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On the Chambers–Mallows–Stuck method for simulating skewed stable random variables

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Abstract

In this note, we give a proof to the equality in law of a skewed stable variable and a nonlinear transformation of two independent uniform and exponential variables. The lack of an explicit proof of this formula has led to some inaccuracies in the literature. The Chambers et al. (1976) method of computer generation of a skewed stable random variable is based on this equality.

Keywords: Stable distribution; Characteristic function; Random variable generation

1. Introduction

The Central Limit Theorem, which offers the fundamental justification for approximate normality, points to the importance of α-stable distributions: they are the only limiting laws of normalized sums of independent, identically distributed random variables. Gaussian distributions, the best-known member of the stable family, have long been well-understood and widely used in all sorts of problems. However, they do not allow for large fluctuations and are thus inadequate for modeling high variability. In the last two or three decades, data which seem to fit the non-Gaussian stable model have been collected in fields as diverse as economics, telecommunications, hydrology and physics. For a review see Janicki and Weron (1994), Samorodnitsky and Taqqu (1994) and Zolotarev (1986).

This situation calls for a fast generator of stable random variables. However, with a few exceptions, there are no analytic expressions for the inverse F^{-1} of the stable distribution function and the *inverse transform method* cannot be used. Chambers et al. (1976) constructed a direct method based on (3.2) and (3.3). In what follows, we give proofs to these equalities (Theorem 3.1).

The importance of this result may be emphasized also by the fact that it is used, for example, to generate Linnik's random variables (see Devroye, 1990) and discrete stable and Linnik's random variables (see

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Devroye, 1993). Moreover, analytical properties of the functions used in this equality have been extensively studied by Buckle (1994).

2. α-Stable distributions

The stable distribution can be most conveniently described by its *characteristic function* (c.f.). The following formula is derived from the so-called Lévy representation of the c.f. of an infinitely divisible law, given in Lévy (1934) (for details see Hall, 1981).

Definition 2.1. A random variable X is α-stable if and only if its characteristic function is given by

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - \mathrm{i}\beta\, \mathrm{sign}(t) \tan\frac{\pi\alpha}{2}\} + \mathrm{i}\mu t, & \alpha \neq 1, \\ -\sigma |t| \{1 + \mathrm{i}\beta\, \mathrm{sign}(t) \frac{2}{\pi} \log|t|\} + \mathrm{i}\mu t, & \alpha = 1, \end{cases} \tag{2.1}$$

where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$, $\mu \in R$.

Since (2.1) is characterized by four parameters we will denote α-stable distributions by $S_\alpha(\sigma, \beta, \mu)$ and write

$$X \sim S_\alpha(\sigma, \beta, \mu) \tag{2.2}$$

to indicate that X has the stable distribution with the *characteristic exponent* (index) α, *scale parameter* σ, *skewness* β, and *location parameter* μ. When $\sigma = 1$ and $\mu = 0$ the distribution is called *standard stable*.

The canonical representation (2.1) has one disagreeable feature. The functions $\phi(t)$ are not continuous functions of the parameters determining them, they have discontinuities at all points of the form $\alpha = 1$, $\beta \neq 0$. However, as Zolotarev (1986) remarks, setting

$$\mu_1 = \begin{cases} \mu + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \mu, & \alpha = 1, \end{cases} \tag{2.3}$$

yields the expression

$$\log \phi(t) = \begin{cases} -\sigma^\alpha \{|t|^\alpha - \mathrm{i}t\beta(|t|^{\alpha-1} - 1) \tan\frac{\pi\alpha}{2}\} + \mathrm{i}\mu_1 t, & \alpha \neq 1, \\ -\sigma |t| \{1 + \mathrm{i}\beta\, \mathrm{sign}(t) \frac{2}{\pi} \log|t|\} + \mathrm{i}\mu_1 t, & \alpha = 1, \end{cases} \tag{2.4}$$

which is a function jointly continuous in α and β. The drawback of this form is that μ_1 does no longer have the natural interpretation as a location parameter. Most authors, therefore, use the form (2.1) of the c.f.

Another form of the c.f., whose use can be justified by considerations of an analytic nature (see Zolotarev, 1986), is the following.

