Model-free Analysis of Dynamic Trading Strategies

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Abstract

We introduce a model-free approach based on *excursions* of trading signals for analyzing the risk and return for a broad class of dynamic trading strategies, including pairs trading and other statistical arbitrage strategies.

Our analysis does not require any probabilistic description of the underlying assets or factors but a pathwise description via *excursions* of a trading signal away from a reference level. We propose a mathematical framework for the risk analysis of such strategies, based on a description in terms of price excursions in a pathwise setting, without probabilistic assumptions.

We introduce the notion of δ*-excursion*, defined as a path which deviates by δ from a reference level before returning to this level. We show that every continuous path has a unique decomposition into δ-excursions, which is useful for the scenario analysis of dynamic trading strategies, leading to simple expressions for the number of trades, realized profit, maximum loss and drawdown. We show that the high frequency limit, which corresponds the the case where δ decreases to zero, is described by the ($p-$th order) local time of the signal. In particular, our results yield a financial interpretation of the local time as the profit of a certain high-frequency mean-reversion trading strategy.

Finally, we describe a non-parametric scenario simulation method for generating paths whose excursion properties match those observed in empirical data.

KEYWORDS: excursion theory, local time, mean-reversion strategies, rough processes, drawdown risk, statistical arbitrage.

Contents

1 Introduction

A broad class of trading strategies may be described in terms of the relation between the *market price* P_t of an asset –a stock, bond, commodity, a spread between two such assets, or a basket of assets– and a *reference* level A_t, which may refer to an assessment of the portfolio's fundamental value by an analyst, or a forecast of the portfolio's value based on 'technical' indicators, such as moving average estimators used in pairs trading [23] or 'technical indicators' used in statistical arbitrage strategies [1, 2, 4, 16]. The deviation of the market price from the reference value then represents a trading signal:

$$S_t = P_t - A_t.$$

If S falls below some negative threshold $-\delta < 0$, this represents a buying opportunity, while if S exceeds a positive threshold $\delta > 0$, this represents an opportunity for entering a short position. A wide range of trading strategies – pairs trading [13, 23], mean-reversion strategies [4, 20], statistical arbitrage strategies based on cointegration [1], index arbitrage [2] and other statistical arbitrage strategies [4, 16]– fall under this description. The reference level A_t is computed differently in each of these examples, but once the signal $S = P - A$ is constructed all these strategies follow the description given above.

Regardless of how the reference value A_t is arrived at, e.g. using fundamental valuation principles, or statistical forecasts, this leads to similar features across all such trading strategies: a long position is entered when the signal S crosses $-\delta$ and held until S crosses 0; similarly, a short position is entered when S crosses δ and held until S crosses zero. The holding periods of positions thus coincide with *excursions* of the signal S above (or below) certain levels.

This remark has interesting implications: it implies that the risk and return profile of such trading strategies may be described in terms of the properties of excursions of the process S. For example, the profit of such a strategy is linked to the number of the excursions alluded to above, while the magnitude of drawdown risk may be linked to the height of the excursions.

In the present work, we show that the concept of excursion is very relevant for the model-free analysis of dynamic trading strategies, and illustrate this through the example of pairs trading strategies.

The construction and empirical performance of pairs trading [23, 13] and 'mean-reversion' trading strategies [4, 20] considered in this paper have been studied by Avellaneda & Lee [4], Gatev et al. [13] and others [23, 16]. Leung and Li [20] study mean-reversion strategies from the perspective of optimal control, in the setting of the Ornstein-Uhlenbeck model. The connection between statistical arbitrage and cointegration has been discussed by many authors, including Alexander [1] and Alexander & Dimitriu [2]. Our approach provides a different perspective on these results through the angle of excursion theory and explains the common features observed across the variety of strategies considered in these studies.

Excursion theory has also been applied in mathematical finance, for the pricing of certain path-dependent options involving barrier crossings of a price process, such as Parisian options [9, 11], barrier options [22] or 'occupation time derivatives' [7]. These studies focus on analytical results for special models such as Brownian motion [9] or certain Lévy processes [7, 22].

A related topic is the modeling of *drawdown risk* for trading strategies [15]. The literature on this topic has focused on the analytical study of drawdown risk and optimal investment under drawdown constraints in specific models. Zhang [26] uses excursion theory for one-dimensional diffusion models to derive formulas for drawdown risk of static portfolios. On the other hand empirical studies of drawdown risk indicate that commonly used stochastic models do not correctly quantify drawdown risk even for passive index portfolios [17], suggesting that better, more flexible models are needed.

Outline We propose a *model-free* framework for the analysis of such dynamic trading strategies, based on a description in terms of *excursions* of the underlying trading signal.

We start in Section 2 by describing how properties of a large class of trading strategies may be expressed in terms of excursions of a *trading signal* away from zero. We then introduce in Section 3 the notion of δ-*excursion*, defined as a path which deviates by δ from a reference level before returning to this level. We show that every continuous path has a unique decomposition into such δ-excursions, which turns out to be useful for the scenario analysis of dynamic trading strategies, leading to simple expressions for the number of trades, realized profit, maximum loss and drawdown (Section 3.3). In the case of irregular paths which possess a local time, we describe in Section 4.1 the relation between δ-excursions and local time at zero of the path.

In Section 6 we propose a non-parametric scenario simulation method for generating paths whose excursions match those observed in a data set.

2 Mean-reversion strategies

2.1 Trading signals

Many trading strategies are based on the assumption that the market price P_t of a reference asset reverts to a 'target value' or forecast A_t over a certain horizon, although it may deviate from it in the short term. The examples below illustrate the generality of this concept.

Example 2.1 (Value trading). An investor who believes that the price of the asset will eventually revert to a 'fundamental' value $A > 0$ will choose to buy the asset when P_t drops below A and short the asset when P_t exceeds A. This 'fundamental' value can be a book value or a valuation by a financial analyst. The deviation $S_t = P_t - A$ from the fundamental value then plays the role of trading signal.

Example 2.2 (Pairs trading). Pairs trading is a relative-value trading strategy which looks for pairs of assets whose prices P^1, P^2 are cointegrated [1], i.e. there exists a stationary combination $P_t = P_t^1 - wP_t^2$. w is typically estimated using regression techniques [13]. If A is the stationary mean of P_t then the deviation $S_t = P_t^1 - wP_t^2 - A$ is expected to revert to zero and is used as a trading signal. In practice this mean A is estimated via a moving average [23] which leads to a time-dependent but slowly varying reference level A_t.

Example 2.3 (Mean-reversion strategies). Many *statistical arbitrage strategies* [4, 16] are based on identifying combinations of assets (portfolios) whose market price follows a stationary, mean-reverting process [2, 4], using methods such as index tracking or cointegration [2].

The market price $P_t = \sum w_i P_t^i$ of such a stationary combination is then expected to revert to its mean A, which may be estimated using for instance a moving average estimator A_t, leading to the trading signal $S_t = \sum w_i P_t^i - A_t$ which is expected to revert to zero.

These strategies, while distinct in their design, share a common feature: they are based on the assumption that a trading signal $S_t = P_t - A_t$, defined as the deviation of the market price P_t of a reference asset from a target value A_t, reverts to zero over some time horizon. This assumption implies that if $S_t < 0$ (resp. $S_t > 0$) one should take a long (resp. short) position in the portfolio P.

In presence of transaction costs, such transactions will be entered only if the signal reaches some threshold $\pm\delta$, leading to the following strategy:

(i) Enter a long position in the reference portfolio when S_t drops below $-\delta$; unwind the long position when S_t crosses zero;

(ii) Enter a short position in the portfolio when S_t exceeds δ; unwind the short position when S_t crosses zero.

Such a strategy may be implemented through limit orders placed at the appropriate price levels, resulting in transactions when the market price P_t crosses these levels.

We now describe the associated trading strategies and their properties in more detail.

2.2 Representation of trading strategies in terms of excursions

Regardless of how the signal S is constructed, the trading strategies in the above examples share some common features, which may be described in terms of the *level crossings* of the signal S.

We define the following level crossing times of S: we set $\tau_0^+ = 0$, $\theta_0^+ = 0$ and

$$\forall i \geq 1, \quad \tau_i^+ = \inf\{t > \theta_{i-1}^+, S_t \geq \delta\} \qquad \theta_i^+ = \inf\{t > \tau_i^+, S_t \leq 0\}. \tag{1}$$

The intervals (τ_i^+, θ_i^+), $(\theta_i^+, \tau_{i+1}^+)$ are the down-crossing and up-crossing intervals of the interval $[0, \delta]$. Each interval $[\theta_i^+, \theta_{i+1}^+]$, corresponds to an *excursion* of S from 0 to δ and back to zero.

It is readily observed that the intervals (τ_i^+, θ_i^+), $(\theta_i^+, \tau_{i+1}^+)$ form a partition of $[0, \infty)$ and, if the path is continuous, they are all non-empty.[1] One can also define similar quantities for downward excursions:

$$\tau_0^- = \theta_i^- = 0, \quad \text{and} \quad \forall i \geq 1, \quad \tau_i^- = \inf\{t > \theta_{i-1}^-, S_t \leq -\delta\} \qquad \theta_i^- = \inf\{t > \tau_i^-, S_t \geq 0\}. \tag{2}$$

A mathematical description of the trading strategies described in Section 2.1 can now be given in terms of the level crossing times defined above:

- buy the reference portfolio when the trading signal drops below $-\delta$, sell when it returns to 0:

$$\phi^- = \sum_{k \geq 1} 1_{[\tau_k^-, \theta_k^-)}. \tag{3}$$

- short the reference portfolio when the signal exceeds δ, unwind the position when it reaches 0:

$$\phi^+ = -\sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+)}. \tag{4}$$

We refer to ϕ^+, ϕ^- as one-sided strategies.

