ECE102, Fall 2023 — **Homework #3**

Signals & Systems — Prof. J.C. Kao

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Due Monday, 30 Oct 2023, by 11:59pm to Gradescope.
Covers material up to Lecture 7.
100 points total.

1. (20 points) **Linear systems** Determine whether each of the following systems is linear or not. Explain your answer.

   (a) $y(t) = sin(t)x(t)$

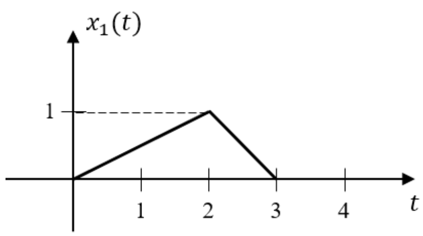
   (b) $y(t) = \frac{d}{dt}(\frac{1}{3}x(t)^3)$

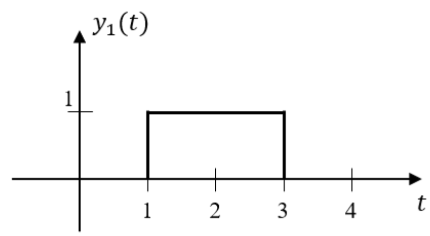
   (c) $y(t) = x(t)e^{-jwt}$

   (d) $y(t) = x(t) + 3u(t+1)$

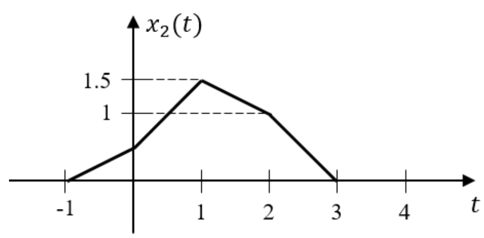
2. (13 points) **LTI systems**

   (a) (7 points) Consider an LTI (linear time-invariant) system whose response to $x_1(t)$ is $y_1(t)$, where $x_1(t)$ and $y_1(t)$ are illustrated as follows:





   Sketch the response of the system to the input $x_2(t)$.



   (b) (6 points) Assume we have a linear system with the following input-output pairs:

   - the output is $y_1(t) = cos(t)u(t)$ when the input is $x_1(t) = u(t)$;
   - the output is $y_2(t) = cos(t)(u(t+1) - u(t))$ when the input is $x_2(t) = \text{rect}(t + \frac{1}{2})$.

   Is the system time-invariant?

3. (38 points) **Convolution**

1. (20 points) **Linear systems** Determine whether each of the following systems is linear or not. Explain your answer.

(a) $y(t) = sin(t)x(t)$

(b) $y(t) = \frac{d}{dt}(\frac{1}{3}x(t)^3)$

(c) $y(t) = x(t)e^{-jwt}$

(d) $y(t) = x(t) + 3u(t+1)$

system is linear if = $S(ax) = aS(x)$ and $S(x+\tilde{x}) = S(x) + S(\tilde{x})$

a) $z(t) = ax(t) + b\tilde{x}(t)$

then, $S[z(t)] = \sin(t)\, z(t)$

$\Rightarrow \sin(t)[ax(t) + b\tilde{x}(t)]$

$\Rightarrow a\sin(t)\,x(t) + b\sin(t)\,\tilde{x}(t)$

$\Rightarrow aS[x(t)] + bS[\tilde{x}(t)]$

$\Rightarrow$ two conditions hold then it is linear.

b) $y(t) = \frac{d}{dt}(\frac{1}{3}x(t)^3)$ $\qquad m(t) = ax(t) + b\tilde{x}(t)$

$S(m(t)) = \frac{d}{dt}(\frac{1}{3}m(t)^3) = \frac{d}{dt}(\frac{1}{3}[ax(t) + b\tilde{x}(t)]^3)$

$\neq aS(x(t)) + bS(\tilde{x}(t))$

NOT Linear

homogeneity and superposition both fail.

c) $y(t) = x(t)e^{-jwt} \Rightarrow S(z(t))$ $\qquad z(t) = ax(t) + b\tilde{x}(t)$

$= e^{-jwt}[ax(t) + b\tilde{x}(t)] = ae^{-jwt}x(t) + be^{-jwt}\tilde{x}(t)$

$aS(x(t)) + bS(\hat{x}(t))$

$\Rightarrow$ Linear

d) $y(t) = x(t) + 3u(t+1)$ $\qquad z(t) = ax(t) + b\hat{\tilde{x}}(t)$

$S(z(t)) = z(t) + 3u(t+1)$

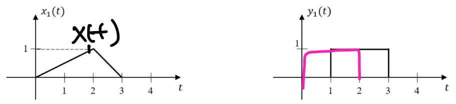
$\Rightarrow ax(t) + b\tilde{x}(t) + 3u(t+1)$

$\neq aS(x(t)) + bS(\tilde{x}(t))$ $\qquad$ Not Linear.

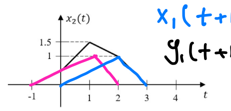
2)

2. (13 points) **LTI systems**

(a) (7 points) Consider an LTI (linear time-invariant) system whose response to $x_1(t)$ is $y_1(t)$, where $x_1(t)$ and $y_1(t)$ are illustrated as follows:



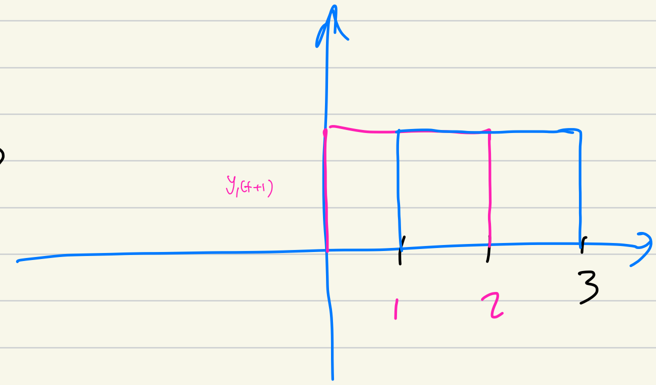
Sketch the response of the system to the input $x_2(t)$.



