A Holistic Approach to Macroeconomic Fundamentals: Joint Estimates of Natural Rates

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Abstract

We develop a method to jointly estimate natural rates—or 'stars'—from long-run macroeconomic data. The approach embeds prior information about natural rates into a time-varying parameter VAR with stochastic volatility. It explicitly accounts for measurement error and outliers, making it well suited for historical analysis, including episodes like the COVID-19 pandemic and the post-pandemic inflation surge.
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1 Introduction

A central question facing policymakers is what part of the fluctuations in key macroeconomic variables such as unemployment, real interest rates, and inflation reflects temporary fluctuations versus persistent shifts. The recent debate over whether the post-COVID inflation surge was transitory is one example, to which we return later in the paper.

How can economists disentangle these features of incoming economic data? We propose to use a flexible vector autoregressive (VAR) model (Cogley & Sargent 2005, Primiceri 2005) that allows for changes in parameter values and volatilities to answer this question. There is no universally accepted definition of a trend or natural rate. We follow Beveridge & Nelson (1981) and Hamilton (2018) and define trends as long-run forecasts, which we in turn equate with natural rates, following a substantial precedent in the literature. This is a useful way to operationalize the notion of natural rates because temporary effects will by definition not have an effect on long-run forecasts (our definition of "long-run" is ten years). The class of models that we use has been shown to forecast well D'Agostino et al. (2013) and has been used for forecasts of inflation (D'Agostino & Surico 2011), so it is a natural fit for our application.[1]

The first key innovation in this paper is the way we set priors for the amount of time variation in the parameters. Previous work (Amir-Ahmadi et al. 2020) has shown that this prior can have substantial effects on posterior moments such as forecasts. We propose an empirical Bayes approach to setting this prior so that most of the volatility of these long-run

[1]This class of models has been used, for example, to estimate long-run trends in real interest rates (Lubik & Matthes 2015).

forecasts is at frequencies lower than business cycle frequencies. We use historical data starting in the 19th century to infer long-time series of natural rates for the United States and show that our prior can lead to substantially different conclusions than standard choices.

To incorporate both long historical series and recent COVID-era data, we must address two issues: measurement error in early data and potential outliers during the pandemic. We handle pre-WWII measurement error following Amir-Ahmadi et al. (2016), but introduce a new treatment for COVID-related outliers. Although structurally similar to the measurement error component, the outlier process requires different prior assumptions. We propose two alternatives, forming the second main innovation of the paper.

Our approach is related to previous literature that study one of the natural rates in isolation. In particular, the natural real rate of interest or r^*[2] has received much attention in policy making and academic circles. For evidence of the former, see, for example, Jerome Powell's 2018 speech at the annual Jackson Hole symposium[3]. Laubach & Williams (2003), Holston et al. (2017) and Lubik & Matthes (2015) have featured prominently in these discussions. Econometrically, we add to this discussion a richer time series model that allows us to use historical data, and explicitly models outliers so that we can tackle the COVID pandemic.

We are particularly interested in how natural rates co-move. The natural rate of unemployment, or u^* (a term coined by Friedman (1968)), has long played a role in both macroeconomic research and policy discussions (see, e.g., Gordon 1997, Staiger et al. 1996, Barnichon & Matthes 2017). The

[2]We drop time subscripts for the natural rates in the introduction for the sake of brevity. We do not mean to imply that these objects are time-invariant.

[3]https://www.federalreserve.gov/newsevents/speech/powell20180824a.htm

third element in our analysis is π^*, trend inflation or the natural rate of inflation (Cogley & Sbordone 2008). Sustained deviations of π^* from the central bank's stated inflation target—the inflation gap of Cogley et al. (2010)—is a closely watched object in monetary policy. While Cogley & Sargent (2005) study π^* and u^* jointly, we extend their framework to include r^* and extend their time series model.

Our approach uses a small set of observables including inflation and unemployment data to infer long-run forecasts. This choice reflects both a preference for parsimony and the prominence of Phillips Curve-based narratives in academic (King & Watson 1994) and policy discussions. That said, our framework is general: it can accommodate a broader set of variables to inform natural rates, and it can be used to study long-run forecasts for other macroeconomic outcomes.

The next section presents the model, with an emphasis on what distinguishes our approach from the existing literature. We then report results for the United States. The appendices contain details on the data, computation, robustness checks, and Monte Carlo evidence on the method's performance.

2 Our Model: An Overview

In richly parameterized classes of models such as the one we will discuss below, it is easy to loose track of the key features. In this section, we thus give a high level overview, highlighting the innovations in this paper.

We will model the evolution of inflation and unemployment using a nonlinear state space model where we first link our vector of observables denoted $\tilde{y}_t$ with a tilde with measurement errors m_t, outliers o_t, and the

’cleaned’ variables y_t.

$$\tilde{y}_t = y_t + o_t + m_t \tag{1}$$

While there is substantial work on modeling outliers in applied time series work in light of the recent COVID pandemic (which we discuss in detail below), our approach to modeling these outliers, which we discuss in detail in the next section, is specifically geared towards uncovering estimates of trends in macroeconomic time series.

We assume that the ’cleaned’ variables y_t follow a Vector Autoregression (VAR) with time-varying parameters (Cogley & Sargent 2005, Primiceri 2005), a model we discuss in detail in the next section as well. Our contribution to that class of models is best understood in light of already existing work: The previous literature has made it clear that in standard sample sizes in macroeconomics, these models depend crucially on how much time variation in parameters researchers think is reasonable *a priori*, as encoded in prior distributions. We formulate an empirical Bayes approach that a researcher can directly use to incorporate prior information on paths of natural rates in the estimation of such models.

2.1 A Nonlinear VAR

In this section we describe an extension of the VAR model time-varying parameters and stochastic volatility (Primiceri (2005) and Cogley & Sargent (2005)). This extension adds measurement error in the observables to a VAR model with time-varying parameters and stochastic volatility, following Amir-Ahmadi et al. (2016), which in turn was inspired by the modeling of inflation in Cogley et al. (2015). Measurement error is pervasive in historical data and thus we choose to incorporate it into our study of the natural rates. Furthermore, we explicitly add outliers to our model to better capture the recent COVID pandemic. To do so, we first define the vector of observables y_t, which in our application equals:

$$\tilde{y}_t = \begin{bmatrix} u_t \\ \pi_t \\ r_t \end{bmatrix} \tag{2}$$

where u_t is the unemployment rate, π_t the inflation rate, and r_t a measure of the real interest rate. Details of our data construction can be found in the appendix. We assume that this vector of observables is linked to an unobserved vector of measurement error-free and outlier-corrected values of inflation, unemployment, and real rates y_t via Equation (1). We assume that measurement errors and outliers are only relevant in mutually exclusive periods so that either m_t or o_t (or both) are equal to 0 in a given period. We can thus specialize this measurement equation further.

We assume that before the end of WWII, both data series were measured with error:

$$\tilde{y}_t = y_t + m_t \tag{3}$$

where m_t is a vector of measurement errors. The measurement errors are independent across variables, but can be persistent so that the law of motion for element i of m_t is given by:

$$m_t^i = \rho^i m_{t-1}^i + \sigma^i \varepsilon_t^{m,i} \tag{4}$$

Furthermore, we assume that all ρ and σ coefficients in the measurement error processes are 0 after WWII.

The outlier dynamics are modeled in a similar fashion. For pre-specified periods where outliers can play a role, we define the following measurement equation linking observables $\tilde{y}_t$ and the underlying outlier and measurement error-free variables y_t:

$$\tilde{y}_t = y_t + o_t \tag{5}$$

The dynamics of the ith element of o_t is given by:

$$o_t^i = \rho^{i,o} o_{t-1}^i + \sigma^{i,o} \varepsilon_t^{o,i} \tag{6}$$

Outside of the pre-specified outlier periods, $\sigma^{i,o}$ and $\rho^{i,o}$ are equal to 0. We assume that time periods with outliers and those with measurement error are mutually exclusive, so we have three scenarios for the measurement equation linking observables $\tilde{y}_t$ and y_t: (i) equation (3) before and during WWII, (ii) the identity $\tilde{y}_t = y_t$ after WWII except for the COVID pandemic, when we use equation (5). The previous literature on outlier corrections for the COVID pandemic has broadly focused on either (i) changing the variance of shocks during the pandemic (Lenza & Primiceri 2020, Carriero et al. 2021), or (ii) using dummy variables (Cascaldi-Garcia 2022).[4] Our approach is closer to the dummy variable assumption, but additionally impose the autoregressive structure in Equation (6). The reader might wonder why we do not just lean on the standard stochastic volatility specification (Primiceri 2005) (which we use to model stochastic volatility throughout our sample) to deal with the COVID period. As highlighted by Hartwig (2022), such a modeling choice leads

[4]A third approach would be to directly include information on COVID-related variables such as hospitalizations in our model. This has been proposed by Ng (2021).

to inferior fit because the standard specification implies that log volatilities follow a random walk *and* that this assumption is valid throughout the sample. Volatility increases due to COVID, instead, were short-lived. In particular, we know that changes in the unemployment rate due to COVID were short-lived and the dynamics of the unemployment rate during COVID and immediately after was very different from the usual unemployment dynamics described in Hall & Kudlyak (2022), as highlighted by Ramey (2021). Our assumption closely mirrors how we model measurement error, even though the priors we use (discussed below) are very different across these two sets of variables.

