Chapter I
Background

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1 Euclidean Spaces

1.1.1

Prove the intersection of an arbitrary collection of convex sets is convex. Deduce that the convex hull of a set $D \subseteq \mathbb{E}$ is well-defined as the intersection of all convex sets containing D.

Proof: Let C_i, $i \in \mathcal{I}$ be a collection of convex sets. Then for all $x, y \in \cap_{i\in\mathcal{I}} C_i$, and all $\lambda \in [0,1]$, we have

$$\lambda x + (1-\lambda)y \in C_i \quad \forall i \in \mathcal{I}, \text{ since } C_i \text{ is convex and } x, y \in C_i.$$

Thus $\cap_{i\in\mathcal{I}} C_i$ is convex. The rest is clear.

1.1.2

- Prove that if the set $C \subseteq \mathbb{E}$ is convex and if

$$x^1, \cdots, x^m \in C, 0 \leq \lambda_!, \cdots, \lambda_m \in \mathbb{R},$$

  and $\sum \lambda_i = 1$ then $\sum \lambda_i x^i \in C$. Prove, furthermore, that if $f : C \to \mathbb{R}$ is a convex function then $f(\sum \lambda_i x^i) \leq \sum \lambda_i f(x^i)$.

- We know that $-\log$ is convex. Deduce, for any strictly positive reals $x^1, \cdots, x^m$, and any nonnegative reals $\lambda_1, \cdots, \lambda_m$ with sum 1, the *arithmetic-geometric* mean inequality

$$\prod_i (x^i)^{\lambda_i} \leq \sum_i \lambda_i x^i.$$

- Prove that for any set $D \subseteq \mathbb{E}$, *convD* is the set of all convex combinations of elements of D.

Proof:

- Obvious induction.

- Since $-\log$ is convex, we have

$$-\log(\sum_i \lambda_i x^i) \leq \sum \lambda_i(-\log(x^i)) \Rightarrow \sum \lambda_i(\log(x^i)) \leq \log(\sum_i \lambda_i x^i).$$

  So,

$$\log(\prod_i (x^i)^{\lambda_i}) \leq \log(\sum_i \lambda_i x^i) \Rightarrow \prod_i (x^i)^{\lambda_i} \leq \sum_i \lambda_i x^i..$$

- Easy.

1.1.3

Prove that a convex set $D \subseteq \mathbb{E}$ has convex closure, and deduce that $cl(\text{conv}\, D)$ is the smallest closed convex set containing D.

Proof:

Let $x, y \in cl(D)$ and suppose $x_i \to x$ and $y_i \to y$ with $x_i, y_i \in D$. Then for $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y = \lim_i \lambda x_i + (1-\lambda)y_i.$$

Thus $\lambda x + (1-\lambda)y$ belongs to the closure of D, and thus $cl(D)$ is convex.

Now for $D \subseteq \mathbb{E}$, if C is the smallest closed convex set containing it, then $cl(conv(D)) \subseteq C$ as first C is convex and contains D and so contains $conv(D)$, also C is closed and hence contains $cl(conv(D))$. On the other hand, since $conv(D)$ is convex, $cl(conv(D))$ is also convex. However, C is the smallest closed convex set containing D, and therefore $C = cl(conv(D))$.

1.1.4. Randstorm cancellation
Suppose sets $A, B, C \subseteq \mathbb{E}$ satisfy

$$A + C \subseteq B + C.$$

If A, B are convex, B is closed, and C is bounded, prove

$$A \subseteq B.$$

Show this result can fail if B is not convex.

Proof: Since A is convex we have $2A = A + A$. In fact, $A \subseteq \frac{1}{2}(A + A)$ as $a = \frac{1}{2}(a + a)$. On the other hand, for $a, b \in A$, $\frac{1}{2}(a + b) \in A$, by definition of convexity. Similarly, $2B = B + B$. Thus, we have

$$2A + C = A + A + C = A + (A + C) \subseteq A + (B + C) = (A + C) + B \subseteq (B + C) + B = 2B + C$$

.

