Signals & Systems

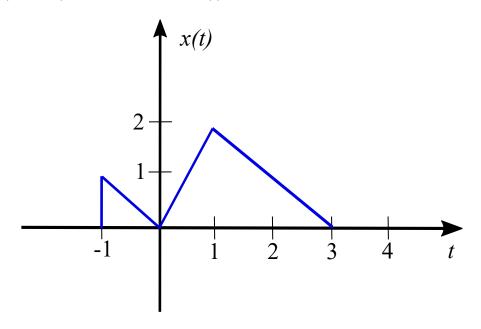
University of California, Los Angeles; Department of ECE

Prof. J.C. Kao TAs: Yang, Bruce, Shreyas

Due Monday, 23 Oct 2023, by 11:59pm to Gradescope. Covers material up to Lecture 5. 100 points total.

## 1. (22 points) Elementary signals.

(a) (9 points) Consider the signal x(t) shown below. Sketch the following:



i. 
$$y(t) = x(t) [1 - u(t-1) + u(t-2)]$$

ii. 
$$y(t) = \int_{-\infty}^{t} [\delta(\tau + 1) - \delta(\tau - 1) + \delta(\tau - 2)] x(\tau) d\tau$$

ii. 
$$y(t) = x(t) [1 - u(t-1) + u(t-2)]$$
  
iii.  $y(t) = \int_{-\infty}^{t} [\delta(\tau+1) - \delta(\tau-1) + \delta(\tau-2)] x(\tau) d\tau$   
iii.  $y(t) = x(t) + r(t+1) - u(t) - 3r(t) + 3r(t-1) - r(t-3)$ 

(b) (9 points) Evaluate these integrals:

i. 
$$\int_{-\infty}^{\infty} f(t+1)\delta(t+1)dt$$

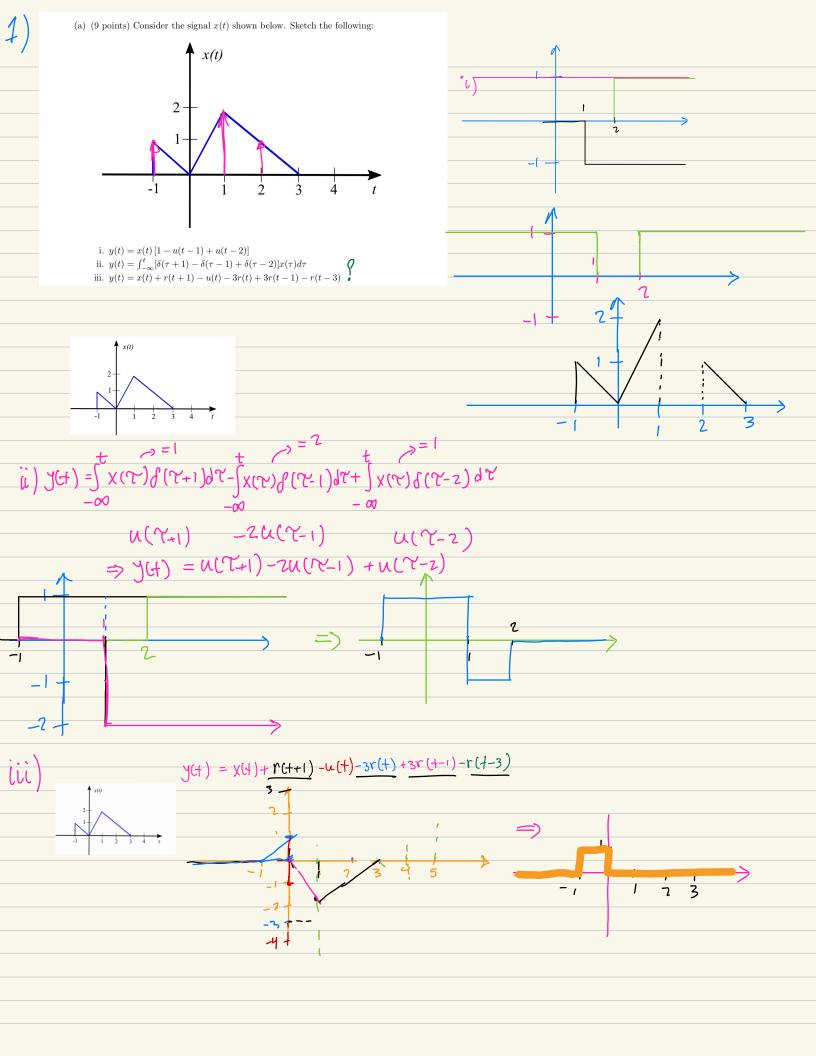
ii. 
$$\int_{t}^{\infty} e^{-2\tau} u(\tau - 1) d\tau$$