The dynamics of y_t are modeled as a VAR with lag length 2

$$y_t = \mu_t + A_{1,t} y_{t-1} + A_{2,t} y_{t-2} + e_t$$

For quarterly models in this class a lag length of 2 is standard (see, for example, Primiceri (2005), Cogley & Sargent (2005)). To concisely state the dynamics of y_t, we define $X_t' \equiv I \otimes (1, y_{t-1}', y_{t-2}')$ and rewrite equation (2.1)

$$y_t = X_t' \theta_t + e_t \tag{7}$$

We follow Cogley & Sargent (2005) in our choice of the law of motion for the time-varying parameters, which is given by

$$\theta_t = \theta_{t-1} + u_t \tag{8}$$

Note that Cogley & Sargent (2005) impose that

$$A_t = \begin{bmatrix} A_{1,t} & A_{2,t} \\ I & 0 \end{bmatrix}$$

has all eigenvalues less than one in absolute value, which is something that we do not impose and will come back to below. It will be useful for later sections to explicitly introduce the prior for the variance of u_t, Σ_u. We will call that prior $p(\Sigma_u|\kappa)$,where we have made the dependence on a hyperparameter κ explicit. As is common in the literature, we assume that this prior is of the inverse Wishart form. We discuss the details in the next section. We introduce stochastic volatility by decomposing the covariance matrix of the one-step ahead forecast errors e_t:

$$e_t = \Lambda_t^{-1}\Sigma_t\varepsilon_t \tag{9}$$

where $\varepsilon_t \sim N(0, I)$ and Λ_t is a lower triangular matrix with ones on the main diagonal and representative non-fixed element λ_t^i. Σ_t is a diagonal matrix with representative non-fixed element σ_t^j. The dynamics of the non-fixed elements of Λ_t and Σ_t are given by:

$$\lambda_t^i = \lambda_{t-1}^i + \zeta_t^i \tag{10}$$

$$\log \sigma_t^j = \log \sigma_{t-1}^j + \eta_t^j \tag{11}$$

The residuals in all equations are jointly Gaussian with covariance matrix

$$V = Var\left[\begin{pmatrix} \varepsilon_t \\ u_t \\ \zeta_t \\ \eta_t \end{pmatrix}\right] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & W \end{bmatrix} \tag{12}$$

S is further restricted to be block diagonal, which simplifies inference. The priors are chosen along the lines discussed in Amir-Ahmadi et al.

(2016), with the exception of the hyperparameter that governs the prior on time variation in θ. We discuss below how we set this prior.[5]

Next, we turn to our definition of a *natural rate*. It is commonplace to associate natural rates with estimates of trends (Laubach & Williams 2003, Lubik & Matthes 2015).

2.2 Natural Rates: Definition and Discussion

We want to trace out the low frequency behavior of the unemployment rate, which we equate to the natural rate of unemployment. In this section, we first show how objects computed with our VAR are closely linked to standard concepts of trends in the time series literature. As a first step, it will be useful to answer the question of what a long-run forecast for y_t would look like if there was no time variation after time t[6] and if the VAR coefficients (under the assumption of no future time variation in the coefficients) implied stationarity of y_t so that all eigenvalues of A_t are less than one in absolute value:

$$\overline{y}_t = (I - A_{1,t} - A_{2,t})^{-1}\mu_t$$

We now briefly discuss the relationship between $\overline{y}_t$ and the Beveridge-Nelson decomposition. For that purpose it is useful to define $c_t^{VAR} = (y_t - \overline{y}_t) = A_{1,t}(y_{t-1} - \overline{y}_t) + A_{2,t}(y_{t-2} - \overline{y}_t) + e_t$. Thus a VAR with time-varying parameters and stochastic volatility can be used to decompose the dynamics of y_t into a slow moving trend $\overline{y}_t$ and deviations from that trend c_t^{VAR}.

[5] As discussed in Amir-Ahmadi et al. (2020), priors for Q can matter even in long samples so that prior choice is potentially important.

[6] Cogley & Sargent (2005) have used the same constructs.

A Beveridge-Nelson decomposition for a vector of time series x_t is given by

$$x_t = \tau_t + c_t \tag{13}$$

$$\tau_t = \tau_{t-1} + \varepsilon_t \tag{14}$$

where c_t is an ergodic, stationary process such as multivariate ARMA. We can see that our VAR has a similar structure to the state space systems used to calculate the Beveridge Nelson trend τ_t.[7] The VAR-based trend $\overline{y}_t$ does not follow exactly a random walk (which is what is assumed for τ_t), but the components that generate $\overline{y}_t$ do (in other words, a first-order Taylor-series approximation would look like a random walk). The c_t^{VAR} component features time variation in the dynamics around $\overline{y}_t$, which generalizes the standard Beveridge-Nelson setup that commonly does not feature stochastic volatility. Finally, we generally propose to calculate smoothed estimates of $\overline{y}_t$ (i.e. estimates using parameter estimates conditional on the entire sample rather than real time estimates as is common), but other papers such as Oh et al. (2006) and Proietti & Harvey (2000) have also considered smoothed estimates of the Beveridge-Nelson trend.

While the calculation above is useful in building intuition about the link between the standard Beveridge-Nelson approach and approaches based on time-varying parameter models, the calculation rests on a restriction on the law of motion of A_t. In our application we choose *not* to restrict the dynamics implied by A_t, and instead of studying infinite horizon forecasts (which could only be computed under the restriction that all eigen-

[7] Another approach that uses forecasts to identify trends is discussed in Hamilton (2018).

values of A_t are less than one in absolute value, as discussed in Cogley & Sargent (2005)), we compute 10 year forecasts, where we keep the parameters constant at the value implied by the current parameter draw, in line with the computation for infinite horizon forecasts outlined above.[8] To be more specific, for a given posterior parameter draw A_t and data at time t, the ten year forecast we focus on in this paper is given by

$$\mathbf{F}_t \equiv \mathbf{Select} * A_t^{40} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \mathbf{Select} * \sum_{j=0}^{39} A_t^j \begin{bmatrix} \mu_t \\ 0 \end{bmatrix} + \begin{bmatrix} \rho^{1,o} & 0 \\ 0 & \rho^{2,o} \end{bmatrix}^{40} o_t, \quad (15)$$

where Select is a selection matrix that selects the first three elements (forecasts of 'cleaned' unemployment, inflation, and the real rate). Note that $\mathbf{F}_t$ depends on potentially unobservable data because y_t is the 'cleaned' unobserved data and o_t is the value of the outlier at time t.[9] We forecast the variables including outliers since in theory outliers could play a role even in long-horizon forecasts. Our estimates of $\rho^{1,o}$ and $\rho^{2,o}$ are small enough that in practice the inclusion of the o_t terms does not matter quantitatively.

For a given set of parameters we compute $\mathbf{F}_t$ using *smoothed* estimates of y_t and y_{t-1}, following papers such as Cogley & Sargent (2005), Oh et al. (2006), and Proietti & Harvey (2000). We extend the Gibbs sampler proposed by Amir-Ahmadi et al. (2016) to infer the underlying noise-free data from the measurement error-ridden data that we have prior to the end

[8]In a different application, Amir-Ahmadi et al. (2016) show that this eigenvalue restriction can easily be violated with historical data. The choice of using finite horizon forecasts can also be motivated by the fact that VAR models with time-varying parameters and stochastic volatility generally do well in forecasting macroeconomic aggregates, as shown by D'Agostino et al. (2013).
Lubik & Matthes (2015) also use a finite horizon forecast as their measure of the natural real interest rate, but use a five year horizon. The choice of horizon is akin to choosing a smoothing parameter.

[9]When there are outliers, $y_t + o_t$ is observable, but not y_t and o_t separably.

of WWII and the outlier-ridden data during the COVID pandemic.

We are now in a position to define the natural rate of unemployment $\mathbf{N}_t$:

Definition (Natural Rates)**.** The natural rates $\mathbf{N}_t = [\mathbf{N}_t^u \; \mathbf{N}_t^\pi \; \mathbf{N}_t^r]'$ at each point in time t are the posterior median of the ten year-ahead forecast $\mathbf{F}_t$ as defined in Equation (15).

2.3 Priors and hyperparameters

This section discusses how we set priors in our model. We divide the discussion into two large parts: First, we discuss how we set priors for the outlier and measurement error processes. The second big part of our model is a TVP-VAR for the 'cleaned' data. We describe priors for that part of the model next, highlight the key parameter κ_Q that governs the prior view on time variation in parameters, before discussing how we set that parameter.

2.3.1 Priors for outlier and measurement error processes

Earlier, we devised the law of motion for the outliers and measurement errors as the following:

$$m_t^i = \rho^i m_{t-1}^i + \sigma^i \varepsilon_t^{m,i}, \tag{16}$$

$$o_t^i = \rho^{i,o} o_{t-1}^i + \sigma^{i,o} \varepsilon_t^{o,i}, \tag{17}$$

where m_t is a vector of measurement errors and o_t is that of outliers. Also, we assumed that the coefficients of the processes, $\rho^i, \rho^{i,o}, \sigma^i$ and $\sigma^{i,o}$ were zero outside the predetermined periods. In this section, we describe specific priors of those coefficients for Bayesian estimation.

The basic specification of the processes are following Cogley et al.

(2015) and Amir-Ahmadi et al. (2016b). First of all, the priors for AR coefficients in the processes, ρ^i and $\rho^{i,o}$ are independent of each other and the break dates. They are Gaussian and have mean 0 and standard deviation 0.15. Secondly, the prior for the variance of the innovation in the processes, σ^i and $\sigma^{i,o}$ follow inverse-gamma and independent of each other and the break dates. We set the parameters of the inverse-gamma distribution for σ^i as Amir-Ahmadi et al. (2016b) did: The scale parameter of the inverse-gamma distribution is fixed at two, and its mode equals to the square of 25% of the training sample standard deviation. The shape parameter for inverse-gamma distribution can then be calculated as a function of the scale parameter and mode[10].

For the outlier process, we set the prior for $\sigma^{i,o}$ to capture the idea that during COVID, a substantial fraction of economic outcomes were directly determined by COVID and hence the outlier o_t. To achieve this, we use two assumptions: First, for simplicity and parsimony, we assume that the standard deviation of the prior and its mode are equal to each other.[11] Under this assumption, the shape parameter of inverse-gamma distribution is approximately fixed at 4.48[12]. The second assumption is that the mode of the innovation variance equals the square of half of each observables' maximum value during an outlier period i.e. mode $[(\sigma^{i,o})^2] = \left(max \left[\{y_t^i\}_{t=\text{start period of } i}^{\text{end period of } i} \right] /2 \right)^2$. With the shape parameter

[10]The inverse-gamma distribution has two parameters, shape(α), and scale(β). The mode of inverse-gamma distribution is $\frac{\beta}{\alpha+1}$, so β can be calculated when α and the mode are given. Likewise, α can be calculated when β and the mode are given.

[11]This assumption implies a relatively uninfomative prior in the sense that a one-standard deviation change can move substantially away from the mode.

[12]*Suppose* $x \sim IG(\alpha, \beta)$ *where the shape parameter* $\alpha > 2$*, the scale* $\beta > 0$ *and* $\alpha, \beta \in \mathbb{R}^+$. *Then* $\alpha \approx 4.48$ *when mode*$(x) =$ *std*(x).

