INF367 Mandatory Assignment 1

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1 Coding part

Can be found in **notebook**

2 Manual Tasks

2.1 Quantum States and Quantum Gates

2.1.1 Express the state and show entanglement

Rewrite $|\phi\rangle$ in the standard basis:

$$\begin{aligned}
|\phi\rangle &= \frac{1}{\sqrt{3}}(|\phi^+\rangle + |\phi^-\rangle - i\,|\psi^-\rangle) \\
&= \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) - \frac{i}{\sqrt{2}}(|01\rangle - |10\rangle)) \\
&= \frac{1}{\sqrt{3}}(\frac{2}{\sqrt{2}}\,|00\rangle - \frac{i}{\sqrt{2}}(|01\rangle - |10\rangle)) \\
&= \frac{1}{\sqrt{6}}(2\,|00\rangle - i(|01\rangle - |10\rangle))
\end{aligned}$$

Show that it is entangled:

$$\begin{aligned}
|\phi\rangle &= \alpha\gamma\,|00\rangle + \alpha\delta\,|01\rangle + \beta\gamma\,|10\rangle + \beta\delta\,|11\rangle \\
\frac{2}{\sqrt{6}}\,|00\rangle - \frac{i}{\sqrt{6}}\,|01\rangle + \frac{i}{\sqrt{6}}\,|10\rangle + 0\,|11\rangle &\neq \alpha\gamma\,|00\rangle + \alpha\delta\,|01\rangle + \beta\gamma\,|10\rangle + \beta\delta\,|11\rangle
\end{aligned}$$

Either β or δ needs to be zero, but that contradicts the other terms. Thus $|\phi\rangle$ is entangeled.

2.1.2 Apply a Hadamard layer and compute the new state in the standard basis

$$
\begin{aligned}
H \otimes H = &\frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\
&\otimes \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\
H \otimes H = &\frac{1}{2}\,|00\rangle\langle 00| + |00\rangle\langle 01| + |00\rangle\langle 10| + |00\rangle\langle 11| \\
&+ |01\rangle\langle 00| - |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11| \\
&+ |10\rangle\langle 00| + |10\rangle\langle 01| - |10\rangle\langle 10| - |10\rangle\langle 11| \\
&+ |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11|
\end{aligned}
$$

$$
\begin{aligned}
|\phi\rangle &= \frac{1}{\sqrt{6}}(2\,|00\rangle - i(|01\rangle - |10\rangle)) \\
(H \otimes H)\,|\phi\rangle &= \frac{1}{2}(|00\rangle\langle 00|\,|\phi\rangle \quad \rightarrow 2\,|00\rangle \\
&+ |00\rangle\langle 01|\,|\phi\rangle \quad \rightarrow -i\,|00\rangle \\
&+ |00\rangle\langle 10|\,|\phi\rangle \quad \rightarrow i\,|00\rangle \\
&+ |00\rangle\langle 11|\,|\phi\rangle \quad \rightarrow 0 \\
&+ |01\rangle\langle 00|\,|\phi\rangle \quad \rightarrow 2\,|01\rangle \\
&- |01\rangle\langle 01|\,|\phi\rangle \quad \rightarrow i\,|01\rangle \\
&+ |01\rangle\langle 10|\,|\phi\rangle \quad \rightarrow i\,|01\rangle \\
&- |01\rangle\langle 11|\,|\phi\rangle \quad \rightarrow 0 \\
&+ |10\rangle\langle 00|\,|\phi\rangle \quad \rightarrow 2\,|10\rangle \\
&+ |10\rangle\langle 01|\,|\phi\rangle \quad \rightarrow -i\,|10\rangle \\
&- |10\rangle\langle 10|\,|\phi\rangle \quad \rightarrow -i\,|10\rangle \\
&- |10\rangle\langle 11|\,|\phi\rangle \quad \rightarrow 0 \\
&+ |11\rangle\langle 00|\,|\phi\rangle \quad \rightarrow 2\,|11\rangle \\
&- |11\rangle\langle 01|\,|\phi\rangle \quad \rightarrow i\,|11\rangle \\
&- |11\rangle\langle 10|\,|\phi\rangle \quad \rightarrow -i\,|11\rangle \\
&+ |11\rangle\langle 11|\,|\phi\rangle \quad \rightarrow 0) \\
&= \frac{1}{2\sqrt{6}}(2\,|00\rangle + 2(1+i)\,|01\rangle + 2(1-i)\,|10\rangle + 2\,|11\rangle) \\
&= \frac{1}{\sqrt{6}}(|00\rangle + (1+i)\,|01\rangle + (1-i)\,|10\rangle + |11\rangle)
\end{aligned}
$$