Combining the two strategies we obtain what is usually called a 'mean-reversion strategy' or 'convergence trade' based on the trading signal S:

$$\phi^0(t) = \phi^+(t) + \phi^-(t) = \sum_{k \geq 1} 1_{[\tau_k^-, \theta_k^-)} - \sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+)}. \tag{5}$$

One may also considering a position size modulated as a function of S. For example, (5) has unbounded exposure to price movements and in most cases portfolios are subject to position limits or exposure limits ('stop loss'). A maximum exposure limit of M on short positions in (4) leads to unwinding the position if S reaches $\delta + M$ during the holding period:

$$\phi_M^+(t) = -\sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+ \wedge \kappa_k)} \qquad \kappa_k = \inf\{t > \tau_k^+, S(t) \geq \delta + M\}. \tag{6}$$

Throughout the paper we will assume that the target price A_t is piecewise constant between transaction times i.e. revised only at transaction times:

[1] In the case of càdlàg paths a finite number of these intervals could be empty, i.e. $\tau_i^+ = \theta_i^+$, if the process jumps across the interval $(0, \delta)$. In this paper we focus on the case of continuous trajectories.

Assumption 2.4 (Trading signal)**.** *The trading signal S has the form $S_t = P_t - A_t$ where*

- *P_t is the market price of a (basket of) traded asset(s), and*
- *A_t is a 'target' (forecast) value which is revised only at transaction dates.*

In the above examples, this means A_t is piecewise constant on the partition defined by the level crossing times $\{\tau_k^+, \theta_k^+, \theta_k^-, \tau_k^-, k \geq 1\}$. In this case for any predictable process ϕ we have

$$\int_0^{\cdot} \phi(t) 1_{[\tau_k^+, \theta_k^+)}(t) dS_t = \int_0^{\cdot} \phi(t) 1_{[\tau_k^+, \theta_k^+)} dP_t \qquad \int_0^{\cdot} \phi(t) 1_{[\tau_k^-, \theta_k^-)}(t) dS_t = \int_0^{\cdot} \phi(t) 1_{[\tau_k^-, \theta_k^-)}(t) dP_t$$

In addition to the position in the risky asset(s), each portfolio has a cash component, which is adjusted to reflect the gains and losses from trading, so that the strategy is self-financing. Denoting by $V_t(\phi)$ the sum of the cash holdings and the market value of a position ϕ in the risky asset, we have

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi(u-) dS_u. \tag{7}$$

Note that as the portfolios considered above are piece-wise constant, no particular assumption on S is required to define the integral in (7).

As the sets $\cup_{k\geq 1}[\tau_k^-, \theta_k^-]$ and $\cup_{k\geq 1}[\tau_k^+, \theta_k^+]$ are disjoint we may study the properties of ϕ^+, ϕ^- separately. In the following sections we will focus on ϕ^+, but it is clear that properties of ϕ^- are analogously obtained by replacing S by $-S$.

Let us now examine further the properties of the one-sided strategy (4). Each transaction cycle $[\theta_{k-1}^+, \theta_k^+]$ is decomposed into a *waiting period* $[\theta_{k-1}^+, \tau_k^+]$ followed by a *holding period* $[\tau_k^+, \theta_k^+]$. The strategy generates a profit of δ over each transaction cycle, leading to a portfolio value

$$V_t(\phi^+) = V_0(\phi^+) + \delta\ D_t^\delta(S) + S(t \wedge \tau^+_{D_t^\delta(S)+1}) - S(t \wedge \theta^+_{D_t^\delta(S)+1}) \quad \text{where} \quad D_t^\delta(S) = \sum_{i\geq 1} 1_{\theta_i^+ \leq t} \tag{8}$$

represents the number of transactions in $[0, t]$. The first term $\delta\ D_t^\delta(S)$ represents the *realized profit* while the second term corresponds to the market value of the current position. If the path of S wanders high above δ then the portfolio can incur a large market loss. It is therefore clear that the gains and losses of the trading strategy ϕ^+ are linked to the frequency, duration and amplitude of positive *excursions* of S which exceed the level δ. Similarly, one can readily observe that the gains and losses of ϕ^- are linked to the frequency, duration and height of negative *excursions* of S which reach $-\delta$. In the following sections, we build on this insight and study in more detail the structure of such excursions in order to model the risk and return profile of such portfolios.

3 Pathwise results and scenario analysis

3.1 Excursions and $\delta-$excursions

Let $\mathcal{E} = C^0([0,\infty),\mathbb{R})$ be the space of continuous functions equipped with the Borel measurable structure induced by the uniform norm and $\mathcal{E}_0 = \{f \in \mathcal{E}, f(0) = 0\}$. Denote, for $f \in \mathcal{E}$,

$$T^x(f) = \inf\{t > 0, f(t) = x\}, \qquad T^x_t(f) = \inf\{u > t, f(u) = x\}. \tag{9}$$

Let $\delta \in \mathbb{R}$. We will call an *excursion* from 0 to δ a path which starts from zero, reaches δ in a finite time, and stops when it reaches δ:

$$\mathcal{E}_{0,\delta} = \{f : C^0([0,\infty) \to \mathbb{R},\ f(0) = 0,\ T^\delta(f) < \infty;\ \forall t \geq T^\delta(f), f(t) = \delta\}. \tag{10}$$

Note that by this definition an excursion from 0 to δ is stopped at the first time it reaches δ. In particular, $\mathcal{E}_{0,0}$ is the space of *excursions* from 0 to 0.

Define the *concatenation* at $T > 0$ of two paths $u, v \in \mathcal{E}$ as the element

$$(u \underset{T}{\oplus} v)(t) := u(t)\ 1_{[0,T)} + v\,(t - T)\ 1_{[T,\infty)}. \tag{11}$$

Note that if $u \in \mathcal{E}_{0,a}, v \in \mathcal{E}_{a,0}$ then for $T \geq T^a(u)$, $u \oplus_T v \in \mathcal{E}_0$.

We define a $\delta-$*excursion* as an excursion from 0 to δ, followed by an excursion from δ back to 0:

Definition 3.1 (δ-excursion)**.** A $\delta-$excursion is a path $f \in \mathcal{E}$ such that

$$\exists(u,v) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0},\ \ f = u \underset{T^\delta(u)}{\oplus} v, \qquad \text{i.e.} \quad f(t) = u(t)\ 1_{[0,T^\delta(u))} + v\left(t - T^\delta(u)\right) 1_{[T^\delta(u),\infty)} \tag{12}$$

The decomposition (12) is then unique and we denote $\Lambda(f) = T^\delta(u) + T^0(v)$ the *duration* of f.

We denote by $\mathcal{U}_\delta$ the set of $\delta-$excursions. The map $f \in \mathcal{U}_\delta \mapsto (u, v, \Lambda(f)) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0} \times [0,\infty)$ is measurable.

Examples of $\delta-$excursions are excursions from 0 to 0 which reach δ:

$$\Gamma_\delta = \Big\{f \in \mathcal{E}_{0,0}:\ \max(f) \geq \delta\Big\} = \mathcal{U}_\delta \cap \mathcal{E}_{0,0}. \tag{13}$$

But the inclusion $\Gamma_\delta \subset \mathcal{U}_\delta$ is strict, as a typical $\delta-$excursion may reach zero (infinitely) many times before reaching δ and we may have $\Lambda(f) > T^0(f)$ for $f \in \mathcal{U}_\delta$. In particular $\mathcal{U}_\delta$ is *not* a subset of $\mathcal{E}_{0,0}$. However, each path in $\mathcal{U}_\delta$ contains *exactly* one excursion of type Γ_δ:

Lemma 3.2 (Last exit decomposition of δ-excursions)**.** *Any δ-excursion $f \in \mathcal{U}_\delta$ has a unique decomposition into a path from 0 to 0 which does not reach δ followed by an excursion $\gamma \in \Gamma_\delta$ from 0 to 0 which reaches δ:*

$$\forall f \in \mathcal{U}_\delta, \quad \exists!(T, g, \gamma) \in [0,\infty) \times \mathcal{E}_0 \times \Gamma_\delta, \quad f = g \underset{T}{\oplus} \gamma \quad \text{with} \quad g(0) = g(T) = 0, \quad \max(g) < \delta. \tag{14}$$

Proof. Consider a δ-excursion $f \in \mathcal{U}_\delta$. Then f has a decomposition (12) for some $(u,v) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0}$ and $f(t) = 0$ for $t \geq \Lambda(f) = T^\delta(u) + T^0(v)$. Now define T as the last zero of u before $T^\delta(u)$:

$$T = \sup\{t < T^\delta(u),\ u(t) = 0\}.$$

Then by continuity of u, $T < T^\delta(u)$ and therefore $\max\{u(t), 0 \leq t \leq T\} < \delta$. Setting $g = f 1_{[0,T]}$ and $\gamma(t) = f((t-T)_+)$, it is readily verified that $\gamma \in \Gamma_\delta$ and g satisfy the required conditions. $\square$

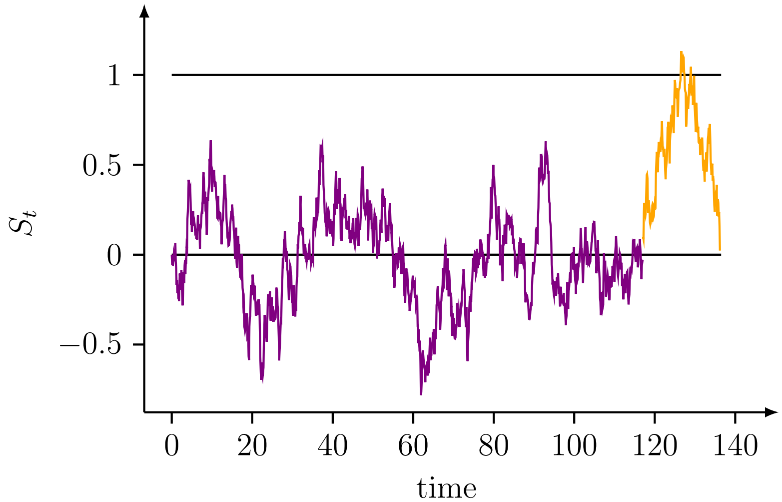


Figure 1: Example of last exit decomposition of a δ-excursion $f \in \mathcal{U}_\delta$ with $\delta = 1$: $f = g \oplus_T \gamma$ where g is in purple and $\gamma \in \Gamma_\delta$ (in orange) is the last excursion.