$\because$ The mode of $x \sim IG(\alpha, \beta)$ where $\alpha > 2$ and $\beta > 0$ is $\frac{\beta}{\alpha+1}$, and its standard deviation of is $\sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}}$. When mode$(x) =$ std(x), $\frac{\beta}{\alpha+1} = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} \iff \alpha^3 - 5\alpha^2 + 3\alpha - 3 = 0$.
A real valued solution for the equation, $\alpha^* = 4.4798157....$ can be derived numerically.

and the mode of inverse-gamma distribution in hand, we can calculate the scale parameter immediately. In Appendix C we introduce the inverse Nakagami distribution as an alternative prior that can directly use information on $\mathbb{E}(\sigma^{i,o})$ and show that our findings are robust to this alternative.

In sum, priors for the coefficients and innovations in outlier and measurement error processes are as follows:

$$
\begin{aligned}
\rho^i &\sim N(0, 0.15^2) \\
\rho^{i,o} &\sim N(0, 0.15^2) \\
\left(\sigma^i\right)^2 &\sim \text{IG}\left(\underbrace{scaling_i \times \widehat{\sigma}^2_{i,train}}_{mode}, \underbrace{2}_{scale}\right) \\
\left(\sigma^{i,o}\right)^2 &\sim \text{IG}\left(\underbrace{\left\{max\left[\left\{x_t^i\right\}_{t=\text{start period of } i}^{\text{end period of } i}\right]/2\right\}^2}_{mode}, \underbrace{4.48}_{shape}\right)
\end{aligned}
$$

2.3.2 Setting hyperparameters

In Bayesian estimation, prior elicitation is often not straightforward. This is particularly evident in models with many parameters such as VARs and even more striking once we allow for time variation in parameters. In these settings, researchers often model priors as themselves depending on few parameters (usually called hyperparameters). There are two common approaches to setting the hyperparameters for priors in Bayesian estimation. Both are commonly used in practice, one estimates the hyperparameters as additional parameters in a fully Bayesian fashion by realizing that the hyperparameters induce a hierarchical model. This is the approach used (and made operational for VARs with time-varying parameters and stochastic volatility) in Amir-Ahmadi et al. (2020). The second approach is usually called an empirical Bayes approach. There the hyperparameter is set to a value that maximizes the marginal likelihood. Both approaches are used for fixed coefficient VARs in Giannone et al. (2015). These approaches are unfortunately not directly applicable in our setting, as we want to impose some prior information on a highly nonlinear function of the parameters of the model - the natural rate of unemployment. The mapping between this function and the hyperparameter of interest is not only also nonlinear, but furthermore not available in closed form.

In order to be able to describe how we choose the hyperparameters in our model, we first have to revisit the existing literature (Primiceri 2005) to discuss how exactly hyperparamters enter the prior distributions for various parameters in this class of models.

2.4 Priors and the hyperparameters they depend on

To analyze the TVP-VAR in a Bayesian way, we use similar priors as Primiceri (2005), Cogley & Sargent (2014), Lubik & Matthes (2015) and many others. As Lubik & Matthes (2015) discuss, we can separate TVP-VAR parameters in two groups. The first set contains initial parameters associated with the coefficients and volatilities. The other set has parameters governing the law of motion of the time-varying terms. Specifically, $\theta_0, \Lambda_0, \log \sigma_0$ belong to the first, and Q, W, S to the second in our model. Our priors for these parameters are borrowed from Primiceri (2005). First, the priors for θ_0 and Λ_0 are normal with mean equal to the OLS point estimates of a fixed coefficient VAR from a training sample of 40 periods (10 years for quarterly data), and variance equal to four times of the variance of OLS estimates on the training sample. Similarly, the prior for $\log \sigma_0$ is normal with mean 0, and variance of four times the identity matrix. Next, we assume the priors for the second group follow Inverse-Wishart distribution with similar structure but different means, degrees of freedom and the scaling factors, κ. For the innovation variance for the lag coefficient matrices Q, mean is $\kappa_Q^2 \times df_Q \times V(\widehat{\theta}_{ols})$ where the degrees of freedom, df_Q is 40 which is the size of training sample. The prior for the error variance W has mean of $\kappa_W^2 \times df_W \times I_n$, where df_W is 3, which is $1+$dimension of W. Lastly, the mean of the prior for the loading matrix, S is $\kappa_S^2 \times df_S \times V(\widehat{\Lambda}_{ols})$ and df_S is 2, $1+$ dimension of S. Thus, the priors we use are:

$$\begin{aligned}
\theta_0 &\sim N\left(\widehat{\theta}_{OLS}, 4 \times V(\widehat{\theta}_{OLS})\right), \\
\Lambda_0 &\sim N\left(\widehat{\Lambda}_{OLS}, 4 \times V(\widehat{\Lambda}_{OLS})\right), \\
\log \sigma_0 &\sim N\left(\log \widehat{\sigma}_{OLS}, 4 \times I_n\right), \\
Q &\sim IW\left(\kappa_Q^2 \times 40 \times V(\widehat{\theta}_{OLS}), 40\right), \\
W &\sim IW\left(\kappa_W^2 \times 3 \times I_n, 3\right), \\
S &\sim IW\left(\kappa_S^2 \times 2 \times V(\widehat{\Lambda}_{,OLS}), 2\right)
\end{aligned}$$

There are three hyperparameters that need to be set for this prior: Our focus is on the choice of κ_Q, which governs how much time variation we expect a priori for the parameters θ. In this paper, we use standard values in the literature for $\kappa_W = 0.01$ and $\kappa_S = 0.1$. We do so for two reasons: (i) this helps speed up the computations, and (ii) more importantly, the estimation of hyperparameters matters most for the hyperparameter associated with the evolution of θ, as highlighted by Amir-Ahmadi et al. (2020).

The only priors we have not discussed yet are those for the measurement error and outlier processes. We will return to those priors in Section 2.3.1.

2.5 An Empirical Bayes Prior for Natural Rates

The standard empirical Bayes approach would set κ_Q to maximize

$$\int p(y|\kappa_Q)p(\kappa_Q)d\kappa_Q$$

where

$$p(y|\kappa_Q) = \int p(y|\kappa_Q, A, \Lambda, \Sigma, \mu)p(A, \Lambda, \Sigma, \mu|\kappa_Q)dAd\Lambda d\mu d\Sigma$$

where variables without a time subscript denote the entire sample of the corresponding variable with a time subscript. This approach works well when $p(y|\kappa_Q)$ is available in closed form, as in the fixed coefficient VAR models used in Giannone et al. (2015).[13]

We propose that instead of trying to maximize the marginal likelihood, we maximize a function f of $\mathbf{N}_t$ where, as defined before, $\mathbf{N}_t$ is a function of all parameters in our model.[14] We then choose the hyperparameter κ_Q according to

$$\kappa^* = argmax_{\kappa_Q} f\left(\mathbf{N}_t\right) \tag{18}$$

The natural rate $\mathbf{N}_t$ depends on the entire posterior distribution of pa-

[13]While algortihms are available to numericalluy approximate the marginal likelihood of models in this class (Chan & Eisenstat 2018), a key issue why this is not done more often is that common priors in this class of models are quite loose, so small changes in prior settings can have large effects on marginal likelihoods, even though point estimates and standard posterior error bands remain largely unchanged.

[14]This notation is geared towards the application. In general we could make f a general function of our posterior distribution.

rameters $p(A, \Lambda, \Sigma, \mu | y, \kappa_Q)$.

We choose the hyperparameter for θ in this fashion because we then evaluate f on a grid for κ_Q. If we also considered the hyperprameters for Λ and Σ this type of grid search would more become time consuming.

2.5.1 Choice of f

We want to impose a prior that implies that the natural is slow moving - most of its variation should be at low frequencies, not at business cycle frequencies or even higher frequencies. We do so by choosing a function f that maximizes this relative variability. We next describe how to use the frequency domain to construct this function f.

For simplicity we focus on natural unemployment rate N_t^u $\forall t$.[15] Given this estimate of the natural rate, we can compute a nonparametric estimate of its spectrum, which we denote by $s_N(\omega)$. If we denote the frequency cutoff for business cycles by ω^{BC} such that all frequencies that are larger or equal to ω^{BC} are associated with business cycles or even higher frequency fluctuations.[16]

This leads to the following definition of f: $f(\mathrm{N}_t^u) = \left(\frac{\sum_{\omega_j < \omega^{BC}} s_N(\omega_j)}{\sum_{\omega_j > \omega^{BC}} s_N(\omega_j)} \right)$. With that choice of f, we maximize the relative variance of N_t^u at low frequencies relative to high frequencies. The value of the total variance of N_t^u does not matter for f - multiplying the total variance by a constant factor that does not change the distribution of variance contributions across frequencies will cancel out in the computation of f. A researcher could add a penalty term to f to penalize, for example, natural rates that are too volatile. More details can be found in Appendix B.

[15] We could alternatively use the natural rate of inflation or the natural real interest rate, or use a function f that uses all elements of N_t.

[16] In practice we set this cutoff to eight years.

3 Implementation via Gibbs sampling

In practice, we follow the estimation method for a time-varying parameter VAR proposed by Amir-Ahmadi et al. (2016). What is new in our paper (in terms of setting priors) is the choice of the hyperparameter for the evolution of θ. We set a grid for κ_Q, which goes in increments of 0.01 from 0 to 0.1 (which is ten times larger than the value used by Primiceri (2005)). Remember that this hyperparameter scales the variance of the innovation to the VAR parameters. We then approximate the posterior distribution for a given single value of κ_Q by using the Gibbs sampler with 10,000 draws. The detailed algorithm proceeds as follows.

1. Set a sequence of κ_Q as $\kappa_Q = \{0, 0.01, \ldots, 0.1\}$. To reduce computation time, we parallelize the estimations over the grid for κ_Q.

2. Draw model parameters, $\theta^T, \Sigma^T, \Lambda^T, V, S^T, \rho_m, \sigma_m^2, \rho_o, \sigma_o^2,$ and s^T. As discussed in Amir-Ahmadi et al. (2016), the estimation steps follow Primiceri (2005) and the corrigendum of Del Negro & Primiceri (2015), and adding the outlier process does not affect those steps in the Gibbs sampler. Since outliers are isomorphic to measurement errors in our model structure, we can directly borrow the Gibbs sampler of Amir-Ahmadi et al. (2016).

