Communications in Statistics - Simulation and Computation

ISSN: 0361-0918 (Print) 1532-4141 (Online) Journal homepage: http://www.tandfonline.com/loi/lssp20

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Communications in Statistics—Simulation and Computation®, 43: 2614–2627, 2014

ISSN: 0361-0918 print / 1532-4141 online
DOI: 10.1080/03610918.2012.762386

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Using BBPSO Algorithm to Estimate the Weibull Parameters with Censored Data

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This article proposes the maximum likelihood estimates based on bare bones particle swarm optimization (BBPSO) algorithm for estimating the parameters of Weibull distribution with censored data, which is widely used in lifetime data analysis. This approach can produce more accuracy of the parameter estimation for the Weibull distribution. Additionally, the confidence intervals for the estimators are obtained. The simulation results show that the BB PSO algorithm outperforms the Newton–Raphson method in most cases in terms of bias, root mean square of errors, and coverage rate. Two examples are used to demonstrate the performance of the proposed approach. The results show that the maximum likelihood estimates via BBPSO algorithm perform well for estimating the Weibull parameters with censored data.

Keywords Aximum likelihood estimation; Bare bones particle swarm optimization; Censored data; Weibull distribution.

Mathematical Subject Classification 62N01; 62N02.

1. Introduction

The two-parameter Weibull distribution is widely used in reliability engineering. Many authors have proposed various methods in order to obtain these two parameters—the scale parameter, θ, and the shape parameter, β. The maximum likelihood estimator (MLE) is the most general method (Menon, 1963). Cohen (1965) investigated the maximum likelihood equations for the two-parameter Weibull distribution with complete, single-censored, and progressively censored data. Thoman et al. (1969) reported the MLE of the two-parameter Weibull distribution with complete data. Bain and Engelhardt (1991) extended the maximum likelihood estimation procedure to the two-parameter Weibull distribution with right censored data. Keats et al. (1997) reported a computer program for point and interval using maximum likelihood estimation for the two-parameter Weibull distribution with complete, singly censored, and multiply censored data. More discussions about the estimation of the Weibull parameters can be found in Murthy et al. (2004) and Balakrishnan and Kateri (2008).

Received February 15, 2012; Accepted December 12, 2012
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The MLE of the Weibull parameters with censored data is a nontrivial optimization problem in which the absolute maximum of the log-likelihood function cannot be obtained explicitly by analytical methods. Gradient methods, such as the Newton–Raphson method and the method of scoring, require finding the first partial derivatives of the log-likelihood function and the Hessian matrix. In addition, gradient methods may fail to converge if the Hessian matrix is not positive definite. Alternative optimization methods such as genetic algorithms (Thomas et al., 1995), simulated annealing (Abbasi et al., 2006), and a shuffled complex-evolution metropolis algorithm (SCEMA-UA) have been proposed to estimate the Weibull parameters with complete data. Lu et al. (2008) investigated the prediction intervals of the Weibull parameters with censored data. Tan (2009) presented a new approach combined the Weibull-to-exponential transformation and the expectation maximization (EM) algorithm for the Weibull distribution with interval data. The results show that the new approach has similar accuracy as the genetic algorithms. Cheng et al. (2010) presented a new algorithm for MLE with progressive type-I interval censored data. Klakattawi et al. (2011) applied Bayesian and non-Bayesian approaches to estimate the exponentiated Weibull parameters with progressive censored data. Shafay and Balakrishnan (2012) presented the Bayesian prediction intervals for type-I hybrid censored data.

Particle swarm optimization (PSO) is an evolutionary computation technique developed by Eberhart and Kennedy in 1995, which was inspired by social behavior of bird flocking and fish schooling (Kennedy and Eberhart, 2001). PSO is a computational intelligence-based technique that is not largely affected by the size and nonlinearity of the problem, and can converge to optimal solution on many problems where most analytical methods fail to converge. It does not require gradient of the objective function that can avoid complicated procedure of calculation. Therefore, the algorithm becomes simple and easy to implement. Kennedy (2003) proposed the Bare Bones PSO (BBPSO) algorithm in which two Gaussian random number generators are used to sampling the search space. Campos et al. (2009) presented the MLE via PSO algorithm for estimating the parameters of generalized gamma distribution with censored data. BBPSO is applied to estimating the parameters of mixed Weibull distributions (Krohling et al., 2010). Wu and Law (2011) proposed a hybrid strategy that combines the Cauchy mutation operator with Gaussian PSO in parameters selection of support vector machine.

