Washington & Jefferson College
Math 420

Constant Transversal Matrices

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1 Introduction

Choose exactly 1 entry from each row and column from the matrix below (note that you will choose exactly 3 numbers):

$$\begin{bmatrix} 1 & 15 & 14 \\ 10 & 24 & 23 \\ 7 & 21 & 20 \end{bmatrix}$$

I can guarantee that no matter what combinations of numbers you choose, the sum of all of them is 45 . Try for example 1+24+20 or 14+10+21. This fun matrix here brings up a couple of questions: Is this matrix unique? How can I make a matrix like this? What are the properties of a matrix like this?

2 Background

In this section of the paper we will provide some background that will become relevant later in the paper.

2.1 Vector Space

2.1.1 Axioms of a vector space

First, let V be a nonempty set, and let $u, v \in V$ and $c, d \in \mathbb{R}$. As you may recall the axioms for a vector space are as follows:

1. $u + v$ is in V
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a zero vector, 0, such that $u + 0 = u$.

5. For each u in V there exists some $-u$ such that $u + (-u) = 0$.

6. cu is in V

7. $c(u + v) = cu + cv$

8. $(c + d)u = cu + cd$

9. $(cd)u = c(du)$

10. $1u = u$

2.1.2 Definition of a subspace $U \subseteq V$

Secondly, let V be a vector space and $U \subseteq V$ be a nonempty subset. U is a vector space if the following are true

1. U is closed under addition (Axiom 1).

2. U is closed under scalar multiplication (Axiom 6).

2.1.3 Spanning Set

A subset of S of V is a spanning set of V if every element of V can be written as a linear combination of the elements of S.

2.1.4 Linear Independence

A set $S \subseteq V$ is linearly independent if for any linear combination

$$a_1v_1 + a_2v_2 + ... + a_xv_x = 0$$

of all the elements in S the only way to get the zero vector is if every coefficient in the linear combination is zero. That is, if every $a_i = 0$.

2.1.5 Basis and Dimension

A basis B of a vector space V is a linearly independent spanning set of V.

The dimension of a vector space V is the number of elements in a basis for V.

2.1.6 Definition of R_k and C_l

$R_k = [r_{ij}]$ is the $n \times n$ matrix such that all entries in R_k are zeros expect for the k^{th} row which contains all ones. That is $r_{ij} = \begin{cases} 1 & \text{if i=k} \\ 0 & \text{if i} \neq \text{k} \end{cases}$

Example for a 4×4 matrix:

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly for $C_l = [c_{ij}]$ is a $n \times n$ matrix such that all entries in C_l are zeros except for the l^{th} column which contains all ones. That is $c_{ij} = \begin{cases} 1 & \text{if j=l} \\ 0 & \text{if j} \neq \text{l} \end{cases}$

Example for a 4×4 matrix:

$$C_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is important to note that the set $\{R_1, R_2, ..., R_n, C_1, C_2, ..., C_n\}$ is a linearly dependent set, but will be a linearly independent set if one element of the set is removed.

2.2 Permutations

2.2.1 Definition of Permutations on the set $\{1, 2, 3, ..., n\}$

A permutation of a set $\{1, 2, ..., n\}$ is a bijection on $\{1, 2, 3, ..., n\}$. That is, a permutation is a rearrangement of all the elements 1 through n.

Let S_n be the set of all permutations on $\{1, 2, 3, ..., n\}$.

3 Constant Transversal Matrices

3.1 Definitions

Let us begin by defining some essential terms.

Let $A = [a_{ij}]$ be a $n \times n$ matrix, and let $\sigma \in S_n$ be a permutation.

A transversal, T_σ of A, is defined as

$$T_\sigma = \{a_{i\sigma(i)} | i = 1, 2, 3, ..., n\}$$

Thus a transversal of A is a set of n numbers from A with exactly one from each row and column on A.

Let $S(T_\sigma)$ be the sum of all numbers in the transversal T_σ of A. That is,

$$S(T_\sigma) = \sum_{i=1}^{n} a_{i\sigma(i)}.$$

A matrix A is a constant transversal matrix, denoted CT, if there is a constant k such that $S(T_\sigma) = k$ for every $\sigma \in S_n$. In this case, the constant transversal sum of A is denoted $S(A)$.