iii. 
$$\int_0^\infty f(t)(\delta(t-1)+\delta(t+1))dt$$

(c) (4 points) Let b be a positive constant. Show the following property for the delta function:

$$\delta(bt) = \frac{1}{b}\delta(t)$$

Hint: what function is "delta-like"?



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$$\int_0^\infty f(t)(\delta(t-1)+\delta(t+1))dt$$

$$i) \int_{-\infty}^{\infty} f(t+1)\delta(t+1) dt$$

$$= \int_{-\infty}^{\infty} f(u) \delta(u) du = f(0)$$

$$= \int_{-\infty}^{\infty} f(u) \delta(u) du = f(0)$$

$$\begin{cases}
f(t)f(t) dt = f(0) \\
dt = du
\end{cases}$$

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f(t)f(t) dt = f(0) \\
dt = f(0)
\end{cases}$$

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\end{cases}$$

 $\frac{d}{dt} \left( u(t) \right) = \delta(t)$ 

$$\frac{1}{(1-1)} \int_{0}^{\infty} f(t) \left( S(t-1) + S(t+1) \right) dt = \int_{0}^{\infty} f(t) S(t-1) dt + \int_{0}^{\infty} f(t) S(t+1) dt$$

1)

(c) (4 points) Let b be a positive constant. Show the following property for the delta function:

$$\delta(bt) = \frac{1}{b}\delta(t)$$

Hint: what function is "delta-like"?

This was function is define?

$$\frac{d}{dt} \int S(t) dt = \frac{d}{dt} \left( \int \int \int S(u) du \right)$$

$$\frac{d}{dt} \int S(t) = S(t)$$

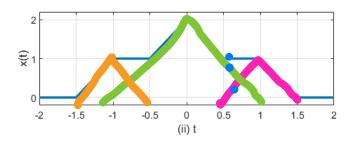
$$= \frac{d}{d+} \left( \frac{1}{b} u(u) \right) \Rightarrow \frac{d}{d+} \left( \frac{1}{b} u(bt) \right) \Rightarrow \frac{1}{b} \delta(t)$$

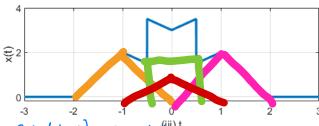
$$0.0 \quad \delta(b+) = \frac{1}{b} \delta(t)$$

# 2. (23 points) Expression for signals.

(a) (15 points) Write the following signals as a combination (sums or products) of unit triangles  $\Delta(t)$  and unit rectangles  $\operatorname{rect}(t)$ .

 $2\Delta(t) + \Delta(2t+2) + \Delta(2t-2)$ 

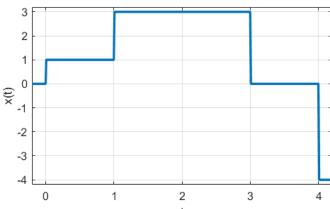




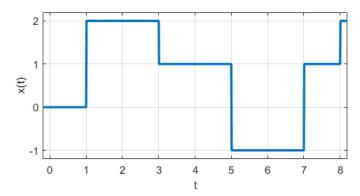
 $X(t) = \Delta(t) + 2\Delta(t+1) + 2\Delta(t-1) + 2rect(4)$ 

(b) (8 points) Express each of the signals shown below as sums of scaled and time shifted unit-step functions.

