Washington & Jefferson College
Math 217

Linear Algebra and Graph Theory

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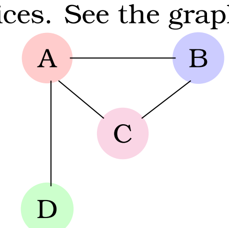
1 Introduction

At first glance graph theory seems impractical, with little real world applications, however with recent advancements in technology and with the complex world we live in today, it has become more and more prevalent and useful. Graph theory has helped in the fields concerning neural networking, GPS systems and much more. Graph theory seems bizarre and overly abstract when first introduced, yet the concepts and theories explored in this paper are simple enough to grasp quickly and still are useful in real life scenarios.

In order to begin this paper it is essential to explain what a graph is, what it consists of and some terms and definitions. (Please note that we will working with simple graphs exclusively, thus there is no need to worry about self loops or multi edges.) A graph is a compilation of vertices and edges. Vertices are commonly represented by a dot, or a small circle, and the edges are represented by lines connecting one vertex to another. The graph can be manipulated in any way imaginable so long as no vertices or edges are added or removed. Two vertices with and edge between them are commonly referred to as neighbors, or adjacent vertices. See the graph, G, below

G



As we can see there is an edge between A and B, A and C, A and D, & B and C. The next important thing to understand in graph theory is the concept of a walk. Going along the edges from A to B to C is a walk; the size of the walk is 2, which is one less than the number of vertices walked through, or the number of edges gone through. Another walk from D to A can be D to A to B to C to A (which is also written as DABCA). It is important here to notice that there are many different options and combinations that can make up a walk. A walk can go over the same edge multiple times and can also go through a vertex multiple times; there are no restrictions other than the fact that there must be an edge between the step from one vertex to another.

Another important concept in graph theory is the idea of the degree of a vertex. The degree of a vertex is determined by how many neighbors it has. Thus the degree of vertex A, denoted by deg(A), is 3, deg(B)=2, deg(C)=2, and deg(D)=1. Another important concept to understand is the idea of the maximum vertex degree of a graph, denoted by $\Delta(G)$. Using the information above concerning the degrees of the different vertices of G we can clearly see that $\Delta(G) = 3$.

2 Undirected Graphs

In graph theory there is a way of representing the edges between vertices using an adjacency matrix. In this matrix each edge has its own row and each edge has its own column, if there is an edge between the vertex in the row and the vertex in the column, then that entry of the matrix is 1, if there is no edge it is 0. Thus for the graph G, the adjacency matrix, P, is $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ where A is the first row and column, B is the second row and column, C is the third row and column, and D is the fourth row and column.

Theorem If $P = (p_{ij})$ is the adjacency matrix of a graph G and we let $P^r = (a^r_{ij})$, i.e., a^r_{ij} is the entry in the ith row and jth column of A^r, then the number of walks of length r between vertex i and vertex j in G is exactly a^r_{ij} .

Example 1 Determine the number of walks of length 1 from A to D of G.

> The first step is to look at the first row, and fourth column of P, from this we get the number of walks of length 1 from A to D is 1.

Example 2 Find the number of walks of length 3 from A to A.

> In order to determine this we must first compute the matrix P to the third power. We get the following
> $$P^3 = \begin{bmatrix} 2 & 4 & 4 & 3 \\ 4 & 2 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$
> In order to see how many walks of length 3 from A to A, we will look at the first row and first column of P^3 whose entry is 2. Looking back at the graph of G we can see that there are the walks ABCA and ACBA. There are no other 3 length walks from A to A.

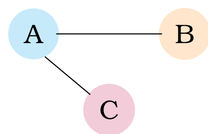
Example 3 Find the number of three length walks from B to A, and list them.

> In order to find how many there are we will look at the second row and first column of P^3. This entry is 4. Thus we know that 4 different 3 length walks from B to A exist in G. If we look at G we see that the walks are BCBA, BADA, BABA, and BACA.

Theorem Let $\lambda_{max}(G)$ be the largest or maximum eigenvalue for the adjacency matrix of the graph of G. Then $\lambda_{max}(G) \leq \Delta(G)$.

Example 1 Consider the graph, H, below. Write an adjacency graph for

H and find the minimum possible degree of $\Delta(H)$.



