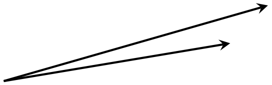
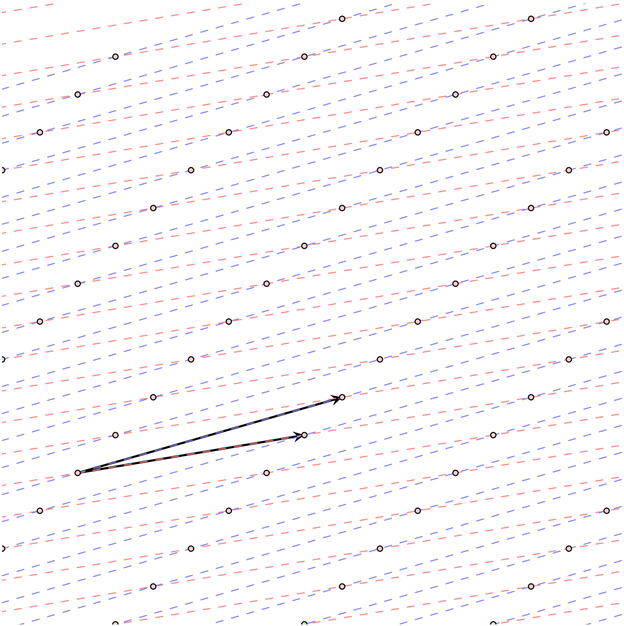
What Is a Lattice?

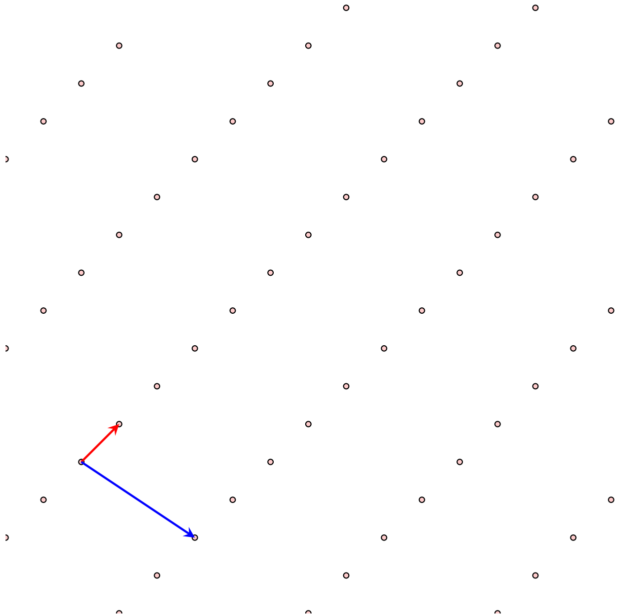
Given n linearly independent vectors $\mathbf{b_1}, \ldots, \mathbf{b_n} \in \mathbb{R}^m$ ($n \leq m$), the lattice generated by them is the set of vectors

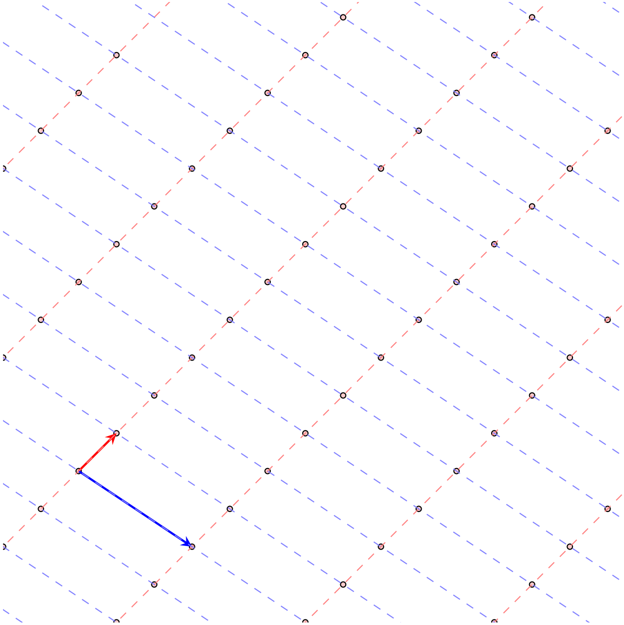
$$L(\mathbf{b_1}, \ldots, \mathbf{b_n}) = \{\sum_{i=1}^{n} x_i \mathbf{b_i} : x_i \in \mathbb{Z}\}$$

The vectors $\mathbf{b_1}, \ldots, \mathbf{b_n}$ form a basis of the lattice.



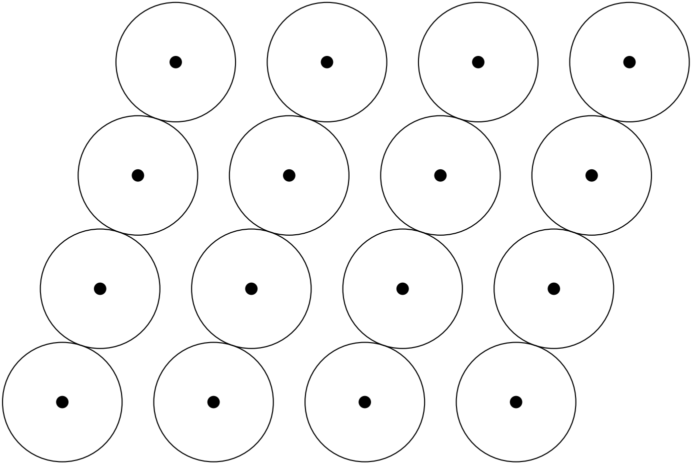






Lattices and Codes

Shortest Vector and Minimum Distance, Closet Vector and Maximum Likelyhood Decoding, Unambiguous decoding, List Decoding and BDD, Deep Holes



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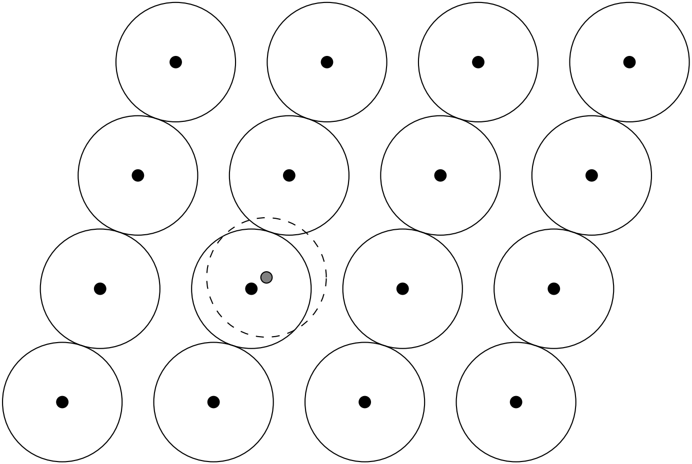

