Washington & Jefferson College
Math 320

Hunting Robbers on Chess Boards

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Abstract

Cops and robber graphs put a fun and game-like spin on graph theory while still providing a real life application. The idea is to have two players, one playing the cops and the other the robber. The goal for the cops is to catch the robber and the goal for the robber is to avoid capture. We present some theorems, lemmas, and examples in the perusal of this topic.

1 A Brief Introduction to Graph Theory

A graph is made up of a set vertices that are connected by edges. Vertices are commonly represented by a dot and the edges are represented by lines connecting one vertex to another. The graph can be manipulated in any way imaginable so long as no vertices or edges are added or removed. The set of all vertices in a graph is denoted by $V(G)$, where G is the name of the graph. The set of all edges in a graph is denoted by $E(G)$.

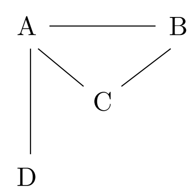

Figure 1: G

Definition 1. Two vertices are *adjacent*, or *neighbors*, if there is an edge connecting them.
For example, in Figure 1 above A is adjacent to B, C, and D, however D is not adjacent to B or C. The set of B's neighbors, or the neighborhood of B is $nbhd(B) = \{A, C\}$.

Definition 2. A *path* is made up of vertices and edges adjacent to each other. A path does not have any repeating vertices or edges and is from one vertex to another, denoted $P(A, B)$.
For example, in Figure 1, ACB is a path from A to B. AB is also a path from A to B. $CABC$, however is not a path because the vertex C is repeated.
The notation $V(P)$ describes all of the vertices in some path $P(x, y)$, where x and y are any vertices. Let us define the path $P(d, b) = \{D, A, B\}$ (though there are other options this is how we have arbitrarily chosen to define $P(d, b)$. We can see from this that A is an element of $P(d, b)$ and so on for any other elements of $P(d, b)$.

Definition 3. The *length of a path* is determined by how many edges lie between the beginning and ending vertex.
For example, the length of AB is 1. The length of BAD is 2.
One important reoccurring idea in cops and robber graphs is the idea of a shortest path. This is the path from one vertex to another with the shortest length possible. The length of the shortest path between two arbitrary vertices, x and y, is denoted by $d(x, y)$. For example the shortest path from B to D, is not $BCAD$, because paths of length 2 exist, such as BAD.

Definition 4. The *degree of a vertex* is the number of elements in its neighborhood. For example, in Figure 1, the degree of D is 1. The degree of C is 2. The degree of A is 3.

Definition 5. The *minimum degree of a graph* is the minimum degree of its vertices. We will be denoting the minimum degree of a graph as $\delta(G)$, where G is the name of the graph. For example, in Figure 1 $\delta(G) = 1$.

Definition 6. The *maximum degree of a graphs* is the maximum degree of its vertices. This is denoted by $\Delta(G)$, were G is the name of a graph. For example, in Figure 1 $\Delta(G) = 3$.

Definition 7. The *domination number* of a graph, the minimum size of a dominating set is the minimum number of vertices required in a set such that every vertex on the graph is either in the set or adjacent to a vertex in the set. We will denote the domination number by $\gamma(G)$, where G is the name of the graph. In Figure 1, $\gamma(G) = 1$ because A can be a dominating set on its own since it is adjacent to every other vertex.

2 Defining Cops and Robber Graphs

For our game of Cops and Robber, it is not restricted to a specific graph. In any game, there is one robber and k cops, where k is a natural number. At the beginning of the game, the cop(s) select which vertex/vertices on which they would like to start. Then the robber chooses a vertex on which to start. Once this has been done, the first round begins and the cops have the option to move to an adjacent vertex or stay put. Then the robber has the option to move or stay put. This concludes round one, and the process continues until the cops or the robber wins. The cops win when a cop is on the same vertex as the robber, and the robber wins when it has a strategy to avoid the cops indefinitely. In our paper, we also explore the idea of lazy cops in the graph. The game plays the same way with lazy cops, however only one cop can move per turn.

Definition 8. A *cycle* is a graph where there exists a path from a vertex back to itself. Figure 2 below is an example of a 6-cycle, cycle with 6 vertices.