H

The adjacency graph of H is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Using methods learned in linear algebra we can determine the eigenvalues of H to be approximately -1.414,0, and 1.414. We see that the highest λ value, or $\lambda_{max}(G) = 1.414$, which is clearly less than 2, which is the degree of the highest degree vertex of G, or $\Delta(G)$.

3 Directed Graphs

Additionally, there are also directed graphs. A directed graph is similar to an undirected graph. However, in a directed graph, the edges have direction from one vertex to another. These kinds of edges are called directed edges and are indicated indicated by a "$\rightarrow$". In directed graphs, there are no self loops, edges such that $P_i \rightarrow P_i$. However, directed graphs can have vertices with edges that are directed towards each other. This is denoted by $P_i \leftrightarrow P_j$. If a directed graph has n vertices, we can represent the graph with a $n \times n$ matrix $D = [d_{ij}]$ such that the entries of D are defined by

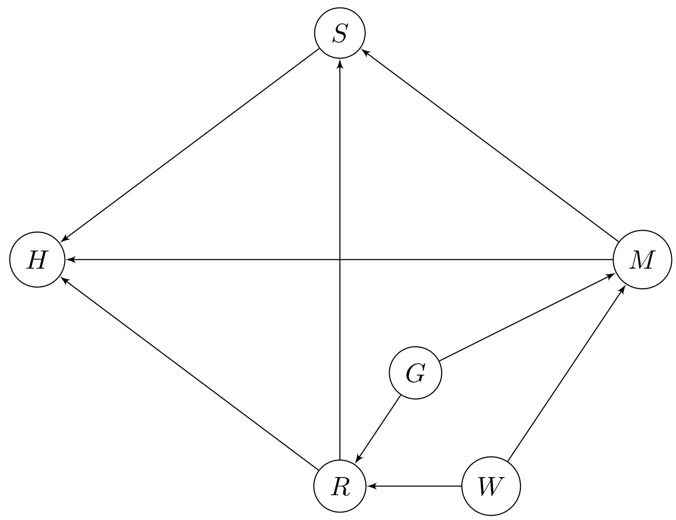
$$d_{ij} = \begin{cases} 1, & \text{if } P_i \rightarrow P_j \\ 0, & \text{if otherwise} \end{cases}.$$

Vertex matrices have the following properties:

1. All entries are either 0 or 1.
2. All diagonal entries are 0 (Anton and Rorres, 564).

We can see that property one holds by the way that a vertex matrix is defined. By the definition of directed graph, a vertex can not be directed towards itself. Thus, property two holds.

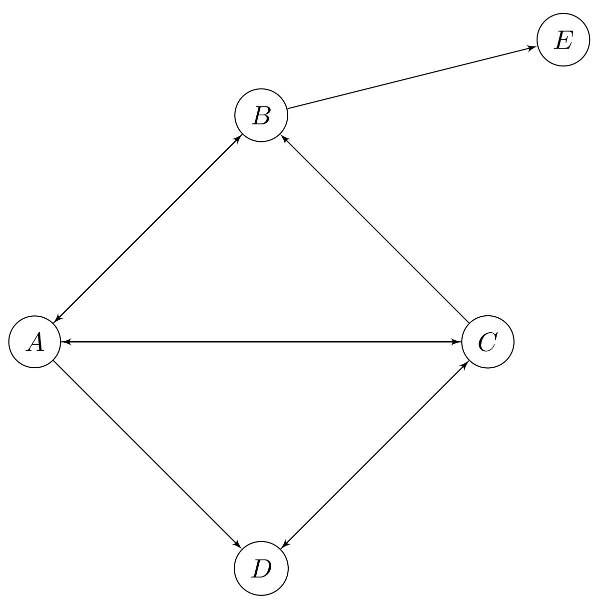
Example 1: Food Webs
Consider a food web that consists of grass, wildflowers, a mouse, a rabbit, and a hawk. The mouse and the rabbit eat the grass and wildflowers. The snake eats the mouse and the rabbit, and the hawk eats the the snake, rabbit,and mouse. We can represent this in a directed graph where $P_i \rightarrow P_j$ represents P_i is eaten by P_j. In this graph, we have represented each organism as a vertex named with the organism's fist letter.