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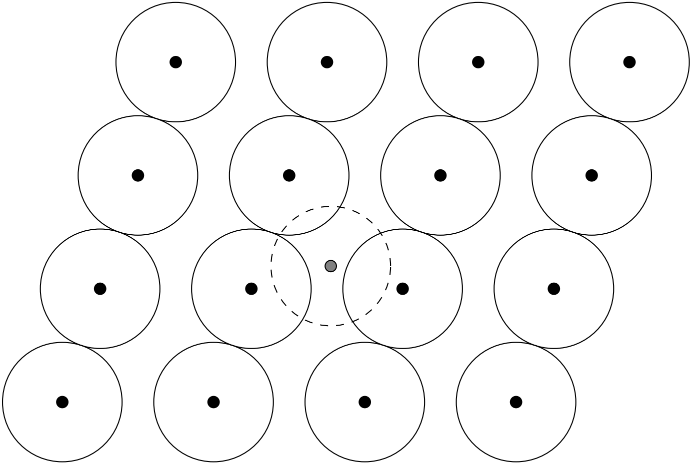

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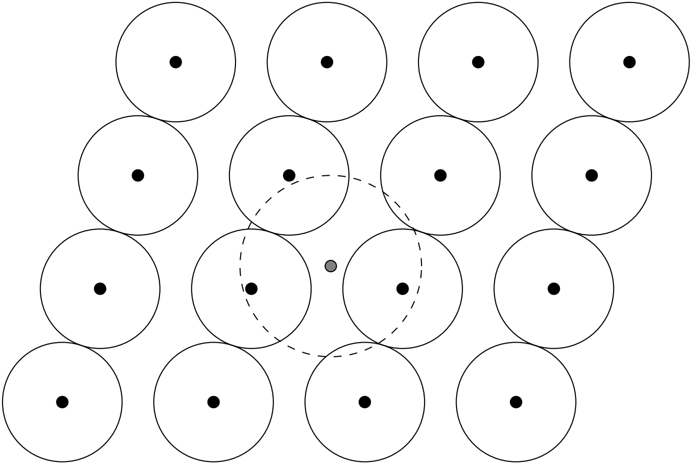

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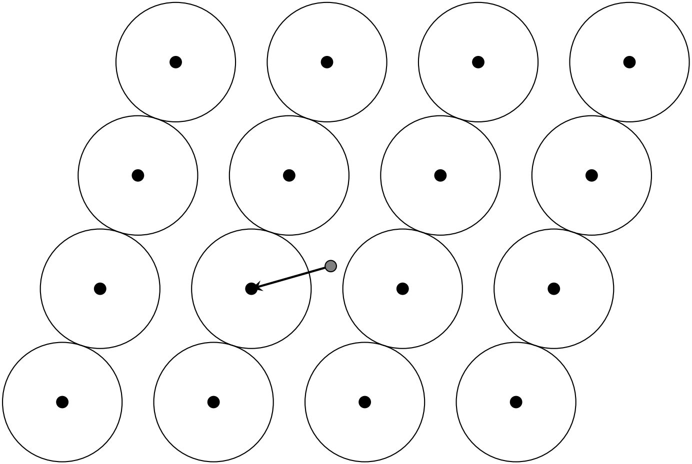

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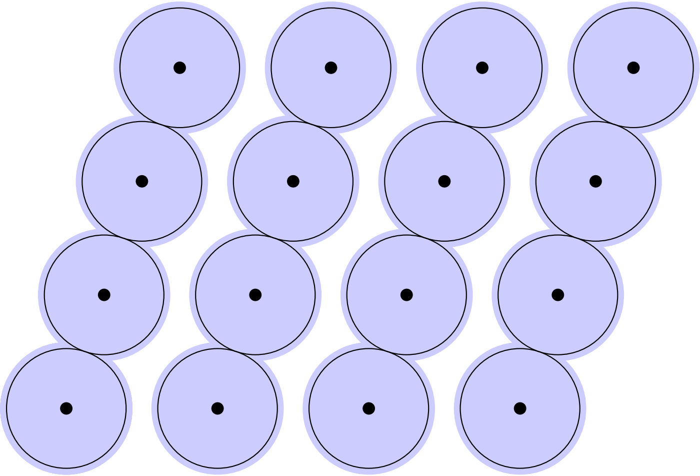

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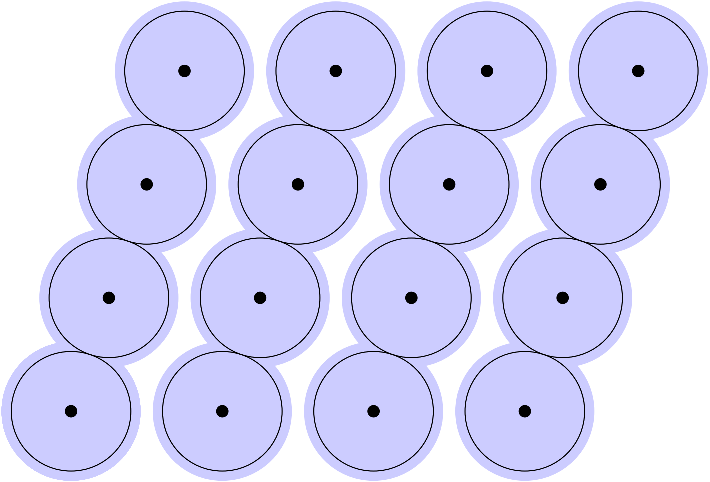

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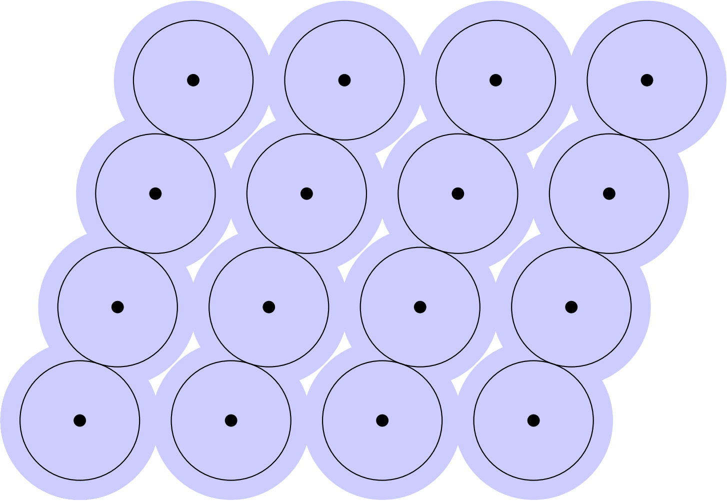

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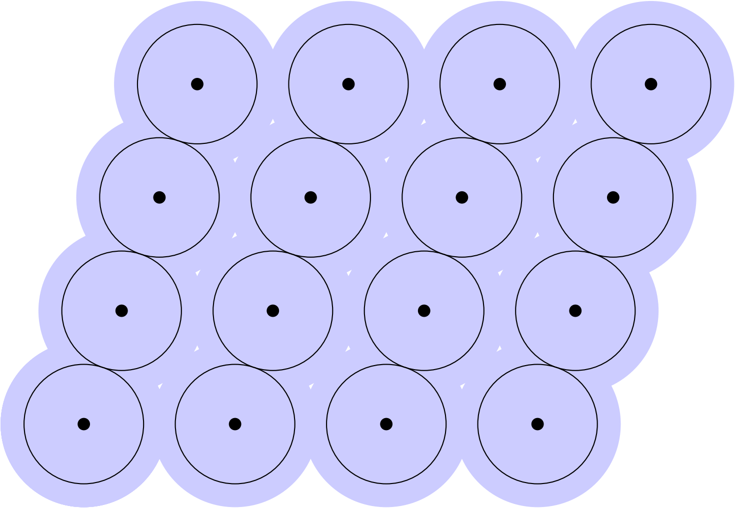

