Boolean Algebra

Logic Design

Boolean Algebra

- Axioms and theorems to **simplify** Boolean equations
- Like regular algebra, but simpler: variables have only two values (1 or 0)
- **Duality** in axioms and theorems:
  - ANDs and ORs, 0's and 1's interchanged

Some Definitions

- **Complement**: variable with a bar over it

  $\bar{A}, \bar{B}, \bar{C}$

- **Literal**: variable or its complement

  $A, \bar{A}, B, \bar{B}, C, \bar{C}$

- **Implicant**: product of literals

  $AB\bar{C}, A\bar{C}, \bar{B}C$

Boolean Axioms

Number	Axiom	Name
A1	$B = 0$ if $B \neq 1$	Binary Field
A2	$\overline{0} = 1$	NOT
A3	$0 \bullet 0 = 0$	AND/OR
A4	$1 \bullet 1 = 1$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	AND/OR

Boolean Axioms

Number	Axiom	Dual	Name
A1	$B = 0$ if $B \neq 1$	$B = 1$ if $B \neq 0$	Binary Field
A2	$\overline{0} = 1$	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	$1 + 0 = 0 + 1 = 1$	AND/OR

Dual: Replace $\bullet$ with $+$
0 with 1

Boolean Theorems of One Variable

Number	Theorem	Name
T1	$B \bullet 1 = B$	Identity
T2	$B \bullet 0 = 0$	Null Element
T3	$B \bullet B = B$	Idempotency
T4	$\overline{\overline{B}} = B$	Involution
T5	$B \bullet \overline{B} = 0$	Complements

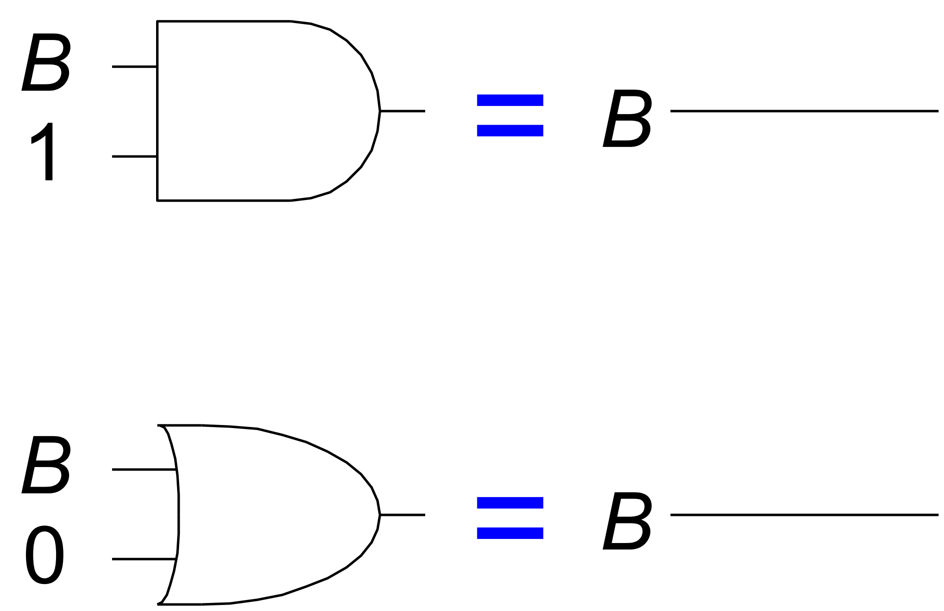
Boolean Theorems of One Variable

Number	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Dual: Replace $\bullet$ with $+$
0 with 1

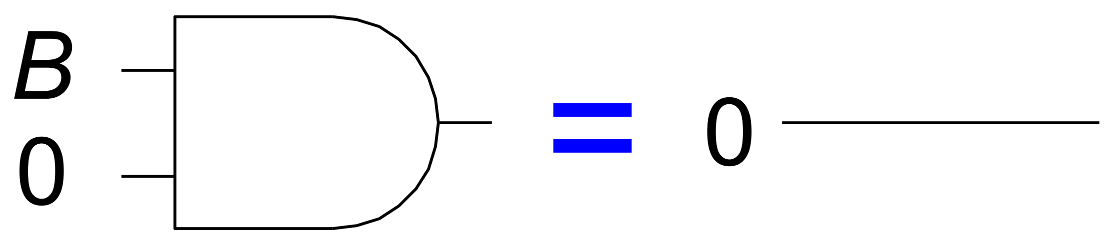
T1: Identity Theorem

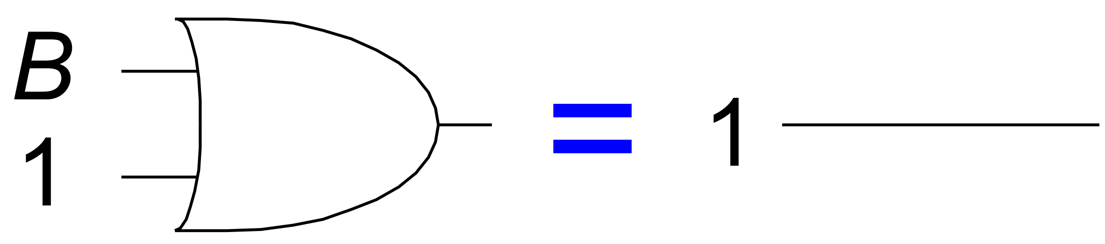
- B • 1 = B
- B + 0 = B



T2: Null Element Theorem

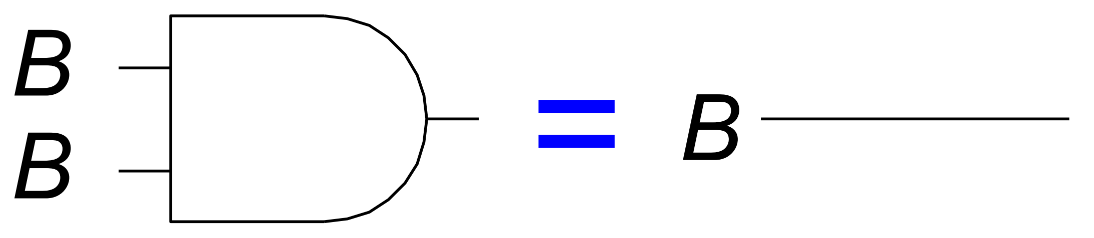
- $B \cdot 0 = 0$
- $B + 1 = 1$

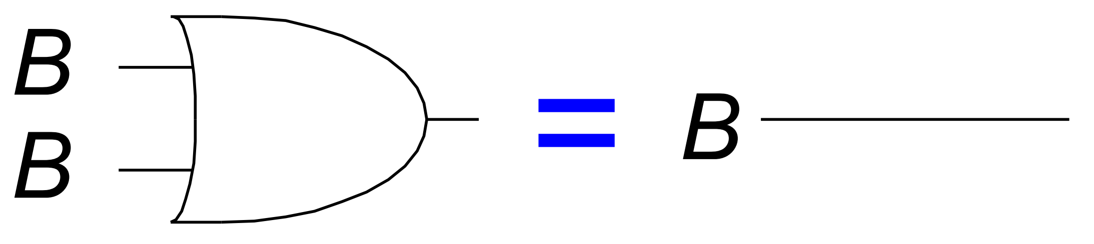




T3: Idempotency Theorem

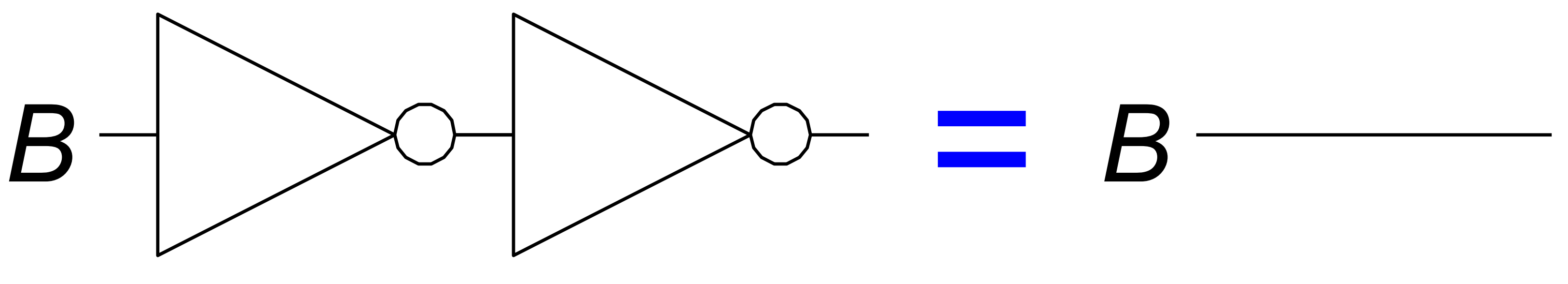
- B • B = B
- B + B = B





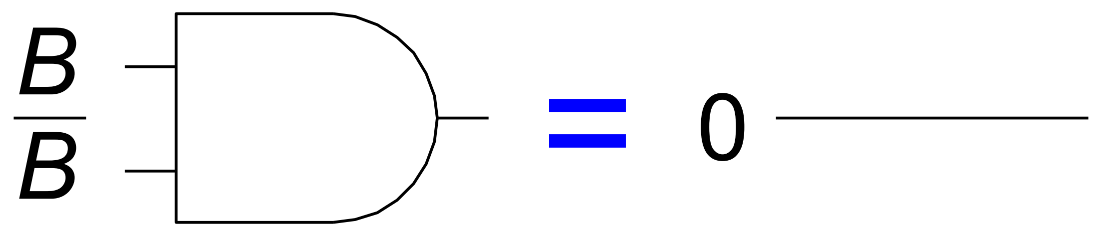
T4: Identity Theorem

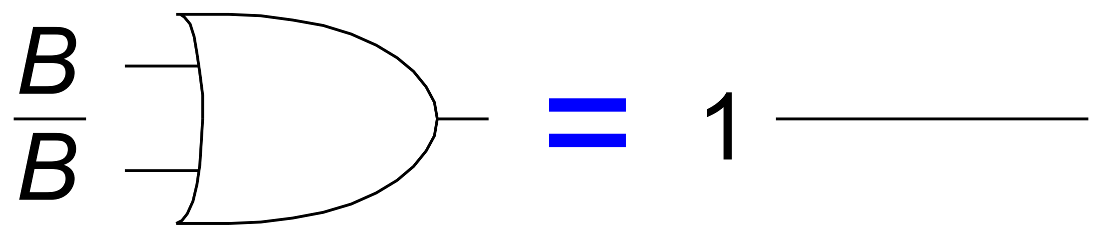
- $\overline{\overline{B}} = B$



T5: Complement Theorem

- $B \bullet \overline{B} = 0$
- $B + \overline{B} = 1$





Recap: Basic Boolean Theorems

Number	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Dual: Replace $\bullet$ with $+$
0 with 1

Boolean Theorems of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B+C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

How to Prove

- **Method 1:** Perfect induction
- **Method 2:** Use other theorems and axioms to simplify the equation
  - Make one side of the equation look like the other

Proof by Perfect Induction

- Check every possible input value
- If two expressions produce the same value for every possible input combination, the expressions are equal

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	B•C = C•B	Commutativity

B	***C***	***BC***	***CB***
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Boolean Theorems of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B+C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B\bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

$B\bullet(B+C) = B\bullet B + B\bullet C$	T8: Distributivity
$= \mathbf{B} + B\bullet C$	T3: Idempotency
$= B\bullet(1 + C)$	T8: Distributivity
$= B\bullet(\mathbf{1})$	T2: Null element
$= B$	T1: Identity

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining

Prove true using other axioms and theorems:

$$B \bullet C + B \bullet \overline{C} = B \bullet (C + \overline{C}) \quad \text{T8: Distributivity}$$

$$= B \bullet (\mathbf{1}) \quad \text{T5': Complements}$$

$$= B \quad \text{T1: Identity}$$

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove.

