Exercise 10.8 (Reversal of order of integration in forward rates)

Forward rate formula is given by

$$f(t,v) = f(0,v) + \int_0^t \alpha(u,v)\mathrm{d}u + \int_0^t \sigma(u,v)\mathrm{d}W(u)$$

Show that

$$\mathrm{d}\left(-\int_t^T f(t,v)\mathrm{d}v\right) = R(t)\mathrm{d}t - \alpha^*(t,T)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W(t)$$

Here

$$\alpha^*(t,T) = \int_t^T \alpha(t,v)\mathrm{d}v$$
$$\sigma^*(t,T) = \int_t^T \sigma(t,v)\mathrm{d}v$$

Proof

We begin by noting that

$$\begin{aligned}
-\int_t^T f(t,v)\mathrm{d}v &= -\int_t^T f(0,v)\mathrm{d}v - \int_t^T \left[\int_0^t \alpha(u,v)\mathrm{d}u + \int_0^t \sigma(u,v)\mathrm{d}W(u)\right]\mathrm{d}v \\
&= -\int_t^T f(0,v)\mathrm{d}v - \int_0^t \underbrace{\int_t^T \alpha(u,v)\mathrm{d}v}_{:=\hat{\alpha}(t,T,u)}\,\mathrm{d}u - \int_0^t \underbrace{\int_t^T \sigma(u,v)\mathrm{d}v}_{:=\hat{\sigma}(t,T,u)}\,\mathrm{d}W(u) \\
&= -\int_t^T f(0,v)\mathrm{d}v - \int_0^t \hat{\alpha}(t,T,u)\mathrm{d}u - \int_0^t \hat{\sigma}(t,T,u)\mathrm{d}W(u)
\end{aligned}$$

Next, we take derivative with respect to t from both sides. Recall the following formula from elementary calculus.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(x,t)\mathrm{d}t = f(x,g(x))\cdot g'(x) + \int_0^{g(x)} \frac{\mathrm{d}f(x,t)}{\mathrm{d}x}\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t} - \int_t^T f(t,v)\mathrm{d}v &= \frac{\partial}{\partial t}\int_T^t f(t,v)\mathrm{d}v \\
&= \frac{\partial}{\partial t}\int_0^t f(t,v)\mathrm{d}v - \frac{\partial}{\partial t}\int_0^T f(t,v)\mathrm{d}v \\
&= f(t,t) + \int_0^t \frac{\partial}{\partial t} f(t,v)\mathrm{d}v - \int_0^T \frac{\partial}{\partial t} f(t,v)\mathrm{d}v \\
&= f(t,t) - \int_t^T \frac{\partial}{\partial t} f(t,v)\mathrm{d}v
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^t \hat{\alpha}(t,T,u)\mathrm{d}u &= \hat{\alpha}(t,T,t) + \int_0^t \frac{\partial}{\partial t}\hat{\alpha}(t,T,u)\mathrm{d}u \\
&= \int_t^T \alpha(t,v)\mathrm{d}v + \int_0^t \frac{\partial}{\partial t}\hat{\alpha}(t,T,u)\mathrm{d}u \\
&= \int_t^T \alpha(t,v)\mathrm{d}v - \int_0^t \frac{\partial}{\partial t}\int_0^t \alpha(u,v)\mathrm{d}v\mathrm{d}u \\
&= \int_t^T \alpha(t,v)\mathrm{d}v - \int_0^t \frac{\partial}{\partial t}\int_0^t \alpha(u,v)\mathrm{d}v\mathrm{d}u \\
&= \int_t^T \alpha(t,v)\mathrm{d}v - \int_0^t \alpha(u,t)\mathrm{d}u
\end{aligned}$$