Digital Signal Processing

Embedded Systems

Based on Lecture Notes from Koen Langendoen

Signals and Frequency Synthesis

Usually signals are composed of signals with many frequencies.

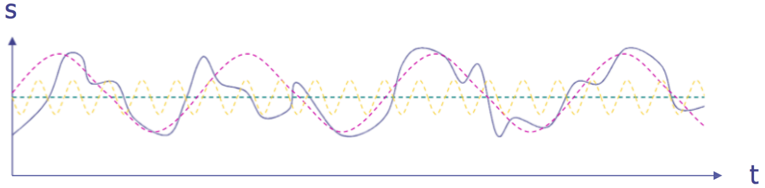


Figure: **s** contains 0 Hz component (green dashed line), lowest freq component (purple dashed line), higher freq component (yellow dashed line), and others.

Fourier

Any periodic signal with base frequency f_b can be constructed from **sine waves** with frequency f_b, $2f_b$, $3f_b$, ...

Sine wave: $s(t) = A\cos(2\pi ft + \theta)$ where A is the *amplitude*, f is the *frequency* and $\theta \in [0, 2\pi]$ is the *phase*.

Frequency Spectrum

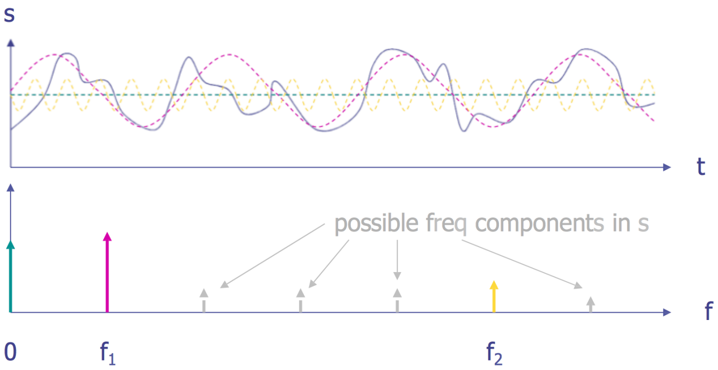


Figure: Frequency spectrum of signal **s**.

Instead of the time waveforms (time-domain), the frequency-domain representation gives the information required to synthesize **s**.

Filter: Frequency Response

Often filters are designed to filter frequency components in a signal.

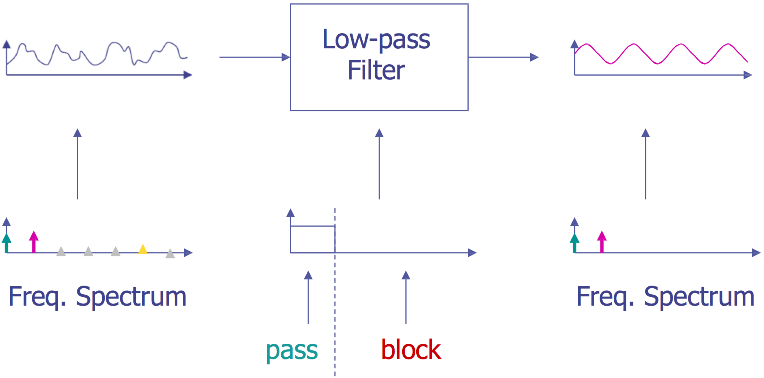


Figure: Frequency response of the filter.

Sampling A Signal

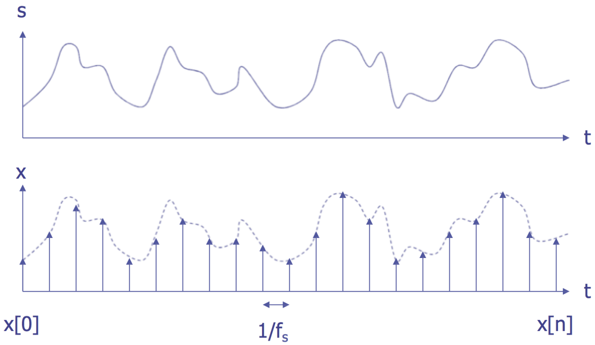
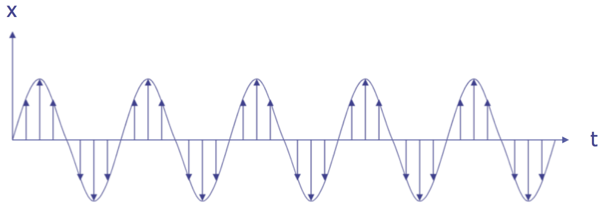