2.1.3 CZ-gate, bottom qubit as control, compute new state in standard basis and X-basis

Let CZ' be the CZ gate with the bottom qubit as the control.

$$\begin{aligned} CZ' &= |0\rangle\langle 0| \otimes Z + |1\rangle\langle 1| \otimes I \\ &= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) + |1\rangle\langle 1| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |00\rangle\langle 00| - |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \end{aligned}$$

Computing in standard basis:

$$\begin{aligned} CZ'|\phi\rangle &= |00\rangle\langle 00| - |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \\ &\qquad \cdot \frac{1}{\sqrt{6}}(|00\rangle + (1+i)|01\rangle + (1-i)|10\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{6}}(|00\rangle - (1+i)|01\rangle + (1-i)|10\rangle + |11\rangle) \end{aligned}$$

Computing in X-basis:

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$|00\rangle = \frac{1}{2}(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle)$

$|01\rangle = \frac{1}{2}(|++\rangle - |+-\rangle + |-+\rangle - |--\rangle)$

$|10\rangle = \frac{1}{2}(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle)$

$|11\rangle = \frac{1}{2}(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)$

$$\begin{aligned} CZ'|\phi\rangle &= \frac{1}{2\sqrt{6}}(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle \\ &\quad - (1+i)(|++\rangle - |+-\rangle + |-+\rangle - |--\rangle) \\ &\quad + (1-i)(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle) \\ &\quad + |++\rangle - |+-\rangle - |-+\rangle + |--\rangle) \\ &= \frac{1}{\sqrt{6}}((1-i)|++\rangle + |+-\rangle - |-+\rangle + (1+i)|--\rangle) \end{aligned}$$

2.2 Measurement Operators

2.2.1 Express the first part of the circuit as one unitary operator U

$$U = (CX \otimes Z) \cdot C(I \otimes X) \cdot (H \otimes X \otimes I)$$

$$U = \left[\begin{array}{cc|cc|cc|cc}
1 & & & & & & & \\
 & -1 & & & & & & \\
\hline
 & & 1 & & & & & \\
 & & & -1 & & & & \\
\hline
 & & & & & & 1 & \\
 & & & & & & & -1 \\
\hline
 & & & & 1 & & & \\
 & & & & & -1 & &
\end{array}\right]$$

$$\cdot \left[\begin{array}{cc|cc|cc|cc}
1 & & & & & & & \\
 & 1 & & & & & & \\
\hline
 & & 1 & & & & & \\
 & & & 1 & & & & \\
\hline
 & & & & & 1 & & \\
 & & & & 1 & & & \\
\hline
 & & & & & & & 1 \\
 & & & & & & 1 &
\end{array}\right]$$

$$\cdot \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc|cc|cc}
 & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 \\
\hline
1 & & & & 1 & & & \\
 & 1 & & & & 1 & & \\
\hline
 & & 1 & & & & -1 & \\
 & & & 1 & & & & -1 \\
\hline
1 & & & & -1 & & & \\
 & 1 & & & & -1 & &
\end{array}\right]$$

$$= \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc|cc|cc}
 & & 1 & & & & 1 & \\
 & & & -1 & & & & -1 \\
\hline
1 & & & & 1 & & & \\
 & -1 & & & & -1 & & \\
\hline
 & 1 & & & & -1 & & \\
-1 & & & & 1 & & & \\
\hline
 & & & 1 & & & & -1 \\
 & & -1 & & & & 1 &
\end{array}\right]$$

2.2.2 Compute the quantum state at the barrier and check for entanglement

$$U\left|000\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc|cc|cc} & & 1 & & & & 1 & \\ & & & -1 & & & & -1 \\ \hline 1 & & & & 1 & & & \\ & -1 & & & & -1 & & \\ \hline & 1 & & & & -1 & & \\ -1 & & & & 1 & & & \\ \hline & & & 1 & & & & -1 \\ & & -1 & & & & 1 & \end{array}\right] \cdot \begin{bmatrix} 1 \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(\left|010\right\rangle - \left|101\right\rangle)$$

