$$
\begin{aligned}
I &= \int_{-\infty}^{+\infty} \frac{x \sin x}{\left(x^2+a^2\right)\left(x^2+b^2\right)} \mathrm{~d}x \\
&= \frac{1}{a^2-b^2} \int_{-\infty}^{+\infty}\left(-\frac{x}{a^2+x^2}+\frac{x}{b^2+x^2}\right) \sin x \mathrm{~d}x \quad \text{(Partial Fraction Decomposition)} \\
&= \frac{1}{a^2-b^2}\left(-\int_{-\infty}^{+\infty} \frac{x \sin x}{a^2+x^2} \mathrm{~d}x+\int_{-\infty}^{+\infty} \frac{x \sin x}{b^2+x^2} \mathrm{~d}x\right) \\
&:= \frac{1}{a^2-b^2}(-I(a)+I(b)) \qquad (1)
\end{aligned}
$$

where we define $I(a)=\int_{-\infty}^{+\infty} \frac{x \sin x}{a^2+x^2} \mathrm{~d}x$ and $I(b)=\int_{-\infty}^{+\infty} \frac{x \sin x}{b^2+x^2} \mathrm{~d}x$, then

$$
\begin{aligned}
I(a) &= \int_{-\infty}^{+\infty} \frac{x \sin x}{a^2+x^2} \mathrm{~d}x \\
&= 2 \int_0^{+\infty} \frac{x \sin x}{a^2+x^2} \mathrm{~d}x \quad \text{(the integrand is an even function)} \\
&\xlongequal{x=at} 2 \int_0^{+\infty} \frac{at \sin (at)}{a^2+a^2 t^2} a \mathrm{~d}t \quad \text{(we need to use that } a>0\text{)} \\
&= 2 \int_0^{+\infty} \frac{t \sin (at)}{1+t^2} \mathrm{~d}t \qquad (2) \\
&= 2 \int_0^{+\infty} \frac{t^2 \sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \\
&= 2 \int_0^{+\infty} \frac{\left(1+t^2-1\right) \sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \\
&= 2 \int_0^{+\infty} \frac{\left(1+t^2\right) \sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t-2 \int_0^{+\infty} \frac{\sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \\
&= 2 \int_0^{+\infty} \frac{\sin (at)}{t} \mathrm{~d}t-2 \int_0^{+\infty} \frac{\sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \\
&= 2 \times \frac{\pi}{2}-2 \int_0^{+\infty} \frac{\sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \\
&= \pi-2 \int_0^{+\infty} \frac{\sin (at)}{t\left(1+t^2\right)} \mathrm{~d}t \qquad (3)
\end{aligned}
$$

Next, the main idea is to differentiate $I(a)$ twice to attain an ODE. However, we have to justify that differentiating under the integral sign is allowable in this case. Indeed,

$$
\int_0^{+\infty} \frac{\partial}{\partial a}\left(\frac{\sin (at)}{t\left(1+t^2\right)}\right) \mathrm{~d}t=\int_0^{+\infty} \frac{t \cos (at)}{t\left(1+t^2\right)} \mathrm{~d}t=\int_0^{+\infty} \frac{\cos (at)}{1+t^2} \mathrm{~d}t
$$

where the RHS uniformly convergences by Weierstrass M-test, since $\left|\dfrac{\cos(at)}{1+t^2}\right| \leq \dfrac{1}{1+t^2}$ and $\displaystyle\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t$ converges. Hence, differentiating (3) by both sides gives us

$$I'(a) = -2\int_0^{+\infty} \frac{\cos(at)}{1+t^2}\ \mathrm{d}t$$

And still we can differentiate this equation, since

$$\int_0^{+\infty} \frac{\partial}{\partial a}\left(\frac{\cos(at)}{1+t^2}\right)\ \mathrm{d}t = -\int_0^{+\infty} \frac{t\sin(at)}{1+t^2}\ \mathrm{d}t$$

where the RHS uniformly converges where $a \in [\delta, +\infty)$ for any $\delta > 0$ by Dirichlet's test, since $\forall A > 0$, $\left|\displaystyle\int_0^A \sin(at)\ \mathrm{d}t\right| = \dfrac{1-\cos aA}{a} \leq \dfrac{2}{a} \leq \dfrac{2}{\delta}$ remains uniformly bounded, and $\dfrac{t}{1+t^2}$ decreasingly approaches 0 when $t \to +\infty$, and so we can differentiate under the integral sign. Thus

$$I''(a) = 2\int_0^{+\infty} \frac{t\sin(at)}{1+t^2}\ \mathrm{d}t$$

holds, because δ is chosen arbitrarily. Noticing that the RHS is the same as 2, we have

$$I''(a) = I(a)$$

The general solution to this ODE is $I(a) = C\mathrm{e}^a + D\mathrm{e}^{-a}$, yet we have

$$I(0) = \int_{-\infty}^{+\infty} \frac{\sin x}{x}\mathrm{d}x = \pi = C + D$$

and

$$I'(0) = -2\int_0^{+\infty} \frac{1}{1+t^2}\ \mathrm{d}t = -\pi = C - D$$

Thus $C = 0$ and $D = \pi$, and so we have

$$I(a) = \pi\mathrm{e}^{-a}$$

Analogously,

$$I(b) = \pi\mathrm{e}^{-b}$$

Finally, by (1), we have the answer

$$I = \frac{1}{a^2-b^2}\left(-\pi\mathrm{e}^{-a} + \pi\mathrm{e}^{-b}\right) = \frac{\pi}{b^2-a^2}\left(\mathrm{e}^{-a} - \mathrm{e}^{-b}\right)$$