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HYPOTHESIS TESTING

INTRODUCTION TO HYPOTHEIS TESTING

- Hypothesis testing is a statistical process of either rejecting or retaining a claim or belief or association related to a business context, product, service, processes, etc
- Hypothesis test consists of two complementary statements called **null hypothesis** and **alternative hypothesis**, and only one of them is true
- Hypothesis is an integral part of many predictive analytics techniques such as multiple linear regression and logistic regression

IMPORTANCE OF HYPOTHESIS TESTING

- Hypothesis is an integral part of many predictive analytics techniques such as multiple linear regression and logistic regression.
- It plays an important role in providing evidence of an association relationship between an outcome variable and predictor variables.

Few examples of such claims are listed below:

- Children who drink the health drink Complan (a health drink owned by the company Heinz in India) are likely to grow taller
- If you drink Horlicks, you can grow taller, stronger, and sharper (3 in 1).
- Beautiful people are likely to have girl child (Miller and Kanazawa, 2007). This is one of my favorite hypotheses since I have a daughter I can claim that I am good looking.

Exploratory Data Analysis and Confirmatory Data Analysis

- Data analysis in general can be classified as exploratory data analysis or confirmatory data analysis

- In exploratory data analysis, the idea is to look for new or previously unknown hypothesis or suggest hypotheses

- In the case of confirmatory data analysis, the objective is to test the validity of a hypothesis (confirm whether the hypothesis is true or not) using techniques such as hypothesis testing and regression.

HYPOTHESIS TESTING STEPS

1. Describe the hypothesis in words. Hypothesis is described using a population parameter (such as mean, standard deviation, proportion, etc.) about which a claim (hypothesis) is made. Few sample claims (hypothesis) are:
   – Average time spent by women using social media is more than men.
   – On average women upload more photos in social media than men.
   – Customers with more than one mobile handsets are more likely to churn.

2) Based on the claim made in step 1, define null and alternative hypotheses. Initially we believe that the null hypothesis is true. In general, null hypothesis means that there is no relationship between the two variables under consideration (for example, null hypothesis for the claim 'women use social media more than men' will be 'there is no relationship between gender and the average time spent in social media'). Null and alternative hypotheses are defined using a population parameter.

3) Identify the test statistic to be used for testing the validity of the null hypothesis. Test statistic will enable us to calculate the evidence in support of null hypothesis. The test statistic will depend on the probability distribution of the sampling distribution; for example, if the test is for mean value and the mean is calculated from a large sample and if the population standard deviation is known, then the sampling distribution will be a normal distribution and the test statistic will be a Z-statistic (standard normal statistic).

4. Decide the criteria for rejection and retention of null hypothesis. This is called **significance value** traditionally denoted by symbol α. The value of α will depend on the context and usually 0.1, 0.05, and 0.01 are used. Significance value α is the Type I error (discussed in section 6.4).

5. Calculate the *p*-value (probability value), which is nothing but the conditional probability of observing the test statistic value when the null hypothesis is true. In simple terms p-value is the evidence in support of the null hypothesis.

6. Take the decision to reject or retain the null hypothesis based on the *p*-value and significance value α. The null hypothesis is rejected when *p*-value is less than α and the null hypothesis is retained when *p*-value is greater than or equal to α.

Description of Hypothesis

Hypotheses are claims that are usually stated in simple words as listed below:

- Average annual salary of machine learning experts is different for males and females.
- On average people with Ph.D. in analytics earn more than people with Ph.D. in engineering.
- The average box-office collection of comedy genre movies is more than that of action movies.
- Average life of vegetarians is more than meat eaters.
- Proportion of married people defaulting on loan repayment is less than proportion of singles defaulting on loan repayment.

Null and Alternative Hypothesis

- **Null hypothesis**, usually denoted as H_0 (*H* zero and *H* naught), refers to the statement that there is no relationship or no difference between different groups with respect to the value of a population parameter.

- **Alternative hypothesis**, usually denoted as H_A (or H_1), is the complement of null hypothesis.