By induction,

$$nA + C \subseteq nB + C \ \forall n \in \mathbb{N}.$$

Now, suppose $a \in A$. Then there exist $b_n \in B$ and $c_n \in C$ such that $na = nb_n + c_n$. Since, C is bounded, we can assume there exists a subsequence c_{n_k} of c_n such that c_{n_k} converges. Now since $a = b_{n_k} + \frac{1}{n_k} c_{n_k}$. Since, c_{n_k} is convergent and so bounded, we deduce, $\lim_k \frac{1}{n_k} c_{n_k} = 0$. So, $a = \lim_k b_{n_k}$. But, B is closed and so $\lim_k b_{n_k}$, if it exists, belongs to B. Hence, $a \in B$ and so $A \subseteq B$.

Now let $A = \{\frac{1}{2}\}, B = \{0, 1\}, C = [0, 1]$. Then $A + C = [\frac{1}{2}, \frac{3}{2}], B + C = [0, 2]$. So, $A + C \subseteq B + C$ and also $A \not\subseteq B$.

1.1.5 Strong separation
Suppose that the set $C \subseteq \mathbb{E}$ is closed and convex, and that the set $D \subseteq \mathbb{E}$ is compact and convex.

- Prove the set $D - C$ is closed and convex.
- Deduce that if in addition D and C are disjoint then there exists a nonzero element a in $\mathbb{E}$ with $\inf_{x \in D} \langle a, x \rangle > \sup_{y \in C} \langle a, y \rangle$. Interpret geometrically.
- Show part (b) fails for the closed convex sets in $\mathbb{R}^2$,

$$D = \{x : x_1 > 0, x_1 x_2 \geq 1\}$$
$$C = \{x : x_2 = 0\}.$$

Proof:

- Note that for $d_1, d_2 \in D$ and $c_1, c_2 \in C$, and $\lambda \in [0, 1]$,

  $$\lambda(d_1 - c_1) + (1 - \lambda)(d_2 - c_2) = (\lambda d_1 + (1 - \lambda) d_2) - (\lambda c_1 + (1 - \lambda) c_2) \in D - C.$$

  Thus, $D - C$ is convex.

  Now, let $d_i \in D$ and $c_i \in C$ such that $d_i - c_i \to x$. We wish to prove that $x \in D - C$. Since D is compact, we may assume $d_i \to d \in D$. So c_i converges to some $c \in \mathbb{E}$. Now since C is closed we have $c \in C$. Thus $x = d - c$ belongs to $D - C$.

- Since $D \cap C \neq \emptyset$, we have $0 \notin D - C$ and so, due to the basic separation, there exists $a \in \mathbb{E}$ such that $\langle a, x \rangle > b > 0$ for all $x \in D - C$ and some fixed $b > 0$. So $\inf_{x \in D} \langle a, x \rangle > \sup_{y \in C} \langle a, y \rangle$.

  Geometrically, it means two disjoint closed, convex sets one of which is compact, can be separated via a hyperplane.

- Let $a \in \mathbb{R}^2$, such that $\inf_{x\in D}\langle a, x\rangle > \sup_{y\in C}\langle a, y\rangle$.. Then $\sup_{y\in C}\langle a, y\rangle = \sup_{y\in C} a_1y_1$, which since it is finite, must be equal to zero and thus $a_1 = 0$. Now,

$$\inf_{x\in D}\langle a, x\rangle = a_2x_2 > 0,$$

which is a contradiction as $x_2 \to 0$ implies $a_2x_2 \to 0$.

1.1.6. Recession cones

Consider a nonempty closed convex set $C \subseteq \mathbb{E}$. We define the *recession cone* of C by

