THE BUCK-PASSING GAME

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Abstract. We consider situations where a finite number of agents want to transfer the responsibility of doing a job (the buck) to their neighbors in a social network. This can be seen as network variation of the public good model. The goal of each agent is to see the buck coming back as rarely as possible. We frame this situation as a game where players are the vertices of a directed graph and the strategy space of each player is the set of her out-neighbors. Nature assigns the buck to a random player according to a given initial distribution. Each player pays a cost that corresponds to the asymptotic expected frequency of times that she gets the buck. We consider two versions of the game. In the deterministic one each player chooses one of her out-neighbors once and for all at the beginning of the game. In the stochastic version a player chooses a probability distribution that determines which of her out-neighbors will be chosen when she passes the buck. We show that in both cases the game admits a generalized ordinal potential whose minimizers provide equilibria in pure strategies, even when the strategy set of each player is uncountable. We also show the existence of equilibria that are prior-free, in the sense that they do not depend on the initial distribution used to initially assign the buck. We provide different characterizations for the potential, we analyze fairness of equilibria, and, finally, we discuss a buck holding variant in which players want to maximize the frequency of times they hold the buck. As an application of the latter we briefly discuss the PageRank game.

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1. Introduction

1.1. **The problem.** Frequently, individuals living in a closed community tend to unload on their neighbors the responsibility of fixing a problem. Each citizen would benefit from having the problem fixed, but, since this involves a personal cost, agents prefer to let somebody else do the job. This can be seen as a network version of the classical free riding problem in which agents want to enjoy a public good and have somebody else pay for it. In our model agents can only *pass the buck* to their neighbors in an existing social network. They would like to be bothered by this issue as little as possible, so they will do their best to not see it coming back in the future. Strategically it is not enough to just pass the buck. The designated neighbor must be chosen in a way that the buck will not cycle back too fast. Thus, the structure of the network plays a key role in the solution of this problem.

1.2. **Our contribution.** We model the situation as a game on a finite directed graph, where agents are vertices of the graph and directed edges represent the possibility for an agent to transfer the buck to another agent. We look at situations where the first agent to hold the buck is drawn at random by Nature. We first consider the deterministic case in which agents designate the neighbor to whom they will transfer the buck, once and for all at the beginning of the game. Their goal is to see the buck coming back to them as rarely as possible, and the cost for an agent is the expected asymptotic frequency of times that she gets the buck. Although the costs depend on the asymptotic behavior of a process over time, our game is static, since players choose their strategy at the beginning of the game and play the same action whenever their turn comes, without any updating based on the history of the game.

To show existence of pure Nash equilibria, we prove that the game has a generalized ordinal potential. A classical result by Monderer and Shapley (1996) then guarantees that its minimizers are pure Nash equilibria. Moreover, there always exist equilibria that are prior-free, i.e., do not depend on the initial distribution according to which Nature makes its draw. In general the game may have multiple Nash equilibria and some of them might be prior-sensitive.

We then look at a stochastic version of the game in which the agents choose the probability with which the buck is passed to each of their neighbors. This gives rise to a Markov chain. When this chain is irreducible its unique stationary distribution is precisely the cost vector of the game, and does not dependent on Nature's initial distribution. In general, we prove the existence of a generalized ordinal potential and we use it to establish the existence of prior-free Nash equilibria. We then use the Markov chain tree theorem to derive an explicit formula for the potential function. We provide two alternative characterizations for the potential, showing that it can be written either as the expected length of unicycles in the graph, or in terms of the eigenvalues of the Laplacian matrix. All of this establishes a new mathematically intriguing bridge between the buck-passing game and Markov chains.

We also investigate fairness of the equilibria in buck-passing games, that is, we study how unevenly the total cost is spread across players in equilibrium in comparison to what could be achieved by a benevolent planner who wants to minimize disparity of treatment. In the spirit of Rawls (2009), we define the social cost function of a strategy profile as the highest cost across all players. Then we use the price of anarchy and the price of stability to measure fairness. Typically these quantities are used to measure efficiency of the worst and the best equilibrium, respectively: the social cost is usually taken to be the sum of the costs of all the players. Since our buck-passing game is a constant-sum game, efficiency is not an issue, but, using a Rawlsian social cost functions, the price of anarchy and the price of stability can be used as a measure of fairness.

In the last section we consider a class of games, called buck-holding games, which have the same structure as buck-passing games, except that the cost becomes a payoff, and the goal of each player is to see the buck coming back as often as possible. We prove the existence of equilibria for this class of games and we show analogies and differences with respect to buck-holding games. We also show that this class of games can be used to model PageRank, a tool to rank webpages.

1.3. **Related literature.** The issue of taking care for fixing a problem that affects an entire community has been considered in different domains such as economics, finance, behavioral science, medicine, political science, law, and philosophy.

Hood (2002) studies the issue of responsibility in the *risk industry*, considering safety and hazard for food, mobile phones, dangerous behavior of people and animals, looking at risk perception and amplification with their political implications. Steffel et al. (2016) discuss several hypotheses concerning delegation of choice, while León et al. (2018) study the ability of individuals to assign responsibility between various levels of government. Gardiner (2006) considers the issue of intergenerational fairness, focusing on climate change and nuclear protection, and proposes a *global core precautionary principle* to prevent one generation of individuals from shifting the negative effects of their choices to the following generations. della Paolera et al. (2011) study one and a half century of the Argentinian economy and model macroeconomic policies as a game between past, present, and future generations of political rulers and economic agents. Sonnenberg (2005) uses a game-theoretic model to analyze the practice of physicians to refer patients with difficult-to-handle pathologies to some other specialist, even when it is clear that the patient will not reap any benefit from seeing a new doctor. The model considered by Bolle (2017) is closer in spirit to ours. In his model a finite number of players decide one after the other whether to pay the cost of an action that is beneficial for the society, or to pass the buck to the next player. He assumes that each player has incomplete information about the preferences of the other players, and studies the Bayesian equilibrium of the game.

Various classes of games on networks have been considered in the literature. In some of them, which go under the name "network games", the payoff of a player depends only on her strategy and on the strategies of her neighbors (see, e.g., Galeotti et al., 2010, Kearns et al., 2001, Parise and Ozdaglar, 2019, among many) This is not the case in our model: the payoff of each player depends on the strategies of all the other players.

Some of the equilibria in the buck-passing game are prior-free. A concept of belief-free equilibrium has been considered by several authors in the framework of extensive form games. The term derives from the fact that a player's belief about his opponent's history is not needed for computing a best-reply. The concept has been studied for the repeated prisoner dilemma Ely and Välimäki (2002), Piccione (2002), and then generalized to larger classes of games by Ely et al. (2005), Miyagawa et al. (2008), Hörner and Lovo (2009), Yamamoto (2009, 2014), and others. In Scarsini and Tomala (2012) belief-free equilibria were studied for repeated congestion games on traffic networks. In Bergemann and Morris (2017) belief-free rationalizability is examined. Heller (2017) shows that belief-free equilibria are not robust, in the sense that only trivial belief-free equilibria may satisfy evolutionary stability. A prior-free approach has also been used in mechanism design. For a survey we refer to Hartline and Karlin (2007), while for recent contributions we cite Chawla et al. (2014), Eden et al. (2018), Roughgarden and Talgam-Cohen (2016).

As mentioned before, our fairness criterion is inspired by the work of Rawls (2009). We adapted to this criterion the typical measures of inefficiency, i.e., the price of anarchy (Koutsoupias and Papadimitriou, 2009, 1999, Papadimitriou, 2001), and the price of stability (Anshelevich et al., 2008, Schulz and Stier Moses, 2003). In most of the literature the social cost is the sum of the costs of all the players. Since our buck-passing game is a constant-sum game, using this social cost produces only trivial results. Other social cost functions were considered, for instance, in Koutsoupias and Papadimitriou (2009, 1999), Vetta (2002), Mavronicolas et al. (2008), Fournier and Scarsini (2019).

The main tools that we use to prove the existence of pure Nash equilibria are the existence of a generalized ordinal potential and the finite improvement property. The relationship between these concepts is studied by Monderer and Shapley (1996). We also rely on classical results for Markov chains, for which we refer to the books of Aldous and Fill (2002), Levin and Peres (2017), Norris (1998). In particular we exploit the celebrated Markov chain tree theorem, attributed to Leighton and Rivest (1986) (see also Anantharam and Tsoucas, 1989), which relates the stationary measure of the chain to the abundance of spanning trees in the underlying graph. Counting the number of spanning trees in an undirected graph in terms of the Laplacian's spectral properties goes back to Kirchhoff (1847) and has been generalized to weighted directed graphs by Brooks et al. (1940), see also Chaiken and Kleitman (1978). Some generalization and variations of these results have been proposed recently, see, e.g., Avena and Gaudillière (2018).

The seminal paper by Brin and Page (1998) introduces the so-called PageRank dynamics as a tool to rank webpages, the ancestor of the algorithm used nowadays by Google to produce an ordered list of pages as output of a query. In the past decade, PageRank was intensively studied in both the theoretical and applied literature, see, e.g. Andersen et al. (2008), Caputo and Quattropani (2019), Chen et al. (2017), Garavaglia et al. (2018), Jeh and Widom (2003), Lee and Olvera-Cravioto (2017). The papers by Avrachenkov and Litvak (2006), de Kerchove et al. (2008) are of particular interest for our work. We refer the reader to Gleich (2015) for an overview of the subject.

Finally, we point out that our model is related to a research stream that connects Markov chains with the classical Hamiltonian cycle problem (see, e.g., Borkar et al., 2004, 2009, 2012, Ejov et al., 2008, 2004, 2011, Filar and Krass, 1994, Litvak and Ejov, 2009).

1.4. **Organization of the paper.** The paper is organized as follows. Section 2 analyzes the deterministic version of the buck-passing game. Section 3 introduces the stochastic model. Section 4 studies fairness of the equilibria. Section 5 Introduces the probabilistic tools which are needed to show the existence of a generalized ordinal potential function. Section 6 deals with the potential nature of the game. Section 7 examines a class of games where players have an interest in seeing the buck as often as possible and explores the PageRank model.

1.5. **Graph terminology and notations.** Throughout this paper we consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which represents a social network, with $\mathcal{V} = \{1, \dots, n\}$

the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. We assume that $\mathcal{G}$ is *simple* with at most one edge between any two vertices, and without loops. The following standard terminology will be used hereafter:

(a) The set of *out-neighbors* of vertex i is denoted by $\mathcal{N}_i^+ = \{j : (i,j) \in \mathcal{E}\}$. Its cardinality $\left|\mathcal{N}_i^+\right|$ is called the *out-degree* of the vertex.
(b) A *path* is a sequence of edges $e_1, \dots, e_k$ where, for all $i \in \{1, \dots, k-1\}$, the head of e_i coincides with the tail of e_{i+1}.
(c) The graph $\mathcal{G}$ is *strongly connected* if for every $i, j \in \mathcal{V}$ there exists a path from i to j.
(d) A *subgraph* is a graph $\mathcal{G}(\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$.
(e) A *cycle* is a strongly connected subgraph where each vertex has out-degree 1.
(f) A *unicycle* is a subgraph where each vertex has out-degree 1 and which contains exactly one cycle.
(g) An *i-rooted tree* is a subgraph that contains no cycles and such that i has out-degree 0 and the other vertices have out-degree 1.

2. The deterministic buck-passing game

2.1. **The game.** We consider a finite game $\Gamma(\mathcal{G}, \mu)$ where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a simple directed graph and $\mu = (\mu_i)_{i \in \mathcal{V}}$ is a given probability distribution over the vertices. Each vertex $i \in \mathcal{V}$ is a player with strategy set $\mathcal{S}_i = \mathcal{N}_i^+$ (assumed nonempty with $i \notin \mathcal{S}_i$), and $\mathcal{S} = \times_{i \in \mathcal{V}} \mathcal{S}_i$ is the set of strategy profiles.

Once each player has chosen an out-neighbor $s_i \in \mathcal{S}_i$, the cost for a player is the asymptotic frequency of times she has the buck, determined by the following process. At time $t = 0$ a buck is given to a vertex $i_0 \in \mathcal{V}$ drawn at random by Nature according to the initial distribution μ. At time $t = 1$, the selected player i_0 passes the buck to her designated neighbor $i_1 = s_{i_0}$, who in turn will pass it at time $t = 2$ to her chosen neighbor, and so on and so forth. Define the random variables

$$\Theta_{i,t}(\boldsymbol{s}) = \begin{cases} 1 & \text{if at time } t \text{ player } i \text{ has the buck,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For a fixed profile $\boldsymbol{s}$, the value of $\Theta_{i,t}$ depends only on the initial draw, with

$$\mathbb{P}(\Theta_{i,0}(\boldsymbol{s}) = 1) = \mu_i. \tag{2.2}$$

After this initial draw, the buck is passed among the players and eventually it will start cycling, so that we can define the cost function $c_i : \mathcal{S} \to \mathbb{R}$ for player i as

$$c_i(\boldsymbol{s}) = \mathbb{E}\left[\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \Theta_{i,t}(\boldsymbol{s})\right], \tag{2.3}$$

where the expectation is taken with respect to the initial measure μ.

This game is denoted $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ and is called a *deterministic buck-passing game* (DBPG). The corresponding set of Nash equilibria is denoted $\mathsf{NE}(\mathcal{S})$. We assume that the graph $\mathcal{G}$, the initial measure μ, and the buck-passing dynamics are common knowledge. We stress that, despite the fact that the costs are defined through a dynamic process, the game is actually static, with strategies fixed once and for all at the beginning of the game. Note also that the costs of all players add up to 1, so that this is equivalent to a zero-sum game. To analyze this game, it is convenient to have a more manageable expression for the costs in (2.3), for which we introduce some additional notation.

Definition 2.1. We call $\mathcal{G}_{\boldsymbol{s}} = (\mathcal{V}, \mathcal{E}_{\boldsymbol{s}})$ the subgraph induced by the strategy profile $\boldsymbol{s}$ with edge set $\mathcal{E}_{\boldsymbol{s}} = \{(i, s_i) : i \in \mathcal{V}\}$. Each vertex has out-degree 1 so that $\mathcal{G}_{\boldsymbol{s}}$ is a union of a finite number $M(\boldsymbol{s})$ of disjoint unicycles (see Fig. 1). We call $\mathcal{A}^{\ell}_{\boldsymbol{s}}$ for $\ell = 1, \dots, M(\boldsymbol{s})$ the vertex sets of these unicycles, so that $\mathcal{V} = \mathcal{A}^1_{\boldsymbol{s}} \dot\cup \cdots \dot\cup \mathcal{A}^{M(\boldsymbol{s})}_{\boldsymbol{s}}$, and we let $\mathcal{C}^{\ell}_{\boldsymbol{s}} \subseteq \mathcal{A}^{\ell}_{\boldsymbol{s}}$ be the vertices in the corresponding cycles.

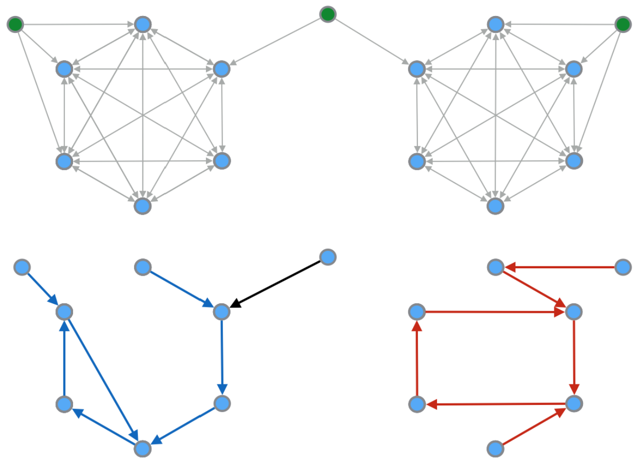

Figure 1. The graph $\mathcal{G}$ on top has 2 strongly connected components and 3 transient vertices. The induced graph $\mathcal{G}_{\boldsymbol{s}}$ on the bottom has two disjoint unicycles with cycles of length 3 and 4.

If the buck is assigned initially to a vertex in $\mathcal{A}^{\ell}_{\boldsymbol{s}}$, then, after finitely many steps, it will reach the cycle $\mathcal{C}^{\ell}_{\boldsymbol{s}}$ and turn around forever. In the long run each player $i \in \mathcal{C}^{\ell}_{\boldsymbol{s}}$ gets the buck a fraction $1/|\mathcal{C}^{\ell}_{\boldsymbol{s}}|$ of the time, while the remaining players are free-riders with a cost of 0. Now, the probability that the buck is assigned initially to a vertex in $\mathcal{A}^{\ell}_{\boldsymbol{s}}$ is

$$\mu^{\ell}_{\boldsymbol{s}} := \sum_{j \in \mathcal{A}^{\ell}_{\boldsymbol{s}}} \mu_j. \tag{2.4}$$

Hence, denoting $\ell(i)$ the label of the unicycle that contains player i and setting

$$\delta_i(\boldsymbol{s}) := \begin{cases} 1 & \text{if } i \in \mathcal{C}^{\ell(i)}_{\boldsymbol{s}}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

the expected cost in (2.3) can be written as

$$c_i(\boldsymbol{s}) = \frac{\mu^{\ell(i)}_{\boldsymbol{s}}}{|\mathcal{C}^{\ell(i)}_{\boldsymbol{s}}|} \delta_i(\boldsymbol{s}). \tag{2.6}$$

2.2. **Ordinal potentials and existence of prior-free equilibria.** As every finite game, the buck-passing game admits equilibria in mixed strategies. However, our main interest here is the existence of Nash equilibria in *pure strategies.* Unless otherwise stated, we always refer to equilibria in pure strategies. We now recall the concepts of profitable deviations and equilibria.