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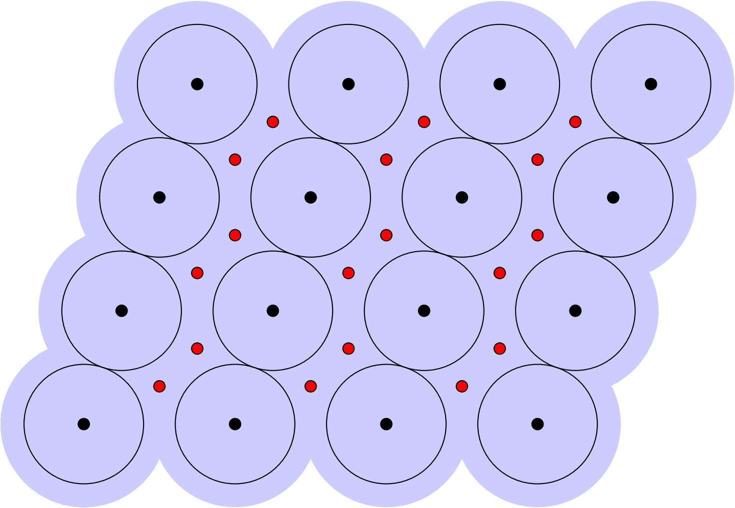

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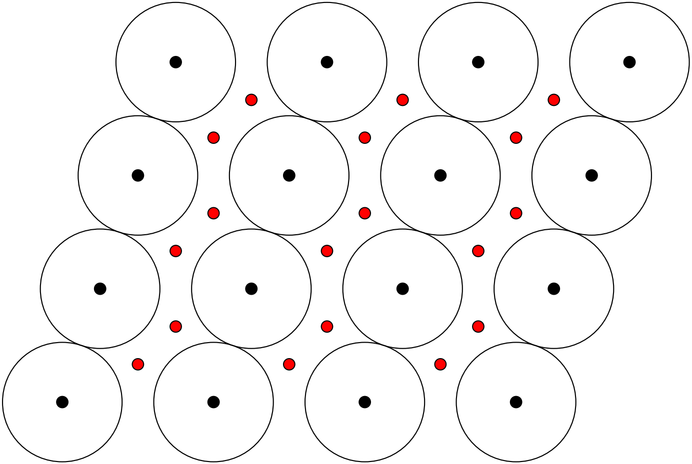
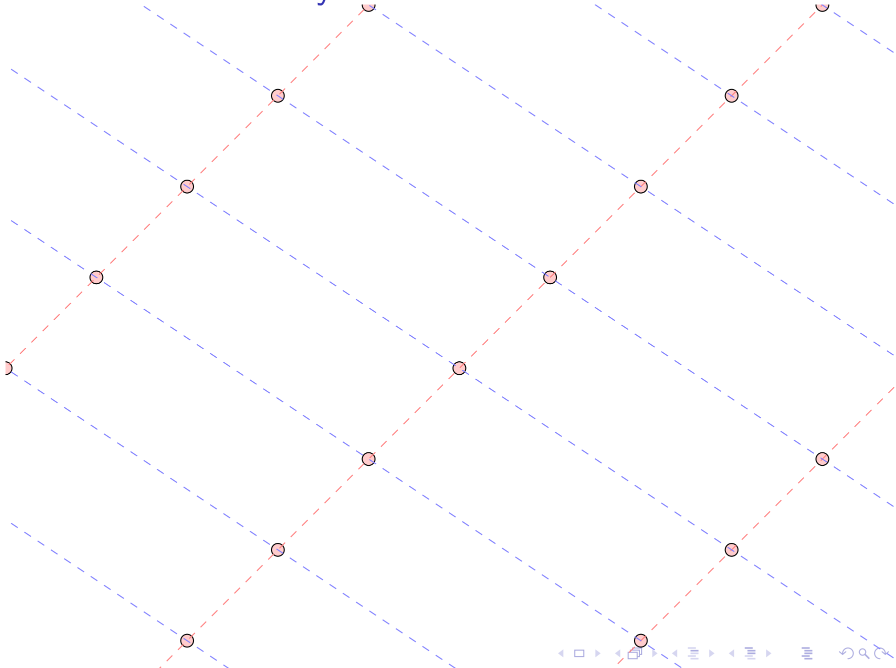


Minkowski Convex Body Theorem



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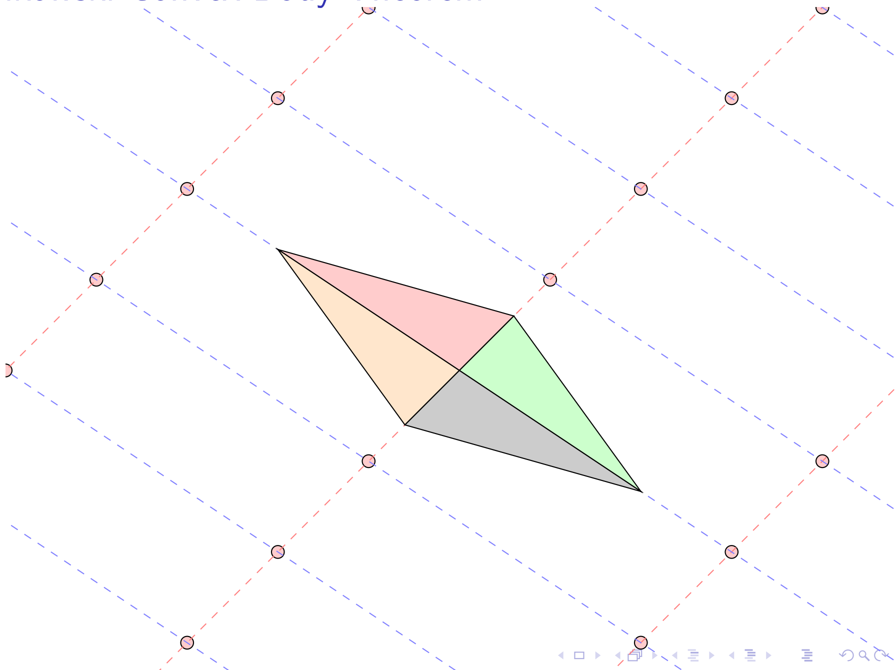

