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A NOTE ON THE MATRIX-VARIATE F DISTRIBUTION*

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SUMMARY. Let S_0 and S_1 be independent random Wishart matrices, $S_i \sim W_p(n_i, \Sigma)$, $n_i \geqslant p$, Σ positive definite. It is shown that the statistic $(S_0+S_1)^{-1/2}S_1S_0^{-1}(S_0+S_1)^{1/2}$ has the usual orthogonally invariant matrix-variate F distribution and is independent of S_0+S_1, these properties holding for all Σ and all choices of the matrix square root $(S_0+S_1)^{1/2}$. This is not true of the statistic $S_0^{-1/2}S_1S_0^{-1/2}$ usually considered as a generalized F variate. The results are extended to the joint distribution of several matrix-variate F-statistics.

1. Introduction and theorem

If X_0 and X_1 are independent chi-square random variables, $X_i \sim \sigma^2\chi^2_{n_i}$, then $X_1/(X_0+X_1)$ has a Beta $\left(\frac{n_1}{2}, \frac{n_0}{2}\right)$ distribution, X_1/X_0 has an (unnormalized) F_{n_1, n_0} distribution, both statistics are independent of X_0+X_1, and these distributions and independence properties do not depend on the value of σ^2.

In the multivariate case, suppose S_0 and S_1 are independent $p\times p$ random Wishart matrices, $S_i \sim W_p(n_i, \Sigma)$, $n_i \geqslant p$, $\Sigma > 0$. [We use the following notation : for any symmetric $p\times p$ matrices C and D, write $C > D$ to indicate that $C-D$ is positive definite; $C^{1/2}$ denotes any square root of C, i.e., a $p\times p$ matrix such that $C^{1/2}(C^{1/2})' = C$, and $C^{-1/2} = (C^{1/2})^{-1}$.] It is well known (Khatri, 1970; Mitra, 1970; Olkin and Rubin, 1964) that the matrix-variate Beta statistic

$$B \equiv (S_0+S_1)^{-1/2}S_1(S_0+S_1)^{-1/2'} \qquad \ldots \quad (1.1)$$

has properties generalizing those of $X_1/(X_0+X_1)$, namely

(i) B has density

$$B(p, n_1, n_0) \equiv \frac{c(n_1, p)c(n_0, p)}{c(n_1+n_0, p)} \, |B|^{(n_1-p-1)/2}|I-B|^{(n_0-p-1)/2}, \qquad \ldots \quad (1.2)$$

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where $0 < B < I$, $B = B'$, and

$$1/c(n, p) = 2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(\frac{n-i+1}{2}\right);$$

(ii) B is independent of S_0+S_1;

(iii) Properties (i) and (ii) hold for all values of Σ and for all selections of the square root $(S_0+S_1)^{1/2}$ (provided the selection is made in a measurable way depending only on S_0+S_1, not on the individual values of S_0 and S_1; cf. Mitra, 1970, Section 6).

We remark that (1.2) shows that the distribution of B is orthogonally invariant, i.e., $B \sim \Gamma B \Gamma'$ for any fixed $p \times p$ orthogonal matrix Γ. Orthogonal invariance of the distribution of a symmetric random matrix B is a desirable property, for it enables one to apply a standard result (Anderson, 1958, Theorem 13.3.1, p. 318) to immediately obtain the joint density of the characteristic roots of B.

However, the analogous properties corresponding to (i), (ii), and (iii) do *not* hold for the statistic

$$F \equiv S_0^{-1/2} S_1 S_0^{-1/2'} \quad \ldots \quad (1.3)$$

usually considered as a matrix-valued generalized F statistic, because the distribution of F and its independence of S_0+S_1 depend heavily on the value of Σ and the selection of the square root $S_0^{1/2}$. For example, Olkin and Rubin (1964) show that if $S_0^{1/2}$ is taken to be the *symmetric* (and, by our definition, always positive definite) square root of S_0, then F has the orthogonally invariant density