Definition 2.2. Consider a cost game.

(a) Given a strategy profile $\boldsymbol{s} \in \mathcal{S}$, a *unilateral deviation* for player i is a strategy $\boldsymbol{s}' \in \mathcal{S}$ that differs from $\boldsymbol{s}$ only in its i-th coordinate. It is a *profitable deviation* if in addition $c_i(\boldsymbol{s}') < c_i(\boldsymbol{s})$, in which case the difference $c_i(\boldsymbol{s}) - c_i(\boldsymbol{s}')$ is called the *improvement* of player i.

(b) A strategy profile $\boldsymbol{s} \in \mathcal{S}$ is a *Nash equilibrium* (NE) if no player has a profitable deviation. Similarly, it is an *ε-Nash equilibrium* (ε-NE) if no player has a profitable deviation with an improvement larger than ε.

In principle, a buck-passing game $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ may have multiple equilibria and they may depend on the initial measure μ. Of special interest are the so-called *prior-free equilibria*, i.e., equilibria that are invariant with respect to the initial measure μ.

Definition 2.3. A *prior-free Nash equilibrium* (PFNE) is a strategy profile that is a Nash equilibrium for every initial distribution μ.

Our main result for the deterministic buck-passing game is the existence of prior-free equilibria. This will be proved by showing the existence of a generalized ordinal potential that does not depend on the initial measure μ. Potential games were introduced in the seminal paper by Rosenthal (1973) and later studied extensively by Monderer and Shapley (1996). A recent account can be found in Lã et al. (2016). We recall these notions for a general cost game $(\mathcal{V}, \mathcal{S}, \boldsymbol{c})$, where $\mathcal{V}$ is a finite set of players, $\mathcal{S} = \times_{i \in \mathcal{V}} \mathcal{S}_i$ is the set of strategy profiles with $\mathcal{S}_i$ the (possibly uncountable) set of strategies for player $i \in \mathcal{V}$, and $\boldsymbol{c}_i : \mathcal{S} \to \mathbb{R}$ is the cost of player $i \in \mathcal{V}$.

Definition 2.4. A function $\Psi : \mathcal{S} \to \mathbb{R}$ is called a *potential* for $(\mathcal{V}, \mathcal{S}, \boldsymbol{c})$ if for each strategy profile $\boldsymbol{s}$ and any unilateral deviation $\boldsymbol{s}' = (s_i', \boldsymbol{s}_{-i})$ by a player i, we have

$$c_i(\boldsymbol{s}') - c_i(\boldsymbol{s}) = \Psi(\boldsymbol{s}') - \Psi(\boldsymbol{s}). \tag{2.7}$$

Similarly, Ψ is said to be an *ordinal potential* if

$$c_i(\boldsymbol{s}') < c_i(\boldsymbol{s}) \Leftrightarrow \Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s}) \tag{2.8}$$

and is called a *generalized ordinal potential* if

$$c_i(\boldsymbol{s}') < c_i(\boldsymbol{s}) \Rightarrow \Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s}). \tag{2.9}$$

Clearly the notion of generalized ordinal potential is the weakest. It is also easy to see that a component-wise minimizer of a generalized ordinal potential is a Nash equilibrium. If Ψ is in fact an ordinal potential, these minimizers coincide with the Nash equilibria. For later reference we record the following direct consequence.

Proposition 2.5. *Every generalized ordinal potential game, which is either finite or, more generally, which has compact strategy sets and a lower semi-continuous potential, admits a Nash equilibrium in pure strategies.*

Definition 2.6.

(a) An *improvement path* is a sequence of strategy profiles $\boldsymbol{s}_0, \boldsymbol{s}_1, \dots$ such that each $\boldsymbol{s}_{k+1}$ is a profitable deviation of $\boldsymbol{s}_k$ for some player i_k. It is called an *ε-improvement path* if the improvement at each stage is at least ε.

(b) A game has the *finite improvement property* (FIP) if every improvement path is finite. Similarly, it has the *ε-finite improvement property* (ε-FIP) if every ε-improvement path is finite.

Monderer and Shapley (1996) showed that a finite game has a generalized ordinal potential if and only if it satisfies the FIP. For a similar characterization of ordinal potentials see Voorneveld and Norde (1997).

Remark 2.7. The FIP can also be described by means of an auxiliary graph—which we call the *meta-graph* of the game to avoid confusion with the other graphs mentioned in the paper—where a vertex represents a strategy profile and a directed edge $(\boldsymbol{s}, \boldsymbol{s}')$ exists iff $\boldsymbol{s}'$ is a profitable deviation from $\boldsymbol{s}$ for some player i. Then, the FIP is equivalent to the acyclicity of the meta-graph. A similar construction is used, for instance, in Candogan et al. (2011). Note that NE are precisely the sinks of the meta-graph, i.e., meta-vertices with out-degree equal to zero. Since every path on an acyclic directed graph is finite and ends in a sink, this shows again that a finite game satisfying the FIP admits a NE.

With these preliminaries, we proceed to establish our main results for the deterministic buck-passing game. We will prove the existence of a generalized ordinal potential, from which we will deduce not only that its minima are Nash equilibria in pure strategies but also that they are prior-free. Furthermore, we establish a bound for the maximum length of an improvement path, which implies that any best response dynamics must reach a prior-free equilibrium in quadratic time.

Since a cost that is already at zero cannot be decreased, it follows from (2.6) that a profitable deviation may only concern a player i who is on a cycle $\mathcal{C}^h_{\boldsymbol{s}}$ with $\mu^h_{\boldsymbol{s}} > 0$. Such a player has two options to reduce her cost: increase the length of the cycle she is currently in, or break the cycle by sending the buck to a different unicycle so that her cost drops to zero. More precisely these options are:

(D_1) shift to $s'_i \in \mathcal{A}^h_{\boldsymbol{s}} \setminus \mathcal{C}^h_{\boldsymbol{s}}$ so that the cycle becomes longer,

(D_2) shift to $s'_i \notin \mathcal{A}^h_{\boldsymbol{s}}$ so that the cycle $\mathcal{C}^h_{\boldsymbol{s}}$ disappears.

These deviations affect only the cycle of the deviating player, either by increasing its length, or by breaking it and merging $\mathcal{A}^h_{\boldsymbol{s}}$ altogether into some other unicycle. All the other cycles remain unaltered, even if their unicycles may change. Our first result identifies a potential function based on the length of these cycles.

Theorem 2.8. *The deterministic buck-passing game $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ is a generalized ordinal potential game with generalized ordinal potential function*

$$\Psi(\boldsymbol{s}) := \sum_{\ell=1}^{M(\boldsymbol{s})} \left(n - |\mathcal{C}^\ell_{\boldsymbol{s}}|\right), \tag{2.10}$$

where $\mathcal{C}^\ell_{\boldsymbol{s}}$ are the cycles in the induced subgraph $\mathcal{G}_{\boldsymbol{s}} = (\mathcal{V}, \mathcal{E}_{\boldsymbol{s}})$ (see Definition 2.1). If the initial measure μ has full support, then Ψ is an ordinal potential.

Proof. Consider a strategy profile $\boldsymbol{s} \in \mathcal{S}$ and let $h = \ell(i)$ be the unicycle containing player i in $\mathcal{G}_{\boldsymbol{s}}$. Consider also a deviation by player i from $\boldsymbol{s}$ to $\boldsymbol{s}' = (s_i', \boldsymbol{s}_{-i})$.

If the deviation is profitable, that is, $c_i(\boldsymbol{s}') < c_i(\boldsymbol{s})$ then, as noted above, there are only two possible cases. After a deviation (D$_1$) the new graph $\mathcal{G}_{\boldsymbol{s}'}$ has the same cycles except for $\mathcal{C}^h_{\boldsymbol{s}}$ which becomes longer, so that only this term in the sum in (2.10) is affected and $\Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s})$. On the other hand, a deviation (D$_2$) removes the cycle $\mathcal{C}^h_{\boldsymbol{s}}$ keeping the other cycles unchanged, so that we lose one term in (2.10) and again $\Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s})$. This proves that Ψ is a generalized ordinal potential.

We next show that, when μ has full support, the reverse implication holds, that is, $\Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s})$ implies $c_i(\boldsymbol{s}') < c_i(\boldsymbol{s})$. The inequality $\Psi(\boldsymbol{s}') < \Psi(\boldsymbol{s})$ conveys a change in the structure of cycles, which can only occur if the deviating player i is on a cycle $\mathcal{C}^h_{\boldsymbol{s}}$. The deviation can only affect this cycle by removing it or changing its length, so again we distinguish these two cases. In the first case a reduction of the potential requires the length of the cycle to increase $|\mathcal{C}^h_{\boldsymbol{s}'}| > |\mathcal{C}^h_{\boldsymbol{s}}|$, which imposes $s_i' \in \mathcal{A}^h_{\boldsymbol{s}} \setminus \mathcal{C}^h_{\boldsymbol{s}}$. In this case $\mathcal{A}^h_{\boldsymbol{s}'} = \mathcal{A}^h_{\boldsymbol{s}}$ so that $\mu^h_{\boldsymbol{s}'} = \mu^h_{\boldsymbol{s}} > 0$ and $c_i(\boldsymbol{s}') < c_i(\boldsymbol{s})$ follows from (2.6). The second case occurs when $s_i' \notin \mathcal{A}^h_{\boldsymbol{s}}$, in which case $c_i(\boldsymbol{s}') = 0 < c_i(\boldsymbol{s})$, where the strict inequality follows from (2.6) by noting that $\delta_i(\boldsymbol{s}) = 1$ and $\mu^h_{\boldsymbol{s}} > 0$. □

Theorem 2.9. *Every deterministic buck-passing game $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ admits a PFNE.*

Proof. Since $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ is a finite game, the existence of a pure Nash equilibrium follows directly from Theorem 2.8 and Proposition 2.5. To prove the existence of a prior-free equilibrium it suffices to note that the expression Ψ in (2.10) does not depend on the initial distribution μ, so that a global minimizer of Ψ is a PFNE. □

We stress that, when μ is fully supported, $\Psi(\boldsymbol{s})$ provides an ordinal potential and NE are exactly its component-wise minimizers. Since $\Psi(\boldsymbol{s})$ does not depend on μ it follows that in this case all NE are prior-free. In other words, prior-sensitive equilibria can appear only when μ is not fully supported.

Example 2.1. Consider the graph $\mathcal{G}$ on the left of Fig. 2, and suppose that μ is a degenerate measure that puts all the mass on the red vertex v in the central figure. Then the strategy profile shown in this central figure is a NE which is not prior-free. Indeed, if we take a different measure μ' which puts some positive mass on one of the green vertices i, j, k, then k has a profitable deviation, which gives rise to the PFNE shown on the right.

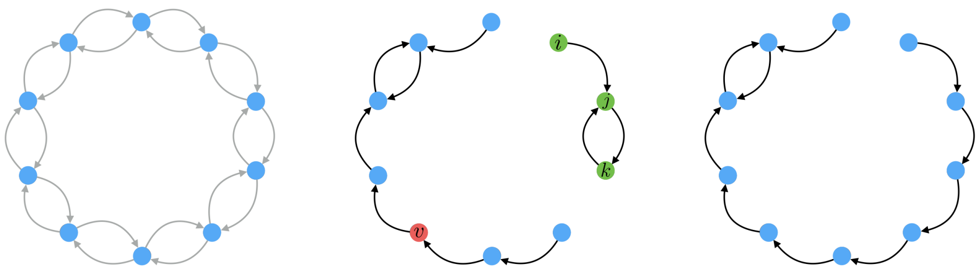

Figure 2. Left: the graph. Middle: a NE that is not prior-free. Right: a PFNE

In view of Theorem 2.8, and according to Monderer and Shapley (1996), it follows that every deterministic buck-passing game has the FIP. Furthermore, considering the two types of profitable deviations (D$_1$) and (D$_2$), we see that, if in a given strategy profile $\boldsymbol{s} \in \mathcal{S}$ two vertices i and j are in the same unicycle, the same holds for every $\boldsymbol{s}' \in \mathcal{S}$ that can be reached by following an improvement path. To be picturesque, we could say that, once the destinies of two players meet, they are doomed to be entangled forever. In the language of probability, the evolution along an improvement path is a *coalescence process.* This observation leads to an explicit bound on the maximum length of an improvement path in terms of the number of players n, and it implies that a best response dynamics will attain a PFNE in quadratic time.

Theorem 2.10. *In every deterministic buck-passing game $\Gamma(\mathcal{G}, \mu, \mathcal{S})$ the length of an improvement path is at most $\frac{1}{4}n^2 - 1$, and this bound is tight.*

Proof. Let ϕ_k be the maximum length of an improvement path when we start from a strategy profile with $M(\boldsymbol{s}) = k$ unicycles. As noted before, there are two types of profitable deviations, (D$_1$) and (D$_2$), in which either the length of a cycle increases or a unicycle merges into another. Since each unicycle contains at least 2 vertices, there can be at most $n - 2k$ deviations of type (D$_1$) before dropping in the next deviation to $k - 1$ unicycles. This yields the bound

$$\phi_k \leq (n - 2k + 1) + \phi_{k-1}$$

and inductively we get

$$\phi_k \leq \sum_{i=2}^{k} (n - 2i + 1) + \phi_1 = kn - k^2 + 1 - n + \phi_1.$$

By the same argument as above we have $\phi_1 \leq n - 2$ which yields

$$\phi_k \leq kn - k^2 - 1.$$

The maximum of the last expression is attained at $k = \lfloor \frac{n}{2} \rfloor$. Ignoring the rounding and maximizing for $k \in \mathbb{R}$ we get $\phi_k \leq \frac{1}{4}n^2 - 1$.

We next show that this bound can be reached. Consider a complete graph K_n with an even number of vertices and a uniform initial measure μ. Take an initial strategy profile with exactly $k = n/2$ cycles $\mathcal{C}_1, \ldots, \mathcal{C}_k$ with 2 players each, and consider the following sequence of unilateral profitable deviations:

(1) Break the cycle $\mathcal{C}_1$ by connecting it as a path to the cycle $\mathcal{C}_2$. This counts as a single step, after which the cycle $\mathcal{C}_2$ has a "tail" composed by the vertices in $\mathcal{C}_1$. Call i the vertex in $\mathcal{C}_2$ to which such tail is connected and let j be the predecessor of i in $\mathcal{C}_2$. Note that j can reduce her cost by connecting to any vertex of the tail: the farther the selected player in the tail, the lower the resulting cost for the deviating player j. Consider the scenario where j chooses the *worst* profitable deviation, by selecting the closest player in the tail as new designated out-neighbor. Repeat this procedure until $\mathcal{C}_1$ and $\mathcal{C}_2$ are merged into a unique cycle $\mathcal{C}_{1,2}$ with 4 vertices. This requires 2 steps.

(2) Break the cycle $\mathcal{C}_{1,2}$ by connecting it as a path to $\mathcal{C}_3$. This requires 1 step. Enlarge the cycle $\mathcal{C}_3$ by collecting one by one the elements of $\mathcal{C}_{1,2}$ as in (1) until merging into a unique cycle $\mathcal{C}_{1,2,3}$ with 6 vertices. This requires 4 steps.

$\vdots$

(k) Break the cycle $\mathcal{C}_{1,\dots,k-1}$ and connect it to $\mathcal{C}_k$. This requires 1 step. Enlarge $\mathcal{C}_k$ by collecting the elements of $\mathcal{C}_{1,\dots,k-1}$ as before. This requires $2(k-1)$ steps. This final cycle is Hamiltonian, hence a NE.

In total we have $\sum_{i=1}^{k-1} 2i = k(k-1)$ steps of type (D1) and $k-1$ steps of type (D2), which altogether give exactly $(k+1)(k-1) = k^2 - 1 = \frac{1}{4}n^2 - 1$ steps. $\square$

Remark 2.11. Notice that the trivial upper bound for the length of a path in the meta-graph of the game is $|\mathcal{S}|$. For a directed graph with minimum out-degree 2, that is, when players can really act strategically, the size of $\mathcal{S}$ is exponentially large with $|\mathcal{S}| \geq 2^n$, which is far worse than the quadratic bound established above.

3. The stochastic buck-passing game

3.1. **The game.** We now extend the deterministic model of Section 2 by allowing players to pass the buck at random to some neighbor. Specifically, the strategy set of player $i \in \mathcal{V}$ is now the simplex of probabilities over $\mathcal{N}_i^+$. This is isomorphic to the set Σ_i of probability vectors $\boldsymbol{\pi}_i$ on $\mathcal{V}$ such that $\boldsymbol{\pi}_i(j) \equiv \pi_{ij} = 0$ for all $j \notin \mathcal{N}_i^+$. We call $\Sigma = \times_{i\in\mathcal{V}}\Sigma_i$ the set of strategy profiles. With a slight abuse of notation, the same symbol $\boldsymbol{\pi}$ will denote the strategy profile $(\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_n) \in \Sigma$ and the stochastic matrix $[\pi_{ij}]_{i,j\in\mathcal{V}}$ whose rows are $\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_n$. We also call $\mathcal{G}_{\boldsymbol{\pi}}$ the induced *weighted* directed graph with vertices in $\mathcal{V}$, edges in $\mathcal{E}_{\boldsymbol{\pi}} \coloneqq \{(i,j) : \pi_{ij} > 0\}$, and weights π_{ij}.

