Brownian

Fractional Brownian

S_n

δ_1 δ_2

Truncated variation

Excursion Risk

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Abstract

The risk and return profiles of a broad class of dynamic trading strategies, including pairs trading and other statistical arbitrage strategies, may be characterized in terms of *excursions* of the market price of a portfolio away from a reference level. We propose a mathematical framework for the risk analysis of such strategies, based on a description in terms of price excursions, first in a pathwise setting, without probabilistic assumptions, then in a Markovian setting.

We introduce the notion of δ-*excursion*, defined as a path which deviates by δ from a reference level before returning to this level. We show that every continuous path has a unique decomposition into δ-excursions, which is useful for the scenario analysis of dynamic trading strategies, leading to simple expressions for the number of trades, realized profit, maximum loss and drawdown. As δ is decreased to zero, properties of this decomposition relate to the local time of the path.

When the underlying asset follows a Markov process, we combine these results with Ito's excursion theory to obtain a tractable decomposition of the process as a concatenation of independent δ-excursions, whose distribution is described in terms of Ito's excursion measure. We provide analytical results for linear diffusions and give new examples of stochastic processes for flexible and tractable modeling of excursions. Finally, we describe a non-parametric scenario simulation method for generating paths whose excursion properties match those observed in empirical data.

Keywords: excursion theory, Markov process, local time, regenerative processes, mean-reversion strategies, diffusion processes, Ornstein-Uhlenbeck process, drawdown risk, statistical arbitrage.

Contents

1 Introduction

A broad class of trading strategies may be described in terms of the relation between the *market price* P_t of an asset –a stock, bond, commodity, a spread between two such assets, or a basket of assets– and a *reference* level A_t, which may refer to an assessment of the portfolio's fundamental value by an analyst, or a forecast of the portfolio's value based on 'technical' indicators, such as moving average estimators used in pairs trading [33] or 'technical indicators' used in statistical arbitrage strategies [1, 2, 3, 20]. The deviation of the market price from the reference value then represents a trading signal:

$$S_t = P_t - A_t.$$

If S falls below some negative threshold $-\delta < 0$, this represents a buying opportunity, while if S exceeds a positive threshold $\delta > 0$, this represents an opportunity for entering a short position. A wide range of trading strategies – pairs trading [33, 17], mean-reversion strategies [3, 27], statistical arbitrage strategies based on cointegration [1], index arbitrage [2] and other statistical arbitrage strategies [3, 20]– fall under this description. The reference level A_t is computed differently in each of these examples, but once the signal $S = P - A$ is constructed all these strategies follow the description given above.

Regardless of how the reference value A_t is arrived at, e.g. using fundamental valuation principles, or statistical forecasts, this leads to similar features across all such trading strategies: a long position is entered when the signal S crosses $-\delta$ and held until S crosses 0; similarly, a short position is entered when S crosses δ and held until S crosses zero. The holding periods of positions thus coincide with *excursions* of the signal S above (or below) certain levels.

It turns out that this remark has interesting implications: it implies that the risk and return profile of such trading strategies may be described in terms of the properties of excursions of the process S. For example, the profit of such a strategy is linked to the number of the excursions alluded to above, while the magnitude of drawdown risk may be linked to the height of the excursions.

Analytical and probabilistic properties of excursions of stochastic processes have been studied in detail, starting with P. Lévy [29] for Brownian motion, K. Ito [21] for Markov processes. Ito's seminal contribution was to note that the excursions of a Markov process from a level may be viewed as a collection of independent random variables in a (infinite-dimensional) space of excursions. This infinite-dimensional viewpoint has proved extremely fruitful for the theoretical study of stochastic processes [6, 14, 31, 36, 35, 43, 45]. In the present work, we show that this approach is also very relevant for applications, in particular for the analysis of dynamic trading strategies.

The construction and empirical performance of pairs trading [33, 17] and 'mean-reversion' trading strategies [3, 27] considered in this paper have been studied by Avellaneda & Lee [3], Gatev et al. [17] and others [33, 20]. Leung and Li [27] study mean-reversion strategies from the perspective of optimal control, in the setting of the Ornstein-Uhlenbeck model. The connection between statistical arbitrage and cointegration has been discussed by many authors, including Alexander [1] and Alexander & Dimitriu [2]. Our approach provides a different perspective on these results through the angle of excursion theory and explains the common features observed across the variety of strategies considered in these studies.

Excursion theory has also been applied in mathematical finance, for the pricing of certain path-dependent options involving barrier crossings of a price process, such as Parisian options [10, 13], barrier options [30] or 'occupation time derivatives' [8]. These studies focus on analytical results for special models such as Brownian motion [10] or certain Lévy processes [8, 30]. Lakner et al [25] apply excursion theory to the study of scaling limits for a limit order book model.

A related topic is the modeling of *drawdown risk* for trading strategies [19]. The literature on this topic has focused on the analytical study of drawdown risk and optimal investment under drawdown constraints in specific models. Zhang [46] uses excursion theory for one-dimensional diffusion models to derive formulas for drawdown risk of static portfolios. On the other hand empirical studies of drawdown risk indicate that commonly used stochastic models do not correctly quantify drawdown risk even for passive index portfolios [23], suggesting that better, more flexible stochastic models are needed. In Section 6 we propose a new approach to stochastic modeling which addresses this issue.

Outline We propose a mathematical framework for the risk analysis of such strategies, based on a description in terms of price excursions. We present our approach, first in a pathwise setting, without probabilistic assumptions, then in a probabilistic setting, when the price is modeled as a Markov process.

We start in Section 2 by describing how properties of a large class of trading strategies may be expressed in terms of excursions of a reference process away from zero. We then introduce in Section 3 the notion of δ-*excursion*, defined as a path which deviates by δ from a reference level before returning to this level. We show that every continuous path has a unique decomposition into such δ-excursions, which turns out to be useful for the scenario analysis of dynamic trading strategies, leading to simple expressions for the number of trades, realized profit, maximum loss and drawdown (Section 3.3). In the case of irregular paths which possess a local time, we describe in Section 3.4 the relation between δ-excursions and local time at zero of the path.

When the underlying asset follows a Markov process, we combine these results with Ito's excursion theory for Markov processes in Section 4 to obtain a tractable decomposition of the process as a concatenation of independent, identically distributed δ-excursions, whose distribution is described in terms of Ito's excursion measure (Theorem 4.7).

In the case of Brownian motion and linear diffusions, the decomposition into $\delta-$excursions may be fruitfully combined with analytical properties of the Ito excursion measure to obtain analytical results on the properties of mean-reversion strategies, such as the distribution of maximum loss or drawdown risk. Examples of such results are given in Section 5.

In Section 6 we propose an approach for building stochastic models based on the properties of their $\delta-$excursions, and provide examples of flexible parametric models for modeling price excursions. Our approach extends the Ito construction [21] and leads to new examples of regenerative processes with asymmetric upward and downward excursions. Finally, we describe in Section 6.2 a non-parametric scenario simulation method for generating paths whose excursions match those observed in a data set.

2 Mean-reversion strategies

2.1 Trading signals

Many trading strategies are based on the assumption that the market price P_t of a reference asset reverts to a 'target value' or forecast A_t over a certain horizon, although it may deviate from it in the short term. The examples below illustrate the generality of this concept.

Example 2.1 (Value trading)**.** An investor who believes that the price of the asset will eventually revert to a 'fundamental' value $A > 0$ will choose to buy the asset when P_t drops below A and short the asset when P_t exceeds A. This 'fundamental' value can be a book value or a valuation by a financial analyst. The deviation $S_t = P_t - A$ from the fundamental value then plays the role of trading signal.

Example 2.2 (Pairs trading)**.** Pairs trading is a relative-value trading strategy which looks for pairs of assets whose prices P^1, P^2 are cointegrated [1], i.e. there exists a stationary combination $P_t = P^1_t - wP^2_t$. w is typically estimated using regression techniques [17]. If A is the stationary mean of P_t then the deviation $S_t = P^1_t - wP^2_t - A$ is expected to revert to zero and is used as a trading signal. In practice this mean A is estimated via a moving average [33] which leads to a time-dependent but slowly varying reference level A_t.

Example 2.3 (Mean-reversion strategies)**.** Many *statistical arbitrage strategies* [3, 20] are based on identifying combinations of assets (portfolios) whose market price follows a stationary, mean-reverting process [2, 3], using methods such as index tracking or cointegration [2].

The market price $P_t = \sum w_i P^i_t$ of such a stationary combination is then expected to revert to its mean A, which may be estimated using for instance a moving average estimator A_t, leading to the trading signal $S_t = \sum w_i P^i_t - A_t$ which is expected to revert to zero.

These strategies, while distinct in their design, share a common feature: they are based on the assumption that a trading signal $S_t = P_t - A_t$, defined as the deviation of the market price P_t of a reference asset from a target value A_t, reverts to zero over some time horizon. This assumption implies that if $S_t < 0$ (resp. $S_t > 0$) one should take a long (resp. short) position in the portfolio P.

In presence of transaction costs, such transactions will be entered only if the signal reaches some threshold $\pm\delta$, leading to the following strategy:

(i) Enter a long position in the reference portfolio when S_t drops below $-\delta$; unwind the long position when S_t crosses zero;

(ii) Enter a short position in the portfolio when S_t exceeds δ; unwind the short position when S_t crosses zero.

Such a strategy may be implemented through limit orders placed at the appropriate price levels, resulting in transactions when the market price P_t crosses these levels.

We now describe the associated trading strategies and their properties in more detail.

2.2 Trades and excursions

Regardless of how the signal S is constructed, the trading strategies in the above examples share some common features, which may be described in terms of the *level crossings* of the signal S.

We define the following level crossing times of S: we set $\tau_0^+ = 0$, $\theta_0^+ = 0$ and

$$\forall i \geq 1, \quad \tau_i^+ = \inf\{t > \theta_{i-1}^+, S_t \geq \delta\} \qquad \theta_i^+ = \inf\{t > \tau_i^+, S_t \leq 0\}. \tag{1}$$

The intervals (τ_i^+, θ_i^+), $(\theta_i^+, \tau_{i+1}^+)$ are the down-crossing and up-crossing intervals of the interval $[0, \delta]$. Each interval $[\theta_i^+, \theta_{i+1}^+]$, corresponds to an *excursion* of S from 0 to δ and back to zero.

It is readily observed that the intervals (τ_i^+, θ_i^+), $(\theta_i^+, \tau_{i+1}^+)$ form a partition of $[0, \infty)$ and, if the path is continuous, they are all non-empty.[1] One can also define similar quantities for downward excursions:

$$\tau_0^- = \theta_i^- = 0, \quad \text{and} \quad \forall i \geq 1, \quad \tau_i^- = \inf\{t > \theta_{i-1}^-, S_t \leq -\delta\} \qquad \theta_i^- = \inf\{t > \tau_i^-, S_t \geq 0\}. \tag{2}$$

A mathematical description of the trading strategies described in Section 2.1 can now be given in terms of the level crossing times defined above:

- buy the reference portfolio when the trading signal drops below $-\delta$, sell when it returns to 0:

$$\phi^- = \sum_{k \geq 1} 1_{[\tau_k^-, \theta_k^-)}. \tag{3}$$

- short the reference portfolio when the signal exceeds δ, unwind the position when it reaches 0:

$$\phi^+ = -\sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+)}. \tag{4}$$

We refer to ϕ^+, ϕ^- as one-sided strategies.

Combining the two strategies we obtain what is usually called a 'mean-reversion strategy' or 'convergence trade' based on the trading signal S:

$$\phi^0(t) = \phi^+(t) + \phi^-(t) = \sum_{k \geq 1} 1_{[\tau_k^-, \theta_k^-)} - \sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+)}. \tag{5}$$

One may also considering a position size modulated as a function of S. For example, (5) has unbounded exposure to price movements and in most cases portfolios are subject to position limits or exposure limits ('stop loss'). A maximum exposure limit of M on short positions in (4) leads to unwinding the position if S reaches $\delta + M$ during the holding period:

$$\phi_M^+(t) = -\sum_{k \geq 1} 1_{[\tau_k^+, \theta_k^+ \wedge \kappa_k)} \qquad \kappa_k = \inf\{t > \tau_k^+, S(t) \geq \delta + M\}. \tag{6}$$

In addition to the position in the risky asset S, each portfolio has a cash component, which is adjusted to reflect the gains and losses from trading, so that the strategy is self-financing. Denoting by $V_t(\phi)$ the sum of the cash holdings and the market value of a position ϕ in the risky asset, we have

$$V_t(\phi) = V_t(0) + \int_0^t \phi(u-) dS_u. \tag{7}$$

[1] In the case of càdlàg paths a finite number of these intervals could be empty, i.e. $\tau_i^+ = \theta_i^+$, if the process jumps across the interval $(0, \delta)$. In this paper we focus on the case of continuous trajectories.

Note that as the portfolios considered above are piece-wise constant, no particular assumption on S is required to define the integral in (7).

As the sets $\cup_{k\geq 1}[\tau_k^-, \theta_k^-]$ and $\cup_{k\geq 1}[\tau_k^+, \theta_k^+]$ are disjoint we may study the properties of ϕ^+, ϕ^- separately. In the following sections we will focus on ϕ^+, but it is clear that properties of ϕ^- are analogously obtained by replacing S by $-S$.

Let us now examine further the properties of the one-sided strategy (4). Each transaction cycle $[\theta_{k-1}^+, \theta_k^+]$ is decomposed into a *waiting period* $[\theta_{k-1}^+, \tau_k^+]$ followed by a *holding period* $[\tau_k^+, \theta_k^+]$. The strategy generates a profit of δ over each transaction cycle, leading to a portfolio value

$$V_t(\phi^+) = V_0(\phi^+) + \delta\ D_t^\delta(S) + S(t \wedge \theta^+_{D_t^\delta(S)+1}) - S(t \wedge \tau^+_{D_t^\delta(S)+1}) \quad \text{where} \quad D_t^\delta(S) = \sum_{i\geq 1} 1_{\theta_i^+ \leq t} \tag{8}$$

represents the number of transactions in $[0, t]$. The first term $\delta\ D_t^\delta(S)$ represents the *realized profit* while the second term corresponds to the market value of the current position. If the path of S wanders high above δ then the portfolio can incur a large market loss. It is therefore clear that the gains and losses of the trading strategy ϕ^+ are linked to the frequency, duration and amplitude of positive *excursions* of S which exceed the level δ. Similarly, one can readily observe that the gains and losses of ϕ^- are linked to the frequency, duration and height of negative *excursions* of S which reach $-\delta$. In the following sections we build on this insight and study in more detail the structure of such excursions in order to model the risk and return profile of such portfolios.

3 Pathwise properties and scenario analysis

3.1 Excursions and $\delta-$excursions

Let $\mathcal{E} = C^0([0,\infty),\mathbb{R})$ be the space of continuous functions equipped with the Borel measurable structure induced by the uniform norm and $\mathcal{E}_0 = \{f \in \mathcal{E}, f(0) = 0\}$. Denote, for $f \in \mathcal{E}$,

$$T^x(f) = \inf\{t > 0, f(t) = x\}, \qquad T_t^x(f) = \inf\{u > t, f(u) = x\}. \tag{9}$$

Let $\delta \in \mathbb{R}$. We will call an *excursion* from 0 to δ a path which starts from zero, reaches δ in a finite time, and stops when it reaches δ:

$$\mathcal{E}_{0,\delta} = \{f : C^0([0,\infty) \to \mathbb{R},\ f(0) = 0,\ T^\delta(f) < \infty;\ \forall t \geq T^\delta(f), f(t) = \delta\,\}. \tag{10}$$

Note that by this definition an excursion from 0 to δ is stopped at the first time it reaches δ. In particular, $\mathcal{E}_{0,0}$ is the space of *excursions* from 0 to 0.

Define the *concatenation* at $T > 0$ of two paths $u, v \in \mathcal{E}$ as the element

$$u \underset{T}{\oplus} v(t) := u(t)\ 1_{[0,T)} + v\,(t - T)\,1_{[T,\infty)}. \tag{11}$$

Note that if $u \in \mathcal{E}_{0,a}, v \in \mathcal{E}_{a,0}$ then for $T \geq T^a(u)$, $u \oplus_T v \in \mathcal{E}_{0,0}$.

We define a $\delta-$*excursion* as an excursion from 0 to δ, followed by an excursion from δ back to 0:

Definition 3.1 (δ-excursion)**.** A $\delta-$excursion is a path $f \in \mathcal{E}$ such that

$$\exists (u, v) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0},\ \ f = u \underset{T^\delta(u)}{\oplus} v, \qquad \text{i.e.} \quad f(t) = u(t)\ 1_{[0,T^\delta(u))} + v\left(t - T^\delta(u)\right) 1_{[T^\delta(u),\infty)} \tag{12}$$

The decomposition (12) is then unique and we denote $\Lambda(f) = T^\delta(u) + T^0(v)$ the *duration* of f.