3.2 Decomposition of a path into $\delta-$excursions

The following proposition gives the decomposition of any path starting from zero into a sequence of excursions from 0 to δ and back to 0:

Proposition 3.3. *Let $S \in C^0([0,\infty),\mathbb{R})$ with $S_0 = 0$. Define the level crossing times*

$$\theta_0^+ = 0, \quad \tau_0^+ = 0, \qquad \tau_i^+ = T^\delta_{\theta_{i-1}^+}(S), \qquad \theta_i^+ = T^0_{\tau_i^+}(S).$$

$$\text{Then} \qquad \forall t \geq 0, \qquad D_t^\delta(S) = \textstyle\sum_{i\geq 1} 1_{\theta_i^+ \leq t} < \infty \qquad \text{and} \tag{15}$$

$$\forall t \geq 0, \qquad S_t = \textstyle\sum_{i=1}^{D_t^\delta(S)+1} \Big[u_i\left(t-\theta_{i-1}^+\right) 1_{[\theta_{i-1}^+,\tau_i^+)} + v_i\left(t-\tau_i^+\right) 1_{[\tau_i^+,\theta_i^+)}\Big], \tag{16}$$

where $u_i \in \mathcal{E}_{0,\delta}$ and $v_i \in \mathcal{E}_{\delta,0}$.

Proof. To prove the first assertion, we first note that S is continuous, thus uniformly continuous on $[0,T]$ for any $T > 0$. If $D_t^\delta = \infty$ for some $t > 0$, then the set $\{k \in \mathbb{N}, \quad \theta_k^+ \leq t\}$ is infinite. Since by construction the intervals (τ_k^+, θ_k^+) are disjoint, we have

$$\sum_{\{k,\theta_k^+\leq t\}} |\theta_k^+ - \tau_k^+| \leq t < \infty, \quad \text{so} \quad \inf_{\theta_k^+\leq t} |\theta_k^+ - \tau_k^+| = 0 \quad \text{while} \qquad |S_{\theta_k^+} - S_{\tau_k^+}| = \delta$$

which contradicts the uniform continuity of S on $[0,t]$. Therefore $D_t^\delta < \infty$ for all $t \geq 0$. Starting from:

$$S_t = \sum_{i=1}^{D_t^\delta(S)+1} \Big[(S_{t\wedge\tau_i^+} - S_{t\wedge\theta_{i-1}^+}) + (S_{t\wedge\theta_i^+} - S_{t\wedge\tau_i^+})\Big].$$

Note that

$$(S_{t\wedge\tau_i^+} - S_{t\wedge\theta_{i-1}^+}) + (S_{t\wedge\theta_i^+} - S_{t\wedge\tau_i^+}) = (S_{t\wedge\tau_i^+} - S_{t\wedge\theta_{i-1}^+})1_{[\theta_{i-1}^+,\tau_i^+)} + (\delta + S_{t\wedge\theta_i^+} - S_{t\wedge\tau_i^+})1_{[\tau_i^+,\theta_i^+)},$$

it remains to set

$$u_i(t) := S_{(t+\theta_{i-1}^+)\wedge\tau_i^+} - S_{(t+\theta_{i-1}^+)\wedge\theta_{i-1}^+}, \; i \geq 1, \qquad v_i(t) := \delta + S_{(t+\tau_i^+)\wedge\theta_i^+} - S_{(t+\tau_i^+)\wedge\tau_i^+}, \; i \geq 1.$$

$\square$

The above results translate into a (measurable) decomposition of any continuous path into δ-excursions:

Proposition 3.4 (Decomposition of a path into $\delta-$excursions)**.** *Let $\delta > 0$ and $S \in C^0([0,\infty),\mathbb{R})$ with $S_0 = 0$, and define $D_t^\delta(S)$ as in* (15)*.*

(i) If $\sup_{t\geq 0} D_t^\delta(S) = \infty$ there exists a unique sequence $(e_k)_{k\geq 1}$ of $\delta-$excursions $e_k \in \mathcal{U}_\delta$ such that

$$\forall t \geq 0, \quad S_t = \sum_{k\geq 1} e_k\left((t-\theta_{k-1}^+)_+\right) \qquad \text{where} \qquad \theta_0^+ = 0, \quad \theta_k^+ = \sum_{i=1}^{k} \Lambda(e_i). \tag{17}$$

(ii) If $d = \sup_{t\geq 0} D_t^\delta(S) < \infty$ then there exist $(e_1, ..., e_d) \in (\mathcal{U}_\delta)^d$ and $e_{d+1} \in \mathcal{E}$ such that

$$S_t = \sum_{k=1}^{d+1} e_k\left((t-\theta_{k-1}^+)_+\right) \qquad \text{where} \qquad \theta_0^+ = 0, \quad \theta_k^+ = \sum_{i=1}^{k} \Lambda(e_i). \tag{18}$$

In both cases the map $S \mapsto (e_k)_{k=1..(d+1)}$ is measurable.

The case (i) corresponds to the 'recurrent' case where the path crosses zero and δ infinitely many times on $[0,\infty)$.

Proof. Set $d = \sup_{t\geq 0} D_t^\delta(S) \in \mathbb{N}\cup\{\infty\}$. Define $(\theta_k^+, k \geq 1)$ as in (1). For $k < d$, set $e_k(t) = S(t-\theta_{k-1}^+)1_{[\theta_{k-1}^+,\theta_k^+)}(t)$. Then it is easily verified, from the definition (1) of θ_k^+, that $e_k \in \mathcal{U}_\delta$ and $\Lambda(e_k) = \theta_k^+ - \theta_{k-1}^+$. Measurability of the map $S \mapsto (e_k)_{k\geq 1}$ follows from the measurability of the hitting times and the shift operator. To show uniqueness, we note that (17) implies that $e_k(\theta_{k-1}^+ + .) = S_{|[\theta_{k-1}^+,\theta_k^+)}$ so it is sufficient to show uniqueness of the sequence $(\theta_k^+)_{k\geq 0}$. As $D_t^\delta(S) < \infty$ for each $t > 0$, the countable set $\{t > 0, \Delta D_t^\delta \neq 0\}$ is discrete and has a unique increasing ordering, which is given by $(\theta_k^+)_{k\geq 0}$. $\square$

Remark 3.5. The above results decompose the path into one-sided δ-excursions i.e. with $\delta > 0$. One can immediately obtain a similar decomposition for $\delta < 0$ by applying the above result to the path $-S$. To obtain a decomposition in terms of *two-sided* δ-excursions, one can iterate these two results: first decompose S into δ-excursions, then decompose each δ-excursion into $(-\delta)-$excursions. One may further show that the resulting decomposition is independent of the order of these two operations.

3.3 Scenario analysis for mean-reversion strategies

The *drawdown* [15] of a portfolio ϕ whose value at time t is $V_t(\phi)$ is defined as

$$\Delta(t) = M_t(\phi) - V_t(\phi) \qquad \text{where} \quad M_t(\phi) = \max_{[0,t]} V_t(\phi)$$

is the running maximum.

The decomposition of the path into $\delta-$excursion given in Proposition 3.4 leads to simple expressions for the portfolio value, the maximum loss and the drawdown of the strategy:

Proposition 3.6. *Along a path S with decomposition* (16)*,*

(i) the gain $V_t(\phi^+) - V_0(\phi^+) = \int_0^t \phi^+ dS$ of the portfolio is given by

$$V_t(\phi^+) - V_0(\phi^+) = \delta \times D_t^\delta(S) + 1_{[\tau^+_{D_t^\delta+1},\theta^+_{D_t^\delta+1}]}\left(\delta - v_{D_t^\delta(S)+1}(t-\tau^+_{D_t^\delta+1})\right) \tag{19}$$

(ii) the worst loss during $[0,t]$ *is given by*

$$\max_{s\in[0,t]}\left(V_0(\phi^+)-V_s(\phi^+)\ \right)=\max_{k=0,\dots,D_t^\delta(S)}\{\max_{[0,(t-\tau_{k+1}^+)+]}(v_{k+1}-(k+1)\delta)\}. \tag{20}$$

(iii) the drawdown of ϕ^+ *is given by*

$$\begin{aligned}\Delta(t) &= \max_{k=0,\dots,D_t^\delta(S)}\{\max_{[0,(t-\tau_{k+1}^+)+]}((k+1)\delta-v_{k+1})\} \\ &\quad -\delta\times D_t^\delta(S)-1_{[\tau^+_{D_t^\delta+1},\theta^+_{D_t^\delta+1}]}(t)\left(\delta-v_{D_t^\delta(S)+1}(t-\tau^+_{D_t^\delta+1})\right).\end{aligned} \tag{21}$$

