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To be filled in

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Preface

To be filled in!

Acknowledgment

To be filled in.

Summary and Conclusions

To be filled in!

Contents

Chapter 1

Introduction

Skrive litt om problemet, layouten og slikt.

The equation

$$\begin{aligned} \dot{y}(t) &= Ay(t) + F(t) \\ y(0) &= y_0 \end{aligned} \tag{1.1}$$

often makes an appearance when solving partial differential equations with numerical methods. The author has earlier observed how the heat equation, discretized with finite difference methods to be on the form of equation (1.1) can be solved with the use of the Krylov projection method(KPM) !!!!Cite rapporten min!!!!. This note will continue on the same track, with more focus on the wave equation, and energy preservation. It will also feature a comparison between Symplectic Lanzcos method(SLM) !!!Cite!!! and KPM. SLM is a projection technique that only works on Hamiltonian matrices. Due to this, SLM (claim to) preserve energy better than KPM.

Chapter 2

Explonation

There will here be a short explanation of all solvers, constants, outdata and expressions used in this text. Matlab notation is used where applicable. Fell free to go straight to next chapter, and use this as a reference later in the note.

!!!!!!!!!!!!!!!Skriv dette sammen til en fin tabell!!!!!!!!!!!!!!!

m	number of points in each spacial direction
n	Restart variable
k	number of points in time
restart	a boolean value. If restart == 1, Arnoldi or symplectinc lanzcos method will restart
convergence criterion	if restart == 1, restarting will commence until the change in the solution is less than the convergence criteria

Table 2.1: Some constants

iterations	number of restarts performed by Arnoldi or symplectic lanzcos method.
Time	time elapsed when solving the problem.
Error	difference between true solution, and estimated.
Energy error	Difference between the largest and smallest energy during the simulated time.
Error2	Difference in error between orthogonalised solution, and the non-orthogonalised solution.
Energy2	Difference in energy between orthogonalised solution, and the non-orthogonalised solution.

Table 2.2: Outdata

The table below shows how the matrix A was made. All test problems where implemented to

`wave`	A = 1/hs^2*gallery('poisson', m-2); A = [0,I;-A,0];
`semirandom`	A = rand(2*(m-2)^2); A = 0.5*[0,I;-I,0]*(A+A'+m^2*I) Keeps the matrix as long as m is the same, but creates a new when not.

Table 2.3: All solvers solve problems of the following form: $\dot{y} = Ay + F$, $y(0) = 0$, (F may depend on t, if need be, but not if the energy is to be kept constant). The matrix A is defined above for each equation. I is the identity matrix of size 2*(m-2)^2.

satisfy the condition $u(t,0,y) = u(t,1,y) = u(t,x,0) = u(t,x,1) = 0$. For `semirandom` all non-edge points where chosen randomly, and kept with the same conditions as for the A.

Three different integrators where used:

Trapezoidal rule
Forward Euler
Implicit midpoint rule

Table 2.4: list of Integrators

All problems on the form $\dot{y}(t) = Ay(t) \quad y(0) = y_0$ was transformed to the form $\dot{u}(t) = Au(t) + F \quad u(0) = 0$, with $u = y - y_0$ and $F = Ay_0$.

First we define the implicit midpoint rule:

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, \frac{1}{2}(y_i + y_{i+1})\right),$$

forward Euler

$$y_{i+1} = y_i + hf(t_i, y_i)$$

and the trapezoidal rule

$$y_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, y_{i+1})).$$

We are solving the system

$$\dot{y}(t) = f(t) = Ay(t) + F(t)$$

A problem with the midpoint rule is also that $F(t)$ is only known in certain points. This can

Implicit midpint rule	$U_i = (I - Ah_t) \backslash (U_{i-2} + h_t A U_{i-2} + 2h_t F_{i-1})$
Forward Euler	$U_i = U_{i-1} + h_t(AU_{i-1} + F_{i-1})$
trapezoidal rule	$U_i = (I - Ah_t/2) \backslash (U_{i-1} + h_t/2 A U_{i-1} + h_t/2(F_i + F_{i-1}))$
The exponential	$U_i = \exp(At_i)U_0$

