INF8245AE(Fall 2025) : Machine Learning — Given on: Aug 27, 05 pm

Assignment #0 — Due on : Sept 10, 10 pm

- The goal of this assignment is a self-assessment on fundamental problems from Probability Theory, Linear Algebra and Calculus. This assignment will NOT be graded.
- This is an individual assignment. Collaborations and discussions with others regarding the problems and solutions are strictly prohibited.
- You have to submit the `pdf` copy of the assignment on gradescope before the deadline. If you handwrite your solutions, you need to scan the pages, merge them into a single `pdf` file and submit.
- Your future assignments will not be graded if you fail to submit assignment-0.
- **Honour Code**: Please write out the following, digitally sign it or type your name under it and return it along with your assignment.
  By enrolling for INF8245AE Machine Learning course, I agree that all the work submitted will be mine and original, and will not be plagiarized. Unless otherwise specifically stated by the instructor or TAs, I will not collaborate with anyone on my assignments or tests. I understand that any violation of this honor code will be strictly dealt with.

Probability Theory

1. (a) (1 point) Two fair dice are rolled. What is the probability that their sum is greater than 4?

   (b) (1 point) A probability experiment has four possible outcomes: e_1, e_2, e_3, e_4. The outcome e_1 is four times as likely as each of the three remaining outcomes. Find the probability of e_1

   (c) (3 points) Which of the following three events is more likely that a person gets ...

      1. exactly 1 six when 6 dice are rolled.
      2. exactly 2 six when 12 dice are rolled.
      3. exactly 3 six when 18 dice are rolled.

2. (3 points) If $P(A) = \dfrac{1}{2}, P(B) = \dfrac{1}{2},$ and $P(B|A) = \dfrac{1}{3}$, find the following:

   (a) $P(A \text{ and } B)$

   (b) $P(A \text{ or } B)$

   (c) $P(A|B)$

3. (a) (3 points) Suppose the probability mass function of the discrete random variable is

$p_x = P(X=x)$

Table 1: Probabilities of the discrete random variable

x	$p(x)$	$g(x)$	$p(x)g(x)$
0	0.2	0	0
1	0.1	5	0,5
2	0.4	14	5,6
3	0.3	27	8,1

What is the value of $\mathbb{E}[3X + 2X^2]$?

(b) (3 points) Let a probability density function of a random variable X be $f(x) = 4x^3$ for $0 < x < 1$. Find $\mathbb{E}[X]$ and $Var(X)$.

4. Let X, Y be two continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} cx + 1 & x, y \geq 0, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) (2 points) Find constant c.

(b) (2 points) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$. Are X and Y independent?

(c) (2 points) Find $P(Y < 2X^2)$.

Linear Algebra

5. (3 points) Find the inverse of the matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 0 \\ -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. (2 points) Determine the component vector of the vector $V = (1, 7, 7)$ in $\mathbb{R}^3$ relative to the desired Basis $B = \{(1, -6, 3), (0, 5, -1), (3, -1, -1)\}$.

7. (5 Points) Find eigenvalues and linearly independent eigenvectors for the matrix $A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$. Using the eigenvectors, diagonalize the matrix A.

8. (2 points) You are given that the eigenvalues of a matrix A are 3, 2 and 2. Is A invertible? Your answer can be "yes","no" or "depends". Give arguments and/or example to support your answer.

9. (4 points) For a matrix A of size n $\times$ n you are told that all its n eigen vectors are independent. Let S denote the matrix whose columns are the n eigen vectors of A.
   (a) Is A invertible?
   (b) Is A diagonalizable?
   (c) If S invertible?
   (d) Is S diagonalizable?

   Your answer can be "yes","no" or "depends". For each one, give arguments and/or examples to support your answer.

Calculus

10. (2 points) Compute $\frac{dy}{dx}$ for the following.
    (a) $y = x^4(sin(x^3) - cos(x^2))$
    (b) $y = ln(x^2)$
11. (2 points) Compute the critical points of the function $f(x) = x^3 - 6x^2 + 9x + 15$ by using first order derivatives. Use second order derivatives to identify which of the critical points are minima and maxima.
12. Compute $\nabla f(x, y)$ and Hessian $Hf(x, y)$ for $f(x, y) = x^2 + xy + y^2$
13. For a matrix A of size $n \times n$ and a vector b of size n, compute the gradient and Hessian of $f(X) = X^\top AX + b^\top X$

1-

a) $6 \times 6 = 36$ possibilities

$\Rightarrow \{1,1\}, \{1,2\}, \{2,1\}, \{2,2\}, \{3,1\}, \{1,3\}$

$\Rightarrow \frac{36-6}{36} = 5/6$

b) e_1, e_2, e_3, e_4

$4e + e + e + e = 1$

$7e = 1$

$e = 1/7$

$\Rightarrow e_1 = 4/7$

c) $\underbrace{\left(\underset{\substack{\uparrow \\ \text{“6”}}}{\frac{1}{6}} \cdot \underbrace{\frac{5}{6} \cdot \ldots \cdot \frac{5}{6}}_{\substack{\text{5 autres rolls} \\ \neq \text{ “6”}}} \right)}_{\text{6 possibilités}} 6 = \frac{3125}{7776} \simeq 0,4$ X