Proof. By definition of the portfolio ϕ^+, we have

$$V_t(\phi^+)=V_0(\phi^+)-\sum_{i=1}^{D_t^\delta(S)+1}\int 1_{[\tau_i^+,\theta_i^+)}dS=\sum_{i=1}^{D_t^\delta(S)}(S_{\tau_i^+}-S_{\theta_i^+})+(S_{t\wedge\tau^+_{D_t^\delta(S)+1}}-S_t)$$

thus $V_t(\phi^+)-V_0(\phi^+)=\delta\times D_t^\delta(S)+S_{t\wedge\tau^+_{D_t^\delta(S)+1}}-S_t$. By the definition of v_i this can be rewritten as

$$V_t(\phi^+)-V_0(\phi^+)=\delta\ D_t^\delta(S)+1_{[\tau^+_{D_t^\delta+1},\theta^+_{D_t^\delta+1}]}\left(\delta-v_{D_t^\delta(S)+1}(t-\tau^+_{D_t^\delta+1})\right),$$

which then implies (20). □

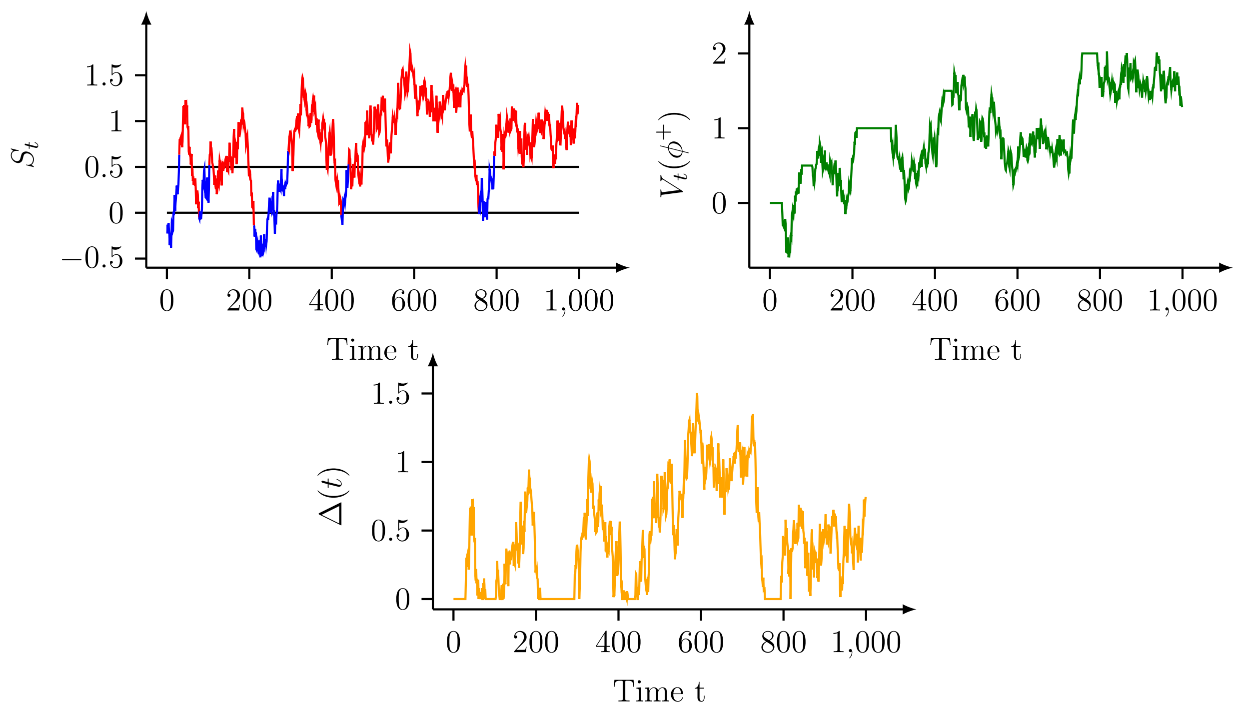


Figure 2: Decomposition of a path into δ-excursions with $\delta=0.5$. Top Left: decomposition into u_i (blue) and v_i (red). Top Right: Value of the portfolio ϕ^+. Bottom: Drawdown $\Delta(t)$.

4 High-frequency asymptotics

4.1 Irregular price paths

Intuitively, decreasing the value of the threshold δ increases the number of level crossings and leads to more transactions but with a lower profit per transaction. The exact behaviour of the trading strategy as $\delta \to 0$ is determined by the *local time* of the path at 0, which measures the time spent by the path in a neighbourhood of zero [14].

Let $S \in C^0([0,\infty),\mathbb{R})$ and $T > 0$. We define the occupation measure of S by

$$\gamma_T(A) := \int_0^T 1_A(S_t)\,dt, \qquad \forall A \in \mathcal{B}(\mathbb{R}). \tag{22}$$

We will say that the path S admits a local time $l_T(S,x)$ if the measure γ_T is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, in which case we denote

$$l_T(S,x) := \frac{d\gamma_T}{dx} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^T 1_{[x-\varepsilon,x+\varepsilon]}(S_t)\,dt.$$

The occupation density is characterized by the *occupation time formula*:

$$\int_0^T h(S_t)\,dt = \int_{-\infty}^{+\infty} h(x)\, l_T(S,x)\,dx,\ \forall h \in C^0(\mathbb{R},\mathbb{R}).$$

Intuitively, the local time $l_T(S,x)$ represents the time S spends at level x during $[0,T]$. We will be interested in particular in the local time at 0, which we denote $\ell_T(S) = l_T(S,0)$.

The map $T \mapsto \ell_T(S)$ is increasing, which allows to define its right-continuous inverse, the *inverse local time* at zero:

$$\forall l > 0, \qquad \tau_l = \inf\{t > 0,\ \ell_t(S) > l\}. \tag{23}$$

We note that $l \mapsto \tau_l$ is an increasing càdlàg function of the variable l. The occupation density at zero $\ell_t(S)$ increases on the set $\{t, S_t = 0\}$ and is constant along any excursion from 0, so the discontinuities of τ correspond to excursions of S, and jump intervals of τ correspond to the complement of the set where S visits 0:

$$\underset{l>0}{\cup} (\tau_{l-}, \tau_l) = \{t \geq 0,\ S_t \neq 0\}.$$

Thus the value l of local time along an excursion may be used as a natural index for labeling excursions of S: the excursion at local time level l is given by

$$e_l(t,S) = \begin{cases} S_{(\tau_{l-}+t)} \mathbf{1}_{(t \leq \tau_l - \tau_{l-})}, & \text{if} \quad \tau_l(\omega) - \tau_{l-}(\omega) > 0 \\ \dagger & \text{if} \quad \tau_l(\omega) = \tau_{l-}(\omega). \end{cases} \tag{24}$$

Points of continuity of τ_l, i.e. points at which $\tau_{l-} = \tau_l$ correspond to 'infinitesimal excursions' which may arise if the path has non-zero local time at 0; we associate such excursions with a 'cemetery' state $e_l(S) = \dagger$. This defines an excursion process $e\colon \mathbb{R}^+ \to \overline{\mathcal{E}_{0,0}} = \mathcal{E}_{0,0} \cup \{\dagger\}$. For a given set $\Gamma \subset \mathcal{E}_{0,0}$, we can define the counting process, which counts excursions of S from 0 which lie in Γ, up to local time l :

$$N_l(\Gamma) := \sum_{\lambda \leq l} 1_\Gamma(e_\lambda). \tag{25}$$

Note that in general $N_l(\Gamma)$ can be infinite. We now establish an important connection between this *excursion point process* N and the decomposition into δ-excursions given by Proposition 3.4. Recall the set Γ_δ of excursions from 0 to 0 which reach a level δ:

$$\Gamma_\delta = \Big\{ f \in \mathcal{E}_{0,0}\colon\ \max(f) \geq \delta \Big\}.$$

Proposition 4.1. *Let $S \in C^0([0,\infty),\mathbb{R})$ be a path with $S_0 = 0$ which admits an occupation density $\ell_t(S)$ at zero, with inverse τ is given by* (1). *For $\delta > 0$ let $D_t^\delta(S)$ be the number of $\delta-$excursions of S on $[0,t]$, defined as in* (17). *Then*

$$\forall \delta > 0, \quad \forall t > 0, \qquad N_t(\Gamma_\delta) < +\infty, \quad \text{ and}$$

$$\forall t > 0, \quad D_t^\delta(S) = N_{\ell_t(S)}(\Gamma_\delta) \quad \text{and} \quad \forall l > 0, \quad D_{\tau_l}^\delta(S) = N_l(\Gamma_\delta). \tag{26}$$

Proof. The condition $N_l(\Gamma_\delta) < +\infty$ is a consequence of the continuity of S. We will now establish a one-to-one correspondence between excursions $e_l \in \Gamma_\delta$ and intervals $(\theta_{i-1}^+, \theta_i^+)$. As in Lemma 3.2, define $i \geq 1$ the 'last exit' from zero in the i-th $\delta-$excursion:

$$\hat{\theta}_i^+ := \sup\{t < \tau_i^+ \colon S_t = 0\}.$$

To show that the two sets of intervals $\{(\hat{\theta}_i^+, \theta_i^+)\}_{i\geq 1}$ and $\{(\tau_{l-}, \tau_l)\}_{e_l \in \Gamma_\delta}$ coincide, we prove the following two claims:

- For each $i \geq 1$ there exist a unique $l_i \geq 0$, such that $(\hat{\theta}_i^+, \theta_i^+) = (\tau_{l_i-}, \tau_{l_i})$.

  Indeed, it is easy to see that on the interval $(\hat{\theta}_i^+, \theta_i^+)$, $S_t > 0$. Furthermore,

  $$\hat{e}_i(t) := S_{t+\hat{\theta}_i^+} 1_{[0,\,\theta_i^+ - \hat{\theta}_i^+]} \in \Gamma_\delta,$$

  since $\hat{e}_i(\tau_i^+ - \hat{\theta}_i^+) \geq \delta$. In particular $(\hat{\theta}_i^+, \theta_i^+)$ is an interval of $\{t > 0,\, S_t \neq 0\}$, thus there exists unique l_i such that $(\hat{\theta}_i^+, \theta_i^+) = (\tau_{l_i-}, \tau_{l_i})$ and $e_{l_i} \in \Gamma_\delta$.

- Conversely, for every $l \geq 0$ such that $e_l \in \Gamma_\delta$, there exists a unique index $i(l) \geq 1$ such that $(\tau_{l-}, \tau_l) = (\hat{\theta}_{i(l)}^+, \theta_{i(l)}^+)$.