At the start of the game each player chooses a strategy $\boldsymbol{\pi}_i \in \Sigma_i$. At time $t = 0$ the buck is given to a player i drawn at random according to the measure μ. This player then passes the buck to a random neighbor sampled according to $\boldsymbol{\pi}_i$, and so on and so forth. Rigorously, the process is a time-homogeneous Markov chain $(X_t)_{t\geq 0}$ with initial measure μ and transition matrix $\boldsymbol{\pi}$, that is,

$$\mathbb{P}_{\boldsymbol{\pi}}(X_0 = i) = \mu_i \quad \text{and} \quad \mathbb{P}_{\boldsymbol{\pi}}(X_{t+1} = j \mid X_t = i) = \pi_{ij}, \tag{3.1}$$

where $\mathbb{P}_{\boldsymbol{\pi}}$ is the probability measure induced by the strategy profile $\boldsymbol{\pi}$. The cost $c_i \colon \Sigma \to \mathbb{R}$ of player i is again defined by Eq. (2.3), but this time the expectation is taken with respect to both the initial measure μ and the transition matrix $\boldsymbol{\pi}$. We call $\Gamma(\mathcal{G}, \mu, \Sigma)$ a *stochastic buck-passing game* (SBPG), and we write $\mathsf{NE}(\Sigma)$ for its set of Nash equilibria.

The analysis of SBPGs requires some standard concepts in the theory of Markov chains. We recall that a *recurrent class* in $\mathcal{G}_{\boldsymbol{\pi}}$ is a strongly connected component $\mathcal{C}$ which is maximal by inclusion. For each $\boldsymbol{\pi}$ the vertex set can be partitioned as

$$\mathcal{V} = \mathcal{V}^0_{\boldsymbol{\pi}} \,\dot\cup\, \mathcal{C}^1_{\boldsymbol{\pi}} \,\dot\cup \ldots \dot\cup\, \mathcal{C}^{M(\boldsymbol{\pi})}_{\boldsymbol{\pi}}, \tag{3.2}$$

where each $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ is a recurrent class in $\mathcal{G}_{\boldsymbol{\pi}}$, $\mathcal{V}^0_{\boldsymbol{\pi}}$ is the set of *transient vertices* that do not belong to any recurrent class, and $\dot\cup$ indicates the disjoint union.

The restriction of the chain to each $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ is itself an *irreducible chain* which supports a unique stationary measure $\rho^\ell_{\boldsymbol{\pi}}$. For convenience we extend this measure to the full vertex set by setting $\rho^\ell_{\boldsymbol{\pi}}(i) = 0$ for $i \notin \mathcal{C}^\ell_{\boldsymbol{\pi}}$. These measures are characterized by the classical ergodic theorem.

Theorem 3.1 (Ergodic Theorem)**.** *Consider an irreducible Markov chain $(X_t)_{t\geq 0}$ on a finite state space $\mathcal{V}$, with transition matrix $\boldsymbol{\pi}$ and initial distribution μ. Let $\rho_{\boldsymbol{\pi}}$ be the unique stationary measure, and $T_i = \inf\{t > 0 | X_t = i\}$ the hitting time of state i. Then*

$$\mathbb{E}\left[\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{X_t = i\}}\right] = \rho_{\boldsymbol{\pi}}(i) = \frac{1}{\mathbb{E}[T_i \mid X_0 = i]}. \tag{3.3}$$

The proof of Theorem 3.1 can be found, for instance, in (Levin and Peres, 2017, Theorem C.1 and Proposition 1.19).

For a general Markov chain (not necessarily irreducible), starting from any state $j \in \mathcal{V}$ the chain X will eventually be absorbed in a recurrent class $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ with probability

$$\mathsf{P}^{j\to\ell}_{\boldsymbol{\pi}} \coloneqq \mathbb{P}(\text{there exists } T \in \mathbb{N} \text{ such that } X_T \in \mathcal{C}^\ell_{\boldsymbol{\pi}} \mid X_0 = j), \tag{3.4}$$

so that the total probability with which the buck is absorbed in $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ is

$$\mu^\ell_{\boldsymbol{\pi}} = \sum_{j\in\mathcal{V}} \mu_j \, \mathsf{P}^{j\to\ell}_{\boldsymbol{\pi}}. \tag{3.5}$$

Thus, applying Theorem 3.1 on each recurrent class $\mathcal{C}^\ell_{\boldsymbol{\pi}}$, the cost for player i in Eq. (2.3) can be finally expressed as

$$c_i(\boldsymbol{\pi}) = \sum_{\ell=1}^{M(\boldsymbol{\pi})} \mu^\ell_{\boldsymbol{\pi}} \, \rho^\ell_{\boldsymbol{\pi}}(i). \tag{3.6}$$

Note that this is equivalent to Eq. (2.6) when $\boldsymbol{\pi}$ is a deterministic strategy profile.

Remark 3.2. A SBPG is *not* the mixed extension of a deterministic game DBPG. Given the cost function in Eq. (2.3), a probability vector $\boldsymbol{\pi}_i \in \Sigma_i$ is not a mixed strategy for DBPG. The difference is that in a SBPG, each time a player receives the buck, a new neighbor is drawn at random, whereas in the mixed extension of DBPG this random neighbor is drawn at the beginning of the game and kept fixed thereafter. To illustrate the difference, let $\mathcal{G}$ be the complete graph on three vertices $\mathcal{V} = \{1, 2, 3\}$ and consider the strategy profile (see Fig. 3):

$$\boldsymbol{\pi}_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \quad \boldsymbol{\pi}_2 = (1, 0, 0), \quad \boldsymbol{\pi}_3 = (0, 1, 0). \tag{3.7}$$

Using (3.6), we can see that in the SBPG the profile $\boldsymbol{\pi}$ has a cost vector equal to the stationary distribution it induces, that is,

$$\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right).$$

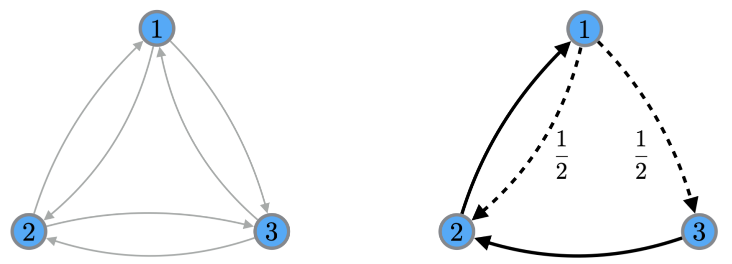

Figure 3. Left: A complete graph with 3 vertices. Right: The strategy (3.7).

If instead we take (3.7) as a mixed strategy in the DBPG, then the cost for player 1 is

$$
\begin{aligned}
c_1(\boldsymbol{\pi}) &= \frac{1}{2}\,\mathbb{E}\left[\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\Theta_{1,t}(2,1,2)\right] + \frac{1}{2}\,\mathbb{E}\left[\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\Theta_{1,t}(3,1,2)\right] \\
&= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{5}{12} \neq \frac{2}{5}.
\end{aligned}
$$

3.2. **Relation between the deterministic and stochastic games.** Even if SBPG is not the mixed extension of DBPG, we next show that—as it happens for mixed extensions—equilibria for the deterministic game are preserved in the stochastic setting. In line with the notation introduced in Section 2, we call $\mathcal{S}_i$ the extreme points of Σ_i, so a strategy in $\mathcal{S}_i$ is a degenerate measure and corresponds to choosing an out-neighbor with probability 1.

Proposition 3.3. *For each buck-passing game $\Gamma(\mathcal{G},\mu)$ we have $\mathsf{NE}(\mathcal{S}) \subseteq \mathsf{NE}(\Sigma)$.*

Proof. By contradiction, suppose there exists some $\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})$ such that $\boldsymbol{s} \notin \mathsf{NE}(\Sigma)$. Notice that

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = i] &= 1 + \sum_{j\in\mathcal{V}} \pi_{ij}\,\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = j] \\
&= 1 + \sum_{j\in\mathcal{V}} \pi_{ij}\,\mathbb{E}_{\boldsymbol{\pi}_{-i}}[T_i \mid X_0 = j],
\end{aligned}
\tag{3.8}
$$

with the obvious meaning of the symbols. In particular, if we consider a deterministic profile $\boldsymbol{s} \in \mathcal{S}$, then

$$
\mathbb{E}_{\boldsymbol{s}}[T_i \mid X_0 = i] = 1 + \mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = s_i]. \tag{3.9}
$$

Consider first the case where $M(\boldsymbol{s}) = 1$. In this setting, if $\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})$, then, thanks to (3.3), for every $i \in \mathcal{V}$,

$$
s_i \in \operatorname*{arg\,max}_{k\in\mathcal{N}_i^+} \mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = k]. \tag{3.10}
$$

Hence, by (3.8), if $\boldsymbol{s} \notin \mathsf{NE}(\Sigma)$, there exists $i \in \mathcal{V}$ and some $\boldsymbol{\pi}_i \in \Sigma_i$ such that

$$
\sum_{j\in\mathcal{V}} \pi_{ij}\,\mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = j] > \mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = s_i]. \tag{3.11}
$$

In particular, there exists some $j \neq s_i$ such that

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = j] > \mathbb{E}_{\boldsymbol{s}_{-i}}[T_i \mid X_0 = s_i]. \tag{3.12}$$

In this case, $s'_i = j$ is a profitable deviation for player i for the deterministic buck-passing game. This contradicts the assumption that $\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})$.

If $\boldsymbol{s}$ induces more than a single recurrent class and $\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})$, then, whenever i lies in a cycle $\mathcal{C}_\ell$ of $\mathcal{G}_{\boldsymbol{s}}$ having $\mu^\ell_{\boldsymbol{s}} > 0$, all her out-neighbors are in the same unicycle. Hence, by the same argument as above, there is no randomized profile $\boldsymbol{\pi} = (\boldsymbol{\pi}_i, \boldsymbol{s}_{-i})$ such that $\rho^\ell_{\boldsymbol{\pi}}(i) < \rho^\ell_{\boldsymbol{s}}(i)$. On the other hand, if $\mu^\ell_{\boldsymbol{s}} = 0$ then, clearly, player i has no improving deviation. Therefore, also in the case of multiple recurrent classes we have $\boldsymbol{s} \in \mathsf{NE}(\mathcal{S}) \Rightarrow \boldsymbol{s} \in \mathsf{NE}(\Sigma)$. □

The stochastic buck-passing game gives each player a richer set of strategies, so it is not surprising that we get a larger set of equilibria. Example 3.1 below shows that the set of equilibria may be strictly larger in the stochastic setting, and illustrates that some players may be favored by the larger strategy sets, whereas others will be affected negatively.

Example 3.1. Consider a graph consisting of a uni-directional cycle with vertices $i = 1, \dots, n-1$ plus a central player n connected bi-directionally to all the other vertices (see Fig. 4). For each player i on the outer cycle denote by i_- and i_+ respectively the predecessor and successor vertices along the cycle.

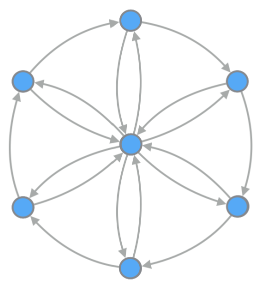

Figure 4. A clockwise wheel, with 6 outer vertices plus a central player.

Consider first the DBPG. Once the central player has designated an out-neighbor i, it is a dominant strategy for the latter to forward the buck along the cycle to her out-neighbor i_+. The same holds for all subsequent players along the cycle, except for player i_-, for whom it is a dominant strategy to return the buck to the central player. This yields a Hamiltonian cycle. There are exactly $n-1$ such cycles, one for each i, and these are exactly the equilibria in $\mathsf{NE}(\mathcal{S})$. In each of these equilibria each player pays $1/n$.

Now consider the strategy profile $\boldsymbol{\pi} \in \Sigma$ in the SBPG in which the central player sends the buck uniformly at random with probability $1/(n-1)$ to each vertex in the outer ring, whereas any other player i plays a deterministic strategy sending the buck to the next player i_+ along the cycle. We claim that $\boldsymbol{\pi} \in \mathsf{NE}(\Sigma)$. The central player is transient and pays 0, so that she has no profitable deviation. Each player i on the outer ring pays $1/(n-1)$. If she deviates to a stochastic strategy by sending to player i_+ with probability p and to the central player with probability $(1-p)$, then her expected return time is

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = i] &= 1 + p\,\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = i_+] + (1-p)\,\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = n] \\ &= 1 + p\,(n-2) + (1-p)\frac{1}{n-1}(1 + 2 + \cdots + (n-1)) \\ &= 1 + p\,(n-2) + (1-p)\frac{n}{2}.\end{aligned}$$

For $n \geq 5$ this expression is strictly increasing with p so that, according to (3.3), the minimum cost is attained for $p = 1$, which proves that $\boldsymbol{\pi}$ is indeed a NE. In this equilibrium the central player pays 0 and is better off than in the DBPG, whereas all the other players are worse off, since their cost is now $1/(n-1)$ rather than $1/n$.

4. Fairness of equilibria

4.1. **Measures of fairness.** As seen in Example 2.1, a buck-passing game may have several equilibria and in some of them the total cost is very unevenly spread among players. We want to compare—in terms of fairness—the equilibrium cost vectors with the optimum cost vectors that could be achieved by a benevolent social planner, whose goal is to minimize disparity in the way players are treated. To this end we adopt a Rawlsian criterion, (see Rawls, 2009), and we define the social cost of a strategy profile as the cost incurred by the player who pays the most

$$\mathsf{SC}(\boldsymbol{s}) = \max_{i \in \mathcal{V}} c_i(\boldsymbol{s}) \tag{4.1}$$

so that minimizing this social cost corresponds somehow to maximizing fairness among the players of the game.

Equilibrium and optimum costs are usually compared in terms of efficiency. Typically the social cost function is taken as the sum of the costs incurred by all players. The standard measures of efficiency in games are the *price of anarchy* (PoA), i.e., the ratio between the social cost of the worst equilibrium and the minimum social cost, and the *price of stability* (PoS), i.e., the ratio between the social cost of the best equilibrium and the minimum social cost. Explicitly, if $\mathsf{SC}\colon \mathcal{S} \to \mathbb{R}$ is the social cost function for a deterministic buck-passing game, we define

$$\mathsf{PoA}(\mathcal{S}) \coloneqq \frac{\max_{\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})} \mathsf{SC}(\boldsymbol{s})}{\min_{\boldsymbol{s} \in \mathcal{S}} \mathsf{SC}(\boldsymbol{s})}, \tag{4.2}$$

$$\mathsf{PoS}(\mathcal{S}) \coloneqq \frac{\min_{\boldsymbol{s} \in \mathsf{NE}(\mathcal{S})} \mathsf{SC}(\boldsymbol{s})}{\min_{\boldsymbol{s} \in \mathcal{S}} \mathsf{SC}(\boldsymbol{s})}. \tag{4.3}$$

For stochastic buck-passing games, $\mathsf{SC}\colon \Sigma \to \mathbb{R}$ and

$$\mathsf{PoA}(\Sigma) \coloneqq \frac{\max_{\boldsymbol{\pi} \in \mathsf{NE}(\Sigma)} \mathsf{SC}(\boldsymbol{\pi})}{\min_{\boldsymbol{\pi} \in \Sigma} \mathsf{SC}(\boldsymbol{\pi})}, \tag{4.4}$$

$$\mathsf{PoS}(\Sigma) \coloneqq \frac{\min_{\boldsymbol{\pi} \in \mathsf{NE}(\Sigma)} \mathsf{SC}(\boldsymbol{\pi})}{\min_{\boldsymbol{\pi} \in \Sigma} \mathsf{SC}(\boldsymbol{\pi})}. \tag{4.5}$$

Since the buck-passing game is a constant sum game, efficiency is not an issue: if social cost is the sum of the player costs, all strategy profiles are equally efficient. On the other hand, by using the social cost function in (4.1), the PoA and PoS can be used to measure fairness. The smaller the PoA (PoS), the fairer the worst (best) equilibrium. We are not the first to use social cost functions that are not the sum of individual costs (see for instance Fournier and Scarsini, 2019, Koutsoupias and Papadimitriou, 2009, 1999, Mavronicolas et al., 2008, Vetta, 2002).

The next proposition establishes tight bounds for both the PoA and the PoS.

Proposition 4.1. *For any buck-passing game* $\Gamma(\mathcal{G}, \mu)$ *we have*

(a) $\mathsf{PoS}(\mathcal{S}) \leq \mathsf{PoA}(\mathcal{S}) \leq n/2$.
(b) $\mathsf{PoS}(\Sigma) \leq \mathsf{PoA}(\Sigma) \leq n/2$.

Moreover, there are instances where $\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = n/2$ *and all these inequalities are satisfied as equalities.*

Proof. By definition PoS is always smaller than PoA so that it suffices to establish the upper bound of $n/2$ in both DBPG and SBPG. In both cases the worse that can happen to a player is to receive the buck every other period so that, for any possible strategy profile, no player pays more than $1/2$. On the other hand, since the sum of costs over all players is 1, the minimum social cost in both settings is at least $1/n$. This implies a bound of $n/2$ in both the deterministic and stochastic cases.

