ECE 102, Fall 2023 **Homework #7**

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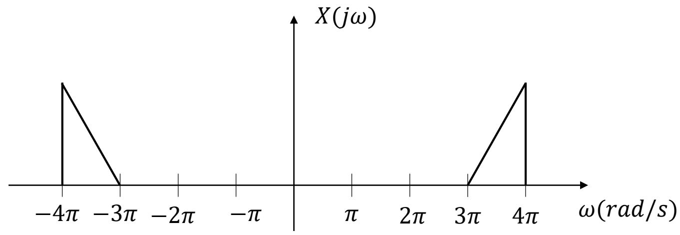
Due Friday, 8 Dec 2023, by 11:59pm to Gradescope.
100 points total.
Covers material on Filters, Sampling and Laplace and Inverse Laplace Transform (up to lecture 18)
100 points total.

1. (14 points) **Bandpass sampling**
The figure below shows the Fourier transform of a real bandpass signal, i.e., a signal whose frequencies are not centered around the origin. We want to sample this signal. Let F_s in Hz



represent the sampling frequency.

   (a) (4 points) One option is to sample this signal at the Nyquist rate. Then to recover the signal, we pass its sampled version through a low pass filter. What is the Nyquist rate of this signal?

   (b) (10 points) Since the signal might have high frequency components, Nyquist rate for this signal can be high. In other words, we need to have a lot of samples of the signal, which means that the sampling scheme is costly. It turns out that for this type of signal, we can sample it at a sampling frequency lower that the Nyquist rate and we can still recover the signal, however in this case, we will use a **bandpass** filter. To see this, we have the following two options for the sampling frequency:

   - $F_s = 0.5$ Hz;
   - $F_s = 1$ Hz;

   For each case, draw the spectrum of the signal after sampling it. To recover the signal, which F_s can we use? How we should choose the frequencies of the bandpass filter? What is the minimum F_s we can use and still recover the signal?

a) Nyquist rate $= T = \frac{1}{2B} \Rightarrow$

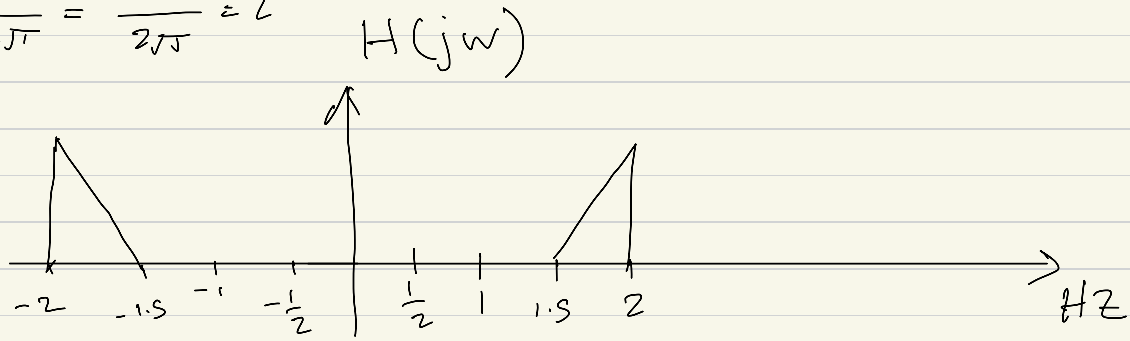
2B nyquist rate

$\Rightarrow 2(2) = 4$ HZ

$W_0 = 2\pi f \Rightarrow$

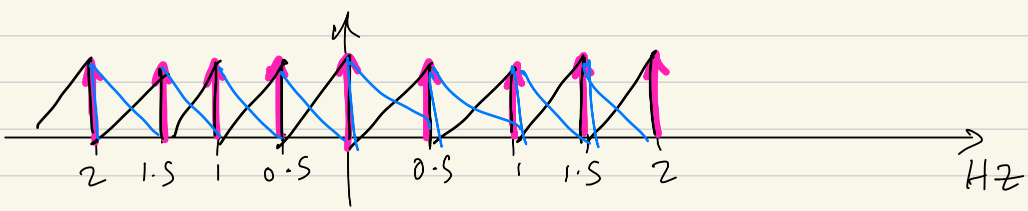
$f = \frac{W_0}{2\pi} = \frac{4\pi}{2\pi} = 2$

b)