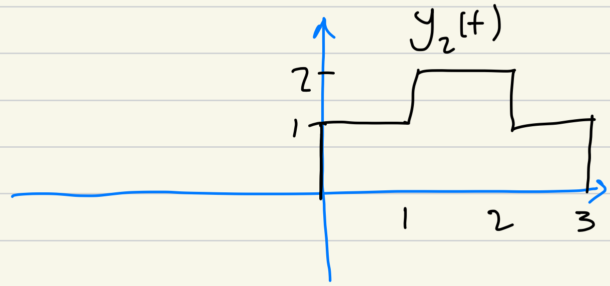
$x_1(t+1) + x_1(t)$

$y_1(t+1) + y_1(t)$

$\Rightarrow$



$\Rightarrow$

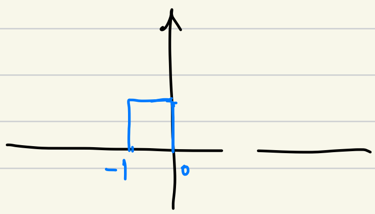


2b)

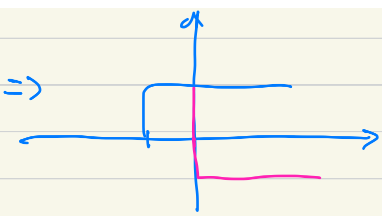
(b) (6 points) Assume we have a linear system with the following input-output pairs:

- the output is $y_1(t) = cos(t)u(t)$ when the input is $x_1(t) = u(t)$;
- the output is $y_2(t) = cos(t)(u(t+1) - u(t))$ when the input is $x_2(t) = \text{rect}(t + \frac{1}{2})$.

Is the system time-invariant?



$\Rightarrow$



$x_2(t) = \text{rect}(t+\frac{1}{2}) \Rightarrow (u(t+1) - u(t))$

$\text{rect}(t+\frac{1}{2}+\alpha) \Rightarrow (u(t+1+\alpha) - u(t+\alpha))$

$\Rightarrow$

$y(t+\alpha) = \cos(t+\alpha)[u(t+1+\alpha) - u(t+\alpha)]$

$\Rightarrow \quad \cos(t)[\text{rect}(t+\frac{1}{2}+\alpha)] = \cos(t)[u(t+\alpha+1) - u(t+\alpha)]$

$\neq$

$y(t+\alpha) = \cos(t+\alpha)[\text{rect}(t+\frac{1}{2}+\alpha)] = \cos(t+\alpha)[u(t+\alpha+1) - u(t+\alpha)]$

NOT time invariant

(a) (10 points) For each pair of the signals given below, compute their convolution using the flip-and-drag technique. Please provide a piecewise formula for $y(t)$.

   i. $f(t) = \delta(t+1) + 2\delta(t-2)$, $\qquad g(t) = e^{-t}u(t)$

   ii. $f(t) = 2\,\mathrm{rect}(t - \frac{3}{2})$, $\qquad g(t) = 2\,r(t-1)\mathrm{rect}(t - \frac{3}{2})$

(b) (10 points) For each of the following, find a function $h(t)$ such that $y(t) = x(t) * h(t)$.

   i. $y(t) = \int_{t-T}^{t} x(\tau)d\tau$

   ii. $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$
   Note: this last operation creates a periodic extension of $x(t)$ where the period is T_s.

(c) (10 points) Use the properties of convolution to simplify the following expressions:

   i. $[\delta(t-3) + \delta(t+2)] * \left[e^{3t}u(-t) + \delta(t+2) + 2\right]$

   ii. $\frac{d}{dt}\left[(u(t) - u(t-1)) * u(t-2)\right]$, *Hint: Show first that $u(t) * u(t) = r(t)$ where $r(t)$ is the ramp function.*

(d) (8 points) Explain whether each of the following statements is true or false.

   i. If $x(t)$ and $h(t)$ are both odd functions, and $y(t) = x(t) * h(t)$, then $y(t)$ is an even function.

   ii. If $y(t) = x(t) * h(t)$, then $y(2t) = h(2t) * x(2t)$.

4. (12 points) **LTI Systems and impulse response**

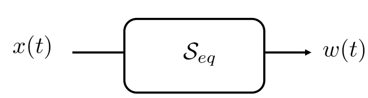
   Consider the following three LTI systems:

   - $\mathcal{S}_1$: $y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)}x(\tau)d\tau$;
   - $\mathcal{S}_2$: $y(t) = \int_{-\infty}^{t-2} x(\tau)d\tau$
   - $\mathcal{S}_3$ is characterized by its impulse response: $h_3(t) = \delta(t-3)$.

   (a) (4 points) Compute the impulse response $h_1(t)$ of $\mathcal{S}_1$.

   (b) (2 points) Define $w(t) = \mathcal{S}_1[x(t)] - \mathcal{S}_3\{\mathcal{S}_2[x(t)]\}$. Represent this relationship using a block diagram where $x(t)$ is the input and $w(t)$ is the output.

