Computer Graphics Assignment #2

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1 Question

1. Draw the Fourier transform of a 2 Dimensional sampling function, whose intervals along with x and y axes are π, on a graph as shown in Fig. 1a.
2. Using the graph, explain the characteristics of the waves, which have the highest frequency in the range of aliasing free.

2 Answer

2.1 Fourier transform of 2-Dimensional Sampling Function

From the Fourier series of 1-dimensional sampling function, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(x-nT,y-mT) &= \left(\sum_{n=-\infty}^{\infty}\delta(x-nT)\right)\left(\sum_{m=-\infty}^{\infty}\delta(y-mT)\right) &\quad (2.1)\\
&= \left(\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{in\omega_{x0}x}\right)\left(\frac{1}{T}\sum_{m=-\infty}^{\infty}e^{im\omega_{y0}y}\right) &\quad (2.2)\\
&= \frac{1}{T^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}e^{in\omega_{x0}x}e^{im\omega_{y0}y} &\quad (2.3)
\end{aligned}$$

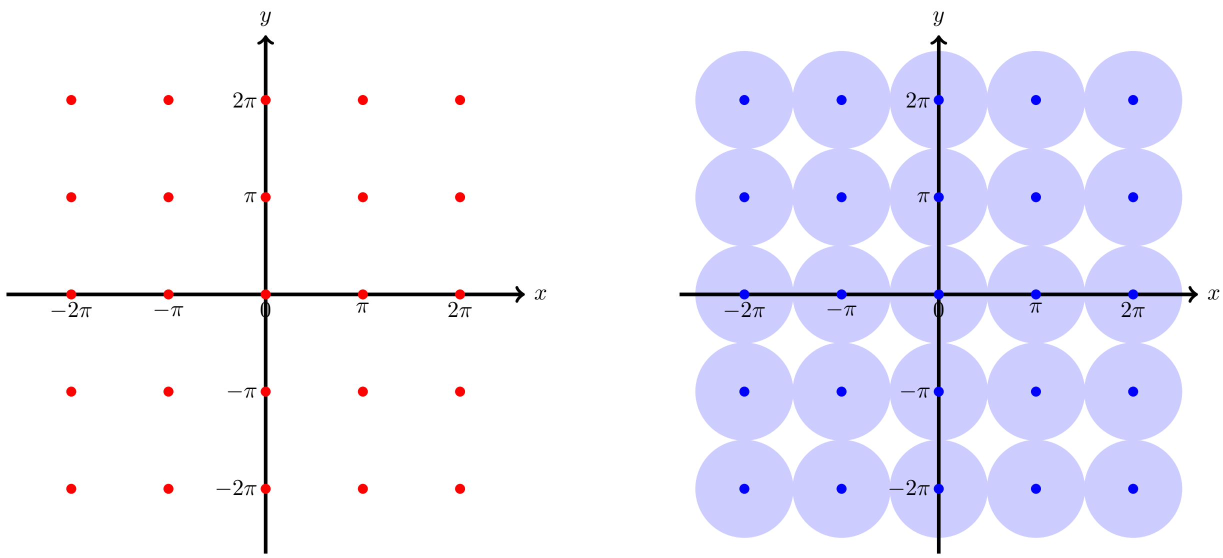
Hence, similar to the Fourier transform of 1-dimensional sampling function, we have

$$\begin{aligned}
\mathscr{F}\{\delta_T(x,y)\} &= \mathscr{F}\left\{\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(x-nT,y-mT)\right\} &\quad (2.4)\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{T^2}e^{in\omega_{x0}x}e^{im\omega_{y0}y}e^{-iw_x x}e^{-iw_y y}\,dx\,dy &\quad (2.5)\\
&= \frac{1}{T^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}e^{-i(\omega_x-n\omega_{x0})x}e^{-i(\omega_y-m\omega_{y0})y} &\quad (2.6)\\
&= \frac{1}{T^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(\omega_x-n\omega_{x0},\omega_y-m\omega_{y0}) &\quad (2.7)
\end{aligned}$$

Since the intervals along x and y axes are π, i.e. $\omega_{x0}=\omega=y0=\pi$, we have

$$\delta_\pi(\omega_x,\omega_y) = \frac{1}{T^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(\omega_x-n\pi,\omega_y-m\pi) \quad (2.8)$$

We can plot the equation (2.8) in the xy-coordinate as shown in Figure 1a. Note that each red point indicate a value of $\frac{1}{T^2}$.



(a) Inverse Fourier transform of sampling function

(b) Aliasing-free frequency interval

Figure 1: Frequency domain (2D)

2.2 Characteristics of Waves

For a signal $f(x, y)$, we have $f(x, y)\delta_T(x, y) = F(\omega_x, \omega_y) * \delta_\omega(\omega_x, \omega_y)$. Now, assuming that the shape of $F(\omega_x, \omega_y)$ a bivariate guassian distribution with variance $\sigma_{\omega_x} = f_x$ and $\sigma_{\omega_y} = f_y$ (for the purpose of visualization), the convolution creates many copies of that shape, centered at each blue point, as illustrated in 1b. If two circles overlap each other, aliasing will occur, making it impossible to recover the original signal. Therefore, recoverable waves must have the highest frequency in the range of aliasing free; that is, $f_x < \frac{\pi}{2}$ and $f_y < \frac{\pi}{2}$. Note that this intuition concurs with the sampling theorem.