  Take the largest $i = i(l) \geq 1$ such that $\theta_{i-1}^+ \leq \tau_{l-}$ then $\tau_{l-} < \theta_i$. Since on $(\hat{\theta}_i^+, \theta_i^+)$, $S_t > 0$, while $S_{\tau_{l-}} = 0$, we get that $\hat{\theta}_i^+ \geq \tau_{l-}$. The condition $e_l \in \Gamma_\delta$ implies that S reaches the level δ in (τ_{l-}, τ_l), by definition $\tau_i^+ > \hat{\theta}_i^+$ is the first such time after θ_{i-1}, hence $\tau_i^+ \in (\tau_{l-}, \tau_l)$. Since the intervals (τ_{l-}, τ_l) and $(\hat{\theta}_i^+, \theta_i^+)$ intersect, we conclude from the first claim that $(\tau_{l-}, \tau_l) = (\hat{\theta}_{i(l)}^+, \theta_{i(l)}^+)$ (we also use the fact that the intervals $\{(\tau_{l-}, \tau_l)\}_{l\geq 0}$ are disjoint).

The correspondence between θ_i^+, $i \geq 1$ and τ_l, $e_l \in \Gamma_\delta$, yields the result:

$$D_t^\delta(S) = \sum_{i\geq 1} 1_{\theta_i^+ \leq t} = \sum_{i\geq 1} 1_{\theta_{i(l)}^+ \leq t} = \sum_{\tau_l \leq t} 1_{\Gamma_\delta}(e_l) = \sum_{l \leq l_t(S)} 1_{\Gamma_\delta}(e_l) = N_{l_t(S)}(\Gamma_\delta).$$

□

4.2 Behaviour of level-crossings as $\delta \to 0$.

The behaviour of the above quantities as $\delta \to 0$ is determined by the 'roughness' of the path. When δ is small, we account for the fact that trading takes places only at prices which are integer multiples of a 'tick', i.e. only at times when S takes such values.

Let $\delta_n = 2^{-n}$ and introduce the partition π_n defined by the hitting times of the grid $\delta_n\mathbb{Z}$:

$$t_0^n := 0, \quad t_{k+1}^n := \inf\left\{t \geq t_k^n \colon S_t \in \delta_n\mathbb{Z}\backslash\{S_{t_k^n}\}\right\}. \tag{27}$$

Then $\sup_{\pi_n} |S(t^n_{k+1}) - S(t^n_k)| \to 0$ as $n \to \infty$. We denote $\pi = (\pi_n)_{n\geq 1}$. Following [10], we will say that $S \in C^0([0,T],\mathbb{R})$ has p-th order variation along π if there exists $[S]^p_\pi \in C^0([0,T],\mathbb{R}_+)$ such that

$$\sum_{\pi_n} |S(t^n_{k+1} \wedge t) - S(t^n_k \wedge t)|^p \to [S]^p_\pi(t).$$

The smallest $p \geq 1$ for which $[S]^p_\pi \neq 0$ then gives an index of 'roughness' for S along π. For example for Brownian paths $p = 2$ while for fractional Brownian motion with Hurst exponent H, $p = 1/H$ [6].

For such a path, the number of down-crossings for levels close to zero is related to a slightly different notion of local time, defined in terms of a *weighted* occupation measure, weighted by the p-th order variation [10]:

Definition 4.2 (Local time of order p [10])**.** *Let $p \geq 1$ and $q \geq 1$. A continuous path $S \in C^0([0,T],\mathbb{R})$ has (L^q-)local time of order p along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ of $[0,T]$ if, for any $t \in [0,T]$, the sequence of functions*

$$L^{\pi_n,p}_t(S,.) : x \in \mathbb{R} \mapsto L^{\pi_n,p}_t(S,x) := \sum_{t^n_j \in \pi_n} \mathbf{1}_{\left[S_{t^n_j \wedge t}, S_{t^n_{j+1}\wedge t}\right)}(x) \left| S_{t^n_{j+1}\wedge t} - x \right|^{p-1}$$

converges in $L^q(\mathbb{R})$ to a limit $L^{\pi,p}_t(S,x) \in L^q(\mathbb{R})$ and the map $t \in [0,T] \mapsto L^{\pi,p}_t(S,x) \in L^q(\mathbb{R})$ is weakly continuous. We call $L^{\pi,p}(S,x)$ the local time of order p of S at level x.

$L^{\pi,p}_t(S,x)$ measures the rate at which the path S accumulates p-th order variation around level x. Note that the local time of order p is non-zero only if S has non-zero p-th order variation along π i.e. $[S]^p_\pi > 0$. If the convergence is uniform in $(t,x) \in [0,T] \times \mathbb{R}$, and the mapping $(x,t) \mapsto L^{\pi,p}_t(S,x)$ is continuous we call it the continuous local time of S [18].

Note that the in the case of $p = 2$ the definitions in [5] and [10, 18] differ by a factor of 2; here we use the latter notation. In the case $p = 2$ we will omit the index p in the notation; $L^\pi := L^{\pi,2}$. The relation between various notions of local time is discussed in [18].

Following the arguments in [10, Lemma 3.4], one can establish a relation between the down-crossings and up-crossings $D^{\delta_n}_t(S), U^{\delta_n}_t(S)$ of the interval $[0,\delta_n]$: for $x \in [0,\delta_n]$,

$$L^{\pi_n,p}_t(S,x) = D^{\delta_n}_t(S)|x|^{p-1} + U^{\delta_n}_t(S)|\delta_n - x|^{p-1} + O(\delta^{p-1}_n).$$

Since the numbers $D^{\delta_n}_t(S), U^{\delta_n}_t(S)$ can differ at most by one, we obtain that

$$L^{\pi_n}_t(S) = L^{\pi_n}_t(S,0) = D^{\delta_n}_t(S)\delta^{p-1}_n + O(\delta^{p-1}_n).$$

If S has a continuous local time $L^{\pi,p}(S,\cdot)$ along the sequence of Lebesgue partitions π, we conclude from above that

$$\lim_{n\to\infty} |\delta_n|^{p-1} D^{\delta_n}_t(S) = L^{\pi,p}_t(S).$$

The following proposition summarizes the behavior of the number of level crossings $D^\delta_t(S)$ (representing the number of trades) and the realized profit $\delta D^\delta_t(S)$ as δ decreases to zero:

Proposition 4.3. *Let $\delta_n = 2^{-n}$ and $p \geq 1$. Assume $S \in C^0([0,T],\mathbb{R})$ has a strictly positive local time $L^{\pi,p}_t > 0$ of order p at zero along the sequence of partitions $(\pi_n)_{n\geq 1}$ defined by (27). Then for any $t \in (0,T]$, as $\delta_n \to 0$,*

(i) if $1 \leq p < 2$ then $\delta_n \; D^{\delta_n}_t(S) \to 0$.

(ii) if $p > 2$ then

$$\delta_n \; D^{\delta_n}_t \overset{\delta_n \to 0}{\to} \infty, \qquad \text{and} \quad D^{\delta_n}_t(S) \overset{\delta_n \to 0}{\sim} \frac{L^{\pi,p}_t(S)}{\delta^{p-1}_n}.$$

(iii) if $p = 2$ then

$$\delta_n \; D_t^{\delta_n} \overset{\delta_n \to 0}{\to} L_t^{\pi,2}(S), \qquad \text{i.e.} \quad D_t^{\delta_n}(S) \overset{\delta_n \to 0}{\sim} \frac{L_t^{\pi,2}(S)}{\delta_n}.$$

In particular, when $p > 2$ the threshold δ should be chosen as small as possible, while for $p \leq 2$ there is an optimal threshold $\delta^*(S) > 0$ which maximizes the realized profit $\delta D_T^{\delta}(S)$.

The assumptions of Proposition 4.3 are satisfied by typical sample paths of many classes of stochastic processes. Typical paths of semimartingales correspond to (iii), while paths of 'rough' processes such as Fractional Brownian motion with Hurst exponent $H < 1/2$ correspond to (ii):

Example 4.4 (Continuous semimartingales)**.** Let $S = M + A$ where M is a continuous martingale and A is a continuous process with bounded variation $\int_0^T |dA_t|$ on $[0, T]$. Denote by $[S] = [M]$ the quadratic variation process of S. Then S admits a local time of order $p = 2$, which corresponds 1/2 of the 'semimartingale local time' of S at 0:

$$L_t^{\pi,2}(S) = \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^T 1_{[-\varepsilon,\varepsilon]}(S_t)\, d[S]_t.$$

Furthermore if for some $q \geq 1$ we have

$$\mathbb{E}\left[[M]_T^{q/2} + \left(\int_0^T |dA_t| \right)^q \right] < \infty,$$

then it was shown by El Karoui [12] that $t \mapsto \delta \; D_t^{\delta}$ is uniformly approximated in $\mathbb{L}^q$ by $L_t^{\pi,2}(S)$ as $\delta \to 0$:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \delta \; D_t^{\delta}(S) - L_t^{\pi,2}(S) \right|^q \right] \quad \overset{\delta \to 0}{\to} \quad 0.$$

Example 4.5 (Fractional Brownian motion)**.** Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Kim [18] has shown that B^H almost surely has a continuous local time $L^{\pi,1/H}(B^H, \cdot)$ of order $p = 1/H$ along the sequence of partitions π defined in (27), and

$$L_t^{\pi,1/H}(B^H, x) = \ell_t(B^H, x) \mathbb{E}\left[|B_1^H|^{\frac{1}{H}} \right],$$