To show that these bounds can be reached consider a graph consisting of two disjoint directed cycles with $n-2$ and 2 players, respectively, and only one pivot player in the longest cycle who has an additional link connecting to the 2-cycle. See Fig. 5.

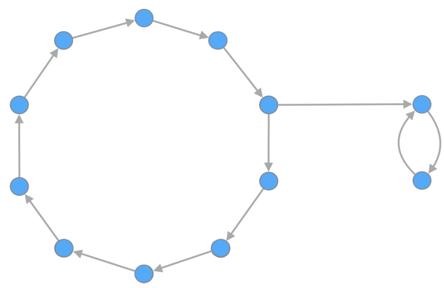

Figure 5. A graph with $\mathsf{PoS} = \mathsf{PoA} = n/2$.

This pivot player is the only one who can randomize by sending the buck to the 2-cycle with probability p and following the long cycle with probability $(1-p)$. For a uniform initial measure μ, the unique strategy that minimizes the social cost is the deterministic one which sets $p = 0$ inducing a social cost of $1/n$ (perfectly fair). This is not an equilibrium though, since the pivot player may deviate to $p \in (0, 1]$ in the SBPG and to $p = 1$ in the DBPG, and in all these equilibria the buck is absorbed in the 2-cycle, whose players pay $1/2$ each. Hence in both cases we get $\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = n/2$. □

A natural question is how PoA and PoS change from the deterministic to the stochastic settings. The comparison is not straightforward since the reference baseline set by the minimal social cost may be different in both cases. Moreover, as seen in Example 3.1, even if the graph is Hamiltonian with the same baseline

$$\min_{\boldsymbol{\pi}\in\Sigma} \mathsf{SC}(\boldsymbol{\pi}) = \min_{\boldsymbol{s}\in\mathcal{S}} \mathsf{SC}(\boldsymbol{s}) = 1/n$$

we may still have

$$\mathsf{PoA}(\Sigma) \geq \frac{n}{n-1} > \mathsf{PoA}(\mathcal{S}) = 1.$$

Proposition 4.2. *For any buck-passing game* $\Gamma(\mathcal{G},\mu)$ *we have*

(a) $\min_{\boldsymbol{\pi}\in\Sigma} \mathsf{SC}(\boldsymbol{\pi}) \leq \min_{\boldsymbol{s}\in\mathcal{S}} \mathsf{SC}(\boldsymbol{s})$
(b) $\mathsf{PoA}(\mathcal{S}) \leq \mathsf{PoA}(\Sigma)$

possibly with strict inequalities.

Proof. Part (a) follows directly by noting that each $\boldsymbol{s}\in\mathcal{S}$ is equivalent to a deterministic strategy in Σ, and then (b) follows from the inclusion $\mathsf{NE}(\mathcal{S}) \subseteq \mathsf{NE}(\Sigma)$ in Proposition 3.3. To see that the inequalities may be strict consider the graph in Fig. 6 composed of two disjoint cycles with 2 vertices each, plus a transient vertex connected to both cycles. If we initially assign the buck to this transient vertex with

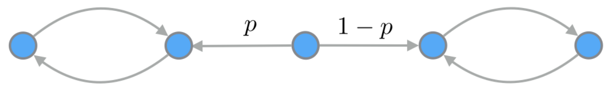

Figure 6. A graph with $\min_{\boldsymbol{\pi}\in\Sigma} \mathsf{SC}(\boldsymbol{\pi}) < \min_{\boldsymbol{s}\in\mathcal{S}} \mathsf{SC}(\boldsymbol{s})$. The initial measure μ is concentrated on the central vertex.

probability one, then the minimum social cost over Σ attained with $p = \frac{1}{2}$, whereas over $\mathcal{S}$ the optimum is attained for $p = 0$ and $p = 1$. Hence

$$\min_{\boldsymbol{\pi}\in\Sigma} \mathsf{SC}(\boldsymbol{\pi}) = \frac{1}{4} < \min_{\boldsymbol{s}\in\mathcal{S}} \mathsf{SC}(\boldsymbol{s}) = \frac{1}{2},$$
$$\mathsf{PoA}(\mathcal{S}) = 1 < \mathsf{PoA}(\Sigma) = 2. \qquad \square$$

If Proposition 4.2(a) holds with equality, the inclusion $\mathsf{NE}(\mathcal{S}) \subseteq \mathsf{NE}(\Sigma)$ implies $\mathsf{PoS}(\Sigma) \leq \mathsf{PoS}(\mathcal{S})$. However, as shown by Example 4.4 in the next section, in general there is no order between $\mathsf{PoS}(\Sigma)$ and $\mathsf{PoS}(\mathcal{S})$. The following special case presents a situation where the deterministic and stochastic buck-passing game have price of stability equal to 1.

Proposition 4.3. *If* $\mathcal{G}$ *is a disjoint union of strongly connected components, then for every buck-passing game* $\Gamma(\mathcal{G},\mu)$ *we have* $\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = 1$.

Proof. It suffices to show that the optimal social cost over Σ is attained at a deterministic strategy $\boldsymbol{s}\in\mathcal{S}$ which is also an equilibrium, namely $\boldsymbol{s}\in\mathsf{NE}(\mathcal{S}) \subseteq \mathsf{NE}(\Sigma)$.

Let $\mathcal{B}_1, \ldots \mathcal{B}_M$ be the strongly connected components in $\mathcal{G}$ and choose a collection of longest cycles $\mathcal{C}_1, \ldots, \mathcal{C}_M$, one in each component. Consider the strategy profile $\boldsymbol{s}$ induced by these cycles and where all the other players are free riders. By construction none of these cycles can be destroyed nor extended, so that $\boldsymbol{s}$ is a prior-free NE for the DBPG, hence also for the SBPG by virtue of Proposition 3.3. It remains to show that, for every initial measure μ, this deterministic strategy profile $\boldsymbol{s}$ minimizes the social cost over Σ (hence also over $\mathcal{S}$). Indeed, call μ_ℓ the initial mass of the component $\mathcal{B}_\ell$. Being $|\mathcal{C}_\ell|$ the length of the maximal cycle in $\mathcal{B}_\ell$, for every $\boldsymbol{\pi} \in \Sigma$ and each player $i \in \mathcal{C}_\ell$ we have $\mathbb{E}_{\boldsymbol{\pi}}[T_i \mid X_0 = i] \leq |\mathcal{C}_\ell|$ so that (3.3) then yields $c_i(\boldsymbol{\pi}) \geq \frac{\mu_\ell}{|\mathcal{C}_\ell|} = c_i(\boldsymbol{s})$. It follows that $\mathsf{SC}(\boldsymbol{\pi}) \geq \mathsf{SC}(\boldsymbol{s})$ completing the proof. □

4.2. **Some examples.** The following examples show that, depending on the structure of the graph $\mathcal{G}$, the inequalities in Proposition 4.1 can be tight or not. The first two examples concern respectively the cases of complete graphs and cycles, with a uniform initial measure μ. These graphs are symmetric in the sense that $(i, j) \in \mathcal{E}$ iff $(j, i) \in \mathcal{E}$, and they have a Hamiltonian cycle which is both an optimal profile and a Nash equilibrium so that the optimal social cost is $1/n$ and $\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = 1$.

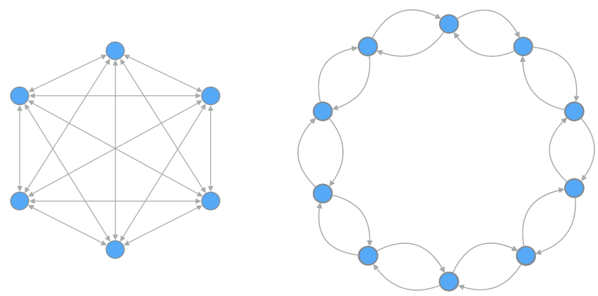

Figure 7. The complete graph K_6 and the bi-directional cycle C_{10}. For ease of representation in the first picture we used bidirectional arrows instead of drawing two arrows for each couple of vertices. These two graphs admit a Hamiltonian cycle.

Example 4.1. For the buck-passing game on the complete graph K_n with n vertices, the Nash equilibria (both in the deterministic and stochastic versions) are exactly the profiles $\boldsymbol{s} \in \mathcal{S}$ such that $\mathcal{G}_{\boldsymbol{s}}$ is a Hamiltonian cycle, and therefore

$$\mathsf{PoS}(\mathcal{S}) = \mathsf{PoA}(\mathcal{S}) = 1 = \mathsf{PoS}(\Sigma) = \mathsf{PoA}(\Sigma).$$

Example 4.2. In the bi-directional cycle C_n on n vertices the game may give rise to very unfair equilibria. Notice that the strategy of each player reduces to a choice between her left and right neighbors, and all cycles are either of length 2 or n. As noted before, a player will never pay more than $1/2$. However, the equilibrium described in Fig. 2 features exactly two players on a cycle of length 2 and each one pays exactly $1/2$, so that in this case

$$\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = 1 < \mathsf{PoA}(\mathcal{S}) = \mathsf{PoA}(\Sigma) = n/2.$$

Example 4.3. In the previous examples we had either $\mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = 1$ or $\mathsf{PoA}(\mathcal{S}) = \mathsf{PoA}(\Sigma) = n/2$. For the graph in Fig. 8, with a uniform initial measure μ, these values are bounded away from these extremes. Indeed, notice that only

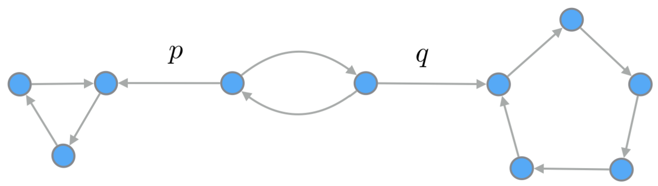

Figure 8. A network with $1 < \mathsf{PoS} < \mathsf{PoA} < \frac{n}{2}$. Here we consider μ as the uniform measure on the vertex set.

the two players on the central cycle have the possibility to randomize. The optimal strategy is attained for $p = q = 0$ with

$$\min_{\boldsymbol{\pi}\in\Sigma} \mathsf{SC}(\boldsymbol{\pi}) = \min_{\boldsymbol{s}\in\mathcal{S}} \mathsf{SC}(\boldsymbol{s}) = \frac{1}{10}.$$

However, this is not an equilibrium and $\mathsf{NE}(\Sigma)$ is precisely characterized by $p+q > 0$. The best equilibrium is achieved with $p = 0, q = 1$, and the worse with $p = 1, q = 0$. These are in fact determinstic equilibria in $\mathsf{NE}(\mathcal{S})$, so that

$$1 < \mathsf{PoS}(\mathcal{S}) = \mathsf{PoS}(\Sigma) = \frac{7}{5} < \frac{5}{3} = \mathsf{PoA}(\mathcal{S}) = \mathsf{PoA}(\Sigma) < \frac{n}{2}.$$

Example 4.4. In general there is no order between $\mathsf{PoS}(\mathcal{S})$ and $\mathsf{PoS}(\Sigma)$, and either one may be larger. Indeed, in Fig. 9 the optimal cost both for $\mathcal{S}$ and Σ is $\frac{1}{6}$, attained with $p = q = 0$. The best equilibrium over $\mathcal{S}$ is attained for $p = 1, q = 0$ (as well as $p = 0, q = 1$) with social cost $\frac{1}{3}$, whereas the best equilibrium over Σ is attained with $p = q = \frac{1}{2}$ for a social cost of $\frac{1}{4}$. Hence $\frac{3}{2} = \mathsf{PoS}(\Sigma) < \mathsf{PoS}(\mathcal{S}) = 2$. Now, in

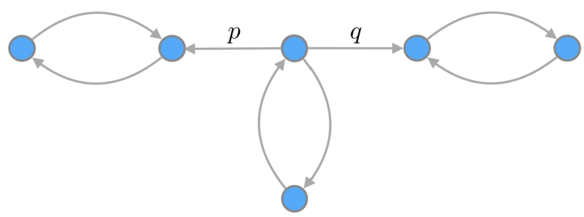

Figure 9. A buck-passing game with $\mathsf{PoS}(\Sigma) < \mathsf{PoS}(\mathcal{S})$. The initial measure μ is uniform across all 6 vertices.

Fig. 10 the deterministic social optimum is $\frac{1}{3}$ attained with $p = 1, q = 0$ and $r = 1$. This is also an equilibrium so that $\mathsf{PoS}(\mathcal{S}) = 1$. In the stochastic case the minimum cost is $\frac{1}{6}$ (attained with $p = q = \frac{1}{2}$ and $r = 0$), while the best equilibrium is achieved for $r = 1, p = \frac{3}{4}, q = \frac{1}{4}$, with social cost $\frac{1}{4}$ and then $\mathsf{PoS}(\Sigma) = \frac{3}{2} > \mathsf{PoS}(\mathcal{S})$.

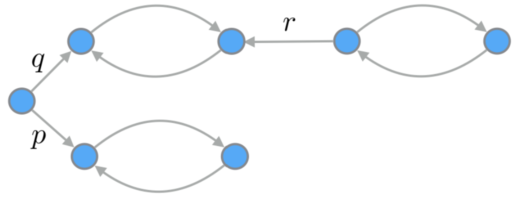

Figure 10. A buck-passing game with $\mathsf{PoS}(\Sigma) > \mathsf{PoS}(\mathcal{S})$. The initial measure μ assigns the buck to the leftmost vertex with probability $\frac{2}{3}$, and to the rightmost vertex with probability $\frac{1}{3}$.

5. Markov chains, spanning trees and spectra

In the next section we will study the existence of equilibria for buck-passing games when the strategy sets of the players are restricted to more general subsets of the whole simplices of probabilities over out-neighbors. We will achieve this by extending the generalized potential function in (2.10), which requires some powerful tools in Markov chains. In particular, we will exploit classical results linking probability and graph theory, which can be traced back to Kirchhoff (1847).

5.1. **The Markov chain tree theorem revisited.** We begin by discussing the Markov chain tree theorem, introduced by Leighton and Rivest (1986) (see also Aldous and Fill, 2002, Anantharam and Tsoucas, 1989). For the sake of completeness we present a short proof that unveils its connection with the buck-passing games. The idea of the proof is to focus on *spanning unicycles* rather than trees. Consider a Markov chain on a finite state space $\mathcal{V}$ with transition matrix $\boldsymbol{\pi} = (\pi_{ij})_{i,j\in\mathcal{V}}$, and assume that it has a unique recurrent class $\mathcal{C} \subseteq \mathcal{V}$, so that all vertices in $\mathcal{V} \setminus \mathcal{C}$ are transient. In this case there is a unique invariant measure $\rho = (\rho(i))_{i\in\mathcal{V}}$, with $\rho(i) > 0$ iff $i \in \mathcal{C}$.

We now define the *weight function* $\omega_{\boldsymbol{\pi}} : 2^{\mathcal{E}} \to [0,1]$ as follows: for every $A \subset \mathcal{E}$

$$\omega_{\boldsymbol{\pi}}(A) = \prod_{(i,j)\in A} \pi_{ij}. \tag{5.1}$$

Note that each strategy profile $\boldsymbol{s} \in \mathcal{S}$ in the deterministic buck-passing game can be identified with the subset of induced edges $\mathcal{E}_{\boldsymbol{s}}$. Thus —with a slight abuse of notation and only for this section— we identify $\mathcal{S}$ with the family of all subsets $\boldsymbol{s} \subset \mathcal{E}$ that contain exactly one outgoing edge for each $i \in \mathcal{V}$.

We then consider the probability space $(\mathcal{S}, \mathbb{Q}_{\boldsymbol{\pi}}, 2^{\mathcal{S}})$, where

$$\mathbb{Q}_{\boldsymbol{\pi}}(\boldsymbol{s}) = \omega_{\boldsymbol{\pi}}(\boldsymbol{s}) = \prod_{(i,j)\in\boldsymbol{s}} \pi_{ij}. \tag{5.2}$$

It is easy to see that $\mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{S}) = 1$. From a probabilistic perspective, the multiplicative form of $\mathbb{Q}_{\boldsymbol{\pi}}$ implies that a random $\boldsymbol{s} \in \mathcal{S}$ sampled according to $\mathbb{Q}_{\boldsymbol{\pi}}$, can be seen as the outcome of n independent draws of an outgoing edge $(i,j) \in \mathcal{N}_i^+$ for each $i \in \mathcal{V}$.

We call $\mathcal{T}_i(\mathcal{V})$ the set of i-rooted spanning trees in the complete graph with vertex set $\mathcal{V}$ and define

$$\Omega_i(\boldsymbol{\pi}) := \sum_{\tau \in \mathcal{T}_i(\mathcal{V})} \omega_{\boldsymbol{\pi}}(\tau) \quad \text{and} \quad \Omega_{\mathcal{V}}(\boldsymbol{\pi}) := \sum_{j \in \mathcal{V}} \Omega_j(\boldsymbol{\pi}). \tag{5.3}$$

Notice that a vertex $i \in \mathcal{V}$ is transient iff $\Omega_i(\boldsymbol{\pi}) = 0$, so that $\mathcal{C} = \{i \in \mathcal{V} : \Omega_i(\boldsymbol{\pi}) > 0\}$. Note also that when computing $\Omega_i(\boldsymbol{\pi})$ it suffices to consider the spanning trees in the induced graph $\mathcal{G}_{\boldsymbol{\pi}}$, which contains only the edges with $\pi_{ij} > 0$, since the remaining trees have weight zero. However, to avoid keeping track of the dependence on the topology of $\mathcal{G}_{\boldsymbol{\pi}}$, it is convenient to consider all spanning trees in the complete graph with vertex set $\mathcal{V}$.