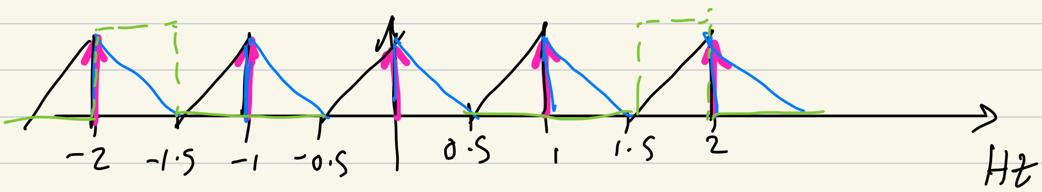


$\Rightarrow f_s = 0.5$ HZ

0.5 HZ



No band pass can get the original signal back.

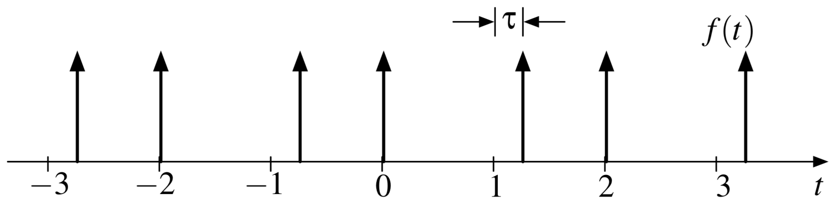


a band Pass at can get the original signal back.

$\Rightarrow 1.5 \le |f_s| \le 2$ HZ

or $3 \le |W_0| \le 4$ (rad/s)

2. (20 points) **Sampling with imperfect sampler**
Imperfections in a sampler cause characteristic artifacts in the sampled signal. In this problem we will look at the case where the sample timing is non-uniform, as shown below: The



sampling function $f(t)$ has its odd samples delayed by a small time τ.

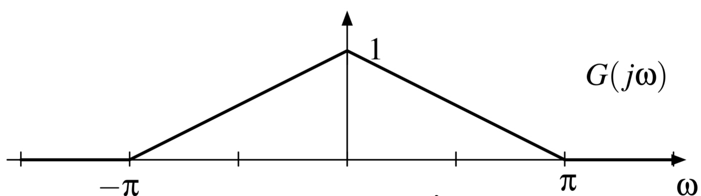
(a) (4 points) Write an expression for $f(t)$ in terms of two uniformly spaced sampling functions.

(b) (4 points) Find $F(j\omega)$, the Fourier transform of $f(t)$. Express the impulse trains as sums, and simplify.

(c) (4 points) Find $F(j\omega)$, for the case where $\tau = 0$, and show that this is aligns with your expectation.

(d) (4 points) Assume the signal we are sampling has a Fourier transform

$\Delta\left(\frac{W}{\pi}\right)$



$\Delta(\pi)$

Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$

(e) (4 points) If we know $g(t)$ is real and even, can we recover $g(t)$ from the non-uniform samples $g(t)f(t)$? .

2) $\delta_T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT) \Rightarrow$

$-3,-1,1,3,5,\dots$ is delayed by $\tau \Rightarrow \delta_2(t-(1+\tau))$

$-2,0,2,4,6,\dots$ is not delayed $\Rightarrow \delta_2(t)$

$\Rightarrow \delta_2(t-(1+\tau))+\delta_2(t)$

$$e^{-jk\pi}=(-1)^k$$
$$=\left(e^{-j\pi}\right)^k$$

b) $F(jw)=\pi\delta_\pi(w)+\pi\delta_\pi(w)e^{-jw(1+\tau)}$

$$=\pi\delta_\pi(w)\left[1+e^{-jw(1+\tau)}\right]$$

$$=\pi\sum_{k=-\infty}^{\infty}\delta(w-k\pi)\left[1+e^{-jk\pi(1+\tau)}\right]$$

$$=\pi\sum_{k=-\infty}^{\infty}\delta(w-k\pi)\left[1+e^{-jk\pi}e^{-jk\pi\tau}\right]$$

$$=\pi\sum_{k=-\infty}^{\infty}\delta(w-k\pi)\left[1+(-1)^k e^{-jk\pi\tau}\right]$$

c) if $\tau=0$, $F(jw)=2\pi\delta_{2\pi}(w) \Rightarrow \pi\sum_{k=-\infty}^{\infty}\delta(w-k\pi)\left[1+(-1)^k\right]$