Table 2.5: Iteration schemes

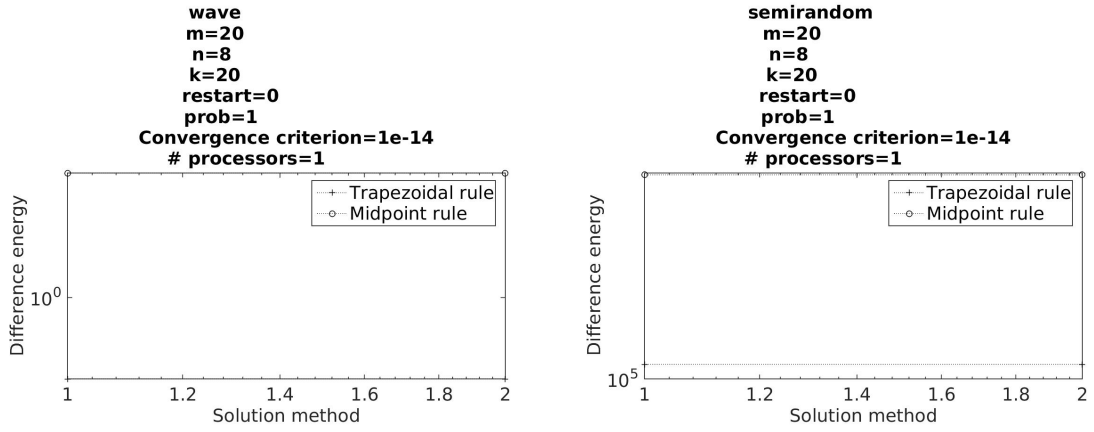
be worked around in two ways, either by approximating $F(t_n + h/2) = 1/2(F(t_n) + F(t_{n+1})$, or doubling the step length. The iteration scheme is given in the table below.

If F is constant, then trapezoidal rule is equivalent to the midpoint rule. Since we are mainly interested in the case where the energy is constant, this becomes a good approximation. The problem arises when using the restart with a projection method, because in this case F is never constant.

`compareintegrateenergy.jpg`

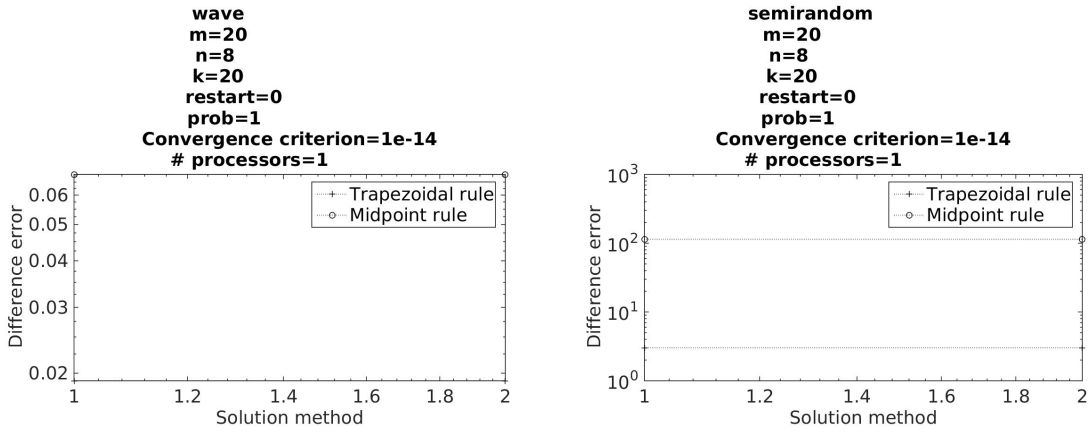
The figure below shows the difference between the different time-integration methods

As we can see from the figure above the trapezoidal rule and midpoint rule is about equally good at approximating the error, while forward Euler is just terrible.



(a) (b)

Figure 2.1: Figure of the difference in energy as a function of number of points in time. In the figure on the left, forward Euler is included, but because of it's poor convergence it is dropped from the rest of the figures in this chapter. The figure to the left shows that there is minimal difference between the trapezoidal rule and the midpoint rule.


wave
m=20
n=8
k=20
restart=0
prob=1
Convergence criterion=1e-14
processors=1
Trapezoidal rule
Midpoint rule
Difference error
Solution method
semirandom
m=20
n=8
k=20
restart=0
prob=1
Convergence criterion=1e-14
processors=1
Trapezoidal rule
Midpoint rule
Difference error
Solution method

(a) (b)

Figure 2.2

Chapter 3

Some results and explanation

We will now show some numerical experiments using a randomly generated Hamiltonian matrix(`semirandom`).

