Proof Portfolio

Frances Sinclair

December 7, 2018

My Favorite Proofs

Verifying the Triangle Inequaltiy Find an efficient proof for all the cases at once by demonstrating that $(a+b)^2 \leq (|a|+|b|)^2$.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+2ab+b^2\\ &\leq a^2+2|a||b|+b^2 && \text{*since we know that } 2ab \leq 2|a||b|\\ &\leq |a|^2+2|a||b|+|b|^2\\ &\leq (|a|+|b|)^2\end{aligned}$$

Finally taking the root of both sides, which is possible because any real numbers squared have real roots, we can clearly see that $|a+b| \leq |a|+|b|$.

Proving if $\mathbb{I}$ is closed under multiplication and addition Let $s = \sqrt[3]{4}$, and let $t = \sqrt[3]{2}$. s and t are both irrational, however

$$\begin{aligned}st &= \sqrt[3]{2} \times \sqrt[3]{4}\\ &= \sqrt[3]{8}\\ &= 2\end{aligned}$$

Thus st is a rational number and $\mathbb{I}$ is not closed under multiplication.
Let us examine another example $s = -\sqrt{2}$ and $t = \sqrt{2}$. $s+t = 0$, $0 \in \mathbb{Q}$. Thus $\mathbb{I}$ is not closed under addition or multiplication.

Closure of sets Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$. Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Let us begin by proving that if E is closed then $\overline{E} = E$.
If E is closed, then E contains is limit points, that is $L_E \subseteq E$. Thus

$$\begin{aligned}L_E \cup E &= \overline{E}\\ E &= \overline{E}.\end{aligned}$$

Next let us prove that if $\overline{E} = E$ then E is closed.
If $\overline{E} = E$ that means that $E \cup L_E = E$, which is to say that $L_E \subseteq E$. This means that E contains all of its limit points. Hence E is closed.
Next let us prove that if $E^\circ = E$ then E is open by way of contradiction.
Let us assume that E is not open. Thus E contains some limit point of itself. Thus since E° does not contain any limit points, $E^\circ \neq E$.

Finally let us prove that if E is open then $E^\circ = E$ by way of contradiction.
Let us assume that $E^\circ \neq E$. Then E contains some $x \notin E^\circ$. Thus x is a limit point and E is not open.
Let L be the set of limit points of E throughout this problem. Let $x \in \overline{E}^c$ then we know that $x \in (E \cup L)^c$. Since $E \cup L$ is closed we know that $(E \cup L)^c$ is open. Thus any x in $(E \cup L)^c$ is in $E^c \cap L^c$, which is to say the set of E, but not including any limit points. Thus $x \in (E^c)^\circ$. Hence we can conclude that $\overline{E}^c \subseteq (E^c)^\circ$.
Let y be any element in $(E^c)^\circ$. Thus since y is in the set of the compliment of E, not including the limit points of that compliment, we know that $y \in (E \cup L)^c$. That is to say that $y \in \overline{E}^c$. Thus we can conclude that $(E^c)^\circ \subseteq \overline{E}^c$, and $\overline{E}^c = (E^c)^\circ$.
Let z be any element in $(E^\circ)^c$, which is the compliment of E not including its limit points. Hence z is not in E°, but rather in L or any other elements not in E°. Thus $z \in L \cup E^c \subseteq \overline{E^c}$. Hence we can conclude that $(E^\circ)^c \subseteq \overline{E^c}$.
We know that for any element $y \in \overline{E^c}$, y must also be in $(E^c \cup L_{E^c})$, where L_{E^c} is the limit point of E^c. This set includes everything not in E as well as the limit points of E. This is a subset of $(E^\circ)^c$. Thus $\overline{E^c} \subseteq (E^\circ)^c$. Hence we can conclude that $(E^\circ)^c = \overline{E^c}$.

Main Topics

Proof of lemma Assume that $A \subseteq \mathbb{R}$ is nonempty and bounded above. Let $c \in \mathbb{R}$. Define the set:

$$c + A := \{c + a | a \in A\}$$

Then sup$(c + A)$ = c+sup$A$.