   (c) (2 points) Determine the impulse response $h_{eq}(t)$ of the above system.



   (d) (4 points) Determine the response of the overall system to $\delta(t) + 2\delta(t-3)$.

5. (17 points) **Python tasks**

   We provide a helper function `nconv()` as defined below:

3 (a) (10 points) For each pair of the signals given below, compute their convolution using the flip-and-drag technique. Please provide a piecewise formula for $y(t)$.

i. $f(t) = \delta(t+1) + 2\delta(t-2)$, $\quad g(t) = e^{-t}u(t)$

ii. $f(t) = 2\,\text{rect}(t-\frac{3}{2})$, $\quad g(t) = 2\,r(t-1)\text{rect}(t-\frac{3}{2})$

i) $f(t) = \delta(t+1) + 2\delta(t-2) \qquad g(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} [\delta(t-\tau+1) + 2\delta(t-\tau-2)]\, e^{-\tau} u(\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\delta(t-\tau+1)\, d\tau + \int_{-\infty}^{\infty} 2\delta(t-\tau-2) e^{-\tau} u(\tau)\, d\tau$$

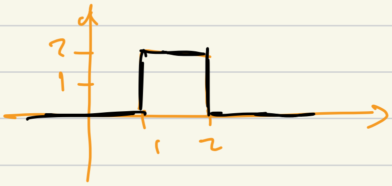
$$\Rightarrow e^{-(t+1)} u(t+1) + 2e^{-(t-2)} u(t-2)$$

$$\Rightarrow \begin{cases} t \le -1, & 0 \\ -1 < t \le 2, & e^{-(t+1)} u(t+1) \\ t > 2, & e^{-(t+1)} u(t+1) + 2e^{-(t-2)} u(t-2) \end{cases}$$

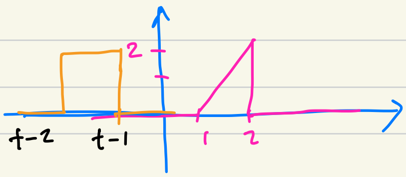
$t > 2$, $e^{-(t+1)} + 2e^{-(t-2)}$

$$\Rightarrow \begin{cases} t \le -1, & 0 \\ -1 < t \le 2, & e^{-(t+1)} \\ t > 2, & e^{-(t+1)} + 2e^{-(t+2)} \end{cases}$$

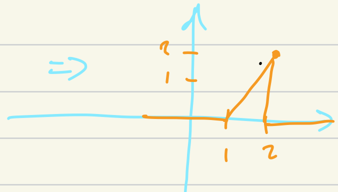
3a ii) $f(t) = 2\,\text{rect}(t - \frac{3}{2})$ $\qquad$ $g(t) = 2r(t-1)\,\text{rect}(t - \frac{3}{2})$



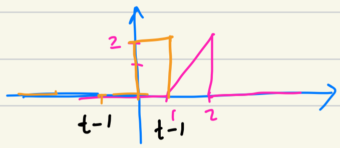
$\Rightarrow$



$\Rightarrow$

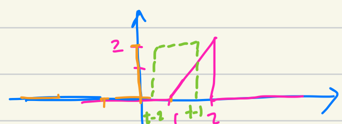


① for $t < 1$, 0



$$\int_{-\infty}^{\infty} 2\,\text{rect}(t - \tfrac{3}{2} - \tau) \cdot 2r(\tau - 1)\,\text{rect}(\tau - \tfrac{3}{2})\, d\tau \Rightarrow$$

$$\int_{1}^{t-1} 2\,\text{rect}(t - \tfrac{3}{2} - \tau)\, 2r(\tau - 1)\, d\tau$$



$r = t$ and $t < 0 : 0$

$$4\int_{1}^{t-1} r(\tau - 1)\, d\tau \Rightarrow 4\int_{1}^{t-1} \tau - 1\, d\tau$$

$t - 1 > 1$
$t > 2$

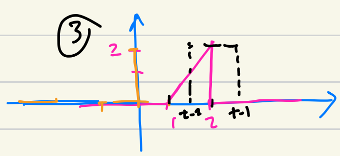
$\Rightarrow$ $t-1>1 \Rightarrow t>2$, $t-2<1 \Rightarrow t<3$

②

$$\Rightarrow 4\frac{\tau^2}{2} - 4\tau\Big|_{1}^{t-1} = \left[2\tau^2 - 4\tau\right]_{1}^{t-1} \Rightarrow 2(t-1)^2 - 4(t-1) - 4[2-1]$$

$$= 2t^2 - 8t + 8$$

③



③ $t - 2 < 2 \Rightarrow t < 4$, $t - 1 > 2 \Rightarrow t > 3$ $\Rightarrow 4\int_{t-2}^{2} r(\tau - 1)\, d\tau$ $\quad t-1$

$$\Rightarrow 4\int_{t-2}^{2} \tau - 1\, d\tau \Rightarrow 4\frac{\tau^2}{2} - 4\tau\Big|_{t-2}^{2} \Rightarrow 2\tau^2 - 4\tau\Big|_{t-2}^{2} \Rightarrow \cancel{2(2)^2 - 4(2)} - \left[2(t-2)^2 - 4(t-2)\right]$$

$$\Rightarrow -2(t-2)^2 + 4(t-2) = -2t^2 + 12t - 16$$

$$f(t) * g(t) = \begin{cases} t < 1, & 0 \\ 2 < t < 3, & 2t^2 - 8t + 8 \\ 3 < t < 4, & -2t^2 + 12t - 16 \\ t > 4, & 0 \end{cases}$$

