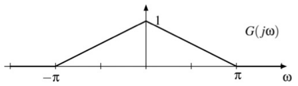
(d) (4 points) Assume the signal we are sampling has a Fourier transform



Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$

Solution: The sampled signal is $f(t)g(t)$, which has a Fourier transform

$$\begin{aligned} G_s(j\omega) &= \frac{1}{2\pi} F(j\omega) * G(j\omega) \\ &= \frac{1}{2\pi}\left(\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau})\right) * \Delta(\omega/\pi) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \Delta\left(\frac{(\omega - n\pi)}{\pi}\right)(1 + (-1)^n e^{-jn\pi\tau}) \end{aligned}$$

We are interested in the baseband replica (n=0) , and the replicas at $\pm\pi(n = \pm 1)$. For $n = 0$,

$$G_{s,0}(j\omega) = \frac{1}{2}\Delta\left(\frac{\omega}{\pi}\right)(1+1) = \Delta\left(\frac{\omega}{\pi}\right)$$

which is the same as $G(j\omega)$. For $n = 1$,

$$G_{s,1}(j\omega) = \frac{1}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)(1 + (-1)^1 e^{-j\pi\tau}) = \Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - e^{-j\pi\tau})$$

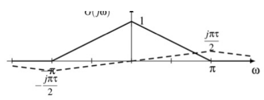
at $\omega = \pi$, if we approximate $e^{j\omega\tau} \simeq 1 + j\pi\tau$

$$\begin{aligned} G_{s,1} &= \Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - (1 - j\pi\tau)) \\ &= \frac{j\pi\tau}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right) \end{aligned}$$

This is a replica of $G(j\omega)$ centered at $\omega = \pi$, multiplied by $j\pi\tau/2$. It is imaginary, and proportional to τ that as τ goes to zero, this replica disappears as we'd expect.
For $n = -1$ we get the same type of term, but with the negative sign,

$$G_{s,-1}(j\omega) = \frac{-j\pi\tau}{2}\Delta\left(\frac{\omega + \pi}{\pi}\right)$$

This is a replica of $G(j\omega)$ centered at $\omega = -\pi$, and scaled by $-j\pi\tau/2$.
If we sketch these three terms, the result is as shown below.



(e) (4 points) If we know $g(t)$ is real and even, can we recover $g(t)$ from the non-uniform samples $g(t)f(t)$?

Solution: We know that in the limit at τ goes to zero, that we can perfectly reconstruct $g(t)$, since we will then be sampling at the Nyquist rate. From the answer to the previous part, this does indeed happen. The replicas at $\pm\pi$ are proportional to τ , and will go to zero. Looking at the solution for the previous part, we can see that the part of the spectrum we want is all in the real component. If $g(t)$ is real and even, then $G(j\omega)$ is real and even. Hence, if we lowpass filter, and take the real part of the spectrum, we can recover $G(j\omega)$. .

(18 points) **Sampling with alternating impulse train**

The figure shown below gives a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.

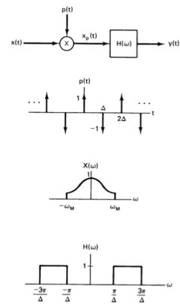


Figure 1: Sampling with alternating impulse train

(a) (6 points) For $\Delta < \frac{\pi}{2\omega_m}$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

Solution:

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x(t)\delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} x(t)\delta(t - \Delta - 2\Delta n) \\ &= x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} \delta(t - \Delta - 2\Delta n)\right] \end{aligned}$$

Then, by convolution theorem

$$\begin{aligned} X_p(\omega) &= \left[\frac{1}{2\pi}X(\omega) * \frac{2\pi}{2\Delta}\sum_{n=-\infty}^{\infty}\delta(\omega - n\frac{2\pi}{2\Delta})\right] - \left[\frac{1}{2\pi}X(\omega) * \frac{2\pi}{2\Delta}\sum_{n=-\infty}^{\infty}\delta(\omega - n\frac{2\pi}{2\Delta})e^{-j\omega\Delta}\right] \\ &= X(\omega) * \left[\frac{1}{2\Delta}\sum_{n=-\infty}^{\infty}(1 - e^{-j\pi n})\delta(\omega - n\frac{\pi}{\Delta})\right] \\ &= X(\omega) * \left[\frac{1}{2\Delta}\sum_{n=-\infty}^{\infty}(1 - (-1)^n)\delta(\omega - n\frac{\pi}{\Delta})\right] \end{aligned}$$

$X_p(\omega)$ and $Y(\omega)$ are sketched below

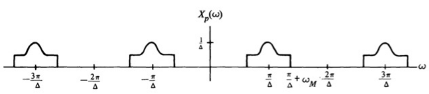


Figure 2: $X_p(\omega)$

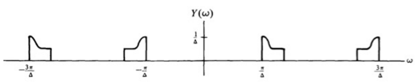


Figure 3: $X_p(\omega)$

(b) (4 points) For $\Delta < \frac{\pi}{2\omega_m}$, determine a system that will recover $x(t)$ from $x_p(t)$.