Proof. Let sup$A$=q. We can see that if $q > a$ for all $a \in A$, it follows that $c + q > c + a$, and we can see that c+sup$A$ is an upper bound for the set $c + A$.
Let us define $b = c$+sup$A$. Now let g =sup$(A + c)$ And let $g < b$.

$$\begin{aligned} &b > g \\ &b > sup(A + c) \\ &b > a + c \\ &b > supA + c \end{aligned}$$

however we already know that $b = supA + c$, thus $b \leq g$ by contradiction and we have that $c + supA$ is less than or equal to any other upper bound b for $c + A$. ☐

proof concerning supremum and constants Given sets A and B, define $A + B = \{a + b : a \in A$ and $b \in B\}$. Prove that if A and B are nonempty and bounded above then sup($A + B$) = sup$A$ + sup$B$.

Proof. Let s = sup$A$ and t=sup$B$. We know that for all $a \in A$ and for all $b \in B$, $a \leq s$ and all $b \leq t$. Thus $a + b \leq s + b$. Since $b \leq t$ we can sub in t for where b is and we get the inequality $a + b \leq s + t$ which demonstrates the first condition of a supremum.
Let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Since u is an upper bound we know that for all $a \in A$ and for all $b \in B$, $a + b \leq u$. Thus $b \leq u - a$. From this inequality we can conclude that $u - a$ is an upper bound for B. Thus since t is the supremum of B we can state that $t \leq u - a$.
Since we know that t+s is an upper bound for A+B the first condition of a supremum is already satisfied. Let there be some other upper bound u of $A + B$. We already know that $t \leq u - a$ and thus $a \leq u - t$. From this we can conclude that $u - t$ is an upper bound of A. Thus we know that $s \leq u - t$ since s is the supremum of A. Using simple manipulation we can see that $s + t \leq u$. Thus $s + t$ is the supremum of $A + B$. ☐

proof on convergence Prove that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, ...$$

converges and find the limit.

Proof. We know that $\sqrt{2}$ is always positive. Thus every element in the sequence is always positive. Hence we can conclude that the sequence above is bounded below by 0.
We can clearly see that $\sqrt{2} < \sqrt{2+\sqrt{2}}$ thus the idea that the sequence is monotone and increasing holds for the base case. Let us assume that this holds up to some point k in the sequence. The term $x_k = \sqrt{2+x_{k-1}}$. Now let us look at the term $x_{k+1} = \sqrt{2+x_k} \geq \sqrt{2+x_{k-1}} = x_k$. Thus the sequence is monotone and increasing and bounded below.
Let us consider the possible upper bound 4. We see immediately that $\sqrt{2} < 4$ and thus this holds for the base case. Let us then assume that this is true for all x_1 through x_k. We know that $x_k = \sqrt{2+x_{k+1}} < 4$.

$$\sqrt{2+x_{k-1}} < 4$$
$$2+x_{k-1} < 16.$$

Now let us look at $x_{k+1} = \sqrt{2+x_k} = \sqrt{2+\sqrt{2+x_{k-1}}}$.

$$\sqrt{2+\sqrt{2+x_{k-1}}} < \sqrt{2+4} < 4.$$

Hence the sequence is bounded above and below and is monotone and increasing, thus the sequence converges.
Using 2.4.1 we know that lim(x_n)=lim(x_{n+1}). Let $\lim x_n = x$
$x_1 = \sqrt{2}$

$$\begin{aligned}
x_{n+1} &= \sqrt{2+x_n} \\
\lim(x_{n+1}) &= \lim(\sqrt{2+x_n}) \\
x &= \lim(\sqrt{2+x_n}) \\
x \times \lim(\sqrt{2+x_n}) &= \lim(\sqrt{2+x_n}) \times \lim(\sqrt{2+x_n}) \\
x \times \lim(x_{n+1}) &= \lim(\sqrt{2+x_n}) \times \lim(\sqrt{2+x_n}) \\
x^2 &= \lim(2+x_n) \\
x^2 - 2 - x = 0 \\
x = 2, -1
\end{aligned}$$