Let V_n be the set of all $n \times n$ CT matrices. We shall see that V_n is indeed a $(2n - 1)$-dimensional subspace of $M_{n \times n}$.

3.2 Special CT Matrices

1. Constant Matrices

   A constant matrix is a $n \times n$ matrix, $A = [a_{ij}]$ such that all entries in the matrix are the same number, e. That is $a_{ij} = e$ for some constant e, for every i, j. For example,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is a constant matrix. We can see that no matter what transversal σ gives, you will always end up with a transversal sum of $S(A) = ne$. Thus the transversal is always a constant, and a constant matrix is a CT matrix.

2. Constant row matrices

   A constant row matrix is a $n \times n$ matrix with constant rows. For example,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$$

   is a constant row matrix. Let $A = [a_{ij}]$ be a matrix with constant rows. Let $a_{ij} = k_i$ for every j. Regardless of the transversal, one number is chosen from each row, and hence $T_\sigma = \{k_1, k_2, ..., k_n\}$ for every σ. Thus we can see that $S(A) = \sum_{i=1}^{n} k_i$ which will always be the same constant regardless of the transversal you choose. Thus a matrix with constant rows is a CT matrix.

3. Constant Column Matrix

   The explanation is almost identical to the explanation of a constant column matrix.

4. Counting Matrices

   A counting matrix is a $n \times n$ matrix, A, consisting of $1, 2, ..., n^2$ in natural order. They are of the form

$$A=\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & n+3 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n \\ \dots & & & & \\ (n-1)n+1 & (n-1)n+2 & (n-1)n+3 & \dots & n^2 \end{bmatrix}.$$

For example,

$$\text{B=}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is a counting matrix. Consider a transversal of $T_\sigma = \{a_{i\sigma(i)} | i = 1, 2, 3, \dots, n\}$. Note that the element in row i and column j is $(i-1)n + j$. Thus

$$\begin{aligned} S(T_\sigma) &= \sum_{i=1}^{n} a_{i\sigma(i)} \\ &= \sum_{i=1}^{n}((i-1)n + \sigma(i)) \\ &= n\sum_{i=1}^{n}(i-1) + \sum_{i=1}^{n}\sigma(i) \\ &= n(0+1+2+\dots+n-1) + \sum_{i=1}^{n} i \\ &= n\frac{(n-1)n}{2} + \frac{n(n+1)}{2} \\ &= \frac{n^2(n-1)+n(n+1)}{2} \\ &= \frac{n(n^2+1)}{2}, \end{aligned}$$

which is a constant. Thus we can conclude that A is a constant transversal matrix and $S(A) = \frac{n(n^2+1)}{2}$.

Let us consider the 3×3 matrix above. According to the formula S(B)=15. We can see that this is true for 1+5+9, 2+6+7 and so on.

3.3 Properties of CT matrices

1. Property 1: If one row of a CT matrix is a constant row, they all must be constant rows.

   Proof: Let $A \in V_n$, and let row k be a constant row, let $l \neq k$ be any row of A, and let i, j be any columns of A such that $i \neq j$. Let us consider a transversal of A, T_{σ_1}, and let $a_{ki}, a_{lj} \in T_{\sigma_1}$. Now let us define T_{σ_2} such that it is the same as T_{σ_1}, expect $a_{ki}, a_{lj} \notin T_{\sigma_2}$, and $a_{li}, a_{kj} \in T_{\sigma_2}$. Since A is CT we know that the sum of all the elements of T_{σ_1} must be equal to the sum of $T_{\sigma_{12}}$. From this we can see that

   $$a_{ki} + a_{lj} = a_{kj} + a_{li}$$

   since all other elements of T_{σ_1} and T_{σ_1} are the same. Now since we know that row k is a constant row we also know that $a_{ki} = a_{kj}$ and we can see that $a_{lj} = a_{li}$ and thus if any row in A is a constant row we can conclude that every row in A must be a constant row.