Chapter 4

Compare time integration methods

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!Citing til integrasjomsmetodene må ordnes!!!!!!!!!!!!!!!!!!!!!!!!!!

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!Steglengder må forklares i forklarings kapittelet !!!!!!!!!!!!!!!!!!

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!Skrive om at midpoint rule er litt teit side du trenger halve steglengder!!!!!!!!!!!

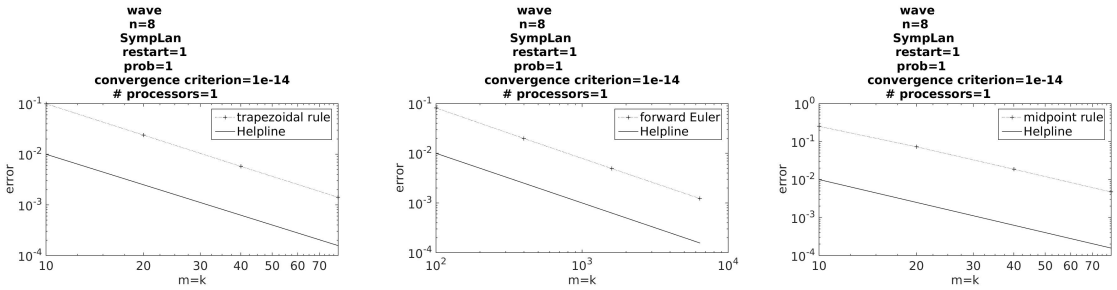
This section will be concerned with comparing the integrators. How well they estimate the error and energy will be the primary concern.

The relevant integrators are trapezoidal rule, forward euler, and midpoint rule. The definition of the different methods are given in the table below.

Convergence of the different methods are shown in figure 5.2

Trapezoidal rule	$y_{i+1} = y_i + h_t f\left(\frac{1}{2}(t_n + t_{n+1}), \frac{1}{2}(y_n + y_{n+1})\right)$	$y_{i+1} = (I - \frac{Ah_t}{2}) \backslash \left(y_i + \frac{h_t}{2}\left(Ay_i + (F_{i+1} + F_i)\right)\right)$
Forward Euler	$y_{i+1} = y_i + h_t f(t_i, y_i)$	$y_{i+1} = y_i + h_t\left(Ay_i + F_i\right)$
Midpoint rule	$y_{i+2} = y_i + h_t + f\left(t_{i+1}, \frac{1}{2}(y_i + y_{i+2})\right)$	$y_{i+2} = (I - Ah_t) \backslash \left(y_i + 2h_t\left(\frac{Ay_i}{2} + F_{i+1}\right)\right)$

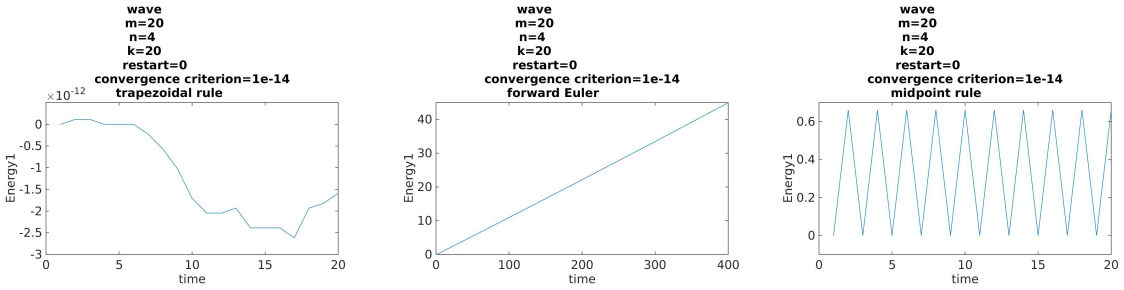
Table 4.1: SKRIV NOE HER! Husk å cite alle disse metodene! Er denne tabellen fin nok?



(a) Tekst her! (b) Tekst her! (c) Tekst her!

Figure 4.1: Figure of the convergence for the different integration methods. !!!!!!!Skrive inn det somstår på toppen av figuren her!!!!!

Figure 5.2 shows that all figures converge with the expected rate.



(a) (b) (c)

Figure 4.2: Figure of the difference in energy at a point in time. It is clear that only trapezoidal rule gives a suitable approximation of the error. Forward euler has an ever increasing energy, while the midpoint rule gives periodic energy. Skrive inn det som står på toppen av figuren her

After looking at figure 4.2 it is easy to conclude that trapezoidal rule outperforms the other two methods.