3 (b) (10 points) For each of the following, find a function $h(t)$ such that $y(t) = x(t) * h(t)$.

i. $y(t) = \int_{t-T}^{t} x(\tau)d\tau$

ii. $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$

Note: this last operation creates a periodic extension of $x(t)$ where the period is T_s.

i) $y(t) = \int_{t-T}^{t} x(\tau)\, d\tau \Rightarrow \int_{t-\tau}^{t} \delta(\tau)\, d\tau \Rightarrow u(\tau)\Big|_{t-\tau}^{t}$

$\Rightarrow u(t) - u(t-\tau)$

ii) $y(t) = \sum_{n=-\infty}^{\infty} x(t-nT_s) \Rightarrow \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$

$x(t-1T_s) + x(t-2TS) + \ldots$

3c) (c) (10 points) Use the properties of convolution to simplify the following expressions:

i. $[\delta(t-3) + \delta(t+2)] * [e^{3t}u(-t) + \delta(t+2) + 2]$

ii. $\frac{d}{dt}[(u(t) - u(t-1)) * u(t-2)]$, *Hint: Show first that $u(t) * u(t) = r(t)$ where $r(t)$ is the ramp function.*

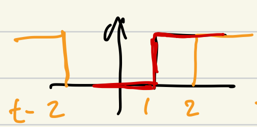
i) $[\delta(t-3)+\delta(t+2)] * [e^{3t}u(-t)+\delta(t+2)+2]$

$\Rightarrow \delta(t-3)*e^{3t}u(-t) + \delta(t-3)*\delta(t+2) + \underbrace{2*\delta(t-3)}_{2} + \delta(t+2)\, e^{3t}u(-t) + \delta(t+2)*\delta(t+2) + \underbrace{2*\delta(t+2)}_{2}$

$\delta(t-3)*e^{3t}u(-t) + 4 + \delta(t+2)*e^{3t}u(-t) + \delta(t-1) + \delta(t+4)$

$\Rightarrow e^{3(t-3)}u(-(t-3)) + 4 + e^{3(t+2)}u(-(t+2)) + \delta(t-1) + \delta(t+4)$

ii) $\frac{d}{dt}[(u(t)-u(t-1)) * u(t-2)]$



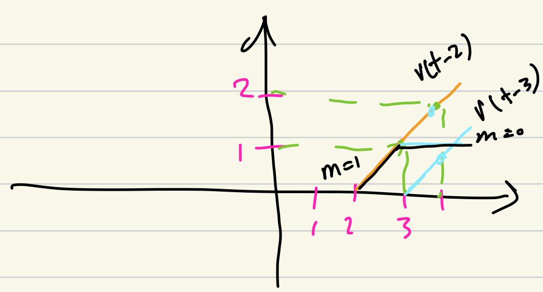
$t-2>1$

$t-2<1$

$t<3$

$u(t) * u(t) = r(t)$ $\qquad r(t) = \begin{cases} t>0, & t \\ t\le 0, & 0 \end{cases}$

$\Rightarrow \int_0^t 1\, d\tau \Rightarrow \tau\Big|_0^t = t$



$\Rightarrow \frac{d}{dt}[(u(t)*u(t-2) - u(t-1)*u(t-2)]$

$\frac{d}{dt}[r(t-2) - r(t-3)]$

$= u(t-2) - u(t-3)$

(d) (8 points) Explain whether each of the following statements is true or false.

i. If $x(t)$ and $h(t)$ are both odd functions, and $y(t) = x(t) * h(t)$, then $y(t)$ is an even function. True

ii. If $y(t) = x(t) * h(t)$, then $y(2t) = h(2t) * x(2t)$. False.

i) from hw, we know that product of two odd signal is even.

$-u = t - \tau$

$-du = -d\tau$

$du = d\tau$

$\Rightarrow y(t) = x(t) * h(t)$ odd $\Rightarrow h(\underbrace{t-\tau}_{-u}) = -h(-(\underbrace{t+\tau}_{\phi})) = -h(\phi) =$

$\Rightarrow \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$ odd

$h(-\tau) = -h(\tau)$ odd

$\Rightarrow \int_{-\infty}^{\infty} x(\tau) h(t+\tau)\, d\tau$

$y(-t) = x(-t) * h(-t) \Rightarrow \int_{-\infty}^{\infty} (-x(-\tau))(-h(t+\tau))\, d\tau$

$\Rightarrow \int_{-\infty}^{\infty} x(\tau) h(t+\tau)\, d\tau$

$\Rightarrow$ they are equal

$y(t) = y(-t)$

$\Rightarrow$ convolution is even.

ii) $y(2t) = \int_{-\infty}^{\infty} x(\tau) h(2t-\tau)\, d\tau$

$x(2t) * h(2t) = \int_{-\infty}^{\infty} x(2\tau) h(2(t-\tau))\, d\tau$

$\Rightarrow \int_{-\infty}^{\infty} x(2\tau) h(2t-2\tau)\, d\tau$

$2\tau = u$

$2d\tau = du$

$d\tau = \frac{du}{2}$

$\Rightarrow \int_{-\infty}^{\infty} x(u) h(2t-u) \frac{du}{2} \neq y(2t) = \int_{-\infty}^{\infty} x(\tau) h(2t-\tau)\, d\tau$

So flase.

