COMP 251 - Fall 2017 - Assignment 1

Due: 11:59pm Sept 22

General rules: In solving these questions you may consult your book; You can discuss high level ideas with each other, but each student must find and write his/her own solution. You should upload the pdf file (either typed, or a clear scan) of your solution to mycourses.

1. (30 points)

   (a) Give an example (say with 4 men and 4 women), where after running the stable matching algorithm that was discussed in the class, every man is paired with his most preferred woman, while every woman is paired with her least preferred man.

   (b) Is it possible to have the opposite? That is on some input with $n > 1$, after running the same algorithm, every woman is paired with her most preferred man, while every man is paired with his least preferred woman.

   (c) Consider the same-sex version of the stable matching problem. We have $2n$ people of the same sex, and each person has a ranking of everybody else. We want to obtain a stable pairing. Does the algorithm discussed in class (i.e. at each iteration a free person proposes to the highest ranked person that he/she has not proposed to yet, etc) leads to a stable matching? Either prove that it always does, or give an example to show that in some cases it might fail. (Clarification: As the question states, when we pick a free person, he/she proposes to the highest ranked person that he/she has not proposed to yet. In other words if x proposes to a person y, then x will never proposes to y again, even if x was the one who ended their possible engagement).

2. (20 points) Recall that for $f, g : \mathbb{N} \to \mathbb{R}^+$:

   - we say $f = O(g)$ if and only if there exists $c, n_0 > 0$ such that for all $n > n_0$ we have $f(n) \leq cg(n)$.

- We say $f = \Omega(g)$ if and only if there exists $c, n_0 > 0$ such that for all $n > n_0$ we have $cg(n) \leq f(n)$.
- We say $f = \Theta(g)$ if and only if $f = O(g)$ and $f = \Omega(g)$.
- We say $f = o(g)$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

- We say $f = \omega(g)$ if and only if $g = o(f)$.

Prove the following statements:

(a) $\sqrt{n} + n\sqrt{n} = O(n^2)$.

(b) $(n + \log_2 n)^5 = \Theta(n^5)$.

(c) $n! = 1 \times 2 \times \ldots \times n = o(n^n)$.

(d) $\log_2 n = o(n^{1/100})$.

3. (20 points) True of False? (Prove or Disprove).

(a) $2^{2^{n+1}} = O(2^{2^n})$.

(b) $(\log_2 n)^5 = O\left(\log_2 n^5\right)$.

(c) $n^{1/n} = \Theta(1)$.

(d) $2^{(\log_2 n)^2} = O(n^{100})$.

4. (15 points) Prove or disprove each one of the following statements:

- $O(f(n) + g(n)) = f(n) + O(g(n))$ if $f(n)$ and $g(n)$ are strictly positive for all n. (Clarification: This questions is asking whether every function that is of $O(f(n) + g(n))$ can be written as $f(n)$ plus another function that is of $O(g(n))$).
- $O(f(n) \times g(n)) = f(n) \times O(g(n))$ if $f(n)$ and $g(n)$ are strictly positive for all n. (Clarification: Similarly this questions is asking whether every function that is of $O(f(n)g(n))$ can be written as $f(n)$ times another function that is of $O(g(n))$).

5. (15 points) What is the error in the following "proof" for the wrong statement $n^2 = O(n)$?

" We use induction on n. The base case $n = 1$ is trivial as obviously $1 = O(1)$. To use induction consider the induction hypothesis $n^2 =$

$O(n)$. To make the induction work, we need to prove that the induction hypothesis implies $(n+1)^2 = O(n+1)$. But indeed this is easy as

$$(n+1)^2 = n^2 + 2n + 1 = O(n) + 2n + 1 = O(n+1),$$

where the second equality uses the induction hypothesis $n^2 = O(n)$."