3. Compute the natural rates $\mathbf{N}_t$ for a given hyperparameter value of κ_Q.

After completing the estimations for the sequence of κ_Q, we have a number of candidate natural unemployment rates $\mathbf{N}_t$. We then construct a function $f(\mathbf{N}_t)$ and choose the natural unemployment rate that maximizes the function $f(\mathbf{N}_t)$.[17] In Appendix E we show in a simulation study that our approach outperforms standard hyperparameter choices when the object of interest is trend estimation.

4 Results

We now estimate our model on US data on the unemployment rate and inflation. Details on the data can be found in the appendix. Our sample runs from 1891:Q4 to 2023:Q2 with 527 observations. We use the following periods for measurement error and outliers: The break date for the parameters of the measurement error process is 1948:Q1, so the period for measurement error is 1891:Q4-1947:Q4. And the period for the outliers (COVID-19 pandemic) is 2020:Q1–2020:Q2.

Using our benchmark prior, we first highlight the importance of prior hyperparameters for the estimation of our natural rate (and thus also more generally for forecasts obtained using this class of models). Figure 1 shows the natural rates of unemployment obtained using all values of κ_Q on our grid. Our optimal value is $\kappa_Q = 0.06$, which thus a priori allows for more time variation in parameters relative to Primiceri (2005). The corresponding natural rate estimate obtained by maximizing f is depicted in red, whereas the green line is obtained using a value of $\kappa_Q = 0.01$, which has become a default value used in many studies and follows Primiceri (2005). We see dramatic differences in the natural rate of unemployment, especially around 1940 and the Great Inflation period around 1980 - estimates range from around 4 percent to over 7.5 percent. Furthermore, differences are substantial even in the immediate pre-pandemic

[17] We recommend that researchers should increase the upper bound of the grid for κ_Q if the maximum of $f(\mathbf{N}_t)$ is achieved at the upper bound of the original grid.

period, for example, so large differences do not only emerge in periods of large economic turmoil. The recent inflation period leads to substantial variation in estimated natural rates across hyperparameters, with some hyperparameters associating the increase in inflation with a substantial increase in long-run unemployment.

Figure 2 shows the same object for inflation. The same patterns emerge. Here the behavior at the end of the sample is particularly striking, with some hyperparameters estimating a higher natural rate of inflation than during the late 1970s.

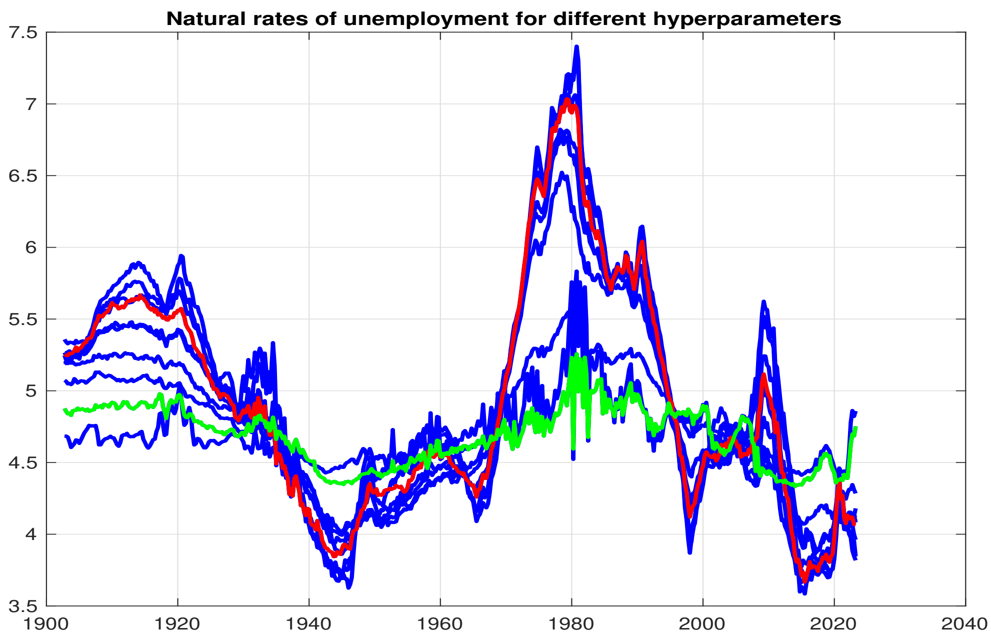

Figure 1: The natural rate of unemployment for all values of κ_Q on our grid. The red line is our benchmark natural rate that maximizes f ($\kappa_Q = 0.06$), whereas green represents the natural rate obtained using a default value of $\kappa_Q = 0.01$ based on Primiceri (2005).

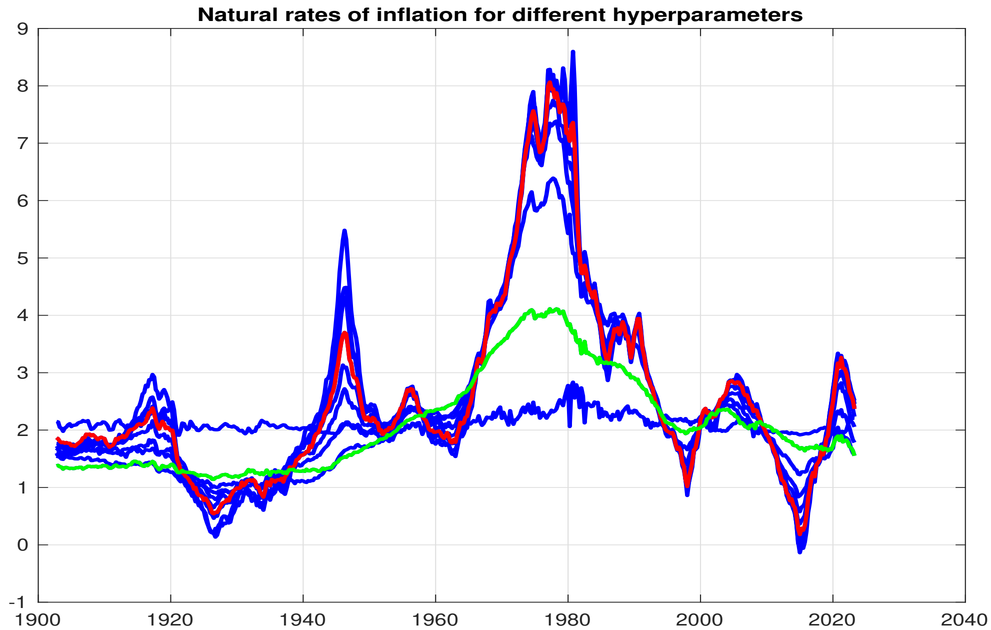

Figure 2: The natural rate of inflation for all values of κ_Q on our grid. The red line is our benchmark natural rate that maximizes f ($\kappa_Q = 0.06$), whereas green represents the natural rate obtained using a default value of $\kappa_Q = 0.01$ based on Primiceri (2005).

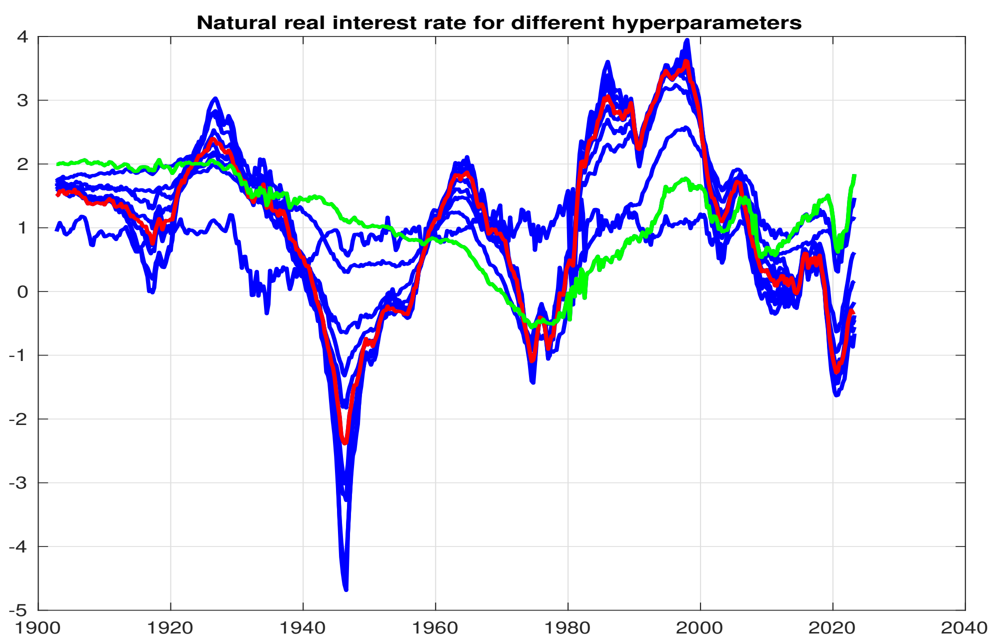

Figure 3: The natural rate of real interest rate for all values of κ_Q on our grid. The red line is our benchmark natural rate that maximizes f ($\kappa_Q = 0.06$), whereas green represents the natural rate obtained using a default value of $\kappa_Q = 0.01$ based on Primiceri (2005).

Next, we compare our natural rate with the actual data. Figure 4 plots our data series as well as the natural rate and associated 90 percent posterior bands.[18] Note the much larger fluctuations in actual data compared to our natural rate. As expected many fluctuations in actual data do not translate into fluctuation of our natural rate.

As Figures 4 and 12 show, statistical uncertainty in the natural rates is generally small relative to fluctuations in the observed data. A striking feature can be observed once we zoom in to the last six years of data in Figure 12: While the COVID pandemic and the associated dramatic increase in unemployment have not lead to a dramatic increase in either the natural rate of employment (approximately a one percentage point increase since 2015) or the associated uncertainty, the recent bout of inflation paints a very different picture for the trend in inflation in that a *relatively small* increase in the natural rate of inflation is associated with a *substantial* increase in uncertainty, with the 95th percentile of the long-run trend in inflation now reaching 6 percent of annual inflation. What does this mean for the discussion on temporary inflation after COVID that we mentioned in the introduction? Our model does agree with those policymakers that were optimistic about inflation coming down that this is the median outcome path. The upside risk to inflation in the long-run is substantial, though, and needs to be monitored closely.

[18] We plot the 5th and 95th posterior percentiles of $\mathbf{F}_t$ in addition to the median.