4. (12 points) **LTI Systems and impulse response**

Consider the following three LTI systems:

- $\mathcal{S}_1$: $y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)} x(\tau) d\tau$;
- $\mathcal{S}_2$: $y(t) = \int_{-\infty}^{t-2} x(\tau) d\tau$ $\longrightarrow$ $h_2 = \int_{-\infty}^{t-2} \delta(\tau) d\tau = u(\tau) \Big|_{-\infty}^{t-2}$
- $\mathcal{S}_3$ is characterized by its impulse response: $h_3(t) = \delta(t-3)$.

h_2

$u(t-2) - \cancel{u(-\infty)}_{0} = u(t-2)$

(a) (4 points) Compute the impulse response $h_1(t)$ of $\mathcal{S}_1$.

(b) (2 points) Define $w(t) = \mathcal{S}_1[x(t)] - \mathcal{S}_3\{\mathcal{S}_2[x(t)]\}$. Represent this relationship using a block diagram where $x(t)$ is the input and $w(t)$ is the output.

(c) (2 points) Determine the impulse response $h_{eq}(t)$ of the above system.

$$x(t) \longrightarrow \boxed{\mathcal{S}_{eq}} \longrightarrow w(t)$$

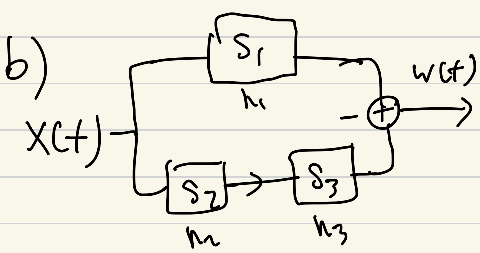
(d) (4 points) Determine the response of the overall system to $\delta(t) + 2\delta(t-3)$.

a) $\Rightarrow S(x(t)) = \int^{t} e^{-3(t-\tau)} x(\tau)$ $\Rightarrow \int_{-\infty}^{t} e^{-3(t-\tau)} \delta(\tau) d\tau$

$$= \int_{-\infty}^{t} e^{-3(\tau)} \delta(t-\tau) d\tau$$

$$= e^{-3(t)} u(t)$$

b)



c) $w(t) = x(t) * h_1(t) - x(t) * h_2(t) * h_3(t)$

$$= x(t) * \underbrace{(h_1(t) - h_2(t) * h_3(t))}_{h_{eq}} \qquad u(t-2) * \delta(t-3)$$

$$e^{-3t} u(t) - u(t-5) = h_{eq}(t)$$

d) $x(t) = \delta(t) + 2\delta(t-3)$

$$\Rightarrow [\delta(t) + 2\delta(t-3)] * e^{-3t} u(t) - u(t-5)$$

$$\Rightarrow e^{-3t} u(t) - u(t-5) + 2e^{-3(t-3)} u(t-3) - 2u(t-8)$$

```python
import numpy as np

def nconv(x, tx, h, th):
    y = np.convolve(x, h) * (th[1] - th[0])
    ty = np.linspace(tx[0] + th[0], tx[-1] + th[-1], len(y))
    return y, ty
```

where the inputs are:

`x` : input signal vector
`tx`: times over which `x` is defined
`h` : impulse response vector
`th`: times over which `h` is defined

and the outputs are:

`y` : output signal vector
`ty`: times over which `y` is defined.

The function is implemented using numpy's `convolve()` function Link.

(a) (10 points) Use `nconv()` to check your result for problem 3(a)(ii) and plot the output. Use the same step size for `tx` and `th` and label the plots.

(b) (7 points) Use `nconv()` to convolve two unit rectangles: $\mathrm{rect}(t) * \mathrm{rect}(t)$. Plot the result and label the axes.

In [3]:
```
import matplotlib.pyplot as plt
import numpy as np
import math
plt.style.use('ggplot')
```