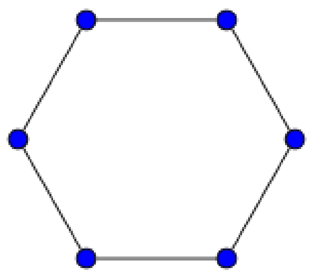

Figure 2:

Definition 9. The *cop number* of a graph is the minimum number of cops necessary to capture the robber on a specific graph. We will be denoting the cop number of a graph as $c(G)$, where G represents the graph name. The lazy cop number of a graph will be denoted as $c_L(G)$.

Proposition 1. *The cop number for a cycle is 2.*

Proof. If we place one cop on the graph for any given cycle, the robber always has a strategy to escape the cop because there will always be a safe vertex to which the robber can move, since the degree of each vertex is 2. The cop can chase the robber around the graph forever, but will never be able to catch the robber. If we place two cops on the graph for any given cycle, the cops have a winning strategy. If the cops continuously move in the opposite direction of each other, eventually they will have the robber pinched between them, and the robber will have no safe vertex for escape. $\square$

Definition 10. A *tree* is an undirected graph on which there exists exactly one path between any two vertices.

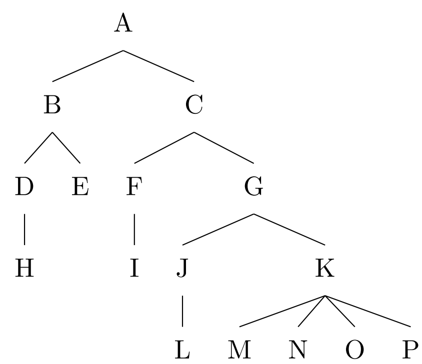

Figure 3: Tree

Proposition 2. *The cop number for any finite tree is 1.*

We encourage the reader to try a game or two before continuing onto the proof of the proposition.

Proof. Let there be only one cop, c, and let the robber be r. Let c be on some vertex higher up in the graph that r. We note that this is possible because c can chase r until this condition is true. As c moves down the branch towards r, the set of safe vertices decreases by at least 1 for every subbranch or ancestor now cut off from r. We use the term "cut off" to mean that there exists no path from r to some point not through c.
Let the set of accessible vertices (the vertices in $nbhd(r)$ that are not in $nbhd(c)$) be denoted by R.
After the cop moves the set of safe vertices, R', is $R' = R - s$, where s is the set of vertices that were cut off by the cop's movement. Thus, we can see that R' is a smaller subset of R. This process repeats as the cop moves strategically. Since the graph is finite, we can see that eventually $R' = 0$ and the robber will have been caught with only 1 cop. Thus, $c(G) = 1$. □

3 Some Lemmas and Theorems

In this section, we will explore simple proofs, lemmas, and examples pertaining to the cop number of a graph.
Lazy Cop Inequality. *For any arbitrary graph, G,* $c(G) \leq c_L(G) \leq \delta(G)$*.* [1, page 325]

Proof. Since the movement of lazy cops is more limited than regular cops, we need at least as many lazy cops as we would regular cops to have a winning strategy for the cops. Thus, $c(G) \leq c_L(G)$. If you can position the cops so they guard every vertex, a victory is guaranteed in one move even if they are lazy. Therefore, $c_L(G) \leq \gamma(G)$. □

Lemma 3. *Let G be any graph where* $u, v \in V(G), u \neq v$ *and* $P(u, v) = \{u, v_1, ..., v_i = v\}$ *a shortest path between u and v. Assuming there are at least 2 cops in the game. Then a single cop c on* $P(u, v)$ *can, after a finite number of moves, prevent r from entering P. That is r will be immediately caught if he moves onto* $P(u, v)$*. [2, Lemma 4]*

Proof. Let us begin by defining a term: the length of the shortest path between x and y is denoted by $d(x, y)$, and d satisfies the triangle inequality.
Let $P(u, v) = \{u = 0, 1, 2, ..., t = v\}$. Suppose after c's move the cop of on vertex c is in the set of vertices in the path $P(u, v)$ and the robber is on r, where r is some vertex on the graph. Let us assume the following:

$$d(r, z) \geq d(c, z) \forall z \in V(P). \tag{1}$$

That is, the distance of the smallest path between the robber and some arbtrirary point in the path $P(u, v)$ is greater than or equal to the distance of the smallest path between the cop and that same arbitrary point; we also assume that this is true for every vertex in the path $P(u, v)$
claim: No matter what the robber does, the cop, by moving in the appropriate direction on P can preserve condition (1). (Essentially the robber should be caught should he enter P.)