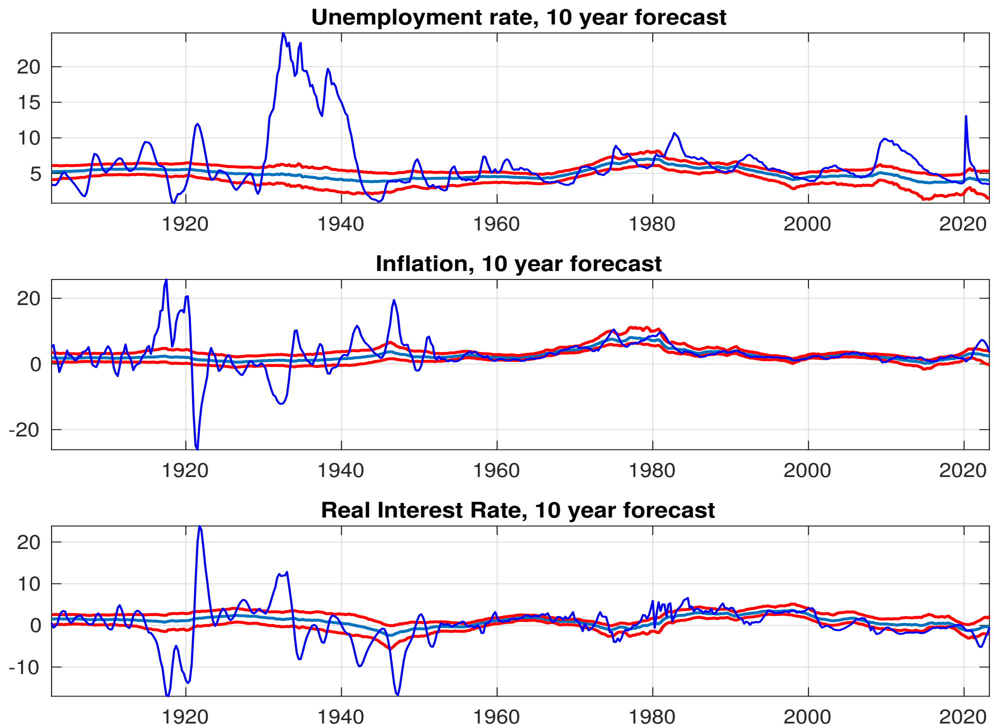

Figure 4: Natural rates and error bands, entire sample.

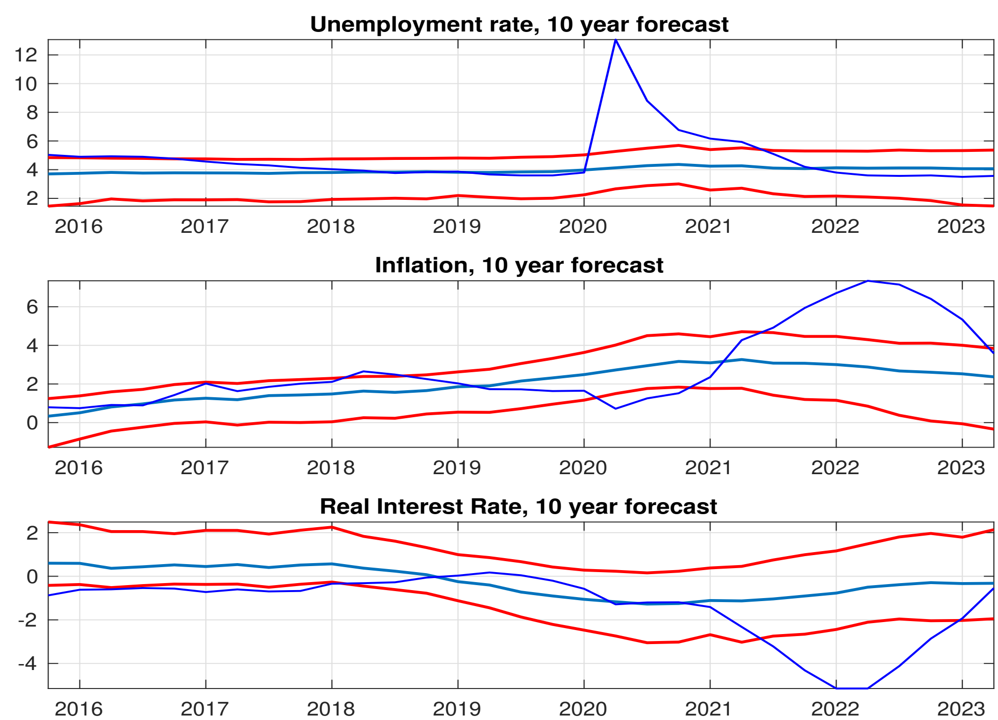

Figure 5: Natural rates and error bands, last six years.

4.1 Do natural rates move together in the long run?

The short answer is: To a large extent, yes. To arrive at this conclusion, we compute the posterior distribution of the (in-sample) correlation of 10-year ahead forecasts for inflation and unemployment, which are two of the elements of $\mathbf{F}_t$. Figure 6 plots this posterior distribution for the correlation. While there is substantial uncertainty, most of the posterior mass points towards a substantial positive (and large) correlation. At the risk of oversimplification, a standard taxonomy of economic shocks implies that demand shocks tend to move unemployment and inflation in opposite directions, whereas supply shocks move them in the same direction. Our results could then be interpreted as saying that supply shocks have longer lasting effects on the inflation-unemployment trade-off than demand shocks. While the positive comovement between these long-run forecasts has been noted before (Cogley & Sargent (2002) use 30 year forecasts and produce a scatterplot of their counterparts of N_t), we highlight that there is substantial uncertainty about the strength of this comovement.

What about the other natural rates? Figure 7 shows the same correlation for r^* and u^*. Interestingly, we do not find much comovement there. Finally, the correlation between r^* and π^* is overwhelmingly negative, implying that periods with low real interest rates such as the 1970s are often associated with high inflation even at lower frequencies.

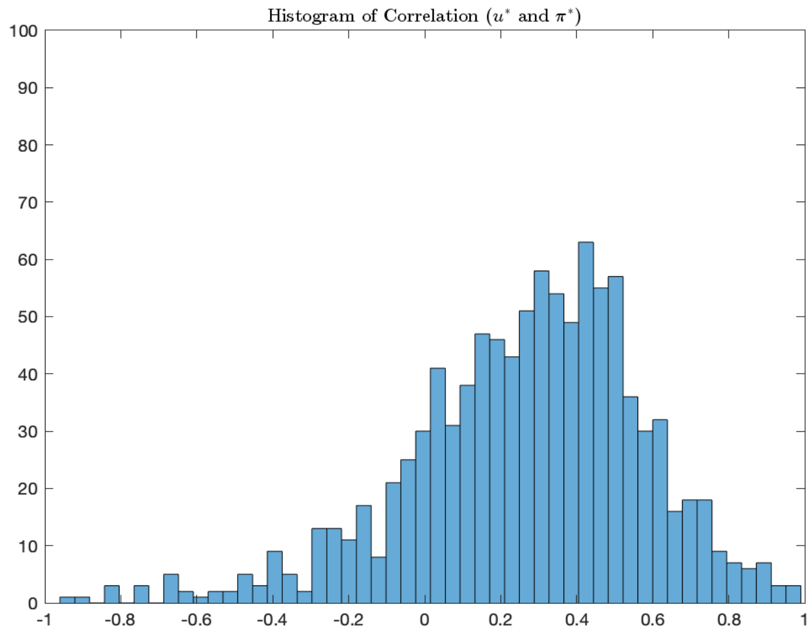

Figure 6: Posterior Correlation of Elements of $\mathbf{F}_t$ (u^* and π^*).

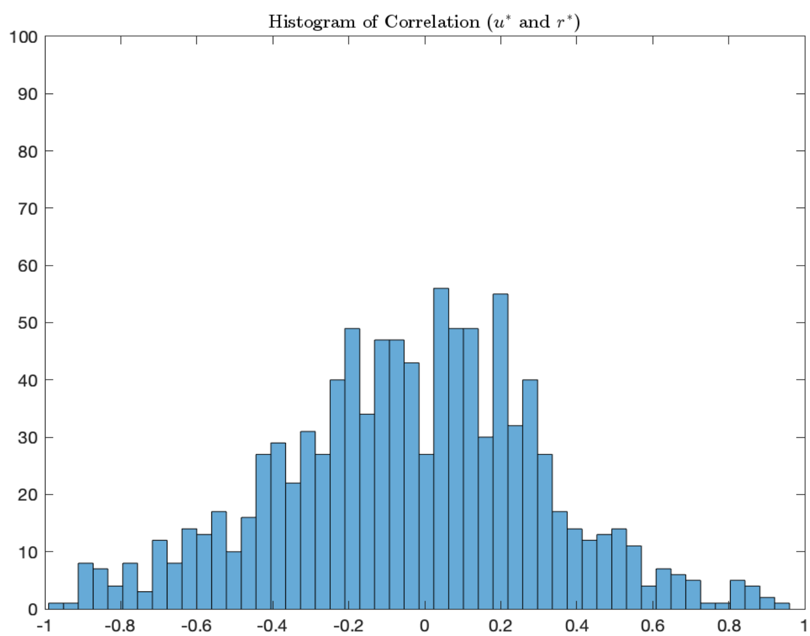

Figure 7: Posterior Correlation of Elements of $\mathbf{F}_t$ (u^* and r^*).

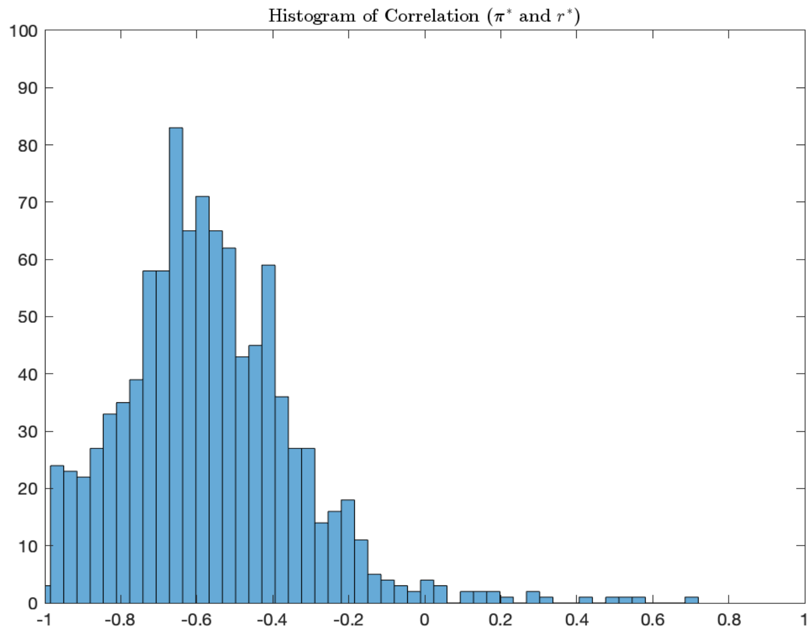

Figure 8: Posterior Correlation of Elements of $\mathbf{F}_t$ (π^* and r^*).