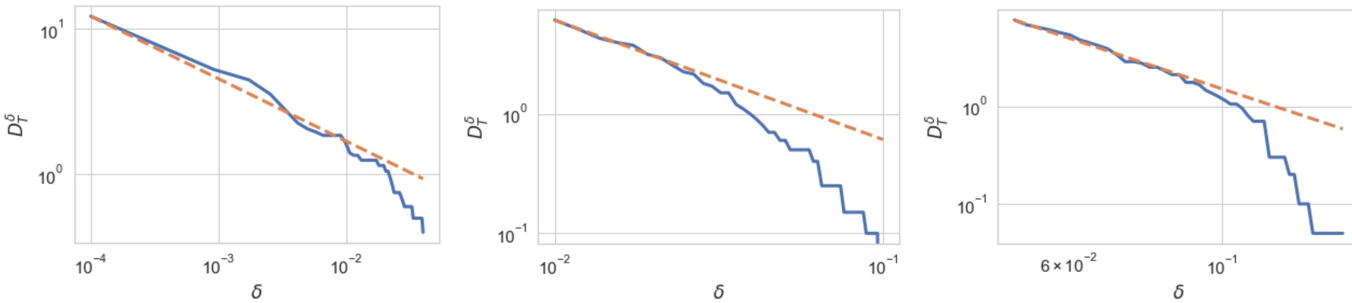
where $\ell_t(B^H, \cdot)$ is the occupation time density of B^H. Denoting $\ell_t(B^H) = \ell_t(B^H, 0)$, this implies

$$\lim_{n \to \infty} |\delta_n|^{\frac{1-H}{H}} D_t^{\delta_n}(B^H) = \ell_t(B^H) \underbrace{\mathbb{E}\left[|B_1^H|^{\frac{1}{H}} \right]}_{c_H}, \quad \text{so} \quad D_t^{\delta_n}(B^H) \overset{\delta_n \to 0}{\sim} \frac{c_H \; \ell_t(B^H)}{|\delta_n|^{\frac{1-H}{H}}}.$$

We illustrate the relevance of Proposition 4.3 using a fractional Ornstein-Uhlenbeck process [8]:

$$dS_t = -\lambda S_t dt + \gamma dB_t^H, \tag{28}$$

where B^H is a fractional Brownian motion with Hurst exponent H. Figure 3 shows, as a function of the threshold δ, the number of $\delta-$excursions estimated from values of S_t on a discrete grid of $N = 28,800$ points (which corresponds to the number of seconds in one trading day). The empirical estimator closely follows the asymptotics described in Proposition 4.3, suggesting that this asymptotic regime is indeed a relevant description of excursions at such frequencies.



(a) $H = 0.7$. (b) $H = 0.5$. (c) $H = 0.3$.

Figure 3: Behavior of D_T^δ when $\delta \to 0$ for a fractional Ornstein-Uhlenbeck process (28) with $\lambda = 5$ and $\gamma = 0.1$. Dotted line: asymptotic behavior $\delta^{\frac{H-1}{H}}$ described in Proposition 4.3.

5 Application to pairs trading

Pairs trading [23, 13] is a trading strategy based on identifying a stationary linear combination of two stock prices and using this linear combination as a trading signal for generating buy/sell transactions in the pair. In most applications the signal is then modeled as an AR(1)/Ornstein-Uhlenbeck process [20, 23]. However, such model-based methods usually rely on strong (and often unrealistic) model assumptions hence may suffer from potential financial losses.

We illustrate the limitation of model-based methods with the empirical distributions of two pairs trading signals, both of which are recognized to have co-movements.

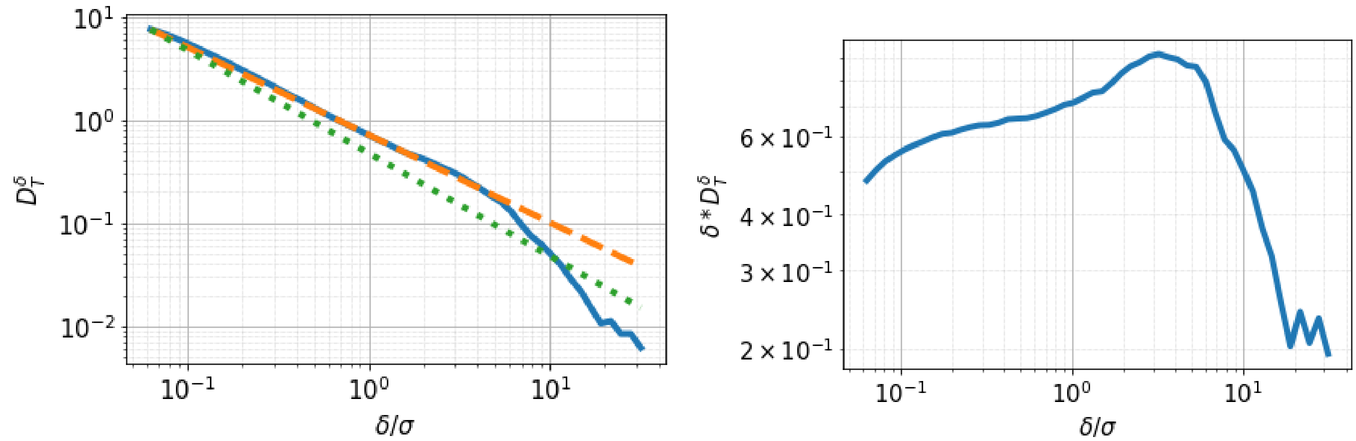
The first pairs trading signal is constructed with CocaCola (KO) and PepsiCola (PEP) and the second one is constructed with two ETFs ProShares Short S&P500 (SH) and ProShares UltraShort S&P500 (SDS). In both examples, as we shall see below, we observe poor performance of the fitted Ornstein-Uhlenbeck process by comparing the empirical distributions with the real pairs trading signals.

We use second-by-second NYSE price records of KO, PEP, SH, SDS shares during trading hours 09:30AM-4:00PM for the period 07/01/2013 - 07/01/2020 to construct pair-trading signals. Denote $P_1(t)$ (resp. $P_2(t)$) the mid-price of the first stock (resp. the second stock) of the pairs trading strategy. The signal is constructed as $S_t := P_1(t) - a_t\, P_2(t) + b_t$ where the coefficients a_t and b_t piece-wise constant, updated on each trading day by an ordinary least square regression of $P_1(\cdot)$ on $P_2(\cdot)$ over the previous 5 days. For the KO and PEP pair, we regress KO (i.e., the first stock) with respect to PEP (i.e., the second stock) to construct the signal. Similarly, we treat SH as the first stock and SDS as the second stock to construct the corresponding signal.

Number of level crossings. To assess the roughness of the signal S, we analyze the number of level crossings as a function of δ and apply Proposition 4.3. Recall that $\log(D_t^\delta(S)) \sim -(p-1)\log(\delta) + \text{constant}$ as $\delta \to 0$, where p measures the roughness of the path. We estimate the exponent p by linear regression of $\log(D_t^\delta(S))$ on $\log(\delta)$. As shown in Figure 4a, the estimated exponent of the trading signal S with KO and PEP is around $p = 1.85$, which implies that the path is slightly smoother than the Brownian motion (for which $p = 2$). On the other hand, 5a shows that the estimated exponent of the trading signal S with SH and SDS is around $p = 1.25$ implying that S is much smoother than Brownian motion.

There are two types of crossings with different time-scales along the path: crossings due to the mean reverting phenomenon on a longer time-scale and crossings with small magnitudes due to the roughness of the path once the signals revert to level 0. The crossings of the first type could be captured by all δ with appropriate choices. Crossings of the second type show up as $\delta \to 0$. For KO-PEP signals, empirical estimates seem to indicate a non-zero limit of the realized profit δD_T^δ as $\delta \to 0$. This is consistent with the result in Proposition 4.3 for $p = 2$, indicating that it is not more profitable to use smaller thresholds δ for trading this pair (see Figure 4b). The realized profit is maximized at $\delta \simeq 3.2\sigma$. For SH-SDS signals, empirical estimates imply a limit of the realized profit δD_T^δ at zero as $\delta \to 0$. This is consistent with the result in Proposition 4.3 for $p < 2$. Similar to the previous case, this indicates that it is also not profitable to use small thresholds δ for trading this pair (see Figure 5b). The realized profit is maximized at $\delta \simeq 0.3\sigma$.

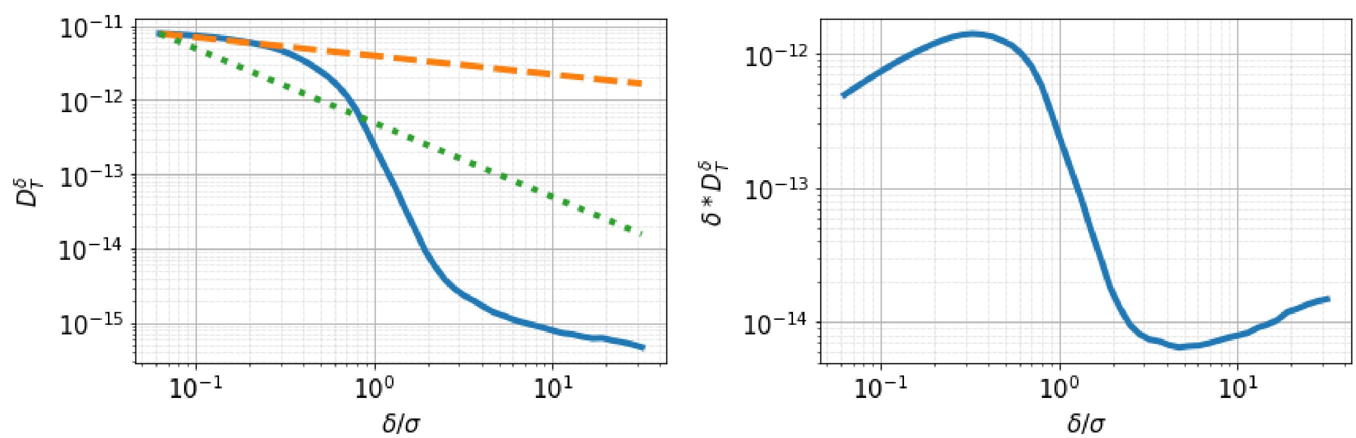
Comparing the results in Figures 4a and (5a), we see that different pairs trading signals have very different behaviors in terms of the roughness of the path and the optimal thresholds. This is different from the results assuming Ornstein-Uhlenbeck models, where realized profit is always maximized at $\delta \lesssim \sigma$ [21].