Figure: **s** sampled at discrete time intervals (sample frequency f_s): $x[n]$

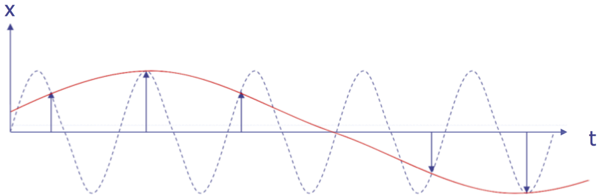
A sine wave: $x(t) = Acos(2\pi ft)$
Sampled wave: $x[n] = x(nT_s) = x(n/f_s) = Acos(n2\pi f/f_s)$

normalized frequency: f/f_s

Sampling: Avoid Aliasing



$f_s \geq 2 * f_{max}$ in **s**: **OK**



$f_s < 2 * f_{max}$ in **s**: you see non-existing low-freq signal(s)

Shannon Sampling Theorem: $f_{max}/f_s \leq 1/2$

Example Filter: Moving Average

Figure: $y[n] = 1/3x[n] + 1/3x[n-1] + 1/3x[n-2]$

MA Filter

```
x[0] = get_sample();
y[0] = (x[0]+x[1]+x[2])/3;
put_sample(y[0]);
x[2] = x[1]; x[1] = x[0];
```

MA filter filters (removes) signals of certain frequency.

Question: Input to MA is x with frequency f and amplitude 1. What is the frequency and amplitude of output y ?

Frequency Behavior MA

lower frequency x: amplitude y = 0.77

x = 0.00, 0.33, 0.66, 1.00, 0.66, 0.33, 0.00, -0.33, -0.66, -1.00, -0.66, -0.33, 0.00

y = 0.00, 0.11, 0.33, 0.66, 0.77, 0.66, 0.33, 0.00, -0.33, -0.66, -0.77, -0.66, -0.33

transient steady-state

higher frequency x: amplitude y = 0.33

x =0.00, 1.00, 0.00, -1.00, 0.00, 1.00, 0.00, -1.00, 0.00, 1.00, 0.00, -1.00, 0.00

y = 0.00, 0.33, 0.33, 0.00, -0.33, 0.00, 0.33, 0.00, -0.33, 0.00, 0.33, 0.00, -0.33

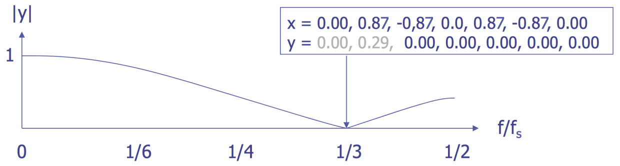


Figure: $y[n] = 1/3x[n] + 1/3x[n-1] + 1/3x[n-2]$

f/f_s: normalized frequency

Z Transform: Definition

Let $x[n]$ be a signal in the time domain (n).

The Z transform of $x[n]$ is given by

$$X(z) = \sum_n x[n]z^{-n}$$

where z is a complex variable.

Example:

$x = 0.00, 0.33, 0.66, 1.00, 0.66, \ldots$

$X = 0 + 0.33z^{-1} + 0.66z^{-2} + z^{-3} + 0.66z^{-4} + \ldots$

Z Transform: Properties

Let $y[n] = x[n-1]$ *(delayed signal)*. Then $Y(z) = z^{-1}X(z)$.

Example:

$x = 0.00, 0.33, 0.66, 1.00, 0.66, \ldots$
$X = 0 + 0.33z^{-1} + 0.66z^{-2} + z^{-3} + 0.66z^{-4} + \ldots$
$y = 0.00, 0.00, 0.33, 0.66, 1.00, \ldots$
$Y = 0 + 0z^{-1} + 0.33z^{-2} + 0.66z^{-3} + z^{-4} + \ldots = z^{-1}X$

Z transform of $Ka[n] = KA(z)$.
Z transform of $a[n] + b[n] = A(z) + B(z)$.