Definition 2.2. A random variable X is α-stable iff its characteristic function is given by

$$\log \phi(t) = \begin{cases} -\sigma_2^\alpha |t|^\alpha \exp\{-\mathrm{i}\beta_2\, \mathrm{sign}(t) \frac{\pi}{2} K(\alpha)\} + \mathrm{i}\mu t, & \alpha \neq 1, \\ -\sigma_2 |t| \{\frac{\pi}{2} + \mathrm{i}\beta_2\, \mathrm{sign}(t) \log|t|\} + \mathrm{i}\mu t, & \alpha = 1, \end{cases} \tag{2.5}$$

where

$$K(\alpha) = \alpha - 1 + \mathrm{sign}(1 - \alpha) = \begin{cases} \alpha, & \alpha < 1, \\ \alpha - 2, & \alpha > 1. \end{cases} \tag{2.6}$$

The parameters σ_2 and β_2 are related to σ and β, from the representation (2.1), as follows. For $\alpha \neq 1$, β_2 is such that

$$\tan\left(\beta_2 \frac{\pi K(\alpha)}{2}\right) = \beta \tan\frac{\pi\alpha}{2}, \tag{2.7}$$

and the new scale parameter

$$\sigma_2 = \sigma\left(1 + \beta^2 \tan^2\frac{\pi\alpha}{2}\right)^{1/(2\alpha)}. \tag{2.8}$$

For $\alpha = 1$, $\beta_2 = \beta$ and $\sigma_2 = \frac{2}{\pi}\sigma$.

The *probability density functions* (p.d.f.) of stable random variables exist and are continuous but, with a few exceptions, they are not known in closed form. The exceptions are:

- *the Gaussian distribution*: $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$,
- *the Cauchy distribution*: $S_1(\sigma, 0, \mu)$,
- *the Lévy distribution*: $S_{1/2}(\sigma, 1, \mu)$,
- *and the case obtained from the latter by using* (2.13): $S_{1/2}(\sigma, -1, \mu)$.

Zolotarev (1986) gives integral representations of the p.d.f.'s for all values of the parameters α and β. We present his result in terms of the *distribution functions* (d.f.).

Proposition 2.1 (Zolotarev, 1986, Remark 1, page 78). *Let*

$$\varepsilon(\alpha) = \operatorname{sign}(1 - \alpha),$$

$$\gamma_0 = -\frac{\pi}{2}\beta_2 \frac{K(\alpha)}{\alpha},$$

$$C(\alpha, \beta_2) = 1 - \tfrac{1}{4}(1 + \beta_2 K(\alpha)/\alpha)(1 + \varepsilon(\alpha)),$$

$$U_\alpha(\gamma, \gamma_0) = \left(\frac{\sin\alpha(\gamma - \gamma_0)}{\cos\gamma}\right)^{\alpha/(1-\alpha)} \frac{\cos(\gamma - \alpha(\gamma - \gamma_0))}{\cos\gamma}, \tag{2.9}$$

and

$$U_1(\gamma, \beta_2) = \frac{\frac{\pi}{2} + \beta_2\gamma}{\cos\gamma} \exp\left(\frac{1}{\beta_2}\left(\frac{\pi}{2} + \beta_2\gamma\right)\tan\gamma\right). \tag{2.10}$$

Then the d.f. $F(x, \alpha, \beta_2)$ *of a standard stable random variable, whose c.f. is of the form* (2.5), *can be written as follows*

- *if* $\alpha \neq 1$ *and* $x > 0$ *then*

$$F(x, \alpha, \beta_2) = C(\alpha, \beta_2) + \frac{\varepsilon(\alpha)}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma, \gamma_0)]\,d\gamma, \tag{2.11}$$

- *if* $\alpha = 1$ *and* $\beta_2 > 0$ *then*

$$F(x, 1, \beta_2) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \exp[-e^{-x/\beta_2} U_1(\gamma, \beta_2)]\,d\gamma. \tag{2.12}$$

The cases $\alpha \neq 1, x < 0$ and $\alpha = 1, \beta_2 < 0$ can be reduced to the corresponding cases $\alpha \neq 1, x > 0$ and $\alpha = 1$, $\beta_2 > 0$ with the help of the following equality:

$$F(-x, \alpha, \beta_2) + F(x, \alpha, -\beta_2) = 1, \tag{2.13}$$

which is valid, for both forms: (2.1) and (2.5), for any real x and any admissable parameters α and β_2 (or β).

3. Computer generation of α-stable random variables

The complexity of the problem of simulation of sequences of stable random variables results from the fact that there are no analytic expressions for the inverse F^{-1} of the d.f. The only exceptions are the Gaussian, the Cauchy and the Lévy distributions, for which simple methods of simulation have been found.

A solution to the problem was found by a path started in the article by Kanter (1975), in which a direct method was given for simulating $S_\alpha(1,1,0)$ random variables, for $\alpha < 1$. It turned out that this method was easily adapted to the general case. Chambers et al. (1976) were the first to give the formulas. However, they did not supply a proof and only gave reference to an article by Zolotarev (1966) where expressions of the type (2.11) and (2.12) could be found. The lack of explicit proofs of these formulas has led to some inaccuracies in the literature. In this note, we want to clarify the situation and present the proofs.

Lemma 3.1. *Let γ_0 and $U_\alpha(\gamma, \gamma_0)$ be defined as in Proposition* 2.1. *For $\alpha \neq 1$ and $\gamma_0 < \gamma < \frac{\pi}{2}$, X is a $S_\alpha(1, \beta_2, 0)$ random variable* (*in representation* (2.5)) *iff for $x > 0$*,

$$\frac{1}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma, \gamma_0)]\,\mathrm{d}\gamma = \begin{cases} P(0 < X \leqslant x), & \alpha < 1, \\ P(X \geqslant x), & \alpha > 1. \end{cases} \tag{3.1}$$

Proof. *Case of* $0 < \alpha < 1$: From (2.11) we have

$$\begin{aligned} F(x, \alpha, \beta_2) &= P(X \leqslant x) \\ &= \frac{1-\beta_2}{2} + \frac{1}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma, \gamma_0)]\,\mathrm{d}\gamma \\ &= \frac{1-\beta_2}{2} + P(0 < X \leqslant x), \end{aligned}$$

because for $\alpha < 1$, $(1-\beta_2)/2 = P(X \leqslant 0)$ (see Zolotarev, 1986, Remark 2, p. 79).

Case of $1 < \alpha \leqslant 2$: From (2.11) we have

$$\begin{aligned} F(x, \alpha, \beta_2) &= P(X \leqslant x) \\ &= 1 - \frac{1}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma, \gamma_0)]\,\mathrm{d}\gamma \\ &= 1 - P(X \geqslant x). \end{aligned}$$

This completes the proof. $\square$

Theorem 3.1. *Let γ_0 be defined as in Proposition 2.1. Let γ be uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and W be an independent exponential random variable with mean* 1. *Then*

- *for* $\alpha \neq 1$

$$X = \frac{\sin\alpha(\gamma - \gamma_0)}{(\cos\gamma)^{1/\alpha}} \left(\frac{\cos(\gamma - \alpha(\gamma - \gamma_0))}{W}\right)^{(1-\alpha)/\alpha}, \tag{3.2}$$

  is $S_\alpha(1, \beta_2, 0)$ *and*

- *for* $\alpha = 1$

$$X = \left(\frac{\pi}{2} + \beta_2\gamma\right)\tan\gamma - \beta_2 \log\left(\frac{W\cos\gamma}{\frac{\pi}{2} + \beta_2\gamma}\right) \tag{3.3}$$

  is $S_1(1, \beta_2, 0)$

for the representation (2.5).

Proof. When $\gamma > \gamma_0$ then the right-hand side of (3.2) is positive and can be expressed as

$$\left(\frac{a(\gamma)}{W}\right)^{(1-\alpha)/\alpha}, \tag{3.4}$$

where

$$a(\gamma) = \left(\frac{\sin\alpha(\gamma - \gamma_0)}{\cos\gamma}\right)^{\alpha/(1-\alpha)} \frac{\cos(\gamma - \alpha(\gamma - \gamma_0))}{\cos\gamma}. \tag{3.5}$$

Case of $0 < \alpha < 1$: Eq. (3.2) implies that $X > 0$ iff $\gamma > \gamma_0$. Since $(1 - \alpha)/\alpha > 0$, we can write

$$\begin{aligned} P(0 < X \leqslant x) &= P(0 < X \leqslant x, \gamma > \gamma_0) \\ &= P(0 < (a(\gamma)/W)^{(1-\alpha)/\alpha} \leqslant x, \gamma > \gamma_0) \\ &= P(W \geqslant x^{\alpha/(\alpha-1)} a(\gamma), \gamma > \gamma_0) \\ &= \boldsymbol{E}_\gamma \exp[-x^{\alpha/(\alpha-1)} a(\gamma)] 1_{\{\gamma > \gamma_0\}} \\ &= \frac{1}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} a(\gamma)]\,\mathrm{d}\gamma, \end{aligned}$$

where $\boldsymbol{E}_\gamma$ denotes the expectation with respect to γ. From Lemma 3.1 and (2.13) we conclude that $X \sim S_\alpha(1, \beta_2, 0)$.

Case of $1 < \alpha \leqslant 2$: Since $(\alpha - 1)/\alpha > 0$, for $x > 0$ we can write

$$\begin{aligned} P(X \geqslant x) &= P(X \geqslant x, \gamma > \gamma_0) \\ &= P((a(\gamma)/W)^{(1-\alpha)/\alpha} \geqslant x, \gamma > \gamma_0) \\ &= P((W/a(\gamma))^{(\alpha-1)/\alpha} \geqslant x, \gamma > \gamma_0) \\ &= P(W \geqslant x^{\alpha/(\alpha-1)} a(\gamma), \gamma > \gamma_0) \\ &= \boldsymbol{E}_\gamma \exp[-x^{\alpha/(\alpha-1)} a(\gamma)] 1_{\{\gamma > \gamma_0\}} \\ &= \frac{1}{\pi}\int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} a(\gamma)]\,\mathrm{d}\gamma. \end{aligned}$$

From Lemma 3.1 and (2.13) we conclude that $X \sim S_\alpha(1, \beta_2, 0)$.