(a) Number of level crossings D_T^δ. (Orange dashed line: δ^{p-1} with $p = 1.85$; Green dotted line: asymptotics for $p = 2.0$; $T = 1$ day).

(b) Realized profit δD_T^δ ($T = 1$ day).

Figure 4: Number of crossings D_T^δ and realized profit δD_T^δ as a function of threshold δ for Coca-Pepsi pairs trading signal.



(a) Number of level crossings D_T^δ. (Orange dashed line: δ^{p-1} with $p = 1.25$; Green dotted line: asymptotics for $p = 2.0$; $T = 1$ day).

(b) Realized profit δD_T^δ ($T = 1$ day).

Figure 5: Number of crossings D_T^δ and realized profit δD_T^δ as a function of threshold δ for SH-SDS pairs trading signal.

Comparison with Ornstein-Uhlenbeck models The most widely used model for pairs trading is the Ornstein-Uhlenbeck (OU) model [13, 20], mainly due to its mean-reversion properties and analytical tractability:

$$dS_t = \gamma dB_t + \alpha(\mu - S_t)dt. \tag{29}$$

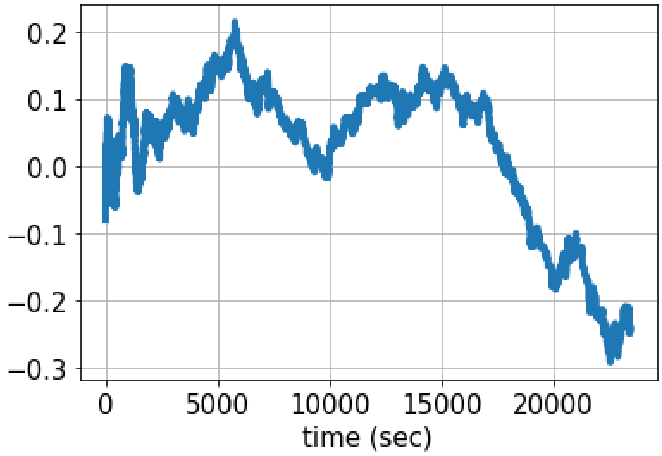
In the stationary case, the signal has a standard deviation $\sigma = \gamma/\sqrt{2\alpha}$.

Recall that the trading strategy ϕ^+ consists in shorting the pair when S_t crosses the threshold δ from below and unwinds the position when S_t returns to 0. Denote by σ the (sample) standard deviation of S_t. In Figure 6, we provide the empirical distributions of the durations for waiting period $(\tau_k^+ - \theta_{k-1}^+)$, holding period $(\theta_k^+ - \tau_k^+)$ and the maximum loss during the holding period, when $\delta = \sigma$ is the intraday standard deviation of the signal, a common choice for mean-reversion strategies [23].

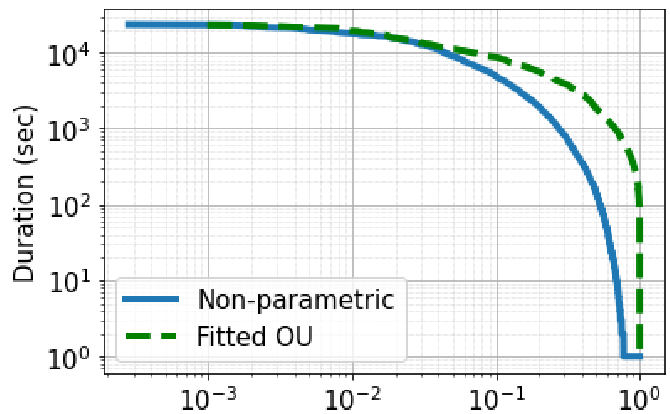
As seen from the semi-logarithmic plots in Figures 6b and 6c, the durations of the holding period and the waiting period are approximately exponentially distributed. The maximum loss has a Pareto tail with exponent $k = 1$, which is very heavy tailed and indicates infinite mean and variance, as shown by the log-log plot in Figure 6d. This combination of a Pareto tail for the excursion height and an exponential duration for $\delta-$excursions corresponds neither to the Brownian case nor to the case of the Ornstein-Uhlenbeck process.

For comparison, we estimate the OU model (29) using a method of moments, leading the following parameter estimates for the KO-PEP pair (time is measured in seconds):

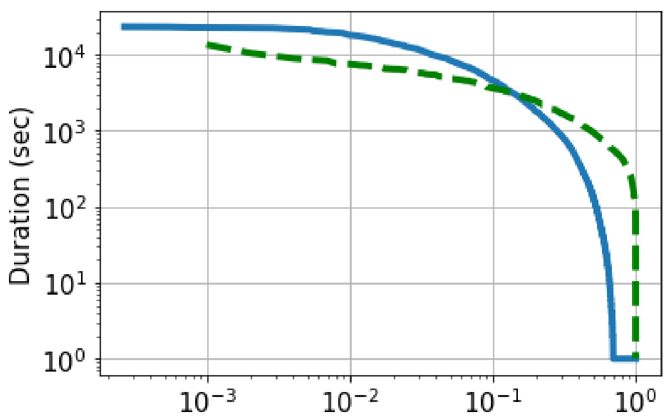
$$\widehat{\alpha} = 0.00064, \quad \widehat{\mu} = -8.634 \times 10^{-5}, \text{ and } \widehat{\gamma} = 0.0014.$$



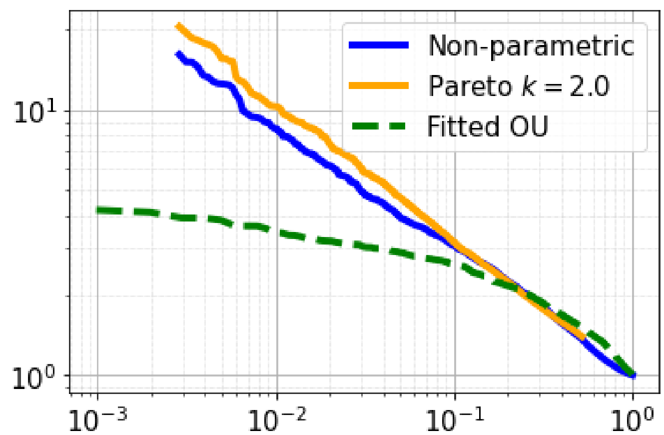
(a) Coca-Pepsi pair trading signal: 07/16/2013.



(b) Rank-frequency plot for the waiting period $(\tau_k^+ - \theta_{k-1}^+)$, semi-logarithmic scale.



(c) Rank-frequency plot for the holding period $(\theta_k^+ - \tau_k^+)$, semi-logarithmic scale.



(d) Rank-frequency plot for maximum loss during the holding period (log-log scale). Orange dotted line: Pareto distribution with exponent $k = 2.0$.

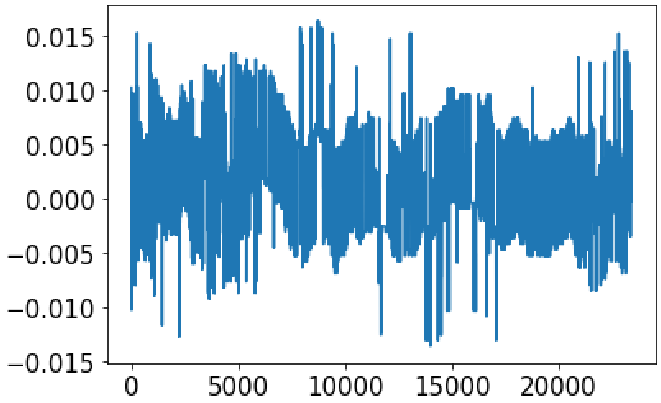
Figure 6: KO-PEP pair trading signal $\mathrm{KO}(t) - a_t\mathrm{PEP}(t) + b_t$ at one-second frequency, for $\delta = \sigma$ (2007-2020). Data (blue) vs O-U model (dotted).

The corresponding model-based distributions for the duration of the holding period, the waiting period and the worst loss during the holding period are displayed (green dotted lines) alongside the empirical distributions of these quantities in Figure 6. The discrepancy between the green dotted lines and the blue solid lines in Figures 6b, 6c, and 6d illustrates that the distributions computed using the Ornstein-Uhlenbeck model give a poor approximation of the corresponding empirical distributions, leading to an inaccurate representation of the risk and return profile of the strategy. In particular, the fitted OU process underestimates duration of the holding period (see Figure 6c) and the maximum loss during the holding period (see Figure 6d). This is a strong indication of the risk of model mis-specification in such mean-reversion strategies.

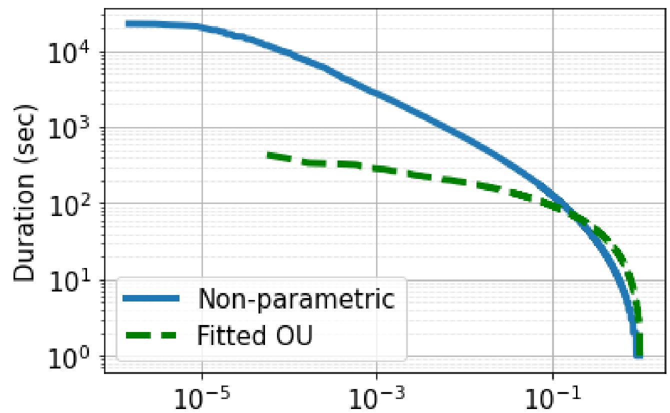
For the SH-SDS pair, we have the following estimates of the Ornstein-Uhlenbeck process:

$$\widehat{\alpha} = 0.08, \quad \widehat{\mu} = -8.634 \times 10^{-5}, \text{ and } \widehat{\gamma} = 0.0014.$$

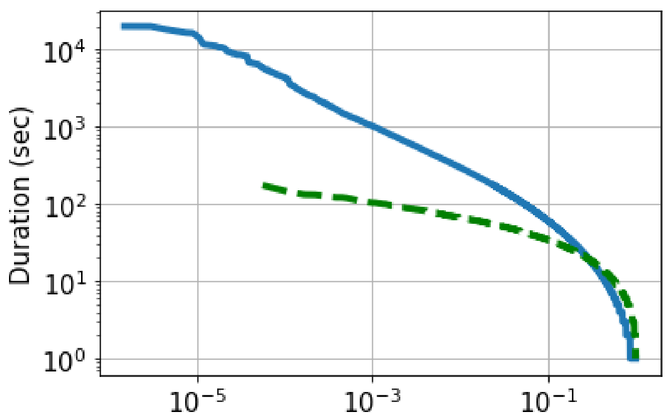
The discrepancy between the green dotted lines and the blue solid lines in Figures 7b, 7c, and 7d illustrates that the distributions computed using the Ornstein-Uhlenbeck model give a poor approximation of all three empirical distributions including the duration of the holding period, the waiting period and the worst loss during the holding period, leading to an inaccurate representation of the risk and return profile of the strategy. This is yet another indication of the risk of model mis-specification in such mean-reversion strategies.