X(t) = n(t) + 2u(t-1) - 3u(t-3) - 4u(t-4)



X(t) = 2u(t-1) - u(t-3) - 2u(t-5) + 2u(t-7) + u(t-8)



### 3. (30 points) System properties.

(a) (20 points) A system with input x(t) and output y(t) can be time-invariant, causal or stable. Determine which of these properties hold for each of the following systems. Explain your answer.

i. 
$$y(t) = |x(t)| + x(t^2)$$

ii. 
$$y(t) = \int_{t-T}^{t+T} x(\lambda) d\lambda$$
, where T is positive and constant.

iii. 
$$y(t) = (t+1) \int_{-\infty}^t x(\lambda) d\lambda$$

iv. 
$$y(t) = 1 + e^{x(t)}$$

v. 
$$y(t) = \frac{1}{1+x^2(t)}$$

(b) (6 points) Consider the following three systems:

$$\mathcal{S}_1: w(t) = x(t/2)$$

$$S_2: z(t) = \int_{-\infty}^t w(\tau) d\tau$$

$$\mathcal{S}_3: y(t) = \mathcal{S}_3(z(t))$$

The three systems are connected in series as illustrated here:

3. (30 points) System properties. (a) (20 points) A system with input x(t) and output y(t) can be time-invariant, causal or stable. Determine which of these properties hold for each of the following systems.

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$$y(t) = (t+1) \int_{-\infty}^t x(\lambda) d\lambda$$

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$$y(t) = 1 + e^{x(t)}$$

v. 
$$y(t) = \frac{1}{1+x^2(t)}$$

output  

$$()$$
 TI =>  $y(t+x) = |x(t+x)| + x((t+x))^2$  not lime  
 $= |x(t+x)| + x(t+x)$  invariant

it is not causal.

(XH) input and relies on only t.

stable.

-A 5 X(+) 5 A

$$t+T$$
(i)  $Y(t) = \int x(\lambda) d\lambda$ 

$$\lambda = t+\alpha \qquad t-T$$

not causal because defends on value t+T which is future.

$$d\lambda = dt$$
 T

$$J(t) = \int_{-\infty}^{\infty} X(t+\alpha) dt$$

time invariant.

y(t+d) = ) x(t+d) d+

$$J(t+d) = S(X(t+d))$$

$$\begin{array}{c|c}
t+T \\
\int x(\lambda) d\lambda \leq \beta = 0
\end{array}$$

$$\int X(\lambda) d\lambda = \alpha = J(t)$$
  
 $t-T$ 

d 6/3

Bounded. then stable.

(ii) 
$$y(t) = (t+1) \int_{-\infty}^{t} x(\lambda) d\lambda$$
 cousal  $\int_{-\infty}^{t} x(\lambda) d\lambda$  covers

that  $x(\lambda) d\lambda$  covers

 $y(t+\alpha) = (t+\alpha+1) \int_{-\infty}^{t} x(\lambda) d\lambda$  covers

current values.

And  $x(\lambda) d\lambda$  NoT dime invariant

 $x(\lambda) d\lambda$ 

Assume 
$$\int x(\lambda) d\lambda | \leq \beta$$
  
 $y(+) = (++1) \int \int x(\lambda) d\lambda$   
 $= (++1) \int B = not BIBO Stable$ 

causal because it only relies on Past and current t's.

Assume 
$$X(+) \le Mx \le 00$$
  
 $Y(+) = 1 + e^{Mx}$   
 $\Rightarrow Y \le My$ 

it is BIBO Stable

$$\forall ) \forall (+) = \frac{1}{1 + x^{1}(+)} \implies$$

in Put = 
$$\frac{1}{1+x^2(t+a)}$$
  
outlut  $y(t+a) = \frac{1}{1+x^2(t+a)}$ 

of t.

$$\Rightarrow x^2(+) \leq M^2 \times \leq \infty$$

Causal since only relies

on Present and Post values

$$=) if  $\chi^{2}(t) z^{0}$ 
then  $\frac{1}{1+0} z^{1}$$$

and as 
$$M^2 \times \rightarrow \infty$$
  
it will go to zero

$$y \cdot y \cdot y - 1 + x^2(t)$$

(b) (6 points) Consider the following three systems:

$$\mathcal{S}_1: w(t) = x(t/2)$$

$$S_2: z(t) = \int_{-\infty}^t w(\tau)d\tau$$

$$\mathcal{S}_3: y(t) = \mathcal{S}_3(z(t))$$

The three systems are connected in series as illustrated here:

$$\Rightarrow S_{1} : w(t) = x(t_{2})$$

$$S_{2} : Z(t) = \int w(r) dr$$

$$S_{3} : y(t) = S_{3} (2(t))$$

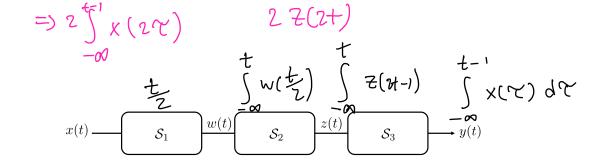
$$Z' = \frac{\tau}{2}$$

$$dr' = \frac{1}{2}d\tau \Rightarrow 2d\tau' = d\tau$$

$$\Rightarrow Z(t) = \int x(\tau) d\tau \qquad \text{when } \tau = t \Rightarrow \tau' = t_{2}$$

$$\Rightarrow Z(t) = 2\int x(\tau) d\tau' \Rightarrow 2(t-1) = \tau'$$

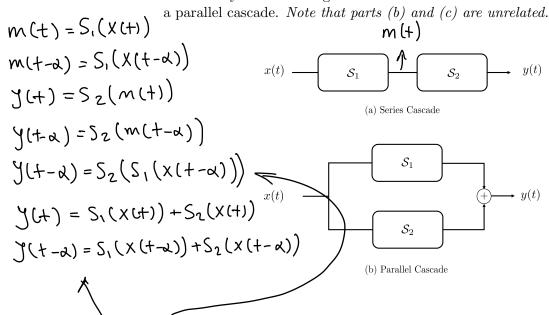
$$y(t) = \frac{1}{2}Z(2(t-1))$$



Choose the third system  $S_3$ , such that overall system is equivalent to the following system:

$$y(t) = \int_{-\infty}^{t-1} x(\tau)d\tau$$

(c) (4 points) In part (b), you saw an example of three systems connected in series. In general, systems can be interconnected in series or in parallel to form what we call cascaded systems. The figure below shows the difference between a series cascade and a parallel cascade. Note that parts (b) and (c) are unrelated.



- i. (2 points) Show that the series cascade of any two time-invariant systems is also time-invariant.
- ii. (2 points) Show that the parallel cascade of any two time-invariant systems is also time-invariant.
- iii. (Optional) Can you think of two **time-variant** systems, whose series cascade is **time-invariant**? Can you think of two **time-variant** systems, whose parallel cascade is **time-invariant**?
- 4. (10 points) Power and energy of complex signals
  - (a) (5 points) Let

$$x(t) = Ae^{j\omega t} + Be^{-j\omega t}$$

where A and B are complex numbers expressed in polar form

$$A = r_1 e^{j\phi}$$
$$B = r_2 e^{j\phi}$$

Is x(t) a power or energy signal? If it is an energy signal, compute its energy. If it is a power signal, compute its power. (Hint: Use the fact that the square magnitude of a complex number v is:  $|v|^2 = v^*v$ , where  $v^*$  is the complex conjugate of the complex number v.)

(b) (5 points) Is  $x(t)=e^{-(2+j\omega_1+j\omega_2)t}u(t+3)$  an energy signal or power signal? Again, if it is an energy signal, compute its energy. If it is a power signal, compute its power.

$$\Rightarrow \Gamma_1 + \Gamma_2^2 + \lim_{T \to \infty} \frac{r_1 r_2}{T} \left( \frac{2jwT}{2jw} - 2jwT \right) = \Gamma_1^2 + \Gamma_2^2$$

 $number\ v.)$ 

(b) (5 points) Is  $x(t) = e^{-(2+j\omega_1+j\omega_2)t}u(t+3)$  an energy signal or power signal? Again, if it is an energy signal, compute its energy. If it is a power signal, compute its power.

$$= \frac{(-2-j(w_1+w_2))t}{e}$$

$$=) \int_{e}^{-2-j(W_1+W_2)} t^2 dt$$

$$=) \int_{e}^{-2-j(W_1+W_2)} t^2 dt$$

$$\Rightarrow \int_{e}^{\infty} (-2 - jk)t (-2 + jk)t \otimes dt$$

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$$\Rightarrow P_{X} = \lim_{T \to \infty} \frac{1}{2T} \int_{-3}^{T} e^{-4t} dt \Rightarrow \lim_{T \to \infty} \frac{1}{2T} \left[ \frac{e^{4T}}{4} \right]_{-3}^{\infty}$$

and after 
$$+=-3$$
 (1) is 1.

where A and B are complex numbers expressed in polar form

$$A = r_1 e^{j\phi}$$

$$B = r_2 e^{j\phi}$$

Is x(t) a power or energy signal? If it is an energy signal, compute its energy. If it is a power signal, compute its power. (Hint: Use the fact that the square magnitude of a complex number v is:  $|v|^2 = v^*v$ , where  $v^*$  is the complex conjugate of the complex number v.)