Scenario 1: If the robber stays put then so does the cop (we assume that there is at least on other cop somewhere who now makes a move).
Scenario 2: Suppose the robber goes from r to s, where s is any vertex adjacent to r, not necessarily on the path $P(u, v)$. Consider that the distance of the shortest path between s and z and the distance between r and z can only be one off because the robber can only move one vertex at a time, either closer of further from z. Thus after the robber's move the robber either moves one vertex away from z or it moves one vertex towards z. Thus we see that

$$d(s, z) = d(r, z) \pm 1,$$

which leads to the inequality

$$d(s, z) \geq d(r, z) - 1.$$

Let us also consider the inequality

$$d(r, z) \geq d(c, z)$$

if we subtract one from each side we see that

$$d(r, z) - 1 \geq d(c, z) - 1.$$

If we string these two inequalities together we can see that the following equation holds true for any vertex z in the path $P(u, v)$

$$d(s, z) \geq d(r, z) - 1 \geq d(c, z) - 1.$$

If some vertex z_0 is in the set of vertices $V(P)$ exists such that $d(s, z_0) = d(c, z_0) - 1$, then c, by moving toward z_0 also reduces the distance by 1, and (1) still holds.
Hence for the robber to really be threatened, there must be vertices x, y in the set of vertices of the path $P(u, v)$ such that x comes earlier in the path than c and c comes earlier in the path than y this creates a situation such that

$$d(s, x) = d(c, x) - 1 \text{ and } d(s, y) \leq d(c, y)$$
$$\text{or}$$
$$d(s, x) \leq d(c, x) \text{ and } d(s, y) = d(c, y) - 1$$

However this is impossible since

$$\begin{aligned} d(x, y) &\leq d(s, x) + d(s, y) \\ &\leq d(c, x) + d(c, y) - 1 \\ &= d(x, y) - 1 \end{aligned}$$

by the triangle inequality and minimality of P.
It remains to be shown after a finite number of moves, the cop c can force condition (1).
First c moves onto some q in $V(P)$. By the same argument as before, the inequality $d(r, z) \leq d(q, z)$ can hold for z's on P on a specific side of q. By moving in the direction of z, (1) is clearly eventually. [2, page 8] □

Definition 11. A graph is planar if it can be drawn in 1 plane (i.e., on paper) without any intersecting edges.

If we look at graph G, we see that no matter how you rearrange the vertices, there is no way to make it so that the vertices do not intersect. Thus G is not planar.

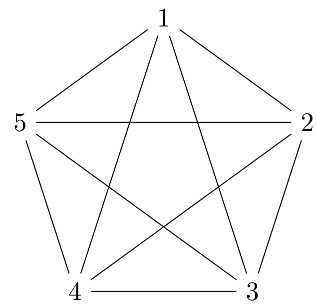

Figure 4: G

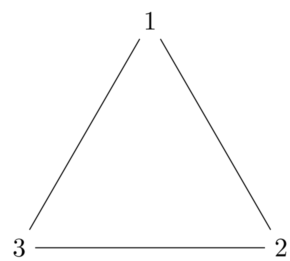

Figure 5: H

If we look at graph H, we see that it can be drawn in 1 plane such that there are no intersecting edges. Thus Hc is planar.