Hypothesis statement to definition of null and alternative hypothesis

S. No.	Hypothesis Description	Null and Alternative Hypothesis
1	Average annual salary of machine learning experts is different for males and females. (In this case, the null hypothesis is that there is no difference in male and female salary of machine learning experts)	H0: $\mu m = \mu f$ HA: $\mu m \neq \mu f$ μm and μf are average annual salary of male and female machine learning experts, respectively.
2	On average people with Ph.D. in analytics earn more than people with Ph.D. in engineering.	H0: $\mu a \leq \mu e$ HA: $\mu a > \mu e$ μa = Average annual salary of people with Ph.D. in analytics. μe = Average annual salary of people with Ph.D. in engineering. It is essential to have the equal sign in null hypothesis statement.

Hypothesis Testing

- ***Hypothesis test checks the validity of the null hypothesis based on the evidence from the sample.***
- ***At the beginning of the test, we assume that the null hypothesis is true.***
- ***Since the researcher may believe in alternative hypothesis, she/he may like to reject the null hypothesis.***
- ***However, in many cases (such as goodness of fit tests), we would like to retain or fail to reject the null hypothesis.***

Test Statistic

- Test statistic is the standardized difference between the estimated value of the parameter being tested calculated from the sample(s) and the hypothesis value (that is standardized difference between $\bar{x}$ and μ) in order to establish the evidence in support of the null hypothesis.

- It measures the standardized distance (measured in terms of number of standard deviations) between the value of the parameter estimated from the sample(s) and the value of the null hypothesis.

P - Value

- The *p*-value is the conditional probability of observing the statistic value when the null hypothesis is true.

- For example, consider the following hypothesis:

Average annual salary of machine learning experts is at least 100,000. The corresponding null hypothesis is H_0: $\mu_m \leq$ 100,000. Assume that estimated value of the salary from a sample is 1,10,000 (that is $\bar{X} = 1{,}10{,}000$ and assume that the standard deviation of population is known and standard error of the sampling distribution is 5000 (that is, $\sigma / \sqrt{n} = 5000$ where *n* is the sample size using which was calculated).

Hypothesis Testing

- The standardized distance between estimated salary from hypothesis salary is (1,10,000 – 1,00,000)/5000 = 2.

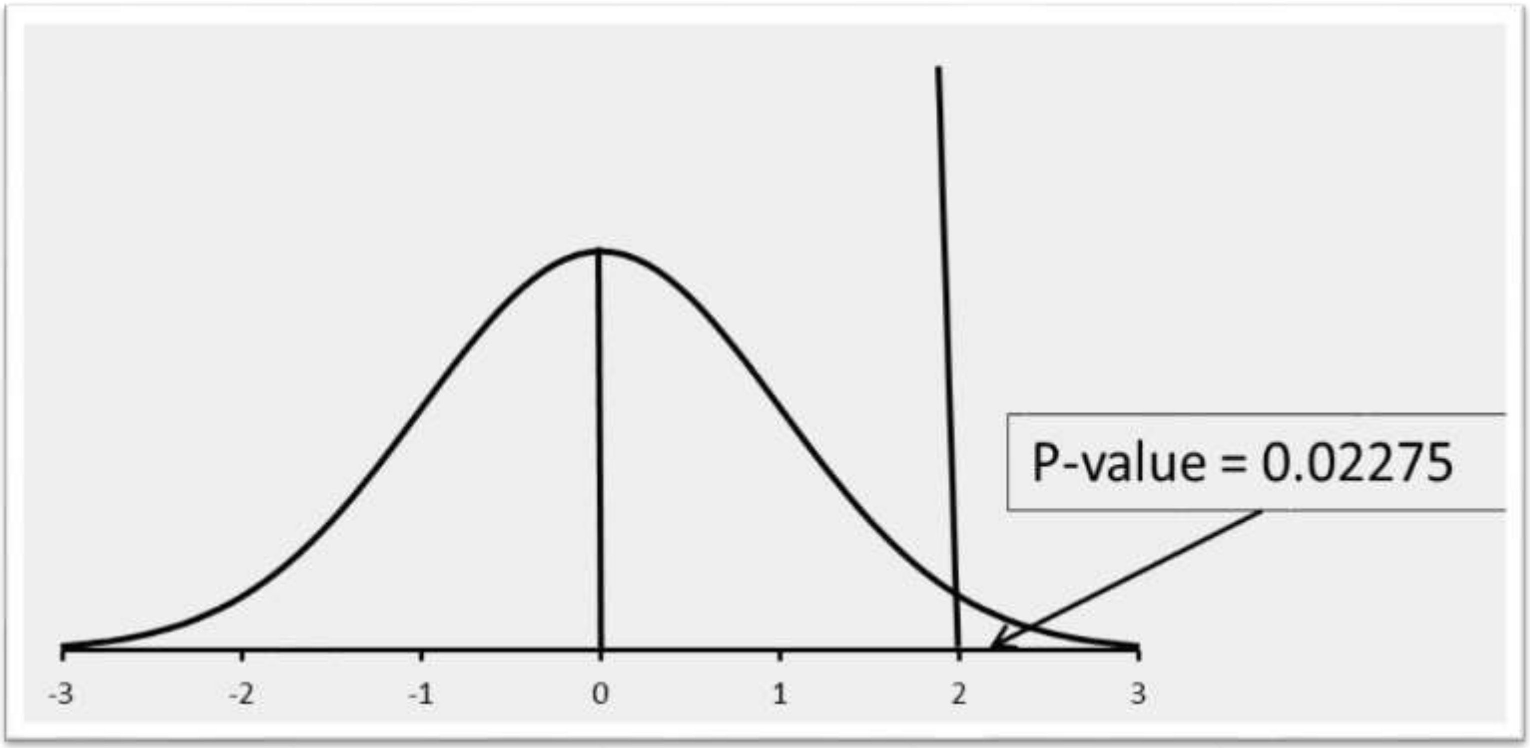
- That is, the standardized distance between estimated value and the hypothesis value is 2 and we can now find the probability of observing this statistic value from the sample if the null hypothesis is true (that is if $\mu_m \leq$ 100,000).

