PDE Derivation and Finite Difference Scheme for Arithmetic Average Asian Call Option

1 PDE Derivation for Arithmetic Average Fixed-Strike Asian Option

We consider a fixed-strike Asian call option whose payoff at maturity T is given by:

$$\max\left(\bar{A} - K, 0\right) \quad \text{where} \quad \bar{A} = \frac{1}{T}\int_0^T S(u)\,du$$

Let $V(S, A, t)$ denote the price of the option at time t, where:

- S is the current asset price,
- A is the running arithmetic average up to time t,
- $t \in [0, T]$ is the current time.

The underlying asset follows a geometric Brownian motion:

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

The average A_t evolves as:

$$A_t = \frac{1}{t}\int_0^t S(u)\,du \quad \Rightarrow \quad dA_t = \frac{S_t - A_t}{t}dt$$

Applying Itô's Lemma

Applying Itô's lemma to $V(S, A, t)$, we obtain:

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial A}dA + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{S - A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt + \sigma S\frac{\partial V}{\partial S}dW \end{aligned}$$

Risk-Neutral Valuation

In the risk-neutral world, replace $\mu \to r$, and since $e^{-rt}V$ is a martingale, the drift must vanish:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{S - A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

Backward Time Transformation

Introduce the backward time variable $\tau = T - t$, so $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau}$. Substituting:

$$-\frac{\partial V}{\partial \tau} + rS\frac{\partial V}{\partial S} + \frac{S-A}{T-\tau}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Rewriting:

$$\frac{\partial V}{\partial \tau} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{S-A}{T-\tau}\frac{\partial V}{\partial A} - rV$$

2 Finite Difference Scheme (Explicit)

Let $V_{i,j}^n \approx V(S_i, A_j, \tau_n)$ on the discrete grid:

$$S_i = i\Delta S, \quad A_j = j\Delta A, \quad \tau_n = n\Delta\tau$$

We approximate derivatives using central differences in space and forward difference in time.

Discretization

$$\begin{aligned}
\frac{\partial V}{\partial \tau} &\approx \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta\tau} \\
\frac{\partial V}{\partial S} &\approx \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta S} \\
\frac{\partial^2 V}{\partial S^2} &\approx \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{(\Delta S)^2} \\
\frac{\partial V}{\partial A} &\approx \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta A}
\end{aligned}$$

Explicit Scheme

$$V_{i,j}^{n+1} = V_{i,j}^n + \Delta\tau\left[rS_i \cdot \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta S} + \frac{1}{2}\sigma^2 S_i^2 \cdot \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{\Delta S^2} + \frac{S_i - A_j}{T - \tau_n} \cdot \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta A} - rV_{i,j}^n\right]$$

Final and Boundary Conditions

At maturity ($\tau = 0$, i.e., $t = T$):

$$V_{i,j}^0 = \max(A_j - K, 0)$$

Boundary conditions:

$$V(0, A, \tau) = 0, \quad V(S, 0, \tau) = 0, \quad V(S_{\max}, A, \tau) \approx S_{\max} - K$$