Primality Testing

With Artificial Feedforward Neural Networks

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CS 5033, Computer Science, University of Oklahoma

Bonus Project Presentation, Fall 2022

Overview

1. Introduction: We'll introduce and motivate some key ideas.
2. Data: We need to be careful.
3. Results: We'll look at some figures.
4. Conclusion & References

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All prime numbers are *odd*, and 2, is the *oddest* of them all.

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Bitwise Complexity For a b-bit x, division takes $\mathcal{O}(b^2)$ machine steps.

$$\begin{array}{ccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\ 2^2-2 & 2^2-1 & 2^2 & 2^3-3 & 2^3-2 & 2^3-1 & 2^3 & 2^4-7 & \ldots \\ 4 & 4 & 4 & 9 & 9 & 9 & 9 & 16 & \ldots \end{array}$$

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Polynomial Time Algorithm exists, but no proof of optimality.

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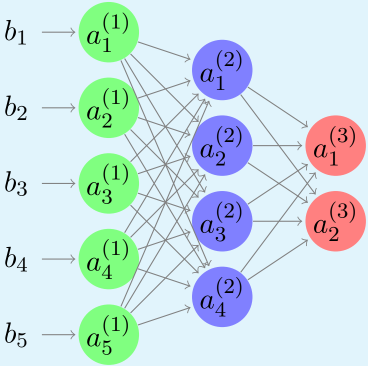
Universal Approximation Theorem (UAT)

There exists an $n \in \mathbb{N}$ such that a feed-forward neural network (FFNN) with only one hidden layer of n units can approximate any continuos function between euclidean spaces as closely as possible. [2]



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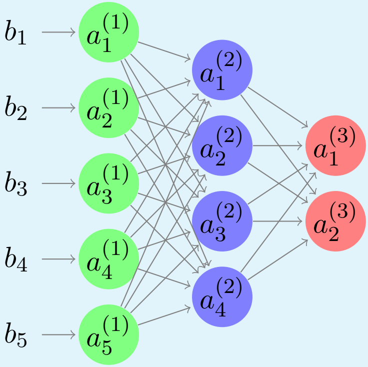

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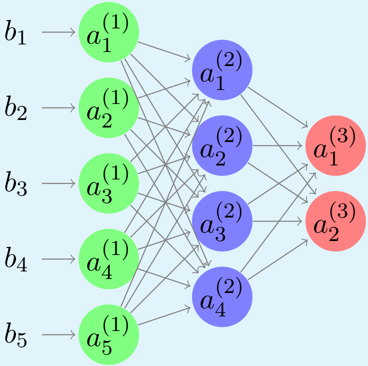

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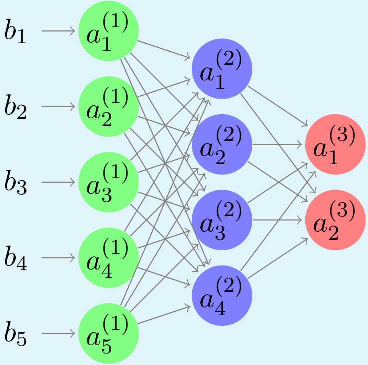
3. How does n change as b increases?



Approach: Base Plan

1. Let $n \to 1, b \to 7, \varepsilon > 0.5$,
2. If $b > N$, end.
3. Train a FFNN with n hidden units on b bit numbers till log-loss does not improve.
4. If accuracy $< \varepsilon$, $n \to n + 1$. Goto step 3.
5. If accuracy $> \varepsilon$, $n \to 1, b \to b + 1$. Note (b, n) Goto step 2.

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$$\pi(2^b) > \frac{2^b}{\ln(2^b)} > \frac{2^b}{\lg(2^b)} = \frac{2^b}{b} = \frac{2^b}{2^{\lg(b)}} = 2^{b-\lg(b)}$$

Then, $1 - (2^{b-\lg(b)}/2^b) = 1 - 2^{-\lg(b)} = \frac{b-1}{b}$ and for 10 bits, we can achieve 90% accuracy by just classifying as composites.

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6. Loop and generate once then store and load.

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$$\frac{2^{64-6} \times 16 \text{ bits}}{2^{50} \text{ petabits}} = 2^{12} \text{ petabits} \Rightarrow \frac{2^{12} \text{ petabits}}{8 \text{ bits}} = 512 \text{ petabytes.}$$

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2. We work with only 32 bit primes. This still takes days of training time and somewhere around 120 gigbytes of memory when the primes and composites' stored as a bit matrix for training.

Results: How it feels.

Figure: At this point, I just ran the experiment on a super computer.

Results: Number of Neurons versus Bits.

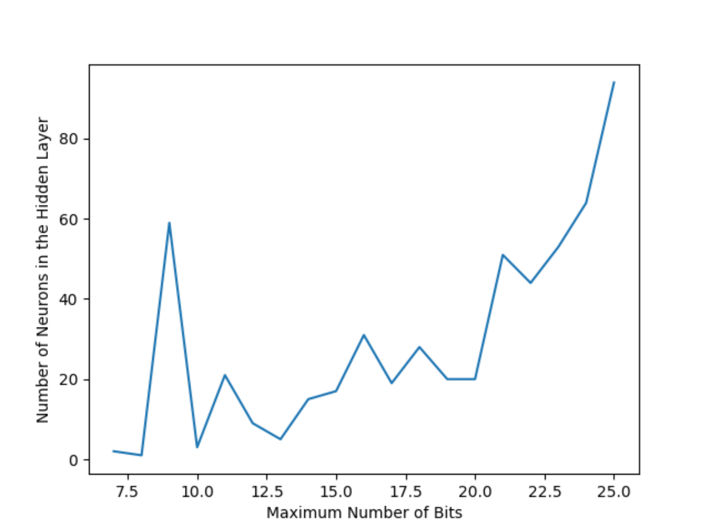

Figure: n vs. b.

Results: Accuracy vs Bits and Neurons.

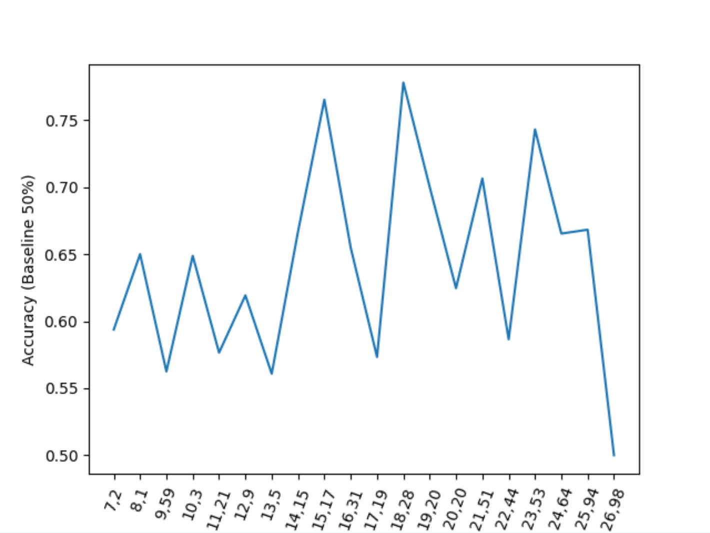

Figure: Accuracy vs (b, n).

Conclusion

1. We can't blindely apply machine learning on number theory problems.
2. Fawzi et al. improved matrix multiplication bound using neural networks [1].

References

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Thank You!
Questions?