Case of $\alpha = 1$: For $\beta_2 = 0$, the right-hand side of (3.3) reduces to $\frac{\pi}{2}\tan\gamma$ whose distribution is Cauchy (in representation (2.5)). When $\beta_2 \neq 0$, it can be expressed as

$$\beta_2 \log\left(\frac{a_1(\gamma)}{W}\right), \tag{3.6}$$

where

$$a_1(\gamma) = \frac{\frac{\pi}{2} + \beta_2\gamma}{\cos\gamma} \exp\left(\frac{1}{\beta_2}\left(\frac{\pi}{2} + \beta_2\gamma\right)\tan\gamma\right). \tag{3.7}$$

We have for $\beta_2 > 0$,

$$\begin{aligned} P(X \leqslant x) &= P(\beta_2 \log(a_1(\gamma)/W) \leqslant x) \\ &= P(W \geqslant \mathrm{e}^{-x/\beta_2} a_1(\gamma)) \\ &= \boldsymbol{E}_\gamma \exp[-\mathrm{e}^{-x/\beta_2} a_1(\gamma)] \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \exp[-\mathrm{e}^{-x/\beta_2} a_1(\gamma)]\,\mathrm{d}\gamma. \end{aligned}$$

From (2.12) and (2.13) we conclude that for all β_2, $X \sim S_1(1, \beta_2, 0)$. This completes the proof. $\square$

Applying this theorem we can easily construct a method of computer generation of a skewed random variable $X \sim S_\alpha(1, \beta, 0)$, in the representation (2.1). For $\alpha \in (0, 2]$ and $\beta \in [-1, 1]$:

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;
- for $\alpha \neq 1$ compute

$$X = S_{\alpha,\beta} \times \frac{\sin(\alpha(V + B_{\alpha,\beta}))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_{\alpha,\beta}))}{W}\right)^{(1-\alpha)/\alpha}, \tag{3.8}$$

  where

$$B_{\alpha,\beta} = \frac{\arctan(\beta \tan\frac{\pi\alpha}{2})}{\alpha},$$

$$S_{\alpha,\beta} = \left[1 + \beta^2 \tan^2\frac{\pi\alpha}{2}\right]^{1/(2\alpha)};$$

- for $\alpha = 1$ compute

$$X = \frac{2}{\pi}\left[\left(\frac{\pi}{2} + \beta V\right)\tan V - \beta \log\left(\frac{W\cos V}{\frac{\pi}{2} + \beta V}\right)\right]. \tag{3.9}$$

$B_{\alpha,\beta}$ accounts for the parameter change from β_2 to β and takes place of γ_0 in (3.2). $S_{\alpha,\beta}$ accounts for the parameter change from σ_2 to σ (see (2.8)).

Formula (3.8) was presented by Janicki and Weron (1994). However, they gave an incorrect form for $C_{\alpha,\beta}$ (the denominator is $1 - |1 - \alpha|$ instead of α, Formula (3.5.2), p. 50), which corresponds to our $B_{\alpha,\beta}$, and a computationally more complicated form for $D_{\alpha,\beta}$ (our $S_{\alpha,\beta}$). They also did not provide the formula for $\alpha = 1$.

Chambers et al. (1976) gave a formula ((2.3) on p. 341) for $\alpha \neq 1$ equivalent to (3.8), in the representation (2.5). Their formula for $\alpha = 1$ ((2.4) on p. 341) has a slightly incorrect form: under the logarithm is

$(\frac{\pi}{2}W\cos V)/(\frac{\pi}{2}+\beta V)$ whereas it should be $(W\cos V)/(\frac{\pi}{2}+\beta V)$. However, this has no impact on their numerical algorithm RSTAB, since it uses a continuous representation, equivalent to (2.4), instead of (2.5) and thus uses only a reparametrization of (3.8).

We have given formulas for simulation of standard stable random variables. Using the following property, which follows from the form of the c.f., we can generate a stable random variable for all admissable values of the parameters α, σ, β and μ:

If $X \sim S_\alpha(1,\beta,0)$ then

$$Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1, \\ \sigma X + \frac{2}{\pi}\beta\sigma\log\sigma + \mu, & \alpha = 1, \end{cases}$$

is $S_\alpha(\sigma,\beta,\mu)$.

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