We denote by $\mathcal{U}_\delta$ the set of $\delta-$excursions. The map $f \in \mathcal{U}_\delta \mapsto (u, v, \Lambda(f)) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0} \times [0,\infty)$ is measurable.

Examples of $\delta-$excursions are excursions from 0 to 0 which reach δ:

$$\Gamma_\delta = \Big\{ f \in \mathcal{E}_{0,0} \colon \ \max(f) \geq \delta \Big\} = \mathcal{U}_\delta \cap \mathcal{E}_{0,0}. \tag{13}$$

But the inclusion $\Gamma_\delta \subset \mathcal{U}_\delta$ is strict, as a typical $\delta-$excursion may reach zero (infinitely) many times before reaching δ and we may have $\Lambda(f) > T^0(f)$ for $f \in \mathcal{U}_\delta$. In particular $\mathcal{U}_\delta$ is *not* a subset of $\mathcal{E}_{0,0}$. However, each path in $\mathcal{U}_\delta$ contains *exactly* one excursion of type Γ_δ:

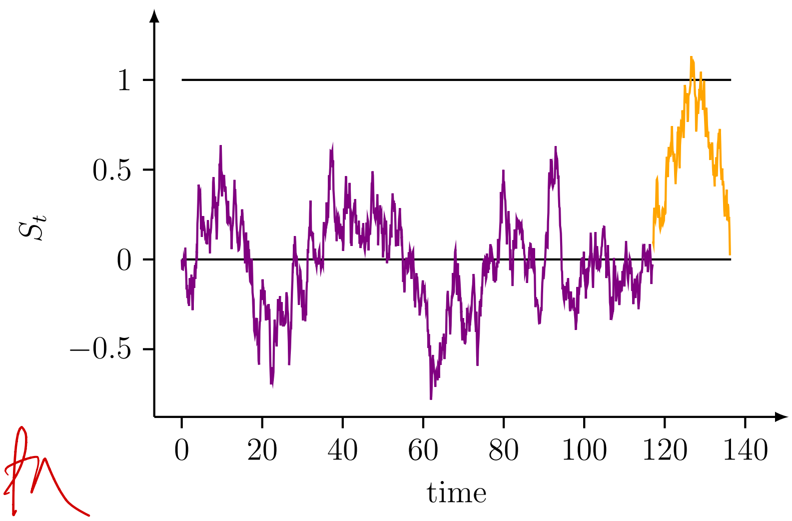
Lemma 3.2 (Last exit decomposition of δ-excursions)**.** *Any δ-excursion $f \in \mathcal{U}_\delta$ has a unique decomposition into a path from 0 to 0 which does not reach δ followed by an excursion $\gamma \in \Gamma_\delta$ from 0 to 0 which reaches δ:*

$$\forall f \in \mathcal{U}_\delta, \quad \exists!(T, g, \gamma) \in [0,\infty) \times \mathcal{E}_0 \times \Gamma_\delta, \quad f = g \underset{T}{\oplus} \gamma \quad \text{with} \quad g(0) = g(T) = 0, \quad \max(g) < \delta. \tag{14}$$

Proof. Consider a δ-excursion $f \in \mathcal{U}_\delta$. Then f has a decomposition (12) for some $(u,v) \in \mathcal{E}_{0,\delta} \times \mathcal{E}_{\delta,0}$ we have $f(t) = 0$ for $t \geq \Lambda(f) = T^\delta(u) + T^0(v)$. Now define T as the last zero of u before $T^\delta(u)$:

$$T = \sup\{t < T^\delta(u), \ u(t) = 0\}.$$

Then by continuity of u, $T < T^\delta(u)$ and therefore $\max\{u(t), 0 \leq t \leq T\} < \delta$. Setting $g = f 1_{[0,T]}$ and $\gamma(t) = f((t-T)_+)$, it is readily verified that $\gamma \in \Gamma_\delta$ and g satisfy the required conditions. □



Calculate delta

Figure 1: Example of last exit decomposition of a δ-excursion $f \in \mathcal{U}_\delta$ with $\delta = 1$: $f = g \oplus_T \gamma$ where g is in purple and $\gamma \in \Gamma_\delta$ (in orange) is the last excursion.

3.2 Decomposition of a path into $\delta-$excursions

The following proposition gives the decomposition of any path starting from zero into a sequence of excursions from 0 to δ and back to 0:

Proposition 3.3. *Let $S \in C^0([0,\infty), \mathbb{R})$ with $S_0 = 0$. Define the level crossing times*

$$\theta_0^+ = 0, \quad \tau_0^+ = 0, \qquad \tau_i^+ = T^\delta_{\theta^+_{i-1}}(S), \qquad \theta_i^+ = T^0_{\tau_i^+}(S).$$

$$\text{Then} \qquad \forall t \geq 0, \qquad D_t^\delta(S) = \textstyle\sum_{i \geq 1} 1_{\theta_i^+ \leq t} < \infty \qquad \text{and} \tag{15}$$

$$\forall t \geq 0, \qquad S_t \ = \textstyle\sum_{i=1}^{D_t^\delta(S)+1} \Big[u_i\left(t - \theta^+_{i-1}\right) 1_{[\theta^+_{i-1}, \tau_i^+)} \ + v_i\left(t - \tau_i^+\right) 1_{[\tau_i^+, \theta_i^+)} \Big], \tag{16}$$

where $u_i \in \mathcal{E}_{0,\delta}$ and $v_i \in \mathcal{E}_{\delta,0}$.

Proof. To prove the first assertion, we first note that S is continuous, thus uniformly continuous on $[0,T]$ for any $T>0$. If $D_t^\delta=\infty$ for some $t>0$, then the set $\{k\in\mathbb{N},\quad \theta_k^+\le t\}$ is infinite. Since by construction the intervals (τ_k^+,θ_k^+) are disjoint, we have

$$\sum_{\{k,\theta_k^+\le t\}}|\theta_k^+-\tau_k^+|\le t<\infty,\quad \text{so}\quad \inf_{\theta_k^+\le t}|\theta_k^+-\tau_k^+|=0\quad\text{while}\qquad |S_{\theta_k^+}-S_{\tau_k^+}|=\delta$$

which contradicts the uniform continuity of S on $[0,t]$. Therefore $D_t^\delta<\infty$ for all $t\ge 0$. Starting from:

$$S_t=\sum_{i=1}^{D_t^\delta(S)+1}\Big[(S_{t\wedge\tau_i^+}-S_{t\wedge\theta_{i-1}^+})+(S_{t\wedge\theta_i^+}-S_{t\wedge\tau_i^+})\Big].$$

Note that

$$(S_{t\wedge\tau_i^+}-S_{t\wedge\theta_{i-1}^+})+(S_{t\wedge\theta_i^+}-S_{t\wedge\tau_i^+})=(S_{t\wedge\tau_i^+}-S_{t\wedge\theta_{i-1}^+})1_{[\theta_{i-1}^+,\tau_i^+)}+(\delta+S_{t\wedge\theta_i^+}-S_{t\wedge\tau_i^+})1_{[\tau_i^+,\theta_i^+)},$$

it remains to set

$$u_i(t):=S_{(t+\theta_{i-1}^+)\wedge\tau_i^+}-S_{(t+\theta_{i-1}^+)\wedge\theta_{i-1}^+},\ i\ge 1,\qquad v_i(t):=\delta+S_{(t+\tau_i^+)\wedge\theta_i^+}-S_{(t+\tau_i^+)\wedge\tau_i^+},\ i\ge 1.$$

□

The above results translate into a (measurable) decomposition of any continuous path into δ-excursions:

Proposition 3.4 (Decomposition of a path into $\delta-$excursions)**.** *Let $\delta>0$ and $S\in C^0([0,\infty),\mathbb{R})$ with $S_0=0$, and define $D_t^\delta(S)$ as in (15).*

(i) If $\sup_{t\ge0}D_t^\delta(S)=\infty$ there exists a unique sequence $(e_k)_{k\ge1}$ of $\delta-$excursions $e_k\in\mathcal{U}_\delta$ such that

$$\forall t\ge 0,\quad S_t=\sum_{k\ge1}e_k\left((t-\theta_{k-1}^+)_+\right)\qquad\text{where}\qquad \theta_0^+=0,\quad \theta_k^+=\sum_{i=1}^k\Lambda(e_i).\tag{17}$$

(ii) If $d=\sup_{t\ge0}D_t^\delta(S)<\infty$ then there exist $(e_1,...,e_d)\in(\mathcal{U}_\delta)^d$ and $e_{d+1}\in\mathcal{E}$ such that

$$S_t=\sum_{k=1}^{d+1}e_k\left((t-\theta_{k-1}^+)_+\right)\qquad\text{where}\qquad \theta_0^+=0,\quad \theta_k^+=\sum_{i=1}^k\Lambda(e_i).\tag{18}$$

In both cases the map $S\mapsto(e_k)_{k=1..(d+1)}$ is measurable.

The case (i) corresponds to the 'recurrent' case where the path crosses zero and δ infinitely many times on $[0,\infty)$.

Proof. Set $d=\sup_{t\ge0}D_t^\delta(S)\in\mathbb{N}\cup\{\infty\}$. Define $(\theta_k^+,k\ge1)$ as in (1). For $k<d$, set $e_k(t)=S(t-\theta_{k-1}^+)1_{[\theta_{k-1}^+,\theta_k^+)}(t)$. Then it is easily verified, from the definition (1) of θ_k^+, that $e_k\in\mathcal{U}_\delta$ and $\Lambda(e_k)=\theta_k^+-\theta_{k-1}^+$. Measurability of the map $S\mapsto(e_k)_{k\ge1}$ follows from the measurability of the hitting times and the shift operator. To show uniqueness, we note that (17) implies that $e_k(\theta_{k-1}^++.)=S_{|[\theta_{k-1}^+,\theta_k^+)}$ so it is sufficient to show uniqueness of the sequence $(\theta_k^+)_{k\ge0}$. As $D_t^\delta(S)<\infty$ for each $t>0$, the countable set $\{t>0,\Delta D_t^\delta\neq0\}$ is discrete and has a unique increasing ordering, which is given by $(\theta_k^+)_{k\ge0}$. □

Remark 3.5. The above results decompose the path into one-sided δ-excursions i.e. with $\delta>0$. One can immediately obtain a similar decomposition for $\delta<0$ by applying the above result to the path $-S$. To obtain a decomposition in terms of *two-sided* δ-excursions, one can iterate these two results: first decompose S into δ-excursions, then decompose each δ-excursion into $(-\delta)-$excursions. One may further show that the resulting decomposition is independent of the order of these two operations.

3.3 Scenario analysis for mean-reversion strategies

The *drawdown* [19] of a portfolio ϕ whose value at time t is $V_t(\phi)$ is defined as

$$\Delta(t) = M_t(\phi) - V_t(\phi) \qquad \text{where} \quad M_t(\phi) = \max_{[0,t]} V_t(\phi)$$

is the running maximum.

The decomposition of the path into $\delta-$excursion given in Proposition 3.4 leads to simple expressions for the portfolio value, the maximum loss and the drawdown of the strategy:

Proposition 3.6. *Along a path S with decomposition (16),*

(i) the gain $V_t(\phi^+) - V_0(\phi^+) = \int_0^t \phi^+ dS$ of the portfolio is given by

$$V_t(\phi^+) - V_0(\phi^+) = \delta \times D_t^\delta(S) + 1_{[\tau^+_{D_t^\delta+1}, \theta^+_{D_t^\delta+1}]} \left(\delta - v_{D_t^\delta(S)+1}(t - \tau^+_{D_t^\delta+1})\right) \tag{19}$$

(ii) the worst loss during $[0,t]$ is given by

$$\max_{s\in[0,t]} \left(V_0(\phi^+) - V_s(\phi^+)\ \right) = \max_{k=0,\ldots,D_t^\delta(S)} \{ \max_{[0,(t-\tau^+_{k+1})+]} (v_{k+1} - (k+1)\delta)\}. \tag{20}$$

(iii) the drawdown of ϕ^+ is given by

$$\begin{aligned}\Delta(t) \quad &= \quad \max_{k=0,\ldots,D_t^\delta(S)} \{ \max_{[0,(t-\tau^+_{k+1})+]} ((k+1)\delta - v_{k+1})\} \\ &\quad -\delta \times D_t^\delta(S) - 1_{[\tau^+_{D_t^\delta+1}, \theta^+_{D_t^\delta+1}]}(t) \left(\delta - v_{D_t^\delta(S)+1}(t - \tau^+_{D_t^\delta+1})\right). \end{aligned} \tag{21}$$

Proof. By definition of the portfolio ϕ^+, we have

$$V_t(\phi^+) = V_0(\phi^+) - \sum_{i=1}^{D_t^\delta(S)+1} \int 1_{[\tau_i^+, \theta_i^+)} dS = \sum_{i=1}^{D_t^\delta(S)} (S_{\tau_i^+} - S_{\theta_i^+}) + (S_{t\wedge\tau^+_{D_t^\delta(S)+1}} - S_t)$$

thus $V_t(\phi^+) - V_0(\phi^+) = \delta \times D_t^\delta(S) + S_{t\wedge\tau^+_{D_t^\delta(S)+1}} - S_t$. By the definition of v_i this can be rewritten as

$$V_t(\phi^+) - V_0(\phi^+) = \delta\ D_t^\delta(S) + 1_{[\tau^+_{D_t^\delta+1}, \theta^+_{D_t^\delta+1}]} \left(\delta - v_{D_t^\delta(S)+1}(t - \tau^+_{D_t^\delta+1})\right),$$

which then implies (20). □

3.4 Irregular price paths

Intuitively, decreasing the value of the threshold δ increases the number of level crossings and leads to more transactions but with a lower profit per transaction. The exact behaviour of the trading strategy as $\delta \to 0$ is determined by the *local time* of the path at 0, which measures the time spent by the path in a neighbourhood of zero [18].

Let $S \in C^0([0,\infty), \mathbb{R})$ and $T > 0$. We define the occupation measure of S by

$$\gamma_T(A) := \int_0^T 1_A(S_t)\, dt, \qquad \forall A \in \mathcal{B}(\mathbb{R}). \tag{22}$$

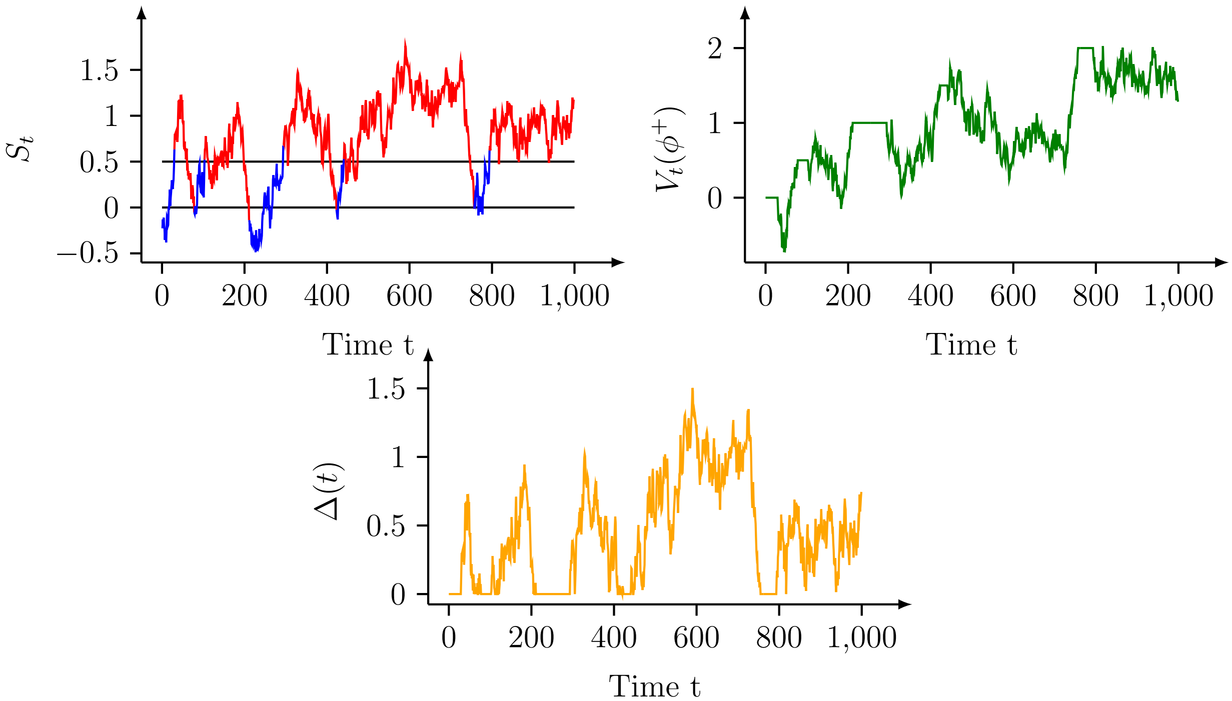


Figure 2: Decomposition of a path into δ-excursions with $\delta = 0.5$. Top Left: decomposition into u_i (blue) and v_i (red). Top Right: Value of the portfolio ϕ^+. Bottom: Drawdown $\Delta(t)$.