$$F(p, n_1, n_0) \equiv \frac{c(n_1, p)c(n_0, p)}{c(n_1+n_0, p)} \frac{|F|^{(n_1-p-1)/2}}{|I+F|^{(n_1+n_0)/2}}, \quad \ldots \quad (1.4)$$

where $F > 0$, $F = F'$, provided that $\Sigma = aI$ for some positive number a, but that F is *not* independent of S_0+S_1 (Olkin and Rubin, 1964, Theorems 3.1 and 7.2). If $\Sigma \neq aI$ then the density of F is not given by (1.4), in fact is not expressible in closed form. On the other hand, if $S_0^{1/2}$ is taken to be the *lower triangular* square root of S_0, then F is independent of S_0+S_1 for all values of Σ (this follows from the proof of Theorem 3.2 in Olkin and Rubin (1964), although not explicitly stated there) and the density of F is expressible in closed form (see Equation (3.2) of Olkin and Rubin (1964), p. 262) not depending on Σ. However, in this case the density of F is *not* given by (1.4) and is not orthogonally invariant.

The main purpose of this note is to point out that the matrix-variate F-statistic

$$\begin{aligned}\tilde{F} &\equiv (S_0+S_1)^{-1/2}S_1S_0^{-1}(S_0+S_1)^{1/2}\\ &= (S_0+S_1)^{1/2\prime}S_0^{-1}\,S_1(S_0+S_1)^{-1/2\prime}\\ &= \tilde{F}', \qquad \ldots \quad (1.5)\end{aligned}$$

like F, a symmetric positive definite random matrix, should be considered a more natural generalization of the univariate F-statistic X_1/X_0, possessing properties analogous to (i), (ii) and (iii), namely

Theorem 1 : (i)* : *$\tilde{F}$ has density $F(p, n_1, n_0)$ given by* (1.4);

(ii)* : *$\tilde{F}$ is independent of S_0+S_1 ;*

(iii)* : *Properties* (i)* *and* (ii)* *hold for all values of Σ and for all selections of the square root $(S_0+S_1)^{1/2}$ (provided the selection is made in a measurable way depending only on S_0+S_1, not on the individual values of S_0 and S_1).*

Proof: Simply note that (1.5) is equivalent to

$$\tilde{F} = B(I-B)^{-1} = (I-B)^{-1}B \qquad \ldots \quad (1.6)$$

so that (dropping the tilde for the remainder of the proof)

$$B = F(I+F)^{-1} = (I+F)^{-1}F, \qquad \ldots \quad (1.7)$$

where B is given by (1.1). The Jacobian of this transformation is easily found by taking differentials

$$\begin{aligned}dB &= (dF)(I+F)^{-1}-F(I+F)^{-1}(dF)(I+F)^{-1}\\ &= (I+F)^{-1}(dF)(I+F)^{-1}\end{aligned}$$

(using the fact that $AA^{-1} = I$ implies $dA = -A^{-1}(dA)A^{-1}$), so that the Jacobian is given by

$$\left|\frac{\partial B}{\partial F}\right| = \left|\frac{\partial(dB)}{\partial(dF)}\right| = |I+F|^{-p-1} \qquad \ldots \quad (1.8)$$

(see Anderson, 1958, Lemma 7.2.1, p. 156). Substitute (1.7) into (1.2) and multiply by the Jacobian to obtain (1.4), verifying (i)*. Then (ii)* and (iii)* follow immediately from (ii) and (iii).

2. Extensions

Let $X_0, X_1, \ldots, X_k$ be independent univariate chi-square random variables, $X_i \sim \sigma^2\chi^2_{n_i}$. Then it is well known that

$$\frac{X_1}{X_0+X_1}, \frac{X_2}{X_0+X_1+X_2}, \ldots, \frac{X_k}{X_0+X_1+\ldots+X_k}, X_0+X_1+\ldots+X_k \quad \ldots \quad (2.1)$$

are $k+1$ mutually independent random variables, the first k having Beta distributions, and these distributions and independence properties do not depend on the value of σ^2. Olkin and Rubin (1964, Theorem 7.3) show that these facts generalize in the multivariate case as follows. Let $S_0, S_1, \ldots, S_k$ be independent $p \times p$ random Wishart matrices, $S_i \sim W_p(n_i, \Sigma)$, $n_i \geqslant p, \Sigma > 0$. Let

$$B_j = (S_0+\ldots+S_j)^{-1/2}S_j(S_0+\ldots+S_j)^{-1/2'}, \quad \ldots \quad (2.2)$$