$$0^+(C) = \{d \in \mathbb{E} : C + \mathbb{R}_+d \subseteq C\}.$$

- Prove $0^+(C)$ is a closed convex cone.
- Prove $d \in 0^+(C)$ if and only if $x + \mathbb{R}_+d \subseteq C$ for some point x in C. Show this equivalence can fail if C is not closed.
- Consider a family of closed convex sets C_γ $(\gamma \in \Gamma)$ with nonempty intersection. Prove $0^+(\cap C_\gamma) = \cap 0^+(C_\gamma)$.
- For a unit vector u in $\mathbb{E}$, prove $u \in 0^+(C)$ if and only if there is a sequence x^r in C satisfying $||x^r|| \to \infty$ and $||x^r||^{-1}x^r \to u$. Deduce C is unbounded if and only if $0^+(C)$ is nontrivial.
- If Y is a Euclidean space, the map $A : \mathbb{E} \to Y$ is linear, and $N(A) \cap 0^+(C)$ is a linear subspace, prove AC is closed. Show this result can fail without the last assumption.
- Consider another nonempty closed convex set $D \subseteq \mathbb{E}$ such that $0^+(C) \cap 0^+(D)$ is a linear subspace. Prove $C - D$ is closed.

Proof:

- Let $d_1, d_2 \in 0^+(C)$ and $\lambda > 0$, then $C + \mathbb{R}_+(\lambda d_1) = C + \mathbb{R}_+d_1 \subseteq C$. Also, $C + \mathbb{R}_+(d_1 + d_2) \subseteq C + \mathbb{R}_+d_1 + \mathbb{R}_+d_2 \subseteq C + \mathbb{R}_+d_2 \subseteq C$.
- Let $C_\infty(x) = \{d \in \mathbb{E} : x + td \in C, \forall t > 0\}$. Now let $d \in \mathbb{C}_\infty(x)$ and also fix $\bar{y} \in C$. We wish to show that $\bar{y} + d \in C$. Since $d \in C_\infty(x)$ for every $\bar{t} > 0$ we have $x + \bar{t}d \in C$. Thus, for $\lambda > 0$ we have

  $$\bar{y}_\lambda = \lambda\bar{y} + (1-\lambda)(x + \frac{1}{1-\lambda}d) \in C,$$

  as C is convex. But $\bar{y}_\lambda = \lambda\bar{y} + (1-\lambda)x + d \in C$. Clearly, $\lim_{\lambda\to 1^-}\bar{y}_\lambda = \bar{y} + d$ and since C is closed we conclude that $\bar{y} + d \in C$. Thus, $C_\infty(x) \subseteq 0^+(C)$.

  Conversely, let $d \in 0^+(C)$. Then, by definition, $x \in C_\infty(x)$.

  Example: Take $C = \{(x, y) : y > 0\} \cup \{(0, 0)\}$.
- Let $x \in \cap C_\gamma$. Then $d \in 0^+(\cap C_\gamma)$ if and only if $x + \mathbb{R}_+d \subseteq \cap C_\gamma$, and this holds, if and only if $x + \mathbb{R}_+d \subseteq C_\gamma$ for all $\gamma \in \Gamma$ or equivalently $d \in 0^+(C_\gamma)$ for all $\gamma \in \Gamma$.
- Let $x \in C$ and $u \in 0^+(C)$. Then $x^r := x + ru \in C$ for $r \in \mathbb{N}$. Note that $\langle x^r, u\rangle = \langle x, u\rangle + r$ and thus $||x^r|| \to +\infty$. We have

  $$\lim_r \frac{x^r}{||x^r||} = \lim_r \frac{x^r||x^r||}{||x^r||^2} = \lim_r \frac{x^r||x^r||}{r^2 + 2r\langle x, u\rangle + ||x||^2} = \lim_r \frac{x^r||x^r||}{r^2} = \lim_r (x/r + u)||x/r + u|| = u.$$

  Conversely, suppose $u^r := ||x^r||^{-1}x^r \to u$ for some $||x^r|| \to +\infty$. Now fix $t \geq 0$,

  $$x + tu = x + t\lim_r ||x^r||^{-1}x^r = \lim_r (1 - t||x^r||^{-1})x + \lim_r t||x^r||^{-1}x^r = \lim_r [(1 - t||x^r||^{-1})x + t||x^r||^{-1}x^r].$$

  But, $(1 - t||x^r||^{-1})x + t||x^r||^{-1}x^r \in C$ and thus the above limit lies in C as C is closed.

  Now if C is bounded then there is no such sequence x^r in C and hence $0^+(C)$. Now suppose that C is unbounded and thus there is a sequence $x^r \in C$ with $||x^r|| \to +\infty$. By passing to a subsequence, we can suppose $||x^r||^{-1}x^r \to u$ for some $||u|| = 1$.