Theorem 4. *If G is any planar graph then $c(G) \leq 3$. [2, Theorem 6]*

Proof. At the start c_1, c_2, and c_3, the three cops, occupy some arbitrary vertex e_0. R makes his move to some arbitrary vertex r_0. The robber territory (where R can move safely in 1 move) is denoted $R_i = G(V) - e_0$ (which contains r_0), that is the set of safe vertices for the robber is all vertices in the graph, but those currently occupied by the cops.
Suppose inductively at the stage i after R moves, one of two situations arise:
Situation 1: Some C is on some vertex u and R is on some vertex r. Looking ot the robber territory we see that $R_i = G - u$ (same as opening). Suppose $N(u) = \{v\}$ then C moves to v, that is the cop moves to its only neighbor. If $r = v$ then the cops win the game.
Otherwise $R_{i(new)} = R_{i(old)} - v = G - u - v$ and $R_{i+1} \subsetneqq R_i$ and the cops can win by step by step diminishing the number of save vertices in the robber territory until the robber is caught.
Suppose $N(u) = \{a, b, ...\}$. Let $P(a, b)$ be the shortest path between a and b. By Lemma 1, one cop can control the path. Thus $V(P)$ are no longer safe and we have $R_{i+1} \subseteq R_i - V(P) \subsetneqq R_i$ and the cops can win.
Situation 2: $P_1(u, v)$ and $P_2(u, v)$ are two u, v paths of length ≥ 1, disjoint except for u and v. By the planarity of G, $P_1(u, v)$ and $P_2(u, v)$ partitions G into $P_1(u, v) \cup P_2(u, v)$ interior and exterior regions. Without loss of generality, $P(u, v)$ occupies some vertex r in the exterior region E.
$P_1(u, v)$ is the shortest u, v path in $P_1(u, v) \cup P_2(u, v) \cup E$, and $P_2(u, v)$ is the shortest u, u path in $P_1(u, v) \cup P_2(u, v) \cup E$ among all such paths which are disjoint from $P_1(u, v)$.
Cop c_1 on $c_1 \in V(P_1)$ controls $P_1(u, v)$ by Lemma 1; similarly c_2 on $c_2 \in V(P_2)$ controls $P_2(u, v)$. At this point $R_1 = E$, and two possibilities arise from this:
Possibility A: There is no path $R_i \cup P_1 \cup P_2$ from u to v other than P_1 and P_2. R_i then consists of the disjoint components of A, B, C. The last free cop can move to (WLOG) a while C_1 and C_2 maintain control of P_1 and P_2. This way there is a winning strategy for the cop.
Possibility B: There are further paths u, v in $R_i \cup P_1(u, v) \cup P_2(u, v)$. Let Z be the shortest such path. Let $x, y \in V(P)$ and $P(x, y)$ is a subpath from x to y. Let w be the first vertex on Z after u and $w \in V(P_1 \cup P_2)$. If $w \in V(P_1)$ (without loss of generality) then $P_3 = Z(u, w) \cup P_1(w, u)$. P_3 is also a shortest path, which is

disjoint from $P_2(u,v)$. c_3 can then move to P_3. The paths and cops,$P_2(u,v)$, P_3, c_2, c_3, then give rise to the situation $R_{i+1} \subsetneqq R_i$, and the we can see inductively that the cops can win. [2, page 8] $\square$

Theorem 5. *Let G be a graph with minimum degree, $\delta(G) \geq n$, which contains no 3 or 4 cycles. Then $c(G) \geq n$.[2, Theorem 3]*

Proof. Let there be $n-1$ cops.
Let $v_1, \dots, v_{n-1}$ be any $n-1$ vertices of G.
Let $W \notin \{v_j, \dots, v_{n-1}\}$.
Consider the neighborhood of W.
$N(W) = \{v_1, \dots, v_k, w_l, w_{l-k}\}$
and $W_i \notin \{v_j, \dots, v_{n-1}\}$
Then $l \geq n, k \leq n+1 thus l - k \geq 1$
Since there are no 3 or 4 cycles
$N(W_i) \cap N(W_j) = \{W\}$ for $i \neq j$.
If $\{v_1, v_2, \dots, v_{n-1}\}$ were a point cover for G then $N(W_i$ would have to contain at leas at least one $V_j, j \geq k+1$ accounting together with $v_1, \dots, v_k$ For at least $l \geq n$ vertices. This is a contradiction.
Thus, $\{v_1, \dots, v_{n-1}\}$ is not a vertex cover.
After the cops make their opening move, say $\{c1, \dots, c_{n-1}\}$, the robber is able to place himself on some vertex which is not occupied by a cop and is not adjacent to a cop. After every time the cops move at most $n-1$ of the vertices adjacent to the robber are occupied by a cop and the robber is allowed to go to a free neighbor. [2, page 5] $\square$

Example: A dodecahedron has a cop number of 3.

