# Do return series have power-law tails with the same index?

Thomas Mikosch, Casper de Vries, Xie Xiaolei

University of Copenhagen

*xie@math.ku.dk*

June 29, 2017

# Questions that we want to answer

**We know**

If assumed stationary, equity return series are often seen to be heavy-tailed – a stylized fact of econometrics.

**But**

- Given the wide confidence bands of estimated tail indices, are they actually the same?
- Are tail parameters of different equity return series in the same market related to each other?
- How are investors' preferences over an equity affected by tail parameters?

# Motivation

- Curiosity
- Many multivariate GARCH processes have a stationary distribution with power-law tails:
  1. CCC-GARCH of Bollerslev [1] and Jeantheau [2].
  2. Orthogonal GARCH of Alexander and Chibumba [3],
  3. GO-GARCH by van der Weide [4] which generalizes Orthogonal GARCH.
  4. Full Factor GARCH of Vrontos et al. [5]
  5. Generalized Orthogonal Factor GARCH of Lanne and Saikkonen [6]

# CCC-GARCH, Kesten's theorem & power-law tail

A 2D CCC-GARCH model reads

$$
\begin{aligned}
\begin{pmatrix} \sigma_{1,t}^2 \\ \sigma_{2,t}^2 \end{pmatrix} &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} X_{1,t-1}^2 \\ X_{2,t-1}^2 \end{pmatrix} + \begin{pmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix} \begin{pmatrix} \sigma_{1,t-1}^2 \\ \sigma_{2,t-1}^2 \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \alpha_{1,1} Z_{1,t-1}^2 + \beta_{1,1} & \alpha_{1,2} Z_{2,t-1}^2 + \beta_{1,2} \\ \alpha_{2,1} Z_{1,t-1}^2 + \beta_{2,1} & \alpha_{2,2} Z_{2,t-1}^2 + \beta_{2,2} \end{pmatrix}}_{A_t} \underbrace{\begin{pmatrix} \sigma_{1,t-1}^2 \\ \sigma_{2,t-1}^2 \end{pmatrix}}_{Y_{t-1}} + \underbrace{\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}}_{B}
\end{aligned}
$$

where $X_{1,t}, X_{2,t}$ are return series, whose conditional variances are $\sigma_{1,t}^2, \sigma_{2,t}^2$. When $\forall i, j, \alpha_{i,j} > 0, \beta_{i,j} > 0$ and certain conditions are satisfied, Kesten's theory [7] gives

$$\lim_{u \to \infty} u^{\alpha} \mathbb{P}(\langle \mathbf{v}, Y \rangle > u) = e_{\alpha}(\mathbf{v})$$

where $e_{\alpha}(\cdot) : \mathbb{S}^1 \to \mathbb{R}_+$. **Each component of $Y_t$ shares the same tail index $\alpha$.**

# Heavy-tailedness of equity returns

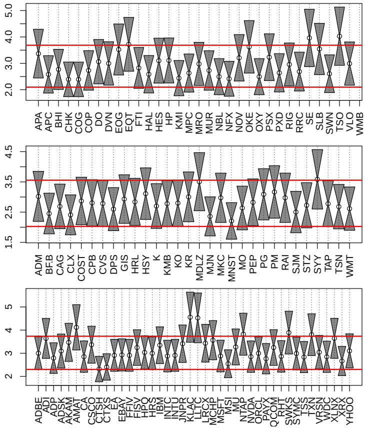



Figure: Hill estimates daily return series in sectors of the S&P 500 index. The data span from 1 January 2010 to 31 December 2014 and comprise $n = 1304$ observations. The graphs from top to bottom correspond to the "Energy", "Consumer Staples" and "Information Technology" sectors.

## Interesting observations

- Hill estimates of equity tail indices are often found between $2.5 \sim 4.5$
- The estimates are contained within each other's confidence bands
- Different sectors have different levels of variability in the tail indices.

# Test for equal tail index: using Hill estimator

- Two return series $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$, with lower-tail indices $\alpha_X$ and $\alpha_Y$.
- We want to test the null hypothesis $H_0 : \alpha_X = \alpha_Y$.

Under $H_0$, we have

$$\sqrt{k}[\hat{\alpha}_X(k) - \hat{\alpha}_Y(k)] \xrightarrow{d} N(0, \alpha_X^2 + \alpha_Y^2)$$

$\hat{\alpha}_X(k)$: Hill estimate of $\alpha_X$ using $k$ upper order statistics.