We can write a statevector with 3 qubits like this:

$\left|\phi\right\rangle = (a_1\left|0\right\rangle + b_1\left|1\right\rangle) \otimes (a_2\left|0\right\rangle + b_2\left|1\right\rangle) \otimes (a_3\left|0\right\rangle + b_3\left|1\right\rangle)$

For the state $\left|\phi\right\rangle$ these are the coefficients. The state is entangeled as none of the a's and b's can be 0, but most products are 0, thus there is a contradiction and the state is entangeled.

$$\begin{cases} a_1a_2a_3 = 0 \\ a_1a_2b_3 = 0 \\ a_1b_2a_3 = \frac{1}{\sqrt{2}} \\ a_1b_2b_3 = 0 \\ b_1a_2a_3 = 0 \\ b_1a_2b_3 = -\frac{1}{\sqrt{2}} \\ b_1b_2a_3 = 0 \\ b_1b_2b_3 = 0 \end{cases}$$

2.2.3 Define a measurement operator M. List eigenspaces, dimensionalities and measurement probabilities

Defining the measurement operator M:

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|\mathbf{2}\rangle - |\mathbf{5}\rangle) \\
U^\dagger &= (H \otimes Y \otimes I) \cdot (I \otimes (Swap \cdot CX \cdot Swap)) \\
&= \frac{i}{\sqrt{2}}(|\mathbf{2}\rangle\langle\mathbf{0}| + |\mathbf{6}\rangle\langle\mathbf{0}| - |\mathbf{1}\rangle\langle\mathbf{1}| - |\mathbf{5}\rangle\langle\mathbf{1}| \\
&- |\mathbf{0}\rangle\langle\mathbf{2}| - |\mathbf{4}\rangle\langle\mathbf{2}| + |\mathbf{3}\rangle\langle\mathbf{3}| + |\mathbf{7}\rangle\langle\mathbf{3}| \\
&+ |\mathbf{2}\rangle\langle\mathbf{4}| - |\mathbf{6}\rangle\langle\mathbf{4}| - |\mathbf{1}\rangle\langle\mathbf{5}| + |\mathbf{5}\rangle\langle\mathbf{5}| \\
&- |\mathbf{0}\rangle\langle\mathbf{6}| + |\mathbf{4}\rangle\langle\mathbf{6}| + |\mathbf{3}\rangle\langle\mathbf{7}| - |\mathbf{7}\rangle\langle\mathbf{7}|)
\end{aligned}$$

Map the standard basis with $U^\dagger$:

$$\begin{aligned}
U^\dagger|\mathbf{0}\rangle &= \frac{i}{\sqrt{2}}(|\mathbf{2}\rangle + |\mathbf{6}\rangle) = \frac{i}{\sqrt{2}}|{+10}\rangle = |{+10}\rangle \\
U^\dagger|\mathbf{1}\rangle &= \frac{i}{\sqrt{2}}(-|\mathbf{1}\rangle - |\mathbf{5}\rangle) = -\frac{i}{\sqrt{2}}|{+01}\rangle = |{+01}\rangle \\
U^\dagger|\mathbf{2}\rangle &= \frac{i}{\sqrt{2}}(-|\mathbf{0}\rangle - |\mathbf{4}\rangle) = -\frac{i}{\sqrt{2}}|{+00}\rangle = |{+00}\rangle \\
U^\dagger|\mathbf{3}\rangle &= \frac{i}{\sqrt{2}}(|\mathbf{3}\rangle + |\mathbf{7}\rangle) = \frac{i}{\sqrt{2}}|{+11}\rangle = |{+11}\rangle \\
U^\dagger|\mathbf{4}\rangle &= \frac{i}{\sqrt{2}}(|\mathbf{2}\rangle - |\mathbf{6}\rangle) = \frac{i}{\sqrt{2}}|{-10}\rangle = |{-10}\rangle \\
U^\dagger|\mathbf{5}\rangle &= \frac{i}{\sqrt{2}}(-|\mathbf{1}\rangle + |\mathbf{5}\rangle) = -\frac{i}{\sqrt{2}}|{-01}\rangle = |{-01}\rangle \\
U^\dagger|\mathbf{6}\rangle &= \frac{i}{\sqrt{2}}(-|\mathbf{0}\rangle + |\mathbf{4}\rangle) = -\frac{i}{\sqrt{2}}|{-00}\rangle = |{-00}\rangle \\
U^\dagger|\mathbf{7}\rangle &= \frac{i}{\sqrt{2}}(|\mathbf{3}\rangle - |\mathbf{7}\rangle) = \frac{i}{\sqrt{2}}|{-11}\rangle = |{-11}\rangle \\
M &= 1 \cdot |{+00}\rangle\langle{+00}| \\
&+ 1 \cdot |{+01}\rangle\langle{+01}| \\
&+ 2 \cdot |{+10}\rangle\langle{+10}| \\
&+ 2 \cdot |{+11}\rangle\langle{+11}| \\
&+ 3 \cdot |{-00}\rangle\langle{-00}| \\
&+ 3 \cdot |{-01}\rangle\langle{-01}| \\
&+ 4 \cdot |{-10}\rangle\langle{-10}| \\
&+ 4 \cdot |{-11}\rangle\langle{-11}|
\end{aligned}$$