This article presents an alternative algorithm called BBPSO, which can be applied to the MLE for the Weibull parameters with censored data. The rest of this paper is organized as follows. Section 2 provides the model under multiply censored data. Also, we provide how to calculate the confidence interval of estimates. In Section 3, we demonstrate the MLE via BBPSO algorithm to estimate the Weibull parameter with multiply censored data. A simulation study in Section 4 is conducted to assess the performance of the MLE via Newton–Raphson method and BBPSO algorithm for different parameters setting, sample sizes, and censoring levels. Two real examples were implemented to assess the performance of the MLE via two approaches in Section 5. Finally, we make the concluding remarks.

2. Model Under Multiply Censored Data

The cumulative distribution function (CDF) and probability density function (PDF) of the two-parameter Weibull distribution are given by

$$F(t;\theta,\beta) = 1 - e^{-(\frac{t}{\theta})^{\beta}} \tag{1}$$

and

$$f(t;\theta,\beta) = \frac{\beta t^{\beta-1}}{\theta^{\beta}} e^{-(\frac{t}{\theta})^{\beta}}, \tag{2}$$

where $t \geq 0, \theta > 0, \beta > 0$. The shape parameter, β, is also called the "characteristic life." The hazard function of the Weibull distribution is decreasing for $0 < \beta < 1$, increasing for $\beta > 1$, and constant for $\beta = 1$.

2.1. *The Multiply Censored Data*

Multiply censored means that items were censored at different times, with failure times intermixed with those censoring times. Multiply censored data are more common in the field, where units go into service at different times. Singly censored data are more common in controlled studies. With r failure terminated data $t_{1,f}, t_{2,f}, \ldots, t_{r,f}$ and m multiply censored data $t_{1,s}, t_{2,s}, \ldots, t_{m,s}$, the likelihood function is given by

$$L = f(t_{1,z}, t_{2,z} \ldots, t_{n,z}) = C \prod_{i=1}^{r} f(t_{i,f}) \prod_{j=1}^{m} [1 - F(t_{j,s})]. \tag{3}$$

Then, we have the log-likelihood function:

$$\ln L = \ln C + r \ln \beta - r\beta \ln \theta + (\beta - 1) \sum_{i=1}^{r} \ln t_{i,f} - \frac{\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}}{\theta^{\beta}}. \tag{4}$$

Maximum likelihood estimator (MLE) is one of the most popular methods for estimating the parameters of continuous distributions because of its attractive properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality. The maximum likelihood estimates of θ and β are obtained by setting the first partial derivatives of Eq. (4) to zero with respect to θ and β, respectively. These simultaneous equations are

$$\frac{\partial \ln L}{\partial \theta} = -\frac{r\beta}{\theta} + \beta\theta^{-\beta-1} \left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta} \right) = 0 \tag{5}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} &= \frac{r}{\beta} - r \ln \theta + \sum_{i=1}^{r} \ln t_{i,f} + \frac{\ln \theta}{\theta^{\beta}} \left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta} \right) \\ &\quad - \frac{1}{\theta^{\beta}} \left(\sum_{i=1}^{r} t_{i,f}^{\beta} \ln t_{i,f} + \sum_{j}^{m} t_{j,s}^{\beta} \ln t_{j,s} \right) = 0. \end{aligned} \tag{6}$$

Solving (5), we obtain the restricted MLE of θ for a given $\hat{\beta}$ as

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}}}{r} \right)^{1/\hat{\beta}}. \tag{7}$$

Substituting $\hat{\theta}$ into (6) we can obtain the MLE of β which is the solution of

$$\frac{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln t_{i,f} + \sum_{j}^{m} t_{j,s}^{\hat{\beta}} \ln t_{j,s}\right)}{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}}\right)} - \frac{1}{\hat{\beta}} = \frac{1}{r}\sum_{i=1}^{r} \ln t_{i,f}. \tag{8}$$

Newton–Raphson iteration is employed to solve Eq. (8).

When all test units are failure times, it becomes a complete data. Also, the singly censored data (Type-I or Type-II) is a special case of the multiply censored data. Here, we propose the initial values of θ and β which can be obtained by

$$\hat{\theta}_0 = \left(\frac{\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}_0} + \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}_0}}{r}\right)^{1/\hat{\beta}_0} \tag{9}$$

$$\hat{\beta}_0 = \left\{\frac{\frac{6}{\pi^2}\left[\sum_{i=1}^{r} \ln^2 t_{i,f} + \sum_{j=1}^{m} \ln^2 t_{j,s} - \frac{\left(\sum_{i=1}^{r} \ln t_{i,f} + \sum_{j=1}^{m} \ln t_{j,s}\right)^2}{n}\right]}{n-1}\right\}^{-1/2} \tag{10}$$

where $\hat{\beta}_0$ is modified from Menon's method (Menon, 1963).