Since we know that the sequence is bounded below by 0, we can rule out -1 by OLT, and we can see that $\lim(x_n) = 2$. ☐

proof on convergence Consider the function g defined by the power series:

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - .... \text{ Does } g \text{ converge where } -1 < x < 1?$$

Proof. We can see that $g = \sum_{n=1}^{\infty} \frac{x^n(-1)^{n-1}}{n}$. Let us consider the absolute value of g. $|g| = \sum_{n=1}^{\infty} \frac{x^n}{n}$. We can easily see that $\sum_{n=1}^{\infty} \frac{x^n}{n} < \sum_{n=1}^{\infty} x^n$. We know that $\sum_{n=1}^{\infty} x^n$ converges by the geometric series test since $|x| < 1$. Thus by the comparison test for series we can conclude that $|g(x)|$ also converges. Finally we can conclude that g does converge absolutely. ☐

proof on functional limits Prove $lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.

Proof. Let $\epsilon > 0$ be arbitrary. Let $\delta > \frac{-3\epsilon}{1-\epsilon}$ and let $|x-3| < \delta$. Consider the inequality

$$\begin{aligned}
|x-3| &< \delta \\
|x| + |3| &< \delta \\
|x| &< \delta - 3
\end{aligned}$$

which will come in handy later in the proof. Then we have...

$$\begin{aligned}|\frac{1}{x}-\frac{1}{3}| &= |\frac{3-x}{3x}| \\ &< \delta|\frac{1}{3x}| \\ &< \frac{1}{3}\delta|\frac{1}{x}| \\ &< \delta|\frac{1}{x}| \\ &< \delta\frac{1}{\delta-3} \\ &< \frac{\delta}{\delta-3} \\ &< \epsilon\end{aligned}$$

as required and we can conclude that $lim_{x\to 3}\frac{1}{x}=\frac{1}{3}$. □

proof on the closure of sets Prove that $\overline{A\cup B}=\overline{A}\cup\overline{B}$

Proof. Let x be any element in $\overline{A\cup B}$, and let $\overline{X}$ be the smallest closed set containing x. Thus $x\in X\cup L_X$, where L_X is the set of limit points of X. Let $L_{A\cup B}$ be the limit points of $A\cup B$. so $x\in X\cup L_X$ and $x\in A\cup B\cup L_{A\cup B}$. Let us consider $x\in L_X$, then x is the sup or inf of $A\cup B$. Let us consider without loss of generality that $x=sup(A\cup B)$. That is to say that $x=max(sup(A),sup(B))$. Let us assume without loss of generality that $x=sup(A)$, this means that $x=lim(a_n)$ where a_n is any subsequence of A. It follows that $x=L_a$, where L_A is the set of limit points of A (note that L_B is the set of limit points of B). So if we take a step back we can see that x is either in L_A or L_B or in A or in B. Thus we know that $x\in A\cup L_A\cup B\cup L_B$. That is to say that $x\in\overline{A}\cup\overline{B}$. Thus $\overline{A\cup B}\subseteq\overline{A}\cup\overline{B}$
Let $y\in\overline{A}\cup\overline{B}$, and let L_A be the set of limit points of A and let L_B be the set of limit points of B. We know that $y\in A\cup L_A\cup B\cup L_B$. We know that if y is a limit point of A or B that means that it is the limit of (a_n), where a_n is any subsequence of A, or it is the limit of (b_n), where b_n is any subsequence of B. Without loss of generality, let $y=lim(a_n)$ this means that either y is an infimum or supremum of A. Thus we can conclude that y is the min of the infimums of A and B if they exist or y is the max of the supremums of A and B if they exist. Thus we can see that y is in the set of limit points of the union of A and B, that is $y\in L_{A\cup B}$. Thus $y\in A\cup B\cup L_{A\cup B}$. Which can be rearranged to say $y\in\overline{A\cup B}$. Thus $\overline{A}\cup\overline{B}\subseteq\overline{A\cup B}$.
Finally we can conclude that $\overline{A}\cup\overline{B}=\overline{A\cup B}$. □

proof of difference quotient Prove that $(f/g)'(c)=\frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2}$