Eigenspaces:

Eigenvalue	*Eigenspace*	*Dim*
$\lambda_1 = 1$	$\{\|+00\rangle, \|+01\rangle\}$	2
$\lambda_2 = 2$	$\{\|+10\rangle, \|+11\rangle\}$	2
$\lambda_3 = 3$	$\{\|-00\rangle, \|-01\rangle\}$	2
$\lambda_4 = 4$	$\{\|-10\rangle, \|-11\rangle\}$	2

Measurement probabilities:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|\mathbf{2}\rangle - |\mathbf{5}\rangle) \\
|\psi\rangle &= \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle) = \frac{1}{2}(|+10\rangle + |-10\rangle + |-01\rangle - |+01\rangle) \\
P_m[|\psi\rangle \to 1] &= |\langle +00|\psi\rangle|^2 + |\langle +01|\psi\rangle|^2 = |-\frac{1}{2}\langle +01| + 01\rangle|^2 = \frac{1}{4} \\
P_m[|\psi\rangle \to 2] &= |\langle +10|\psi\rangle|^2 + |\langle +11|\psi\rangle|^2 = |\frac{1}{2}\langle +10| + 10\rangle|^2 = \frac{1}{4} \\
P_m[|\psi\rangle \to 3] &= |\langle -00|\psi\rangle|^2 + |\langle -01|\psi\rangle|^2 = |\frac{1}{2}\langle -01| - 01\rangle)|^2 = \frac{1}{4} \\
P_m[|\psi\rangle \to 4] &= |\langle -10|\psi\rangle|^2 + |\langle -11|\psi\rangle|^2 = |\frac{1}{2}\langle -10| - 10\rangle)|^2 = \frac{1}{4}
\end{aligned}
$$

2.2.4 Compute the expectation value and the posterior states

Expectation value:

$$\begin{aligned}\langle M_\psi \rangle &= 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} \\ &= \frac{10}{4} = 2.5\end{aligned}$$

Posterior states:

$|\psi\rangle = \frac{1}{2}(|{+}10\rangle + |{-}10\rangle + |{-}01\rangle - |{+}01\rangle)$

for λ_1:

$$\begin{aligned}\pi_1 &= |{+}00\rangle \langle{+}00| + |{+}01\rangle \langle{+}01| \\ |\phi\rangle_{\lambda_1} &= \frac{\pi_1 |\psi\rangle}{||\pi_1 |\psi\rangle ||} \\ &= \frac{-\frac{1}{2} |{+}01\rangle}{|| -\frac{1}{2} |{+}01\rangle ||} = \frac{-\frac{1}{2} |{+}01\rangle}{\frac{1}{2}} \\ &= |{+}01\rangle\end{aligned}$$

for λ_2:

$$\begin{aligned}\pi_2 &= |{+}10\rangle \langle{+}10| + |{+}11\rangle \langle{+}11| \\ |\phi\rangle_{\lambda_2} &= \frac{\pi_2 |\psi\rangle}{||\pi_2 |\psi\rangle ||} \\ &= \frac{\frac{1}{2} |{+}10\rangle}{||\frac{1}{2} |{+}10\rangle ||} = \frac{\frac{1}{2} |{+}10\rangle}{\frac{1}{2}} \\ &= |{+}10\rangle\end{aligned}$$

for λ_3:

$$\begin{aligned}\pi_3 &= |{-}00\rangle \langle{-}00| + |{-}01\rangle \langle{-}01| \\ |\phi\rangle_{\lambda_3} &= \frac{\pi_3 |\psi\rangle}{||\pi_3 |\psi\rangle ||} \\ &= \frac{\frac{1}{2} |{-}01\rangle}{||\frac{1}{2} |{-}01\rangle ||} = \frac{\frac{1}{2} |{-}01\rangle}{\frac{1}{2}} \\ &= |{-}01\rangle\end{aligned}$$

for λ_4:

$$
\begin{aligned}
\pi_4 &= |{-10}\rangle \langle{-10}| + |{-11}\rangle \langle{-11}| \\
|\phi\rangle_{\lambda_4} &= \frac{\pi_4 |\psi\rangle}{||\pi_4 |\psi\rangle ||} \\
&= \frac{\frac{1}{2} |{-10}\rangle}{||\frac{1}{2} |{-10}\rangle ||} = \frac{\frac{1}{2} |{-10}\rangle}{\frac{1}{2}} \\
&= |{-10}\rangle
\end{aligned}
$$

2.2.5 Use the measurement operator $\hat{M}$. List eigenspaces, dimensionalities and measurement probabilities

Eigenstate:

Eigenvalue	*Eigenspace*	*Dimension*
$\lambda_1 = 1$	$\{\lvert L\Phi^+\rangle, \lvert R\Phi^+\rangle\}$	2
$\lambda_2 = 2$	$\{\lvert L\Phi^-\rangle, \lvert R\Phi^-\rangle\}$	2
$\lambda_3 = 3$	$\{\lvert L\Psi^+\rangle, \lvert R\Psi^+\rangle\}$	2
$\lambda_4 = 4$	$\{\lvert L\Psi^-\rangle, \lvert R\Psi^-\rangle\}$	2

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle) = \frac{1}{\sqrt{2}}(|\mathbf{2}\rangle - |\mathbf{5}\rangle)$$

$$|L\Phi^+\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \\ \\ 1 \\ -i \\ \\ \\ -i \end{bmatrix}, |R\Phi^+\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \\ \\ 1 \\ i \\ \\ \\ i \end{bmatrix}, |L\Phi^-\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \\ \\ -1 \\ -i \\ \\ \\ i \end{bmatrix}, |R\Phi^-\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \\ \\ -1 \\ i \\ \\ \\ -i \end{bmatrix},$$

$$|L\Psi^+\rangle = \frac{1}{2}\begin{bmatrix} \\ 1 \\ 1 \\ \\ \\ -i \\ -i \\ \\ \end{bmatrix}, |R\Psi^+\rangle = \frac{1}{2}\begin{bmatrix} \\ 1 \\ 1 \\ \\ \\ i \\ i \\ \\ \end{bmatrix}, |L\Psi^-\rangle = \frac{1}{2}\begin{bmatrix} \\ 1 \\ -1 \\ \\ \\ -i \\ i \\ \\ \end{bmatrix}, |R\Psi^-\rangle = \frac{1}{2}\begin{bmatrix} \\ 1 \\ -1 \\ \\ \\ i \\ -i \\ \\ \end{bmatrix}$$

$$\begin{aligned} M = {} & 1 \cdot (|\mathbf{0}\rangle\langle\mathbf{0}| + |\mathbf{3}\rangle\langle\mathbf{0}| + |\mathbf{0}\rangle\langle\mathbf{3}| + |\mathbf{3}\rangle\langle\mathbf{3}| + |\mathbf{4}\rangle\langle\mathbf{4}| + |\mathbf{4}\rangle\langle\mathbf{7}| + |\mathbf{7}\rangle\langle\mathbf{4}| + |\mathbf{7}\rangle\langle\mathbf{7}|) \\ & + 2 \cdot (|\mathbf{0}\rangle\langle\mathbf{0}| - |\mathbf{3}\rangle\langle\mathbf{0}| - |\mathbf{0}\rangle\langle\mathbf{3}| + |\mathbf{3}\rangle\langle\mathbf{3}| + |\mathbf{4}\rangle\langle\mathbf{4}| - |\mathbf{4}\rangle\langle\mathbf{7}| - |\mathbf{7}\rangle\langle\mathbf{4}| + |\mathbf{7}\rangle\langle\mathbf{7}|) \\ & + 3 \cdot (|\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{1}\rangle\langle\mathbf{2}| + |\mathbf{2}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}| + |\mathbf{5}\rangle\langle\mathbf{5}| + |\mathbf{5}\rangle\langle\mathbf{6}| + |\mathbf{6}\rangle\langle\mathbf{5}| + |\mathbf{6}\rangle\langle\mathbf{6}|) \\ & + 4 \cdot (|\mathbf{1}\rangle\langle\mathbf{1}| - |\mathbf{1}\rangle\langle\mathbf{2}| - |\mathbf{2}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}| + |\mathbf{5}\rangle\langle\mathbf{5}| - |\mathbf{5}\rangle\langle\mathbf{6}| - |\mathbf{6}\rangle\langle\mathbf{5}| + |\mathbf{6}\rangle\langle\mathbf{6}|) \end{aligned}$$

Measurement probabilities:

$$
\begin{aligned}
P_m[|\psi\rangle \to 1] &= |\langle \mathbf{0}|\psi\rangle|^2 + |\langle \mathbf{3}|\psi\rangle|^2 + |-i\langle \mathbf{4}|\psi\rangle|^2 + |-i\langle \mathbf{7}|\psi\rangle|^2 \\
&+ |\langle \mathbf{0}|\psi\rangle|^2 + |\langle \mathbf{3}|\psi\rangle|^2 + |i\langle \mathbf{4}|\psi\rangle|^2 + |i\langle \mathbf{7}|\psi\rangle|^2 \\
&= 0+0+0+0+0+0+0+0 \\
&= 0 \\
P_m[|\psi\rangle \to 2] &= |\langle \mathbf{0}|\psi\rangle|^2 + |-\langle \mathbf{3}|\psi\rangle|^2 + |-i\langle \mathbf{4}|\psi\rangle|^2 + |i\langle \mathbf{7}|\psi\rangle|^2 \\
&+ |\langle \mathbf{0}|\psi\rangle|^2 + |-\langle \mathbf{3}|\psi\rangle|^2 + |i\langle \mathbf{4}|\psi\rangle|^2 + |-i\langle \mathbf{7}|\psi\rangle|^2 \\
&= 0+0+0+0+0+0+0+0 \\
&= 0 \\
P_m[|\psi\rangle \to 3] &= |\langle \mathbf{1}|\psi\rangle|^2 + |\langle \mathbf{2}|\psi\rangle|^2 + |-i\langle \mathbf{5}|\psi\rangle|^2 + |-i\langle \mathbf{6}|\psi\rangle|^2 \\
&+ |\langle \mathbf{1}|\psi\rangle|^2 + |\langle \mathbf{2}|\psi\rangle|^2 + |i\langle \mathbf{5}|\psi\rangle|^2 + |i\langle \mathbf{6}|\psi\rangle|^2 \\
&= 0 + |\frac{1}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{2}|\mathbf{2}\rangle - \langle \mathbf{2}|\mathbf{5}\rangle)|^2 + |\frac{-i}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{5}|\mathbf{2}\rangle - \langle \mathbf{5}|\mathbf{5}\rangle)|^2 + 0 \\
&+ 0 + |\frac{1}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{2}|\mathbf{2}\rangle - \langle \mathbf{2}|\mathbf{5}\rangle)|^2 + |\frac{i}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{5}|\mathbf{2}\rangle - \langle \mathbf{5}|\mathbf{5}\rangle)|^2 + 0 \\
&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{2} \\
P_m[|\psi\rangle \to 4] &= |\langle \mathbf{1}|\psi\rangle|^2 + |-\langle \mathbf{2}|\psi\rangle|^2 + |-i\langle \mathbf{5}|\psi\rangle|^2 + |\langle \mathbf{6}|\psi\rangle|^2 \\
&+ |\langle \mathbf{1}|\psi\rangle|^2 + |-\langle \mathbf{2}|\psi\rangle|^2 + |i\langle \mathbf{5}|\psi\rangle|^2 + |-i\langle \mathbf{6}|\psi\rangle|^2 \\
&= 0 + |\frac{-1}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{2}|\mathbf{2}\rangle - \langle \mathbf{2}|\mathbf{5}\rangle)|^2 + |\frac{-i}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{5}|\mathbf{2}\rangle - \langle \mathbf{5}|\mathbf{5}\rangle)|^2 + 0 \\
&+ 0 + |\frac{-1}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{2}|\mathbf{2}\rangle - \langle \mathbf{2}|\mathbf{5}\rangle)|^2 + |\frac{i}{\sqrt{2}\sqrt{2}\sqrt{2}}(\langle \mathbf{5}|\mathbf{2}\rangle - \langle \mathbf{5}|\mathbf{5}\rangle)|^2 + 0 \\
&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{2}
\end{aligned}
$$