2.2. *Confidence Interval*

The asymptotic variance-covariance matrix of θ and β is obtained by inverting the Fisher information matrix, $I = E[-\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}]$, $i, j = 1, 2$, where $\theta_1 = \theta, \theta_2 = \beta$ (Nelson, 1990). Thus, we have

$$\begin{bmatrix} \mathrm{Var}(\hat{\theta}) & \mathrm{Cov}(\hat{\theta}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\theta}, \hat{\beta}) & \mathrm{Var}(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} -\left.\frac{\partial^2 \ln L}{\partial\theta^2}\right|_{\hat{\theta},\hat{\beta}} & -\left.\frac{\partial^2 \ln L}{\partial\theta\partial\beta}\right|_{\hat{\theta},\hat{\beta}} \\ -\left.\frac{\partial^2 \ln L}{\partial\theta\partial\beta}\right|_{\hat{\theta},\hat{\beta}} & -\left.\frac{\partial^2 \ln L}{\partial\beta^2}\right|_{\hat{\theta},\hat{\beta}} \end{bmatrix}^{-1}. \tag{11}$$

The second derivates of the log-likelihood function for the multiply censored data with respect to β and θ are given as follows:

$$\frac{\partial^2 \ln L}{\partial\theta^2} = \frac{r\beta}{\theta^2} - \frac{\beta(\beta+1)}{\theta^{(\beta+2)}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}\right) \tag{12}$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -\frac{r}{\beta^2} - \frac{(\ln\theta)^2}{\theta^{\beta}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}\right) + 2\frac{\ln\theta}{\theta^{\beta}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} \ln t_{i,f} + \sum_{j=1}^{m} t_{j,s}^{\beta} \ln t_{j,s}\right)$$
$$- \frac{1}{\theta^{\beta}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta}(\ln t_{i,f})^2 + \sum_{j=1}^{m} t_{j,s}^{\beta}(\ln t_{j,s})^2\right) \tag{13}$$

$$\frac{\partial \ln L}{\partial\theta\partial\beta} = -\frac{r}{\theta} + \frac{1-\beta\ln\theta}{\theta^{\beta+1}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}\right) + \frac{\beta}{\theta^{\beta+1}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta}\ln t_{i,f} + \sum_{j}^{m} t_{j,s}^{\beta}\ln t_{j,s}\right). \tag{14}$$

Therefore, an approximate (1–α)100% confidence intervals for θ and β are obtained as

$$\hat{\theta} \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\theta})} \text{ and } \hat{\beta} \pm z_{1-\alpha/2}\sqrt{\mathrm{var}(\hat{\beta})}, \tag{15}$$

where $z_{1-\alpha/2}$ is the $[100(1-\alpha/2)]$ percentile of a standard normal distribution.

In addition, an alternative formula for interval estimation can be found in Murthy et al. (2004). The lower and upper confidence limits, confidence intervals for θ and β are given by $[\hat{\theta}/e^{(\frac{z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\theta})}}{\hat{\theta}})}, \hat{\theta}\times e^{(\frac{z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\theta})}}{\hat{\theta}})}]$ and $[\hat{\beta}/e^{(\frac{z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\beta})}}{\hat{\beta}})}, \hat{\beta}\times e^{(\frac{z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\beta})}}{\hat{\beta}})}]$, respectively.

3. The BBPSO Algorithm

Particle swarm optimization (PSO) is a population-based algorithm. It is initialized with a population of candidate solution. The swarm consists of m particles: each particle has a position $X_i = \{x_{i1}, x_{i2}, \ldots, x_{id}\}$, and a velocity $V_i = \{v_{i1}, v_{i2}, \ldots, v_{id}\}$, where $i = 1, 2, \ldots, n$ and moves through a n-dimensional search space, which is associated with the best solution (fitness) called personal best. Another best value tracked by the global version of the particle swarm optimizer is the overall best value, global best, and its location, obtained so far by any particle in the population. The updating of velocity and particle position can be obtained by using the following equations:

$$V_{id}^{k+1} = wV_{id}^{k} + c_1\cdot r_1\cdot\left(p_{id} - x_{id}^{k}\right) + c_2\cdot r_2\cdot\left(p_{gd} - x_{id}^{k}\right) \tag{16}$$

$$x_{id}^{k+1} = x_{id}^{k} + V_{id}^{k+1}, \tag{17}$$

where w is an inertia weight; c_1, c_2 are two positive constants called acceleration coefficients and r_1, r_2 are random number uniformly distributed in [0,1]. The inertia weight can be obtained by $w = \frac{2}{|2-c-\sqrt{c^2-4c}|}$, where $c = c_1 + c_2 > 4$ (Clerc and Kennedy, 2002). In the BBPSO algorithm, two Gaussian random number generators are used to sampling the search space based on the global best (gbest) and the personal best (pbest) particle are given as follows:

$$\theta_{I,J} = \mathrm{Normal}\left(\mu_{I,J}, \sigma_{I,J}^{2}\right), \tag{18}$$

where $\mu_{I,J} = (\mathrm{gbest}_J + \mathrm{pbest}_{I,J})/2$ and $\sigma_{I,J} = |\mathrm{gbest}_J - \mathrm{pbest}_{I,J}|$ for each variables $J = 1, 2, \ldots, k$ of the particle I.

The estimation of Weibull parameters using the BBPSO algorithm is given as follows:

Step 1: Set parameters: particle size = 20, maximum iterations = 100

Step 2: Using uniform distribution to initialize the position $\Omega_i = (\theta, \beta)_i$ of particles according to Eqs. (9)–(10) and $\underline{\Omega}_i$, and $\bar{\Omega}_i$, are the lower and upper bounds for the position, which are three times standard deviations of parameters from by Eq. (11). That is, we have

$$\Omega_i = \underline{\Omega}_i + (\bar{\Omega}_i - \Omega_i)U_i(0,1)$$

$$p_i = \Omega_i$$

Step 3: Compute $L(p_i)$ based on Eq. (4) and $p_g = \arg\max L(p_i)$.
Step 4: Do Iteration $= 1$. If the maximum iteration number is obtained, go to step 7, otherwise go to step 5.
Step 5: Update the position of each particle according to Eq. (17). Compute $L(\Omega_i)$ based on Eq. (4). If $L(\Omega_i) > L(p_i)$ then $p_i = \Omega_i$. If $L(\Omega_i) > L(p_g)$ then $p_g = \Omega_i$.
Step 6: iteration $+1$, and go to step 4.
Step 7: Output $p_g = \Omega^* = (\theta^*, \beta^*)$.

4. Simulation Study

Monte Carlo simulations have been implemented to assess the performance of two approaches: the Newton–Raphson method and the BBPSO algorithm. Complete and censored samples are randomly generated from a two-parameter Weibull distribution with the specified values of θ and β. Procedures for generating multiply censored samples are given as follows:

(1) Generate n random samples, $t_i, \forall i = 1, 2, \ldots, n$, from a Weibull distribution with the specified values of θ and β.
(2) Randomly select $r = n \times$ CL numbers from $t_i, \forall i = 1, 2, \ldots, n$ as failure samples, $t_i, \forall i \in S$, where CL is the censoring level; the remaining (n–r) numbers as censored samples $t_j, \forall j \in S^c$.
(3) Generate n random numbers w_i from a uniform distribution.
(4) Let $w_i = \{ \begin{matrix} 1, i \in S \\ w_i, i \in S^c \end{matrix}$. Then, set $t_i = t_i \times w_i$ to produce a multiple-censored sample.

The simulation included the conditions of sample size, parameter true value and censoring level. 10,000 replications were generated for each simulation run. We consider three major measures: the bias (bias), the root mean squared error (RMSE) and the coverage rate (CR). These measures are defined as following:

(1) $\text{bias}(\hat{\theta}) = \bar{\theta} - \theta$ and $\text{bias}(\hat{\beta}) = \bar{\beta} - \beta$,
(2) $\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}$ and $\text{RMSE}(\hat{\beta}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta)^2}$,
(3) The coverage rates of the 95% confidence interval for the parameters (θ, β) based on N replications,

where $\bar{\theta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$, $\beta = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i$ and $N = 10{,}000$.

For complete data, the simulation included the following conditions: sample size $n = 25, 50$; parameter true values $\theta = 1, 5$ and $\beta = 0.5, 1, 2$. For multiply censored data, the simulation included the following conditions: sample size $n = 25, 50$; parameter true values $\theta = 1, 5$ and $\beta = 0.5, 1, 2.0$; censoring level CL $= 0.2, 0.4, 0.6$, where CL represents the proportion of censored data. All simulation programs were written by MATLAB (2008). The simulation results using the MLE via Newton–Raphson method was obtained from the MATLAB's function. In addition, the formula of interval estimation is the same as Murthy et al. (2004). The simulation results for complete data are presented in Table 1. The following conclusions from the simulation study were observed:

The simulation results for complete data are presented in Table 1. The following conclusions were observed:

(1) In most simulation combinations for parameters θ and β, the BBPSO algorithm has similar accuracy as the Newton-Raphson method in terms of bias and RMSE.