Binomiale

$\binom{n}{k} p^k (1-p)^{n-k}$

$\Rightarrow p = 1/6$: prob succès

$\Rightarrow k = 1$: nb succès à avoir

$\Rightarrow n = 6$: nb fois on fait expérience

$\frac{1}{6} \cdot \frac{1}{6} \cdot \underbrace{\frac{5}{6} \cdot \ldots \cdot \frac{5}{6}}_{\text{10 fois}} \cdot 11 \cdot 12 \simeq 0,59$ X

$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \underbrace{\frac{5}{6} \cdot \ldots \cdot \frac{5}{6}}_{\text{15 fois}} \cdot 18 \cdot 17 \cdot 16 =$ X

2-

a) $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A \cap B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$

$= 5/6$

c) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = 1/3$

3-

a) $E[3X+2X^2] = E[3X] + E[2X^2] \Rightarrow E_X[g]$ où $g(x) = 3x+2x^2$

$= 3E[X] + 2E[X^2]$ $\Rightarrow \sum_{x=0}^{3} (3x+2x^2)\cdot p(x)$

$= 3\sum_{x=0}^{3} g(x)\,p(x)$ where $g(x) = x$

$+ 2\sum_{x=0}^{3} g_2(x)\,p(x)$ where $g_2(x) = x^2$ $\Rightarrow 14,2$

$= 3\,[0\cdot 0{,}2 + 1\cdot 0{,}1 + 2\cdot 0{,}4 + 3\cdot 0{,}3]$

$+ 2\,[0^2\cdot 0{,}2 + 1^2\cdot 0{,}1 + 2^2\cdot 0{,}4 + 3^2\cdot 0{,}3]$

$= 3\,[1{,}8] + 2\,[4{,}4] = 14{,}2$

b) $E[X] = \int_0^1 x\,f(x)\,dx$

$= \int_0^1 x\,4x^3\,dx$

$= \int_0^1 4x^4\,dx$

$= \left[\frac{4}{5}x^5\right]\Big|_0^1$

$= \frac{4}{5}$

$Var[X] = E[X^2] - E[X]^2$

$= \int_0^1 x^2\,4x^3\,dx - \left(\frac{4}{5}\right)^2$

$= \frac{2}{3} - \frac{16}{25}$

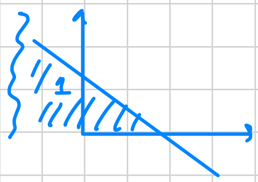
$= 2/75$

4- $f_{X,Y}(x,y) = \begin{cases} cx+1 & x,y \geq 0,\ x+y<1 \\ 0 & \text{otherwise} \end{cases}$

$\rightarrow x \geq 0$ et $y \geq 0$

$\rightarrow y < -x+1$

$\Rightarrow$



4-

$f_{X,Y}(x,y) = \begin{cases} cx+1 & x,y \geq 0,\ x+y<1 \\ 0 & \text{otherwise} \end{cases}$ $\left\{ 3x+1 \right.$

Domaine : $x+y<1$ où $x,y \geq 0$

$y \geq 0$ $\quad x<1 \Rightarrow x \in [0,1]$

$\int_0^1 \int_0^{-x+1} f_{X,Y}(x,y)\,dy\,dx$

$\Rightarrow \underbrace{\iint (cx+1)\,dy\,dx}$

$\Rightarrow \int_0^1 (cxy+y)\Big|_0^{-x+1}$

$\Rightarrow \int_0^1 (cx(-x+1) - x+1)\,dx$

$\Rightarrow \int_0^1 (-cx^2+cx-x+1)\,dx$

$\Rightarrow \left[\frac{-cx^3}{3} + \frac{cx^2}{2} - \frac{x^2}{2} + x\right]\Big|_0^1$

$\Rightarrow \frac{-c}{3} + \frac{c}{2} - \frac{1}{2} + 1$

$\Rightarrow \frac{c}{6} + \frac{1}{2} = 1$

$\Rightarrow c = 3$

b)

c)

Linear Algebra

5-

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 0 \\ -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{L3+2L1}{\overset{L2+L1}{\Longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \underset{L2-L3}{\overset{L1-\frac{1}{3}L3}{\Longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 6 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & \underline{4} & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{L4/2}{\overset{L3/3}{\Longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/4 \end{array}\right] \underset{L3-2L4}{\overset{L2+4L4}{\Longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/3 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/4 \end{array}\right]$$

$$A^{-1} = \begin{array}{c} \begin{bmatrix} 1/3 & 0 & -1/3 & 0 \\ -1 & 1 & -1 & 1 \\ 2/3 & 0 & 1/3 & -1/2 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ \\ 1 \end{array}$$

Voir Corrigé

Notes Eigenvectors / Eigenvalues