Theorem 5.1 (Markov chain tree theorem)**.** *Consider a Markov chain with transition matrix $\boldsymbol{\pi}$ and with a single recurrent class. Then, the unique invariant measure ρ is given by*

$$\rho(i) = \frac{\Omega_i(\boldsymbol{\pi})}{\Omega_{\mathcal{V}}(\boldsymbol{\pi})}. \tag{5.4}$$

As mentioned before, each $\boldsymbol{s} \in \mathcal{S}$ can be identified with a pure strategy profile in the deterministic buck-passing game, so the induced graph $\mathcal{G}_{\boldsymbol{s}}$ is a disjoint union of unicycles. Let $\mathcal{U}$ denote the set of all $\boldsymbol{s} \in \mathcal{S}$ inducing a single spanning unicycle. Moreover, let $\mathcal{U}_i$ be the spanning unicycles that have i in the cycle, and $\mathcal{U}_{ij}$ those in which the edge (i, j) is part of the cycle.

Proof of Theorem 5.1. Call $x_i := \Omega_i(\boldsymbol{\pi})$ and $x := \Omega_{\mathcal{V}}(\boldsymbol{\pi})$. We observe that each $\boldsymbol{s} \in \mathcal{U}_{ij}$ is of the form $\boldsymbol{s} = \tau \cup \{(i, j)\}$ for a unique i-rooted tree $\tau \in \mathcal{T}_i(\mathcal{V})$ so that $\omega_{\boldsymbol{\pi}}(\boldsymbol{s}) = \omega_{\boldsymbol{\pi}}(\tau)\, \pi_{ij}$, and therefore $\mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_{ij}) = x_i\, \pi_{ij}$. Now, the set $\mathcal{U}_i$ can be expressed as a disjoint union $\mathcal{U}_i = \dot{\cup}_{j \in \mathcal{V}}\ \mathcal{U}_{ij}$, so that

$$\mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_i) = \sum_{j \in \mathcal{V}} \mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_{ij}) = \sum_{j \in \mathcal{V}} x_i\, \pi_{ij} = x_i. \tag{5.5}$$

Similarly, if we focus on the edge (k, i) preceeding i, we may write $\mathcal{U}_i = \dot{\cup}_{k \in \mathcal{V}}\ \mathcal{U}_{ki}$, so that

$$x_i = \mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_i) = \sum_{k \in \mathcal{V}} \mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_{ki}) = \sum_{k \in \mathcal{V}} x_j\, \pi_{ki}. \tag{5.6}$$

This shows that $(x_i)_{i \in \mathcal{V}}$ is a left eigenvector of $\boldsymbol{\pi}$ with eigenvalue 1, so it is collinear with the invariant measure ρ. The conclusion follows dividing each x_i by x. $\square$

As a by-product of the previous proof, we observe that $\Omega_i(\boldsymbol{\pi})$ can be expressed as the expected length of spanning unicycles. Indeed, consider the random variable $\mathbb{1}_{\{\boldsymbol{s} \in \mathcal{U}_i\}}$ whose expected value is the probability that vertex i lies on the cycle of a spanning unicycle, that is,

$$\mathbb{E}_{\mathbb{Q}_{\boldsymbol{\pi}}}[\mathbb{1}_{\mathcal{U}_i}] = \mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_i) = \Omega_i(\boldsymbol{\pi}). \tag{5.7}$$

Moreover, let

$$\Lambda(\boldsymbol{s}) := \sum_{i \in \mathcal{V}} \mathbb{1}_{\{\boldsymbol{s} \in \mathcal{U}_i\}} \tag{5.8}$$

be the length of the cycle if $\boldsymbol{s} \in \mathcal{S}$ is a spanning unicycle, and 0 otherwise.

Corollary 5.2. *Consider a Markov chain with transition matrix $\boldsymbol{\pi}$ and with a single recurrent class $\mathcal{C} \subseteq \mathcal{V}$. Then $\Omega_{\mathcal{V}}(\boldsymbol{\pi}) = \mathbb{E}_{\mathbb{Q}_{\boldsymbol{\pi}}}[\Lambda]$ and, in particular, $\Omega_{\mathcal{V}}(\boldsymbol{\pi}) \leq |\mathcal{C}|$.*

Remark 5.3. Note that $\Lambda(\boldsymbol{s})$ is the total length of the cycle in the graph $\mathcal{G}_{\boldsymbol{s}}$, which appears in the potential (2.10) for the deterministic buck-passing game. In the next section, see Eq. (6.6), we introduce a potential for the stochastic buck-passing game which involves the expected value $\mathbb{E}_{\mathbb{Q}_{\boldsymbol{\pi}}}[\Lambda]$.

5.2. **A spectral perspective.** We next provide a representation of $\Omega(\boldsymbol{\pi})$ in terms of the spectrum of the Laplacian matrix $L_{\boldsymbol{\pi}} := I - \boldsymbol{\pi}$. This is essentially a consequence of the celebrated *matrix-tree theorem*, which generalizes the original work of Kirchhoff (1847) (see, e.g., Brooks et al., 1940, Chaiken and Kleitman, 1978).

For a matrix L, let $L_{(i|j)}$ denote the matrix obtained from L by removing row i and column j, and consider the *adjugate* matrix, whose entries are given by

$$\operatorname{adj}(L)_{ij} = (-1)^{i+j} \det L_{(i|j)}. \tag{5.9}$$

Theorem 5.4 (matrix-tree theorem)**.** *For a Markov chain with a unique recurrent class and Laplacian L we have*

$$\sum_{\tau \in \mathcal{T}(i)} \omega_{\boldsymbol{\pi}}(\tau) = \det L_{(i|i)}. \tag{5.10}$$

Hence,

$$\Omega_{\mathcal{V}}(\boldsymbol{\pi}) = \operatorname{tr}[\operatorname{adj}(L_{\boldsymbol{\pi}})]. \tag{5.11}$$

From this result we obtain the following alternative formula for $\Omega_{\mathcal{V}}(\boldsymbol{\pi})$ in terms of the nonzero eigenvalues of the Laplacian matrix.

Theorem 5.5. *Let $\boldsymbol{\pi}$ be the transition matrix of a Markov chain with a unique recurrent class. Call $\{\lambda_{\boldsymbol{\pi}}^{(1)}, \dots, \lambda_{\boldsymbol{\pi}}^{(n)}\}$ a specific ordering of the spectrum of the Laplacian matrix $L_{\boldsymbol{\pi}}$, such that $\lambda_{\boldsymbol{\pi}}^{(1)} = 0$, whereas all the other (possibly complex) eigenvalues have a non-zero modulus. Then*

$$\Omega_{\mathcal{V}}(\boldsymbol{\pi}) = \prod_{i=2}^{n} \lambda_{\boldsymbol{\pi}}^{(i)}. \tag{5.12}$$

Proof. Consider the spectral decomposition of $L_{\boldsymbol{\pi}} = UJU^{-1}$ with J the Jordan matrix having the eigenvalues $\lambda_{\boldsymbol{\pi}}^{(i)}$ on the diagonal in increasing order of i, and U an orthogonal matrix. We have

$$\begin{aligned}
\operatorname{tr}\operatorname{adj}(L_{\boldsymbol{\pi}}) &= \operatorname{tr}(\operatorname{adj}(U)\operatorname{adj}(J)\operatorname{adj}(U^{-1})) && (5.13)\\
&= \operatorname{tr}(\det(U)U^{-1}\operatorname{adj}(J)\det(U^{-1})U) && (5.14)\\
&= \operatorname{tr}(U^{-1}\operatorname{adj}(J)U) && (5.15)\\
&= \operatorname{tr}\operatorname{adj}(J) && (5.16)\\
&= \prod_{i=2}^{n} \lambda_{\boldsymbol{\pi}}^{(i)}, && (5.17)
\end{aligned}$$

where the first equality is due to the fact that adjugate and product commute; the second equality stems from the fact that the adjugate of a full rank matrix is the inverse of that matrix times its determinant; the third equality derives from $\det(M^{-1}) = [\det(M)]^{-1}$, when M is full rank; the fourth is just the invariance of trace with respect to change of basis; for the last one consider that $L_{\boldsymbol{\pi}}$ is the Laplacian of an irreducible chain, so the only nonzero element on the diagonal of $\operatorname{adj}(J)$ is the cofactor $(1|1)$. Indeed, the Laplacian of an irreducible chain has a kernel of dimension 1. Hence, the Jordan Matrix J has the first row and the first column equal to zero. This immediately implies that all the cofactors $(i|i)$ of J are zero except for $i = 1$. ☐

6. The constrained buck-passing game

We consider next a generalized version of the stochastic buck-passing game, in which player i's strategy set is a subset $\Xi_i \subset \Sigma_i$. Accordingly, we define $\Xi := \times_{i\in\mathcal{V}}\Xi_i$. We call this game $\Gamma(\mathcal{G}, \mu, \Xi)$ a *constrained buck-passing game* (CBPG), and we denote $\mathsf{NE}(\Xi)$ its set of Nash equilibria. We will show that CBPGs are generalized ordinal potential games.

For the sake of simplicity, consider first a strategy profile $\boldsymbol{\pi}$ inducing an irreducible Markov chain. In this case, by Theorem 5.1, the cost for player i is simply

$$c_i(\boldsymbol{\pi}) = \rho_{\boldsymbol{\pi}}(i) = \frac{\Omega_i(\boldsymbol{\pi})}{\Omega_{\mathcal{V}}(\boldsymbol{\pi})}. \tag{6.1}$$

Since the numerator $\Omega_i(\boldsymbol{\pi})$ in (6.1) does not depend on π_i, a profitable deviation for player $i \in \mathcal{V}$ can only be achieved by increasing the denominator $\Omega_{\mathcal{V}}(\boldsymbol{\pi})$. This suggests to take the map $\Psi(\boldsymbol{\pi}) = -\Omega_{\mathcal{V}}(\boldsymbol{\pi})$ as a generalized ordinal potential. Since Ψ does not depend on μ, any of its minimizers provides a PFNE. This is indeed the case if every strategy profile $\boldsymbol{\pi} \in \Xi$ gives rise to an irreducible Markov chain.

6.1. A generalized ordinal potential.

To get a workable expression for the costs (3.6), we use the Markov chain tree formula. To this end, consider the *transient closures* of the recurrent classes

$$\mathcal{A}^{\ell}_{\boldsymbol{\pi}} := \left\{ j \in \mathcal{V} : \mathsf{P}^{j\to\ell}_{\boldsymbol{\pi}} = 1 \right\} \quad \forall\, \ell = 1, \dots, M(\boldsymbol{\pi}) \tag{6.2}$$

and the *residual transient class* that contains the remaining vertices

$$\mathcal{R}_{\boldsymbol{\pi}} := \left\{ j \in \mathcal{V} : \mathsf{P}^{j\to\ell}_{\boldsymbol{\pi}} < 1,\ \forall \ell = 1, \dots, M(\boldsymbol{\pi}) \right\}, \tag{6.3}$$

where $\mathsf{P}^{j\to\ell}_{\boldsymbol{\pi}}$ is defined as in (3.4).

Each set $\mathcal{A}^{\ell}_{\boldsymbol{\pi}}$ is closed with respect to the Markov chain and $\mathcal{C}^{\ell}_{\boldsymbol{\pi}} \subseteq \mathcal{A}^{\ell}_{\boldsymbol{\pi}}$. Therefore, the restriction of the original Markov chain to $\mathcal{A}^{\ell}_{\boldsymbol{\pi}}$ is itself a Markov chain having $\mathcal{C}^{\ell}_{\boldsymbol{\pi}}$ as its unique recurrent class, so that Theorem 5.1 gives

$$\rho^{\ell}_{\boldsymbol{\pi}}(i) = \begin{cases} \dfrac{\Omega^{\ell}_i(\boldsymbol{\pi})}{\Omega^{\ell}(\boldsymbol{\pi})} & \text{if } i \in \mathcal{A}^{\ell}_{\boldsymbol{\pi}} \\ 0 & \text{if } i \in \mathcal{V} \setminus \mathcal{A}^{\ell}_{\boldsymbol{\pi}} \end{cases} \tag{6.4}$$

where

$$\Omega^{\ell}_i(\boldsymbol{\pi}) = \sum_{\tau \in \mathcal{T}_i(\mathcal{A}^{\ell}_{\boldsymbol{\pi}})} \omega_{\boldsymbol{\pi}}(\tau) \quad \text{and} \quad \Omega^{\ell}(\boldsymbol{\pi}) = \sum_{i \in \mathcal{A}^{\ell}_{\boldsymbol{\pi}}} \Omega^{\ell}_i(\boldsymbol{\pi}), \tag{6.5}$$

with $\mathcal{T}_i(\mathcal{A}^\ell_{\boldsymbol{\pi}})$ the set of i-rooted spanning trees in the complete graph over $\mathcal{A}^\ell_{\boldsymbol{\pi}}$.

With these preliminaries, we may now state our main result for CBPGs.

Theorem 6.1. *The constrained buck-passing game* $\Gamma(\mathcal{G}, \mu, \Xi)$ *admits the generalized ordinal potential*

$$\Psi(\boldsymbol{\pi}) := \sum_{\ell=1}^{M(\boldsymbol{\pi})} (n - \Omega^\ell(\boldsymbol{\pi})), \tag{6.6}$$

with $\Omega^\ell(\boldsymbol{\pi})$ *defined by* (6.5).

Proof. Consider a profitable deviation by a player i from $\boldsymbol{\pi}$ to $\boldsymbol{\pi}' := (\pi'_i, \boldsymbol{\pi}_{-i})$, with $c_i(\boldsymbol{\pi}') < c_i(\boldsymbol{\pi})$. This conveys the fact that $c_i(\boldsymbol{\pi}) > 0$, so that from Eq. (3.6) it follows that player i must belong to a recurrent class $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ with $\mu^\ell_{\boldsymbol{\pi}} > 0$. We distinguish two possible scenarios, depending on whether the player remains recurrent or becomes transient after deviating.

Case 1: $\pi'_{ij} = 0$ for all $j \notin \mathcal{A}^\ell_{\boldsymbol{\pi}}$. In this case i remains recurrent and, although $\mathcal{C}^\ell_{\boldsymbol{\pi}}$ may change, we have $\mathcal{A}^\ell_{\boldsymbol{\pi}'} = \mathcal{A}^\ell_{\boldsymbol{\pi}}$ and $\mu^\ell_{\boldsymbol{\pi}'} = \mu^\ell_{\boldsymbol{\pi}} > 0$. It then follows from (3.6) that $c_i(\boldsymbol{\pi}') < c_i(\boldsymbol{\pi})$ is equivalent to $\rho^\ell_{\boldsymbol{\pi}'}(i) < \rho^\ell_{\boldsymbol{\pi}}(i)$. Now, since the weight of any i-rooted tree $\tau \in \mathcal{T}_i(\mathcal{A}^\ell_{\boldsymbol{\pi}}) = \mathcal{T}_i(\mathcal{A}^\ell_{\boldsymbol{\pi}'})$ does not depend on π_i nor π'_i, we get $\Omega^\ell_i(\boldsymbol{\pi}) = \Omega^\ell_i(\boldsymbol{\pi}')$. Then, from (6.4) it follows that $\rho^\ell_{\boldsymbol{\pi}'}(i) < \rho^\ell_{\boldsymbol{\pi}}(i)$ is equivalent to $\Omega^\ell(\boldsymbol{\pi}') > \Omega^\ell(\boldsymbol{\pi})$, which implies $\Psi(\boldsymbol{\pi}') < \Psi(\boldsymbol{\pi})$.

Case 2: $\pi'_{ij} > 0$ for some $j \notin \mathcal{A}^\ell_{\boldsymbol{\pi}}$. In this case i together with all vertices in $\mathcal{A}^\ell_{\boldsymbol{\pi}}$ become transient and their costs drop to zero. We distinguish two subcases depending whether i becomes residual transient or it is absorbed into a different class.

Case 2.1: $i \in \mathcal{R}_{\boldsymbol{\pi}'}$. In this case the class $\mathcal{A}^\ell_{\boldsymbol{\pi}}$ becomes part of $\mathcal{R}_{\boldsymbol{\pi}'}$, while the remaining classes $\mathcal{A}^h_{\boldsymbol{\pi}}$ remain unchanged. Hence, we lose the ℓ-th term in the sum of (6.6), while the other terms do not change. From Corollary 5.2 we have $\Omega^\ell(\boldsymbol{\pi}) \leq |\mathcal{C}^\ell_{\boldsymbol{\pi}}| < n$, so that the removed term is strictly positive and $\Psi(\boldsymbol{\pi}') < \Psi(\boldsymbol{\pi})$.