Table 1
Bias, RMSE, and CR of parameter estimates for complete data

Parameter setting			Approach			
			Newton–Raphson		BBPSO	
n	θ	β	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
25	1	0.5	0.0585	0.0299	0.0584	0.0298
			0.4479	0.0881	0.4479	0.0880
			0.9299	0.9309	0.9062	0.9485
		1	0.0069	0.0597	0.0068	0.0595
			0.2113	0.1760	0.2113	0.1761
			0.9299	0.9309	0.9203	0.9485
			–0.0021	0.1194	–0.0021	0.1190
		2	0.1050	0.3522	0.1050	0.3522
			0.9299	0.9309	0.9278	0.9485
	5	0.5	0.2923	0.0299	0.2918	0.0298
			2.2396	0.0881	2.2395	0.0880
			0.9299	0.9309	0.9062	0.9485
		1	0.0344	0.0597	0.0341	0.0595
			1.0566	0.1761	1.0566	0.1761
			0.9299	0.9309	0.9203	0.9485
		2	–0.0104	0.1194	–0.0105	0.1190
			0.5249	0.3522	0.5249	0.3522
			0.9299	0.9309	0.9278	0.9485
50	1	0.5	0.0291	0.0144	0.0291	0.0143
			0.3076	0.0584	0.3075	0.0584
			0.9399	0.9430	0.9250	0.9506
			0.0034	0.0287	0.0034	0.0287
		1	0.1492	0.1169	0.1492	0.1169
			0.9399	0.9430	0.9349	0.9506
		2	–0.0011	0.0574	–0.0011	0.0573
			0.0744	0.2337	0.0744	0.2338
			0.9399	0.9430	0.9374	0.9506
		0.5	0.1456	0.0144	0.1431	0.0143
	5		1.5379	0.0584	1.5377	0.0584
			0.9399	0.9430	0.9250	0.9506
		1	0.0171	0.0287	0.0170	0.0287
			0.7462	0.1168	0.7462	0.1169
			0.9399	0.9430	0.9349	0.9506
		2	–0.0053	0.0574	–0.0054	0.0573
			0.3720	0.2337	0.3720	0.2338
			0.9399	0.9430	0.9374	0.9506

(2) The BBPSO algorithm gives more coverage rates compared with the Newton-Raphson method in all combinations of simulation conditions. That is, the formula of interval estimation should be based on Eq. (15).
(3) The accuracy of the estimations of θ and β are improved as expected as the sample size increases.

The simulation results for multiply censored data are presented in Tables 2–4. The following conclusions were observed:

(1) With respect to the performance of bias and RMSE for parameters θ and β, the BBPSO algorithm is slightly better than the Newton–Raphson method in most simulation combinations.
(2) The BBPSO algorithm gives more coverage rates compared with the Newton–Raphson method in all combinations of simulation conditions. That is, the formula of interval estimation should be based on Eq. (15).
(3) The accuracy of the estimations of θ and β are improved as expected as the sample size increases.
(4) It appears that the censoring level affects the estimates of θ and β. That is, the bias and RMSE of parameters θ and β using both approaches in all simulation combinations increase when the censoring level increases.
(5) The coverage rates of parameter β using BBPSO algorithm under different censoring levels are always close to the nominal value in all simulation combinations.

5. Illustrative Examples

To demonstrate and validate the proposed method, we apply it to published Weibull life data and compare the derived estimates with published results.

Example 1. *We consider a sample of $n = 30$ machining centers with $r = 20$ failures (Dai et al., 2003) and the lifetimes data is shown in Table 5.*

The maximum likelihood estimates of the two-parameter Weibull distribution using both algorithms have the same results which are and $\hat{\theta} = 376.99$ and $\hat{\beta} = 1.2009$ with $\ln L = -138.5883$ (see Table 6). The 95% confidence intervals for parameters θ and β using the Newton–Raphson method are obtained as (261.5128, 543.4589), and (0.8278, 1.7422), respectively. However, the 95% confidence intervals for parameters θ and β using the BBPSO algorithm are obtained as (239.1088, 514.8771), and (0.7541, 1.6477), respectively. The width of confidence intervals using the BBPSO algorithm is smaller than that of the Newton–Raphson method.