Proof.

$$\begin{aligned}
(f/g)'(c) &= lim_{x\to c}\frac{(f/g)(x)-(f/g)(c)}{x-c}\\
&= lim_{x\to c}\frac{f(x)}{g(x)(x-c)}-\frac{f(c)}{g(c)(x-c)}\\
&= lim_{x\to c}\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= lim_{x\to c}\frac{f(x)g(c)+f(c)g(c)-f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= lim_{x\to c}g(c)\frac{f(x)-f(c)}{g(x)g(c)(x-c)}-f(c)\frac{g(x)-g(c)}{g(x)g(c)(x-c)}\\
&= lim_{x\to c}\frac{f'(c)}{g(x)}-\frac{f(c)g'(c)}{g(x)g(c)}\\
&= lim_{x\to c}\frac{g(c)f'(c)-f(c)g'(c)}{g(x)g(c)}
\end{aligned}$$

That is to say that $(f/g)'(c)=\frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2}$ using the definition of continuity. □

problems on differentiability Let $f_a(x)=\begin{cases}x^a & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$. For which values of a is f continuous at zero? For which values of a is f differentiable at zero? In this case, is the derivative function continuous? For which values of a is f twice-differentiable?

Proof. f is continuous at zero for all $a>0$. Since as x approaches 0 from the right, $lim_{x\to 0}x^a=0$ expect for when a is negative, which would produce $f(x)=\frac{1}{x^{|a|}}$ or when a is zero, which would produce $f(x)=1$.
Let us look at $f'(x)$

$$\begin{aligned}
f'(x) &= lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\
&= lim_{x\to c}\frac{x^a-c^a}{x-c}\\
&= lim_{x\to c}\frac{(x-c)(x^{a-1}+cx_c^{a-22}x^{a-3}+\ldots+c^{a-1})}{x-c}\\
&= lim_{x\to c}x^{a-1}+cx_c^{a-22}x^{a-3}+\ldots+c^{a-1}\\
&= c^{a-1}+c^{a-1}+c^{a-1}+\ldots+c^{a-1}\\
&= ac^{a-1}
\end{aligned}$$

which is continuous for any a. Thus we can see that $f'(x)=ax^{a-1}$ is continuous for any $a>0$ since the restrictions from $f(x)$ must carry over.

$$
\begin{aligned}
f''(x) &= lim_{x\to c}\frac{ax^{a-1}-ac^{a-1}}{x-c} \\
&= lim_{x\to c}a\frac{x^{a-1}-c^{a-1}}{x-c} \\
&= lim_{x\to c}a\frac{(x-c)(x^{a-2}+cx^{a-3}+c^2x^{a-4}+...+c^{a-2})}{x-c} \\
&= lim_{x\to c}a(x^{a-2}+cx^{a-3}+c^2x^{a-4}+...+c^{a-2}) \\
&= a(c^{a-1}+c^{a-2}+c^{a-2}+...+c^{a-2}) \\
&= a(a-1)c^{a-2} \\
&= a^2c^{a-2}-ac^{a-2}
\end{aligned}
$$

Thus we can conclude that $f''(x)=a^2x^{a-2}-ax^{a-2}$ which has no restrictions. Thus $f''(x)$ is continuous on all $a>0$ (since the restrictions from $f'(x)$ carry over). □

proof of Reimann Integrals compute $\int_0^1 x$ exactly.

