NON-REPEATING NIM

SINAN CEPEL (SCEPEL@ANDREW.CMU.EDU)

This paper deals with a variation of Nim where if any player took k stones from a pile, k stones may not be taken by either player until a different move is made on that pile. The rest of the rules are identical to that of Nim.

We analyze the nimbers for a single pile of stones in this variation of Nim. By Sprague-Grundy, the analysis of one pile is sufficient to understand the complete game.

Definition For any n and $k \in \mathbb{N} \cup \{0\}$, e define (n, k) to be the Nim pile with n stones remaining, where the last move removed k stones from the pile. Let $U = \mathbb{N} \times \mathbb{N}$ denote the universe of Nim piles.

Definition Define $N : U \to \mathbb{N} : (n, k) \mapsto \text{nimber}(n, k)$ be the function that maps a Nim pile to it's nimber.

Theorem If n odd, $N(n, n) = 0$.

Proof. Since the first player cannot remove all stones, he must make a move $m < n$, reducing the pile to $(n-m, m)$. Since n odd, $n-m \neq m$, and the second player can make the move $n - m$, reducing the pile to 0. Hence the game is a second-player min, implying that $N(n, n) = 0$.

Fact 1 If $k \geq 12$, $N(22, k) = 12$.

Fact 2 If $n < 22$, for all k, $N(n, k) \neq 12$.

Fact 3 If $n \geq 39$, and $k \neq n$, $N(n, k) > 11$.

Fact 4 The only $n < 39$ that satisfy $N(n, k) = 12$ and $k \neq n - 22$ are $n = 24, k = 1$ and $n = 32, k = 5$.

Claim If $n \geq 39$, $N(n, k) = 12$ only if $k = n - 22$.

Proof. Suppose $k \neq n - 22$. Then, by Fact 3, $N(n, k) \geq 12$. Since $n \geq 39$, $n-22 \geq 17$, so we can use Fact 1, implying that $N(22, n-22) = 12$. Since $k \neq n - 22$, we can make this move, implying that $N(n, k) > 12$, proving the claim. □

Claim If $n \geq 39$, n odd, and $k = n - 22$, $N(n, k) = 12$.

Proof. By Fact 3, $N(n, k) \geq 12$, so we need to show we cannot achieve a pile of nimber 12.

Since $n \geq 39$, by Fact 4, we cannot reach the states (32, 5) or (24, 1), and $k = n-22$, so we cannot reach $(22, n-22)$. Hence, to reach a pile of nimber 12, we have to make a move $m \neq k$ such that $N(n-m,m) = 12$, but by the previous claim, $m = n - m - 22$.

But look, $n - m + n - m - 22 = 2(n - m - 11)$ is even, while n is odd, so such a move does not exist. Hence, $N(n,k) = 12$. $\square$

Claim Let a be an arbitrary odd number. All Nim piles of the form $(2^b a, 2^b a)$ are N-positions if and only if b is odd. Hence, if b is even, $(2^b a, 2^b a)$ is a P-position.

Proof. By induction on b.

The base case, where $b = 0$, holds by our first theorem.

For the inductive step, suppose the proposition holds for some b, and consider $b + 1$.

Our initial position is $(2^{b+1}a, 2^{b+1}a)$.

If $b + 1$ odd, then b was even, so $(2^b a, 2^b a)$ is a P-position, and the removing $2^b a$ stones from the pile reduces the state to $(2^{b+1}a - 2^b a, 2^b a) = (2^b a, 2^b a)$. We have a move that takes the game to a P-position, so $(2^{b+1}a, 2^{b+1}a)$ is an N-position in this case.

Otherwise, $b + 1$ even. The current player cannot take all stones. If the current player makes any move $m \neq 2^b a$, then the other player can make the move $2^{b+1}a - m \neq m$, reducing the pile to 0 stones. If the current player makes the move $2^b a$, the game is reduced to a state $(2^b a, 2^b a)$, but since b odd, this is an N-position for the next player.

Hence, no matter how the current player moves, this pile is a P-position, and the claim holds. $\square$