Tower of Hanoi

Chinchuthakun Worameth

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1 Tower of Hanoi

The Tower of Hanoi consists of 3 pegs, refered as peg 1 to peg 3, and n disks, refered as disk 1 to disk n. The initial state of the problem refers to the state which every disks is in peg 1 such that disk i is above disk $i+1, \forall 1 \leq i \leq n$.

The objective of the problem is to move all disks from peg 1 to peg 3 while preserve the original configuration; that is disk i is above disk $i+1$. There are 3 rules concern moving disks:

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the pegs and placing it on top of another peg.
3. No higher indexed disk may be placed on top of a lower indexed disk.

2 Derivation of recurrence relation

Definition 1. Let $P = \{1, 2, 3\}$ and Let $P^2 = \{(s, t) : s, t \in P \text{ where } s \neq t\}$.

Definition 2. The function $f : \mathbb{N} \times P^2 \to \mathbb{N}$ defined as $f(n, (s, t))$ is the number of moves to move the first n disks from the top of peg s to peg t.

Theorem 1. $f(n, (s, t)) = f(n, (i, j)) \forall (s, t), (i, j) \in P^2$ such that $(s, t) \neq (i, j)$ Denote $f(n, (s, t)) = a_n$ for arbitrary $(s, t) \in P^2$.

Proof. Let $a_n = f(n, (0, 2))$. Consider $f(n, (s, t))$ for arbitrary $(s, t) \in P^2$. Rename peg s to peg 1 and peg t to peg 3. Thus, it can be concluded that $f(n, (s, t)) = f(n, (0, 2)) = a_n, \forall (s, t) \in P^2$. □

Now consider the initial state i.e. all disks is in peg 1. In order to move disk $n-1$ to peg 3, we need to move disk 0 to disk $n-2$ to somewhere else, but obviously not peg 3. (Otherwise, in order to move disk $n-1$ to peg 3, we have to move those disks to another peg again; leading to the unnecessary moves.) Therefore, the only possible option is peg 1. This process takes $f(n-1,0,1)$ moves.

After spending one more move to move disk $n-1$ to peg 3, we have to move disk 0 to disk $n-2$ to peg 3, on top of disk $n-1$; using another $f(n-1,1,2)$ moves. Thus, we have the recurrence relation $f(n,0,2) = f(n-1,0,1)+f(n-1,1,2)+1$ which can be simplified into $a_n = 2a_{n-1}+1$ from theorem 1.

Theorem 2. *$a_n = 2a_{n-1}+1$ where $a_1 = 1$ is the minimum number of moves required to solve Tower of Hanoi with n disks.*

Proof. Let $P(n)$ denotes the above statement. Proceed the proof with mathematical induction.

Consider when $n = 1$. Suppose that the minimum number of moves required to solve Tower of Hanoi with 1 disk is x. It is obvious that $x \geq 1$. Since a possible solution is to move the disk from peg 1 to peg 3; using only one move, it implies that $x \geq 1$. Thus, it can be conclude that $x = 1 = a_1$ i.e. $P(1)$ holds.

Suppose that $P(n)$ where $n \in \mathbb{N}$ holds i.e. a_n is the minimum number of moves required to solve Tower of Hanoi with n disks. By using the same argument used when in the derivation of recurrence relation, it can be concluded that the minimum number of moves required to solve Tower of Hanoi with $n+1$ disks is $a_{n+1} = 2a_n + 1$ i.e. $P(n+1)$ holds.

Thus, by using mathematical induction, it can be concluded that the statement holds for all $n \in \mathbb{N}$. □

3 Analysis of recurrence relation

Since $a_n = 2a_{n-1}+1$ is a linear non-homogeneous recurrence relation, closed form of recurrence relation is $a_n = a_n^{h(n)} + a_n^{p(n)}$ where $a_n^{h(n)}$ and $a_n^{p(n)}$ are homogeneous solution and particular solution respectively.

Consider the recurrence relation, $a_n = 2a_{n-1}$. The characteristic equation of the recurrence relation is $x^2 - 2x = 0$; giving $x = 0, 2$. Therefore, the general solution of the recurrence relation $a_n^{h(n)} = c_1(0^n) + c_2(2^n) = c_2 2^n$.

Next, Consider the recurrence relation,$a_n = 2a_{n-1}+1$. Assume that $a_n^{p(n)} = k \in \mathbb{R}$. Plug the value of $a_n^{p(n)}$ in the relation, we have $k = 2k+1$. Solving the linear equation, we have $k = -1$. Therefore, $a_n^{p(n)} = -1$.

Consider the general solution $a_n = a_n^{h(n)} + a_n^{p(n)} = c_2 2^n - 1$ with the base case $a_1 = 1$. We have $a_1 = 2c_2 - 1 = 1$; leading to $c_2 = 1$. Thus, it can be concluded that the solution of $a_n = 2a_{n-1}+1$ is $a_n = 2^n - 1$.

4 Implementation of the algorithm

The recursive implementation is quite straightforward. Note that the indices of pegs and disks are slightly different from above; they begin at 0 instead of 1.

Algorithm 1 Tower of Hanoi

function HANOI(d, n, s, t)
    if $n \geq 0$ **then**
        HANOI$(d, n-1, s, u)$
        move the disk from s to t
        HANOI$(d, n-1, u, t)$
    end if
end function

5 Restricted Tower of Hanoi

This section considers a variation of the original Tower of Hanoi; adding another rule concern moving disks. The disk cannot be moved from peg 0 to peg 2 and vice versa. In other words, the disk can be moved a peg to its adjacent pegs only. The algorithm for solving this problem is implemented in the attached file. Note that the explanation for the algorithm is not included in the report.