Proof. For every $n\in\mathbb{N}$, let P_n be the partition

$$P_n=\{0,\tfrac{1}{n},\tfrac{2}{n},\tfrac{3}{n},...,\tfrac{n-1}{n},1\}.$$

For each k, $M_k=\{\frac{1}{n},\frac{2}{n},\frac{3}{n},...,\frac{n-1}{n},1\}$ and $m_k=\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},...,\frac{n-1}{n}\}$. A closed form for $U(f,P_n)$ and $L(f,P_n)$ are defined as follows:

$$
\begin{aligned}
L(f,P) &= (1+\tfrac{1}{n}+\tfrac{2}{n}+\tfrac{3}{n}+...+\tfrac{n-1}{n})(\tfrac{1}{n}) = (1+2+3+...+(n-1))(\tfrac{1}{n^2} = \tfrac{(n-1)n}{2n^2} = \tfrac{1}{2}-\tfrac{1}{2n} \\
U(f,P) &= (\tfrac{1}{n}+\tfrac{2}{n}+\tfrac{3}{n}+...+\tfrac{n}{n})(\tfrac{1}{n}) = (1+2+3+...+n)(\tfrac{1}{n^2}) = \tfrac{(n+1)n}{2n^2} = \tfrac{1}{2}+\tfrac{1}{2n}
\end{aligned}
$$

From this we can easily see that $inf\{U(f,P_n)|n\in\mathbb{N}\}=\frac{1}{2}$ and $sup\{L(f,P_n)|n\in\mathbb{N}\}=\frac{1}{2}$. Thus we can see that $\frac{1}{2}\le L(f)\le U(f)\le\frac{1}{2}$ and hence we can conclude that $\int_0^1 x=\frac{1}{2}$. □

proof on integral properties If $m\le f(x)\le M$ on $[a,b]$, then

$$m(b-a)\le\int_a^b f\le M(b-a).$$

Proof. First recall how $L(f)=sup\{L(f,P)|p\in P$, thus $L(f,P)\le L(f)$. Similarly we know that $U(f)\le U(f,P)$. Thus we can see that

$$L(f,P)\le\int_a^b f\le U(f,P)$$

for every partition P. Let us consider the partition $[a,b]$ where m and M are already defined. We can then see that

$$m(b-a)\le\int_a^b f\le M(b-a).$$

□

supporting topics

Proving the density of $\mathbb{Q}$ in $\mathbb{R}$ Let $a,b\in\mathbb{R}$ be such that $a<b$. Then there exists some $r\in\mathbb{Q}$ such that $a<r<b$.

Proof. Since $b - a \in \mathbb{R}$ then by the archimedean property of $\mathbb{N}$ in $\mathbb{R}$ we know that for some $n \in \mathbb{N}$, $\frac{1}{b-a} < n$ and thus $\frac{1}{n} < b - a$. Now let $m \in \mathbb{N}$ be the smallest natural number such that $na < m$. From this we can show that $a < \frac{m}{n}$. We can also show that since $\frac{1}{n} < b - a$, $a + \frac{1}{n} < b$, and $a < b - \frac{1}{n}$. Since m is the smallest natural number such that $m > na$, we know that $na + 1 \geq m$.
Let us now play with the inequality

$$\begin{aligned} an + 1 &\geq M \\ (b - \frac{1}{n})(n) + 1 &> m \\ bn - 1 + 1 &> m \\ bn &> m \end{aligned}$$

Thus we can see that $a < \frac{m}{n}$ and $b > \frac{m}{n}$ which means that we can conclude with $a < \frac{m}{n} < b$, where $r = \frac{m}{n}$. □

Problem on the Algebraic Limit Theorem$\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

Proof. Let us consider briefly $\lim(1 + 2a_n) = \lim(1) + \lim(2a_n) = 1$. Thus we can conclude that $\lim(1+2a_n)$ is defined. Let us also consider $\lim(1+3a_n-4a_n^2) = \lim(1)+\lim(3a_n)+\lim(-4a_n^2) = 1$. Thus we can conclude that $\lim(1 + 3a_n - 4a_n^2)$ is defined and is not equal to zero.

$$\begin{aligned} \lim \left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2} \right) &= \frac{\lim(1 + 2a_n)}{\lim(1 + 3a_n - 4a_n^2)} \\ &= \frac{1}{1} \\ &= 1. \end{aligned}$$

□

Alternating Series Comparison Tests determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n^2}$ converges conditionally or absolutely, or if it diverges.