$j = 1, \ldots, k$. Then

$$B_1, B_2, \ldots, B_k, \; S_0+S_1+\ldots+S_k \quad \ldots \quad (2.3)$$

are $k+1$ mutually independent random matrices, B_j has the orthogonally invariant distribution

$$B_j \sim B(p, n_j, n_0+\ldots+n_{j-1}), \quad \ldots \quad (2.4)$$

and these distributions and independence properties are valid for all values of Σ and all selections of the square roots $(S_0+\ldots+S_j)^{1/2}$ in (2.2) (subject to the usual proviso).

In the univariate case, the above facts have their immediate counterparts in terms of F-variates :

$$\frac{X_1}{X_0}, \frac{X_2}{X_0+X_1}, \ldots, \frac{X_k}{X_0+\ldots+X_{k-1}}, X_0+X_1+\ldots+X_k \quad \ldots \quad (2.5)$$

are mutually independent, the first k having (unnormalized) F distributions, and these properties do not depend on σ^2. In the multivariate case, however, the usual matrix-valued F-statistics

$$F_j = (S_0+\ldots+S_{j-1})^{-1/2}S_j(S_0+\ldots+S_{j-1})^{-1/2'}, \quad \ldots \quad (2.6)$$

($j = 1, \ldots, k$) do not enjoy the corresponding properties. That is, the joint distribution of the $k+1$ random matrices

$$F_1, F_2, \ldots, F_k, \; S_0+S_1+\ldots+S_k \quad \ldots \quad (2.7)$$

depend both on the value of Σ and the selection of square roots in (2.6). For example, if the lower triangular square roots are selected, then the $k+1$ random matrices in (2.7) are mutually independent for all Σ (cf. Olkin and

Rubin, 1964, Theorem 7.1), but the distribution of F_j, though free of Σ, is not $F(p, n_j, n_0+\ldots+n_{j-1})$. If the symmetric square roots are selected in (2.6), however, then the matrices are not independent for any value of Σ (cf. Olkin and Rubin, 1964, Theorem 7.2), and the distribution of F_j depends on Σ, reducing to $F(p, n_j, n_0+\ldots+n_{j-1})$ only if $\Sigma = aI$ for some $a > 0$.

However, if instead of F_j in (2.6) we consider

$$\widetilde{F}_j \equiv (S_0+\ldots+S_j)^{-1/2}S_j[S_0+\ldots+S_{j-1}]^{-1}(S_0+\ldots+S_j)^{1/2}, \qquad \ldots \quad (2.8)$$

$(j = 1, \ldots, k)$ where $\widetilde{F}_j > 0$, $\widetilde{F}_j = \widetilde{F}'_j$, then

$$\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_k, \; S_0+S_1+\ldots+S_k \qquad \ldots \quad (2.9)$$

enjoy properties extending those of (2.5) :

Theorem 2 : *The $k+1$ random matrices in* (2.9) *are mutually independent, $\widetilde{F}_j$ has the orthogonally invariant distribution*

$$\widetilde{F}_j \sim F(p, n_j, n_0+\ldots+n_{j-1}), \qquad \ldots \quad (2.10)$$

and these properties hold for all values of Σ and all selections of the square roots $(S_0+\ldots+S_j)^{1/2}$ (provided the selection is made in a measurable way depending only on the value of $S_0+\ldots+S_j$).

Proof : The proof is similar to that of Theorem 1, using the fact that

$$\widetilde{F}_j = B_j(I-B_j)^{-1} \qquad \ldots \quad (2.11)$$

and the properties of (2.3) described above.

The results of Section 1 can be generalized in a second way. First, returning to the univariate case, for every value of σ^2 and k the random variables

$$Z_j = \frac{X_j}{X_0+X_1+\ldots+X_k}, \qquad \ldots \quad (2.12)$$

$(j = 1, \ldots, k)$ are independent of $X_0+X_1+\ldots+X_k$ and $(Z_1, \ldots, Z_k)$ has joint density

$$\frac{\Gamma(n_0+\ldots+n_k)}{\Gamma(n_0)\ldots\Gamma(n_k)} \left(\prod_{j=1}^{k} z_j^{(n-2)/2} \right) \left(1-\sum_{j=1}^{k} z_j \right)^{(n_0-2)/2} \qquad \ldots \quad (2.13)$$

where $z_j > 0$ and $\Sigma z_j < 1$, a Dirichlet distribution (cf. Wilks, 1962, Section 7.7). In the multivariate case, this generalizes as follows (cf. Olkin and Rubin, 1964, Theorem 3.3)

Let $$W_j = (S_0+\ldots+S_k)^{-1/2}S_j(S_0+\ldots+S_k)^{-1/2'}, \qquad \ldots \quad (2.14)$$

$(j = 1, \ldots, k)$. Then for all Σ and all selections of the square roots in (2.14), $(W_1, \ldots, W_k)$ is independent of $S_0+S_1+\ldots+S_k$ and has joint density

$$c\left(\prod_{j=1}^{k} |W_j|^{(n_j-p-1)/2}\right)\left|I-\sum_{j=1}^{k} W_j\right|^{(n_0-p-1)/2}, \qquad \ldots \quad (2.15)$$

a generalized Dirichlet distribution, where $W_j = W_j'$, $W_j > 0$, $\Sigma W_j < I$,

and
$$c = \left[\prod_{j=0}^{k} c(n_j, p)\right] / c(n_0+\ldots+n_k, p). \qquad \ldots \quad (2.16)$$

The distribution of $(W_1, \ldots, W_k)$ is invariant under a common orthogonal transformation, i.e., $(W_1, \ldots, W_k) \sim (\Gamma W_1 \Gamma', \ldots, \Gamma W_k \Gamma')$.

Now consider the counterparts of the facts in the preceding paragraph for F-variates. In the univariate case, for every value of σ^2 the k random variables

$$Y_j = \frac{X_j}{X_0}, \qquad \ldots \quad (2.17)$$

$(j = 1, \ldots, k)$ are independent of $X_0+X_1+\ldots+X_k$, and $(Y_1, \ldots, Y_k)$ has joint density

$$\frac{\Gamma(n_0+\ldots+n_k)}{\Gamma(n_0)\ldots\Gamma(n_k)} \frac{\prod_{j=1}^{k} y_j^{(n_j-2)/2}}{\left(1+\sum_{j=1}^{k} y_j\right)^{n/2}} \qquad \ldots \quad (2.18)$$

where $y_j > 0$ and $n = n_0+n_1+\ldots+n_k$ ((2.18) is occasionally called an inverted Dirichlet, or multivariate-F, distribution). In the multivariate case, the usual matrix-variate F-statistics

$$V_j = S_0^{-1/2} S_j S_0^{-1/2'} \qquad \ldots \quad (2.19)$$

$(j = 1, \ldots, k)$ again do not enjoy properties corresponding to those of Y_j, $j = 1, \ldots, k$, in (2.17). If the symmetric square root $S_0^{1/2}$ is selected in (2.19) and if $\Sigma = aI$, then $(V_1, \ldots, V_k)$ has the orthogonally invariant joint density

$$c\,\frac{\prod_{j=1}^{k} |V_j|^{(n_j-p-1)/2}}{\left|I+\sum_{j=1}^{k} V_j\right|^{n/2}}, \qquad \ldots \quad (2.20)$$

where $V_j = V_j'$, $V_j > 0$, and c is given by (2.16), but this fails if $\Sigma \neq aI$; furthermore, $(V_1, \ldots, V_k)$ is not independent of $S_0+S_1+\ldots+S_k$ (cf. Olkin and Rubin, 1964, Theorems 3.1 and 7.2). If $S_0^{1/2}$ is taken to be the lower triangular square root in (2.19), then $(V_1, \ldots, V_k)$ is independent of $S_0+S_1+\ldots+S_k$ and the distribution of the former does not depend on Σ, but is not given by (2.20) and is not orthogonally invariant (cf. Olkin and Rubin, 1964, Theorem 3.2; note that it follows from the proof of that theorem, although not explicitly stated there, that $Y \equiv S_0+S_1+\ldots+S_k$ is independent of $(U_1, \ldots, U_k) \equiv (V_1, \ldots, V_k)$, the latter our notation).