2.2.6 Compute the expectation value and the posterior states

Expectation value: $\langle M\rangle_{\psi} = 0\cdot 1 + 0\cdot 2 + \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 4 = 7/2 = 3.5$

Posterior states:

$$\pi_3 = |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}| + i\,|\mathbf{5}\rangle\langle\mathbf{5}| + i\,|\mathbf{6}\rangle\langle\mathbf{6}| + |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}| - i\,|\mathbf{5}\rangle\langle\mathbf{5}| - i\,|\mathbf{6}\rangle\langle\mathbf{6}|$$
$$\pi_4 = |\mathbf{1}\rangle\langle\mathbf{1}| - |\mathbf{2}\rangle\langle\mathbf{2}| - i\,|\mathbf{5}\rangle\langle\mathbf{5}| + i\,|\mathbf{6}\rangle\langle\mathbf{6}| + |\mathbf{1}\rangle\langle\mathbf{1}| - |\mathbf{2}\rangle\langle\mathbf{2}| + i\,|\mathbf{5}\rangle\langle\mathbf{5}| - i\,|\mathbf{6}\rangle\langle\mathbf{6}|$$
$$\pi_3 : |\psi\rangle = \frac{1}{\sqrt{8}}(|\mathbf{2}\rangle + |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)$$
$$\pi_4 : |\psi\rangle = \frac{1}{\sqrt{8}}(-|\mathbf{2}\rangle - |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)$$
$$|\psi\rangle \to \frac{\pi_3\,|\psi\rangle}{||\pi_3\,|\psi\rangle\,||} = \frac{\frac{1}{\sqrt{8}}(|\mathbf{2}\rangle + |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)}{\frac{\sqrt{4}}{\sqrt{8}}} = \frac{(|\mathbf{2}\rangle + |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)}{\sqrt{4}} = \frac{(2\cdot|\mathbf{2}\rangle)}{\sqrt{4}} = |0\rangle\otimes|1\rangle\otimes|0\rangle$$
$$|\psi\rangle \to \frac{\pi_4\,|\psi\rangle}{||\pi_4\,|\psi\rangle\,||} = \frac{\frac{1}{\sqrt{8}}(-|\mathbf{2}\rangle - |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)}{\frac{\sqrt{4}}{\sqrt{8}}}$$
$$= \frac{(-|\mathbf{2}\rangle - |\mathbf{2}\rangle + i\,|\mathbf{5}\rangle - i\,|\mathbf{5}\rangle)}{\sqrt{4}} = \frac{(-2\cdot|\mathbf{2}\rangle)}{\sqrt{4}} = -(|0\rangle\otimes|1\rangle\otimes|0\rangle)$$

2.2.7 Realization of $\hat{M}$ on a quantum computer with only (partial) standard measurements

To realize $\hat{M}$ on a quantum computer with only (partial) standard measurements, we need a unitary operator U that maps the eigenspaces of $\hat{M}$ onto the standard basis.

$Eig(\hat{M}, \lambda_k) \overset{U}{\mapsto} Eig(M_{std}, \lambda_k)$

After measuring and calculating the posterior states in M_{std}, we then revert back by $U^{\dagger}$.