PDE Derivation for Floating-Strike Arithmetic Average Asian Option

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Payoff Definition

For a floating strike arithmetic average Asian option, the payoff is:

$$\max\left(S_T - \frac{1}{T}\int_0^T S(u)\,du,\ 0\right) = (S_T - A_T)^+$$

PDE in 2 Dimensions

Let the value function be $V(t, S, I)$, where:

- S is the current stock price,
- $I = \int_0^t S(u)\,du$ is the accumulated average,
- $A = \frac{I}{t}$ is the average at time t.

Then the PDE becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial I} = 0$$

Similarity Reduction: 2D $\rightarrow\longrightarrow$ 1D

Let us define a new variable $x = \frac{I}{S}$, and define the transformed function:

$$W(t, x) = V(S, I, t)$$

We then compute the derivatives using the chain rule:

$$\begin{aligned}
V_t &= W_t + \frac{\partial W}{\partial x}\frac{\partial x}{\partial t} \\
V_S &= \frac{\partial W}{\partial x}\frac{\partial x}{\partial S} = -\frac{I}{S^2}W_x = -\frac{x}{S}W_x \\
V_I &= \frac{\partial W}{\partial x}\frac{\partial x}{\partial I} = \frac{1}{S}W_x \\
V_{SS} &= \frac{\partial}{\partial S}\left(-\frac{x}{S}W_x\right) = \frac{2x}{S^2}W_x + \frac{x^2}{S^2}W_{xx}
\end{aligned}$$

Substitute back into the PDE:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial I} = 0 \\
\Rightarrow\ & W_t + \frac{1}{2}\sigma^2 x^2 W_{xx} + (r-q)xW_x - qW = 0
\end{aligned}$$

Final PDE in $W(x,t)$**W(x,t)**

We now have the one-dimensional PDE:

$$W_t + \frac{1}{2}\sigma^2 x^2 W_{xx} + (r-q)xW_x - qW = 0$$

Time Reversal Transformation

To solve backward in time, define $\tau = T - t$, then:

$$\frac{\partial W}{\partial t} = -\frac{\partial W}{\partial \tau}$$

The PDE becomes:

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} + (r-q)x\frac{\partial W}{\partial x} - qW$$

Initial and Boundary Conditions

- Terminal payoff: at $\tau = 0$ (i.e., $t = T$):

$$W(x,0) = \left(1 - \frac{x}{T}\right)^+$$

- As $x \to 0$: the PDE simplifies to:

$$-W_\tau + W_x - qW = 0$$

- As $x \to x_{\max}$, $W \to 0$

Finite Difference Scheme Outline

We use an explicit scheme with central differences in space:

$$\begin{aligned}
&\text{Forward in time:} && \frac{W_i^{n+1} - W_i^n}{\Delta\tau} \\
&\text{Central in space:} && W_x \approx \frac{W_{i+1}^n - W_{i-1}^n}{2\Delta x}, \quad W_{xx} \approx \frac{W_{i+1}^n - 2W_i^n + W_{i-1}^n}{\Delta x^2}
\end{aligned}$$