If, however, we consider the alternative matrix-variate F-statistics

$$\begin{aligned}\tilde{V} &\equiv W_0^{-1/2} W_j W_0^{-1/2} \\ &\equiv \left[\left(\sum_{j=0}^{k} S_j\right)^{-1/2} S_0 \left(\sum_{i=0}^{k} S_i\right)^{-1/2'}\right]^{-1/2} \left(\sum_{i=0}^{k} S_i\right)^{-1/2} S_j \left(\sum_{i=0}^{k} S_i\right)^{-1/2'} \\ &\qquad \times \left[\left(\sum_{i=0}^{k} S_i\right)^{-1/2} S_0 \left(\sum_{i=0}^{k} S_i\right)^{-1/2'}\right]^{-1/2} \\ &\equiv \left(I - \sum_{j=1}^{k} W_j\right)^{-1/2} W_j \left(I - \sum_{j=1}^{k} W_j\right)^{-1/2}, \qquad \ldots \quad (2.21)\end{aligned}$$

$(j = 1, \ldots, k)$, where $\left(\sum_0^k S_i\right)^{1/2}$ is an arbitrary square root but $W_0^{1/2}$ is the *symmetric* square root of $W_0 \equiv I - \sum_1^k W_j$, then $(\tilde{V}_1, \ldots, \tilde{V}_k)$ does enjoy properties extending those of $(Y_1, \ldots, Y_k)$:

Theorem 3 : (a) $(\tilde{V}_1, \ldots, \tilde{V}_k)$ *is independent of* $\sum_0^k S_i$ *and has the orthogonally invariant joint density given by* (2.20). *These properties hold for all values of* $\Sigma > 0$ *and all selections of the square root* $\left(\sum_0^k S_i\right)^{1/2}$ (*provided the selection depends measurably on* $\sum_0^k S_i$ *alone*).

(b) *If* $\tilde{V}_j$ *is defined as* $W_0^{-1/2} W_j W_0^{-1/2'}$ *where* $W_0^{1/2}$ *is now taken to be an arbitrary* (*measurable, etc.*) *square root, then all statements in part* (a) *continue to hold, except that the joint density of* $(\tilde{V}_1, \ldots, \tilde{V}_k)$ *depends on the selection of the square root function* $W_0^{1/2}$ *and is not necessarily orthogonally invariant, although still free of* Σ.

Proof: Since $(\tilde{V}_1, \ldots, \tilde{V}_k)$ is a function of $(W_1, \ldots, W_k)$, it is independent of $\sum_0^k S_i$ for all Σ and its distribution does not depend on Σ. Therefore, all statements in (a) and (b) follow, except for those concerning the joint density of $(\tilde{V}_1, \ldots, \tilde{V}_k)$. The derivation of the joint density that follows is essentially a reversal of the steps of the proof of Theorem 3.4 in Olkin and Rubin (1964). We present some detail in order to demonstrate why the symmetric square root $W_0^{1/2}$ is required to yield the joint density (2.20). [For their Theorem 3.4, Olkin and Rubin state (just above Theorem 3.3) that—in their notation—$\left(I+\sum_1^k V_j\right)^{1/2}$ may be an arbitrary square root and $(Z_1, \ldots, Z_k)$ still will have the joint density (3.10) of their paper. This is not the case, however, for essentially the same reason that we require the symmetric square root $W_0^{1/2}$ to obtain the joint density (2.20).]