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#### 5. (15 points) **Python**

(a) (3 points) Task 1

A complex sinusoid is denoted:

$$y(t) = e^{(\sigma + j\omega)t}$$

First compute a vector representing time from 0 to 10 seconds in about 500 steps (You can use **np.linspace**). Use this vector to compute a complex sinusoid with a period of 2 seconds, and a decay rate that reduces the signal level at 10 seconds to 1/3 its original value. What  $\sigma$  and  $\omega$  did you choose? Evaluate (y(t)) as shown above i.e.

$$y(t) = e^{(\sigma + j\omega)t}$$

Hint: to define complex number  $e^{(5+6j)}$ 

$$y = np.exp(5 + 1j*6)$$

(b) (7 points) Task 2

Use the np.real(y) and np.imag(y) Python functions to extract the real and imaginary parts of the complex exponential.

- i. (5 points) Plot them as a function of time (plot them separately, you can use subplot for this task). This should look more reasonable. Label your axes, and check that your signal has the required period and decay rate.
- ii. (2 points) Plot the imaginary component of y(t) as a function of real component of y(t). What can be inferred from the plot? Hint: Comment on the shape of the plot and what we can infer about the envelope of y(t) from the shape?
- (c) (5 points) Task 3

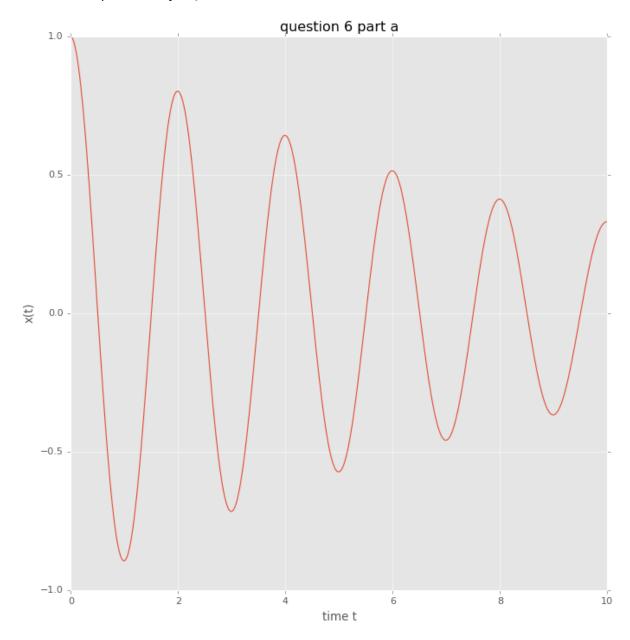
Use the np.abs() and np.angle() functions to plot the magnitude and phase angle of the complex exponential (plot them in the same figure). Scale the np.angle() plot by dividing it by 2\*pi so that it fits well on the same plot as the np.abs() plot (i.e. plot the angle in cycles, instead of radians, the function np.angle(x) returns the angle in radians).

Feel free to also explore and visualize the change in the wave-forms for different sigma and omega values.

In [11]: t= np.linspace(0,10,500)

```
In [10]: import matplotlib.pyplot as plt
import numpy as np
import math
plt.style.use('ggplot')
```

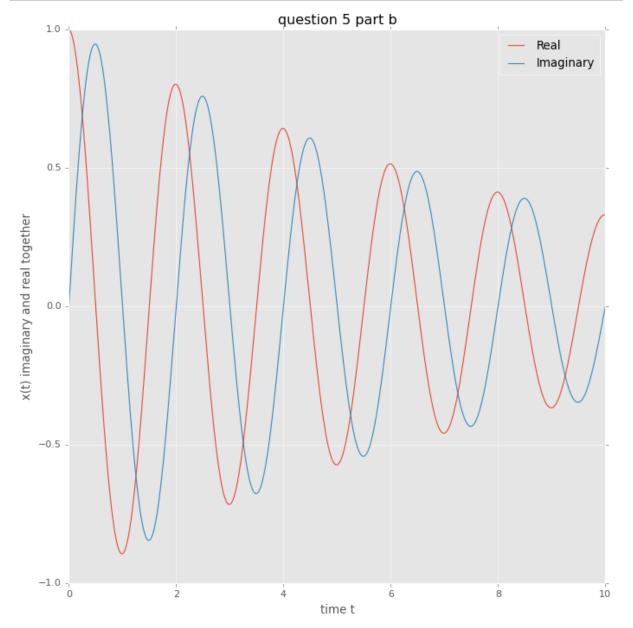
/Applications/anaconda3/lib/python3.11/site-packages/matplotlib/cbook/\_
\_init\_\_.py:1335: ComplexWarning: Casting complex values to real discard
s the imaginary part
 return np.asarray(x, float)



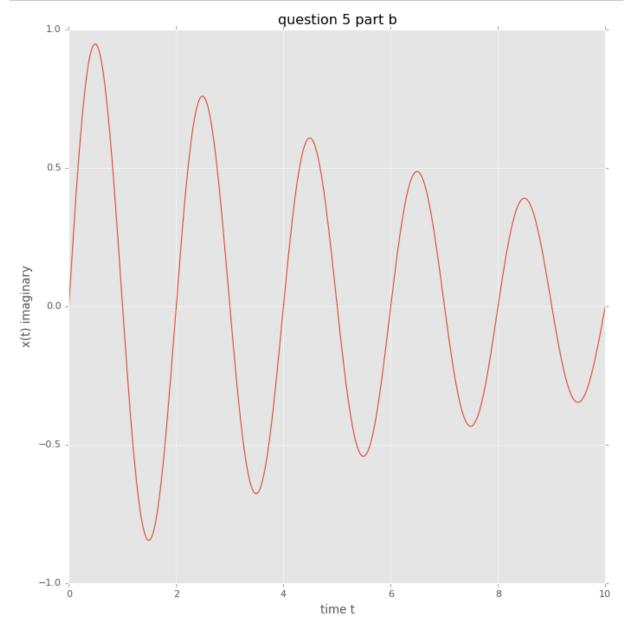
Question 5 Part b

```
In [13]: real = np.real(y)
imaginary = np.imag(y)
```

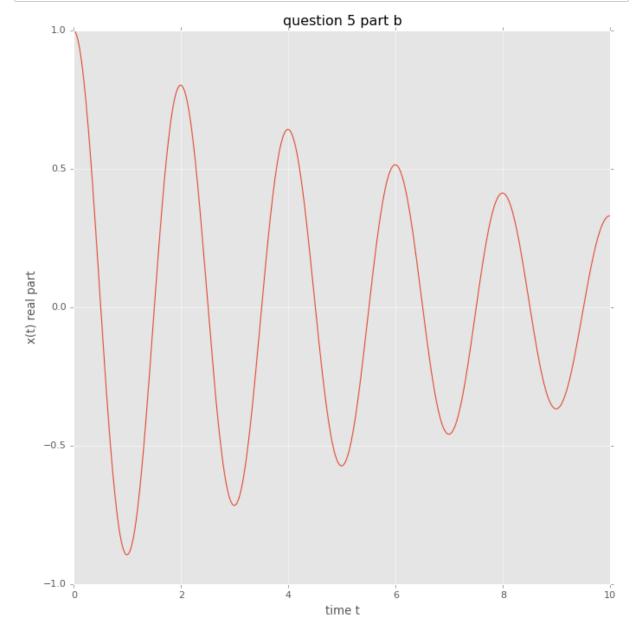
```
In [23]: fig, q5a = plt.subplots(figsize=(10,10))
    q5a.plot(t, real, label='Real')
    q5a.plot(t,imaginary, label='Imaginary')
    q5a.set_xlabel("time t")
    q5a.set_ylabel("x(t) imaginary and real together")
    q5a.set_xlim([0, 10])
    q5a.set_ylim([-1,1])
    q5a.set_title("question 5 part b")
    plt.legend()
    plt.show()
```