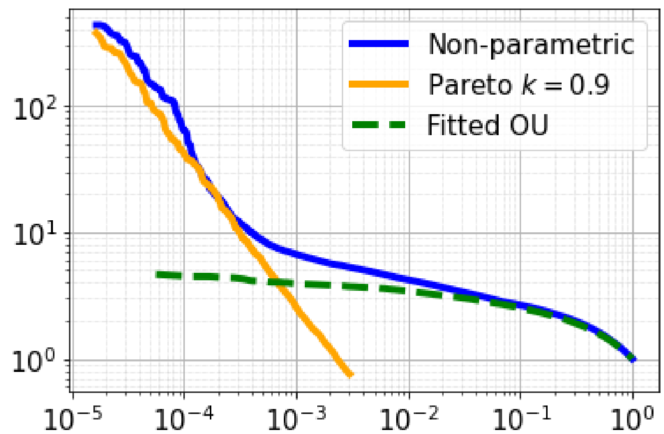
(a) SH-SDS pair trading signal: 07/16/2013.



(b) Rank-frequency plot for the waiting period ($\tau_k^+ - \theta_{k-1}^+$), semi-logarithmic scale.



(c) Rank-frequency plot for the holding period ($\theta_k^+ - \tau_k^+$), semi-logarithmic scale.



(d) Rank-frequency plot for maximum loss during the holding period (log-log scale). Orange dotted line: Pareto distribution with exponent $k = 0.9$.

Figure 7: SH-SDS pair trading signal $\mathrm{SH}(t) - a_t\mathrm{SDS}(t) + b_t$ at one-second frequency, for $\delta = \sigma$ (2007-2020). Data (blue) vs O-U model (dotted).

It is worth pointing out that the distribution of the rank-frequency plot has two regimes (Figure 7d): small losses, associated with excursions of small amplitude, behave as in the Ornstein-Uhlenbeck model whereas large losses, associated with excursions of large amplitudes exhibit a heavy tail following a Pareto distribution with exponent $k = 0.9$, which resembles the case oif Brownian excursions [3]. The excursions of these trading signals thus seem to interpolate between the OU model for small amplitudes and the (driftless) Brownian case for large amplitudes, as observed in [3].

6 Model-free scenario simulation

To tackle the shortcomings of the paremetric models in representing key features of price excursions, we propose a non-parametric scenario simulation method. The idea is to resample from the set of empirical excursions, by pasting shuffled historical excursions and develop the corresponding theoretical guarantees for a broad class of regenerative processes.

Construction of regenerative processes by concatenation of independent excursions has been studied by Lambert and Simatos [19] and Yano [25]. Our $\delta-$excursion concept is related to, but slightly different from, the concept of 'big' excursion in [19]; elements of Γ_δ may be seen as 'big excursions' in the sense of [19] and relate to $\delta-$excursions through the last exit decomposition, as noted in Lemma 3.2. The above examples illustrate that, to correctly reflect the risk and return of dynamic trading strategies, a model needs to adequately reflect the excursion properties of trading signals. We now show how Proposition 3.3 may be used as a *non-parametric* method for simulating paths whose excursions match those observed in data.

Define the *shift* operator

$$\begin{aligned} \theta_t : \mathcal{E} \quad &\mapsto \quad \mathcal{E} \\ f \quad &\to \theta_t(f) := f(\cdot + t) \end{aligned} \tag{30}$$

Consider a data set consisting of observations on the path $(S_t, t \in [0,T])$ of a $\mathbb{P}-$regenerative process which is recurrent at 0. We make the following assumption: for some $\delta > 0$,

Assumption 6.1 (Regenerative and Recurrent Process)**.** *There exists a measure* $\mathbf{P}_0$ *such that* S *with law* $\mathbb{P}$ *satisfies:*

$$\mathbb{P}(\theta_\tau(S) \in \cdot | \mathcal{F}_\tau) = \mathbf{P}_0,\ \mathbb{P} - \textit{almost surely on } \{\tau < \infty, S(\tau) = 0\}, \tag{31}$$

where $\mathcal{F}$ *is the natural filtration of* S*. In addition, assume that*

$$\mathbb{P}(T_0^0(S) < \infty) = 1. \tag{32}$$

Applying Proposition 3.3, we may decompose the path S into a sequence $(e_k)_{k=1..D_T^\delta}$ of $\delta-$excursions as in (17). By Assumption 6.1, e_k are IID variables with values in $\mathcal{U}_\delta$, whose law we denote Π^δ. Denoting by ϵ_x a unit point mass at x, we define the *empirical* δ*-excursion measure*

$$\Pi_T = \frac{1}{D_T^\delta(S)} \sum_{k=1}^{D_T^\delta(S)} \epsilon_{e_k},$$

which defines a probability measure on $\mathcal{U}_\delta$.

Under Assumption 6.1, $D_T^\delta(S) \to \infty$ as $T \to \infty$, so by the law of large numbers the empirical excursion measure Π_T provides a good approximation of Π^δ for large T and for any Glivenko-Cantelli class $\mathcal{F}$ of functions [24] on $\mathcal{U}_\delta$, representing properties of δ-excursions, we have

$$\forall H \in \mathcal{F}, \qquad \int H(f)\Pi_T(df) \overset{T\to\infty}{\to} \int H(f)\Pi^\delta(df). \tag{33}$$

For the moment, (33) is only proved to hold when S is regenerative and recurrent (i.e. under Assumption 6.1) which include Brownian motion and Ornstein-Uhlenbeck processes as special cases.

The point is that it is fairly easy to simulate samples from Π_T and evaluate $\int F(f)\Pi_T(df)$: this may be done by randomly resampling from the empirical sequence of excursions (e_k). This leads to an approach for *non-parametric scenario simulation* based on $\delta-$excursions (see Table 1).

Non-parametric scenario simulation by pasting of excursions.

Input data : sample path $(S_t, t \in [0,T])$

- Decompose $(S_t, t \in [0,T])$ into $\delta-$excursions $e_1, ...e_N \in \mathcal{U}_\delta$ using Proposition 3.4.
- Generate an IID sequence of integers $(k_1(\omega), k_2(\omega), ..)$ where $k_i \sim \text{UNIF}(\{1,2,..,N\})$.
- Construct a path $X(\omega)$ as in (17) by concatenating the excursions in the order given by $(k_i, i \geq 1)$:

$$X_t(\omega) = \sum_{i\geq 1} e_{k_i(\omega)}((t - \theta^+_{i-1}) \qquad \text{where} \qquad \theta^+_0 = 0, \quad \theta^+_i = \sum_{j=1}^{i} \Lambda(e_{k_i}).$$

Output : simulated sample path X

Table 1: Non-parametric scenario simulation by pasting of excursions.

If $N = D^\delta_T(S)$ is large, the paths X generated in this way have $\delta-$excursions whose properties mimic those of S.

The shortcomings of the model-based methods could be addressed by using our non-parametric scenario simulation approach to evaluate the quantities of interest as described in Table 1: we decompose the empirical path into $\delta-$excursions and generate scenarios by random pasting of such empirical $\delta-$excursions. Figure 8 shows a typical sample path generated in this manner, for $\delta = \sigma$. By construction, paths generated in this way retain the roughness properties of the observe path as well as the empirical distribution of heights and durations of $\delta-$excursions.

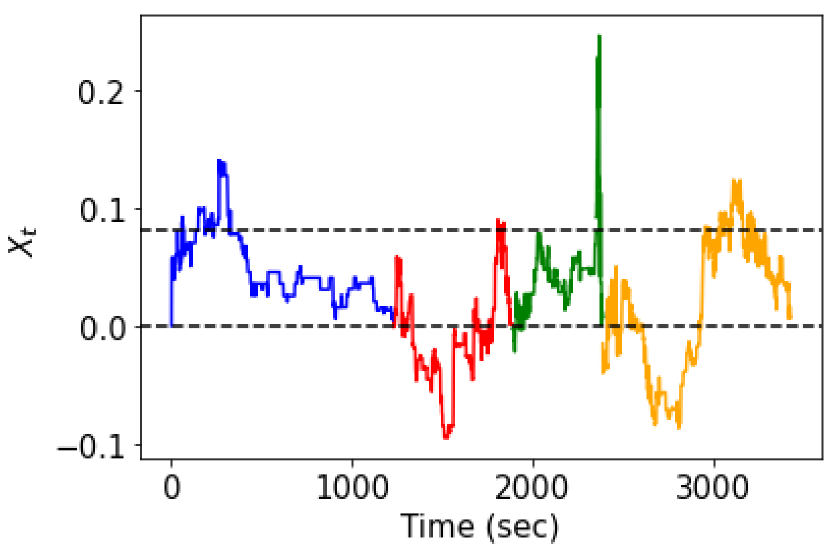


Figure 8: Example of sample path generated via the non-parametric method, using KO-PEP signals.

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