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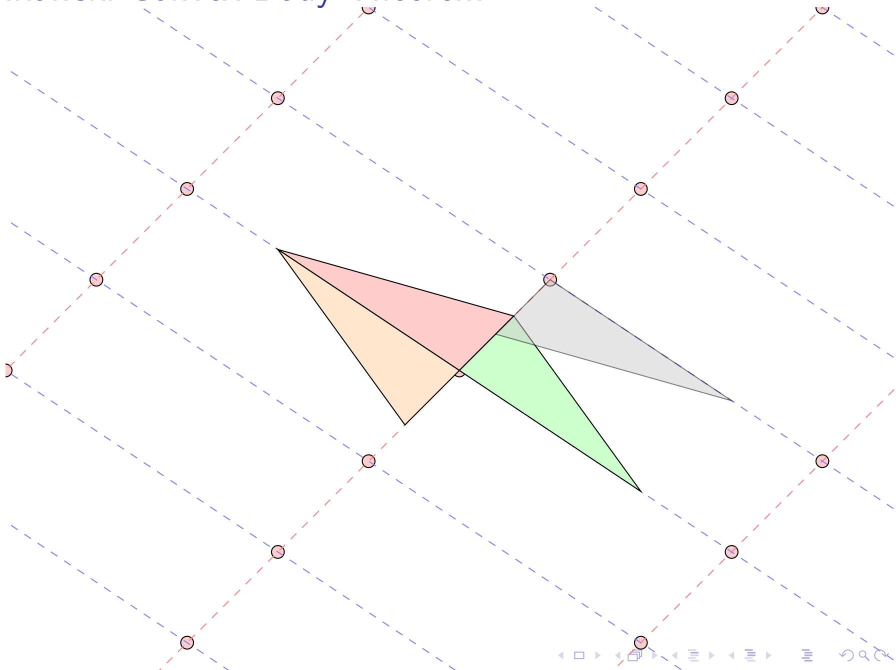

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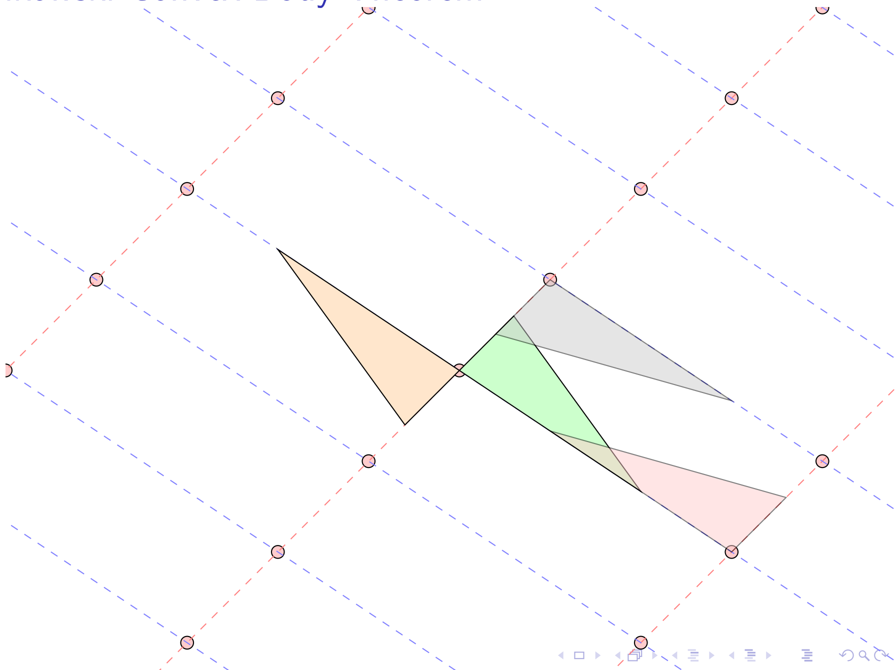

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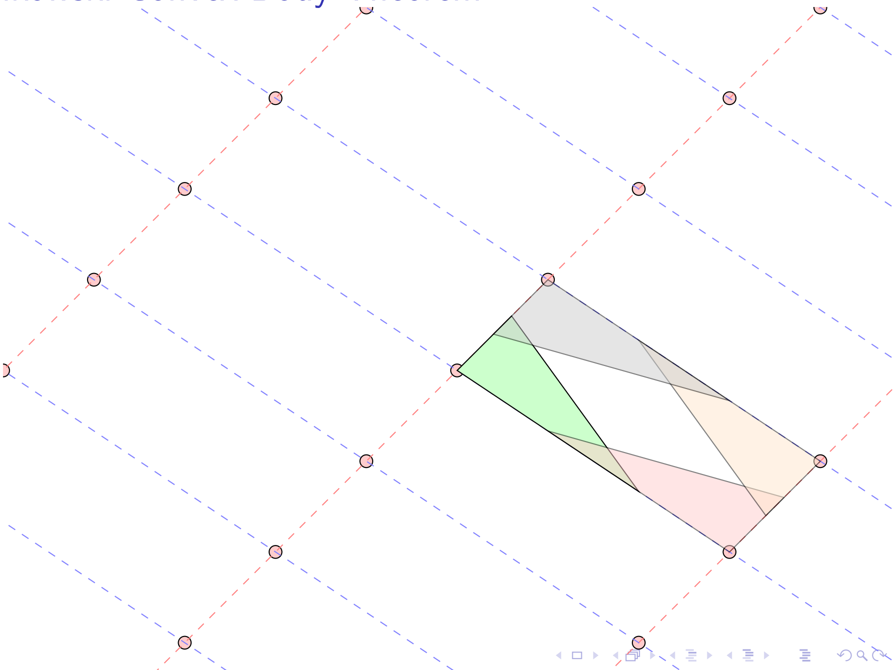

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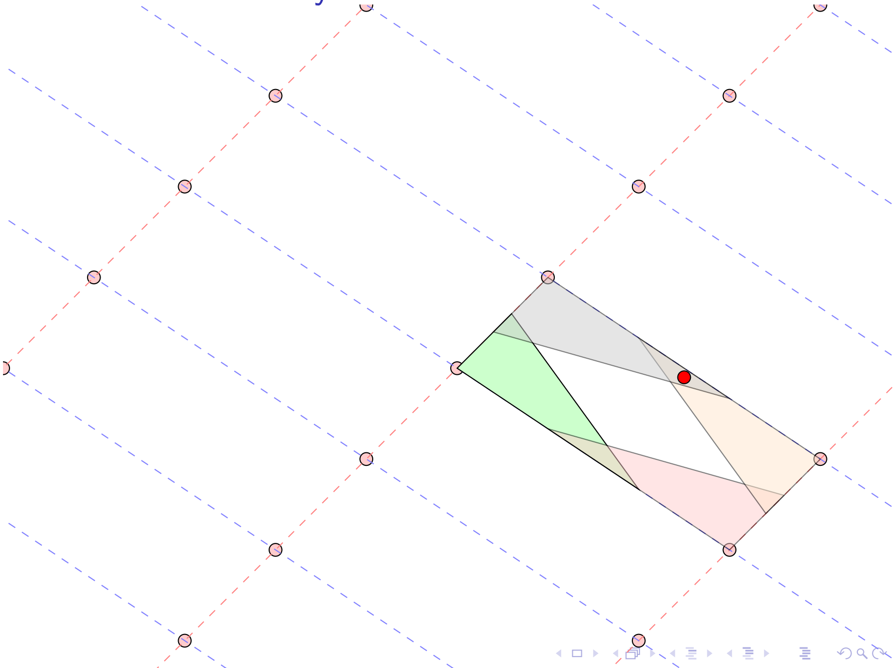