d) $g_s(t) = g(t) \cdot f(t)$

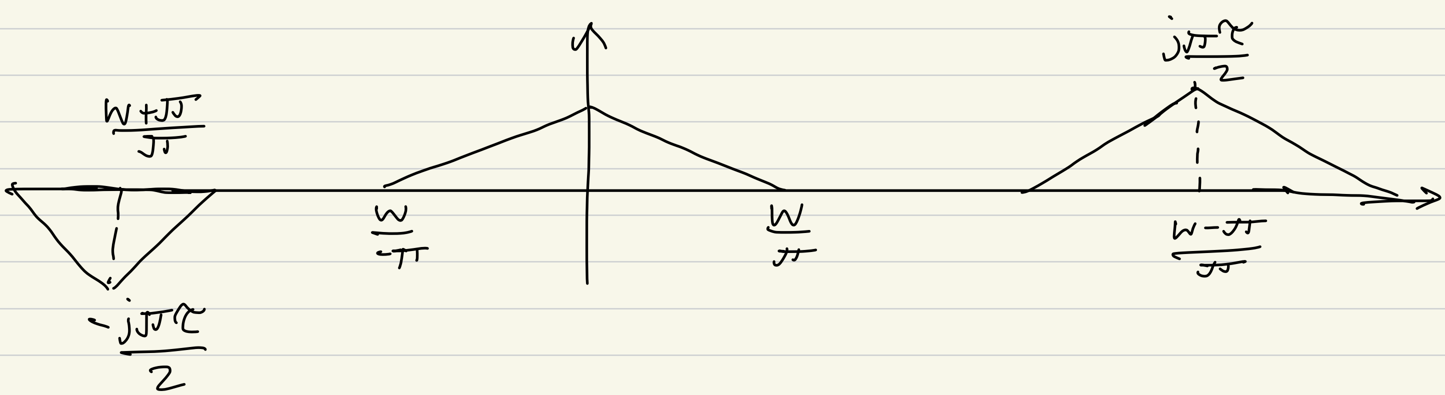
$$G_s(j\omega) = \frac{1}{2\pi} F(j\omega) * G(j\omega)$$

$$= \frac{1}{2\pi} \pi \left[\sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)\left[1+(-1)^k e^{-jk\pi\tau}\right]\right] * \Delta\left(\frac{\omega}{\pi}\right)$$

$$= \frac{1}{2}\left[\delta(\omega)[1+1] * \Delta\left(\frac{\omega}{\pi}\right) + \delta(\omega-\pi)\left(1-(1-j\pi\tau)\right) * \Delta\left(\frac{\omega}{\pi}\right) + \delta(\omega+\pi)\left(1-(1+j\pi\tau)\right) * \Delta\left(\frac{\omega}{\pi}\right)\right.$$

$$\Rightarrow \frac{1}{2}\left[2\Delta\left(\frac{\omega}{\pi}\right) + j\pi\tau\,\Delta\left(\frac{\omega-\pi}{\pi}\right) - j\pi\tau\,\Delta\left(\frac{\omega+\pi}{\pi}\right)\right]$$

e) $\Delta\left(\frac{\omega}{\pi}\right) + \frac{j\pi\tau}{2}\Delta\left(\frac{\omega-\pi}{\pi}\right) - \frac{j\pi\tau}{2}\Delta\left(\frac{\omega+\pi}{\pi}\right)$



$\Rightarrow$ by Low Pass filtering the Signal we can get the $\Delta\left(\frac{\omega}{\pi}\right)$

then

3. (18 points) **Sampling with alternating impulse train**

The figure shown below gives a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.

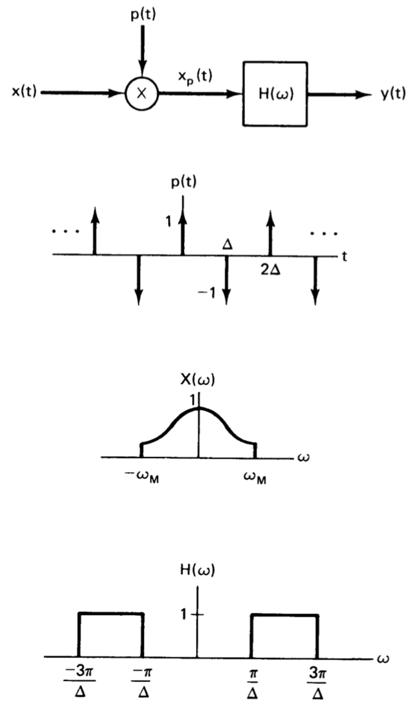


Figure 1: Sampling with alternating impulse train

(a) (6 points) For $\Delta < \frac{\pi}{2\omega_m}$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

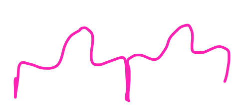
(b) (4 points) For $\Delta < \frac{\pi}{2\omega_m}$, determine a system that will recover $x(t)$ from $x_p(t)$.