- Define $L = N(A) \cap 0^+(C)$. Let $c_i \in C$ and $y_i := Ac_i \to y$. Note that if $||c_i||$ is bounded then, passing to a subsequence, we can assume $c_i \to c$ and since C is closed we have $c \in C$. Hence, $Ac_i \to Ac \in AC$. Thus, suppose that $||c_i|| \to +\infty$ and also $||c_i||^{-1}c_i \to u$ for some $||u|| = 1$. Then, according to the above part, $u \in 0^+(C)$. Now note that

$$Au = \lim_i ||c_i||^{-1}Ac_i = 0 \text{ as } Ac_i \text{ is bounded and } ||c_i|| \to +\infty.$$

  Thus, $u \in N(A) \cap 0^+(C) = L$. Hence, if $L = \{0\}$, AC will be closed.

  Note that $C + L \subseteq C + 0^+(C) \subseteq C$. Define $\tilde{C} := C \cap L^\perp$. Then $\tilde{C}$ is nonempty as for $c \in C$, write $c = c' + c''$ wherein $c' \in L, c'' \in L^\perp$. Then, $c'' = c - c' \in C + L \subseteq C$. Hence, $c'' \in C \cap L^\perp \neq \emptyset$. Note that we also proved that $C \subseteq \tilde{C} + L$. Note that also $C = \tilde{C} + L$ as in fact, $\tilde{C} + L \subseteq C + L \subseteq C$, thus $C = \tilde{C} + L$. However, $AC = A\tilde{C} + AL = A\tilde{C}$.

  Further, $\tilde{C}$ is closed and convex as it is the intersection of two closed convex sets in $\mathbb{E}$. Note that $0^+(\tilde{C}) \subseteq 0^+(C)$ as $\tilde{C} \cap C = \tilde{C}$ and thus $0^+(\tilde{C}) = 0^+(\tilde{C}) \cap 0^+(C)$ and so $0^+(\tilde{C}) \subseteq 0^+(C)$.

  Now, we claim that $0^+(\tilde{C}) \cap N(A) = \{0\}$. In fact, let $d \in 0^+(\tilde{C}) \cap N(A)$, then $d \in 0^+(\tilde{C}) \cap N(A) \subseteq 0^+(C) \cap N(A) = L$. Let $c \in \tilde{C}$, then $c + d \in \tilde{C} \subseteq L^\perp$ and so $d \in L^\perp - c \subseteq L^\perp - L^\perp$. Hence, $d \in L^\perp$ and therefore $d \in L \cap L^\perp = \{0\}$. Now according to the above discussion $A\tilde{C}$ is closed. However, we have $AC = A\tilde{C}$.

- Let $A : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ with $A(x, y) = x - y$. Then $A(C \times D) = C - D$. However, $N(A) = \{(x, x) : x \in \mathbb{E}\}$ and since $0^+(C \times D) = 0^+(C) \times 0^+(D)$, we have $N(A) \cap 0^+(C \times D) = 0^+(C) \cap 0^+(D) = \{0\}$. Thus, based on the previous part, $A(C \times D) = C - D$ is closed.

1.1.7

For any set of vectors $a^1, \cdots, a^m$ in $\mathbb{E}$, prove the function $f(x) = \max_i \langle a^i, x \rangle$ is convex on $\mathbb{E}$.

Proof: We prove if $f : \mathbb{E} \to (-\infty, +\infty]$ are convex then $f(x) = \max f_i(x)$ is convex. Then for $x, y \in \mathbb{E}$ and $\lambda \in [0, 1]$ we have

$$f_i(\lambda x + (1 - \lambda)y) \leq \lambda f_i(x) + (1 - \lambda)f_i(y) \leq \lambda f(x) + (1 - \lambda)f(y) \ \forall i.$$

Thus $f(x) = \max_i f_i(x) \leq \lambda f(x) + (1 - \lambda)f(y)$.

Note that $f_i(x) = \langle a^i, x \rangle$ is obviously linear.