5 Conclusion

We present a modified version of the time-varying-parameter VAR model that is specifically constructed to estimate natural rates/long-run forecasts. We propose priors for both the time-variation in VAR parameters and outlier processes that are new in the literature. In terms of economic conclusions, we find that the recent increase in inflation has has dramatically increased uncertainty surrounding long-run prognoses even if the increase in median forecast (our natural rate) is modest - an increase of around 1.5 percent for annualized inflation from 2015 to 2022.

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A Data description

A.1 Unemployment rate

Our unemployment series is obtained by connecting two different unemployment series encompassing different samples. From 1891:Q4 to 1947:Q4, we employ the historical unemployment rate series constructed by Ramey & Zubairy (2018). The data series has multiple sources including the Federal Reserve Bank of St. Louis FRED database, NBER Macrohistory Database, and Weir (1992). The details of data sources and adjustments are documented in their data appendix. For the periods from 1948:Q1 to 2023:Q2, the data series comes from the St. Louis Fed FRED database. We use the monthly unemployment rate (UNRATE), averaging them quarterly.

A.2 Inflation rate

We use two non-overlapping annual inflation series derived from the GDP deflator. From 1891:Q4 to 2011:Q2, we use the data from Amir-Ahmadi et al. (2016). The data prior to 1947 is from Balke & Gordon (1986), and the rest of the periods come from the St. Louis Fed FRED database. To increase the length of the inflation series, we utilize the GDP deflator from the U.S. Bureau of Economic Analysis (BEA). We use year-over-year inflation rates throughout.

A.3 Real interest rate

For the real interest rate, we use the difference between the 90-day T-bill rate and the moving average of the inflation rate described in the previous section. We use the 90-day T-bill rates from the two data sources. For the period from 1891:Q4 to 2011:Q2, we use the data from Amir-Ahmadi et al. (2016), and add the recent series from the St. Louis Fed FRED database for the period from 2011:Q3 to 2023:Q2. To ensure the consistency of the data, we use the 3-month average rates from the secondary market (TB3MS) for the newly added series following Amir-Ahmadi et al. (2016). One remark is that Amir-Ahmadi et al. (2016) back-cast the 90-day T-bill rates prior to 1920:Q1 using the sample from February 1920 to April 1934. The detail of the back-casting method is described in the supplement document of the article.

We use the 4-quarter moving average of the inflation rate to approximate inflation expectations following Laubach & Williams (2003), Holston et al. (2017), which we use in turn to calculate our measure of the real interest rate. For example, instead of using the inflation rate of 2008:Q4 directly, we use the average of inflation rates from 2008:Q1 to 2008:Q4.

B More Details on f

Suppose that the natural rate of unemployment $\mathbf{N}_t^u$ is a covariance-stationary process with mean $E(N_t) = \mu_{\mathbf{N}_t^u}$ and jth autocovariance $\gamma_j \equiv E(\mathbf{N}_t^u - \mu_{\mathbf{N}_t^u})(\mathbf{N}_{t-j} - \mu_{\mathbf{N}_t^u})$.[19] Then, we can define the population spectrum at frequency ω to be

$$s_N(\omega) = \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j \mathbf{cos}(\omega j)\right],$$

Since the population spectrum is usually unknown, we use the sample periodogram as an estimate of the spectral density $s_u(\omega)$. The sample periodogram can be expressed as

$$\hat{s}_N(\omega) = \frac{1}{2\pi}\left[\hat{\gamma}_0 + 2\sum_{j=1}^{T-1}\hat{\gamma}_j \mathbf{cos}(\omega j)\right],$$

where

$$\hat{\gamma}_j = \begin{cases} T^{-1}\sum_{t=j+1}^{T}(\mathbf{N}_t^u - \bar{\mathbf{N}}_t^u)(\mathbf{N}_{t-j}^u - \bar{\mathbf{N}}_t^u) & \text{for } j = 0, 1, \ldots, T-1 \\ \hat{\gamma}_{-j} & \text{for } j = -1, \ldots, -T+1 \end{cases}$$

and

$$\bar{\mathbf{N}}_t^u = T^{-1}\sum_{t=1}^{T}\mathbf{N}_t^u.$$

To calculate the value of the sample periodogram at frequency ω_j, we use the following fact that the portion of the sample variance of any observed series can be expressed as

$$\frac{1}{2}(\hat{\alpha}_j^2 + \hat{\delta}_j^2) = \frac{4\pi}{T}\hat{s}_N(\omega_j),$$

[19]While our model imparts random walk behavior on its parameters, it is best viewed as a flexible modeling device rather than the true data-generating process. When confronted with stationary data, our model will generally produce forecasts that reflect this stationarity.

where $\hat{s}_u(\omega_j)$ is the above sample periodogram at frequency $\omega_j (= 2\pi j/T)$ for $j = 1, \ldots, (T-1)/2$ when T is an odd integer,

$$\hat{\alpha}_j = \frac{2}{T}\sum_{t=1}^{T}(\mathbf{N}_t^u - \bar{\mathbf{N}}_t^u)\mathbf{cos}(\omega_j(t-1)) \ \ \mathbf{for}\, j = 1, \ldots, (T-1)/2,$$

and

$$\hat{\delta}_j = \frac{2}{T}\sum_{t=1}^{T}(\mathbf{N}_t - \bar{\mathbf{N}}_t^u)\mathbf{sin}(\omega_j(t-1)) \ \ \mathbf{for}\, j = 1, \ldots, (T-1)/2.$$

From the above results, the sample periodogram of $\mathbf{N}_t^u$ can then be represented as

$$\hat{s}_N(\omega_j) = \frac{1}{2\pi T}\left\{\left[\sum_{t=1}^{T}(\mathbf{N}_t^u - \bar{\mathbf{N}}_t^u)\mathbf{cos}(\omega_j(t-1))\right]^2 + \left[\sum_{t=1}^{T}(\mathbf{N}_t^u - \bar{\mathbf{N}}_t^u)\mathbf{sin}(\omega_j(t-1))\right]^2\right\}.$$

C An alternative prior for the innovation to outliers

We also consider an alternative prior for $\sigma^{i,o}$. The new assumption is similar in spirit to our original assumption, but focuses on the mean of $\sigma^{i,o}$ – we assume that this mean equals half of an observables' maximum value during the period when the outlier is active, i.e. $\mathbb{E}\left[\sigma^{i,o}\right] = max\left[\left\{y_t^i\right\}_{t=\text{start period of }i}^{\text{end period of }i}\right]/2$. $\left(*\mathbb{E}\left[\sigma_\ell^{i,o}\right] = max\left[\left\{y_{t,\ell}^i\right\}_{t=\text{start period of }\ell}^{\text{end period of }\ell}\right]/2\right)$. We use the fact that when $\left(\sigma^{i,o}\right)^2$ follows an inverse-gamma distribution, this implies that $\sigma^{i,o}$ follows an inverse-Nakagami distribution (Louzada et al. 2018)[20]. Using the relationship between the inverse-gamma and inverse Nakagami distributions, and the assumption that $mode((\sigma^{i,o})^2)$ equals to $std((\sigma^{i,o})^2)$, we can calculate the scale parameter for the inverse-gamma distribution directly. To summarize, this alternative prior is

$$\sigma^{i,o}_{alternative} \sim \text{INK}\left(\underbrace{max\left[\left\{y_t^i\right\}_{t=\text{start period of }i}^{\text{end period of }i}\right]/2}_{mean}, \underbrace{4.48}_{shape}\right)$$

$$\left(*\sigma^{i,o}_{\ell,alternative} \sim \text{INK}\left(\underbrace{max\left[\left\{y_{t,\ell}^i\right\}_{t=\text{start period of }\ell}^{\text{end period of }\ell}\right]/2}_{mean}, \underbrace{4.48}_{shape}\right)\right)$$

Figure C plots our benchmark prior and the inverse Nagakami prior. We see that there are substantial differences.

[20] *If* $x \sim IG(\alpha,\beta)$, $\sqrt{x} \sim INK(\mu,\Omega)$ *where* $\alpha = \mu$ *and* $\Omega = \alpha/\beta$. *When* $\sqrt{x} \sim INK(\mu,\Omega)$, $\mathbb{E}\left(\sqrt{x}\right) = \frac{1}{\Gamma(\mu)}\left(\frac{\mu}{\Omega}\right)^{\frac{1}{2}}\Gamma\left(\mu - \frac{1}{2}\right)$.

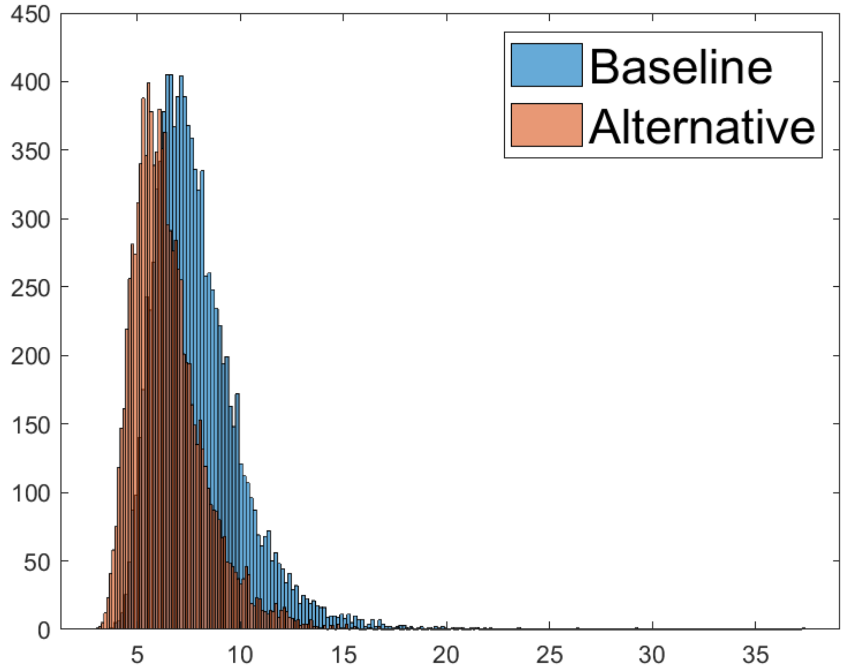

Figure 2. Histogram for baseline and alternative prior, $\sigma^{i,o}$

We now show that our results are robust to using the inverse-Nakagami distribution, as outlined above, for the volatility of the outliers.