(c) (4 points) For $\Delta < \frac{\pi}{2\omega_m}$, determine a system that will recover $x(t)$ from $y(t)$.

(d) (4 points) What is the maximum value of Δ in relation to ω_m for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$.

$\Delta = \frac{\pi}{2W_m}$

$\frac{2\Delta}{\pi} = W_m$



$\frac{2\Delta}{\pi} > W_m$

$\Delta > \frac{W_m \pi}{2}$

a) $x_p(t) = x(t) \cdot \left[\delta_{2\Delta}(t) - \delta_{2\Delta}(t-\Delta) \right]$

$$P(jw) = \frac{\pi}{\Delta} \delta_{\frac{\pi}{\Delta}}(w) - \frac{\pi}{\Delta} \delta_{\frac{\pi}{\Delta}}(w) e^{-jw\Delta} \Rightarrow \frac{\pi}{\Delta} \delta_{\frac{\pi}{\Delta}}(w) \left[1 - e^{-jw\Delta} \right]$$

$$X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw)$$

$$e^{-jk\pi} = (-1)^K$$

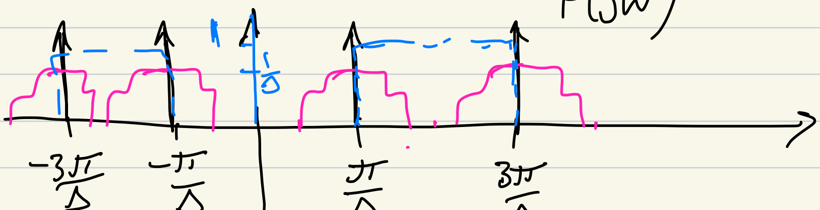
$$= \frac{1}{2\pi} X(jw) * \frac{\pi}{\Delta} \delta_{\frac{\pi}{\Delta}}(w) \left[1 - e^{-jw\Delta} \right]$$

$$= \frac{1}{2\pi} X(jw) * \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta\left(w - k\frac{\pi}{\Delta}\right) \left[1 - e^{-jk\frac{\pi}{\cancel{\Delta}}\cancel{\Delta}} \right]$$

$$= \frac{1}{2\pi} X(jw) * \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta\left(w - k\frac{\pi}{\Delta}\right) \left[1 - (-1)^k \right]$$

$$= \frac{1}{\Delta} X(jw) * \sum_{k \text{ odd}} \delta\left(w - k\frac{\pi}{\Delta}\right) \Rightarrow$$