## Test for equal tail index: testing for a changed extreme quantile

Given a series $X_1, X_2, \ldots, X_n$ where $X_i \sim F_i$, Hoga [8] proposed a test for the hypothesis $H_0$: $1 < \exists k < n$ such that

- $F_i^{-1}(1-p) = F_j^{-1}(1-p)$, for all $1 \leq i < j < k$
- $F_{k-1}^{-1}(1-p) \neq F_k^{-1}(1-p)$ and $F_i^{-1}(1-p) = F_k^{-1}(1-p)$ for all $i \geq k$.

where $p = p_n \to 0$ as $n \to \infty$. The test statistic is

$$T_n = \sup_{s\in[t_0,1-t_0]} \frac{\left[s(1-s)\log\left(\hat{x}_p(0,s)/\hat{x}_p(s,1)\right)\right]^2}{\int_{t_0}^{s}\left[r\log\left(\hat{x}_p(0,r)/\hat{x}_p(0,s)\right)\right]^2 dr + \int_{s}^{1-t_0}\left[(1-r)\log\left(\hat{x}_p(r,1)/\hat{x}_p(s,1)\right)\right]^2 dr}$$

where

$$\hat{x}_p(s,t) = X_{k,s,t}\left(\frac{np}{k}\right)^{-1/\hat{\alpha}}$$

- $X_{k,s,t}$ is the $k$-th largest value among $X_{\lfloor ns\rfloor+1}, \ldots, X_{\lfloor nt\rfloor}$.
- The **asymptotic distribution of $T_n$ under $H_0$ is not available** explicitly, but given as the distribution of a function of a standard Brownian motion – Computed by simulation.

# Results of the tests on S&P 500 sectors

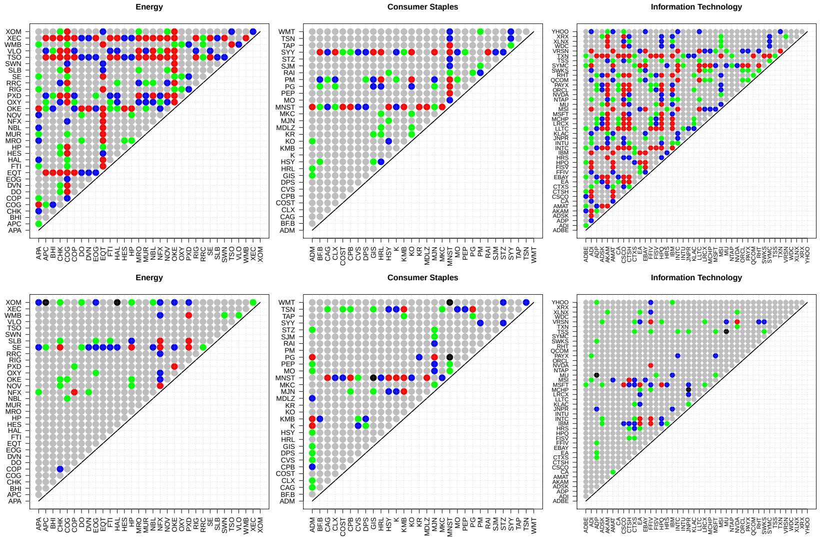



Figure: *Top row*: Hill-based test. The green, blue and red points correspond to pairs of stock in a sector when the test statistic is outside the intervals $[q_{0.075}, q_{0.925}]$, $[q_{0.05}, q_{0.95}]$, $[q_{0.025}, q_{0.975}]$ *Bottom row:* Hoga's test. The green, blue and red points correspond to pairs of stock in a sector when the test statistic $T_n$ exceeds the 85%-, 90%-, 95%-quantile of the limit distribution . The same number (50) of upper order statistics is used for both tests.

# The scale parameter

Assume the equity return $X_t$ is stationary and follows Pareto distribution when $X < -K$:

$$\mathbb{P}(X_t < -x) = \frac{K^\alpha}{x^\alpha}, \quad x > K$$

Hill's [9] maximum likelihood estimator for $K$ is

$$\hat{K}_k = \left(\frac{k}{n}\right)^{1/\hat{\alpha}_k} X_{(k)}$$

where

- $n$ is the sample size
- $\hat{\alpha}_k$ is Hill's estimator of the tail index of $k$ upper order statistics.
- $X_{(k)}$ is the $k$-th upper order statistic in the sample

By asymptotic normality of upper order statistics,

$$\sqrt{k}(\hat{K}_k - K) \xrightarrow{d} N(0, K^2/\alpha^2)$$

# Hill estimates of the scale parameter for S&P 500 sectors

S&P 500: Standard & Poor's stock index. A weighted average of prices of 500 or so stocks listed on NYSE and NASDAQ. Stocks are grouped into 10 sectors. The **Energy, Consumer Staples, IT** sectors are shown here.