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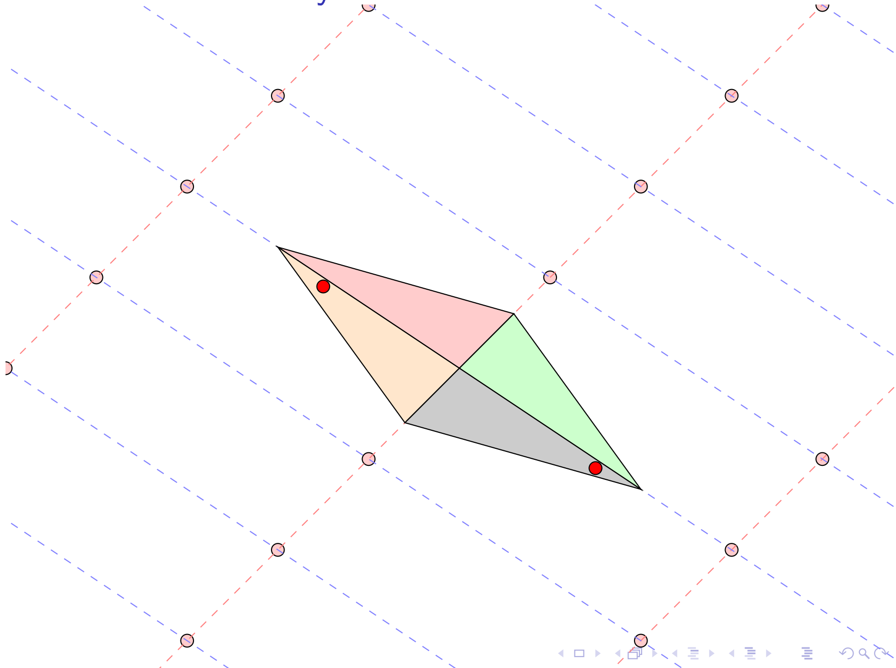

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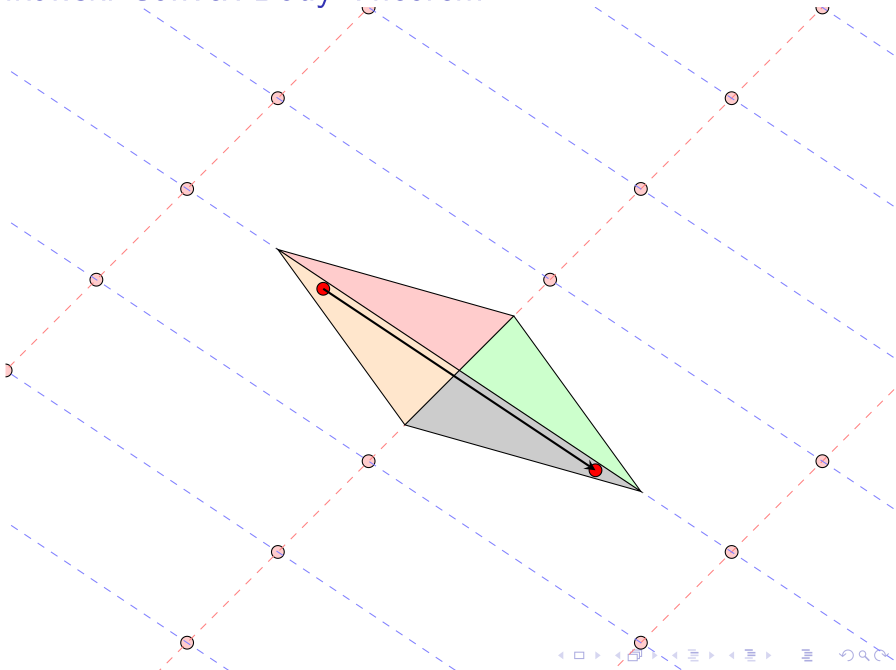
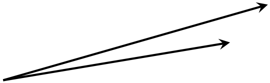


The shortest vector

- Hermite bound: $\sqrt{n} det(L)^{1/n}$ (uniform)
- On average has length $(1+o(1))\sqrt{\frac{n}{2e\pi}} det(L)^{1/n}$ (Gauss Heuristic)
- Must have length less than $(1+o(1))\sqrt{\frac{2n}{e\pi}} det(L)^{1/n}$. (The Minkowski Convex Body Theorem)

Lattice reduction at dimension 2



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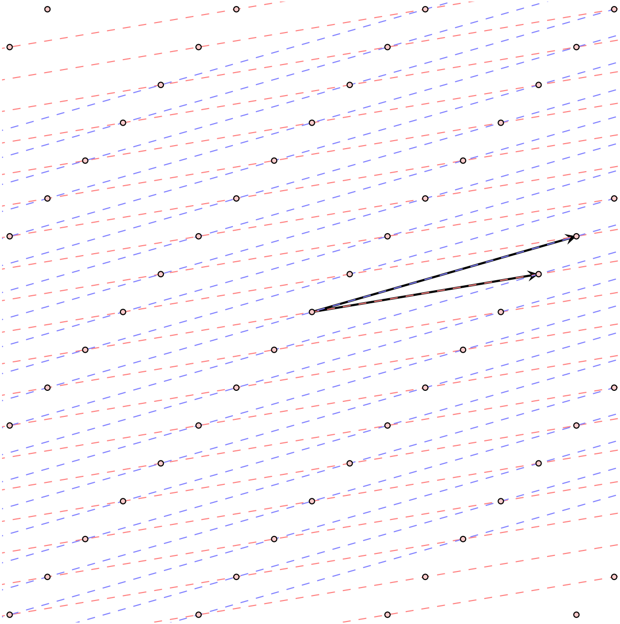

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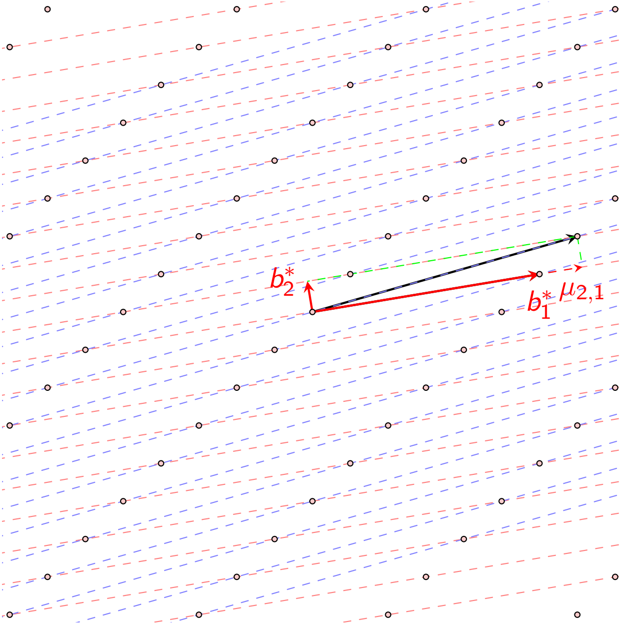