$P(jw)$



$X(jw)$

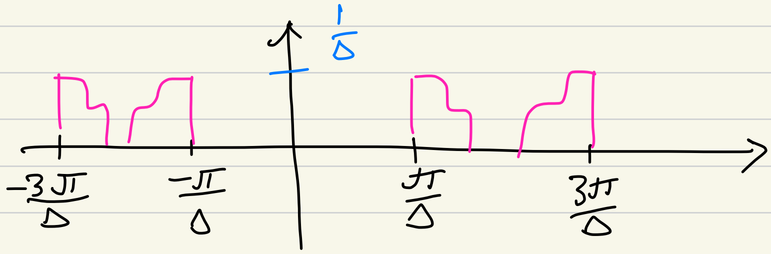
$$\Delta < \frac{\pi}{2w_m}$$

$$\frac{3\pi}{\frac{\pi}{2w_m}} = 6w_m$$

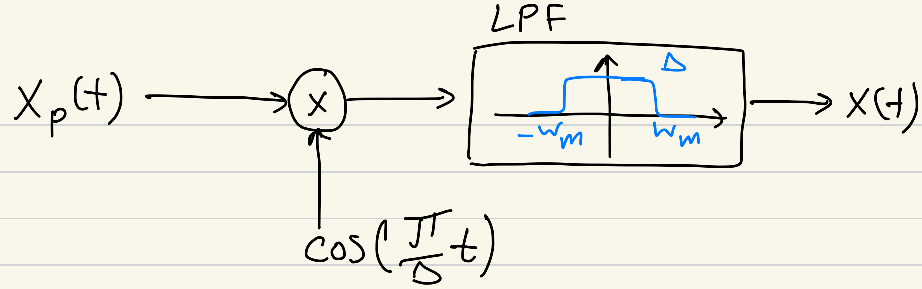
$$2w_m < \frac{\pi}{\Delta}$$

$$w_m < \frac{\pi}{2\Delta}$$

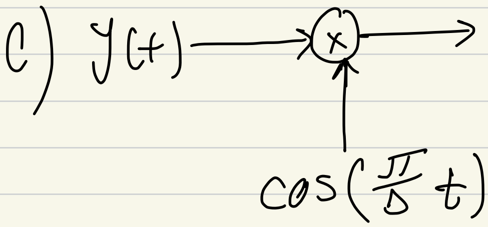
$Y(jw) =$



b)



c)

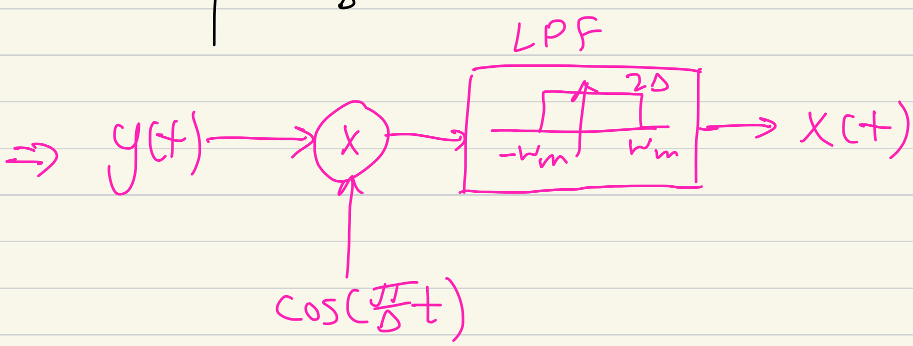


$$y(t)\cdot\cos(w_c t)$$

$$= \frac{1}{2\pi} Y(jw) * \pi\left(\delta(w+w_c) + \delta(w-w_c)\right)$$

$$= \frac{1}{2}\left(Y j(w+w_c) + Y(j(w-w_c))\right)$$

$$\Rightarrow \frac{1}{2}\left(Y(j(w_m + \frac{\pi}{\Delta}) + Y(j(w_m - \frac{\pi}{\Delta}))\right)$$



d) $w_m < \frac{\pi}{2\Delta}$ so if $\Delta = \frac{\pi}{2w_m}$ the signal will still be recoverable.

$\frac{2\Delta}{\pi} > w_m$ then signal will not be recoverable.

$\Delta > \frac{\pi w_m}{2}$

4. (20 points) **Laplace Transform**

   (a) Find the Laplace transforms of the following signals and determine their region of convergence.

      i. (5 points) $f(t) = te^{-at}(\sin \omega_0 t)^2 u(t)$

      ii. (5 points) $f(t) = e^{-b|t|}$ where $b \leq 0$

   (b) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: Re}\{s\} > -1$$

   Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

      i. (5 points) $\mathcal{F}\{x(t)e^{\frac{t}{2}}\}$

      ii. (5 points) $\mathcal{F}\{x(t)e^{2t}\}$

5. (12 points) **Inverse Laplace Transform**

   Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

   (a) (6 points) $F(s) = \dfrac{e^{-s}(s+1)}{(s-2)^2(s-3)}$

   (b) (6 points) $F(s) = \dfrac{s+4}{s^3 + 4s}$

6. (16 points) **LTI system**

   Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

   where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

   (a) (5 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

   (b) (5 points) Find the output $y(t)$ when the input is $x(t) = u(t)$.

   (c) (6 points) The system $\mathcal{S}_1$ is linearly cascaded with another causal LTI system $\mathcal{S}_2$. The system $\mathcal{S}_2$ is given by the following input-output pair:

$$\mathcal{S}_2 \quad \text{input} : u(t) - u(t-1) \rightarrow \text{output} : r(t) - 2r(t-1) + r(t-2)$$

   Find the overall impulse response.

