Weekly Assignment 1

22.08.2024 to 28.08.2024

The book for the course is Algorithm Design by Kleinberg and Tardos and this week, we will cover exercises $1, 2, 5$, and 8 from Chapter 1 in this book:

1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

2. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

5. The Stable Matching Problem, as discussed in the text, assumes that all men and women have a fully ordered list of preferences. In this problem we will consider a version of the problem in which men and women can be indifferent between certain options. As before we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that w prefers m to m' if m is ranked higher than m' on her preference list (they are not tied). With indifferences in the rankings, there could be two natural notions for stability. And for each, we can ask about the existence of stable matchings, as follows.

   (a) A strong instability in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability; or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

   (b) A weak instability in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

      – m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or

      – w prefers m to m', and m either prefers w to w' or is indifferent between these two choices. In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

8. For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m'? (We can ask the same question for men, but will focus on the case of women for purposes of this question.) Resolve this question by doing one of the following two things:

   (a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

   (b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.