1.1.8

Prove the Weiestrass theorem: Suppose that the set $D \subseteq \mathbb{E}$ is nonempty and closed and that all the level sets of the continuous function $f : D \to \mathbb{R}$ are bounded. Then f has a global minimizer.

Proof:

Let $\alpha \in \mathbb{R}$ such that $\{x \in D : f(x) \leq \alpha\}$ is nonempty. Then there exists $r > 0$ such that $\{x \in D : f(x) \leq \alpha\} \subseteq B_r$. However, $\{x \in D : f(x) \leq \alpha\} = D \cap f^{-1}(-\infty, \alpha]$ is closed and thus it is compact as well. Now if f is unbounded below then there exists $x_i \in D$ such that $f(x_i) \to -\infty$. Then $x_i \in \{x \in D : f(x) \leq \alpha\} = D \cap f^{-1}(-\infty, \alpha]$ and thus $x_i \to x$ and hence $f(x_i) \to f(x)$ and so $f(x) = \infty$. which is a contradiction. So f is bounded below and so if $c = \inf_{x \in \mathbb{E}}$ then there exists $f(x_i) \to c$. Let $x_i \to x$ and hence $f(x) = c$. x is a global minimizer.

1.1.10. Convex growth conditions

- Find a function with bounded level sets which does not satisfy the growth condition:

$$\liminf_{||x|| \to +\infty} \frac{f(x)}{||x||} > 0.$$

- Prove that any function satisfying the above condition has bounded level sets.

- Suppose the convex function $f : C \to \mathbb{R}$ has bounded level sets but the growth condition fails. Deduce the existence of a sequence (x^m) in C with $f(x^m) \le ||x^m||/m \to +\infty$. For a fixed point $\bar{x}$ in C, derive a contradiction by considering the sequence

$$\bar{x} + \frac{m}{||x^m||}(x^m - \bar{x}).$$

Hence, prove that for a convex function f, it has bounded level sets if and only if it satisfies the growth condition.

Proof:

- Let $f(x) = x^3$ for $f : \mathbb{R} \to \mathbb{R}$.
- Assume that f satisfies the growth condition and does not have bounded level sets. Then there exists $x_1, x_2, \cdots$ such that $f(x_i) \le M$ for some $M > 0$ and $||x_i|| > i$. But then,

$$0 < \liminf_{||x|| \to +\infty} \frac{f(x)}{||x||} \le \lim_i \frac{f(x_i)}{||x_i||} \le 0.$$

- If f does not satisfy the growth condition then

$$\liminf_{||x|| \to +\infty} \frac{f(x)}{||x||} \le 0,$$

and hence there exists x^m with $||x^m|| \ge m^2$ such that $f(x^m) \le ||x^m||/m$. Hence, $f(x^m) \le ||x^m||/m \to +\infty$. We have

$$f(\bar{x} + \frac{m}{||x^m||}(x^m - \bar{x})) \le \frac{m}{||x^m||} f(x^m) + (1 - \frac{m}{||x^m||}) f(\bar{x}) \le 1 + (1 - \frac{m}{||x^m||}) f(\bar{x}) \le 1 + |f(\bar{x})|.$$

However, $\bar{x} + \frac{m}{||x^m||}(x^m - \bar{x})$ is not bounded as in fact $||\frac{m}{||x^m||}(x^m - \bar{x})|| = m$. Hence, we have proved that if f has bounded level sets then f satisfies the growth condition. We proved the opposite for general functions.

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2 Symmetric Matrices

1.2.1

Prove $\mathbb{S}^n_+$ is a closed convex cone with interior $\mathbb{S}^n_{++}$.

Proof: It is clear that $\mathbb{S}^n_+$ is convex and a cone. However, let $X \notin \mathbb{S}^n_+$ then $\lambda_{\min}(X) < 0$ with $x^T X x \le -\delta$ for some $||x|| = 1$. Then let $A \in \mathbb{S}^n$ with $||A|| \le \frac{1}{2}\delta$ then

$$x^T(X + A)x \le x^T X x + ||x|| ||Ax|| = x^T X x + ||Ax|| \le x^T X x + ||A|| \le -\delta + \frac{1}{2}\delta < 0.$$

Thus, $A + X$ can't be positive semidefinite.