Solution:

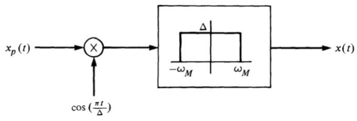


Figure 4: System for recovering $x(t)$ from $x_p(t)$

(c) (4 points) For $\Delta < \frac{\pi}{2\omega_m}$, determine a system that will recover $x(t)$ from $y(t)$.

Solution:

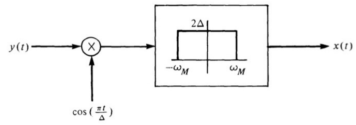


Figure 5: System for recovering $x(t)$ from $y(t)$

(d) (4 points) What is the maximum value of Δ in relation to ω_m for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$.

Solution: Δ is maximum when $\frac{\pi}{\Delta}$ is minimum. From part (a) we can clearly see that aliasing is avoided in $X_p(\omega)$ if $\omega_m \leq \frac{\pi}{\Delta}$. Hence, $\Delta_{max} = \frac{\pi}{\omega_m}$

4. (20 points) **Laplace Transform**

(a) (10 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. (5 points) $f(t) = te^{-at}(\sin\omega_0 t)^2 u(t)$

Solution: We can equivalently write $f(t)$ as:

$$f(t) = te^{-at}(\sin\omega_0 t)^2 u(t) = te^{-at}\frac{1 - \cos(2\omega_0 t)}{2}u(t) = \frac{1}{2}te^{-at}u(t) - \frac{1}{2}te^{-at}\cos(2\omega_0 t)u(t)$$

We have the following:

$$\begin{aligned} \cos(2\omega_0 t)u(t) &\rightarrow \frac{s}{s^2 + (2\omega_0)^2} \\ t\cos(2\omega_0 t)u(t) &\rightarrow -\frac{d}{ds}\frac{s}{s^2 + (2\omega_0)^2} = -\frac{s^2 + 4\omega_0^2 - 2s^2}{(s^2 + 4\omega_0^2)^2} = \frac{s^2 - 4\omega_0^2}{(s^2 + 4\omega_0^2)^2} \\ e^{-at}t\cos(2\omega_0 t)u(t) &\rightarrow \frac{(s+a)^2 - 4\omega_0^2}{((s+a)^2 + 4\omega_0^2)^2} \end{aligned}$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{(s+a)^2} - \frac{1}{2}\frac{(s+a)^2 - 4\omega_0^2}{((s+a)^2 + 4\omega_0^2)^2}, \quad \text{Re}\{s\} > -a$$

ii. (5 points) $f(t) = e^{-b|t|}$ where $b \leq 0$

Solution: We can write $f(t) = e^{-bt}u(t) + e^{bt}u(-t)$. We know that the Laplace transform for the two parts of the function are:

$$e^{-bt}u(t) \rightarrow \frac{1}{s+b}, Re\{s\} > -b$$

and the bilateral Laplace transform of $e^{bt}u(-t)$ is

$$e^{bt}u(-t) \rightarrow \frac{-1}{s-b}, Re\{s\} < b$$

Although the Laplace transforms of each of the individual terms have a region of convergence, there is no common region of convergence as $b \leq 0$, and thus, for those values of b, $x(t)$ has no Laplace transform.

Alternative solutions: If students use unilateral Laplace transform, they still get full credits if they acknowledge the result is actually the LT of $e^{-bt}u(t)$. Unilateral LT cannot fully represent a non-causal signal. The LT of $e^{-bt}u(t)$ is $\frac{1}{s+b}$ $(Re\{s\} > -b)$

(b) (10 points) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: Re}\{s\} > -1$$

Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

i. (5 points) $\mathcal{F}\{x(t)e^{\frac{t}{2}}\}$

Solution: Let $y(t) = x(t)e^{\frac{t}{2}}$, then $Y(s) = X(s - \frac{1}{2})$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s - \frac{1}{2}\} > -1 \implies \text{Re}\{s\} > -\frac{1}{2}$$