Case 2.2: $\mathsf{P}^{i \to h}_{\boldsymbol{\pi}'} = 1$ for some $h \neq \ell$. Here the full class $\mathcal{A}^\ell_{\boldsymbol{\pi}}$ is absorbed into the h-th class, that is, $\mathcal{A}^h_{\boldsymbol{\pi}'} = \mathcal{A}^h_{\boldsymbol{\pi}} \cup \mathcal{A}^\ell_{\boldsymbol{\pi}}$, so that the ℓ-th and h-th terms in the sum $\Psi(\boldsymbol{\pi})$ are merged into the single h-th term in $\Psi(\boldsymbol{\pi}')$, while the other terms do not change. Hence, $\Psi(\boldsymbol{\pi}') < \Psi(\boldsymbol{\pi})$ is equivalent to

$$n - \Omega^h(\boldsymbol{\pi}') < [n - \Omega^\ell(\boldsymbol{\pi})] + [n - \Omega^h(\boldsymbol{\pi})],$$

which follows by noting that $\Omega^h(\boldsymbol{\pi}') > 0$ and using Corollary 5.2 once again, which gives

$$\Omega^\ell(\boldsymbol{\pi}) + \Omega^h(\boldsymbol{\pi}) \leq |\mathcal{C}^\ell_{\boldsymbol{\pi}}| + |\mathcal{C}^h_{\boldsymbol{\pi}}| \leq n.$$

In all scenarios we have that $c_i(\boldsymbol{\pi}') < c_i(\boldsymbol{\pi})$ implies $\Psi(\boldsymbol{\pi}') < \Psi(\boldsymbol{\pi})$, which proves that Ψ is a generalized ordinal potential. □

We stress the analogy between the potential function Ψ in (6.6) and the one for the deterministic game in (2.10). Indeed, the quantity $|\mathcal{C}^\ell_{\boldsymbol{s}}|$ in the latter is simply the number of rooted spanning trees for the ℓ-th unicycle and, since in the deterministic case the weight of each tree is 1, we have $|\mathcal{C}^\ell_{\boldsymbol{s}}| = \Omega^\ell(\boldsymbol{s})$. However, in contrast with the deterministic case, Ψ may fail to provide an ordinal potential even when μ is fully supported.

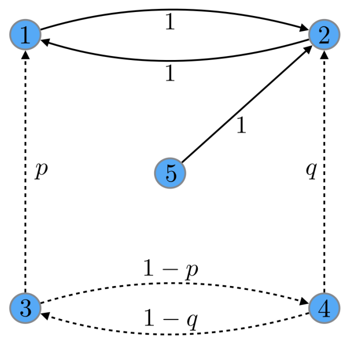

Figure 11. The strategy profile considered in Example 6.1.

Example 6.1. Consider the graph in Fig. 11. The set of players in the corresponding SBPG is $\mathcal{V} = \{1, 2, 3, 4, 5\}$ and the initial distribution μ is assumed to be uniform. Fix $p, q \in (0, 1)$ and consider the following strategy profile

$$\begin{aligned}
&\pi_{12} = \pi_{21} = 1,\\
&\pi_{31} = p, \quad \pi_{34} = 1 - p,\\
&\pi_{42} = q, \quad \pi_{43} = 1 - q,\\
&\pi_{52} = 1.
\end{aligned}$$

This strategy profile induces a unique recurrent class $\mathcal{C}_{\boldsymbol{\pi}} = \{1, 2\}$ and we have

$$\Omega_1(\boldsymbol{\pi}) = \Omega_2(\boldsymbol{\pi}) = p + q - pq,$$

so that

$$\Psi(\boldsymbol{\pi}) = 5 - 2(p + q - pq).$$

Note that players 3 and 4 can decrease the potential by increasing p and q, respectively, although their cost remains 0, since they are transient. Therefore, Ψ is not an ordinal potential, even though μ is fully supported. Note also that $\Psi(\boldsymbol{\pi}) \to 5$, as p and q tend to 0, whereas, for $p = q = 0$, there are two recurrent classes $\mathcal{C}^1_{\boldsymbol{\pi}} = \{1, 2\}$ and $\mathcal{C}^2_{\boldsymbol{\pi}} = \{3, 4\}$ and the value of the potential is 6. Therefore Ψ is not continuous, and not even lower semicontinuous.

Remark 6.2. Theorem 6.1 can be connected to some literature that looks at the Hamiltonian cycle problem from the perspective of Markov chains. Namely, considering the set of transition matrices that are compatible with a given graph $\mathcal{G}$, this literature focuses on the class of functionals whose global minimum is attained on a permutation matrix which corresponds to a Hamiltonian cycle, provided it exists (see, e.g., Borkar et al., 2012, Ejov et al., 2011, Filar, 2006, Filar and Krass, 1994). Theorem 6.1 shows that the potential function Ψ in (6.6) belongs to this class.

6.2. **Existence of equilibria.** We now address the existence of a PFNE for general constrained buck-passing games. The fact that $\Gamma(\mathcal{G},\mu,\Xi)$ has a generalized ordinal potential guarantees the existence of ε-equilibria.

Proposition 6.3. *For each $\varepsilon > 0$ the constrained buck-passing game $\Gamma(\mathcal{G},\mu,\Xi)$ has an ε-NE which is also prior-free.*

Proof. The existence of an ε-NE is a consequence of Monderer and Shapley (1996, Lemmata 4.1 and 4.2). See also Lã et al. (2016, Section 2.2.2.2). Prior-freeness is due to the fact that the potential function Ψ in (6.6) does not depend on μ. $\square$

Since Ψ is not lower semicontinuous, even if Ξ is compact, we cannot invoke Proposition 2.5 to establish existence of equilibria, so we develop an *ad hoc* argument that requires some additional notation and preliminary results.

Let Ξ^0 denote the set of strategy profiles $\boldsymbol{\pi} \in \Xi$ with a minimal number of recurrent classes $M^0 = \min_{\boldsymbol{\pi}\in\Xi} M(\boldsymbol{\pi})$, and let $\Xi(\boldsymbol{\pi})$ be the set of all unilateral deviations $\boldsymbol{\pi}' = (\pi_i', \boldsymbol{\pi}_{-i})$ by recurrent players $i \in \mathcal{C}^1_{\boldsymbol{\pi}} \dot\cup \cdots \dot\cup \mathcal{C}^{M^0}_{\boldsymbol{\pi}}$. Our first result is the following simple observation.

Lemma 6.4. *For each $\boldsymbol{\pi} \in \Xi^0$ and $\boldsymbol{\pi}' \in \Xi(\boldsymbol{\pi})$ we have $M(\boldsymbol{\pi}') = M^0$ and $\mathcal{A}^\ell_{\boldsymbol{\pi}'} = \mathcal{A}^\ell_{\boldsymbol{\pi}}$ for all $\ell = 1,\dots,M^0$. Moreover, if Ξ is compact then $\inf_{\boldsymbol{\pi}'\in\Xi(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi}')$ is attained.*

Proof. By the minimality of $M(\boldsymbol{\pi}) = M^0$, a recurrent player $i \in \mathcal{C}^\ell_{\boldsymbol{\pi}}$ cannot become transient after a unilateral deviation $\boldsymbol{\pi}' = (\pi_i', \boldsymbol{\pi}_{-i})$, so that $\pi'_{ij} = 0$ for all $j \notin \mathcal{A}^\ell_{\boldsymbol{\pi}}$. Hence, every such deviation preserves the number of classes $M(\boldsymbol{\pi}') = M(\boldsymbol{\pi}) = M^0$, as well as all the transient closures $\mathcal{A}^\ell_{\boldsymbol{\pi}'} = \mathcal{A}^\ell_{\boldsymbol{\pi}}$. It follows that, for all $\boldsymbol{\pi}' \in \Xi(\boldsymbol{\pi})$,

$$\Psi(\boldsymbol{\pi}') = \sum_{\ell=1}^{M(\boldsymbol{\pi})} (n - \Omega^\ell(\boldsymbol{\pi}')) \quad \text{with} \quad \Omega^\ell(\boldsymbol{\pi}') = \sum_{i\in\mathcal{A}^\ell_{\boldsymbol{\pi}}} \sum_{\tau\in\mathcal{T}_i(\mathcal{A}^\ell_{\boldsymbol{\pi}})} \omega_{\boldsymbol{\pi}'}(\tau). \tag{6.7}$$

For fixed $\boldsymbol{\pi}$ these functions are continuous with respect to $\boldsymbol{\pi}'$ and the set $\Xi(\boldsymbol{\pi})$ is compact, since it is a section of a compact set. Therefore, the minimum of $\Psi(\boldsymbol{\pi}')$ over $\Xi(\boldsymbol{\pi})$ is attained. $\square$

Our next step is less trivial and requires the notion of *skeleton* of a transient closure $\mathcal{A}^\ell_{\boldsymbol{\pi}}$, defined as any rooted tree

$$\widehat{\tau}^\ell_{\boldsymbol{\pi}} \in \bigcup_{i\in\mathcal{A}^\ell_{\boldsymbol{\pi}}} \mathcal{T}_i(\mathcal{A}^\ell_{\boldsymbol{\pi}}) \tag{6.8}$$

having maximal weight $\omega_{\boldsymbol{\pi}}(\tau)$. Note that

$$\Omega^\ell(\boldsymbol{\pi}) \le R_n\, \omega_{\boldsymbol{\pi}}(\widehat{\tau}^\ell_{\boldsymbol{\pi}}), \tag{6.9}$$

where R_n is the number of rooted trees on n vertices.

Theorem 6.5. *Suppose that Ξ is compact. Then there exists $\boldsymbol{\pi} \in \Xi^0$ such that $\Psi(\boldsymbol{\pi}) \le \Psi(\boldsymbol{\pi}')$ for all $\boldsymbol{\pi}' \in \Xi(\boldsymbol{\pi})$.*

Proof. Fix $\boldsymbol{\pi}^0 \in \Xi^0$ and, for $\ell = 1, \ldots, M^0$, let $\mathcal{A}^\ell = \mathcal{A}^\ell_{\boldsymbol{\pi}^0}$ be the corresponding transient closures. Consider a sequence defined inductively by

$$\boldsymbol{\pi}^{k+1} \in \underset{\boldsymbol{\pi}' \in \Xi(\boldsymbol{\pi}^k)}{\arg\min}\, \Psi(\boldsymbol{\pi}'), \tag{6.10}$$

so that $M(\boldsymbol{\pi}^k) = M^0$ and $\mathcal{A}^\ell_{\boldsymbol{\pi}^k} \equiv \mathcal{A}^\ell$ for $\ell = 1, \ldots, M^0$ and all $k \in \mathbb{N}$.

Since $\boldsymbol{\pi}^k \in \Xi(\boldsymbol{\pi}^k)$ we have $\Psi(\boldsymbol{\pi}^{k+1}) \leq \Psi(\boldsymbol{\pi}^k)$. If equality holds for some k, then the conclusion follows by taking $\boldsymbol{\pi} = \boldsymbol{\pi}^k$. Consider then the case where $\Psi(\boldsymbol{\pi}^{k+1}) < \Psi(\boldsymbol{\pi}^k)$ for all $k \in \mathbb{N}$. Note that along the iterations we have

$$\Psi(\boldsymbol{\pi}^k) = \sum_{\ell=1}^{M^0} (n - \Omega^\ell(\boldsymbol{\pi}^k)) \quad \text{with} \quad \Omega^\ell(\boldsymbol{\pi}^k) = \sum_{i \in \mathcal{A}^\ell} \sum_{\tau \in \mathcal{T}_i(\mathcal{A}^\ell)} \omega_{\boldsymbol{\pi}^k}(\tau), \tag{6.11}$$

so that $\sum_{\ell=1}^{M^0} \Omega^\ell(\boldsymbol{\pi}^k)$ increases with k. Moreover, $\boldsymbol{\pi}^{k+1}$ is obtained from $\boldsymbol{\pi}^k$ by a deviation of a player i_k in some recurrent class $\mathcal{C}^{\ell^k}_{\boldsymbol{\pi}^k}$, so that only the ℓ^k-th term in the sum changes and therefore $\Omega^\ell(\boldsymbol{\pi}^k)$ is nondecreasing in k for each $\ell = 1, \ldots, M^0$. In particular $\Omega^\ell(\boldsymbol{\pi}^k)$ remains bounded away from 0, and then, using (6.9), we may find $\varepsilon > 0$ such that,

$$\text{for all } k \in \mathbb{N}, \quad \omega_{\boldsymbol{\pi}^k}(\widehat{\tau}^\ell_{\boldsymbol{\pi}^k}) \geq \varepsilon. \tag{6.12}$$

Take a convergent subsequence $\boldsymbol{\pi}^{k_m} \to \boldsymbol{\pi} \in \Xi$, and extract a further subsequence along which the skeletons are constant $\widehat{\tau}^\ell_{\boldsymbol{\pi}^{k_m}} \equiv \widehat{\tau}^\ell$ for $\ell = 1, \ldots, M^0$. Passing to the limit in (6.12) along this subsequence we get $\omega_{\boldsymbol{\pi}}(\widehat{\tau}^\ell) \geq \varepsilon$, which implies that $\mathcal{A}^\ell$ is still connected in the limit and therefore $M(\boldsymbol{\pi}) = M^0$ and $\mathcal{A}^\ell_{\boldsymbol{\pi}} = \mathcal{A}^\ell$. From these facts, using (6.11) and the continuity of the polynomials $\boldsymbol{\pi} \mapsto \omega_{\boldsymbol{\pi}}(\tau)$, we obtain $\Psi(\boldsymbol{\pi}^{k_m}) \to \Psi(\boldsymbol{\pi})$. Since $\Psi(\boldsymbol{\pi}^k)$ is decreasing, we conclude in fact that the full sequence of potential values converges $\Psi(\boldsymbol{\pi}^k) \to \Psi(\boldsymbol{\pi})$.

We now show that the limit point $\boldsymbol{\pi}$ satisfies the claim of the theorem. Indeed, we already proved that $M(\boldsymbol{\pi}) = M^0$, so that $\boldsymbol{\pi} \in \Xi^0$. Now, consider a player $i \in \mathcal{C}^\ell_{\boldsymbol{\pi}}$ and a deviation $\boldsymbol{\pi}' = (\pi'_i, \boldsymbol{\pi}_{-i})$. Since $i \in \mathcal{C}^\ell_{\boldsymbol{\pi}}$, it follows that, for each $j \in \mathcal{A}^\ell$, there is a path from j to i whose edges have positive probability under $\boldsymbol{\pi}$. Since $\boldsymbol{\pi}^{k_m} \to \boldsymbol{\pi}$ this is also the case for $\boldsymbol{\pi}^{k_m}$ for m large enough. Hence, $i \in \mathcal{C}^\ell_{\boldsymbol{\pi}^{k_m}}$ and then the definition of the sequence $\boldsymbol{\pi}^k$ implies that

$$\Psi(\boldsymbol{\pi}^{k_m+1}) \leq \Psi(\pi'_i, \boldsymbol{\pi}^{k_m}_{-i}). \tag{6.13}$$

From Lemma 6.4 we have that $(\pi'_i, \boldsymbol{\pi}^{k_m}_{-i})$ has the same transient closures $\mathcal{A}^\ell$ as $\boldsymbol{\pi}^{k_m}$, so we may write explicitly

$$\Psi(\boldsymbol{\pi}^{k_m+1}) \leq \Psi(\pi'_i, \boldsymbol{\pi}^{k_m}_{-i}) = \sum_{\ell=1}^{M^0} \left(n - \sum_{i \in \mathcal{A}^\ell} \sum_{\tau \in \mathcal{T}_i(\mathcal{A}^\ell)} \omega_{(\pi'_i, \boldsymbol{\pi}^{k_m}_{-i})}(\tau) \right).$$

Letting $m \to \infty$, we conclude

$$\Psi(\boldsymbol{\pi}) \leq \sum_{\ell=1}^{M^0} \left(n - \sum_{i \in \mathcal{A}^\ell} \sum_{\tau \in \mathcal{T}_i(\mathcal{A}^\ell)} \omega_{(\pi'_i, \boldsymbol{\pi}_{-i})}(\tau) \right) = \Psi(\pi'_i, \boldsymbol{\pi}_{-i}), \tag{6.14}$$

where in the last equality we used once again Lemma 6.4, according to which $M(\pi_i', \boldsymbol{\pi}_{-i}) = M^0$ and $\mathcal{A}^{\ell}_{(\pi_i', \boldsymbol{\pi}_{-i})} = \mathcal{A}^{\ell}$. This shows that $\Psi(\boldsymbol{\pi}) \leq \Psi(\boldsymbol{\pi}')$ for all $\boldsymbol{\pi}' \in \Xi(\boldsymbol{\pi})$, completing the proof. ☐

With these preliminaries, we may now prove the existence of prior-free equilibria.