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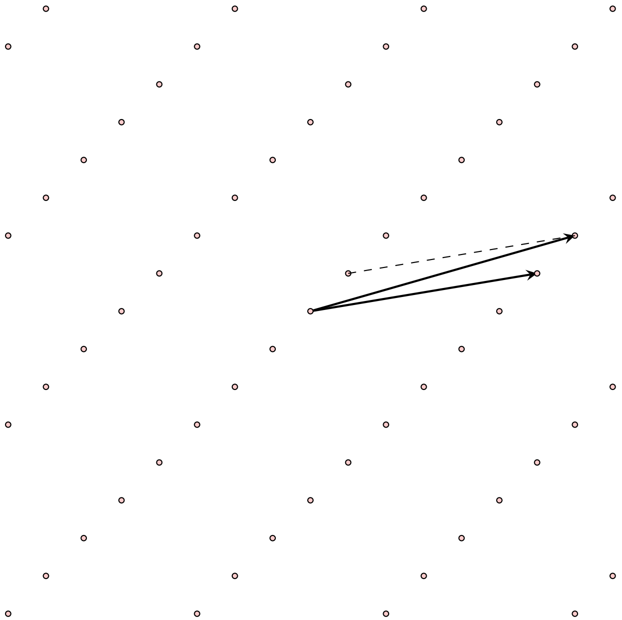

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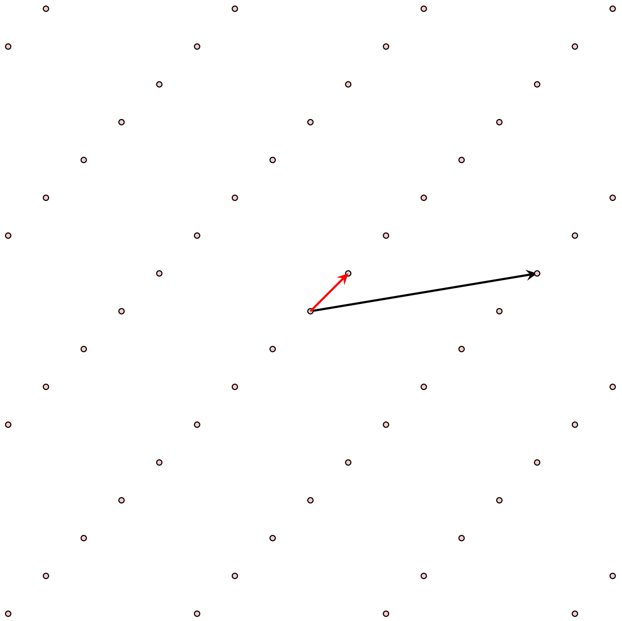

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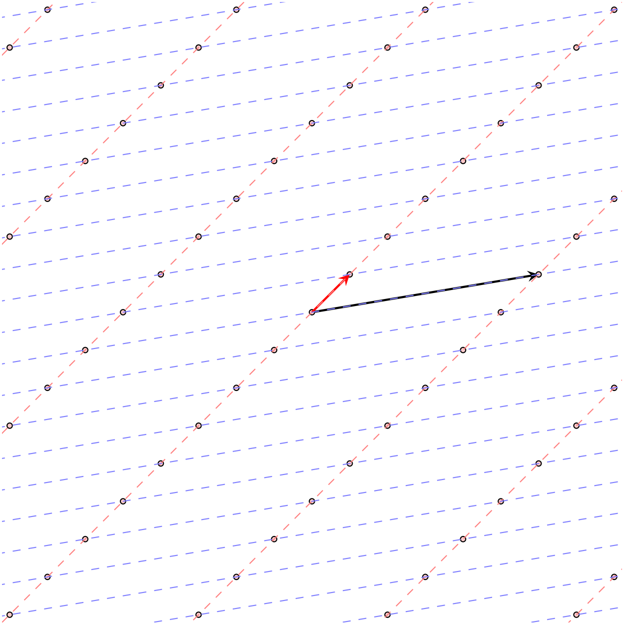

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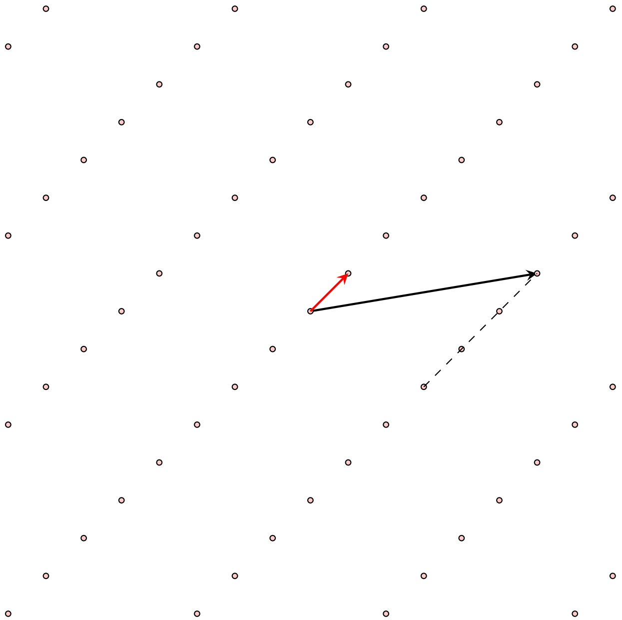

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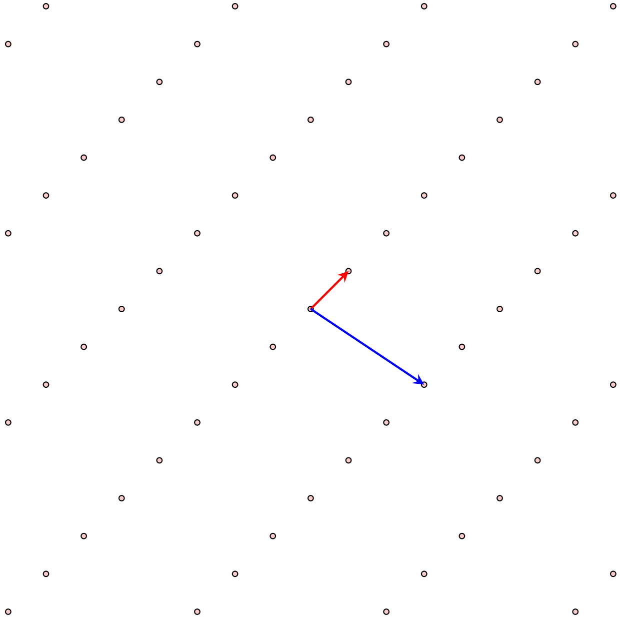