Since the ROC of $Y(s)$ includes the $j\omega$-axis, we have:

$$Y(j\omega) = Y(s)|_{s=j\omega} = \frac{1}{(j\omega - \frac{1}{2})^2 + 2(j\omega - \frac{1}{2}) + 5}$$

ii. (5 points) $\mathcal{F}\{x(t)e^{2t}\}$

Solution: Let $y(t) = x(t)e^{2t}$, then $Y(s) = X(s-2)$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s - 2\} > -1 \implies \text{Re}\{s\} > 1$$

Since the ROC of $Y(s)$ does not include the $j\omega$-axis, we cannot determine $Y(j\omega)$ from $Y(s)$.

5. (12 points) **Inverse Laplace Transform**

Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

(a) (6 points) $F(s) = \frac{e^{-s}(s+1)}{(s-2)^2(s-3)}$

Solution: Full credit for all students irrespective of work shown.
Let us first focus on $\frac{(s+1)}{(s-2)^2(s-3)}$. It can be equivalently written as:

$$\frac{s+1}{(s-2)^2(s-3)} = \frac{r_1}{(s-2)^2} + \frac{r_2}{s-2} + \frac{r_3}{s-3}$$

Using the cover-up method,

$$\begin{aligned} r_1 &= \left.\frac{s+1}{s-3}\right|_{s=2} = \frac{3}{-1} = -3 \\ r_3 &= \left.\frac{s+1}{(s-2)^2}\right|_{s=3} = \frac{4}{1} = 4 \end{aligned}$$

Therefore,

$$\frac{s+1}{(s-2)^2(s-3)} = \frac{-3}{(s-2)^2} + \frac{r_2}{s-2} + \frac{4}{s-3}$$

Now to find r_2, we will evaluate the above equation at $s = 0$, we then have:

$$\frac{1}{-12} = \frac{-3}{4} - \frac{r_2}{2} - \frac{4}{3}$$

Then

$$\frac{r_2}{2} = \frac{1}{12} - \frac{3}{4} - \frac{4}{3} = \frac{1 - 9 - 16}{12} = \frac{-24}{12} = -2 \implies r_2 = -4$$

Therefore,

$$F(s) = e^{-s}\left(-\frac{3}{(s-2)^2} - \frac{4}{s-2} + \frac{4}{s-3}\right)$$

then

$$f(t) = \left(-3(t-1)e^{2(t-1)} - 4e^{2(t-1)} + 4e^{3(t-1)}\right)u(t-1)$$

(b) (6 points) $F(s) = \frac{s+4}{s^3 + 4s}$

Solution:

$$F(s) = \frac{s+4}{s(s^2+4)} = \frac{r_1}{s} + \frac{r_2}{s+j2} + \frac{r_3}{s-j2}$$

Using the cover-up procedure:

$$r_1 = \left.\frac{s+4}{(s^2+4)}\right|_{s=0} = \frac{4}{4} = 1$$

and

$$r_2 = \left.\frac{s+4}{s(s-j2)}\right|_{s=-j2} = \frac{-2j+4}{(-j2)(-j2-j2)} = \frac{j-2}{4}$$

Then,

$$r_3 = r_2^* = \frac{-j-2}{4}$$

We thus have

$$F(s) = \frac{s+4}{s^3+4s} = \frac{1}{s} + \frac{1}{4}\frac{j-2}{s+j2} - \frac{1}{4}\frac{j+2}{s-j2}$$

The inverse Laplace transform is:

$$f(t) = \left(1 + \frac{1}{4}(j-2)(e^{-j2t} - \frac{1}{4}(j+2)e^{j2t})\right)u(t) = (1 + 0.5\sin(2t) - \cos(2t))\,u(t)$$

Alternatively, we can find the inverse as follows:

$$F(s) = \frac{s+4}{s(s^2+4)} = \frac{r_1}{s} + \frac{As+B}{s^2+4}$$

Using the cover-up method, we can determine $r_1 = 1$ (as previously obtained). Therefore,

$$As + B = \frac{s+4}{s} - \frac{s^2+4}{s} = \frac{s - s^2}{s} = 1 - s$$

Therefore,

$$F(s) = \frac{1}{s} - \frac{s-1}{s^2+4} = \frac{1}{s} - \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$$

Therefore,

$$f(t) = \left(1 - \cos(2t) + \frac{1}{2}\sin(2t)\right)u(t)$$

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)\,g(x)) = f'(x)\,g(x) + f(x)\,g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}$, for real numbers n
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$