Theorem 6.6. *Every constrained buck-passing game $\Gamma(\mathcal{G}, \mu, \Xi)$ in which Ξ is compact admits a PFNE.*

Proof. Consider $\boldsymbol{\pi} \in \Xi^0$ as in Theorem 6.5. We will show that this $\boldsymbol{\pi}$ is a NE for every initial μ. Suppose by contradiction that there exists a player i and a deviation $\boldsymbol{\pi}' = (\pi_i', \boldsymbol{\pi}_{-i})$ such that $c_i(\boldsymbol{\pi}') < c_i(\boldsymbol{\pi})$. As noted in the proof of Theorem 6.1, player i must belong to some recurrent class $\mathcal{C}^{\ell}_{\boldsymbol{\pi}}$ with $\mu^{\ell}_{\boldsymbol{\pi}} > 0$. Moreover, $\boldsymbol{\pi} \in \Xi^0$ so that Lemma 6.4 implies that $\mathcal{A}^{\ell}_{\boldsymbol{\pi}'} = \mathcal{A}^{\ell}_{\boldsymbol{\pi}}$ and, *a fortiori*, $\mu^{\ell}_{\boldsymbol{\pi}'} = \mu^{\ell}_{\boldsymbol{\pi}}$. Arguing as in Case 1 in the proof of Theorem 6.1, the cost reduction must come from a decrease in the stationary probability $\rho^{\ell}_{\boldsymbol{\pi}'}(i) < \rho^{\ell}_{\boldsymbol{\pi}}(i)$. This is in turn equivalent to $\Omega^{\ell}(\boldsymbol{\pi}') > \Omega^{\ell}(\boldsymbol{\pi})$ and implies $\Psi(\boldsymbol{\pi}') < \Psi(\boldsymbol{\pi})$, which contradicts the choice of $\boldsymbol{\pi}$. ☐

7. Variations on the theme

7.1. **The buck-holding game.** In this section we consider a different game, called *buck-holding game* (BHG) which we denote $\hat{\Gamma}(\mathcal{G}, \mu)$. This game is similar to the buck-passing game described in the previous sections, but now the goal of each player is to maximize the fraction of time in which she has the buck. Hence, the cost in (2.3) becomes a payoff. The definitions of improvement and Nash equilibrium change accordingly.

Definition 7.1. Consider a game with payoffs $(c_i)_{i \in \mathcal{V}}$.

(a) Given a strategy profile $\boldsymbol{s} \in \mathcal{S}$, a *unilateral deviation* for player i is a strategy $\boldsymbol{s}' \in \mathcal{S}$ which differs from $\boldsymbol{s}$ only in its i-th coordinate. It is a *profitable deviation* if in addition $c_i(\boldsymbol{s}') > c_i(\boldsymbol{s})$, in which case the difference $c_i(\boldsymbol{s}') - c_i(\boldsymbol{s})$ is called the *improvement* of player i.

(b) A strategy profile $\boldsymbol{s} \in \mathcal{S}$ is a *Nash equilibrium* (NE) if no player has a profitable deviation. Similarly, it is an *ε-Nash equilibrium* (ε-NE) if no player has a profitable deviation with an improvement larger than ε.

Definition 2.4 still holds, but now the goal is to maximize the potential.

In a *deterministic buck-holding game* (DBHG) $\hat{\Gamma}(\mathcal{G}, \mu, \mathcal{S})$ each player chooses a single out-neighbor. The following definition is needed to analyze these games.

Definition 7.2. A strategy profile is called a *weakly prior-free Nash equilibrium* (WPFNE) if it is an equilibrium for every fully supported initial distribution μ.

Proposition 7.3. *Let μ be a fully supported initial measure on $\mathcal{V}$. Then*

(a) *A deterministic buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \mathcal{S})$ is an ordinal potential game, with the ordinal potential Ψ as in (2.10).*

(b) *Every deterministic buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \mathcal{S})$ admits a WPFNE.*

(c) *In a deterministic buck-holding game every improvement path has length $\mathcal{O}(n^2)$. There exist instances with improvement paths of length $\Theta(n^2)$.*

Proof. Given a profile $\boldsymbol{s} \in \mathcal{S}$, consider a player $i \in \mathcal{V}$ who has a profitable deviation s_i' and let $\boldsymbol{s}' := (s_i', \boldsymbol{s}_{-i})$. Since $c_i(\boldsymbol{s}') > c_i(\boldsymbol{s})$, one of the following two scenarios occurs:

(C_1) Player $i \in \mathcal{C}_{\boldsymbol{s}}^{\ell}$ for some $\ell \leq M(\boldsymbol{s})$ and $s_i' \in \mathcal{A}_{\boldsymbol{s}}^{\ell}$ in such a way that the cycle where i lies becomes shorter. Notice that in this case $\mathcal{A}_{\boldsymbol{s}}^{\ell} = \mathcal{A}_{\boldsymbol{s}'}^{\ell}$, hence $\mu_{\boldsymbol{s}}^{\ell} = \mu_{\boldsymbol{s}'}^{\ell}$ and the new payoff of player i is given by

$$c_i(\boldsymbol{s}') = \frac{|\mathcal{C}_{\boldsymbol{s}}^{\ell}|}{|\mathcal{C}_{\boldsymbol{s}'}^{\ell}|} c_i(\boldsymbol{s}). \tag{7.1}$$

Notice that s_i' can be a vertex in $\mathcal{C}_{\boldsymbol{s}}^{\ell}$ as well as a vertex in $\mathcal{A}_{\boldsymbol{s}}^{\ell} \setminus \mathcal{C}_{\boldsymbol{s}}^{\ell}$.

(C_2) Player i is transient in $\boldsymbol{s}$ and becomes recurrent in $\boldsymbol{s}'$, creating a new class. In particular, assume $i \in \mathcal{A}_{\boldsymbol{s}}^{h}$ for some $h \leq M(\boldsymbol{s})$. We will have

$$\mathcal{C}_{\boldsymbol{s}}^{\ell} = \mathcal{C}_{\boldsymbol{s}'}^{\ell}, \quad \text{for all } \ell \leq M(\boldsymbol{s}) \tag{7.2}$$

and $\boldsymbol{s}'$ has a new cycle $\mathcal{C}_{\boldsymbol{s}'}^{M(\boldsymbol{s}')} \ni i$, with $M(\boldsymbol{s}') = M(\boldsymbol{s}) + 1$. Moreover, we have

$$\mathcal{A}_{\boldsymbol{s}}^{\ell} = \mathcal{A}_{\boldsymbol{s}'}^{\ell}, \quad \text{for all } \ell \leq M(\boldsymbol{s}),\ \ell \neq h, \tag{7.3}$$

and

$$\mathcal{A}_{\boldsymbol{s}'}^{M(\boldsymbol{s}')} \cup \mathcal{A}_{\boldsymbol{s}'}^{h} = \mathcal{A}_{\boldsymbol{s}}^{h}. \tag{7.4}$$

Notice that in this case the deviation is profitable for i because, by the assumption that μ is fully supported, we have $\mu_{\boldsymbol{s}'}^{M(\boldsymbol{s}')} > 0$. It is worth noting that full support of initial distribution μ is needed for the arguments that follows. Our assumption on μ is enough to guarantee that every deviation of type (C_2) is improving for the deviating player, which is what we need to show that Ψ is a ordinal potential function.

The claims of the theorem now follow straightforwardly.

(a) We argue as in Theorem 2.8. Let Ψ be defined as in (2.10). If $c_i(\boldsymbol{s}') > c_i(\boldsymbol{s})$, then $\Psi(\boldsymbol{s}') > \Psi(\boldsymbol{s})$, both under ($C_1$) and ($C_2$). Indeed, if a deviation of type (C_1) takes place, then

$$\Psi(\boldsymbol{s}') - \Psi(\boldsymbol{s}) = |\mathcal{C}_{\boldsymbol{s}}^{\ell}| - |\mathcal{C}_{\boldsymbol{s}'}^{\ell}| > 0. \tag{7.5}$$

On the other hand, if a deviation of type (C_2) takes place, then

$$\Psi(\boldsymbol{s}') - \Psi(\boldsymbol{s}) = n - |\mathcal{C}_{\boldsymbol{s}'}^{h}| + n - |\mathcal{C}_{\boldsymbol{s}'}^{M(\boldsymbol{s}')}| - (n - |\mathcal{C}_{\boldsymbol{s}}^{h}|) > 0. \tag{7.6}$$

This proves that the game is generalized ordinal potential. To prove that it is, indeed, ordinal potential, we have to check that $\Psi(\boldsymbol{s}') > \Psi(\boldsymbol{s})$ implies a deviation of type (C_1) or (C_2). This, again, follows the line of Theorem 2.8. Indeed, if $\boldsymbol{s}' = (s_i', \boldsymbol{s}_{-i})$ and $\Psi(\boldsymbol{s}') > \Psi(\boldsymbol{s})$, then, either $\boldsymbol{s}'$ has one extra cycle, which implies that the deviation $\boldsymbol{s}$ to $\boldsymbol{s}'$ is of type (C_2) and is improving for the deviating player, or the cycle in which i lies in $\boldsymbol{s}$ has been shortened in $\boldsymbol{s}'$, which implies that the deviation is of type (C_1), therefore $c_i(\boldsymbol{s}') > c_i(\boldsymbol{s})$.

(b) To prove the existence of WPFNE, notice that the strategies in which Ψ is maximized are NE for every initial distribution μ which is fully supported. It is worth noting that, since we can claim the potential nature of the game only if μ is

fully supported, weak prior-freeness is the best we can achieve with the techniques developed in the previous sections.
(c) To show that the uniform upper bound of Theorem 2.10 applies also to DBHGs, consider the following: Deviations of type (C_2) can occur at most $\lfloor n/2 \rfloor - 1$ times, since the number of cycles is between 1 and $\lfloor n/2 \rfloor$. On the other hand, each cycle can be shrunk at most $n-2$ times. Hence, we have a quadratic upper bound. On the other hand, a lower bound of the same order of magnitude can be obtained using a complete graph with an even number of vertices, as in the proof of Theorem 2.10. The starting configuration is now a Hamiltonian cycle and the improvement steps are the same of the DBPG, but in reverse order. □

Remark 7.4. We say that $\mathcal{G}$ admits a perfect matching if there exists a partition $\mathcal{V}^1 \dot{\cup} \mathcal{V}^2 = \mathcal{V}$ and a subset $\widetilde{\mathcal{E}} \subset \mathcal{E}$ of cardinality n such that, for each $i \in \mathcal{V}^1$ and $j \in \mathcal{V}^2$, both $(i,j) \in \widetilde{\mathcal{E}}$ and $(j,i) \in \widetilde{\mathcal{E}}$. In a DBPG, if $\mathcal{G}$ admits a Hamiltonian cycle, then the strategy profile $\boldsymbol{s}$ in which players play along such cycle is a NE. In a DBHG, if $\mathcal{G}$ admits a perfect matching, then the strategy profile that realizes this matching is a NE. Indeed, in this case, for any possible deviation, the payoff of the deviating player would drop to zero. More generally, we saw in Section 2 that for a connected graph $\mathcal{G}$, every unicyclic strategy in which the cycle cannot be extended by a unilateral deviation is a PFNE. Similarly, in a DBHG, a subgraph $\boldsymbol{s} \in \mathcal{S}$ in which every player is in a cycle that she cannot unilaterally shorten is a PFNE.

We now show the *holding analogue* of Theorem 6.1. Given a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, an initial distribution μ, and an arbitrary set of strategy profiles Ξ as in Section 6, we consider the constrained buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \Xi)$.

Proposition 7.5. *Every constrained buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \Xi)$ in which μ is fully supported is generalized ordinal potential with generalized ordinal potential function Ψ as in* (2.10).

Proof. Consider a profitable deviation for player i from $\boldsymbol{\pi}$ to $\boldsymbol{\pi}' := (\pi_i', \boldsymbol{\pi}_{-i})$, with $c_i(\boldsymbol{\pi}') > c_i(\boldsymbol{\pi})$. As for the deterministic case, we distinguish two possible scenarios, depending on whether the player i is recurrent or transient before the deviation. Notice that player i cannot improve her payoff by moving some probability mass out of her own transient closure class.

Case 1: i is recurrent both in $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$. This is the stochastic version of (C_1). Although $\mathcal{C}^{\ell}_{\boldsymbol{\pi}}$ may change, we have $\mathcal{A}^{\ell}_{\boldsymbol{\pi}'} = \mathcal{A}^{\ell}_{\boldsymbol{\pi}}$ and $\mu^{\ell}_{\boldsymbol{\pi}'} = \mu^{\ell}_{\boldsymbol{\pi}} > 0$. It then follows from (3.6) that $c_i(\boldsymbol{\pi}') > c_i(\boldsymbol{\pi})$ is equivalent to $\rho^{\ell}_{\boldsymbol{\pi}'}(i) > \rho^{\ell}_{\boldsymbol{\pi}}(i)$. Now, since the weight of any i-rooted tree $\tau \in \mathcal{T}_i(\mathcal{A}^{\ell}_{\boldsymbol{\pi}}) = \mathcal{T}_i(\mathcal{A}^{\ell}_{\boldsymbol{\pi}'})$ does not depend on π_i nor π_i', we get $\Omega^{\ell}_i(\boldsymbol{\pi}) = \Omega^{\ell}_i(\boldsymbol{\pi}')$. Then, from (6.4) it follows that $\rho^{\ell}_{\boldsymbol{\pi}'}(i) > \rho^{\ell}_{\boldsymbol{\pi}}(i)$ is equivalent to $\Omega^{\ell}(\boldsymbol{\pi}') < \Omega^{\ell}(\boldsymbol{\pi})$, which implies $\Psi(\boldsymbol{\pi}') > \Psi(\boldsymbol{\pi})$.

Case 2: i transient in $\boldsymbol{\pi}$ but recurrent in $\boldsymbol{\pi}'$. This is the stochastic version of (C_2). In this case i creates a new recurrent class. Hence, using the notation of

Proposition 7.3, we have

$$\begin{aligned}\Psi(\boldsymbol{\pi}') - \Psi(\boldsymbol{\pi}) &= n - \Omega^h(\boldsymbol{\pi}') + n - \Omega^{M(\boldsymbol{\pi}')}(\boldsymbol{\pi}') - (n - \Omega^h(\boldsymbol{\pi}))\\ &= n + \Omega^h(\boldsymbol{\pi}) - \Omega^{M(\boldsymbol{\pi}')}(\boldsymbol{\pi}') - \Omega^h(\boldsymbol{\pi}')\\ &\geq n + \Omega^h(\boldsymbol{\pi}) - |\mathcal{A}_{\boldsymbol{\pi}'}^{M(\boldsymbol{\pi}')}| - |\mathcal{A}_{\boldsymbol{\pi}'}^h|\\ &= n + \Omega^h(\boldsymbol{\pi}) - |\mathcal{A}_{\boldsymbol{\pi}}^h| > 0,\end{aligned}$$

where the inequality stems from $\Omega^h(\boldsymbol{\pi}) > 0$ and $|\mathcal{A}_{\boldsymbol{\pi}}^h| \leq n$. □

Corollary 7.6. *For every $\varepsilon > 0$, a constrained buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \Xi)$ has the ε-FIP and admits a weakly prior-free ε-NE.*

Unfortunately, mimicking the argument of Theorems 6.5 and 6.6 and Lemma 6.4 is not enough to prove the existence of PFNE for general compact strategy space Ξ. Indeed, the notion of *skeleton* introduced in Section 6.2 does not guarantee that the transient closures are retained in the limit. Nonetheless, the following proposition easily follows by the previous analysis.

Proposition 7.7. *Every constrained buck-holding game $\hat{\Gamma}(\mathcal{G}, \mu, \Xi)$ in which one of the following holds admits a WPFNE:*

(i) *The set Ξ is finite and μ is fully supported.*
(ii) *For every $\boldsymbol{\pi} \in \Xi$ the associated Markov chain is irreducible.*

7.2. **The PageRank game.** The PageRank dynamics was introduced by Brin and Page (1998) as a tool to rank webpages. From a mathematical perspective, PageRank is a Markov chain on the state space $\mathcal{V}$ of web-pages, where two webpages $i, j \in \mathcal{V}$ are connected by a directed edge $(i, j) \in \mathcal{E}$ if there exists a weblink on page i leading to page j. A websurfer visiting a given page i at time t clicks at random on a link $(i, j) \in \mathcal{E}$ and moves to page j at time $t + 1$. Alternatively, with small probability, she chooses one of the billion webpages in $\mathcal{V}$ according to ν. This describes a Markov chain having a unique stationary measure ρ, according to which webpages are then ranked.

We now consider a game-theoretic version of this problem, where players are webmasters whose strategies are the out-links of their webpages and the payoff is the ranking of their pages. This model lies in the realm of BHGs. In particular, the game can be framed as follows: we identify the web pages with the set $\mathcal{V} = \{1, \dots, n\}$ and we consider $\mathcal{G} = K_n$, the complete graph. We fix a dumping factor $\alpha \in (0, 1)$ and a fully supported probability measure ν on $\mathcal{V}$, which can be seen as a column vector of size n. For each player i, the strategy set $\mathcal{B}_i \subset 2^{\mathcal{V} \setminus \{i\}}$, i.e., a strategy b_i of player i is a subset of $\mathcal{V} \setminus \{i\}$ that satisfies some constraints. For instance, a page cannot connect to more than a fixed number of other pages; alternatively, if the page is about some topic it must link to at least another page with a related content, etc.

Given a strategy profile $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathcal{B}$, define the transition matrix Q with entries

$$Q_{ij} := \frac{\mathbb{1}_{j \in b_i}}{|b_i|}. \tag{7.7}$$

According to the transition matrix Q player i chooses uniformly at random one of the players in b_i. We then consider the perturbation given by

$$\boldsymbol{\pi} := (1-\alpha)Q + \alpha \mathbf{1}\nu^\top, \tag{7.8}$$

where $\mathbf{1}$ is a column vector whose components are all 1, and $\nu^\top$ is a row vector. The transition matrix $\boldsymbol{\pi}$ is irreducible; its unique stationary measure is called *PageRank*.

The above game can be framed as a constrained buck-holding game as follows: The set $\mathcal{B}_i$ corresponds to the strategy set Ξ_i of vectors $\boldsymbol{\pi}_i$ such that

$$\pi_{ij} = (1-\alpha)\frac{\mathbb{1}_{j\in b_i}}{|b_i|} + \alpha\nu_j. \tag{7.9}$$

Since every $\boldsymbol{\pi} \in \Xi$ is irreducible, the payoff vector does not depend on the initial measure μ. Therefore, Proposition 7.7(ii) applies and all the equilibria of this game are prior-free.