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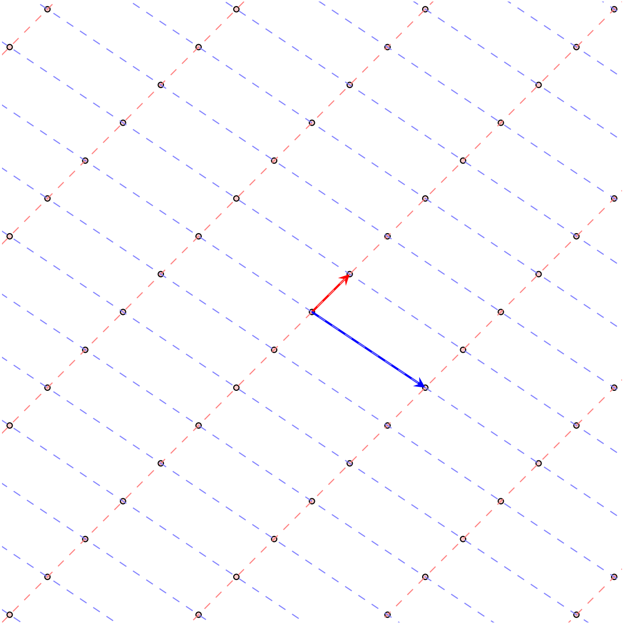
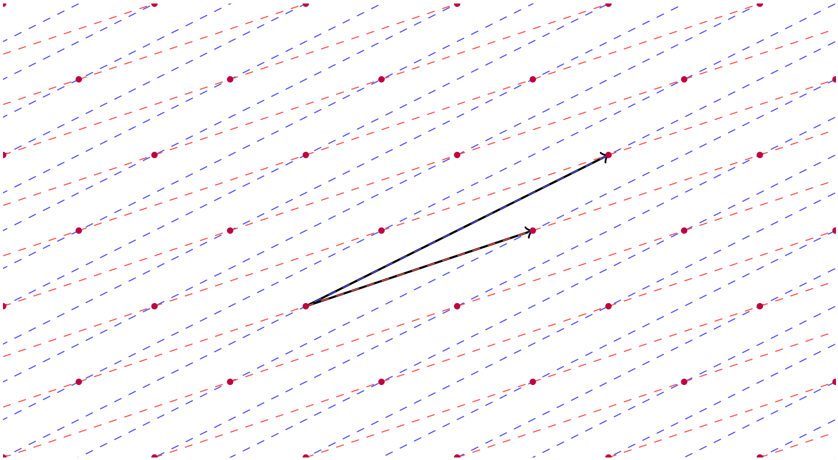


Closest Vector Problem

For one dimensional lattices, rounding works.

Closest Vector Problem–Dimension two



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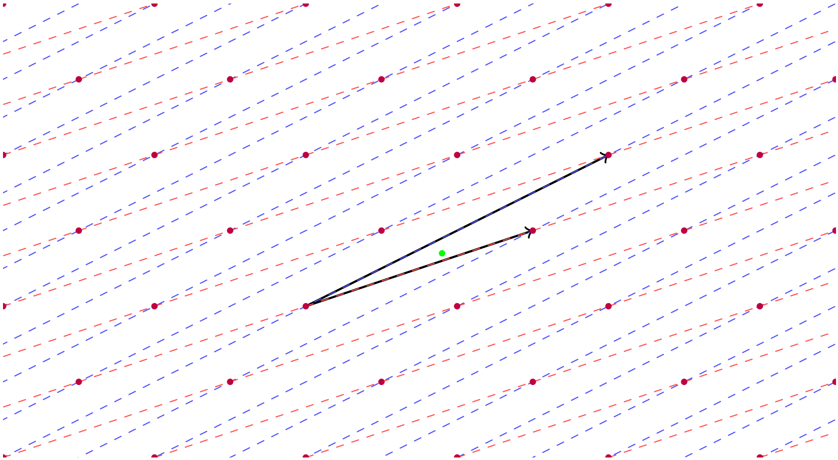

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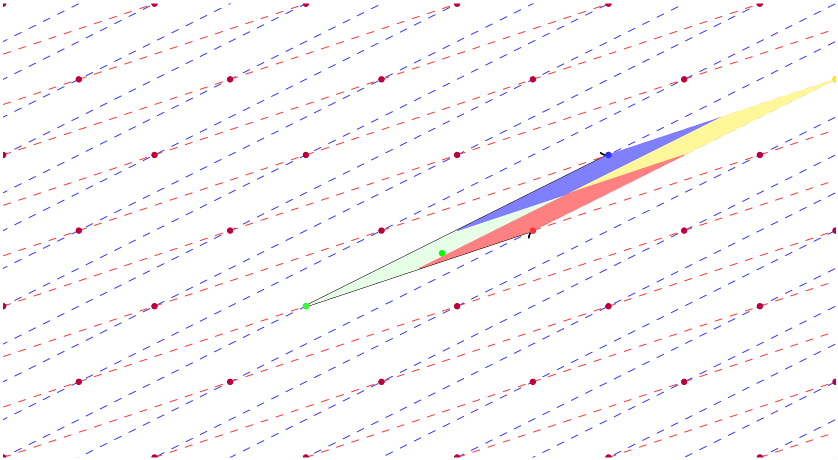

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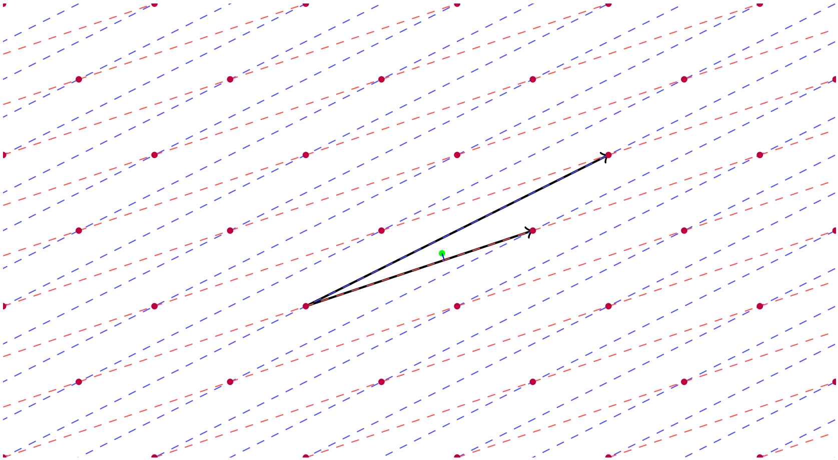

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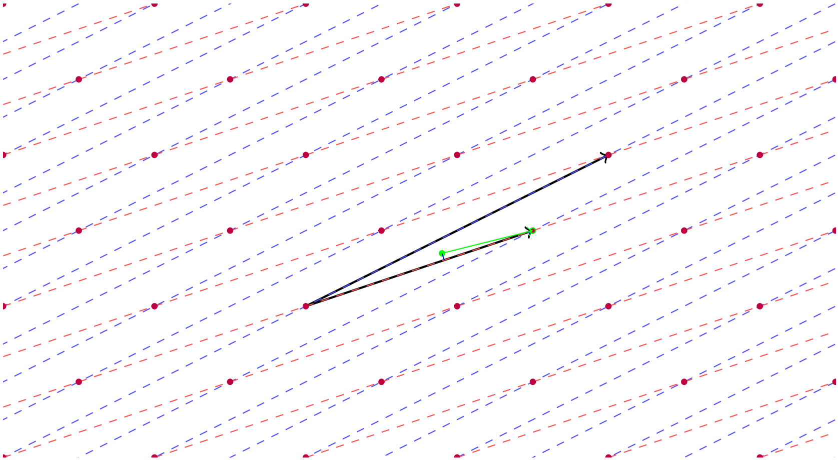