4. (20 points) **Laplace Transform**

(a) Find the Laplace transforms of the following signals and determine their region of convergence.

i. (5 points) $f(t) = te^{-at}(\sin\omega_0 t)^2 u(t)$

ii. (5 points) $f(t) = e^{-b|t|}$ where $b \le 0$

(b) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: } \mathrm{Re}\{s\} > -1$$

Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

i. (5 points) $\mathcal{F}\{x(t)e^{\frac{t}{2}}\}$

ii. (5 points) $\mathcal{F}\{x(t)e^{2t}\}$

a) $f(t) = te^{-at}(\sin W_0 t)^2 u(t)$

$= te^{-at}\left[\frac{1-\cos(2W_0 t)}{2}\right]u(t)$

$= \frac{te^{-at}}{2}u(t) - \frac{te^{-at}\cos(2W_0 t)}{2}u(t)$

$t\,u(t) \Longleftrightarrow \frac{1}{s^2}$

$\frac{te^{-at}u(t)}{2} \Longleftrightarrow \frac{1}{2(s+a)^2}$

$\cos(2W_0 t)u(t) \Longleftrightarrow \frac{s}{s^2+(2W_0)^2}$

$t\cos(2W_0 t)u(t) \Longleftrightarrow -\frac{d}{ds}\left[\frac{s}{s^2+(2W_0)^2}\right]$

$\Rightarrow -\frac{(s^2+4W_0^2)(1) - s(2s)}{(s^2+(4W_0^2))^2} = \frac{-(-s^2+4W_0^2)}{(s^2+4W_0^2)^2}$

$= \frac{s^2-4W_0^2}{(s^2+4W_0^2)^2}$

$\frac{te^{-at}\cos(2W_0 t)u(t)}{2} = \frac{(s+a)^2-4W_0^2}{2((s+a)^2+4W_0^2)^2}$

$\Rightarrow \sigma = \mathrm{Re}\{s\} > 0$

$\sigma > 0$

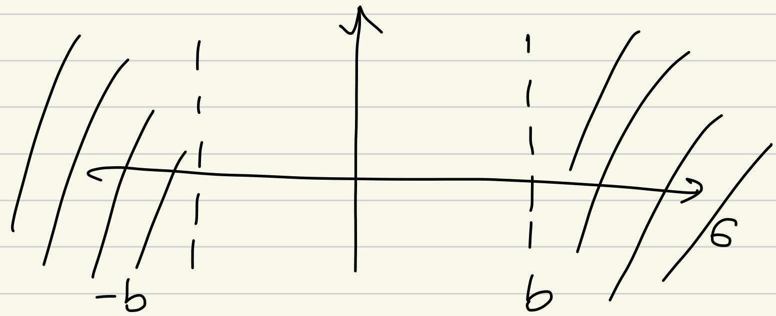
$\sigma > -a$

b ii) $f(t) = e^{-b|t|}$ $\quad b \leq 0$

$= e^{-bt} u(t) + e^{bt} u(-t)$

$\Updownarrow$

$F(s) = \frac{1}{s+b} + \frac{-1}{s-b}$



ROC: $\sigma > -b$ $\quad \sigma < b$

No Laplace Transform

b) $\frac{1}{s^2 + 2s + 5} = X(s)$

ROC: $Re\{s\} > -1$

$\mathcal{F}(x(t)e^{\frac{t}{2}})$

$\mathcal{L}(x(t)e^{\frac{t}{2}})$

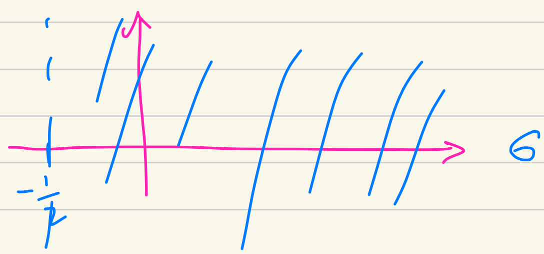
$\Rightarrow \frac{1}{(s-\frac{1}{2})^2 + 2(s-\frac{1}{2}) + 5}$

$\sigma > -1$

$\Rightarrow \sigma > -1 + \frac{1}{2}$

$\sigma > -\frac{1}{2}$

$\sigma > -\frac{1}{2}$

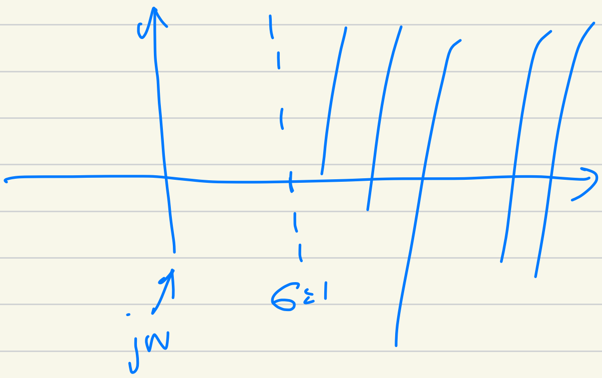


fourier transform exist !