Boolean Theorems of Several Vars

#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B+C = C+B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B+C)\,(B+D)$	Distributivity
T9	$B \bullet (B+C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B+C) \bullet (B+\overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	$(B+C) \bullet (\overline{B}+D) \bullet (C+D) =$ $(B+C) \bullet (\overline{B}+D)$	Consensus

Dual: Replace $\bullet$ with $+$
0 with 1

Simplifying an Equation

Reducing an equation to the **fewest number of implicants**, where each implicant has the **fewest literals**

Recall:

- **Implicant:** product of literals

  $A\bar{B}C, \bar{A}C, \bar{B}C$

- **Literal:** variable or its complement

  $A, \bar{A}, B, \bar{B}, C, \bar{C}$

*Also called **minimizing** the equation*

Simplification methods

- **Distributivity (T8, T8')** $B(C+D) = BC + BD$

  $B + CD = (B+C)(B+D)$
- **Covering (T9')** $A + AP = A$
- **Combining (T10)** $P\overline{A} + PA = P$
- **Expansion** $P = P\overline{A} + PA$

  $A = A + AP$
- **Duplication** $A = A + A$
- **"Simplification" theorem** $P\overline{A} + A = P + A$

  $PA + \overline{A} = P + \overline{A}$

Proving the "Simplification" Theorem

"Simplification" theorem

$$PA + \overline{A} = P + \overline{A}$$

Method 1: $PA + \overline{A}$

	$= PA + (\mathbf{\overline{A} + \overline{A}P})$	**T9' Covering**
	$= PA + \mathbf{P\overline{A}} + \overline{A}$	**T6 Commutativity**
	$= \mathbf{P(A + \overline{A})} + \overline{A}$	**T8 Distributivity**
	$= P\mathbf{(1)} + \overline{A}$	**T5' Complements**
	$= P + \overline{A}$	**T1 Identity**

Proving the “Simplification” Theorem

“Simplification” theorem

$$PA + \overline{A} = P + \overline{A}$$

Method 2: $PA + \overline{A}$ $= (\overline{A} + A)(\overline{A} + P)$ **T8’ Distributivity**

$= 1(\overline{A} + P)$ **T5’ Complements**

$= \mathbf{\overline{A} + P}$ **T1 Identity**

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove using other theorems and axioms:

$B \bullet C + \overline{B} \bullet D + C \bullet D$

$= BC + \overline{B}D + (CDB + CD\overline{B})$	**T10: Combining**
$= BC + \overline{B}D + BCD + \overline{B}CD$	**T6: Commutativity**
$= BC + BCD + \overline{B}D + \overline{B}CD$	**T6: Commutativity**
$= (BC + BCD) + (\overline{B}D + \overline{B}CD)$	**T7: Associativity**
$= BC + \overline{B}D$	**T9’: Covering**

Simplifying Boolean Equations

Example 1:

$Y = AB + A\overline{B}$

Simplifying Boolean Equations
Example 2:

$Y = A(AB + ABC)$

Simplifying Boolean Equations Example 3:

$Y = A'BC + A'$ **Recall:** $A' = \overline{A}$

Simplifying Boolean Equations
Example 4:

$Y = AB'C + ABC + A'BC$

Simplifying Boolean Equations
Example 5:

$Y = AB + BC + B'D' + AC'D'$

Method 1:

$Y = AB + BC + B'D' + (ABC'D' + AB'C'D')$	T10: Combining
$= (AB + ABC'D') + BC + (B'D' + AB'C'D')$	T6: Commutativity
	T7: Associativity
$= AB + BC + B'D'$	T9: Covering

Method 2:

$Y = \mathbf{AB} + BC + \mathbf{B'D'} + AC'D' + \mathbf{AD'}$	T11: Consensus
$= AB + BC + B'D' + AD'$	T9: Covering
$= AB + BC + B'D'$	T11: Consensus

Simplifying Boolean Equations
Example 6:

$Y = (A + BC)(A + DE)$