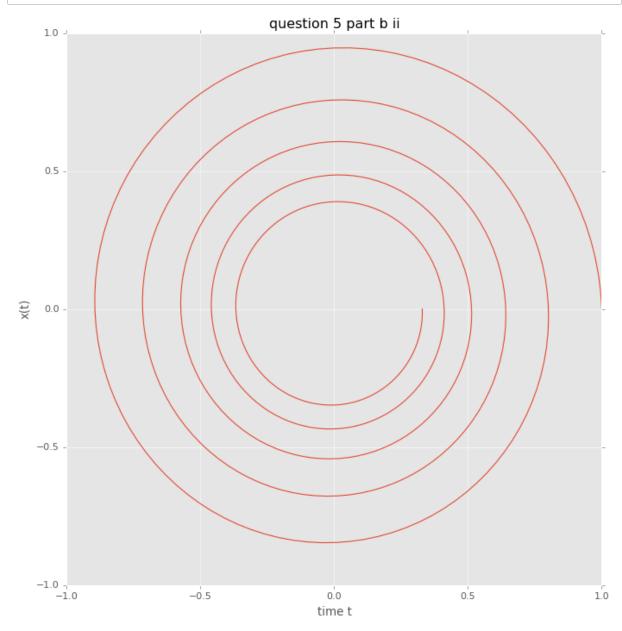
```
In [18]: fig, q5b_j = plt.subplots(figsize=(10,10))
q5b_j.plot(t, imaginary)
q5b_j.set_xlabel("time t")
q5b_j.set_ylabel("x(t) imaginary")
q5b_j.set_xlim([0, 10])
q5b_j.set_ylim([-1,1])
q5b_j.set_title("question 5 part b")
plt.show()
```



```
In [17]: fig, q5b_r = plt.subplots(figsize=(10,10))
    q5b_r.plot(t, real)
    q5b_r.set_xlabel("time t")
    q5b_r.set_ylabel("x(t) real part")
    q5b_r.set_xlim([0, 10])
    q5b_r.set_ylim([-1,1])
    q5b_r.set_title("question 5 part b")
    plt.show()
```

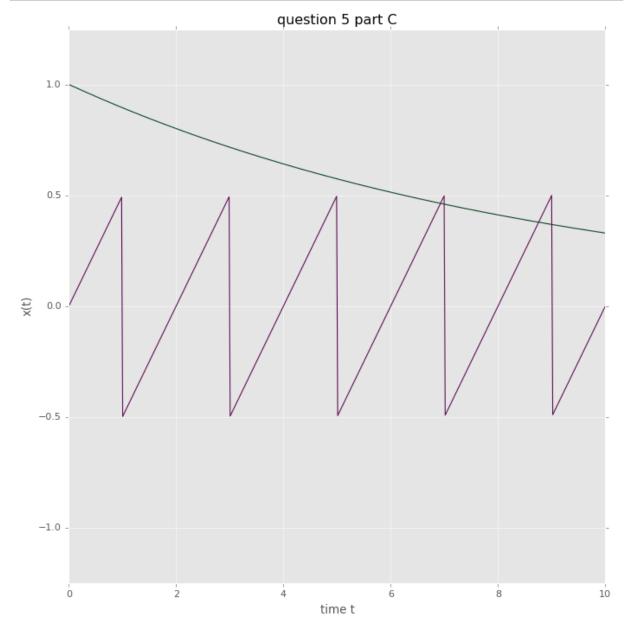


```
In [25]: ig, q5b_ii = plt.subplots(figsize=(10,10))
    q5b_ii.plot(real, imaginary)
    q5b_ii.set_xlabel("time t")
    q5b_ii.set_ylabel("x(t)")
    q5b_ii.set_xlim([-1, 1])
    q5b_ii.set_ylim([-1,1])
    q5b_ii.set_title("question 5 part b ii")
    plt.show()
# This function shows that by plotting the real numbers in x axis and image the real numbers are continually decreasing since the sigma is negative.
```



#### Question 5 part C

```
In [26]: fig, q5a = plt.subplots(figsize=(10,10))
    abso = np.abs(y)
    anglo = np.angle(y)/(2*np.pi)
    q5a.plot(t, abso, label='Magnitude', color='#0a4522')
    q5a.plot(t, anglo, label='Phaser', color='#5e0d55')
    q5a.set_xlabel("time t")
    q5a.set_ylabel("x(t)")
    q5a.set_xlim([0, 10])
    q5a.set_ylim([-1.25,1.25])
    q5a.set_title("question 5 part C")
    plt.show()
```



In [ ]:	
In [ ]:	