2. The same can be said for constant column matrices and the proof is similar.

4 V_n is a subspace of $M_{n\times n}$

Let $A, B \in V_n$, where $A = [a_{ij}]$ and $B = [b_{ij}]$ with $S(A) = q$ and $S(B) = r$. Let $\sigma \in S_n$. The following proofs will demonstrate that V_n is a subspace:

V_n is closed under addition.

Proof. Consider $A+B = [a_{ij}+b_{ij}]$ so any T_σ of $A+B$ is equal to $\{a_{i\sigma(i)}+b_{i\sigma(i)} | i = 1, 2, ..., n\}$.

So then we have

$$
\begin{aligned}
S(A+B) &= \sum_{i=1}^{n} a_{i\sigma(i)} + b_{i\sigma(i)} \\
&= \sum_{i=1}^{n} a_{i\sigma(i)} + \sum_{i=1}^{n} b_{i\sigma(i)} \\
&= S(A) + S(B) \\
&= q + r.
\end{aligned}
$$

As we can see $S(T_\sigma)$ of $A+B$ is a constant number and so we can conclude that $A+B$ is a CT matrix. Thus CT matrices are closed under addition. □

V_n closed under scalar multiplication.

Proof. Since A is a constant transversal matrix we know that $S(T_A) = \sum_{i=1}^{n} a_{i\sigma(i)} = q$ where q is a constant. Now multiply A by $c \in \mathbb{R}$ to get $cA = [ca_{ij}]$. So a transversal of cA would be $T_\sigma = \{ca_{i\sigma(i)} | i = 1, 2, ..., n\}$, and $S(cA) = \sum_{i=1}^{n} ca_{i\sigma(i)} = c\sum_{i=1}^{n} a_{i\sigma(i)} = cq$ which is a constant for every $\in S_n$. Thus we can conclude that cA is a constant transversal matrix, and constant transversal matrices are closed under scalar multiplication. □

Thus since V_n is closed under addition and scalar multiplication we know that V_n is a subspace of $M_{n\times n}$

5 Dimension of V_n

We will prove that you can find a basis with $(2n-1)$ elements for V_n.

Let $A = [a_{ij}] \in V_n$.

$$
\text{A=} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}
$$

Let B be the CT matrix consisting of constant columns using the values from the first row of A. That is,

$$B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \\ a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

Then the matrix $A - B$ has a constant row of 0 in row 1. That is,

$$A - B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ (a_{21} - a_{11}) & (a_{22} - a_{12}) & \dots & (a_{2n} - a_{1n}) \\ (a_{31} - a_{11}) & (a_{32} - a_{12}) & \dots & (a_{3n} - a_{1n}) \\ \dots & & & \\ & & & \\ (a_{n1} - a_{11}) & (a_{n2} - a_{12}) & \dots & (a_{nn} - a_{1n}) \end{bmatrix}$$

By Property 1, since row 1 is a constant row, all rows in A_B are constant rows, so

$$A - B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ (a_{21} - a_{11}) & (a_{21} - a_{11}) & \dots & (a_{21} - a_{11}) \\ (a_{31} - a_{11}) & (a_{31} - a_{11}) & \dots & (a_{31} - a_{11}) \\ \dots & & & \\ & & & \\ (a_{n1} - a_{11}) & (a_{n1} - a_{11}) & \dots & (a_{n1} - a_{11}) \end{bmatrix}$$

Note that

$$A = (A - B) + B$$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 \\ (a_{21}-a_{11}) & (a_{21}-a_{11}) & \dots & (a_{21}-a_{11}) \\ (a_{31}-a_{11}) & (a_{31}-a_{11}) & \dots & (a_{31}-a_{11}) \\ \dots & & & \\ & & & \\ (a_{n1}-a_{11}) & (a_{n1}-a_{11}) & \dots & (a_{n1}-a_{11}) \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \\ a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix} =$$

$$(a_{21}-a_{11}) \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} + (a_{31}-a_{11}) \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} + \dots + (a_{n1}-a_{11}) \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix} +$$

$$a_{11} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & & & \\ 1 & 0 & \dots & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \\ 0 & 1 & \dots & 0 \end{bmatrix} + \dots + a_{1n} \begin{bmatrix} 0 & \dots & 1 \\ 0 & \dots & 1 \\ \dots & & \\ 0 & \dots & 1 \end{bmatrix}$$

$$= (a_{21}-a_{11})R_2 + (a_{31}-a_{11})R_3 + \dots + (a_{n1}-a_{11})R_n + a_{11}C_1 + a_{12}C_2 + \dots + a_{1n}C_n$$