We will say that the path S admits a local time $l_T(S,x)$ if the measure γ_T is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, in which case we denote

$$l_T(S,x) := \frac{d\gamma_T}{dx} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^T 1_{[x-\varepsilon,x+\varepsilon]}(S_t)\, dt.$$

The local time is characterized by the *occupation time formula*:

$$\int_0^T h(S_t)\, dt = \int_{-\infty}^{+\infty} h(x)\, l_T(S,x)\, dx, \ \forall h \in C^0(\mathbb{R},\mathbb{R}).$$

Intuitively, the local time $l_T(S,x)$ represents the time S spends at level x during $[0,T]$. We will be interested in particular in the local time at 0, which we denote $\ell_T(S) = l_T(S,0)$.

The map $T \mapsto \ell_T(S)$ is increasing, which allows to define its right-continuous inverse, the *inverse local time* at zero:

$$\forall l > 0, \qquad \tau_l = \inf\{t > 0,\ \ell_t(S) > l\}. \tag{23}$$

We note that $l \mapsto \tau_l$ is an increasing càdlàg function of the variable l. The local time $\ell_t(S)$ increases on the set $\{t, S_t = 0\}$ and is constant along any excursion from 0, so the discontinuities of τ correspond to excursions of S, and jump intervals of τ correspond to the complement of the set where S visits 0:

$$\bigcup_{l>0} (\tau_{l-}, \tau_l) = \{t \geq 0,\ S_t \neq 0\}.$$

Thus the value l of local time along an excursion may be used as a natural index for labeling excursions of S: the excursion at local time level l is given by

$$e_l(t,S) = \begin{cases} S_{(\tau_{l-}+t)} \mathbf{1}_{(t \leq \tau_l - \tau_{l-})}, & \text{if} \quad \tau_l(\omega) - \tau_{l-}(\omega) > 0 \\ \dagger & \text{if} \quad \tau_l(\omega) = \tau_{l-}(\omega). \end{cases} \tag{24}$$

Points of continuity of τ_l, i.e. points at which $\tau_{l-} = \tau_l$ correspond to 'infinitesimal excursions' which may arise if the path has non-zero local time at 0; we associate such excursions with a 'cemetery' state $e_l(S) = \dagger$. This defines an excursion process $e\colon \mathbb{R}^+ \to \overline{\mathcal{E}_{0,0}} = \mathcal{E}_{0,0} \cup \{\dagger\}$. For a given set $\Gamma \subset \mathcal{E}_{0,0}$, we can define the counting process, which counts excursions of S from 0 which lie in Γ, up to local time l :

$$N_l(\Gamma) := \sum_{\lambda \le l} 1_\Gamma(e_\lambda). \tag{25}$$

Note that in general $N_l(\Gamma)$ can be infinite. We now establish an important connection between this *excursion point process* N and the decomposition into δ-excursions given by Proposition 3.4. Recall the set Γ_δ of excursions from 0 to 0 which reach a level δ:

$$\Gamma_\delta = \Big\{ f \in \mathcal{E}_{0,0} \colon\ \max(f) \ge \delta \Big\}.$$

Proposition 3.7. *Let $S \in C^0([0,\infty),\mathbb{R})$ be a path with $S_0 = 0$ which admits a local time $\ell_t(S)$ at zero, with inverse τ is given by (1). For $\delta > 0$ let $D_t^\delta(S)$ be the number of $\delta-$excursions of S on $[0,t]$, defined as in (17). Then*

$$\forall \delta > 0, \quad \forall t > 0, \qquad N_t(\Gamma_\delta) < +\infty, \quad \text{and}$$

$$\forall t > 0, \quad D_t^\delta(S) = N_{\ell_t(S)}(\Gamma_\delta) \quad \text{and} \quad \forall l > 0, \quad D_{\tau_l}^\delta(S) = N_l(\Gamma_\delta). \tag{26}$$

Proof. The condition $N_l(\Gamma_\delta) < +\infty$ is a consequence of the continuity of S. We will now establish a one-to-one correspondence between excursions $e_l \in \Gamma_\delta$ and intervals $(\theta_{i-1}^+, \theta_i^+)$. As in Lemma 3.2, define $i \ge 1$ the 'last exit' from zero in the i-th $\delta-$excursion:

$$\hat{\theta}_i^+ := \sup\{t < \tau_i^+ \colon S_t = 0\}.$$

To show that the two sets of intervals $\{(\hat{\theta}_i^+, \theta_i^+)\}_{i\ge 1}$ and $\{(\tau_{l-}, \tau_l)\}_{e_l \in \Gamma_\delta}$ coincide, we prove the following two claims:

- For each $i \ge 1$ there exist a unique $l_i \ge 0$, such that $(\hat{\theta}_i^+, \theta_i^+) = (\tau_{l_i-}, \tau_{l_i})$.

  Indeed, it is easy to see that on the interval $(\hat{\theta}_i^+, \theta_i^+)$, $S_t > 0$. Furthermore,

  $$\hat{e}_i(t) := S_{t+\hat{\theta}_i^+} 1_{[0,\,\theta_i^+ - \hat{\theta}_i^+]} \in \Gamma_\delta,$$

  since $\hat{e}_i(\tau_i^+ - \hat{\theta}_i^+) \ge \delta$. In particular $(\hat{\theta}_i^+, \theta_i^+)$ is an interval of $\{t > 0,\ S_t \ne 0\}$, thus there exists unique l_i such that $(\hat{\theta}_i^+, \theta_i^+) = (\tau_{l_i-}, \tau_{l_i})$ and $e_{l_i} \in \Gamma_\delta$.

- Conversely, for every $l \ge 0$ such that $e_l \in \Gamma_\delta$, there exists a unique index $i(l) \ge 1$ such that $(\tau_{l-}, \tau_l) = (\hat{\theta}_{i(l)}^+, \theta_{i(l)}^+)$.

  Take the largest $i = i(l) \ge 1$ such that $\theta_{i-1}^+ \le \tau_{l-}$ then $\tau_{l-} < \theta_i$. Since on $(\hat{\theta}_i^+, \theta_i^+)$, $S_t > 0$, while $S_{\tau_{l-}} = 0$, we get that $\hat{\theta}_i^+ \ge \tau_{l-}$. The condition $e_l \in \Gamma_\delta$ implies that S reaches the level δ in (τ_{l-}, τ_l), by definition $\tau_i^+ > \hat{\theta}_i^+$ is the first such time after θ_{i-1}, hence $\tau_i^+ \in (\tau_{l-}, \tau_l)$. Since the intervals (τ_{l-}, τ_l) and $(\hat{\theta}_i^+, \theta_i^+)$ intersect, we conclude from the first claim that $(\tau_{l-}, \tau_l) = (\hat{\theta}_{i(l)}^+, \theta_{i(l)}^+)$ (we also use the fact that the intervals $\{(\tau_{l-}, \tau_l)\}_{l\ge 0}$ are disjoint).

The correspondence between θ_i^+, $i \ge 1$ and τ_l, $e_l \in \Gamma_\delta$, yields the result:

$$D_t^\delta(S) = \sum_{i\ge 1} 1_{\theta_i^+ \le t} = \sum_{i \ge 1} 1_{\theta_{i(l)}^+ \le t} = \sum_{\tau_l \le t} 1_{\Gamma_\delta}(e_l) = \sum_{l \le l_t(S)} 1_{\Gamma_\delta}(e_l) = N_{l_t(S)}(\Gamma_\delta).$$

$\square$

Behaviour of level-crossings as $\delta \to 0$. The behaviour of the above quantities as $\delta \to 0$ is determined by the 'roughness' of the path. When δ is small, we account for the fact that trading takes places only at prices which are integer multiples of a 'tick', i.e. only at times when S takes such values.

Let $\delta_n = 2^{-n}$ and introduce the partition π_n defined by the hitting times of the grid $\delta_n\mathbb{Z}$:

$$t_0^n := 0, \quad t_{k+1}^n := \inf\left\{t \geq t_k^n \colon S_t \in \delta_n\mathbb{Z}\backslash\{S_{t_k^n}\}\right\}. \tag{27}$$

Then $\sup_{\pi_n} |S(t_{k+1}^n) - S(t_k^n)| \to 0$ as $n \to \infty$. We denote $\pi = (\pi_n)_{n\geq 1}$. Following [12], we will say that $S \in C^0([0,T],\mathbb{R})$ has p-th order variation along π if there exists $[S]_\pi^p \in C^0([0,T],\mathbb{R}_+)$ such that

$$\sum_{\pi_n} |S(t_{k+1}^n \wedge t) - S(t_k^n \wedge t)|^p \to [S]_\pi^p(t).$$

The smallest $p \geq 1$ for which $[S]_\pi^p \neq 0$ then gives an index of 'roughness' for S along π. For example for Brownian paths $p = 2$ while for fractional Brownian motion with Hurst exponent H, $p = 1/H$ [5].

For such a path, the number of down-crossings for levels close to zero is related to a slightly different notion of local time, defined in terms of a *weighted* occupation measure, weighted by the p-th order variation [12]:

Definition 3.8 (Local time of order p [12])**.** *Let $p \geq 1$ and $q \geq 1$. A continuous path $S \in C^0([0,T],\mathbb{R})$ has (L^q-)local time of order p along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ of $[0,T]$ if, for any $t \in [0,T]$, the sequence of functions*

$$L_t^{\pi_n,p}(S,.) : x \in \mathbb{R} \mapsto L_t^{\pi_n,p}(S,x) := \sum_{t_j^n \in \pi_n} \mathbf{1}_{\left[S_{t_j^n \wedge t}, S_{t_{j+1}^n \wedge t}\right)}(x) \left|S_{t_{j+1}^n \wedge t} - x\right|^{p-1}$$

converges in $L^q(\mathbb{R})$ to a limit $L_t^{\pi,p}(S,x) \in L^q(\mathbb{R})$ and the map $t \in [0,T] \mapsto L_t^{\pi,p}(S,x) \in L^q(\mathbb{R})$ is weakly continuous. We call $L^{\pi,p}(S,x)$ the local time of order p of S at level x.

$L_t^{\pi,p}(S,x)$ measures the rate at which the path S accumulates p-th order variation around level x. Note that the local time of order p is non-zero only if S has non-zero p-th order variation along π i.e. $[S]_\pi^p > 0$. If the convergence is uniform in $(t,x) \in [0,T] \times \mathbb{R}$, and the mapping $(x,t) \mapsto L_t^{\pi,p}(S,x)$ is continuous we call it the continuous local time of S [24].

Note that the in the case of $p = 2$ the definitions in [4] and [12, 24] differ by a factor of 2; here we use the latter notation. In the case $p = 2$ we will omit the index p in the notation; $L^\pi := L^{\pi,2}$. The relation between various notions of local time is discussed in [24].

Following the arguments in [12, Lemma 3.4], one can establish a relation between the down-crossings and up-crossings $D_t^{\delta_n}(S), U_t^{\delta_n}(S)$ of the interval $[0,\delta_n]$: for $x \in [0,\delta_n]$,

$$L_t^{\pi_n,p}(S,x) = D_t^{\delta_n}(S)|x|^{p-1} + U_t^{\delta_n}(S)|\delta_n - x|^{p-1} + O(\delta_n^{p-1}).$$

Since the numbers $D_t^{\delta_n}(S), U_t^{\delta_n}(S)$ can differ at most by one, we obtain that

$$L_t^{\pi_n}(S) = L_t^{\pi_n}(S,0) = D_t^{\delta_n}(S)\delta_n^{p-1} + O(\delta_n^{p-1}).$$

If S has a continuous local time $L^{\pi,p}(S,\cdot)$ along the sequence of Lebesgue partitions π, we conclude from above that

$$\lim_{n\to\infty} |\delta_n|^{p-1} D_t^{\delta_n}(S) = L_t^{\pi,p}(S).$$

The following proposition summarizes the behavior of the number of level crossings $D_t^\delta(S)$ (representing the number of trades) and the realized profit $\delta D_t^\delta(S)$ as δ decreases to zero:

Proposition 3.9. *Let $\delta_n = 2^{-n}$ and $p \geq 1$. Assume $S \in C^0([0,T], \mathbb{R})$ has a strictly positive local time $L_t^{\pi,p} > 0$ of order p at zero along the sequence of partitions $(\pi_n)_{n\geq 1}$ defined by (27). Then for any $t \in (0,T]$, as $\delta_n \to 0$,* $1/H$ H: hurst parameter (fractional brownian)

(i) if $1 \leq p < 2$ then $\quad \delta_n \, D_t^{\delta_n}(S) \to 0.$

(ii) if $p > 2$ then

$$\delta_n \, D_t^{\delta_n} \overset{\delta_n \to 0}{\to} \infty, \qquad \text{and} \quad D_t^{\delta_n}(S) \overset{\delta_n \to 0}{\sim} \frac{L_t^{\pi,p}(S)}{\delta_n^{p-1}}.$$

(iii) if $p = 2$ then

$$\delta_n \, D_t^{\delta_n} \overset{\delta_n \to 0}{\to} L_t^{\pi,2}(S), \qquad \text{i.e.} \quad D_t^{\delta_n}(S) \overset{\delta_n \to 0}{\sim} \frac{L_t^{\pi,2}(S)}{\delta_n}.$$

In particular, when $p > 2$ the threshold δ should be chosen as small as possible, while for $p \leq 2$ there is an optimal threshold $\delta^*(S) > 0$ which maximizes the realized profit $\delta D_T^\delta(S)$.

The assumptions of Proposition 3.9 are satisfied by typical sample paths of many classes of stochastic processes. Typical paths of semimartingales correspond to (iii), while paths of 'rough' processes such as Fractional Brownian motion with Hurst exponent $H < 1/2$ correspond to (ii):

Example 3.10 (Continuous semimartingales)**.** Let $S = M + A$ where M is a continuous martingale and A is a continuous process with bounded variation $\int_0^T |dA_t|$ on $[0,T]$. Denote by $[S] = [M]$ the quadratic variation process of S. Then S admits a local time of order $p = 2$, which corresponds 1/2 of the 'semimartingale local time' of S at 0:

$$L_t^{\pi,2}(S) = \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^T 1_{[-\varepsilon,\varepsilon]}(S_t) \, d[S]_t.$$

Furthermore if for some $q \geq 1$ we have

$$\mathbb{E}\left([M]_T^{q/2} + (\int_0^T |dA_t|)^q \right) < \infty,$$

then it was shown by El Karoui [15] that $t \mapsto \delta \, D_t^\delta$ is uniformly approximated in $\mathbb{L}^q$ by $L_t^{\pi,2}(S)$ as $\delta \to 0$:

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \delta \, D_t^\delta(S) - L_t^{\pi,2}(S) \right|^q \right) \quad \overset{\delta \to 0}{\to} \quad 0.$$

Example 3.11 (Fractional Brownian motion)**.** Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Kim [24] has shown that B^H almost surely has a continuous local time $L^{\pi,1/H}(B^H, \cdot)$ of order $p = 1/H$ along the sequence of partitions π defined in (27), and

$$L_t^{\pi,1/H}(B^H, x) = \ell_t(B^H, x) \mathbb{E}\left[|B_1^H|^{\frac{1}{H}} \right],$$

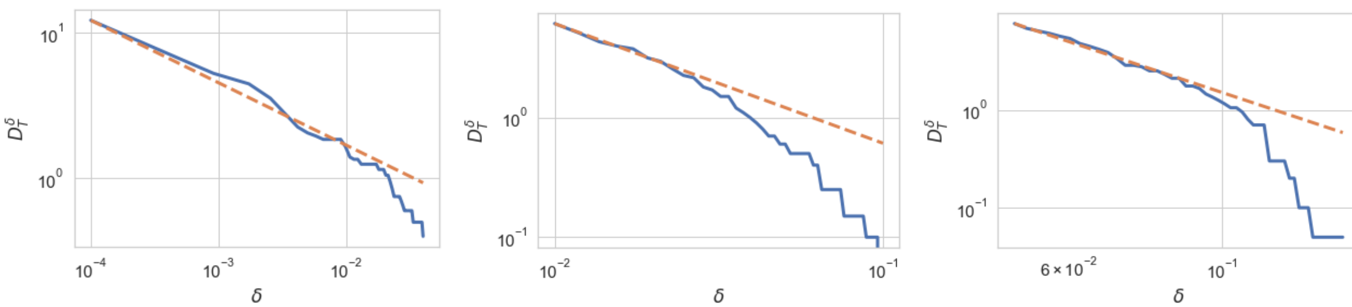
where $\ell_t(B^H, \cdot)$ is the occupation time density of B^H. Denoting $\ell_t(B^H) = \ell_t(B^H, 0)$, this implies

$$\lim_{n \to \infty} |\delta_n|^{\frac{1-H}{H}} D_t^{\delta_n}(B^H) = \ell_t(B^H) \underbrace{\mathbb{E}\left[|B_1^H|^{\frac{1}{H}} \right]}_{c_H}, \quad \text{so} \quad D_t^{\delta_n}(B^H) \overset{\delta_n \to 0}{\sim} \frac{c_H \; \ell_t(B^H)}{|\delta_n|^{\frac{1-H}{H}}}.$$

We illustrate the relevance of Proposition 3.9 using a fractional Ornstein-Uhlenbeck process [9]:

$$dS_t = -\lambda S_t dt + \gamma dB_t^H, \tag{28}$$

where B^H is a fractional Brownian motion with Hurst exponent H. Figure 3 shows, as a function of the threshold δ, the number of $\delta-$excursions estimated from values of S_t on a discrete grid of $N = 28,800$ points (which corresponds to the number of seconds in one trading day). The empirical estimator closely follows the asymptotics described in Proposition 3.9, suggesting that this asymptotic regime is indeed a relevant description of excursions at such frequencies.



(a) $H = 0.7$. (b) $H = 0.5$. (c) $H = 0.3$.

Figure 3: Behavior of D_T^δ when $\delta \to 0$ for a fractional Ornstein-Uhlenbeck process (28) with $\lambda = 5$ and $\gamma = 0.1$. Dotted line: asymptotic behavior $\delta^{\frac{H-1}{H}}$ described in Proposition 3.9.

4 Markovian trading signals

We now consider the case where the trading signal S is described by a (one-dimensional) diffusion process, a situation often encountered in mathematical finance. Excursions of Markov processes were studied in detail by Ito [21], who developed a beautiful description in terms of an infinite-dimensional Poisson point process [6, 21, 36, 42]. We first recall some key results from Ito's theory, then show how they may be used to derive probabilistic properties of $\delta-$excursions.

4.1 Ito's point process of excursions

Let us briefly recall some results from K. Ito's excursion theory for Markov processes [21]. An excellent survey is given by Rogers [36]. For proofs and more detailed accounts we refer to [6, 21] or [37, Ch. XII].

Let $(S_t, t \geq 0)$ be a standard Markov process with continuous paths, whose state space is an interval $I \subset \mathbb{R}$ which includes zero. Then S has a *local time* $\ell_t(S)$ at zero [22], defined as

$$\ell_t(S) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[-\varepsilon,\varepsilon]}(S_u) du.$$

We denote $\mathbb{P}^{(t,x)}$ the law of $(S_u, u \geq t)$ given $S_t = x$, $\mathbb{P}^a = \mathbb{P}^{(0,a)}$ and $\mathbb{P}_0^a$ the law of the process starting from $S_0 = a$ and stopped at $T^0(S)$.

We shall make the following assumption on S:

Assumption 4.1. $[0, \delta] \subset I$ *and*

$$\mathbb{P}^0\left(T^\delta(S) < \infty\right) = 1, \qquad \mathbb{P}^\delta\left(T^0(S) < \infty\right) = 1. \tag{29}$$

These assumptions are satisfied for a wide range of one-dimensional diffusion processes [22].

The process $t \mapsto \ell_t(S)$ is increasing, and increases on $S^{-1}(\{0\}) = \{t \geq 0, \quad S_t = 0\}$. We may define its right-continuous inverse τ, the *inverse local time* at zero:

$$\forall \lambda > 0, \qquad \tau_\lambda = \inf\{t > 0,\ \ell_t(S) > \lambda\}. \tag{30}$$

Then τ is an increasing process whose discontinuities correspond to the excursions of S away from 0:

$$S^{-1}(\{0\}) = \{\tau_l, l \geq 0\} \cup \{\tau_{l-}, \tau_l - \tau_{l-} > 0\}.$$

During an excursion from 0, the local time is constant and τ undergoes a discontinuity whose size $\tau_\lambda - \tau_{\lambda-}$ corresponds to the duration of the excursion. There is thus a one-to-one correspondence between excursions of S from 0 and the discontinuities of τ. Ito [21] defined the process of *excursions of S from* 0, indexed by the local time at 0, as follows:

Definition 4.2 (Excursion process). *The excursion process of S is an $\mathcal{E}-$valued process $(e_\lambda, \lambda > 0)$ defined as*

$$e_\lambda(t,\omega) = \begin{cases} 1_{t \leq \tau_\lambda(\omega) - \tau_{\lambda-}(\omega)} S_{\tau_{\lambda-}(\omega)+t}(\omega) & \textit{if} \quad \tau_\lambda(\omega) - \tau_{\lambda-}(\omega) > 0 \\ \dagger & \textit{if} \quad \tau_\lambda(\omega) = \tau_{\lambda-}(\omega). \end{cases}$$

Here e_λ designates the excursion along which the local time takes the value λ. Note that $T^0(e_\lambda) = \tau_\lambda - \tau_{\lambda-}$.

The following important result, due to Ito [21], characterizes the excursion process of a Markov process as as Poisson point process with values in $\mathcal{E}_{0,0}$:

Theorem 4.3 (Ito's excursion process [21]). *$(e_\lambda, \lambda > 0)$ is an $\mathcal{F}_{\tau_\lambda}-$ Poisson point process: there exists a σ-finite measure ν on $\mathcal{E}$ such that, for any $\Gamma \subset \mathcal{E}$ with $0 < \nu(\Gamma) < \infty$,*

$$N_\lambda(\Gamma) = \sum_{l \leq \lambda} 1_\Gamma(e_l) \tag{31}$$

is a Poisson process with $\mathcal{F}_{\tau_\lambda}-$intensity $\nu(\Gamma)$. ν is called the Ito excursion measure *of S.*

For a proof and detailed discussion we refer to [6, 21, 37]. We have the following representation

$$\tau_l(\omega) = \sum_{\lambda \leq l} T^0(e_\lambda(\omega)), \qquad \tau_{l-}(\omega) = \sum_{\lambda < l} T^0(e_\lambda(\omega)), \text{ and } S_t(\omega) = \sum_{\lambda \leq \ell_t(S)} e_\lambda(t - \tau_{\lambda-}, \omega).$$

The counting process $N_\lambda(\Gamma)$ represents the number of excursions of 'type Γ', up to local time λ. Note that the condition $\nu(\Gamma) < \infty$ is essential: in general ν is not a finite measure, due to the existence of infinitely many 'infinitesimal' excursions. The fact that $(e_\lambda, \lambda > 0)$ is a Poisson point process has many interesting implications, which we may exploit to derive a probabilistic description of $\delta-$excursions.

To explore the connection with δ-excursions, we will be interested in the case where $\Gamma = \Gamma_\delta$ is the set of excursions which reach $\delta > 0$, defined in (13):

Proposition 4.4. *Under Assumption 4.1, we have $0 < \nu(\Gamma_\delta) < \infty$ and*

$$\mathbb{P}^0\left(\forall t \geq 0, D^\delta_t(S) = N_{\ell_t(S)}(\Gamma_\delta)\right) = 1, \qquad \mathbb{P}^0\left(\forall l > 0, \quad N_l(\Gamma_\delta) = D^\delta_{\tau_l}(S)\right) = 1. \tag{32}$$

Furthermore:

(i) $(D^\delta_{\tau_l}(S), l \geq 0)$ *is a Poisson process with* $(\mathcal{F}_{\tau_l})-$*intensity* $\nu(\Gamma_\delta)$ *under* $\mathbb{P}^0$.

(ii) $(D^\delta_t(S), t \geq 0)$ *is a renewal process under* $\mathbb{P}^0$*: the durations of* δ*-excursions are independent and identically distributed.*

Proof. Under Assumption 4.1 we can apply Proposition 3.7 to obtain

$$\forall t \geq 0, D^\delta_t(S(\omega)) = N_{\ell_t(S(\omega),a)}(\Gamma_\delta), \text{ and } \forall l > 0, N_l(\Gamma_\delta) = D^\delta_{\tau_l}(S(\omega)), \text{ for } \mathbb{P}^0 - a.s.$$

(i) then follows from Theorem 4.3 and $\mathbb{P}^0(\forall l > 0, D^\delta_{\tau_l}(S) = N_l(\Gamma_\delta)\,) = 1$. For (ii) we note that is also self-evident that the intervals $\theta^+_{i+1} - \theta^+_i$ are independent and identically distributed. For that note first that θ^+_i are stopping times, by strong Markov property of the law of process S on $t > \theta^+_i$ depends on the past only through the information at time θ_i. Since also $S_{\theta^+_i} = 0$ we conclude from the strong Markov property of S that $\theta^+_{i+1} - \theta^+_i$ is independent of $\theta^+_{j+1} - \theta^+_j$, $j < i$ and is distributed as $\theta^+_1 - \theta^+_0$. □

As stated in Proposition 4.4, the counting process D^δ_t is a renewal process i.e. the durations of $\delta-$excursions are independent and identically distributed. The next result relates the distribution of these durations -through its Laplace transform- to the Ito excursion measure ν:

Proposition 4.5. *Under Assumption 4.1, the duration of* δ*-excursions of* S *are independent random variables with moment generating function given by*

$$\mathbb{E}^0[e^{-\lambda\theta^+_1}] = \frac{\int_{\Gamma_\delta} \nu(df) e^{-\lambda T^0(f)}}{\int_{\Gamma_\delta} \nu(df) e^{-\lambda T^0(f)} + \int_{\mathcal{E}_0} \nu(df)(1 - e^{-\lambda T^0(f)})}. \tag{33}$$

Proof. We use the method of marked excursions described by Rogers & Williams [37, 36]. Let X^λ be a Poisson process with intensity λ, independent from S, with jump times $0 < t_1 < t_2 < \ldots$ where $t_{i+1} - t_i$, $i \geq 0$ (where $t_0 = 0$) are independent $\text{Exp}(\lambda)$ random variables. The discussion below follows the approach described in [37, VI.49]. Let

$$J := \{g \in D([0, \infty), \mathbb{N}), \quad \Delta_t g \in \{0, 1\}\}$$

be the space of càdlàg step functions with unit step size and m_λ be the law of the process X^λ. We consider the process (S_t, X_t), $\quad t \geq 0$ with values in $\mathbb{R} \times \mathbb{Z}^+$ and law $\tilde{\mathbb{P}}^0 := \mathbb{P}^0 \times m_\lambda$, which can be identified with the coordinate process on $\tilde{\Omega} := \mathcal{E} \times J$. Let e_l be the excursion of the process S at local time l defined as in Definition 4.2. In addition, we define the increment function of the process X^λ at local time l:

$$\eta_l(t, \omega) = \begin{cases} X^\lambda_{(\tau_{l-}+t)\wedge\tau_l} - X^\lambda_{\tau_{l-}} & \text{if} \quad \tau_l - \tau_{l-} > 0 \\ \dagger & \text{if} \quad \tau_l = \tau_{l-}. \end{cases}$$

The map $l \mapsto (e_l, \eta_l)$, $\quad l \geq 0$, defines a point process of *marked excursions* $\tilde{N}$ on $[0, \infty) \times \tilde{\mathcal{E}}_0$ where

$$\tilde{\mathcal{E}}_0 := \{(f, g) \in \mathcal{E}_0 \times J \colon g(t) = g(T^0(f)) \quad \forall t \geq T^0(f)\},$$

defined by

$$\tilde{N}_l(\Gamma \times B) := \sum_{\lambda \leq l} 1_{\Gamma \times B}(e_\lambda, \eta_\lambda), \qquad \text{for} \quad \Gamma \times B \subset \tilde{\mathcal{E}}_0.$$

Then by [37, Theorem VI.49.2], $\tilde{N}_l$ is a Poisson point process with intensity measure

$$\tilde{\nu}(\Gamma \times B) := \int_\Gamma \nu(df) m_\lambda \left(\{g \in J : g(\cdot \wedge T^0(f)) \in B\}\right). \tag{34}$$

We define the space of starred excursions as the set of marked excursions which contain a jump of the process X^λ during its lifetime $[0, T^0(f)]$:

$$\tilde{\mathcal{E}}_0^* := \{(f,g) \in \tilde{\mathcal{E}}_0 \colon g(\infty) > 0\}.$$

Then by formula (34) the counting measure for starred excursions is a Poisson process with intensity

$$\tilde{\nu}(\tilde{\mathcal{E}}_0^*) = \int_{\mathcal{E}_0} \nu(df)(1 - e^{-\lambda T^0(f)}).$$

Let $\tilde{\Gamma}_\delta$ be the set of marked excursions $(f,g) \in \tilde{\mathcal{E}}_0$, with $f \in \Gamma_\delta$, and let $\tilde{\Gamma}_\delta^* = \tilde{\Gamma}_\delta \cap \tilde{\mathcal{E}}_0^*$ be its subset of starred excursions, then

$$\tilde{\nu}(\tilde{\Gamma}_\delta) = \tilde{\nu}(\Gamma_\delta \times J) = \nu(\Gamma_\delta)$$

and

$$\tilde{\nu}(\tilde{\Gamma}_\delta^*) = \int_{\Gamma^\delta} \nu(df) m_\lambda \left(\{g \in J : g(T^0(f)) > 0\}\right) = \int_{\Gamma_\delta} \nu(df)(1 - e^{-\lambda T^0(f)}).$$

We are now ready to prove (33). First, note that

$$\tilde{\mathbb{P}}^0 \left(\theta_1^+ < t_1\right) = \tilde{\mathbb{P}}^0 \begin{pmatrix} \text{First excursion in } \tilde{\Gamma}_\delta \text{ happens} \\ \text{before the first starred excursion} \end{pmatrix}. \tag{35}$$

By independence of t_1 from S, the LHS is exactly the moment generating function of θ_1^+:

$$\tilde{\mathbb{P}}^0 \left(\theta_1^+ < t_1\right) = \mathbb{E}^{\mathbb{P}^0}[e^{-\lambda\theta_1^+}].$$

Meanwhile, by the property of the marked excursion point process

$$\tilde{\mathbb{P}}^0 \begin{pmatrix} \text{the first excursion in } \Gamma_\delta \text{ happens} \\ \text{before the first 'starred' excursion} \end{pmatrix} = \frac{\tilde{\nu}(\tilde{\Gamma}_\delta \setminus \tilde{\mathcal{E}}_0^*)}{\tilde{\nu}(\tilde{\Gamma}_\delta \cup \tilde{\mathcal{E}}_0^*)} = \frac{\tilde{\nu}(\tilde{\Gamma}_\delta \setminus \tilde{\mathcal{E}}_0^*)}{\tilde{\nu}(\tilde{\Gamma}_\delta \setminus \tilde{\mathcal{E}}_0^*) + \tilde{\nu}(\tilde{\mathcal{E}}_0^*)}.$$

We have

$$\tilde{\nu}(\tilde{\Gamma}_\delta \setminus \tilde{\mathcal{E}}_0^*) = \tilde{\nu}(\tilde{\Gamma}_\delta) - \tilde{\nu}(\tilde{\Gamma}_\delta^*) = \nu(\Gamma_\delta) - \int_{\Gamma_\delta} \nu(df)(1 - e^{-\lambda T^0(f)}) = \int_{\Gamma_\delta} \nu(df) e^{-\lambda T^0(f)},$$

thus the RHS of (35) is

$$\frac{\int_{\Gamma_\delta} \nu(df) e^{-\lambda T^0(f)}}{\int_{\Gamma_\delta} \nu(df) e^{-\lambda T^0(f)} + \int_{\mathcal{E}_0} \nu(df)(1 - e^{-\lambda T^0(f)})},$$

hence the result. □

Various analytical descriptions have been proposed for Ito's excursion measure ν [6, 45]; one which is particularly relevant for the applications considered here is obtained by 'slicing' the space of excursions according to their maximum height M:

$$\nu(df) = \int_0^\infty F_M(dm)\, \mathbb{Q}_m(df)$$

where F_M is the distribution of the maximum height $M = \max(|f|)$ of an excursion $f \in \mathcal{E}_{0,0}$ and $\mathbb{Q}_m$ is the law of the excursion conditional on the maximum $M = m$. The distribution of many quantities of interest described in Section 3 only involve F_M. In the case where S is Brownian motion, this is known as the 'Williams decomposition' [43] and has been studied in great detail [32, 35, 43, 45].

Example 4.6 (δ-excursions of Brownian motion)**.** Consider the case when S is a Brownian motion and let ν_+ is the restriction of ν to the set of positive excursions. By Williams' characterization of the Ito excursion measure [43],

$$F_M(dm) = \nu_+(\max(f) \in dm) = \frac{dm}{2m^2}$$

and

$$\int_{\{f\in\mathcal{E}_{0,0},\max(f)=m\}} \nu_+(df)e^{-\lambda T^0(f)} = \left(\frac{\sqrt{2\lambda}m}{\sinh(\sqrt{2\lambda}m)}\right)^2.$$

From this we obtain

$$\int_{\Gamma_\delta} \nu(df)e^{-\lambda T^0(f)} = \int_\delta^\infty \left(\frac{\sqrt{2\lambda}m}{\sinh(\sqrt{2\lambda}m)}\right)^2 \frac{dm}{2m^2} = \frac{\sqrt{2\lambda}}{e^{2\sqrt{2\lambda}\delta}-1}.$$

Similarly, we can compute

$$\int_{\mathcal{E}_0} \nu(df)(1-e^{-\lambda T^0(f)}) = \sqrt{2\lambda}.$$

Substituting in formula (33), we obtain

$$\mathbb{E}^0[e^{-\lambda\theta_1^+}] = e^{-2\sqrt{2\lambda}\delta},$$

which corresponds to the Laplace transform of a Lévy distribution i.e. a one-side $1/2-$stable distribution [28]. In particular, the duration $(\theta_{k+1}^+ - \theta_k^+)$ of $\delta-$excursions has infinite mean.

4.2 Decomposition of a Markov process into δ-excursions

One of the difficulties with Ito's representation is the existence of infinitely many 'infinitesimal' excursions, which translate into the existence of a local time at zero and the infinite mass of Ito's excursion measure $\nu(.)$.

Using Assumption 4.1, together with the property $\nu(\Gamma_\delta) < \infty$, we derive a simpler *finite* decomposition of the process in terms of δ-excursions, which is more amenable to applications:

Theorem 4.7. *Let S be Markov process satisfying Assumption 4.1. There exists an IID sequence $(e_k)_{k\geq 1}$ of δ-excursions $e_k \in \mathcal{U}_\delta$ such that*

$$\mathbb{P}^0\left(\forall t\geq 0, \quad S_t = \sum_{k\geq 1} e_k\left((t-\theta_{k-1}^+)\right)\right) = 1 \tag{36}$$

where $\theta_0^+ = 0$ and $\theta_k^+ = \sum_{i\leq k}\Lambda(e_i)$. The distribution Π_δ of e_k is a probability measure concentrated on $\mathcal{U}_\delta$, given by the law of the first δ-excursion of S, i.e. the law of $S(.\wedge\theta_1^+)$ under $\mathbb{P}^0$:

$$\forall H \in C_b^0(\mathcal{E}), \quad \int_{\mathcal{E}} \Pi_\delta(df)H(f) = \mathbb{E}^0\left[\quad H\left(S(.\wedge\theta_1^+)\right)\quad\right]. \tag{37}$$

Proof. The existence and uniqueness of the decomposition is a consequence of Proposition 3.4 and the measurability of the decomposition (17) applied to the sample paths of S. As θ_k^+ are stopping times, the strong Markov property of S implies that $(e_k)_{k\geq 1}$ are independently and identically distributed. We now describe the law Π_δ of e_k in terms of the Ito excursion measure ν given in Theorem 4.3. By Proposition 4.4, $\nu(\Gamma_\delta) < \infty$. In Lemma 3.2, we constructed a measurable map

$$f \in \mathcal{U}_\delta \mapsto (T, g, \gamma_f) \in [0,\infty)\times\mathcal{E}\times\Gamma_\delta \quad \text{such that} \quad f = g\bigoplus_T \gamma_f \quad \text{and} \quad g(0) = g(T) = 0.$$

This map defines a projection of $\mathcal{U}_\delta$ onto Γ_δ, which projects any measurable set $A \subset \mathcal{U}_\delta$, onto the set $\{\gamma_f, f \in A\} \subset \Gamma_\delta$ of their last excursions. This projection simply consists in mapping each path in $\mathcal{U}_\delta$ to its last excursion, which by Lemma 3.2 is always an element of Γ_δ.

The distribution Π_δ of $\delta-$excursions is related to the Ito excursion measure $\nu(.)$ through this projection: for a measurable subset $A \subset \mathcal{U}_\delta$,

$$\Pi_\delta(A) = \frac{\nu\left(\{\gamma_f, f \in A\}\right)}{\nu(\Gamma_\delta)}, \tag{38}$$

where γ_f denotes the last excursion of f. It is then readily verified that Π_δ is a probability measure on $\mathcal{U}_\delta$, satisfying (37). □

This result may also be used as a device for simulating sample paths of the process. Rather than simulating, as is often done, from transition probabilities of the process on a fixed time grid, if we are able to sample from Π_δ then we may simulate S by sampling an IID sequence $e_k \sim \Pi_\delta$ and constructing paths of S by concatenation using (36). This leads to a more efficient computation of path-dependent quantities which have simple expressions in terms of excursions.

4.3 Distributional properties of mean-reversion strategies

Consider now the case where S is a diffusion process whose state space is some interval $I \in \mathbb{R}$ containing $[0, \delta]$, with infinitesimal generator

$$\mathcal{G} := \frac{a(x)}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad a(.) > 0, \quad b \in L^1_{loc}$$

acting on C^2 functions subject to appropriate boundary conditions. $\mathcal{G}$ can be written as

$$\mathcal{G} = \frac{1}{m(x)}\frac{d}{dx}\frac{1}{s'(x)}\frac{d}{dx}, \qquad s'(x) = \exp\left(-\int_0^x \frac{2b(y)}{a(y)}dy\right), \quad m(x) = \frac{2}{s'(x)a(x)}.$$

where $m(x)$ is the speed measure of X and $s'(x)$ is the derivative of the scale function $s(x)$ [22]. The scale function s is defined up to an additive constant and we will use the normalization $s(0) = 0$ in the sequel. For Brownian motion, we have $m(x) = 2$ and $s(x) = x$. The Laplace transform of the hitting time distribution may then be expressed in terms of m and s [7, 22]:

$$\forall (x, y) \in I^2, \forall \lambda \geq 0, \quad \mathbb{E}^x\left[e^{-\lambda T^y(S)}\right] = \begin{cases} \Phi_{\lambda,-}(x)/\Phi_{\lambda,-}(y), & \text{if } x < y, \\ \Phi_{\lambda,+}(x)/\Phi_{\lambda,+}(y), & \text{if } x > y, \end{cases}$$

where $\Phi_{\lambda,-}$ (resp. $\Phi_{\lambda,+}$) is the unique increasing (resp. decreasing), up to multiplicative constant factors, non-negative solution of the differential equation

$$\mathcal{G}\Phi = \lambda\Phi.$$

Using these results, we can now derive various quantities of interest related to the profit and loss of the strategy ϕ^+ defined in (4).

Recall that $(\tau_k^+ - \theta_{k-1}^+)$, $\theta_k^+ - \tau_k^+$ and $(\theta_k^+ - \theta_{k-1}^+)$ denote, respectively, the duration of the IID waiting period, duration of the IID holding period and the lifetime of the IID δ-excursion of S. Let $F_\theta(t) = \mathbb{P}^0(\theta_k^- \theta_{k-1}^+ \leq t) = \mathbb{P}^\delta(T^0(S) \leq t)$ be the distribution of the duration of holding period.

- **Distribution of holding and waiting periods:** The distributions of $(\tau_k^+ - \theta_{k-1}^+)$, $\theta_k^+ - \tau_k^+$ and $(\theta_k^+ - \theta_{k-1}^+)$ may be characterized through their Laplace transforms

$$\mathcal{L}_\tau(\lambda) := \mathbb{E}^0[e^{-\lambda(\tau_k^+ - \theta_{k-1}^+)}] = \frac{\Phi_{\lambda,-}(0)}{\Phi_{\lambda,-}(\delta)}, \qquad \mathbb{E}^\delta[e^{-\lambda(\theta_k^+ - \tau_k^+)}] = \mathbb{E}^\delta[e^{-\lambda T^0(S)}] = \frac{\Phi_{\lambda,+}(\delta)}{\Phi_{\lambda,+}(0)}. \tag{39}$$

By the strong Markov property, these two variables are independent so

$$\mathcal{L}_\theta(\lambda) := \int_0^\infty e^{-\lambda s} F_\theta(ds) = \mathbb{E}^0[e^{-\lambda(\theta_k^+ - \theta_{k-1}^+)}] = \frac{\Phi_{\lambda,-}(0)}{\Phi_{\lambda,-}(\delta)} \times \frac{\Phi_{\lambda,+}(\delta)}{\Phi_{\lambda,+}(0)}. \tag{40}$$

- **Expected number of trades**: Recall that $D_t^\delta(S)$ is the number of transactions on $[0,t]$. The expected number $n_\delta(t) = \mathbb{E}^0[D_t^\delta(S)]$ of transactions up to t is related to F_θ via the *renewal equation* [11]:

$$n_\delta(t) = F_\theta(t) + \int_0^t n_\delta(t-s) F_\theta(ds), \quad \text{i.e.} \quad n_\delta = F_\theta + n_\delta \star F_\theta. \tag{41}$$

From the renewal equation for the Laplace transform of n_δ, we obtain the Laplace transform of n_δ using (33):

$$\mathcal{L}[n_\delta](\lambda) := \int_0^\infty e^{-\lambda t} n_\delta(t) dt = \frac{\mathcal{L}_\theta(\lambda)}{1 - \mathcal{L}_\theta(\lambda)}$$

- **Maximum loss per trade cycle and stop-loss probabilities:** As a consequence of the Markov property, the maximum loss during the holding period $[\tau_k^+, \theta_k^+]$ is independent and identically distributed across trading periods $k = 1, 2, ..$ with distribution given by

$$\mathbb{P}\left(\max_{t \in [\tau_k^+, \theta_k^+]} \left(V_{\tau_k^+}(\phi^+) - V_t(\phi^+)\right) \geq x\right) = \mathbb{P}^\delta(T^{x+\delta}(S) < T^0(S)) = \frac{s(\delta)}{s(x+\delta)}. \tag{42}$$

This quantity also represents the probability of hitting an exposure limit M and may be used for setting such a limit. For example an exposure limit M_q chosen such that the stop-loss is triggered with probability $0 < q < 1$ is given by

$$M_q = s^{-1}\left(\frac{s(\delta)}{q}\right) - \delta$$

- **Expected maximum loss per trade cycle:** The expected maximum loss for the the strategy (6) with stop-loss at level M is given by

$$\mathbb{E}^0\left[\max_{t \in [\tau_k^+, \kappa_k^+]} \left(V_{\tau_k^+}(\phi_M^+) - V_t(\phi_M^+)\right)\right] = \mathbb{E}^\delta\left[\max_{t \in [0, T^0(S) \wedge T^{M+\delta}(S)]} (S_t - \delta)\right] = \int_0^M \frac{s(\delta)\, dx}{s(\delta + x)}.$$

- **Distribution of realized profit**: Let π_k^M be the realized profit of the strategy ϕ_M^+ with stop-loss level M over a trading cycle $[\theta_{k-1}^+, \theta_k^+]$. Then $(\pi_k^M, k \geq 1)$ are IID Bernoulli variables with

$$\mathbb{P}(\pi_k^M = -M) = \frac{s(\delta)}{s(M+\delta)} = 1 - \mathbb{P}(\pi_k^M = \delta).$$

This leads to a binomial profit/loss distribution after n trade cycles. For a fixed horizon t, the expected realized profit is given by

$$\mathbb{E}^0\left[\sum_{k=1}^{D_t^\delta(S)} \pi_k^M\right] = n_\delta(t) \left(\delta - (M+\delta)\frac{s(\delta)}{s(M+\delta)}\right), \tag{43}$$

where n_δ is given by (41).

To compute the Laplace transform of the distribution of the portfolio value $V_t(\phi^+)$, let us introduce

$$U_1(t,\lambda,\delta) := \mathbb{E}^0\left[e^{-\lambda\left(S(t\wedge\tau_1^+)-S(t\wedge\theta_1^+)\right)}\right] = \int_{\mathcal{E}} \Pi_\delta(df) e^{-\lambda\,(f(t)-f(t\wedge T^0(f)))},$$

$$U_2(t,\lambda,\delta) := \mathbb{E}^\delta\left[e^{\lambda\, S(t\wedge T^0(S))}\right], \qquad \tilde{U}_2(z,\lambda,\delta) = \int_0^\infty \left[e^{-z\,t} U_2(t,\lambda,\delta)dt.\right]$$

U_2 is the solution of the following boundary value problem:

$$\begin{cases} \frac{\partial}{\partial t} U_2(t,\lambda,x) = \mathcal{G}_x U_2(t,\lambda,x),\ x\in\mathbb{R}\setminus\{0\}, \\ U_2(0,\lambda,x) = e^{\lambda x},\ \forall x\in\mathbb{R}; \qquad U_2(t,\lambda,0)=1. \end{cases} \tag{44}$$

Proposition 4.8. *Denote by $H^\delta(t,\lambda)$ the Laplace transform of the distribution of $V_t(\phi^+)$:*

$$H^\delta(t,\lambda) := \mathbb{E}^0\left[e^{-\lambda\, V_t(\phi^+)}\right].$$

Then under Assumption 4.1, we have

$$\begin{aligned} H^\delta(t,\lambda) &= e^{-\lambda\delta}\left(H^\delta(\cdot,\lambda) * F_\theta\right)(t) \;+\; U_1(t,\lambda,\delta) - e^{-\lambda\delta} \\ &= e^{-\lambda\delta}\left(H^\delta(\cdot,\lambda) \star F_\theta\right)(t) \;+\; e^{-\lambda\delta}\left(U_2(\cdot,\lambda,\delta)\star F_\tau\right)(t) - e^{-\lambda\delta}. \end{aligned}$$

In particular the Laplace transform of H in the time variable is given by:

$$\mathcal{L}\left[H^\delta(\cdot,\lambda)\right](z) = \int_0^\infty H^\delta(t,\lambda)e^{-zt}dt = \frac{\tilde{U}_2(z,\lambda,\delta)\,\mathcal{L}_\tau(z) - 1/z}{e^{\lambda\delta} - \mathcal{L}_\theta(z)}. \tag{45}$$

For a proof see Appendix A.1.

5 Analytical results for linear diffusions

In this section we provide some explicit calculations for two classical models: Brownian motion and the Ornstein-Uhlenbeck process.

5.1 Brownian δ-excursions

Brownian excursions have been extensively studied starting with Lévy [29], and a great deal of analytical results are available, see e.g. [32] or [34, Ch. XII].

The case where the signal S follows Brownian motion corresponds to *no* mean reversion i.e. a case of model mis-specification for mean-reversion strategies based on the signal S. As the following proposition shows, the strategy ϕ^+ has a heavy-tailed loss distribution and infinite expected holding periods before realized profits materialize:

Proposition 5.1 (δ-excursions of Brownian motion.). *Let $S=\sigma B$ where B is a standard Brownian motion. Then*

(i) The duration $\theta_{k+1}^+ - \theta_k^+$ of δ-excursions are IID variables with

$$\theta_{k+1}^+ - \theta_k^+ \stackrel{\text{(law)}}{=} \frac{\delta^2}{\sigma^2}\left(\frac{1}{Z^2}+\frac{1}{Z'^2}\right),$$

where Z, Z' are independent standard Gaussian variables.

(ii) The distribution of the waiting time $\tau_k^+ - \theta_{k-1}^+$ is given by

$$\mathbb{P}(\tau_k^+ - \theta_{k-1}^+ > t) = \mathbb{P}^0(T^\delta(S) > t) = 1 - 2\ \Phi\left(\frac{\delta}{\sigma\sqrt{t}}\right)$$

where Φ is the standard normal distribution.

(iii) The worst loss during each holding period $[\theta_k^+, \tau_k^+]$ has a Pareto distribution with tail index 1:

$$\mathbb{P}^0\left(\max_{t\in[\tau_k^+,\theta_k^+]} V_{\tau_k^+}(\phi^+) - V_t(\phi^+) \geq M\right) = \frac{\delta}{M+\delta}.$$

In particular, the expected maximum loss for a strategy with exposure limit M is $\delta\log(1+M/\delta)$, which goes to infinity as the exposure limit is removed ($M\to\infty$).

(iv) The distribution of the portfolio value $V_t(\phi^+)$ satisfies

$$\int_0^\infty \mathbb{E}^0\left[e^{-\lambda V_t(\phi^+)}\right] e^{-zt}dt = \frac{\sigma^2\lambda^2 e^{\frac{-2\sqrt{2z}}{\sigma}\delta} - 2ze^{\lambda\delta}\, e^{\frac{-\sqrt{2z}}{\sigma}\delta} - (\sigma^2\lambda^2 - 2z)}{z\,(\sigma^2\lambda^2 - 2z)\left(e^{\lambda\delta} - e^{\frac{-2\sqrt{2z}}{\sigma}\delta}\right)},\ z > \frac{1}{2}\sigma^2\lambda^2.$$

The proof is given in Appendix A.2. These results provide insights for the assessment of *model risk* in mean-reversion strategies: it shows that when mean-reversion is absent, strategies based on this assumption lead to heavy-tailed loss distributions, even if underlying asset returns are not heavy tailed. This example also shows that the distribution of returns for the underlying asset (in this case: Gaussian) does not provide the right benchmark for understanding the risk of dynamic trading strategies, which relate to the height and duration of the excursions (in this case, both heavy-tailed with infinite mean!).

5.2 Linear diffusions

The most widely used example of mean-reverting process in finance is the Ornstein-Uhlenbeck (OU) process, defined as the unique solution $(S_t)_{t\geq 0}$ to the stochastic differential equation

$$dS_t = \gamma dB_t + \alpha(\mu - S_t)dt, \qquad \text{i.e,}\quad S_t = e^{-\alpha t}\left(S_0 + \gamma\int_0^t e^{\alpha s}dB_s\right) + \mu(1-e^{-\alpha t}), \tag{46}$$

where $\alpha\in\mathbb{R}$ and $\gamma\in\mathbb{R}$ and B is a standard Brownian motion. The stationary distribution is $N(\mu, \frac{\gamma^2}{2\alpha})$. We denote $\sigma_\infty^2 = \frac{\gamma^2}{2\alpha}$ the stationary variance.

Distributional properties of excursions for the OU process have been studied in detail by Salminen et al [39]. We now use these results to study δ-excursions of the OU process.

An application of the formulas in Section 4 yields:

Proposition 5.2 (Property of δ-excursions for OU Process)**.** *Let S be the OU process defined in (46). Then the duration $(\theta_k^+ - \theta_{k-1}^+)$ of the δ-excursion satisfies*

$$\mathbb{E}^0[e^{-\lambda(\theta_k^+ - \theta_{k-1}^+)}] = \frac{\Phi_{\lambda,-}(0)}{\Phi_{\lambda,-}(\delta)} \times \frac{\Phi_{\lambda,+}(\delta)}{\Phi_{\lambda,+}(0)},$$

where

$$\Phi_{\lambda,-}(x) = \int_0^\infty u^{\frac{\lambda}{\alpha}-1}\exp\left(\sqrt{\frac{2\alpha}{\gamma^2}}(x-\mu)u - \frac{u^2}{2}\right)du,$$
$$\Phi_{\lambda,+}(x) = \int_0^\infty u^{\frac{\lambda}{\alpha}-1}\exp\left(\sqrt{\frac{2\alpha}{\gamma^2}}(\mu-x)u - \frac{u^2}{2}\right)du.$$

Holding periods $(\theta_k^+ - \tau_k^+)$ are independent variables; for $\mu = 0$ their probability density is given by

$$f(t) = \frac{\delta}{\gamma\sqrt{2\pi}} \exp\left(-\frac{\delta^2 \alpha e^{-\alpha t}}{2\gamma^2 \sinh(\alpha t)} + \frac{\alpha t}{2}\right)\left(\frac{\alpha}{\sinh(\alpha t)}\right)^{3/2} \tag{47}$$

The distribution of the worst loss during the kth holding period $[\tau_k^+, \theta_k^+]$ is given by

$$\mathbb{P}^0\left(\max_{t\in[\tau_k^+,\theta_k^+]} V_0(\phi^+) - V_t(\phi^+) > M\right) = \frac{s(\delta)}{s(M+\delta)}, \quad \text{where} \quad s(x) = \int_0^x \exp\left(\frac{\alpha}{\gamma^2}y^2 - \frac{2\alpha\mu}{\gamma^2}y\right)dy.$$

$$\text{and} \qquad \mathbb{E}^0\left[\max_{t\in[\tau_k^+,\theta_k^+]} V_0(\phi^+) - V_t(\phi^+)\right] = \int_0^\infty \frac{s(\delta)}{s(\delta+x)}dx. \tag{48}$$

Proof. $(\theta_k^+ - \tau_k^+)$ represents the hitting time of level 0 starting from δ for the OU process. The probability density of this hitting time is given [41] by (47). The scale function [22] of the OU Process (46) is given by

$$s(x) = \int_0^x \exp\left(\frac{\alpha}{\gamma^2}y^2 - \frac{2\alpha\mu}{\gamma^2}y\right)dy.$$

Then for any $M > 0$,

$$\mathbb{P}^\delta\left(\max_{t\in[0,T^0(S)]} S_t \geq M + \delta\right) = \mathbb{P}^\delta(T^{M+\delta}(S) < T^0(S)) = \frac{s(\delta)}{s(M+\delta)}.$$

□

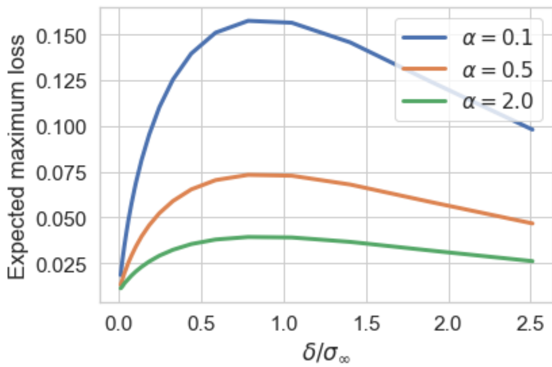
These results allow to compute various quantities of interest for the trading strategy ϕ^+ as a function of model parameters; these are displayed in Figure 4. In most empirical studies the threshold δ is chosen close to the standard deviation of the signal [17]: $\delta \simeq \sigma_\infty$. We observe that the expected maximum loss per trade cycle has a maximum around $\delta \simeq \sigma_\infty$ (Figure 4a), while the expected profit is maximized at a lower threshold $\delta \simeq 0.5\ \sigma_\infty$ (Figure 4c). This implies that the choice $\delta = \sigma_\infty$, may not achieve an optimal risk-return tradeoff.

One may represent this risk-return trade-off in terms of an 'efficient frontier'. Figure 4d shows the realized profit vs the expected maximum loss for various levels of the threshold δ. The profitability of the strategy strongly depends on the mean-reversion rate α, as expected. The maximum profit is obtained for $\delta \simeq 0.5\sigma_\infty$ but the choice $\delta = \sigma_\infty$ is never optimal. These results are consistent with previous studies using the OU model [27].

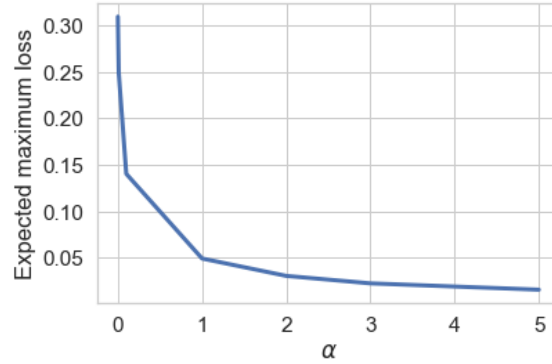
As observed in Figure 4b, the expected worst loss diverges to infinity as the mean reversion rate $\alpha \to 0$, which corresponds to the Brownian case (see Proposition 5.1). This indicates a significant downside risk for such strategies if mean reversion is 'slow' i.e. if α is small.

6 Modeling stochastic processes via δ-excursions

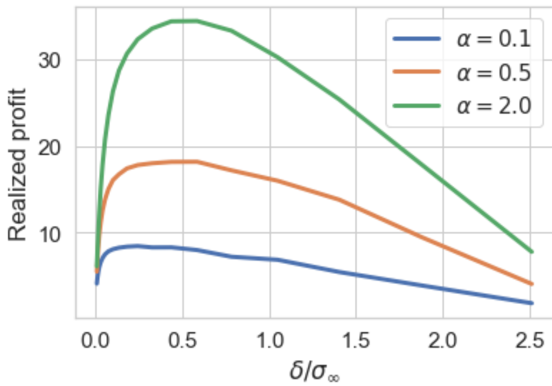
So far we have been using excursions as a way to dissect sample path properties of a process. However one can use this entire apparatus as a method for constructing stochastic models, using excursions as building blocks. This approach was first carried out by Ito, in his construction of a 'recurrent extension' of a Markov process defined up to its first return to zero [21] and explored in depth by Salisbury [38]. Construction of regenerative processes by concatenation of independent excursions has been studied by Lambert and Simatos [26] and Yano [44]. Our $\delta-$excursion concept is related to, but slightly different



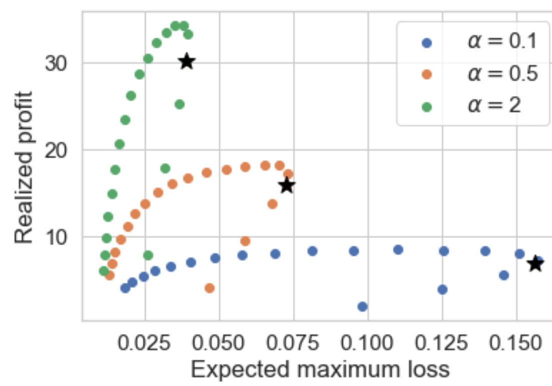
(a) Expected worst loss as a function of δ/σ_∞.



(b) Expected worst loss as a function of mean-reversion parameter α ($\delta = 0.1$).

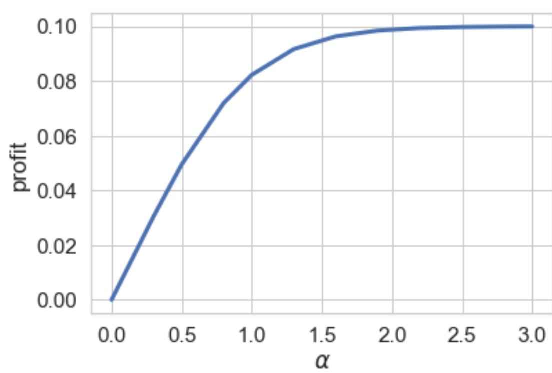


(c) $\delta\mathbb{E}[D_T^\delta]$ as a function of δ/σ_∞.

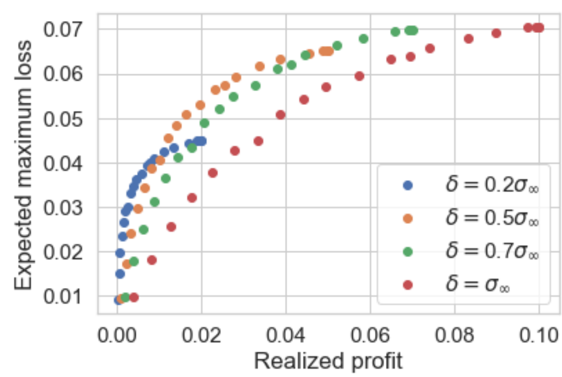


(d) Efficient frontier (Star: $\delta = \sigma_\infty$).

Figure 4: Distributional properties of the trading strategy (4) when the trading signal is an OU process with $\gamma = 0.1$. (No stop-loss.)



(a) Realized profit per cycle; $M = \delta = 0.1$.



(b) Efficient frontier with various choice of $M \in [0.1\sigma_\infty, 4\sigma_\infty]$. ($\alpha = 0.5$ and $\sigma_\infty = 0.1$).

Figure 5: Distributional properties of the trading strategy (6) with stop-loss M and when the trading signal is an OU process with $\gamma = 0.1$.

from, the concept of 'big' excursion in [26]; elements of Γ_δ may be seen as 'big excursions' in the sense of [26] and relate to $\delta-$excursions through the last exit decomposition, as noted in Lemma 3.2.

We present in this section a variation on this theme, which allows to construct processes by concatenation of a sequence of δ-excursions with desired features, and use this approach to provide new constructions as well as a non-parametric approach to scenario simulation based on excursions. Unlike

Ito's original synthesis theorem, our construction leads to Markov processes on an enlarged state space. Compared with [26, 44], we retain a discrete construction and are able to incorporate dependence in the sequence, as shown in the examples below, without losing analytical tractability.

6.1 Construction of processes from δ-excursions

We now present a converse to Theorem 4.7 which allows to construct a stochastic process by pasting together

- excursions from 0 to δ of a Markov process S^+, with
- excursions from δ to 0 of a Markov process S^-.

Theorem 6.1 (Markovian concatenation of δ-excursions)**.** *Let $I \subset \mathbb{R}$ be an interval and $\delta > 0$. Let $(S^+, \mathbb{P}_+^x)$ and $(S^-, \mathbb{P}_-^x)$ be regular Markov processes with state space I satisfying*

$$\mathbb{P}_+^0(S_0^+ = 0) = \mathbb{P}_-^0(S_0^- = 0) = 1, \qquad \mathbb{P}_+^0(0 < T^\delta(S^+) < \infty) = 1, \qquad \mathbb{P}_-^\delta(0 < T^0(S^-) < \infty) = 1.$$

Consider the process e defined by concatenating S^+ and S^- at $T^\delta(S^+)$

$$e = S^+ \underset{T^\delta(S^+)}{\oplus} S^-, \qquad S^+ \sim \mathbb{P}_+^0, \quad S^- \sim \mathbb{P}_-^\delta \tag{49}$$

Then the law of e defines a probability measure Π_δ on $\mathcal{U}_\delta$. Let $(e_k)_{k\geq 1}$ be an IID sequence with law Π_δ and

$$\Delta_0 = 0, \quad \Delta_k = \sum_{i=1}^{k} \Lambda(e_i), \qquad X_t = \sum_{k\geq 1} e_k\left((t - \Delta_{k-1})\right). \tag{50}$$

Then X is unique in law and its $\delta-$excursions are independent with law Π_δ. Π_δ is the law of $X(. \wedge \Delta_1)$ and (50) is the decomposition of X into $\delta-$excursions.

Proof. The proof consists in constructing an auxiliary Markov process Y such that the δ-excursions of X corresponds to the (classical) excursions of Y away from a certain point.

Intuitively, to describe the transitions of the process X, in addition to the current position $x \in I$ we need to know whether we are in an excursion of type $'+'$ or $'-'$. We therefore construct a Markovian lifting of X i.e. a Markov process $Y = (Y_1, Y_2)$ on an enlarged state space where the second component is in $\{+, -\}$ and show that $X = Y_1$ satisfies the requirement of the theorem. Let

$$E = \underbrace{((I \cap (-\infty, \delta]\ ) \times \{+\})}_{E_+} \cup \underbrace{(([0, \infty) \cap I)\ \times \{-\})}_{E_-} \subset I \times \{+, -\}.$$

be state space E shown in Figure 3: E_+ corresponds to the region shaded in blue, while E_- corresponds to the region shaded in green.

One method of proof is to consider e as a Markov process on E absorbed at $(0, -)$ and construct Y as a recurrent extension of e defined by (49). Consider the process $\widehat{Y}$ defined on $[0, \Lambda(e)]$ as

$$\widehat{Y}(0) = (0, -), \quad \widehat{Y}(t) = (e(t), +) \quad 0 < t \leq T^\delta(S^+), \qquad \widehat{Y}(t) = (e(t), -) \quad t > T^\delta(S^+)$$

as an $E-$valued Markov process absorbed at $(0, -)$. Ito's recurrent extension theorem [21, Theorem 6.1] then implies the existence of a unique Markov process $(Y, \mathbb{P}_Y)$ with state space E such that the law of $(Y(. \wedge T^{\partial E}))$ coincides with that of $\widehat{Y}$. From the assumptions on the hitting times of S^+, S^-, this corresponds to the 'discrete visiting case' of Ito's construction [21, Sec. 6] so Y is a concatenation of IID copies of $\hat{Y}$. Let us now describe the Markov process Y through its infinitesimal generator. Denote by $C_0^2(I)=\{f\in C_0(I), \partial f\in C_0(I), \partial^2 f\in C_0(I)\}$. Let L_+ (resp. L_-) be the infinitesimal generator of S^+ (resp. S^-) on $C_0^2(I)$, in the sense that $(S^\pm,\mathbb{P}_\pm)$ is the unique solution of the martingale problem for $L_\pm$. For $f:E\mapsto\mathbb{R}$, denote by $f_{|E_+}$ (resp. $f_{|E_-}$) its restriction to E_+ (resp. E_-). We define

$$\begin{aligned}\mathcal{D}_L &= \Big\{ f:E\to\mathbb{R}, \quad f_{|E_+}(.,+)=f_+\in C_0^2(I\cap(-\infty,\delta]\ ), \\ &\qquad f_{|E_-}(.,-)=f_-\in C_0^2([0,\infty)\cap I), \quad f_+(\delta)=f_-(\delta), \quad f_-(0)=f_+(0)\Big\}. \end{aligned} \tag{51}$$

Define the operator $L:\mathcal{D}_L\to C_0(E)$ by

$$\begin{aligned} Lf(x,+) &= L_+f_+(x), \quad \forall x\in I\cap(-\infty,\delta) \\ Lf(x,-) &= L_-f_-(x), \quad \forall x\in I\cap(0,\infty) \\ Lf(\delta,+)=L_-f_-(\delta), &\qquad Lf(0,-)=L_+f_+(0). \end{aligned} \tag{52}$$

We shall now show that $(Y,\mathbb{P}_Y)$ is the unique solution to the martingale problem for $(L,(0,+))$ on the canonical space $D([0,\infty),E)$ [16, Ch.4, Sec. 5]. Denote by S the canonical process and let $f\in\mathcal{D}_L$. $\mathbb{P}_+^x$ (resp. $\mathbb{P}_-^x$)) is the unique solution to the martingale problem for (L_+,x) (resp. (L_-,x)) on $D([0,\infty),I)$, so using the notation in (52), $M_t^+=f_+(S_t)-\int_0^t L_+f_+(S_u)du$ is a $\mathbb{P}_+-$martingale, and $M_t^-=f_-(S_t)-\int_0^t L_-f_-(S_u)du$ is a $\mathbb{P}_--$martingale. Let us now show that $M_t=f(Y_t)-\int_0^t Lf(Y_u)du$ is a $\mathbb{P}_Y-$martingale.

$$\begin{aligned} M_t &= \sum_{i=1}^{D_t^\delta(S)}\left[f(Y_{\tau_i^+})-f(Y_{\theta_{i-1}^+})-\int_{\theta_{i-1}^+}^{\tau_i^+}Lf(Y_u)du+f(Y_{\theta_i^+})-f(Y_{\tau_i^+})-\int_{\tau_i^+}^{\theta_i^+}Lf(Y_u)du\right] \\ &\quad+\left[f(Y_{\tau^+_{D_t^\delta(S)+1}\wedge t})-f(Y_{\theta^+_{D_t^\delta(S)}\wedge t})-\int_{\theta^+_{D_t^\delta(S)}\wedge t}^{\tau^+_{D_t^\delta(S)+1}\wedge t}Lf(Y_u)du\right] \\ &\quad+\left[f(Y_{\theta^+_{D_t^\delta(S)+1}\wedge t})-f(Y_{\tau^+_{D_t^\delta(S)+1}\wedge t})-\int_{\tau^+_{D_t^\delta(S)+1}\wedge t}^{\theta^+_{D_t^\delta(S)+1}\wedge t}Lf(Y_u)du\right] \\ &= \sum_{i=1}^{D_t^\delta(S)}\Big(M^+_{\tau_i^+}-M^+_{\theta^+_{i-1}}+f_+(\delta)-f_-(\delta)\Big)+\sum_{i=1}^{D_t^\delta(S)}\Big(M^+_{\theta_i^+}-M^+_{\tau_i^+}+f_-(0)-f_+(0)\Big) \\ &\quad+M^+_{\tau^+_{D_t^\delta(S)+1}\wedge t}-M^+_{\theta^+_{D_t^\delta(S)}\wedge t}+1_{t\ge\tau^+_{D_t^\delta(S)+1}}(f_+(\delta)-f_-(\delta))+\left(M^-_{\theta^+_{D_t^\delta(S)+1}\wedge t}-M^-_{\tau^+_{D_t^\delta(S)+1\wedge t}}\right) \end{aligned}$$

where θ_k^+,τ_k^+ are the stopping times introduced in (1). The boundary terms $f_-(0)-f_+(0)$ and $f_+(\delta)-f_-(\delta)$ are zero by definition of $\mathcal{D}_L$, and all other terms are martingale differences under $\mathbb{P}_Y$, so M is a $\mathbb{P}_Y$-martingale. Thus $(Y,\mathbb{P}_Y)$ is indeed a solution of the martingale problem for $(L,(0,+))$. □

The behavior of the Markovian lifting Y can be described from the infinitesimal generator (52): while $Y^2=+$, Y_t^1 evolves according to the transition probabilities of S^+ up to its first hitting time of δ, at which point Y^2 jumps to $-$. Thereafter, Y^1 evolves according to the transition probabilities of S^- until it's next hitting time of zero 0 at which point Y^2 jumps back to $+$, and so on.

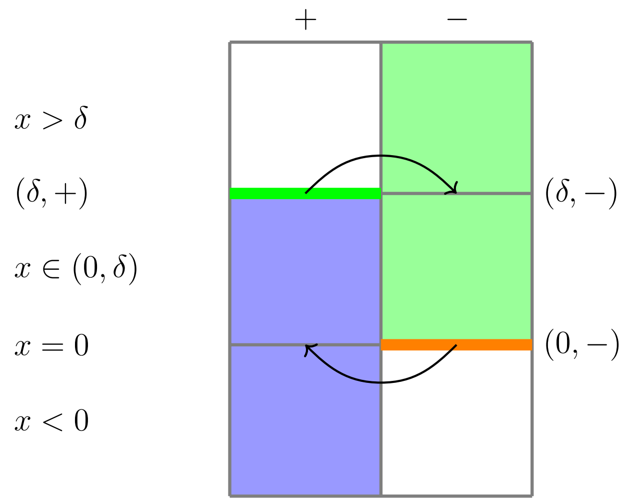


Figure 6: State space E of the Markovian lifting of the process constructed in Theorem 6.1.

Though this construction shares at first glance some features with the Ito's synthesis theorem [21, 38], there are some key differences. When $\delta = 0$ we recover Ito's construction: X is then the recurrent extension of $(S^+(. \wedge T^0(S^+)), \mathbb{P}^0_+)$.

However, for $\delta > 0$, X itself is *not* a Markov process, but the projection of a Markov process Y on an enlarged state space, which may be thought of as a 'regime-switching' model with two regimes $\{+, -\}$. The process X

- behaves like $(S^+, , \mathbb{P}^0_+)$ during its excursions from 0 to δ (+ regime): for measurable $A \subset \mathbb{R}$

$$\forall h > 0, \quad \mathbb{P}(X_{t+h} \in A | X_t = 0, T^\delta_t(X) > h) = \mathbb{P}^0_+(S^+_h \in A);$$

- behaves like $(S^-, \mathbb{P}^\delta_-)$ during its excursions from δ to 0 (− regime): for measurable $A \subset \mathbb{R}$

$$\forall h > 0, \quad \mathbb{P}(X_{t+h} \in A | X_t = \delta, T^0_t(X) > h) = \mathbb{P}^\delta_-(S^-_h \in A).$$

Also, unlike the Ito synthesis theorem, the process is constructed as a discrete concatenation: under Assumption 4.1 we have $\mathbb{P}^0(\forall k \geq 0, \Delta_{k+1} > \Delta_k) = 1$ and local time is not involved in this construction.

Note that any $\delta > 0$ works for this construction, although in some applications δ may have a natural interpretation.

Theorem 6.1 provides a systematic and tractable way of constructing stochastic processes whose excursions have some desired properties. We now provide an example of how this result may be used to construct stochastic processes with *asymmetric* upward and downward excursions.

Example 6.2 (Asymmetric OU process)**.** Recall the OU process defined in (46), which is characterized by parameter (α, μ, γ). Denote by $\mathbb{P}^x_+$ the law of the OU process with parameter $(\alpha_+, \mu_+, \gamma_+)$ and $\mathbb{P}^x_-$ the law of the OU process with $(\alpha_-, \mu_-, \gamma_-)$. We can construct an asymmetric OU-process using asymmetric δ-excursions with formula (49). Figure 7 shows an example of sample path of X.

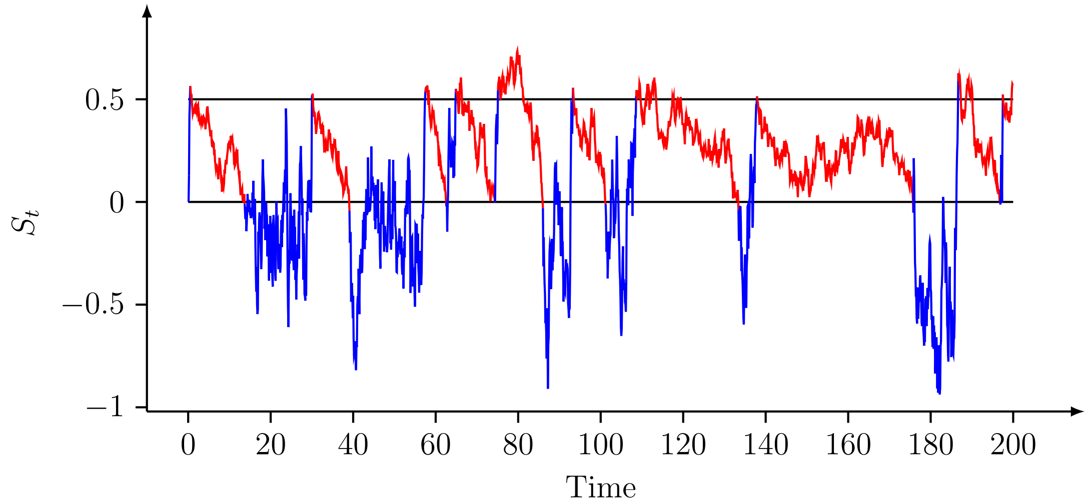


Figure 7: Sample path of an asymmetric OU process with $\delta = 0.5$. Blue segments represent excursions from 0 to δ, generated with $(\alpha_+, \mu_+, \gamma_+) = (0.4, 0, 0.4)$, while red segments represent excursions from δ to 0, generated with $(\alpha_-, \mu_-, \gamma_-) = (0.1, 0, 0.1)$.

If we attempt to model such asymmetric excursions with a (symmetric) OU process (46), this leads to an overestimation or underestimation of excursion heights or durations. For example, based on a sample of size $N = 10000$ generated with parameters $(\alpha_+, \mu_+, \gamma_+) = (0.4, 0, 0.4)$, $(\alpha_-, \mu_-, \gamma_-) = (0.1, 0, 0.1)$ and $\delta = 0.5$, estimating an OU process yields

$$\gamma_- < \widehat{\gamma} = 0.288 < \gamma_+, \text{ and } \alpha_- < \widehat{\alpha} = 0.279 < \alpha_+.$$

We thus observe that the OU model either overestimates or underestimates the volatility and mean-reversion in each regime.

6.2 Non-parametric scenario simulation using excursions

The above examples use as building blocks parametric models of Markov processes to construct other models with desirable excursion properties. More generally, Proposition 3.3 may be used as a *non-parametric* method for simulating paths whose excursions match those observed in data.

Consider a data set consisting of observations on the path $(S_t, t \in [0, T])$ of a $\mathbb{P}-$Markov process satisfying Assumption 4.1 for some $\delta > 0$.

Applying Proposition 3.3, we may decompose the path S into a sequence $(e_k)_{k=1..D^\delta_T}$ of $\delta-$excursions as in (17). By Theorem 4.7, e_k are IID variables with values in $\mathcal{U}_\delta$, whose law we denote Π^δ. Denoting by ϵ_x a unit point mass at x,

$$\Pi_T = \frac{1}{D^\delta_T(S)} \sum_{k=1}^{D^\delta_T(S)} \epsilon_{e_k}$$

defines a probability measure on $\mathcal{U}_\delta$ which we call the *empirical* δ*-excursion measure*. Under Assumption 4.1, $D^\delta_T(S) \to \infty$ as $T \to \infty$, so by the law of large numbers the empirical excursion measure Π_T provides a good approximation of Π^δ for large T and for any Glivenko-Cantelli class $\mathcal{F}$ of functions [40] on $\mathcal{U}_\delta$, representing properties of δ-excursions, we have

$$\forall H \in \mathcal{F}, \qquad \int H(f)\Pi_T(df) \overset{T\to\infty}{\to} \int H(f)\Pi^\delta(df).$$

The point is that it is fairly easy to simulate samples from Π_T and evaluate $\int F(f)\Pi_T(df)$: this may be done by randomly resampling from the empirical sequence of excursions (e_k). This leads to an approach for *non-parametric scenario simulation* based on $\delta-$excursions:

Non-parametric scenario simulation by pasting of excursions.

Input data : sample path $(S_t, t \in [0,T])$

- Decompose $(S_t, t \in [0,T])$ into $\delta-$excursions $e_1, ...e_N \in \mathcal{U}_\delta$ using Proposition 3.4.
- Generate an IID sequence of integers $(k_1(\omega), k_2(\omega), ..)$ where $k_i \sim \text{UNIF}(\{1, 2, .., N\})$.
- Construct a path $X(\omega)$ as in (17) by concatenating the excursions in the order given by $(k_i, i \geq 1)$:

$$X_t(\omega) = \sum_{i \geq 1} e_{k_i(\omega)}((t - \theta_{i-1}^+) \qquad \text{where} \qquad \theta_0^+ = 0, \quad \theta_i^+ = \sum_{j=1}^{i} \Lambda(e_{k_i}).$$

Output : simulated sample path X

Table 1: Non-parametric scenario simulation by pasting of excursions.

If $N = D_T^\delta(S)$ is large, the paths X generated in this way have $\delta-$excursions whose properties mimic those of S.

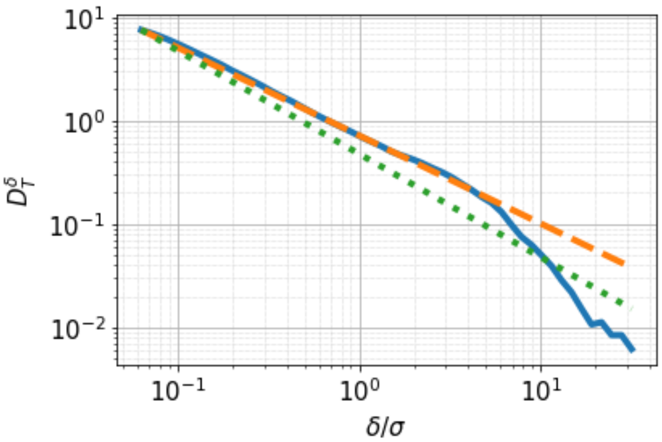
We illustrate the flexibility of this approach in an example based on financial data.

Example 6.3 (Pairs trading)**.** Pairs trading [33, 17] is a trading strategy based on identifying a stationary linear combination of two stock prices and using this linear combination as a trading signal for generating buy/sell transactions in the pair. In most applications the signal is then modeled as an AR(1)/ Ornstein-Uhlenbeck process [27, 33].

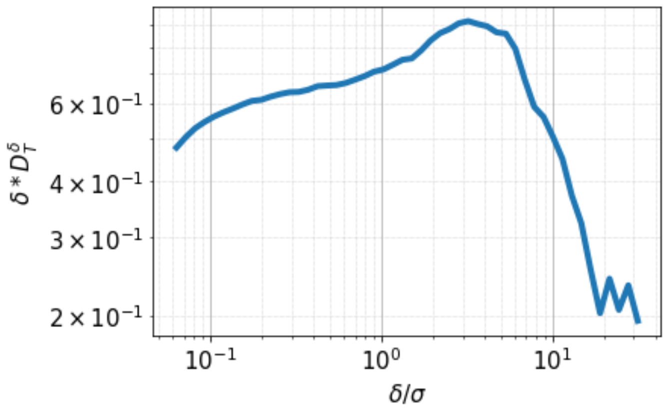
As an example, we use second-by-second NYSE price records of CocaCola (KO) and PepsiCola (PEP) shares during trading hours 10:00AM-4:00PM for the period 07/01/2013 - 07/01/2020 to construct a pair-trading signal. Denote PEP(t) (resp. CO(t)) the mid-price of Pepsi (resp. Coca-Cola). The signal is constructed as $S_t := \text{PEP}(t) - a_t\, \text{CO}(t)$ where the coefficient a_t is piece-wise constant, updated on each trading day by an ordinary least square regression of PEP$(\cdot)$ on CO$(\cdot)$ over the previous 5 days.

To assess the roughness of the signal S, we analyze the number of level crossings as a function of δ and apply Proposition 3.9. Recall that $\log(D_t^\delta(S)) \sim -(p-1)\log(\delta) + \text{constant}$ as $\delta \to 0$, where p measure the roughness of the path. We estimate the exponent p by linear regression of $\log(D_t^\delta(S))$ on $\log(\delta)$. As shown in Figure 8a, the estimated exponent of the trading signal S is around $p = 1.85$, which implies that the path is slightly smoother than the Brownian motion (for which $p = 2$).

There are two types of crossings with different time-scales along the path: crossings due to the mean reverting phenomenon on a longer time-scale and crossings with small magnitudes due to the roughness of the path once the signals revert to level 0. The crossings of the first type could be captured by all δ with appropriate choices. Crossings of the second type show up as $\delta \to 0$. Empirical estimates seem to indicate a non-zero limit of the realized profit δD_T^δ as $\delta \to 0$. This is consistent with the result in Proposition 3.9 for $p = 2$, indicating that it is not use small thresholds δ for trading this pair (see Figure 8b). The realized profit is maximized at $\delta \simeq 3.2\sigma$, which is quite different from the OU model, where realized profit is maximized for $\delta < \sigma$.



(a) Number of level crossings D_T^δ. (Orange dashed line: δ^{p-1} with $p = 1.85$; Green dotted line: asymptotics for $p = 2.0$; $T = 1$ day).



(b) Realized profit δD_T^δ ($T = 1$ day).

Figure 8: Number of crossings D_T^δ and realized profit δD_T^δ as a function of threshold δ.

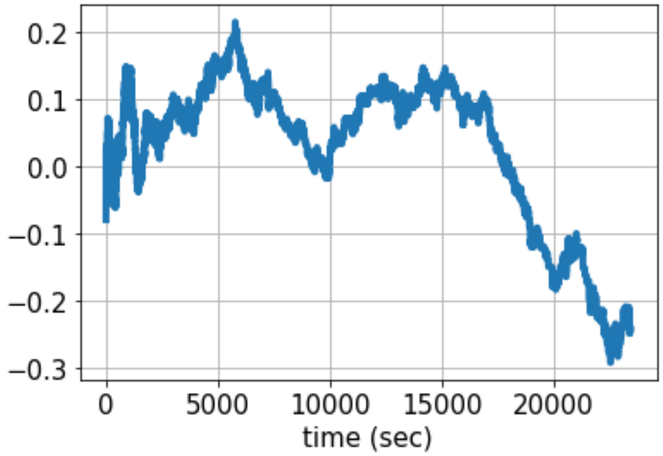
Recall that the trading strategy ϕ^+ consists in shorting the pair when S_t crosses the threshold δ from below and unwinds the position when S_t returns to 0. Denote by σ the standard deviation of S_t. In Figure 9, we provide the empirical distributions of the durations for waiting period waiting period $(\tau_k^+ - \theta_{k-1}^+)$ and holding period $(\theta_k^+ - \tau_k^+)$ and the maximum loss during the holding period, when $\delta = \sigma$ is the intraday standard deviation of the signal, a common choice for mean-reversion strategies [33].

As seen from the semi-logarithmic plots in Figures 9b and 9c, the durations of the holding period and the waiting period are approximately exponentially distributed. The maximum loss has a Pareto tail with exponent $k = 1$, which is very heavy tailed and indicates infinite mean and variance, as shown by the log-log plot in Figure 9d. This combination of a Pareto tail for the excursion height and an exponential duration for $\delta-$excursions corresponds neither to the Brownian case nor to the case of the Ornstein-Uhlenbeck process.

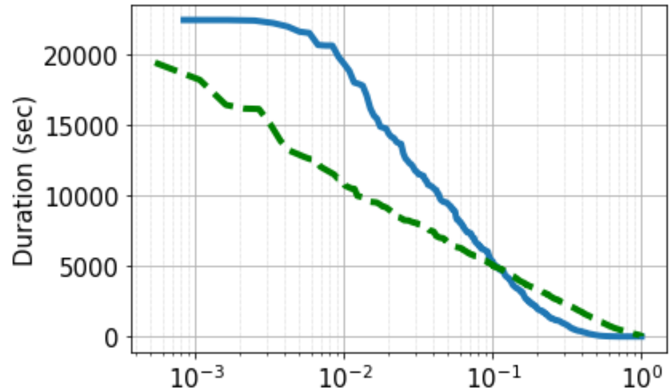
Estimating an Ornstein-Uhlenbeck process as in (46) using the method of moments yields the following parameter estimates (time is measured in seconds):

$$\widehat{\alpha} = 1.3 \times 10^{-3}, \quad \widehat{\mu} = -0.02, \text{ and } \widehat{\gamma} = 4.8 \times 10^{-3}. \tag{53}$$

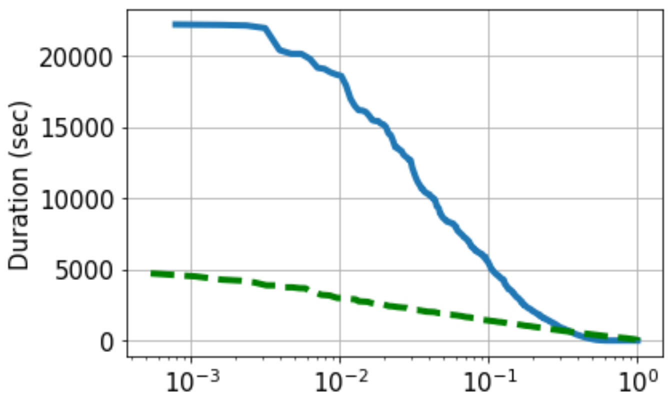
The corresponding distributions for the duration of the holding period, the waiting period and the worst loss during the holding period are displayed (green dotted lines) alongside the empirical distributions of these quantities in Figure 9. The discrepancy between the green dotted lines and the blue solid lines in Figures 9b, 9c, and 9d illustrates that the distributions computed using the OU model give a poor approximation of the corresponding empirical distributions, leading to an inaccurate representation of the risk and return profile of the strategy. This is yet another indication of the risk of model misspecification in such mean-reversion strategies.



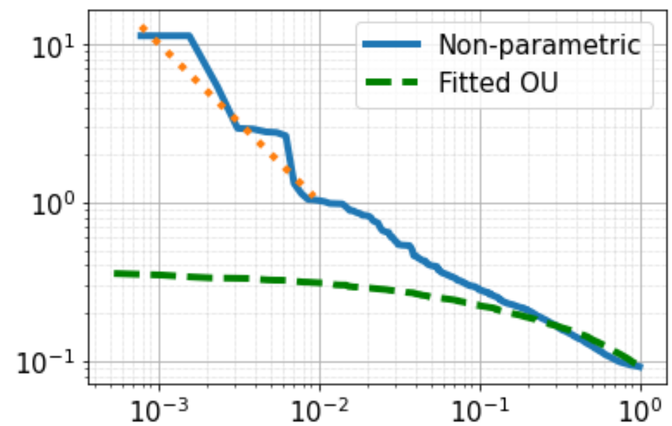
(a) Coca-Pepsi pair trading signal: 07/16/2013.



(b) Rank-frequency plot for the waiting period $(\tau_k^+ - \theta_{k-1}^+)$, semi-logarithmic scale.



(c) Rank-frequency plot for the holding period $(\theta_k^+ - \tau_k^+)$, semi-logarithmic scale.



(d) Rank-frequency plot for maximum loss during the holding period (log-log scale). Orange dotted line: Pareto distribution with exponent $k = 1.0$.

Figure 9: Coca-Pepsi pair trading signal $\mathrm{KO}(t) - a_t\mathrm{PEP}(t)$ at one-second frequency, for $\delta = \sigma$ (2007-2020). Data (blue) vs OU model (dotted).

We tackle the shortcomings of the OU model by using our non-parametric scenario simulation approach to evaluate the quantities of interest as described in Table 1: we decompose the empirical path into $\delta-$excursions and generate scenarios by random pasting of such empirical $\delta-$excursions. Figure 10 shows a typical sample path generated in this manner, for $\delta = \sigma$. By construction, paths generated in this way retain the roughness properties of the observe path as well as the empirical distribution of heights and durations of $\delta-$excursions.

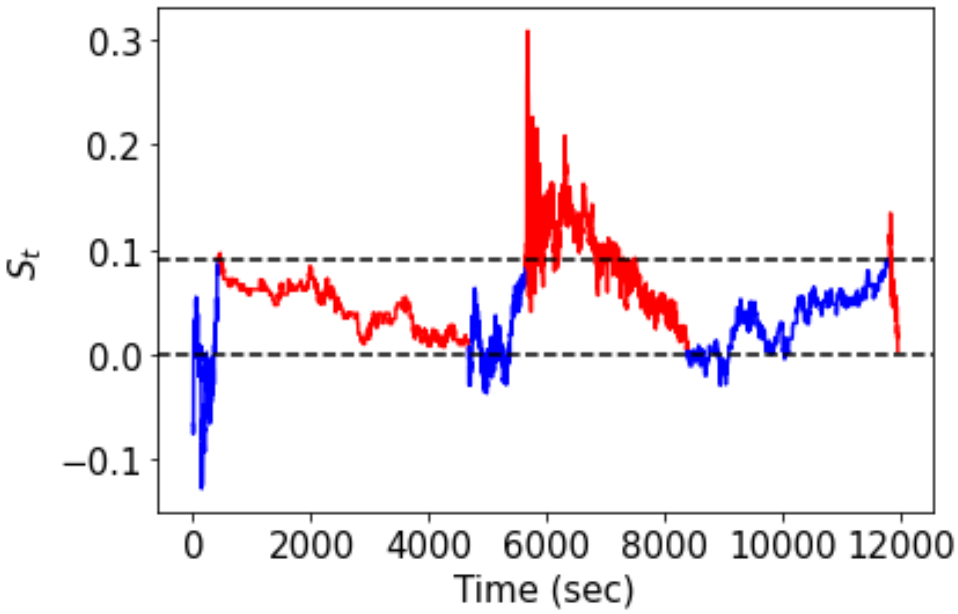


Figure 10: Example of sample path generated using non-parametric method, using the pairs trading signal described in Section 6.3.

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A Technical proofs

A.1 Proof of Proposition 4.8

The strong Markov property of S implies that

$$V_t(\phi^+) = \sum_{i\geq 1} S_{t\wedge\tau_i^+} - S_{t\wedge\theta_i^+} \overset{(law)}{=} S_{t\wedge\tau_1^+} - S_{t\wedge\theta_1^+} + V_{(\theta_1^+)_+}(\phi^+)$$

Depending on whether $\theta_1^+ < t$ or $> t$, we have either $S_{t\wedge\tau_1^+} - S_{t\wedge\theta_1^+} = \delta$ or $V_{(t-\theta_1^+)_+}(\phi^+) = 0$, so

$$e^{-\lambda V_t(\phi^+)} \overset{(law)}{=} e^{-\lambda\left(S_{t\wedge\tau_1^+} - S_{t\wedge\theta_1^+}\right) - \lambda V_{(t-\theta_1^+)_+}(\phi^+)} = e^{-\lambda\delta - \lambda V_{(t-\theta_1^+)_+}(\phi^+)} + e^{-\lambda\left(S_{t\wedge\tau_1^+} - S_{t\wedge\theta_1^+}\right)} - e^{-\lambda\delta}.$$

Hence taking the expectation, we obtain

$$H^\delta(t,\lambda) = e^{-\lambda\delta}\left(H^\delta(\cdot,\lambda) \star F_\theta(ds)\right)(t) \, + \, U_1(t,\lambda,\delta) - e^{-\lambda\delta}.$$

It remains to note that, since for given τ_1^+; $S(t\wedge\theta_1^+) - S(t\wedge\tau_1^+)$ has the same law under $\mathbb{P}^0$ as $S((t-\tau_1^+)_+ \wedge T^0) - \delta$ under $\mathbb{P}^\delta$, then

$$\begin{aligned} U(t,\lambda,\delta) &= \mathbb{E}^0\left[\mathbb{E}^0\left[e^{-\lambda\left(S(t\wedge\tau_1^+) - S(t\wedge\theta_1^+)\right)}\Big|\mathcal{F}_{\tau_1^+}\right]\right] \\ &= \mathbb{E}^0\left[e^{-\lambda\delta} U_2((t-\tau_1^+)_+, \lambda, \delta)\right] = e^{-\lambda\delta}\left(U_2(\cdot,\lambda,\delta) \star F_\tau(ds)\right)(t). \end{aligned}$$

Plugging this in the previous identity, we get

$$H^\delta(t,\lambda) = e^{-\lambda\delta}\left(H^\delta(\cdot,\lambda) \star F_\theta\right)(t) \, + \, e^{-\lambda\delta}\left(U_2(\cdot,\lambda,\delta) \star F_\tau\right)(t) - e^{-\lambda\delta}.$$

Equation (45) follows by applying Laplace transform in t and solving for $\mathcal{L}\left[H^\delta(\cdot,\lambda)\right](\cdot)$.

A.2 Proof of Proposition 5.1

First we note that

$$(\theta_{i+1}^+ - \tau_{i+1}^+) \overset{\text{(law)}}{=} (\tau_{i+1}^+ - \theta_i^+) \overset{\text{(law)}}{=} T^\delta,$$

and

$$\mathbb{P}^0(T^\delta(S) > t) = \mathbb{P}^0\left(\max_{u\in[0,t]} S_u < \delta\right) = \mathbb{P}^0(|S_t| < \delta) = \mathbb{P}^0(\sqrt{t}|S_1| < \delta) = 1 - 2\Phi\left(\frac{\delta}{\sigma\sqrt{t}}\right)$$

where Φ is the standard normal distribution. So

$$\theta_{i+1}^+ - \theta_i^+ = (\theta_{i+1}^+ - \tau_{i+1}^+) + (\tau_{i+1}^+ - \theta_i^+) \overset{\text{(Law)}}{=} \frac{\delta^2}{\sigma^2}\left(\frac{1}{Z^2} + \frac{1}{Z'^2}\right),$$

where Z, Z' are independent $N(0,1)$, which proves (i) and (ii). To show (iii) note that for any $M > 0$,

$$\mathbb{P}^0\left(\max_{t\in[\tau_1^+,\theta_1^+]} V_0(\phi^+) - V_t(\phi^+) \geq M\right) = \mathbb{P}^\delta(T^{M+\delta}(S) < T^0(S)) = \frac{\delta}{M+\delta}.$$

which then implies that the expected maximum loss is infinite. To show (iv) note that when $\mathcal{G}_x = \sigma^2\partial_x^2$ one can solve boundary value problem (44) to obtain

$$\begin{aligned} U_2(t,\lambda,x) \; &= \; 1 \, + \, \frac{1}{\sigma\sqrt{2\pi t}}\int_0^\infty \left(e^{-\frac{(x-y)^2}{2\sigma^2 t}} \, - \, e^{-\frac{(x+y)^2}{2\sigma^2 t}}\right)\left(e^{\lambda y} \, - \, 1\right) dy \\ &= e^{\lambda x + p(\lambda)t}\Phi\left(\frac{x+\lambda\sigma^2 t}{\sigma\sqrt{t}}\right) \, - \, e^{-\lambda x + p(\lambda)t}\Phi\left(\frac{-x+\lambda\sigma^2 t}{\sigma\sqrt{t}}\right) + 2\Phi\left(\frac{-x}{\sigma\sqrt{t}}\right). \end{aligned}$$

where $p(\lambda) := \frac{1}{2}\sigma^2\lambda^2$ and Φ is the standard normal distribution. Hence $U_2(t,\lambda,x)$ grows like $e^{p(\lambda)t}$ as $t \to +\infty$ so the Laplace transform $\tilde{U}_2(z,\lambda,x)$ is well-defined for $z > p(\lambda)$. Taking the Laplace transform in the PDE for U_2, we obtain

$$\tilde{U}_2(z,\lambda,x) \;=\; \frac{p(\lambda)}{z\,(p(\lambda)-z)}\frac{\Phi_{z,+}(x)}{\Phi_{z,+}(0)} - \frac{e^{\lambda x}}{p(\lambda)-z},\; z > p(\lambda).$$

Using (40) and formula (45) in Proposition 4.8, we obtain

$$\mathcal{L}\left[H^{\delta}(\cdot,\lambda)\right](z) = \frac{\frac{p(\lambda)}{z(p(\lambda)-z)}\mathcal{L}_{\theta}(z) - \frac{e^{\lambda\delta}}{p(\lambda)-z}\,\mathcal{L}_{\tau}(z) - 1/z}{e^{\lambda\delta} - \mathcal{L}_{\theta}(z)},\; z > p(\lambda).$$

In this case we have explicit forms for $\Phi_{z,\pm}(x) = c\,e^{\lambda_{\mp}x}$, where $\lambda_{\pm} := \pm\frac{\sqrt{2z}}{\sigma}$. Thus we have

$$\mathcal{L}_{\tau}(z) = e^{-\frac{\sqrt{2z}}{\sigma}\delta},\; \mathcal{L}_{\theta}(z) = e^{\frac{-2\sqrt{2z}}{\sigma}\delta}.$$

Therefore,

$$\mathcal{L}\left[H^{\delta}(\cdot,\lambda)\right](z) = \frac{p(\lambda)e^{\frac{-2\sqrt{2z}}{\sigma}\delta} \;-\; ze^{\lambda\delta}\,e^{\frac{-\sqrt{2z}}{\sigma}\delta} - (p(\lambda)-z)}{z\,(p(\lambda)-z)\left(e^{\lambda\delta} - e^{\frac{-2\sqrt{2z}}{\sigma}\delta}\right)},\; z > p(\lambda).$$

Hence

$$\int_0^{\infty}\mathbb{E}^0\left[e^{-\lambda V_t(\phi^+)}\right]e^{-zt}dt = \frac{\sigma^2\lambda^2 e^{\frac{-2\sqrt{2z}}{\sigma}\delta} \;-\; 2ze^{\lambda\delta}\,e^{\frac{-\sqrt{2z}}{\sigma}\delta} - (\sigma^2\lambda^2 - 2z)}{z\,(\sigma^2\lambda^2-2z)\left(e^{\lambda\delta} - e^{\frac{-2\sqrt{2z}}{\sigma}\delta}\right)},\;\text{for}\quad z > \frac{1}{2}\sigma^2\lambda^2.$$