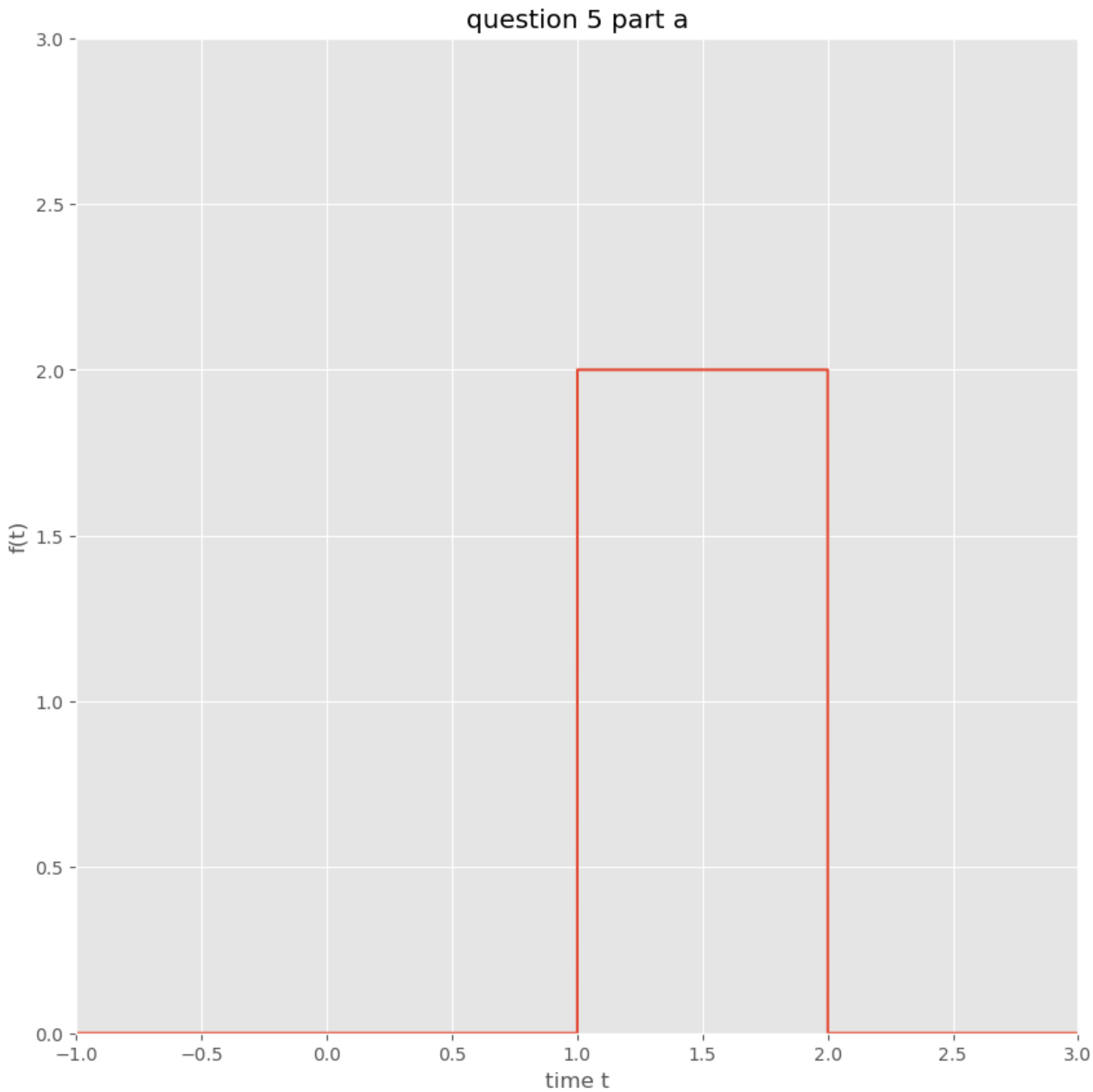
In [4]:
```
def nconv(x,time_x,h,time_h):
    y=np.convolve(x,h) * (time_h[1]-time_h[0])
    time_y= np.linspace(time_x[0] +time_h[0], time_x[-1] + time_h[-1],
    return time_y, y
```

In [5]:
```
t = np.arange(-1,3,0.0001, dtype=float)
```

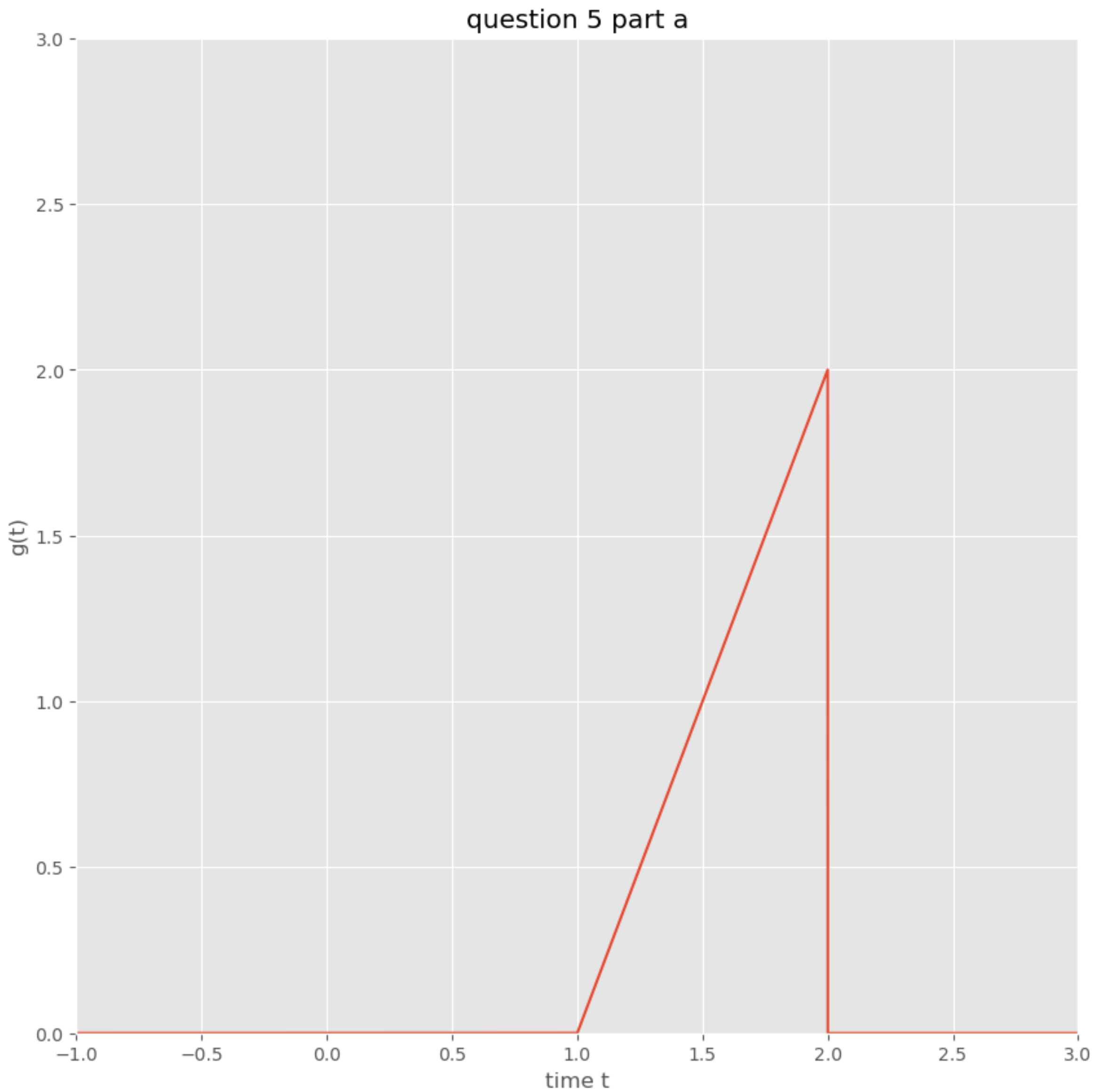
In [6]:
```
def rectangle(t):
    y = np.zeros(len(t))
    for i in range(len(t)):
        if np.absolute(t[i])<=0.5:
            y[i] =1
        else:
            y[i]=0
    return y
```