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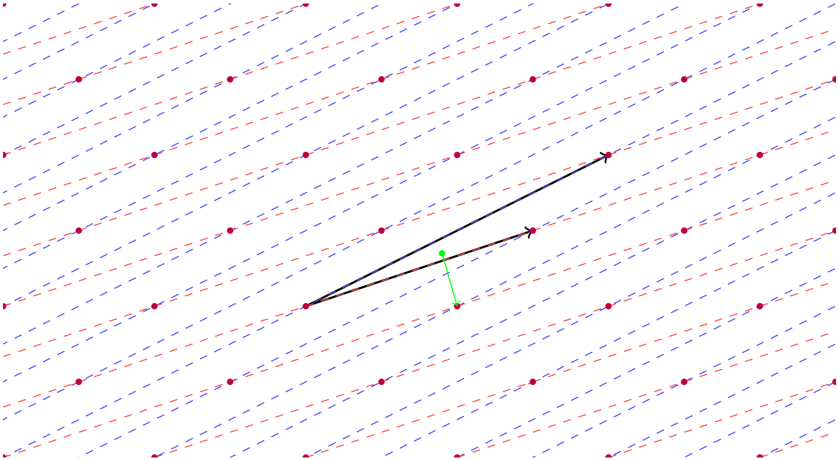

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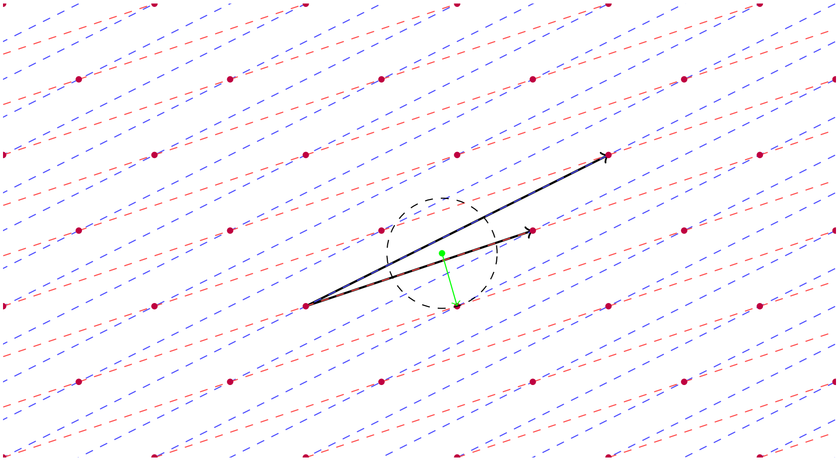

The LLL reduction

There is a polynomial time algorithm to find a vector b_1 in the lattice such that $|b_1| \leq (2/\sqrt{3})^n \lambda_1$, where λ_1 denotes the length of the shortest vector.

LLL explained

$$b_1 = b_1^*$$

$$b_2 = \mu_{2,1} b_1^* + b_2^*$$

$$b_3 = \mu_{3,1} b_1^* + \mu_{3,2} b_2^* + b_3^*$$

$$b_4 = \mu_{4,1} b_1^* + \mu_{4,2} b_2^* + \mu_{4,3} b_3^* + b_4^*$$

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Do

1. Calculate b_i^*
2. Apply integral linear operations so $|\mu_{ij}| \leq 1/2$
3. Swap if $\delta |b_i^*| > |\mu_{i+1,i} b_i^* + b_{i+1}^*|$

Until no swapping in the last step

LLL explained

$$b_1 = b_1^*$$

$$b_2 = \mu_{2,1} b_1^* + b_2^*$$

$$b_3 = \mu_{3,1} b_1^* + \mu_{3,2} b_2^* + b_3^*$$

$$b_4 = \mu_{4,1} b_1^* + \mu_{4,2} b_2^* + \mu_{4,3} b_3^* + b_4^*$$

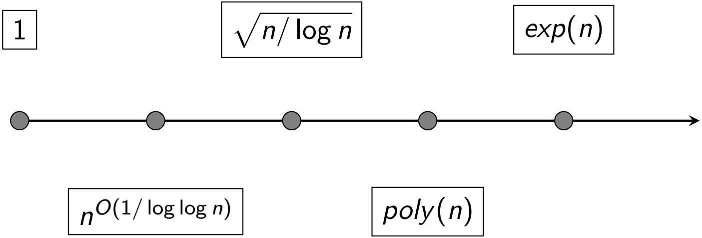
It is δLLL-reduced $(1/4 < \delta < 1)$ if $|\mu_{i,j}| \leq 1/2$ and

$$\begin{aligned}
\delta|b_1^*| &\leq |\mu_{2,1} b_1^* + b_2^*| \\
\delta|b_2^*| &\leq |\mu_{3,2} b_2^* + b_3^*| \\
\delta|b_3^*| &\leq |\mu_{4,3} b_3^* + b_4^*| \\
&\cdots
\end{aligned}$$

More algorithms for SVP

- Can be found in $4^{n+o(1)}$ arithmetic operations deterministically. (Micciancio-Voulgaris 2010),
- or random $2^{(1+\epsilon)n}$ time (Divesh Aggarwal, Daniel Dadush, Oded Regev, Noah Stephens-Davidowitz 2015).
- For approximation factor 2^k, need time $2^{O(n/k)}$.

Complexity of approx-SVP



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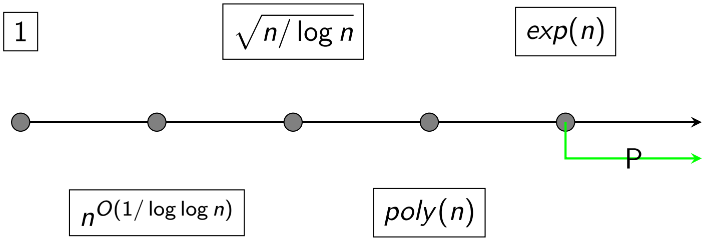

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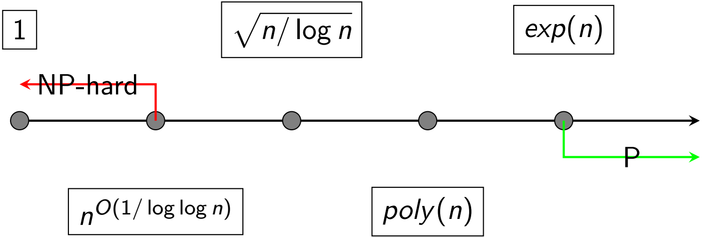

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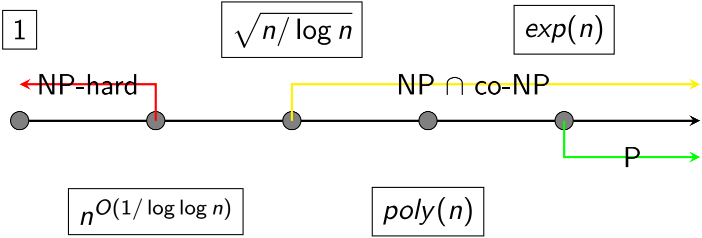

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