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References

Aldous, D. and Fill, J. A. (2002). Reversible Markov chains and random walks on graphs. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/$\sim$aldous/RWG/book.html.

Anantharam, V. and Tsoucas, P. (1989). A proof of the Markov chain tree theorem. *Statist. Probab. Lett.*, 8(2):189–192.

Andersen, R., Chung, F., and Lang, K. (2008). Local partitioning for directed graphs using PageRank. *Internet Math.*, 5(1-2):3–22.

Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, É., Wexler, T., and Roughgarden, T. (2008). The price of stability for network design with fair cost allocation. *SIAM J. Comput.*, 38(4):1602–1623.

Avena, L. and Gaudillière, A. (2018). Two applications of random spanning forests. *J. Theoret. Probab.*, 31(4):1975–2004.

Avrachenkov, K. and Litvak, N. (2006). The effect of new links on Google PageRank. *Stoch. Models*, 22(2):319–331.

Bergemann, D. and Morris, S. (2017). Belief-free rationalizability and informational robustness. *Games Econom. Behav.*, 104:744–759.

Bolle, F. (2017). Passing the buck on the acceptance of responsibility. *Research in Economics*, 71(1):86–101.

Borkar, V. S., Ejov, V., and Filar, J. A. (2004). Directed graphs, Hamiltonicity and doubly stochastic matrices. *Random Structures Algorithms*, 25(4):376–395.

Borkar, V. S., Ejov, V., and Filar, J. A. (2009). On the Hamiltonicity gap and doubly stochastic matrices. *Random Structures Algorithms*, 34(4):502–519.

Borkar, V. S., Ejov, V., Filar, J. A., and Nguyen, G. T. (2012). *Hamiltonian Cycle Problem and Markov chains*. Springer, New York.

Brin, S. and Page, L. (1998). The anatomy of a large-scale hypertextual web search engine. In *Seventh International World-Wide Web Conference (WWW 1998)*.

Brooks, R. L., Smith, C. A. B., Stone, A. H., and Tutte, W. T. (1940). The dissection of rectangles into squares. *Duke Math. J.*, 7:312–340.

Candogan, O., Menache, I., Ozdaglar, A., and Parrilo, P. A. (2011). Flows and decompositions of games: harmonic and potential games. *Math. Oper. Res.*, 36(3):474–503.

Caputo, P. and Quattropani, M. (2019). Mixing time of pagerank surfers on sparse random digraphs. arXiv:1905.04993.

Chaiken, S. and Kleitman, D. J. (1978). Matrix tree theorems. *J. Combinatorial Theory Ser. A*, 24(3):377–381.

Chawla, S., Fu, H., and Karlin, A. (2014). Approximate revenue maximization in interdependent value settings. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*, EC '14, pages 277–294, New York, NY, USA. ACM.

Chen, N., Litvak, N., and Olvera-Cravioto, M. (2017). Generalized PageRank on directed configuration networks. *Random Structures Algorithms*, 51(2):237–274.

de Kerchove, C., Ninove, L., and van Dooren, P. (2008). Maximizing PageRank via outlinks. *Linear Algebra Appl.*, 429(5-6):1254–1276.

della Paolera, G., Irigoin, M. A., and Bózzoli, C. G. (2011). Passing the buck: Monetary and fiscal policies. In della Paolera, G. and Taylor, A. M., editors, *A New Economic History of Argentina, Volume 1*. Cambridge University Press.

Eden, A., Feldman, M., Fiat, A., and Goldner, K. (2018). Interdependent values without single-crossing. arXiv 1806.03865.

Ejov, V., Filar, J. A., Murray, W., and Nguyen, G. T. (2008). Determinants and longest cycles of graphs. *SIAM J. Discrete Math.*, 22(3):1215–1225.

Ejov, V., Filar, J. A., and Nguyen, M.-T. (2004). Hamiltonian cycles and singularly perturbed Markov chains. *Math. Oper. Res.*, 29(1):114–131.

Ejov, V., Litvak, N., Nguyen, G. T., and Taylor, P. G. (2011). Proof of the Hamiltonicity-trace conjecture for singularly perturbed Markov chains. *J. Appl. Probab.*, 48(4):901–910.

Ely, J. C., Hörner, J., and Olszewski, W. (2005). Belief-free equilibria in repeated games. *Econometrica*, 73(2):377–415.

Ely, J. C. and Välimäki, J. (2002). A robust folk theorem for the prisoner's dilemma. *J. Econom. Theory*, 102(1):84–105.

Filar, J. A. (2006). Controlled Markov chains, graphs, and Hamiltonicity. *Found. Trends Stoch. Syst.*, 1(2):77–162.

Filar, J. A. and Krass, D. (1994). Hamiltonian cycles and Markov chains. *Math. Oper. Res.*, 19(1):223–237.

Fournier, G. and Scarsini, M. (2019). Locations games on networks: Existence and efficiency of equilibria. *Math. Oper. Res.*, 44(1):212–235.

Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. *Rev. Econom. Stud.*, 77(1):218–244.

Garavaglia, A., van der Hofstad, R., and Litvak, N. (2018). Local weak convergence for pagerank. *arXiv preprint arXiv:1803.06146*.

Gardiner, S. M. (2006). Protecting future generations: intergenerational buck-passing, theoretical ineptitude and a brief for a global core precautionary principle. In Tremmel, J. C., editor, *Handbook of Intergenerational Justice*. Edward Elgar Publishing.

Gleich, D. F. (2015). PageRank beyond the web. *SIAM Rev.*, 57(3):321–363.

Hartline, J. D. and Karlin, A. R. (2007). Profit maximization in mechanism design. In Nisan, N., Roughgarden, T., and Tardos, E., editors, *Algorithmic Game Theory*, pages 331–361. Cambridge Univ. Press, Cambridge.

Heller, Y. (2017). Instability of belief-free equilibria. *J. Econom. Theory*, 168:261–286.

Hood, C. (2002). The risk game and the blame game. *Gov. Opposition*, 37(1):15–37.

Hörner, J. and Lovo, S. (2009). Belief-free equilibria in games with incomplete information. *Econometrica*, 77(2):453–487.

Jeh, G. and Widom, J. (2003). Scaling personalized web wearch. In *Proceedings of the 12th International Conference on World Wide Web*, WWW '03, pages 271–279, New York, NY, USA. ACM.

Kearns, M. J., Littman, M. L., and Singh, S. P. (2001). Graphical models for game theory. In *Proceedings of the Seventeenth Conference on Uncertainty in Artificial Intelligence (UAI2001)*.

Kirchhoff, G. (1847). Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. *Annalen der Physik*, 148(12):497–508.

Koutsoupias, E. and Papadimitriou, C. (2009). Worst-case equilibria. *Computer Science Review*, 3(2):65–69.

Koutsoupias, E. and Papadimitriou, C. H. (1999). Worst-case equilibria. In *STACS '99: Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science.*

Lã, Q. D., Chew, Y., and Soong, B.-H. (2016). *Potential Game Theory Applications in Radio Resource Allocation.* Springer.

Lee, J. and Olvera-Cravioto, M. (2017). Pagerank on inhomogeneous random digraphs. arXiv:1707.02492.

Leighton, F. and Rivest, R. (1986). Estimating a probability using finite memory. *IEEE Transactions on Information Theory*, 32(6):733–742.

León, S., Jurado, I., and Garmendia Madariaga, A. (2018). Passing the buck? responsibility attribution and cognitive bias in multilevel democracies. *West European Politics*, 41(3):660–682.

Levin, D. A. and Peres, Y. (2017). *Markov Chains and Mixing Times.* American Mathematical Society, Providence, RI. Second edition. With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson.

Litvak, N. and Ejov, V. (2009). Markov chains and optimality of the Hamiltonian cycle. *Math. Oper. Res.*, 34(1):71–82.

Mavronicolas, M., Monien, B., Papadopoulou, V. G., and Schoppmann, F. (2008). Voronoi games on cycle graphs. In *Mathematical Foundations of Computer Science 2008*, volume 5162 of *Lecture Notes in Comput. Science*, pages 503–514. Springer, Berlin.

Miyagawa, E., Miyahara, Y., and Sekiguchi, T. (2008). The folk theorem for repeated games with observation costs. *J. Econom. Theory*, 139(1):192–221.

Monderer, D. and Shapley, L. S. (1996). Potential games. *Games Econom. Behav.*, 14(1):124–143.

Norris, J. R. (1998). *Markov Chains*, volume 2 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge. Reprint of 1997 original.

Papadimitriou, C. H. (2001). Algorithms, games, and the Internet. In *STOC '01: Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing.*

Parise, F. and Ozdaglar, A. (2019). A variational inequality framework for network games: existence, uniqueness, convergence and sensitivity analysis. *Games Econom. Behav.*, 114:47–82.

Piccione, M. (2002). The repeated prisoner's dilemma with imperfect private monitoring. *J. Econom. Theory*, 102(1):70–83.

Rawls, J. (2009). *A Theory of Justice: Revised Edition.* Harvard University Press, Cambridge.

Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria. *Internat. J. Game Theory*, 2:65–67.

Roughgarden, T. and Talgam-Cohen, I. (2016). Optimal and robust mechanism design with interdependent values. *ACM Trans. Econ. Comput.*, 4(3):Art. 18, 34.

Scarsini, M. and Tomala, T. (2012). Repeated congestion games with bounded rationality. *Internat. J. Game Theory*, 41(3):651–669.

Schulz, A. S. and Stier Moses, N. (2003). On the performance of user equilibria in traffic networks. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MD, 2003)*, pages 86–87, New York. ACM.

Sonnenberg, A. (2005). Personal view: passing the buck and taking a free ride –a game-theoretical approach to evasive management strategies in gastroenterology. *Alimentary Pharmacology & Therapeutics*, 22(6):513–518.

Steffel, M., Williams, E. F., and Perrmann-Graham, J. (2016). Passing the buck: Delegating choices to others to avoid responsibility and blame. *Organizational Behavior and Human Decision Processes*, 135:32–44.

Vetta, A. (2002). Nash equilibria in competitive societies with applications to facility location, traffic routing and auctions. In *Proceedings of the 43rd Symposium on Foundations of Computer Science*, pages 416–425, Washington, DC, USA. IEEE Computer Society.

Voorneveld, M. and Norde, H. (1997). A characterization of ordinal potential games. *Games Econom. Behav.*, 19(2):235–242.

Yamamoto, Y. (2009). A limit characterization of belief-free equilibrium payoffs in repeated games. *J. Econom. Theory*, 144(2):802–824.

Yamamoto, Y. (2014). Individual learning and cooperation in noisy repeated games. *Rev. Econ. Stud.*, 81(1):473–500.

8. List of symbols

$\mathcal{A}_{\boldsymbol{s}}^{\ell}$ unicycle induced by $\boldsymbol{s}$, defined in Definition 2.1 and Eq. (6.2)
adj adjugate matrix, defined in (5.9)
b_i strategy of player i in the PageRank game
$\boldsymbol{b}$ strategy profile in the PageRank game
$\mathcal{B}_i$ strategy set of player i in the PageRank game
$\mathcal{B}$ set of strategy profiles in the PageRank game
c_i cost function of player i, defined in Eqs. (2.3) and (3.6)
C_n bi-directional cycle
$\mathcal{C}_{\boldsymbol{s}}^{\ell}$ cycle induced by $\boldsymbol{s}$, defined in Definition 2.1
$\mathcal{E}$ set of edges
$\mathcal{E}_{\boldsymbol{s}}$ subset of edges induced by $\boldsymbol{s}$, defined in Definition 2.1
$\mathcal{E}_{\boldsymbol{\pi}}$ set of weighted edges with weights determined by $\boldsymbol{\pi}$
$\mathcal{G}$ directed graph
$\mathcal{G}_{\boldsymbol{s}}$ subgraph induced by $\boldsymbol{s}$, defined in Definition 2.1
$\mathcal{G}_{\boldsymbol{\pi}}$ weighted graph induced by $\boldsymbol{\pi}$
K_n complete graph
$\ell(i)$ label of the unicycle that contains player i
$L_{\boldsymbol{\pi}}$ Laplacian
$\widetilde{m} := n/2 - 1$
$M(\boldsymbol{s})$ number of unicycles under $\boldsymbol{s}$, defined in Definition 2.1
n number of players
$\mathcal{N}_i^+$ out-neighbors of player i
$\mathsf{NE}(\mathcal{S})$ Nash equilibria in DBPG
$\mathsf{NE}(\Sigma)$ Nash equilibria in SBPG
$\mathsf{NE}(\Xi)$ Nash equilibria in CBPG
$\mathbb{P}_{\boldsymbol{\pi}}$ probability measure induced by $\boldsymbol{\pi}$, defined in (3.1)
$\mathsf{P}_{\boldsymbol{\pi}}^{\ell}$ probability of absorption in $\mathcal{C}_{\boldsymbol{\pi}}^{\ell}$
$\mathsf{P}_{\boldsymbol{\pi}}^{j\to\ell}$ probability of absorption in $\mathcal{C}_{\boldsymbol{\pi}}^{\ell}$ starting from j, defined in (3.4)
PoA price of anarchy, defined in Eqs. (4.2) and (4.4)
PoS price of stability, defined in Eqs. (4.3) and (4.5)
$\mathbb{Q}_{\boldsymbol{\pi}}$ restriction of $\omega_{\boldsymbol{\pi}}$ to $\mathcal{S}$, defined in (5.2)
Q transition matrix in the PageRank game, defined in (7.7)
R_n number of rooted trees on n vertices
$\mathcal{R}_{\boldsymbol{\pi}}$ residual transient class, defined in (6.3)
s_i strategy of player i in DBPG
$\boldsymbol{s}$ strategy profile in DBPG
$\mathcal{S}_i$ strategy set of player i in DBPG
$\mathcal{S}$ set of strategy profiles in DBPG
SC social cost function, defined in (4.1)
t time
T_i hitting time of i
$\mathcal{T}(i)$ set of i-rooted spanning trees
$\mathcal{U}$ set of all $\boldsymbol{s} \in \mathcal{S}$ inducing a single spanning unicycle
$\mathcal{U}_i$ spanning unicycles that have i in the cycle

$\mathcal{U}_{ij}$ spanning unicycles in which the edge (i, j) is part of the cycle
$\mathcal{V}$ set of vertices
$\mathcal{V}_{\boldsymbol{\pi}}^0$ set of transient vertices
W_n wheel graph
$x_i = \mathbb{Q}_{\boldsymbol{\pi}}(\mathcal{U}_i)$, defined in (5.6)
X_t Markov chain
α dumping factor in the PageRank game
Γ game
$\delta_i(\boldsymbol{s})$ indicator of $i \in \mathcal{C}_{\boldsymbol{s}}^{\ell(i)}$, defined in (2.5)
$\Theta_{i,t}(\boldsymbol{s})$ indicator of the event that player i has the buck at time t under profile $\boldsymbol{s}$, defined in (2.1)
$\lambda_{\boldsymbol{\pi}}^{(k)}$ eigenvalue of the Laplacian
$\Lambda(\boldsymbol{s})$ length of the cycle if $\boldsymbol{s} \in \mathcal{S}$ is a spanning unicycle, and 0 otherwise, defined in (5.8)
μ initial measure
$\mu_{\boldsymbol{s}}^{\ell}$ probability that the buck is assigned initially to a vertex in $\mathcal{A}_{\boldsymbol{s}}^{\ell}$, defined in (2.4)
$\mu_{\boldsymbol{s}}^{\ell}$ probability that the buck is absorbed in $\mathcal{C}_{\boldsymbol{s}}^{\ell}$, defined in (3.5)
ν fully supported probability measure in the PageRank game
Ξ_i strategy set of player i in CBPG
Ξ set of strategy profiles in CBPG
$\boldsymbol{\pi}_i$ strategy of player i in SBPG
$\boldsymbol{\pi}$ strategy profile in SBPG
$\boldsymbol{\pi}$ transition matrix
$\rho_{\boldsymbol{\pi}}$ stationary measure induced by $\boldsymbol{\pi}$
$\rho_{\boldsymbol{\pi}}^{\ell}$ stationary measure induced by $\boldsymbol{\pi}$ on the class $\mathcal{C}_{\boldsymbol{\pi}}^{\ell}$
$\rho_{\boldsymbol{\pi}}^{\ell}$ stationary measure on $\mathcal{C}_{\boldsymbol{\pi}}^{\ell}$
Σ set of strategy profiles in SBPG
Σ_i strategy set of player i in SBPG
τ tree
$\widehat{\tau}_{\boldsymbol{\pi}}^{\ell}$ skeleton of $\mathcal{A}_{\boldsymbol{\pi}}^{\ell}$, defined in (6.8)
Ψ potential function, defined in Definition 2.4 and Eq. (2.10)
$\omega_{\boldsymbol{\pi}}$ weight function, defined in (5.1)
$\Omega_i(\boldsymbol{\pi})$ tree-volume of vertex i, defined in (5.3)
$\Omega_{\mathcal{V}}(\boldsymbol{\pi})$ tree-volume of the Markov chain, defined in (5.3)
$|A|$ cardinality of set A
$\mathbb{1}_A$ indicator of set A