Thus A can be written as a linear combination of R_2 through R_n and C_1 through C_n. Hence $\{R_2, R_3, ..., R_n, C_1, C_2, ..., C_n\}$ spans V_n, and since we already know that the set $\{R_2, R_3, ..., R_n, C_1, C_2, ..., C_n\}$ is linearly independent, since R_1 is missing, we can conclude that $\{R_2, R_3, ..., R_n, C_1, C_2, ..., C_n\}$ is a basis for V_n, and V_n has dimension $2n-1$.

6 How to fill in 3×3 and 4×4 CT matrices:

Given the first row and column of a matrix, you can generate a CT matrix yourself by following these simple steps:

6.1 Filling in a 3×3 CT matrix:

Given:

5	1	13
1		
7		

Step 1 fill in the 2nd row using the difference $a_{12} - a_{11}$ and add it to the entry above:

5	1	13
1		
7		

$\rightarrow (1 - 5) = -4 \rightarrow$

5	1	13
1	1-4	13-4
7		

$\rightarrow$

5	1	13
1	-3	9
7		

Step 2 repeat for row 3:

5	1	13
1	-3	9
7		

$\rightarrow (7 - 1) = 6 \rightarrow$

5	1	13
1	-3	9
7	-3+6	9+6

$\rightarrow$

5	1	13
1	-3	9
7	3	15

DONE!! Looking at the matrix above we see that it's constant transversal sum is 17. We see this is true from the transversal sum $5 - 3 + 15$ or $1 + 9 + 7$. Feel free to try other possible transversals to test that this is true.

6.2 Filling in a 4×4 CT matrix:

Given:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline 2 & & & \\ \hline \end{array}$$

Step 1 fill in the 2nd row using the difference $a_{12} - a_{11}$ and add it to the entry above:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline 2 & & & \\ \hline \end{array} \to (3-1) = 2 \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 2+2 & 5+2 & 9+2 \\ \hline 4 & & & \\ \hline 2 & & & \\ \hline \end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & & & \\ \hline 2 & & & \\ \hline \end{array}$$

Step 2 repeat for row 3:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & & & \\ \hline 2 & & & \\ \hline \end{array} \to (4-3) = 1 \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & 4+1 & 7+1 & 11+1 \\ \hline 2 & & & \\ \hline \end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & 5 & 8 & 12 \\ \hline 2 & & & \\ \hline \end{array} \to$$

Step 3 repeat for row 4:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & 5 & 8 & 12 \\ \hline 2 & & & \\ \hline \end{array} \to (2-4) = -2 \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & 5 & 8 & 12 \\ \hline 2 & 5-2 & 8-2 & 12-2 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 & 11 \\ \hline 4 & 5 & 8 & 12 \\ \hline 2 & 3 & 6 & 10 \\ \hline \end{array}$$

DONE!! Looking at the matrix above we see that it's constant transversal sum is 23. We see this is true from the transversal sum $3+2+8+10$ or $3+2+6+12$. Feel free to try other possible transversals to test that this is true.

At this point I would like to encourage the reader to practice this fun trick on their own to impress and baffle your friends at a later time.

7 Conclusion

As seen in this paper we explained what CT matrices are. We looked at a couple different CT matrices and explained why they are CT. We explained the reasoning behind a couple different properties of CT matrices, and showed how to create a basis for V_n, also discovering the dimension of V_n and proving that it is indeed a subspace of $M_{n \times n}$. Finally we looked over a neat little trick for generating CT matrices when given the first row and column of a matrix.

8 References

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