In [7]:
```
def ramp(t):
    y=np.zeros(len(t))
    for i in range(len(t)):
        if t[i]<=0:
            y[i]=0
        else:
            y[i]=t[i]
    return y
```

In [19]:
```
f_t = 2*rectangle(t-1.5)
fig, f_t_plt = plt.subplots(figsize=(10,10))
f_t_plt.plot(t,f_t)
f_t_plt.set_xlabel("time t")
f_t_plt.set_ylabel("f(t)")
f_t_plt.set_xlim([-1, 3])
f_t_plt.set_ylim([0,3])
f_t_plt.set_title("question 5 part a")
plt.show()
```

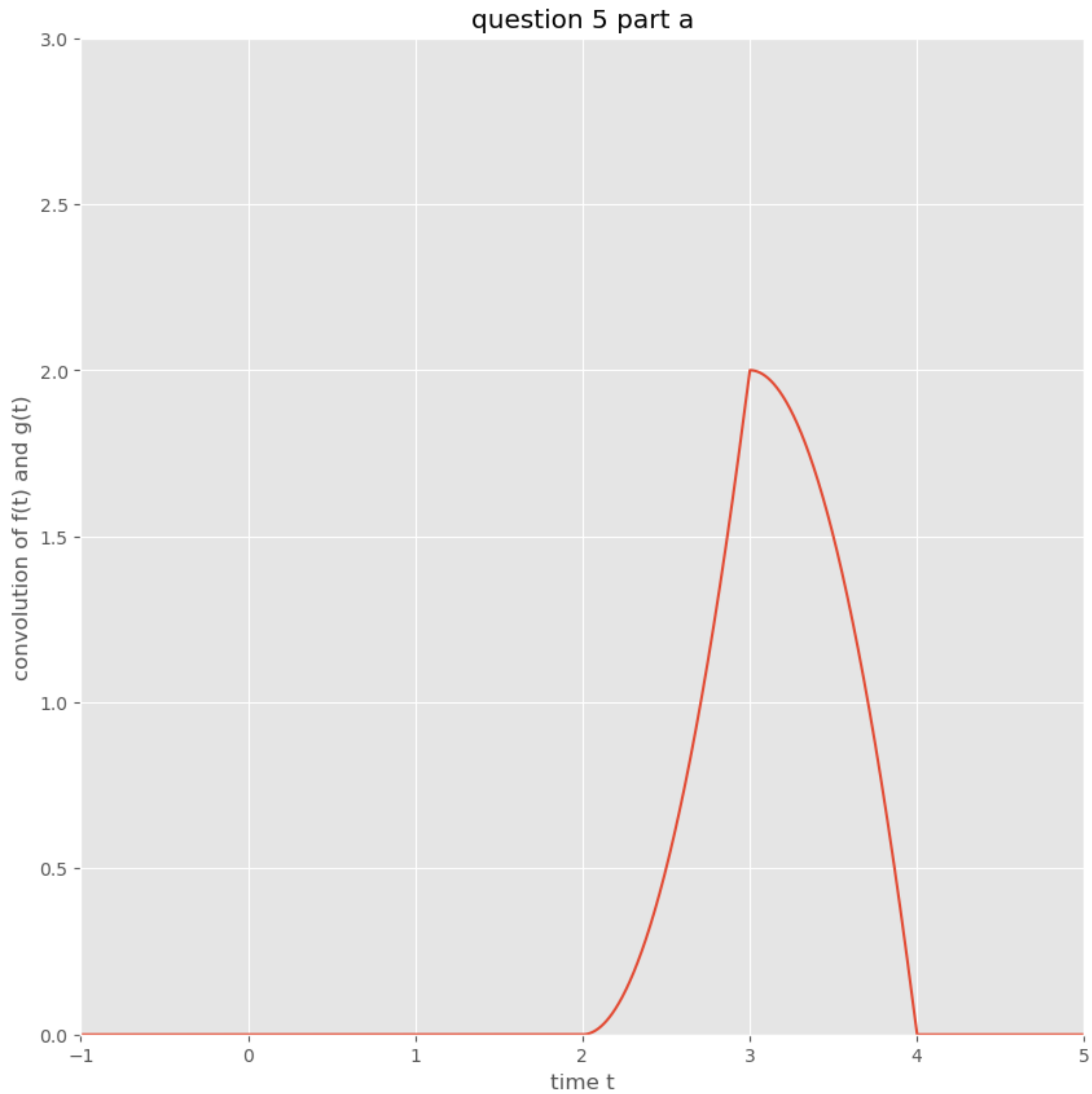


In [18]:
```
g_t = 2*ramp(t-1)*rectangle(t-1.5)
fig, g_t_plt = plt.subplots(figsize=(10,10))
g_t_plt.plot(t,g_t)
g_t_plt.set_xlabel("time t")
g_t_plt.set_ylabel("g(t)")
g_t_plt.set_xlim([-1, 3])
g_t_plt.set_ylim([0,3])
g_t_plt.set_title("question 5 part a")
plt.show()
```