Example 2. *We consider a multiply censored data from Murthy et al. (2004) and the lifetime data shown in Table 7. The maximum likelihood estimates of the two-parameter Weibull distribution using both algorithms have the same results, which are $\hat{\theta} = 26292.85$ and $\hat{\beta} = 1.0584$ with $\ln L = -135.1527$ (see Table 8). The 95% confidence intervals for parameters θ and β using the Newton–Raphson method are obtained as (10552.07, 65534.46), and (0.6441, 1.7394), respectively. However, the 95% confidence intervals for parameters θ and β using the BBPSO algorithm are obtained as (2284.04, 50309.65), and (0.5327, 1.5842), respectively. The width of confidence intervals using the BBPSO algorithm is smaller than that of the Newton–Raphson method.*

Table 2
Bias, RMSE, and CR of parameter estimates for multiply censored data with $CL = 0.2$

Parameter setting			Approach			
			Newton–Raphson		BBPSO	
n	θ	β	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
25	1	0.5	0.4424	0.0496	0.4423	0.0495
			0.6054	0.1018	0.6053	0.1014
			0.8774	0.9150	0.9650	0.9497
			0.1370	0.1059	0.1368	0.1046
		1	0.2426	0.2036	0.2425	0.2013
			0.8963	0.9112	0.9362	0.9513
		2	0.0410	0.2093	0.0390	0.1690
			0.1146	0.4059	0.1141	0.3822
			0.9099	0.9126	0.9300	0.9611
	5	0.5	2.2118	0.0496	2.2113	0.0495
			3.0271	0.1018	3.0267	0.1014
			0.8774	0.9150	0.9650	0.9497
		1	0.6850	0.1059	0.6839	0.1046
			1.2132	0.2036	1.2127	0.2013
			0.8963	0.9112	0.9362	0.9513
		2	0.2052	0.2093	0.1948	0.1690
			0.5729	0.4059	0.5707	0.3822
			0.9099	0.9126	0.9300	0.9611
50	1	0.5	0.4091	0.0313	0.4091	0.0312
			0.4156	0.0662	0.4156	0.0662
			0.8240	0.9249	0.9573	0.9480
		1	0.1352	0.0678	0.1350	0.0674
			0.1706	0.1323	0.1706	0.1318
			0.8701	0.9193	0.9227	0.9450
		2	0.0431	0.1311	0.0402	0.0860
			0.0807	0.2636	0.0801	0.2541
			0.9058	0.9227	0.9366	0.9522
	5	0.5	2.0454	0.0313	2.0454	0.0312
			2.0781	0.0662	2.0780	0.0662
			0.8240	0.9249	0.9573	0.9480
		1	0.6760	0.0678	0.6756	0.0674
			0.8532	0.1323	0.8531	0.1318
			0.8701	0.9193	0.9227	0.9450
		2	0.2154	0.1311	0.2011	0.0860
			0.4033	0.2636	0.4004	0.2541
			0.9058	0.9227	0.9366	0.9522

Table 3
Bias, RMSE, and CR of parameter estimates for multiply censored data with CL = 0.4

			Approach			
Parameter setting			Newton–Raphson		BBPSO	
n	θ	β	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
25	1	0.5	1.1500	0.0751	1.1499	0.0746
			0.9154	0.1242	0.9154	0.1228
			0.6947	0.8916	0.9674	0.9420
			0.3312	0.1678	0.3308	0.1625
		1	0.2941	0.2504	0.2940	0.2401
			0.7776	0.8788	0.9009	0.9460
		2	0.1016	0.3363	0.0992	0.2235
			0.1288	0.4982	0.1282	0.4148
			0.8521	0.8802	0.9124	0.9710
	5	0.5	5.7502	0.0751	5.7495	0.0746
			4.5770	0.1242	4.5768	0.1228
			0.6947	0.8916	0.9674	0.9420
		1	1.6558	0.1678	1.6540	0.1625
			1.4706	0.2504	1.4699	0.2401
			0.7776	0.8788	0.9009	0.9460
		2	0.5082	0.3363	0.4959	0.2235
			0.6438	0.4982	0.6409	0.4148
			0.8521	0.8802	0.9124	0.9710
50	1	0.5	1.1132	0.0520	1.1132	0.0519
			0.6274	0.0797	0.6274	0.0795
			0.4383	0.8921	0.8419	0.9311
		1	0.3335	0.1188	0.3332	0.1159
			0.2071	0.1592	0.2070	0.1550
			0.6102	0.8751	0.7548	0.9271
		2	0.1057	0.2336	0.1022	0.0883
			0.0911	0.3155	0.0903	0.2717
			0.7700	0.8789	0.8669	0.9667
	5	0.5	5.5660	0.0520	5.5658	0.0519
			3.1372	0.0797	3.1371	0.0795
			0.4383	0.8921	0.8419	0.9311
		1	1.6674	0.1188	1.6661	0.1159
			1.0356	0.1592	1.0349	0.1550
			0.6102	0.8751	0.7548	0.9271
		2	0.5287	0.2336	0.5111	0.0883
			0.4553	0.3155	0.4515	0.2717
			0.7700	0.8789	0.8669	0.9667

Table 4
Bias, RMSE, and CR of parameter estimates for multiply censored data with $CL = 0.6$

Parameter setting			Approach			
			Newton–Raphson		BBPSO	
n	θ	β	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
25	1	0.5	2.8602	0.1115	2.8608	0.1093
			1.8658	0.1662	1.8654	0.1595
			0.3979	0.8662	0.9553	0.9413
		1	0.6737	0.2606	0.6743	0.2442
			0.4079	0.3378	0.4077	0.3009
			0.5768	0.8412	0.8319	0.9498
		2	0.1978	0.5452	0.2009	0.3343
			0.1556	0.6825	0.1554	0.4662
			0.7367	0.8356	0.8626	0.9846
	5	0.5	14.3008	0.1115	14.3042	0.1093
			9.3290	0.1662	9.3272	0.1595
			0.3979	0.8662	0.9553	0.9413
		1	3.3684	0.2606	3.3713	0.2442
			2.0397	0.3378	2.0383	0.3009
			0.5768	0.8412	0.8319	0.9498
		2	0.9890	0.5452	1.0044	0.3343
			0.7779	0.6825	0.7769	0.4662
			0.7367	0.8356	0.8626	0.9846
50	1	0.5	2.7810	0.0789	2.7816	0.0783
			1.2524	0.1030	1.2326	0.1015
			0.0753	0.8557	0.7456	0.9182
		1	0.6814	0.1888	0.6823	0.1808
			0.2871	0.2062	0.2872	0.1919
			0.2415	0.8117	0.4752	0.9122
		2	0.2057	0.3872	0.2138	0.1363
			0.1102	0.4074	0.1117	0.2878
			0.5179	0.8167	0.6932	0.9854
	5	0.5	13.9051	0.0789	13.9078	0.0783
			6.2621	0.1030	6.2631	0.1015
			0.0753	0.8557	0.7456	0.9182
		1	3.4071	0.1888	3.4117	0.1802
			1.4357	0.2062	1.4361	0.1919
			0.2415	0.8117	0.4752	0.9122
		2	1.0285	0.3872	1.0688	0.1363
			0.5510	0.4074	0.5583	0.2878
			0.5179	0.8167	0.6932	0.9854

Table 5
Machine center life data (Dai et al., 2003)

1.5+	10.5	32	39	45	50	84+	120	137.06	138.5
165+	176	209.33	224	248	261.25	267+	267.5	283.16+	332.5+
348	353	383+	387+	398	472	478	510	562+	700+

+indicates a censored value.

Table 6
The results for Example 1

Algorithm	Log-likelihood value	$\hat{\theta}$ (95% confidence interval)	$\hat{\beta}$ (95% confidence interval)
Newton-Raphson	–138.5883	376.99 (261.51, 543.456)	1.2009 (0.8278, 1.7422)
BBPSO	–138.5883	376.99 (239.11, 514.88)	1.2009 (0.7541, 1.6477)

Table 7
Multiply censored data (Murthy et al., 2004)

450	460+	1150	1150	1560+	1600	1660+	1850+	1850+	1850+
1850+	1850+	2030+	2030+	2030+	2070	2070	2080	2200+	3000+
3000+	3000+	3000+	3100	3200+	3450	3750+	3750+	4150+	4150+
4150+	4150+	4300+	4300+	4300+	4300+	4600	4850+	4850+	4850+
4850+	5000+	5000+	5000+	6100	6100+	6100+	6100+	6300+	6450+
6450+	6700+	7450+	7800+	7800+	8100+	8100+	8200+	8500+	8500+
8500+	8750	8750+	8750+	9400+	9900+	10100+	10100+	10100+	11500+

+indicates a censored value.

Table 8
The results for Example 2

Algorithm	Log-likelihood value	$\hat{\theta}$ (95% confidence interval)	$\hat{\beta}$ (95% confidence interval)
Newton–Raphson	–135.1527	26296.85 (10552.07, 65534.46)	1.0584 (0.6441.1.7394)
BBPSO	–135.1527	26296.85 (2284.04, 50309.65)	1.0584 (0.5327,1.5842)

6. Conclusions

In this study, we present the performance of the MLE via Newton–Raphson method and BBPSO algorithm for estimating the parameters of Weibull distribution with multiply censored data. In particular, we propose the initial values of θ and β based on Eqs. (9) and (10) which can provide more efficient in computation in terms of convergence rate. The simulation results show that the BBPSO algorithm performs well in most cases in terms of the bias and RMSE for multiply censored data. Also, the coverage rates using the BBPSO algorithm are better than the Newton–Raphson method in most cases. Thus, we could conclude that the formula of interval estimation should be based on Eq. (15). The estimation results of two real examples using the MLE via BBPSO algorithm are the same as the published results. Thus, we could conclude that the MLE via BBPSO algorithm is a good approach for estimating the parameters of the Weibull distribution with multiply censored data.

References

Abbasi, B., Jahromi, A. H. E., Arkat, J., Hosseinkouchack, M. (2006). Estimating the parameters of Weibull distribution using simulated annealing algorithm. *Applied Mathematics and Computation* 183:85–93.

Bain, L. J., Engelhardt, M. (1991). *Statististical Analysis of Reliability and Life-Testing Models*. New York: Marcel Dekker.

Balakrishnan, N., Kateri, M. (2008). On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics and Probability Letters* 78:2971–2975.

Campos, M., Krohling, R. A., Borges, P. (2009). Particle swarm optimization for inference procedures in the generalized Gamma family based on censored data. In: Mehnen, J., Köppen, M., Saad, A., Tiawri, A., eds. *Applications of Soft Computing*, AISC 58. Heidelberg: Springer, pp. 53–60.

Cheng, C., Chen, J., Li, Z. (2010). A new algorithm for maximum likelihood estimation with progressive type-I interval censored data. *Communications in Statistics – Simulation and Computation* 39:750–766.

Clerc, M., Kennedy, J. (2002). The particle swarm-explosion, stability, and convergence in a multidimensional complex space. *IEEE Transactions on Evolutionary Computation* 6:58–73.

Cohen, A. C. (1965). Maximum likelihood estimate in the Weibull distribution based on complete and on censored samples. *Technometrics* 7:579–588.

Dai, Y., Zhou, Y. F., Jai, Y. Z. (2003). Distribution of time between failures of machining center based on type I censored data. *Reliability Engineering and System Safety* 79:377–379.

Gong, Z. (2006). Estimation of mixed Weibull distribution parameters using the SCEM-UA algorithm: Application and comparison with MLE in automotive reliability analysis. *Reliability Engineering and System Safety* 91:915–922.

Keats, J. B., Lawrence, F. P., Wang, F. K. (1997). Weibull maximum likelihood parameter estimates with censored data. *Journal of Quality Technology* 29:105–110.

Kennedy, J. (2003). Bare bones particle swarms. In: *Proceedings of the IEEE Swarm Intelligence Symposium*, pp. 80–87.

Kennedy, J., Eberhart, R. (2001). *Swarm Intelligence*. San Francisco, CA: Morgan Kaufmann Publishers.

Klakattawi, H. S., Baharith, L. A., Al-Dayian, G. R. (2011). Bayesian and non-Bayesian estimations on the exponentiated modified Weibull distribution for progressive censored sample. *Communications in Statistics – Simulation and Computation* 40:1291–1309.

Krohling, R. A., Campos, M., Borges, P. (2010). Bare bones particle swarm applied to parameter estimation of mixed Weibull distribution. In: Gao, X. Z., et al., eds. *Applications of Soft Computing*, AISC 75. Heidelberg: Springer, pp. 411–422.

Lu, H. L., Wu, T. H., Lu, H. W. (2008). Prediction intervals for an ordered observation from Weibull distribution based on censored samples. *Communications in Statistics – Simulation and Computation* 38:288–307.

MATLAB. (2008). *Version 7.7.0*. Natick, MA: The MathWorks.

Menon, M. V. (1963). Estimation of the shape and scale parameter of the Weibull distribution. *Technometrics* 5:175–182.

Murthy, D. N. P., Xie, M., Jiang, R. (2004). *Weibull Models*. Hoboken, New Jersey: John Wiley and Sons.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plan*. New York: John Wiley and Sons.

Shafay, A. R., N. Balakrishnan, N. (2012). One and two sample Bayesian prediction intervals based on type-I hybrid censored data. *Communications in Statistics – Simulation and Computation* 41:65–88.

Thoman, D. R., Bain, L. J., Antle, C. E. (1969). Inferences on the parameters of the Weibull distribution. *Technometrics* 11:445–460.

Thomas, G. M., Gerth, R., Velasco, T., Rabelo, L. C. (1995). Using real-code genetic algorithms for Weibull parameter estimation. *Computers and Industrial Engineering* 29:377–381.

Tan, Z. (2009). A new approach to MLE of Weibull distribution with interval data. *Reliability Engineering and System Safety* 94:394–403.

Wu, Q., Law, R. (2011). Cauchy mutation based on objective variable of Gaussian particle swarm optimization for parameters selection of SVM. *Expert Systems with Applications* 38:6405–6411.