S
H
M
G
R
W

We can represent this directed graph as the following vertex matrix:

$$\begin{array}{c} \\ W \\ G \\ M \\ R \\ S \\ H \end{array} \begin{array}{c} \begin{array}{cccccc} W & G & M & R & S & H \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Example 2: GPS
We can use directed graphs to find optimal routes where the streets and intersections are represented by the edges and vertices. In these kinds of examples, we can use singularly directed edges to represent one-way streets Consider the graph below (Anton and Rorres, 567).



Since B can be reached directly from C, $B \to C$ is a 1-step connection. Intersection E cannot be directly reached from C, $C \to E$ is not true based on our graph. However, C can reach B, and B can reach E. Thus, $C \to B \to E$ is true based on our graph,$C \to B \to E$ is a 2-step connection. This idea can be expanded to r-step connections.

Theorem Let M be the vertex matrix of a directed graph and let $m_{ij}^{(r)}$ be the (i,j)-th element of M^r. Then $m_{ij}^{(r)}$ is equal to the number r-step

connections P_i to P_j (Anton and Rorres, 566).
In Example 2, the matrix

$$G = \begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

represents the graph depicted.
Using the theorem we just defined, we can find the number of r-step connections for any $r \in \mathbb{R}$. By matrix multiplication, we know that the matrices $G^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $G^3 = \begin{bmatrix} 2 & 3 & 3 & 3 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Using this, we can calculate the number of 2-step connections and 3-step connections from any vertex to any other vertex. For example, we can calculate the number of 2 and 3 step connections from C to A. Because $m_{31}^{(2)} = 1$, there is one 2-step connection from C to A. Because $m_{31}^{(3)} = 3$, there are three 3-step connections from C to A.

This can be helpful for a GPS to find an optimal route from one place to another. The GPS can start at $r = 1$. If these options are not available, the GPS can then calculate the number of 2-step connections, and so on. This theorem allows us to know when our options have been exhausted and a new, longer route needs to be found.

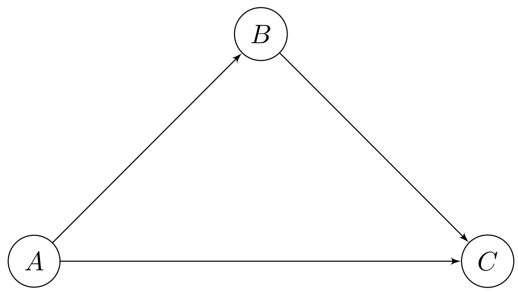
If we put some restrictions on a directed graph, we will have a tournament or "dominance-directed graph". A tournament eliminates the possibility for any two vertices to have a relation such that $P_i \leftrightarrow P_j$. There is a connection between every two vertices in a tournament. In tournaments, there are teams (vertices) that we will call All-Stars. A team is an All-Star if there is a 1-step or 2-step connection to every other team.

Theorem In any tournament, there is at leasts one vertex from which there is a 1-step or 2-step connection to any other vertex (Anton and Rorres, 569).

A vertex with the most 1-step and 2-step connections to other vertices is an All-Star. With the vertex matrix of a graph, we can find the All-Star teams easily. From the first theorem that we covered in this section, we know that the number of r-step connections between any two points is (i, j)-th entry in the r power of the vertex matrix V. Using this, we can see that adding the entries in the ith rows of the vertex matrices when $r = 1$ and $r = 2$ will give us the total number of 1-step and

2-step connections from P_i to every other vertex. The sum of the entries in the rows of $V + V^2$ will result in a vector that contains the number of connections for the vertices corresponding to each row (Anton and Rorres, 569).

Example
Consider the tournament below.



This tournament can be represented by the vertex matrix $V = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,
so $V + V^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The sum of the rows are:

Row one = 3
Row two = 1
Row three = 0.

Since row one has the largest sum, P_1 has the most-step and 2-step connections to other vertices and is the All-Star.

4 Conclusion

From food webs to city maps, graphs can represent almost anything, making graph theory a powerful tool. Graph theory can be used to help solve many real world issues; it is one of the most applicable and digestible fields of mathematics. Used in combination with linear algebra, it becomes even more compelling. In this paper, we have looked at a few of the uses for graph theory, and how linear algebra can be applied to make the lives of graph theorists easier.

5 References

1. Anton, Howard, and Chris Rorres. Elementary Linear Algebra: Applications Version. Wiley, 2010.

2. Cutler. Applications of Linear Algebra to Graph Theory.