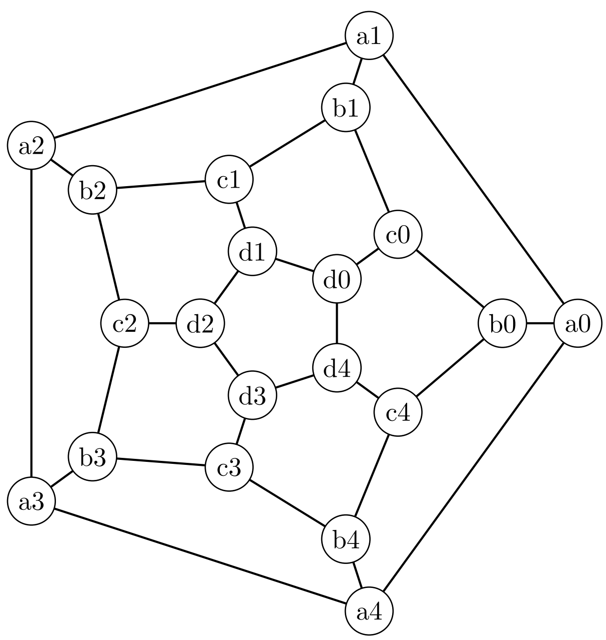

Figure 6: G

Let G be a dodecahedron.We know by theorem 1, $c(G) \leq 3$. Thus in a game with 3 cops, the cops have a winning strategy. The minimum degree vertex of G is 3 (all vertices have at least degree 3). Thus by theorem 2 we know that $c(G) \geq 3$. Since $c(G) \geq 3$ and $c(G) \leq 3$ we can conclude that $c(G) = 3$.

4 Translating Cops and Robber Graphs to Chess

Cops and robber graphs have proved to be yield interesting results when looking at the game on graphical representations for chess pieces. For these graphs, the vertices are any spaces on a chess board. Two vertices are adjacent if the piece we are studying (rooks, knights, queens) can move to that vertex in one step.

4.1 Rook Graphs

The first graph derived from the pieces on a chess board that we will take a look at are rook graphs. On a chess board, rooks can move any number of spaces in a vertical or horizontal direction. In the figure below, the black dots represent the possible moves for a rook on a chess board.

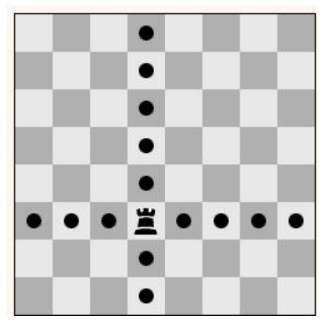

Figure 7: Possible Rook Moves

We can create graphical representations of this by starting with an $n \times n$ graph. We have provided some examples of 3×3 and 4×4 rook graphs below. We will denote an $n \times n$ rook graph as R_n.

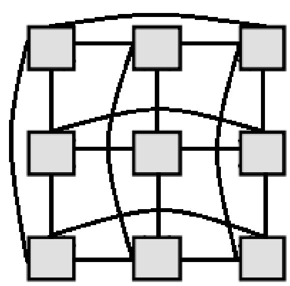

Figure 8: 3×3 Rook Graph

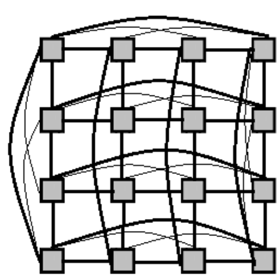

Figure 9: 4×4 Rook Graph

There are a couple theorems that we looked at involving rook graphs.