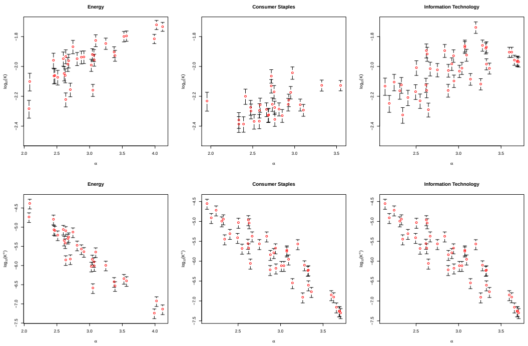



Figure: Estimates of $\hat{K}_k$ (top) and $\hat{K}_k^{\hat{\alpha}}$ (bottom) on $\log_{10}$-scale. The bars are the asymptotic 95%-confidence intervals.

- Estimated tail indices ($\hat{\alpha}$) and scale parameters $\hat{K}$ are positively dependent.
- The positive dependence is stronger for energy and IT stocks than for consumer staple's stocks.
- The scale $\hat{K}^{\hat{\alpha}}$ varies much more across different equities than does the tail index $\alpha$.

# Model of the market & equity returns

**The toy market consists of**

1. A riskless bond that pays $e^r$€ annually for each € invested, $r$ is fixed.
2. A risky equity that pays $e^X$€ annually for each € invested. $X$ has Pareto tails

$$F_X(x) = \begin{cases} p\left(\frac{K}{K-x}\right)^{\alpha} & x \leq 0\,, \\ 1-(1-p)\left(\frac{K'}{K'+x}\right)^{\beta} & x > 0\,, \end{cases}$$

# Model of the investor

1. He has 1 unit of currency for investment
2. His happiness is proportional to a utility function $u(C)$:

$$\begin{aligned} C &= (1-\phi)e^r + \phi e^X \\ u(C) &= -C^{-\xi}/\xi, \quad \xi > 0 \end{aligned}$$

   where
   - $C$ is his monetary amount of consumption
   - $\phi$ is the portion of his asset allocated to the equity
3. His preference over the equity is given by *Generalized Disappointment Aversion*:

$$\tilde{u}(F_X, \phi) = \mathbb{E}u(C) - b\mathbb{E}[u(v) - u(C); C < v]$$

   where
   - $v$ is the level of consumption below which the investor will be disappointed
   - $b$ captures how disappointed the investor will be in case his consumption falls below $v$.

# Analytic results: when $\alpha$ and $\beta$ are independent

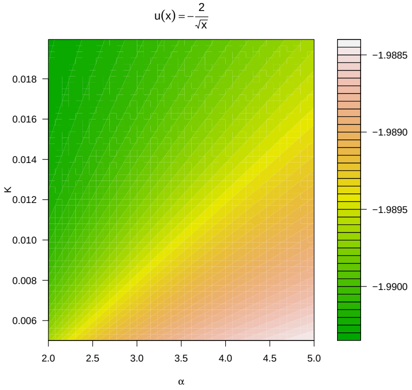
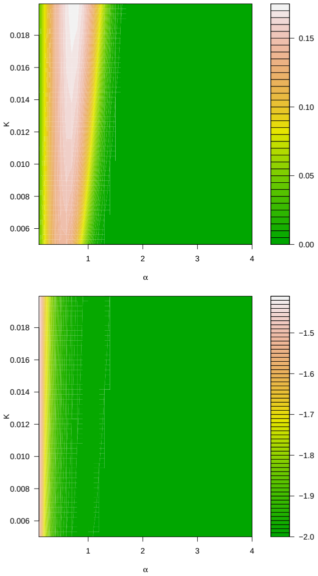



Figure: $\tilde{u}_{\max}$ as a function of $\alpha$ and $K$ in the two-sided Pareto model with $K' = 0.012$, $\beta = 1.4$. $b = 0.01$ in all cases.

- The investor's preference $\tilde{u}$ increases with $\alpha$ and decreases with $K$.
- Moving along a curve of equal preference, if $\alpha$ increases then $K$ also increases.
- At market equilibrium, all actively traded stocks should have nearly the same investor preference – $\alpha$ and $K$ values are expected to have positive dependence. – **consistent with empirical results shown in figure 3**.