Chapter 5

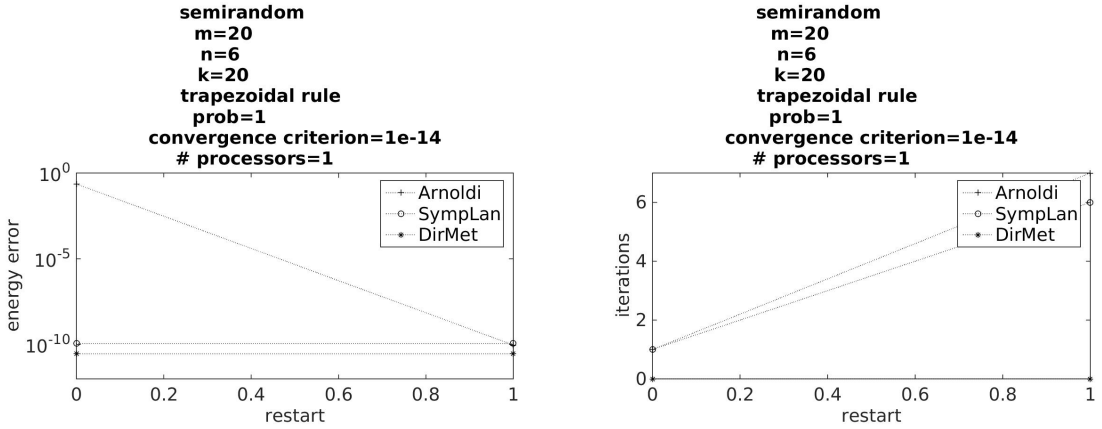
SLM energy preservation

It can be proved that a restart of SLM does not alter the energy. This chapter is devoted to showing that this is true.

The residual energy of the symplectic Lanzcos method is

$$\frac{1}{2}e_r^\top JAe_r + e_r^\top Jr_{n+1}e_{2n}^\top z \tag{5.1}$$

with $e_r = y - y_n$ (analytical solution minus approximated solution) and r_{n+1} is the residual vector given by the symplectic Lanzcos method.



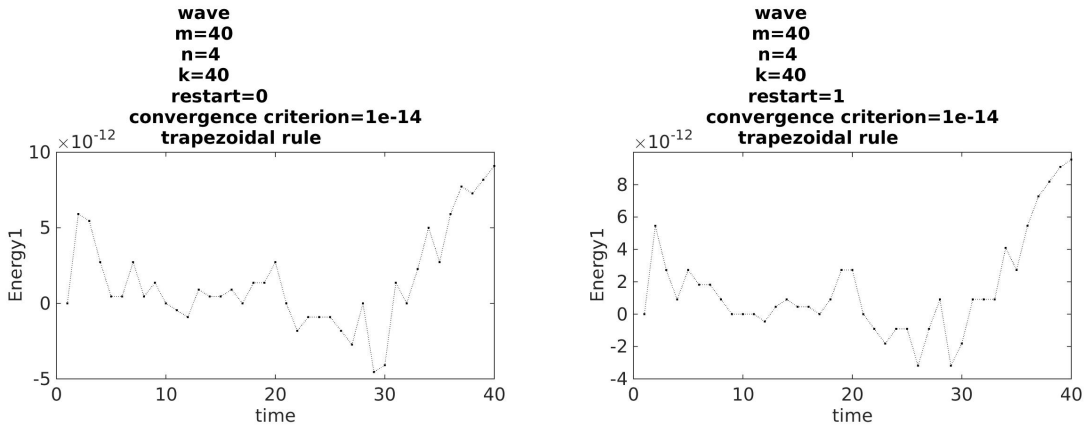
(a) Tekst her!

(b) Tekst her!

Figure 5.1: Tekst her!

The figures above implies that restarting the symplectic Lanzcos method does indeed not change the energy. But for Arnoldi's method it changes quite a bit.

The figures below shows the difference between the symplectic Lanczos method with and without restart.



(a) Without restart. Tekst her! (b) With restart. Tekst her! iter = 47

Figure 5.2: The figures shows the change in energy over time. Mer tekst

The figure above shows very little difference in energy, even with several restarts. !!!!!!!!!!!!!!!!!ARG, hva må jeg ha med for at dette skal bli mer overbevisende?!!!!!!!!!!!!!!!!!!!!!