$\Rightarrow s = 0 + j\omega$ $\quad \Rightarrow \frac{1}{(j\omega - \frac{1}{2})^2 + 2(j\omega - \frac{1}{2}) + 5}$

$X(j\omega) = \frac{1}{-\omega^2 + j\omega + \frac{17}{4}}$

ii) $F\left(x(t)\,e^{2t}\right) \Rightarrow \dfrac{1}{(s-2)^2+2(s-2)+5}$



$\sigma > -1$

$\sigma > -1+2 \Rightarrow \sigma > 1$

No fourier.

since fourier transform happens at $\sigma = 0$

$S = (\sigma = 0) + jw$

5)

5. (12 points) **Inverse Laplace Transform**

Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

(a) (6 points) $F(s) = \dfrac{e^{-s}(s+1)}{(s-2)^2(s-3)}$

(b) (6 points) $F(s) = \dfrac{s+4}{s^3+4s}$

$\Rightarrow$ a) $f(s) = \dfrac{e^{-s}(s+1)}{(s-2)^2(s-3)} \Rightarrow e^{-s}H(s)$

$$\Rightarrow \frac{(s+1)}{(s-2)^2(s-3)} = \frac{r_1}{(s-2)^2} + \frac{r_2}{(s-2)} + \frac{r_3}{(s-3)}$$

$$\Rightarrow (s-2)(s-3)r_2 + (s-3)r_1 + (s-2)^2 r_3 = (s+1)$$

$$(s^2-5s+6)r_2 + (s-3)r_1 + (s^2-4s+4)r_3 = (s+1)$$

$s^2 r_2 + s^2 r_3 = 0$

$-5s r_2 + s r_1 - 4s r_3 = s$

$-5r_2 + r_1 - 4r_3 = 1$

$6r_2 - 3r_1 + 4r_3 = 1$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -5 & 1 & -4 & 1 \\ 6 & -3 & 4 & \end{array}\right.$$

$r_2 = -4$

$r_1 = -3$

$r_3 = 4$

$$\Rightarrow \frac{-3}{(s-2)^2} + \frac{-4}{(s-2)} + \frac{4}{(s-3)} \Rightarrow -3te^{2t} - 4e^{2t} + 4e^{3t}$$

for $t \geq 0$

$$\Rightarrow -3(t-1)e^{2(t-1)} - 4e^{2(t-1)} + 4e^{3(t-1)} \quad \text{for } t \geq 1$$

$$b) \ \frac{S+4}{S^3+4S} \Rightarrow \quad S(S^2+4)$$

$$S^2+4 = (S-2j)(S+2j)$$

$$S^2 - 4j^2 = S+4$$

$$\Rightarrow \frac{S+4}{S(S^2+4)} \Rightarrow$$

$$\Rightarrow \frac{S+4}{(S)(S-2j)(S+2j)} \Rightarrow \frac{r_1}{S} + \frac{r_2}{S-2j} + \frac{r_3}{S+2j} = \frac{S+4}{(S)(S-2j)(S+2j)}$$

$$\Rightarrow^{S=0} \quad \frac{S+4}{S+4} = r_1 \Rightarrow r_1 = 1$$

$$\Rightarrow^{S=2j} \quad \Rightarrow \frac{S+4}{(S)(S+2j)} = \frac{r_2(S-2j)}{(S-2j)} \Rightarrow \frac{4+2j}{(2j)(2j+2j)} = \frac{4+2j}{-8} = r_2$$

$$S = -2j$$

$$\Rightarrow \quad \frac{S+4}{(S)(S-2j)} = \frac{r_3(S+2j)}{(S+2j)} \Rightarrow \frac{4-2j}{(-2j)(-2j-2j)} = \frac{4-2j}{8} = r_3$$