Example:

$x = 0.00, 0.33, 0.66, 1.00, 0.66, \ldots$
$X = 0 + 0.33z - 1 + 0.66z - 2 + z - 3 + 0.66z - 4 + \ldots$
$y = 0.00, 0.66, 1.32, 2.00, 1.32, \ldots$
$Y = 0 + 0.66z - 1 + 1.32z - 2 + 2.00z - 3 + 1.32z - 4 + \ldots = 2X$

Apply Z transform to MA Filter

$$
\begin{aligned}
y[n] &= 1/3x[n] + 1/3x[n-1] + 1/3x[n-2] \\
Y(z) &= 1/3X(z) + 1/3z^{-1}X(z) + 1/3z^{-2}X(z) \\
&= (1/3 + 1/3z^{-1} + 1/3z^{-2})X(z) \\
&= H(z)X(z)
\end{aligned}
$$

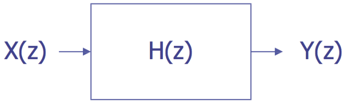


Figure: It holds $Y(z) = H(z)X(z)$, where $H(z)$ is **filter transfer function**

Frequency response of filter can be read from $H(z)$—determines *amplification* of $X(z)$

Frequency Response $H(z)$

The variable z is a *complex* variable and encodes frequency f according to

$$z = e^{j2\pi f} = cos(2\pi f) + jsin(2\pi f)$$

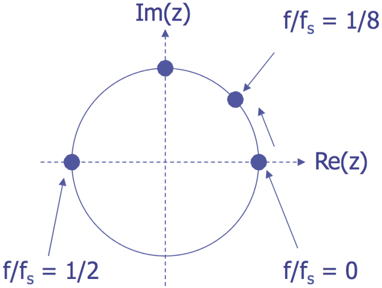


Figure: z-plane and the unit circle.

$H(z)$ reveals frequency response: $H(f) = H(z)|z = e^{j2\pi f}$.

Fourier Interpretation $H(z)$

Recall Z transform of $x[n]$ equals $X(z) = \sum_n x[n]z^{-n}$.

Fourier Transform

$$X(f) = \sum_n x[n]e^{-j2\pi nf}$$

For a filter with transfer function $H(f)$, its frequency response for a signal with frequency f is $|H(f)|$.

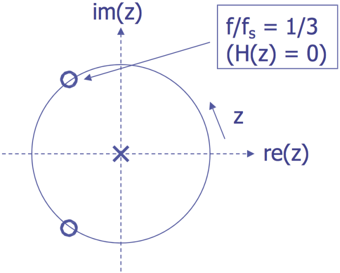
By substituting $z = e^{j2\pi f}$ in $H(z)$ we essentially obtain the Fourier transform $H(f)$ of which we know $|H(f)|$ is the frequency response.

So let $z = e^{j2\pi f}$ and evaluate $|H(z)|$!

Frequency Response MA Filter

The transfer function of the MA filter is given by:

$$H(z) = (1/3 + 1/3z^{-1} + 1/3z^{-2})$$
$$= (1/3z^2 + 1/3z + 1/3)/z^2$$



Determine **poles** and **zeros** of $H(z)$:

zero (= root of numerator):