Theorem 6. *For any $n \geq 2$, $c(R_n) = 2$.*

Proof. Suppose there is only one cop on the rook graph. The robber can always win because there will always be some vertex on his row or column not adjacent to the cop.
Suppose then that there are two cops on the board, c_1 and c_2. If c_1 goes on the same row as the robber and c_2 goes on the same column as the robber, there will be no "safe" vertices adjacent to the robber and the game is over. □

Lemma 7. *For any arbitrary rook graph, when $c_L(R_n) < n$ there exists at least 1 row and column free of cops. Thus there exists at least 1 safe vertex adjacent to the robber.*

Theorem 8. *For any $n \times n$ rook graph, where $n \geq 2$, $c_L(R_n) = \gamma(R_n) = n$.*

This theorem tells us that when we have an $n \times n$ graph, where $n \geq 2$, the minimum number of lazy cops needed is equal to the domination number of the graph.

Proof. First, let us show that the robber has a winning strategy when there are $n-1$ lazy cops on the graph. Assume there are $n-1$ lazy cops on the graph and the robber is not adjacent to any cop at the start. The robber's row and column do not contain any cops. Without loss of generality, let c_1 move adjacent to the robber on his row. Thus the robber's row is no longer safe. The robber's column still contains no cop. Because of Lemma 7, we know that there exists a row somewhere with no cop. Thus there exists a safe vertex using the robber's column and some row. The robber can then move to the safe vertex and we return to our assumption.
Second, let us show that there is a winning strategy for n cops.
If there are n cops, they can position themselves such that they create a dominating set,every vertex in the graph is either occupied by a cop or adjacent to a vertex occupied by a cop. Thus, there are no safe vertices for the robber and the robber loses. □

4.2 Queen Graphs

On a chess board, a queen can move an indefinite number of spaces in any direction. The black dots on the figure below represent the possible moves of a queen on a chess board from a particular position.

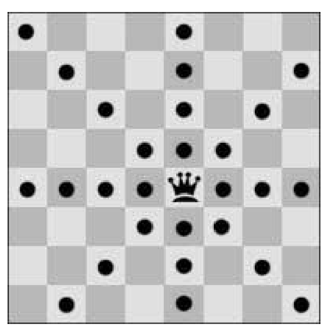

Figure 10: Possible Queen Moves

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We can create a graphical representation of queens on a chess board by looking at each square on a chess board as a vertex in our graph, and connecting each vertex to the vertices corresponding to potential moves available to a queen on a chess board. We have provided some examples of 3×3 and 4×4 rook graphs below. We will denote an $n \times n$ rook graph as R_n.

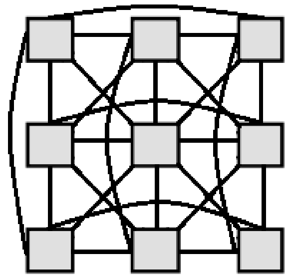

Figure 11: 3×3 Queen Graph

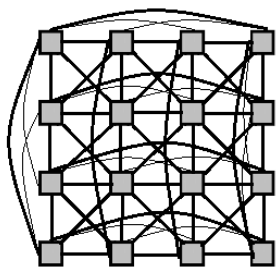

Figure 12: 4×4 Queen Graph

Theorem 9. *For any $n \geq 7$, $c(Q_n) \leq 4$.*

Proof. At the start of the game, the four cops arbitrarily place themselves on the graph, and the robber places himself on some vertex not adjacent to one of the vertices occupied by a cop. On the cops' first turn, one cop is moved to the column, row, and both diagonals adjacent to the robber's vertices. The robber is now completely surrounded by cops, and no matter what his next move is, he will be caught. ☐

4.3 Knight Graphs

Similarly to rook and queen graphs, we can also look at graphs for knights on a chess board. On a chess board, a knight can move 2 positions horizontally and 1 position vertically or 2 positions vertically and 1 position horizontally. Below, the black dots represent the potential moves of a knight from a particular space on a chess board.

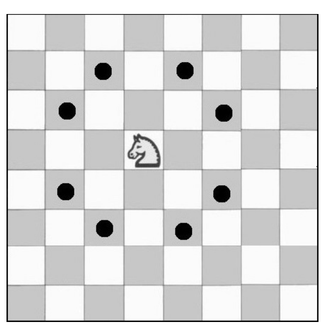

Figure 13: Possible Knight Moves

We can create a graphical representation using the same methods we used for the rook and queen graphs. Below are figures of 3×3 and 4×4 knight graphs.

