Discrete Mathmatics

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Abstract

A compilation of notes based on *Discrete Mathematics and its Applications* (7^{th} Edition) – K. Rosen.

Contents

Chapter 1

Logic and Proofs

1.1 Propositional Logic

1.1.1 Propositions

A proposition is a declarative sentence that has a truth value of true (T) or false (F). It may be denoted by a propositional variable (such as p, q, r, s). Compound propositions can be made from one or more propositions by means of *logical connectives*. The definition of a logical connective can be expressed by its corresponding truth table.

1.1.2 Logical Connectives

A logical connective operates on one or more propositions to yield a new proposition.

Negation $\neg$

Negation is a unary operation that yields the logical complement of a proposition.

p	$\neg p$
T	F
F	T

Table 1.1: Truth table for negation

Conjuction $\wedge$

The conjuction of two propositions is true if both propositions are true, and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: Truth table for conjunction

Disjunction $\vee$

The disjunction of two propositions is true if either or both are true, and is false otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: Truth table for disjunction

Exclusive Disjunction $\oplus$

The exclusive disjunction of two propositions is true if one is the logical complement of the other, and is false otherwise.

1.1.3 Conditional Statements

A conditional statement is a proposition composed from a *premise* p and a *conclusion* q. It represents

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 1.4: Truth table for exclusive disjunction

the implication of q by p, and is denoted

$$p \to q$$

The only case in which this statement is false is when p is true but q is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.5: Truth table for a conditional statement

Converse

The converse of a conditional statement $p \to q$ is given by

$$q \to p$$

Contrapositive

The contrapositive of a conditional statement is given by

$$\neg q \to \neg p$$

It is *logically equivelant* to the conditional statement itself; that is to say, the truth tables for a conditional statement and its contrapositvie are identical.

Inverse

The inverse of a conditional statement is given by

$$\neg p \to \neg q$$

Since a conditional statement is logically equivelant to its contrapositive, and the inverse is the contrapositive of the converse, then the inverse is thus logically equivelant to the converse.

Biconditional Statements

A biconditional statement is the conjucntion of a contitional statement and its converse. It is denoted by

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

and is only true when both p and q have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6: Truth table for a biconditional statement.

1.1.4 Logical & Bit operations

A *bit* (portamentau of binary and digit) is the smallest unit of information representable by a computer. The value of a bit can be either 1 or 0, analagous to the truth value of a propositon, T or F, respectively. One may consider bit operators analogous to logical operators.

Logical Connective	Bit Operator
$\vee$	OR
$\wedge$	AND
$\oplus$	XOR

Table 1.7: Correspondence between logical connectives and bit operators

1.2 Applications of Propositional Logic

1.2.1 Logic Gates

1.3 Propositional Equivelances

1.3.1 Compound Propositions

A compound proposition may be categorised into one of three categories:

Tautology

A *tautology* is a compound proposition that is *true* in all possible cases. An arbitrary tautology is denoted by

$$\top$$

Contradiction

A *contradiction* is a compound propsition that *false* in all possible cases. An arbitrary contradiction is denoted by

$$\perp$$

Contingent

A *contingent* proposition is one that is neither a tautology nor a contradiction. Its truth value is contingent on the particular configuration of truth values of its constituent propositions.

1.3.2 Logical Equivalences

The compound propositions p and q are *logically equivelant* if

$$p \leftrightarrow q \equiv \top \tag{1.1}$$

This statement may be equally expressed by

$$p \equiv q$$

The following are important logical equivelances:

Domination Laws

Disjunction with a tautology is a tautology, and conjunction with a contradiction is a contradiction

$$p \vee \top \equiv \top \tag{1.2a}$$

$$p \wedge \perp \equiv \perp \tag{1.2b}$$

Identity Laws

Conjunction with a tautology, and disjunction with a contradiction, are *identity* operations on a proposition

$$p \wedge \top \equiv p \tag{1.3a}$$

$$p \vee \perp \equiv p \tag{1.3b}$$

Negation laws

Conjunction of a proposition with its negation is a contradiction, and disjunction of a proposition with its negation is a tautology

$$p \wedge \neg p \equiv \perp \tag{1.4a}$$

$$p \vee \neg p \equiv \top \tag{1.4b}$$

Double Negation Law

A proposition is the negation of its negation

$$\neg(\neg p) \equiv p \tag{1.5a}$$

Idempotent Laws

Both the conjunction and disjunction of a proposition with itself yields itself

$$p \wedge p \equiv p \tag{1.6a}$$

$$p \vee p \equiv p \tag{1.6b}$$

Absorption Laws

The former two equivelances may be *absorbed* into each other as in the following

$$p \wedge (p \vee p) \equiv p \tag{1.7a}$$

$$p \vee (p \wedge p) \equiv p \tag{1.7b}$$

Commutative Laws

The commutativity of a binary logical connective can be determined by examining its operation on two propositions with complementary truth values. From Table 1.1.2, one observes that

$$\top \wedge \perp \equiv \perp \wedge \top \equiv \perp$$

Likewise, from Table 1.1.2

$$\top \vee \perp \equiv \perp \vee \top \equiv \top$$

This leads to the *commutative laws*

$$p \wedge q \equiv q \wedge p \tag{1.8a}$$

$$p \vee q \equiv q \vee p \tag{1.8b}$$

A similar argument may be used to demonstrate the commutativity of $\oplus$ and $\leftrightarrow$.

Associative Laws

Both conjunction and disjunction are *associative*

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \tag{1.9a}$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \tag{1.9b}$$

Distributive Laws

Conjunction and disjunction *distribute* as in the following

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \tag{1.10a}$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \tag{1.10b}$$

De Morgan's Laws

De Morgan's laws demonstrate how conjunctions and disjunctions are negated

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1.11a}$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{1.11b}$$

Logical Equivelances with $\rightarrow$

DO TABLE

Logical Equivelances with $\leftrightarrow$

DO TABLE

1.3.3 Propositional Satisfiability

Consider the set of N propositions

$$\{s_1, s_2, \ldots, s_N\} = S_N$$

There exist 2^N possible configurations of truth values for the propositions contained in S_N. Let p be a compound proposition that is some amalgamation of the propositions in S_N through logical connectives. We say that p is *satisfiable* if there exists a configuration of truth values in S_N such that p is true. We call this configuration a *solution* of p. If there exist no solution to p, we say the proposition is *unsatisfiable*. That is, a proposition is unsatisfiable if and only if its negation is a tautology.

1.4 Predicates and Quantifiers

1.4.1 Predicates

A *predicate* is statement regarding a *subject* that becomes a proposition when the subject is specified. A predicate may be condisered as a *propositional function*; that is, a mapping from the subject to a bivalent truth value. We denote a unary predicate as

$$P(x)$$

where P is the predicate condition, and x is a variable that denotes the subject. A predecate may have multiple subjects; the predicate

$$P(x_1, x_2, \ldots, x_n)$$

is refered to as an *n-ary* predicate.

1.4.2 Quantifiers

The variable x in the statement $P(x)$ is a *free* variable. In the previous section we discussed how $P(x)$ becomes a proposotion when x is set to a value. Here, we discuss another way in which $P(x)$ becomes a proposition: *binding* x with a *universal quantifier.*

Universal Quantifier

Consider the set

$$D = \{x_1, x_2, \ldots, x_n\}$$

which we shall call the *domain of discourse.* One can construct a compound proposition that is the conjunction of the predicate $P(x)$ for all x in D. That is

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n) \equiv \bigwedge_{i=1}^{n} P(x_i)$$

Such a prescription is sufficient when the domain of dicource is *finite.* What is the equivelent proposition if D is *not* finite?

$$\forall x P(x)$$

where we use the **universal quantifier** $\forall$, read as "for all".

Existential Quantifier

Analagous to the universal quantifier, the proposition

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n) \equiv \bigvee_{i=1}^{n} P(x_i)$$

may be denoted for a non-finite universe of discourse by

$$\exists x P(x)$$

where we use the **existential quantifier** $\exists$, read as "there exists".

1.4.3 Negating Quantifiers

Using the Associative Laws (1.9), one may generalise De Morgan's Laws (1.11) for n propositions

$$\neg\left(\bigwedge_{i=1}^{n} p_i\right) \equiv \bigvee_{i=1}^{n} \neg p_i \tag{1.12a}$$

$$\neg\left(\bigvee_{i=1}^{n} p_i\right) \equiv \bigwedge_{i=1}^{n} \neg p_i \tag{1.12b}$$

Suppose

$$p_i \equiv P(x_i)$$

for some predicate P where the x_i are members of a finite domain of discourse D. What then if D is *not* finite?

$$\neg\forall x P(x) \equiv \exists x \neg P(x) \tag{1.13a}$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x) \tag{1.13b}$$

Equations 1.13 are known as *De Morgan's Laws for Quantifiers.*

1.4.4 Nested Quantifiers

Consider the n-ary predicate

$$P(x_1, x_2, \ldots, x_n)$$

As discussed previously, this statement becomes a proposition if and only if all the free variables are either *bound* or *set* to a value. Suppose we bind (or set) $s < n$ of the free variables. The result is an $(s-n)$-ary predicate. Thus, the act of *nesting* quantifiers is a way in which a predicate can be made into either a proposition or a predicate of lower *arity.* The total number of nested quantifiers may not exceed the arity of the predicate it quantifies.