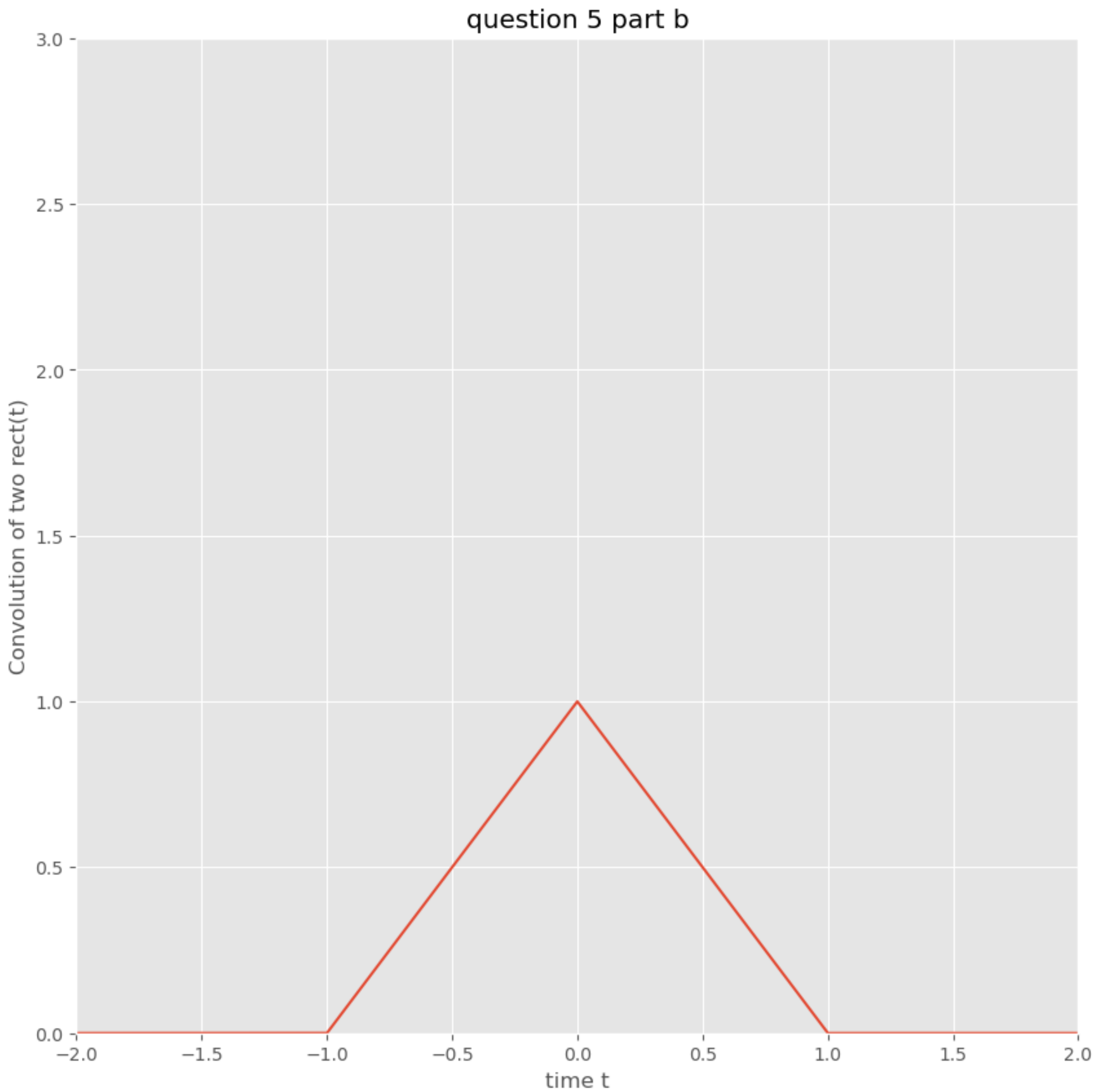
In [10]:
```
y_t = nconv(f_t,t,g_t,t)
```

In [11]:
```
fig, q5a = plt.subplots(figsize=(10,10))
# y =np.exp((-1/9+1j*np.pi)*t)
q5a.plot(y_t[0],y_t[1])
q5a.set_xlabel("time t")
q5a.set_ylabel("convolution of f(t) and g(t)")
q5a.set_xlim([-1, 5])
q5a.set_ylim([0,3])
q5a.set_title("question 5 part a")
plt.show()
```



In [12]:
```
rect_conv = nconv(rectangle(t),t,rectangle(t),t)
```

In [13]:
```
fig, q5a = plt.subplots(figsize=(10,10))
# y =np.exp((-1/9+1j*np.pi)*t)
q5a.plot(rect_conv[0],rect_conv[1])
q5a.set_xlabel("time t")
q5a.set_ylabel("Convolution of two rect(t)")
q5a.set_xlim([-2, 2])
q5a.set_ylim([0,3])
q5a.set_title("question 5 part b")
plt.show()
```



In [ ]: