but it requires fewer observations. We denote that number of observations by n^*. Fig. 1 graphically illustrates how the optimal design with the desired $\mathcal{D}$-criterion value $\mathcal{D}_n^{\text{sub}}$ can be obtained, and how the corresponding sample size n^* can be found. Mathematically, n^* can be determined using the following equation:

$$n^* = \frac{\mathcal{D}_n^{\text{sub}} - \psi}{\omega}. \tag{C.2}$$

The difference $n - n^*$ is a measure for the number of observations that can be saved by using a $\mathcal{D}$-optimal design instead of a suboptimal one. The larger the difference, the poorer the suboptimal design in terms of the $\mathcal{D}$-optimality criterion. We call the difference $n - n^*$ the number of redundant observations.

Table B.2
$\mathscr{D}$-optimal conjoint design involving all 72 candidate profiles in the (2, 3, 3, 4) scenario

Set	Attributes				Set	Attributes				Set	Attributes			
	1	2	3	4		1	2	3	4		1	2	3	4
1	2	1	3	1	9	2	3	3	1	17	2	1	1	3
1	1	2	2	3	9	1	1	1	2	17	2	3	2	4
1	1	3	1	2	9	1	2	2	4	17	1	2	3	1
2	2	1	3	2	10	2	3	3	2	18	2	1	1	4
2	1	2	1	4	10	1	1	2	1	18	2	3	2	3
2	1	3	2	1	10	1	2	1	3	18	1	2	3	2
3	2	1	3	3	11	2	3	3	3	19	2	1	2	1
3	1	2	1	1	11	1	1	1	4	19	2	3	1	2
3	1	3	2	4	11	1	2	2	2	19	1	2	3	3
4	2	1	3	4	12	2	3	3	4	20	2	1	1	2
4	1	2	1	2	12	1	1	1	3	20	2	3	2	1
4	1	3	2	3	12	1	2	2	1	20	1	2	3	4
5	2	2	3	1	13	2	2	1	4	21	2	1	2	2
5	1	1	2	4	13	2	3	2	2	21	2	2	1	3
5	1	3	1	3	13	1	1	3	1	21	1	3	3	1
6	2	2	3	2	14	2	2	2	3	22	2	1	1	1
6	1	1	2	3	14	2	3	1	1	22	2	2	2	4
6	1	3	1	4	14	1	1	3	2	22	1	3	3	2
7	2	2	3	3	15	2	2	2	2	23	2	1	2	4
7	1	1	2	2	15	2	3	1	4	23	2	2	1	1
7	1	3	1	1	15	1	1	3	3	23	1	3	3	3
8	2	2	3	4	16	2	2	2	1	24	2	1	2	3
8	1	1	1	1	16	2	3	1	3	24	2	2	1	2
8	1	3	2	2	16	1	1	3	4	24	1	3	3	4

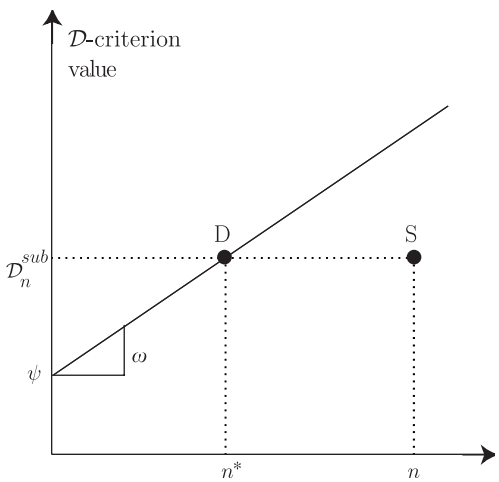

Fig. C.1. Graphical illustration of the determination of the number of redundant observations, $n - n^*$, using a regression line. D and S represent a $\mathscr{D}$-optimal design with sample size n^* and a suboptimal design with sample size n, respectively.

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