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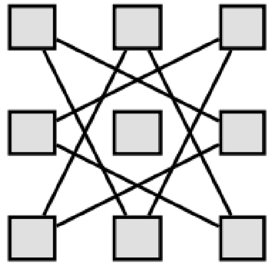

Figure 14: 3×3 Knight Graph

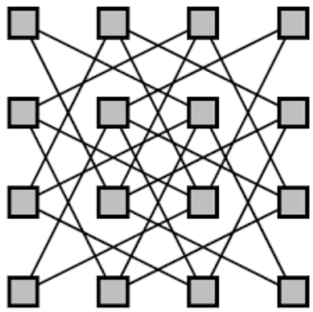

Figure 15: 4×4 Knight Graph

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Definition 12. An isolated vertex is a vertex of degree zero.

Theorem 10. *For $n = 3$, $c(Kn) = 3$.*

Proof. Consider that a K_3 graph can be represented with an 8-cycle and an isolated vertex in the middle after rearranging the vertices. From Proposition 1, we know that the cop number for a cycle is two. Since there is an isolated vertex, we must place a cop there. If no cop is positioned on the isolated vertex, the robber can position himself there at the beginning of the game, making him impossible to capture. Thus, $c(K_3) = 3$. □

Interestingly, we find the same result for lazy cops on a 3×3 knight graph.

Theorem 11. *For $n = 3$, $c_L(K_n) = 3$.*

Proof. Since there is an isolated vertex, a cop must be positioned there. This cop is not affected by being lazy because he can not move. The result we find in Proposition 1 is also not affected. Suppose we have two lazy cops on the cycle in the graph. As the lazy cops take turns moving toward the robber. The number of safe vertices for the robber steadily decreases by one. Eventually, the robber will no longer have any safe vertices left, and he will be caught. □

Definition 13. Let us begin by quickly explaining the complement of a graph. For any graph G, G^c (the compliment of G), is a graph with the same vertices as G, but for any two vertices that are adjacent in G, they are not adjacent in G^c; for any vertices not adjacent in G, they are adjacent in G^c.

Theorem 12. *For $n = 4$, $c(K_n^c) = 2$.*

We encourage the reader to draw the compliment of a 3×3 knight graph and write a proof for the cop number before proceeding.

Proof. Consider an arbitrary $n \times n$ knight graph, where $n \geq 4$. For any of these graphs, $\delta(K_n) = 2$ and $\Delta(K_n) \leq 8$. For every n, we calculate the minimum degree and maximum degree of the complement of the graph, using the following equations:

$$\delta(K_n^c) = n^2 - (\Delta(K_n) + 1$$
$$\Delta(K_n^c) = n^2 - (\delta(K_n) + 1$$

We add one to these equations to account for the vertex itself because there are no self loops, vertices connected to themselves, in our graphs. Since $\delta(K_n) = 2$ for $n \geq 4$, $\Delta(K_n^c) = n^2 - 3$. Assume there are two cops on the graph. Let c_1 be positioned on the vertex of maximum degree. This leaves two vertices uncovered. Based on the way knights move on a chess board, in the original K_n graph, the vertices adjacent to the vertex we selected are one diagonal unit away from each other. These vertices are the vertices left uncovered in our complement graph. Thus, there is a vertex one unit up, down, or on the side of one of the vertices that will be adjacent to both vertices. If we place the second cop, c_2, on this vertex, the two cops will create a dominating set, and there will be no safe vertices for the robber. □

5 Conclusion

We encourage the reader to explore if they can find similar conclusions for the compliment of queen and rook graphs. There are many interesting related conjectures and open problems. Here are some that we considered while researching this topic.

- There has not been much research done on bishop graphs. There may be some interesting phenomena unique to bishop graphs.
- We did not look into the complements of other chess board graphs. Do the other graphs produce the same or similar results?
- We did not look at how lazy cops could potentially affect the cop number for knight graphs. Is the lazy cop number significantly larger for the complement of knight graphs?

We were most drawn to knight graphs because of there complex, unique structure. We encourage the reader to try their hand at working with some of these graphs. The mathematics behind the topic is fairly simple and the concept is a lot of fun, so it is the perfect topic for instructors to present to their students.

6 References

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