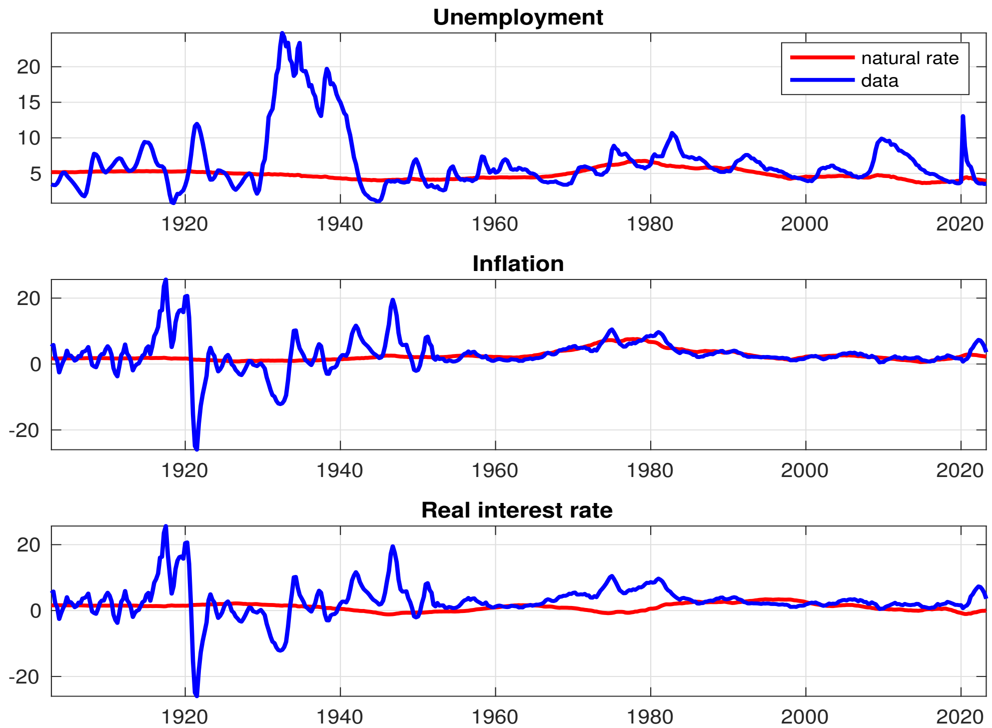

Figure 9: Results with the inverse-Nakagami distribution for the outlier variance.

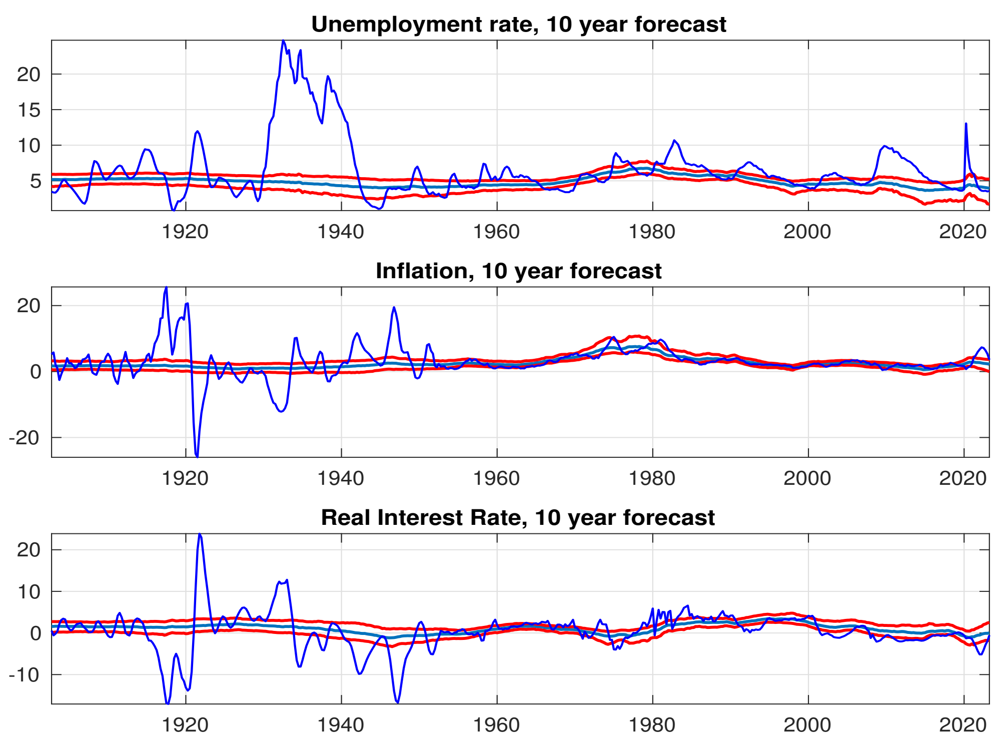

Figure 10: Results with the inverse-Nakagami distribution for the outlier variance.

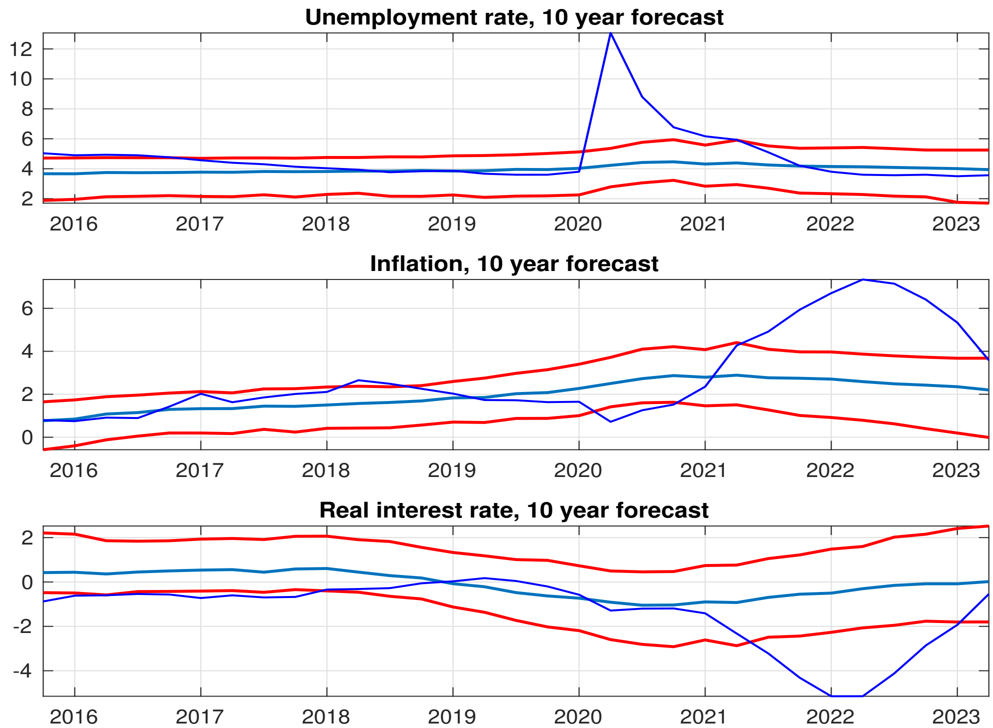

Figure 11: Results with the inverse-Nakagami distribution for the outlier variance.

D Comparison with other Natural Rate Measures

To give a sense of how much our modeling choices matter, we now compare our natural interest rate to existing measures of Laubach & Williams (2003) and Lubik & Matthes (2015). We focus here on the real interest rate since those other measures are readily available. Note that the Lubik & Matthes (2015) measure, while using a similar TVP-VAR model, uses a shorter forecasting horizon for its natural rates (5 years), which explains part of the difference.

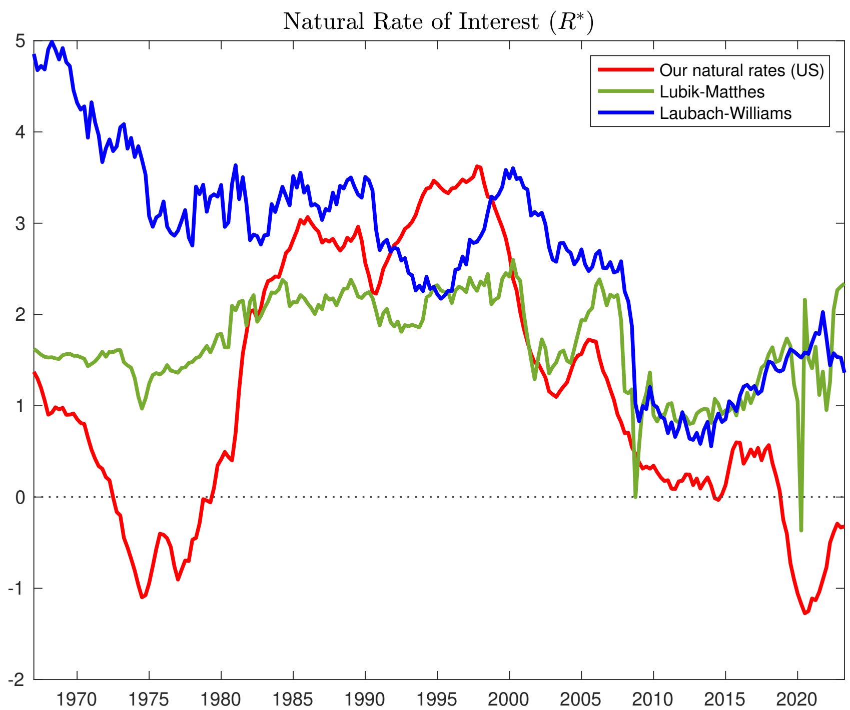

Figure 12: Various Measures of R^*

E Monte Carlo Study

We conduct a Monte Carlo study to assess the robustness of our estimation method. The setup of the simulation study is as follows. First, we generate time series following an AR(1) process. To resemble the movement of the actual data, we extract low-frequency movements from the actual time series using the method of Müller & Watson (2020) and treat these as the intercepts in the DGP. We then compare forecasts based on the true parameter values to those derived following the definition of natural rates in this paper. For estimation, we apply the baseline model but switch off the measurement errors and outlier processes, as they are designed for specific historical windows–namely, the pre-WWII and the COVID pandemic. Finally, we compare the "true " natural rates with the estimated natural rates and measure the accuracy of the estimation.

