Chapter V
Special cases

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1 Functions of Eigenvalues

5.2.11 Semidefinite complementarity Suppose matrices X and Y lie in $\mathbb{S}^n_+$.

- If $\mathrm{Tr}(XY) = 0$, prove $-Y \in \partial\delta_{\mathbb{S}^n_+}(X)$.
- Hence prove the following properties are equivalent:
  1. $\mathrm{Tr}(XY) = 0$.
  2. $XY = 0$.
  3. $XY + YX = 0$.
- Prove for any matrices U and V in $\mathbb{S}^n$

$$(U^2 + V^2)^{\frac{1}{2}} = U + V \iff U, V \succeq 0, \mathrm{Tr}(UV) = 0.$$

Proof:

- Note that $\langle -Y, Z - X\rangle = \langle -Y, Z\rangle = -\langle Y, Z\rangle \leq 0$ for all $Z \in \mathbb{S}^n_+$. Thus, $-Y \in \partial\delta_{\mathbb{S}^n_+}(X)$.
- Suppose $\mathrm{Tr}(XY) = 0$ and hence $-Y \in \partial\delta_{\mathbb{S}^n_+}(X)$. Thus, we have $\lambda(-Y) \in \delta_{\mathbb{R}^+_n}(\lambda(X))$ and so $\langle \lambda(-Y), \lambda(X)\rangle = 0$. Thus, $\mathrm{Tr}(X(-Y)) = \langle \lambda(-Y), \lambda(X)\rangle = 0$. Thus, X and $-Y$ have common spectral decomposition. But, $\lambda(X) \geq 0$ and $\lambda(-Y) \leq 0$ and hence $\langle \lambda(-Y), \lambda(X)\rangle$ along with the fact that X and $-Y$ have common spectral decomposition implies $-XY = 0$. The rest is clear.
- Suppose $(U^2 + V^2)^{\frac{1}{2}} = U + V$ then since $U^2 + V^2 \succeq (U^2)^{\frac{1}{2}}$, we should have $U + V = (U^2 + V^2)^{\frac{1}{2}} \succeq (U^2)^{\frac{1}{2}} \succeq U$. Note that if $U = Q\,\mathrm{Diag}(\lambda)Q^T$ for some $\lambda \in \mathbb{R}^n$ and some $Q \in O(n)$, then $(U^2)^{\frac{1}{2}} = Q\,\mathrm{Diag}(|\lambda|)Q^T$. Hence, $V \succeq 0$ and similarly $U \succeq 0$. Now since $(U^2 + V^2)^{\frac{1}{2}} = U + V$ we have $U^2 + V^2 = U^2 + V^2 + UV + VU$ and hence $\mathrm{Tr}(UV) = 0$. The other way is clear.