$z_1 = -1/2 + 1/2\sqrt{3}j, z_2 = -1/2 - 1/2\sqrt{3}j \implies H(z_{1,2}) = 0$

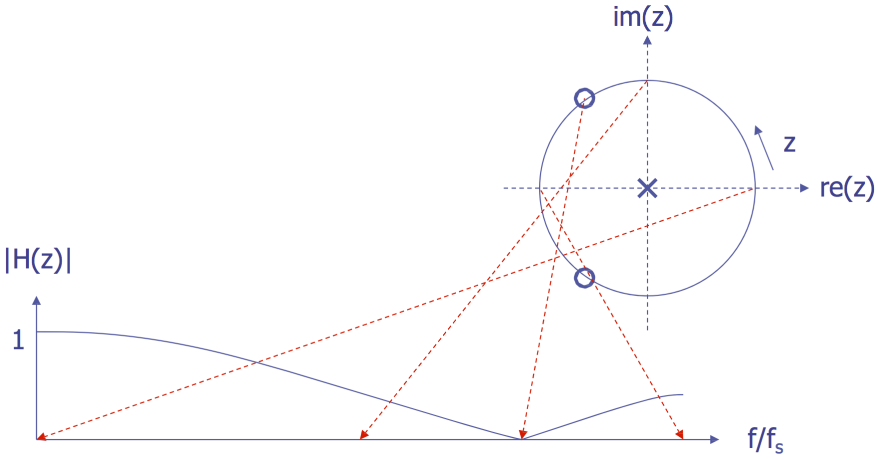
pole (= root of denominator):

$z_3, z_4 = 0 \implies H(z_{3,4}) = \infty$

Simply inspect distance z to poles/zeros.

Frequency Response MA Filter

Interpret $H(z)$ while traversing the unit circle (**upper half only** since $f/f_s \leq 1/2$):



Impulse Response (IR)

Impulse signal $\delta[n] = 1, 0, 0, 0, \ldots.$ (Dirac pulse)
Impulse response (IR) of a filter:

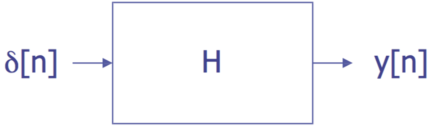


Figure: Characteristics for H

MA filter: $y[n] = 1/3x[n] + 1/3x[n-1] + 1/3x[n-2]$
Let $x[n] = \delta[n]$, then $y[n] = 1/3, 1/3, 1/3, 0, 0, 0\ldots$

Z Transform: $X(z) = 1,\ Y(z) = H(z).1 = H(z) = 1/3 + 1/3z^{-1} + 1/3z^{-2}$

Impulse signal δ reveals $H(z)$ in terms of $h[n]$

Impulse Response (IR)

MA filter: $h[n] = 1/3, 1/3, 1/3, 0, 0, 0 \ldots$
The IR is **finite**.

Filters defined by

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] \ldots$$

always have a finite IR and are therefore called FIR filters

Averaging Filter

$y[n] = 1/Nx[n] + 1/Nx[n-1] + \ldots 1/Nx[n-N-1]$ Suppose we don't want to implement an N-cell FIFO + 2N operationss and experiment with the following "short cut":

$$y[n] = (N-1)/Ny[n-1] + 1/Nx[n].$$

Frequency Response:

$$\begin{aligned} Y(z) &= (N-1)/Nz^{-1}Y(z) + 1/NX(z) \\ H(z) &= (1/N)/(1–(N-1)/Nz^{-1}) \\ &= (z/N)/(z–(N-1)/N) \end{aligned}$$

Frequency Response Comparison

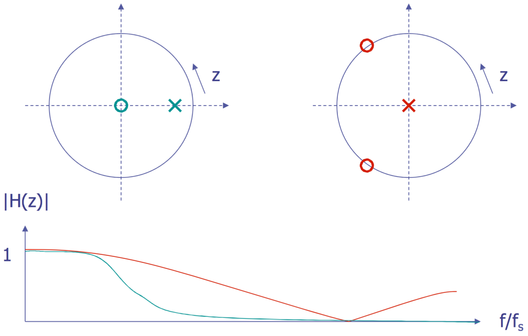


Figure: Averaging vs. MA filter

Pole-zero plot is quite different: now poles not zero
Frequency response is (therefore) more low-pass than MA filter.
The closer the pole is to unit circle (larger N), the sooner is the cut-off (in terms of frequency f).

Impulse Response

Filter equation: $y[n] = (N-1)/Ny[n-1] + 1/Nx[n]$
IR (N = 3): $h[n] = 1/3, (2/3)/3, (2/3)^2/3, \ldots, (2/3)^n/3, \ldots$
The IR is **infinite**.

Filters defined by

$$b_0 y[n] + b_1 y[n-1] + \ldots = a_0 x[n] + a_1 x[n-1] + \ldots$$

always have an infinite IR and are therefore called **IIR filters** (the equation is recursive in y)