Suppose for the moment that $\tilde{V}_j$ is defined in terms of an arbitrary square root $W_0^{1/2}$, so that $\tilde{V}_j = W_0^{-1/2} W_j W_0^{-1/2'}$. The joint density of $(W_1, \ldots, W_k)$ is given by (2.15). First transform from $(W_1, \ldots, W_k)$ to $(\tilde{V}_1, \ldots, \tilde{V}_{k-1}, W_0)$. After some manipulation it is found that the Jacobian is given by

$$J_1 \equiv \left|\frac{\partial(W_1, \ldots, W_k)}{\partial(V_1, \ldots, V_{k-1}, W_0)}\right| = |W_0|^{(k-1)(p+1)/2} \qquad \ldots \quad (2.22)$$

(we omit the tildes for the remainder of the proof).

Since
$$\begin{aligned} W_k &= I - W_0 - \sum_{i=1}^{k-1} W_j \\ &= I - W_0 - \sum_{i=1}^{k-1} W_0^{1/2} V_j W_0^{1/2'} \\ &= W_0^{1/2}\left[W_0^{-1/2} W_0^{-1/2'} - I - \sum_{i=1}^{k-1} V_j\right] W_0^{1/2'}, \end{aligned} \qquad \ldots \quad (2.23)$$

the joint density of $(V_1, \ldots, V_{k-1}, W_0)$ is therefore

$$c\left[\prod_{j=1}^{k-1} |V_j|^{(n_j-p-1)/2}\right] \left|W_0^{-1/2} W_0^{-1/2'} - I - \sum_{i=1}^{k-1} V_j\right|^{(n_k-p-1)/2} |W_0|^{(n-2p-2)/2} \qquad \ldots \quad (2.24)$$

on the appropriate range. Note that, since $W_0 = W_0^{1/2} W_0^{1/2'}$,

$$W_0^{-1} = W_0^{-1/2'} W_0^{-1/2} \neq W_0^{-1/2} W_0^{-1/2'} \qquad \ldots \quad (2.25)$$

in general, unless $W_0^{1/2}$ is symmetric. Next transform from $(V_1, \ldots, V_{k-1}, W_0)$ to $(V_1, \ldots, V_{k-1}, V_k)$, where

$$V_k = W_0^{-1/2} W_k W_0^{-1/2'} = W_0^{-1/2} W_0^{-1/2'} - I - \sum_{i=1}^{k-1} V_j. \qquad \ldots \quad (2.26)$$

At this stage we must specify which square root $W_0^{1/2}$ is to be selected, for the Jacobian $|\partial W_0/\partial V_k|$ depends heavily on this selection. If $W_0^{1/2}$ is taken to be the symmetric square root then (2.26) becomes

$$V_k = W_0^{-1}-I-\sum_{i=1}^{k-1} V_j \quad \ldots \quad (2.27)$$

and
$$\left|\frac{\partial W_0}{\partial V_k}\right| = |W_0|^{p+1} = \left|I+\sum_{i=1}^{k} V_j\right|^{-p-1}. \quad \ldots \quad (2.28)$$

Now substitute (2.27) into (2.24) and multiply by (2.28) to obtain the joint density (2.20) for $(V_1, \ldots, V_k)$. If, instead, we selected $W_0^{1/2}$ to be the lower triangular square root, we would obtain the non-orthogonally invariant joint density given by equation (3.6) of Olkin and Rubin (1964, Theorem 3.2).

Remark 1 : The definition of $\tilde{V}_j$ in (2.21) in terms of the underlying Wishart matrices $S_0, \ldots, S_k$ is not as neat as we would like, since it involves double, or iterated, matrix square roots (compare to the expressions for $\widetilde{F}$ in (1.5) and $\widetilde{F}_j$ in (2.8)). A simple expression for $\tilde{V}_j$ would result if we could define $\tilde{V}_j$ to be $W_jW_0^{-1}$ (compare (1.6) and (2.11)), but $W_jW_0^{-1} \neq W_0^{-1}W_j$ if $k \geqslant 2$, so this definition of $\tilde{V}_j$ does not yield a symmetric matrix.

Remark 2 : The transformations (1.6), (1.7), and (2.21) were introduced by Olkin (1959, Section 4) to evaluate generalized Beta and generalized Dirichlet integrals.

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