Proof. Let us consider the beginning terms denoted $a_1, a_2, a_3, \ldots$. We can see that $a_1 = -2$, $a_2 = \frac{3}{4}$, $a_3 = \frac{-4}{9}$. Using this we can see that $|a_1| > |a_2| > |a_3| > \ldots$ and so on. We can also see that the lower limit of $|a_n|$ is zero, as the numerator will always be greater than zero. Thus we know that the series as a whole converges by the alternating series test. Secondly we must see whether the series converges conditionally or absolutely. Let us consider $\sum_{n=1}^{\infty} |(-1)^n \frac{n+1}{n^2}| = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$. If we consider this second series, we can easily see that $\sum_{n=1}^{\infty} \frac{n+1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. Thus by the comparison test we know that the absolute value of the series converge, and we can conclude that the series converges conditionally. □

compact if and only if closed and bounded Determine if the Cantor set is compact.

Proof. The Cantor set is an intersection of closed sets and thus is closed itself. We also know that the Cantor set is bounded by $[0, 1]$. Thus since it is both bounded and closed, it is compact. □

compact if and only if closed and bounded Determine if $\mathbb{N}$ is compact.

Proof. We know that N is not bounded thus we know from Theorem 3.3.4 that it cannot be compact. Let us consider the set $a_n = x$. This converges to ∞ which is not in the natural numbers. □

Proof on Uniform Continuity Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$, but not on the set $(0, 1]$.

Proof. Let us begin by proving that $f(x)$ is uniformly continuous on the set $[1, \infty)$.
Let $\epsilon > 0$. Three exists a δ such that $\delta < \frac{\epsilon}{2}$. Let $|x - y| < \delta$, where $x, y \in [1, \infty)$. Then we have...
$|f(x) - f(y)| = |\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{y^2 - x^2}{x^2 y^2}| = |\frac{(y-x)(x-y)}{x^2 y^2}| < \delta|\frac{x+y}{x^2 y^2}| = \delta(|\frac{1}{xy^2}| + |\frac{1}{x^2 y}) \leq \delta(|\frac{1}{x}| + |\frac{1}{x^2}|) < \delta(1+1) = 2\delta < \epsilon$ as required.
Let us conclude by proving that $f(x)$ is not uniformly continuous on the set $(0, 1]$.
Let $\epsilon = \frac{1}{8}$. Let $x_n = 2n$ and let $y_n = 2n - 1$. We know that $|x_n - y_n| = 1$. We can also see that $|f(x_n) - f(y_n)| = |\frac{1}{4n^2} - \frac{1}{4n^2 - 4n + 1}| = |\frac{4n^2 - 4n + 1 - 4n^2}{(4n^2)(4n^2 - 4n + 1)}| = |\frac{4n+1}{(4n^2)(4n^2 - 4n + 1)}| \geq |\frac{4+1}{4-4+1}| = 5 > \epsilon$. Thus $|f(x_n) - f(y_n)| > \epsilon$, and we can conclude that $f(x)$ is not uniformly continuous on the set $(0, 1]$. □

Personal Reflection

This course pushed my understanding of math more than any other course I have taken before, and I found myself stuck more often than I would like to admit. My tactics for working with a problem I was stuck with tended to be to try and figure it our from multiple angles, and if everything fails to contact my peers. I felt as though the chapters towards the end of the class were those which came hardest to me. I had to do multiple problems to feel like I really understood definitions and how to prove things; whereas at the beginning of the class everything came very easily to me, and it all seemed intuitive. At the beginning everything seemed easy and I was able to do the proofs and work easily, but as the course moved on things only appeared a little harder while they got a lot harder.

This semester I feel as though I rediscovered the beauty of scratch paper. Thorough out other courses I would solve problems while typing them into LaTex, however with this course I found this much more difficult. The best route for me was to draw a picture or two and then to write my formal answer on paper and to go from them. I wish I had discovered this at the beginning of class!! I thought the idea of having a draft of the homework and then submitting a final version helped me a lot. Often times after I submit a homework assignment I will forget about the content and move on, however this class allowed me the time and required that I go over my mistakes and learn from them, reinforcing ideas and concepts from the homework assignments. I also thought that having examples done in class and working in small groups to do classwork allowed for a better understanding of the content.