$$\Rightarrow \frac{S+4}{S^3+4S} = \frac{1}{S} - \frac{\frac{4+2j}{8}}{S-2j} + \frac{\frac{4-2j}{8}}{S+2j}$$

$$\Rightarrow \frac{1}{S} - \frac{4+2j}{8(S-2j)} + \frac{4-2j}{8(S+2j)}$$

$$\Rightarrow \frac{1}{S} - \frac{(2+j)}{4}\left[\frac{1}{S-2j}\right] + \frac{(2-j)}{4}\left[\frac{1}{S+2j}\right]$$

$$\Rightarrow u(t) - \left(\frac{2+j}{4}\right)\left[e^{2jt}\right] + \frac{2-j}{4}\left[e^{-2jt}\right]$$

6)

6. (16 points) **LTI system**

Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

(a) (5 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

(b) (5 points) Find the output $y(t)$ when the input is $x(t) = u(t)$.

(c) (6 points) The system $\mathcal{S}_1$ is linearly cascaded with another causal LTI system $\mathcal{S}_2$. The system $\mathcal{S}_2$ is given by the following input-output pair:

$$\mathcal{S}_2 \quad \text{input} : u(t) - u(t-1) \rightarrow \text{output} : r(t) - 2r(t-1) + r(t-2)$$

Find the overall impulse response.

$\Rightarrow$ a) $s^2 Y(s) + 5s\,Y(s) + 4Y(s) = aX(s)$

$$H(s) = \frac{Y(s)}{X(s)}$$

$\Rightarrow Y(s)\left[(s+4)(s+1)\right] = aX(s)$

$$\Rightarrow H(s) = \frac{a}{(s+4)(s+1)}$$

$$\Rightarrow e^t \rightarrow \boxed{H(s)} \longrightarrow \frac{1}{2}e^t$$

$$e^{st} \rightarrow \boxed{H(s)} \longrightarrow H(s)e^{st}$$

$$H(1) = \frac{a}{(5)(2)} = \frac{a}{10} = \frac{1}{2} \Rightarrow a = \frac{10}{2} = 5 \quad \Rightarrow H(s) = \frac{5}{(s+4)(s+1)}$$

$x(t) = u(t) \Rightarrow \mathcal{L}(u(t)) = \frac{1}{s}$

$\Rightarrow Y(s) = H(s)X(s)$

$$\Rightarrow \frac{5}{(s+4)(s+1)(s)} = \frac{r_3}{s+4} + \frac{r_2}{s+1} + \frac{r_1}{s}$$

$\Rightarrow r_1(s+1)(s) + r_2(s+4)(s) + r_3(s+1)(s+4) = 5$

$r_1 = \frac{5}{4} \qquad r_2 = -\frac{5}{3} \qquad r_3 = \frac{5}{12}$

$$\Rightarrow \frac{\left(\frac{5}{12}\right)}{(s+4)} - \frac{\left(\frac{5}{3}\right)}{(s+1)} + \frac{\left(\frac{5}{4}\right)}{s} \Rightarrow \frac{5}{4}u(t) - \frac{5}{3}e^{-t}u(t) + \frac{5}{12}e^{-4t}u(t)$$

$t \geq 0$

$$\Rightarrow y(t) = \frac{5}{4} - \frac{5}{3}e^{-t} + \frac{5}{12}e^{-4t}$$

c) $$X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

$$H_2(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}\left(1 - e^{-s}\right)$$

$$H_{eq}(s) = H_1(s)H_2(s) = \frac{5}{(s+4)(s+1)} \cdot \frac{1}{s}\left(1 - e^{-s}\right)$$

$$= \frac{5}{s(s+4)(s+1)} - \frac{5e^{-s}}{s(s+4)(s+1)}$$

$$\underbrace{\frac{5}{4} - \frac{5}{3}e^{-t} + \frac{5}{12}e^{-4t}}_{t \geq 0} - \underbrace{\frac{5}{4} - \frac{5}{3}e^{-(t-1)} + \frac{5}{12}e^{-4(t-1)}}_{t \geq 1}$$

$$h_{eq}(t) = \frac{5}{4}(u(t)) - \frac{5}{3}e^{-t}u(t) + \frac{5}{12}e^{-4t}u(t) - \frac{5}{4}u(t-1) - \frac{5}{3}e^{-(t-1)}u(t-1)$$

$$+ \frac{5}{12}e^{-4(t-1)}u(t-1)$$