On the other hand, let $X \in \mathbb{S}^n_{++}$, then $S^n \to \mathbb{R}$ defined by $x \mapsto x^T X x$ attains its minimum and hence there exists $\delta > 0$ such that $x^T X x \ge \delta$.

Now let $A \in \mathbb{S}^n$ such that $||A|| \le \frac{1}{2}\delta$ then for all x with $||x|| = 1$ we have

$$x^T(X + A)x \le \delta - ||x|| ||A|| ||x|| \ge \delta - \frac{1}{2}\delta \ge \frac{1}{2}\delta.$$

Hence, $X + A \succ 0$.

1.2.1

Explain why $\mathbb{S}^2_+$ is not a polyhedron.

Proof: Suppose that $\mathbb{S}^2_+ = \{x \in \mathbb{R}^3 : Ax \geq b\}$. Then since $\mathbb{S}^2_+$ is a cone we have $b = 0$. Let $A = [A_{11} : A_{12} : A_{21} : A_{22}]$, then $A_{11}, A_{22} \geq 0$. But $A_{12} + A_{21}$ is not less than 0 as $e_{12} + e_{21}$ does not belong to $\mathbb{S}^2_+$.

1.2.4. A nonlattice ordering

Suppose the matrix Z in $\mathbb{S}^2$ satisfies

$$W \succeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } W \succeq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \iff W \succeq Z.$$

- By considering diagonal W, prove
$$Z = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$
for some real a.

- By considering $W = I$, prove $Z = I$.

- Derive a contradiction by considering
$$W = \frac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Proof:

- Let
$$W = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$
Then $x, y \geq 1$ if and only if $W \succeq Z$. Thus, $W \succeq I$ if and only if $W \succeq Z$. Hence, $I \succeq Z$ as well as $Z \succeq I$. Hence, $Z = I$.

- However,
$$W = \frac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \succeq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, W = \frac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
Hence, $W \succeq I$ which is incorrect.

1.2.5 Order preservation

- Prove any matrix X in $\mathbb{S}^n$ satisfies $(X^2)^{\frac{1}{2}} \succeq X$.

- Find matrices $X \succeq Y$ in $\mathbb{S}^2_+$ such that $X^2 \not\succeq Y^2$.

- For matrices $X \succeq Y$ in $\mathbb{S}^n_+$, prove $X^{\frac{1}{2}} \succeq Y^{\frac{1}{2}}$.

Proof:

- Let $X = \sum_i \lambda_i u_i u_i^T$ with u_i form an orthogonal basis for $\mathbb{R}^n$. $X^2 = \sum_i \lambda_i^2 u_i u_i^T$, and hence $(X^2)^{\frac{1}{2}} = \sum_i |\lambda_i| u_i u_i^T$. However, since $|\lambda_i| \geq \lambda_i$, we have $\sum_i |\lambda_i| u_i u_i^T \geq \sum_i \lambda_i u_i u_i^T$.

- Let
$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = Y.$$
However,
$$X^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = Y^2.$$

- Let v be an eigenvector of $X^{\frac{1}{2}} - Y^{\frac{1}{2}}$ with $(X^{\frac{1}{2}} - Y^{\frac{1}{2}})v = \lambda v$. Then
$$\langle (X^{\frac{1}{2}} + Y^{\frac{1}{2}})v, (X^{\frac{1}{2}} - Y^{\frac{1}{2}})v \rangle = \langle (X - Y)v, v \rangle \geq 0.$$
However, $\langle (X^{\frac{1}{2}} + Y^{\frac{1}{2}})v, (X^{\frac{1}{2}} - Y^{\frac{1}{2}})v \rangle = \lambda v^T (X^{\frac{1}{2}} + Y^{\frac{1}{2}})v$. If $(X^{\frac{1}{2}} + Y^{\frac{1}{2}})v = 0$ then $2X^{\frac{1}{2}}v = (X^{\frac{1}{2}} + Y^{\frac{1}{2}})v + (X^{\frac{1}{2}} - Y^{\frac{1}{2}})v = \lambda v$. Hence, $\lambda \geq 0$. Thus, $v^T (X^{\frac{1}{2}} + Y^{\frac{1}{2}})v > 0$.

1.2.6. Square-root iteration
Suppose a matrix A in $\mathbb{S}^n_+$ satisfies $I \succeq A \succeq 0$. Prove that the iteration
$$Y_0 = 0, Y_{n+1} = \frac{1}{2}(A + Y_n^2) \ (n = 0, 1, \cdots)$$
is nondecreasing and converges to the matrix $I - (I - A)^{\frac{1}{2}}$.

Proof: Note that A, Y_0 commute and Y_1 is a polynomial in A and so so forth. Hence, there exists $Q \in O(n)$ such that $Q^T Y_i Q = D_i$ is a diagonal matrix for $i = 0, 1, \cdots, T+1$. Now the below argument complete the proof.

Consider $x_0 = 0$ and $x_{n+1} = \frac{1}{2}(a + x_n^2)$ with $1 \geq a \geq 0$. First, note that $0 \leq x_n \leq 1$ for all n obviously, simple induction.

Then $x_{n+1} - x_n = \frac{1}{2}(a + x_n^2 - 2x_n) = \frac{1}{2}(a + (x_n - 1)^2 - 1)$. Thus, $x_{n+1} \geq x_n$ if and only if $(1 - x_n)^2 \geq 1 - a$ which holds if and only if $(1 - x_n) \geq \sqrt{1-a}$. So assume $1 - \sqrt{1-a} \geq x_t \geq 0$, them $x_{t+1} \leq \frac{1}{2}(a + (1 - \sqrt{1-a})^2) = \frac{1}{2}(a + 1 + 1 - a - 2\sqrt{1-a}) = 1 - \sqrt{1-a}$.

Now let $Y_n \to Y$, then $2Y = A + Y^2$. Hence, $I - A = (Y - I)^2$ and thus $\sqrt{I - A} = I - Y$.

1.2.14 Level sets of perturbed log barriers

- For δ in $\mathbb{R}_{++}$, prove the function
$$t \in \mathbb{R}^n_{++} \to \delta t - \log t$$
has compact level sets.

- For c in R^n_{++}, prove the function
$$x \in \mathbb{R}^n_{++} \mapsto c^T x - \sum_{i=1}^n \log x_i$$
has compact level sets.

- For C in S^n_{++}, prove the function
$$X \in \mathbb{S}^n_{++} \mapsto \langle C, X \rangle - \log \det X$$
has compact level sets.

Proof:

- Since $\delta t - \log t = \delta t - \log \delta t + \log \delta$, without loss of generality suppose, $\delta = 1$. We need to show $\{t \in \mathbb{R}_{++} : t - \log t \leq c\}$ is bounded. If not, $\exists t_n \to +\infty$, s.t. $t_n - \log t_n \leq c$ for some constant c. $\frac{t_n}{\log t_n} - 1 \leq \frac{c}{\log t_n}$. However, $\lim_{t \to +\infty} \frac{t}{\log t} = +\infty$. This contradiction completes the proof.

- Note that $\sum c_i x_i - \sum \log x_i = \sum c_i x_i - \sum \log c_i x_i + \sum \log c_i$, without loss of generality, suppose that $c_i = 1$. But $t - \log t \geq 0$ for all $t > 0$. In fact, for $0 \leq t \leq 1$, $t - \log t \geq t > 0$. Also for $t \geq 1$, $(t -] \log t)' = 1 - \frac{1}{t} \leq 0$ and thus $t - \log t$ is nondecreasing on $t \geq 1$. Note that $t - \log t|_{t=1} = 0$. Now, $t_1 = \log t_1 \leq \sum t_i - \log t_i \leq c$. Thus, to the previous part, t_1 is bounded above. This completes the proof.

- Let $\mu(C)_i = \lambda(C)_{n+1-i}$, then $\mu(C)^T \lambda(X) \leq \langle C, X \rangle$. Thus, $\mu(C)^T \lambda(X) - \sum \log \lambda_i(X) \leq c$.Hence, $\lambda(X)$ is upperbounded and so is $||X|| = \sqrt{\lambda_i(X)^{\frac{1}{2}}}$.
