Quiz #2: Supplementary Document

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1 Question #1

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and $\mathbf{A}$ be an undirected graph and an adjacency matrix associated with $\mathbf{G}$, respectively. Since there exists a walk[1] of length k between node i and j if and only if $\mathbf{A}^k_{ij} = 1$, one might be tempted to conclude that the number of triangles, i.e. cycle of length 3, in the graph is $\mathrm{Tr}\left(\mathbf{A}^k\right)$. However, it is, in fact, false because each cycle is counted 6 times instead of only once. Mathematically, let $e_1, e_2, e_3 \in \mathbf{E}$ be edges in a triangle, every permutation of $\{e_1, e_2, e_3\}$ is equivalent. Therefore, we need to account for it by multiplying $1/6$:

$$\#\text{triangle} = \frac{1}{6}\,\mathrm{Tr}\left(\mathbf{A}^k\right) \tag{1.1}$$

Incidentally, this formula cannot be generalized for cycles of length $n > 3$ because a cycle is defined as a trail, i.e. walk without repeated edges, in which only the first and last nodes are the same. It is only true for triangles because all closed walks of length 3 must be a trail. That is, given that u, A, B, u as a sequence of visited nodes, A and B can neither be node u nor the same node because $\mathbf{G}$ contains no self-loop. Therefore, all nodes are unique, implying that all edges are unique since $\mathbf{G}$ is not a multigraph.

2 Question #2

Suppose that $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is an undirected graph without self-loops and multiedges, we can verify that nodes $u, v, w \in \mathbf{V}$ form a triangle if it is a 3-clique. Therefore, $\mathbf{G}$ contains the maximum possible number of triangle if all combinations of $u, v, w \in \mathbf{V}$ form K_3

$$\#\text{max\_triangle} = \binom{|\mathbf{V}|}{3} \tag{2.1}$$

3 Question #3

According to Turan's theorem, a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ without a $(k+1)$-clique must satisfy the inequality

$$|E| \leq \frac{(k-1)|V|^2}{2k} \tag{3.1}$$

For $k = 2$, graph with $\left\lfloor \frac{(k-1)|V|^2}{2k} \right\rfloor$ edges is a complete bipartite graph such that the two sets of nodes have as equal size as possible. Therefore, we generate a graph of $n = 9$ nodes and 12 edges that contains no triangles by:

1. Divide nodes into 2 groups, $S_1 = \{1, \ldots, \lfloor n/2 \rfloor\}$ and $S_2 = \{\lceil n/2 \rceil, \ldots, n\}$.
2. Add an edge between a pair of nodes (u, v) where $u \in S_1$ and $v \in S_2$ until we cannot do it anymore or $|E| = m$.

[1] Here, I refer to *walk* instead of *path* because nodes and edges may be repeated in this manner of calculation.