# Analytic results: when $\alpha = \beta$




## What's on the left?

- Top: $\hat{\phi}(\alpha, K) = \text{argmax}_{\phi}\, \tilde{u}(\alpha, K, \phi)$: optimal portion of allocation to the equity.
- Bottom: $\tilde{u}_{\max}(\alpha, K) = \max_{\phi} \tilde{u}(\alpha, K, \phi)$: Investor preference with optimal equity allocation.
- utility function $u(x) = -\frac{2}{\sqrt{x}}$

## What I see from the plots

- $\hat{\phi}$ is not monotone w.r.t. $\alpha$ or $K$.
- For a fixed $K$, $\hat{\phi}$ is decreasing with $\alpha$ when $\alpha$ is in the range 2.5 $\sim$ 4.5, typical for real equity return series.
- $\tilde{u}_{\max}(\alpha, K)$ decreases with $\alpha$ but is rather insensitive to $K$.

# When equity returns have t-distribution & $b = 0$

The GDA preference reduces to *expected utility*. A few cases arise depending on

$$a = \frac{(1-\phi)e^r}{\phi}, \quad y_\pm = \frac{a^2 - \xi \pm \sqrt{(a^2-1)(a^2-\xi^2)}}{a(\xi - 1)}$$

1. If $\max\{a, 1\} < \xi$, $\tilde{u}_{\max}$ is monotone increasing with $\alpha$.
2. If $a < \xi < 1$ and $(a + y_-)/(ay_- + 1) < y_-^{(1-\xi)/(1+\xi)}$, $\tilde{u}_{\max}$ is monotone increasing with $\alpha$.
3. If $\xi < a < 1$, $\tilde{u}_{\max}$ is monotone decreasing with $\alpha$.
4. If $1 < \xi < a$ and $(a + y_+)/(ay_+ + 1) > y_+^{(1-\xi)/(1+\xi)}$, $\tilde{u}_{\max}$ is monotone decreasing with $\alpha$.
5. In other case, $\tilde{u}_{\max}$ is not monotone w.r.t. $\alpha$.

# When equity returns have t-distribution & $b > 0$

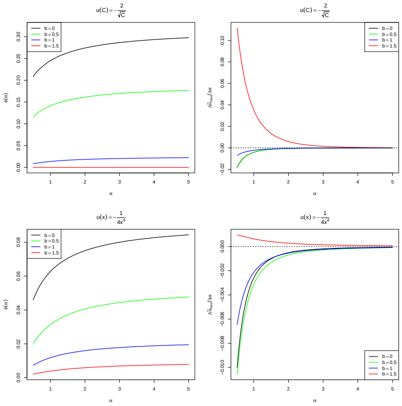



Figure: $\hat{\phi}$ (left) and $\frac{\partial \tilde{u}_{\max}}{\partial \alpha}$ (right).
*top:* $\xi = 1/2$. *bottom:* $\xi = 4$.

**The figure says:**

- $\hat{\phi}$ is monotone increasing for all 4 values of $b$
- $\tilde{u}_{\max}(\alpha)$ is increasing with $\alpha$ when $b$ is relatively large, but decreasing with $\alpha$ when $b$ is small
- A sizable value of $b$ indicates a conservative, risk-averse investor.

# Summary

- The tail index varies to different extent in different sectors/markets.
- The scale parameter is positively dependent on the tail index.
- The scale $K^{\alpha}$ varies much more across different equities than does the tail index $\alpha$.
- An investor's preference over $\alpha$ is dependent on the riskless rate of return, his own utility function and how disappointed he will be if his investment falls below expectation.

# Thank you!

Questions?

T. Bollerslev.
Modelling the coherence in short-run nominal exchange rates: a multivariate generalized arch model.
*Review of Economics and Statistics*, pages 498–505, 1990.

T. Jeantheau.
Strong consistency of estimators for multivariate arch models.
*Econometric theory*, pages 70–86, 1998.

C. Alexander and A. Chibumba.
Multivariate orthogonal factor GARCH.
Technical report, University of Sussex, 1996.
Discussion Papers in Mathematics.

R. van der Weide.
GO-GARCH: a multivariate generalized orthogonal garch model.
*Journal of Applied Econometrics*, pages 549–564, 2002.

I. Vrontos, P. Dellaportas, and D. Politis.
A full-factor multivariate GARCH model.

*Econometrics Journal*, pages 312–334, 2003.

M. Lanne and S. Pentti.
Modelling conditional skewness in stock returns.
*European Journal of Finance*, pages 691–704, 2007.

H. Kesten.
Random difference equations and renewal theory for products of random matrices.
*Acta Mathematica*, pages 207–248, 1973.

Y. Hoga.
Testing for changes in (extreme) VaR.
*Econometrics Journal*, pages 1 – 30, 2017.

B. Hill et al.
A simple general approach to inference about the tail of a distribution.
*Annals of Statistics*, pages 1163–1174, 1975.