E.1 Data generating process

We consider a trivariate DGP for unemployment, inflation, and the real interest rate, $\boldsymbol{X}_t = (u_t,\ \pi_t,\ r_t)^\top$, specified as an AR(1) process:

$$\underset{3\times1}{X_t} - \underset{3\times1}{\mu_t} = \underset{3\times3}{A}\left(\underset{3\times1}{X_{t-1}} - \underset{3\times1}{\mu_t}\right) + \underset{3\times1}{\nu_t}, \quad t = 1, 2, \cdots, T,$$

where the coefficient matrix is

$$A = \rho_A \times I_3$$

and the error vector is

$$\nu_t \sim N\left(\mathbf{0}, \Sigma_{I_3}\right).$$

We set the sample length to $T = 527$, choose $\rho_A = 0.4$ to match the persistence observed in the data, and initialize at $\boldsymbol{X}_0 = \mathbf{0}$.

To align the simulations with slow-moving features of the actual series, we obtain μ_t using the approach of Müller & Watson (2020). For each variable, we regress the observed data on $\boldsymbol{\Psi}_T^0 = [\mathbf{1}_T, \boldsymbol{\Psi}(T)]$, where

$$\underset{T\times Q}{\boldsymbol{\Psi}(T)} = \left[\; f(t,q) \;\right], \; f(t,q) = \sqrt{2}\cos\left(2\pi \times \left(\frac{2T}{q}\right)^{-1} \times t\right),$$

with indices $t = 1, \cdots, T$ and $q = 1, \cdots, Q$, and we take the fitted values from this regression as μ_t. Because the period of the cosine function pins down the frequency of the cycle,[21] we set $Q = 10$, implying a 25-year business cycle in quarterly data.

E.2 "True" natural rates

Given the DGP, we derive the "true" natural rates as the ten-year-ahead forecasts following the definition of natural rates in Section 3.1. First, we observe that

$$X_{t+40} = \left(\sum_{j=0}^{39} A^j m_{t+40-j}\right) + A^{40} X_t + \sum_{j=0}^{39} A^j \nu_{t+40-j}$$

where $m_t = (I - A)\mu_t$. By taking the conditional expectation at time t of both sides, we obtain

[21]For $\cos(2\pi t/P)$, the period is P. Under our normalization $P = 2T/q$, so larger q targets lower frequencies. See Section 3.1 in Müller & Watson (2020) for details.

$$
\begin{aligned}
\mathbb{E}_t X_{t+40} &= m_t \left(\sum_{j=0}^{39} A^j \right) + A^{40} X_t \\
&\approx m_t \left(\sum_{j=0}^{39} A^j \right)
\end{aligned}
$$

since $A^{40} X_t$ is negligible. Note that μ_t is assumed to follow a random walk by the model.

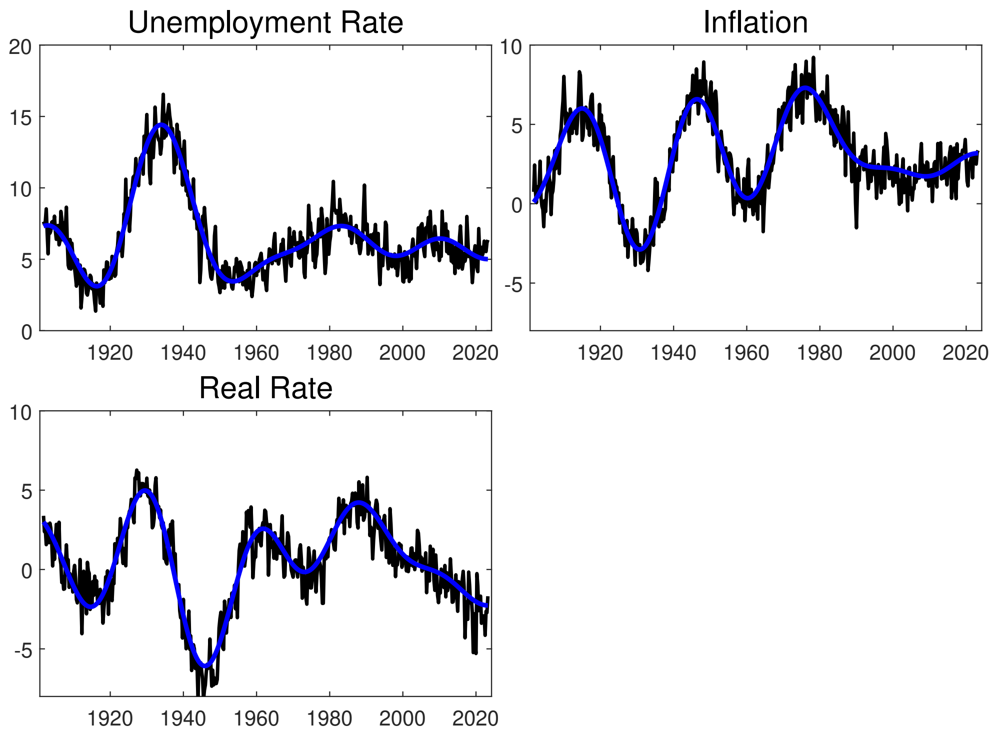

Figure 13: Representative example of the DGP. The black line shows simulated data, while the blue line shows the "true" natural rate.

E.3 Results

We generate $N = 100$ Monte Carlo replications from the DGP and estimate our baseline model for each replication, without the measurement errors and outlier processes. For comparison, we also estimate the model imposing $\kappa_Q = 0.01$ as in Primiceri (2005). Figure 14 displays a representative replication; qualitatively similar patterns are observed across the 100 replications. Relative to the $\kappa_Q = 0.01$ specification, the baseline model more closely aligns with the "true" natural rate paths.

To assess comparative accuracy, we compute the RMSE of the estimated natural rate against the "true" path for each variable and average over the $N = 100$ replications, then compare the resulting mean RMSEs for the baseline and the $\kappa_Q = 0.01$ specification. As reported in Table 1, the baseline yields uniformly lower RMSEs, consistent with Figure 14.

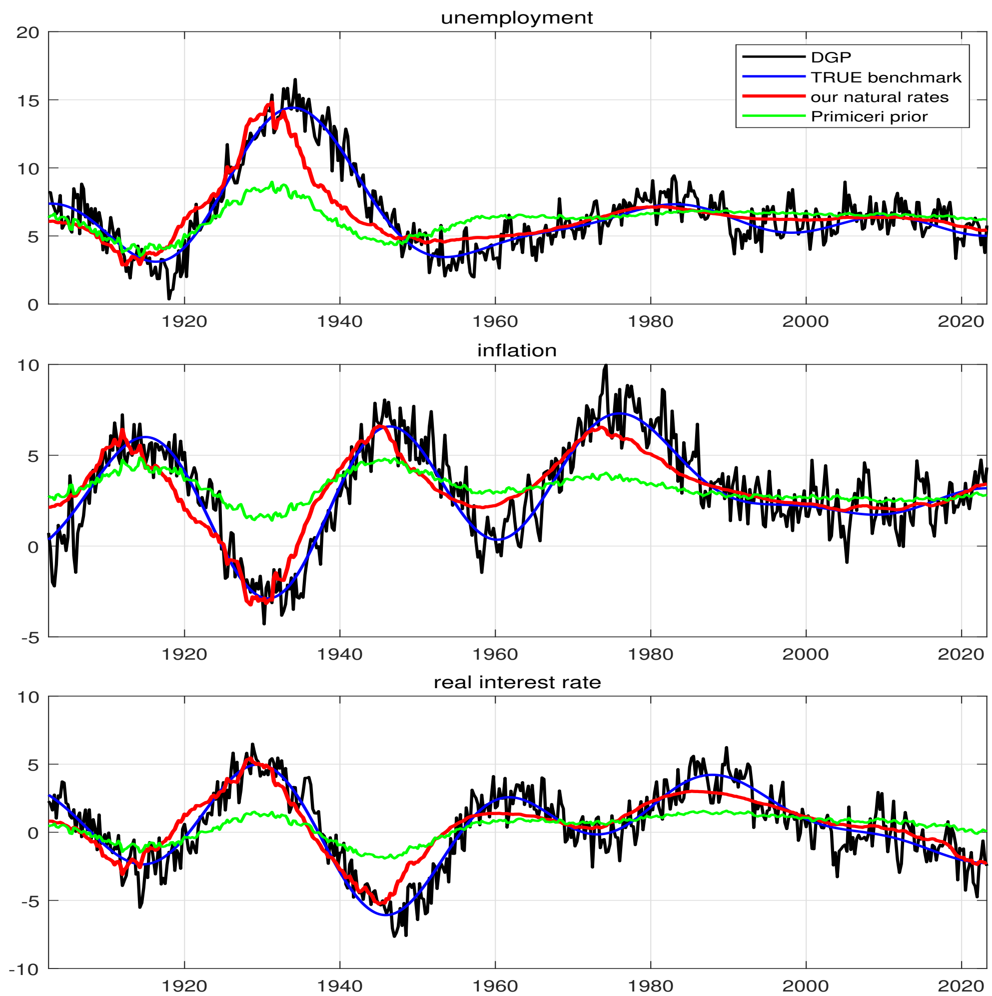

Figure 14: Results for a representative replication. The blue line shows the "true" natural rate path, the red line shows our baseline model, and the green line shows the estimate under the Primiceri prior ($\kappa_Q = 0.01$).

Table 1: Mean RMSE across 100 Monte Carlo replications

Variable	Baseline	Primiceri prior ($\kappa_Q = 0.01$)
Unemployment	**1.61**	2.38
Inflation	**1.40**	1.98
Real interest rate	**1.32**	1.93