- A large standardized distance between the estimated value and the hypothesis value will result in a low *p*-value.

- Note that the value 2 is actually the value under a standard normal distribution since it is calculated from

$$\frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

Standard normal distribution and the *p*-value corresponding to Z = 2 are shown below:



Hypothesis Testing

- Probability of observing a value of 2 and higher from a standard normal distribution is 0.02275.

- That is, if the population mean is 1,00,000 and standard error of the sampling distribution is 5000 then probability of observing a sample mean greater than or equal to 1,10,000 is 0.02275.

- The value 0.02275 is the *p*-value, which is the evidence in support of the statement in the null hypothesis.

p-value = P(Observing test statistics value | null hypothesis is true)

P-value

Note that the p-value is a conditional probability. It is the conditional probability of observing the statistic value given that the null hypothesis is true. P-value is the evidence in support of null hypothesis.

Decision Criteria – Significance Value

- Significance level, usually denoted by α, is the criteria used for taking the decision regarding the null hypothesis (reject or retain) based on the calculated *p*-value.

- The significance value α is the maximum threshold for *p*-value.

- The decision to reject or retain will depend on whether the calculated *p*-value crosses the threshold value α or not

Decision making under hypothesis testing

Criteria	Decision
p-value < α	Reject the null hypothesis
p-value $\geq \alpha$	Retain (or fail to reject) the null hypothesis

- Usually α = 0.05 is used by researchers (recommended by Fisher, 1956); however, values such as 0.1, 0.02, and 0.01 are also frequently used.

- The value of α chosen is very low (0.05) for reason that we start the process of hypothesis testing with an assumption that null hypothesis is true

- The value of statistic for which the probability is α is called the **critical value**.

- The areas beyond the critical values are known as **rejection region**.

Significance Value (α)

The significance value α is the threshold conditional probability of rejecting a null hypothesis when it is true. It is the value of Type I error.

One-Tailed and Two-Tailed Test

Consider the following three hypotheses

1. Salary of machine learning experts on average is at least US $100,000.
2. Average waiting time at the London Heathrow airport security check is less than 30 minutes.
3. Average annual salaries of male and female MBA students are different at the time of graduation.

Example 1

Statement 1 – Salary of machine learning experts on average is at least US $100,000:

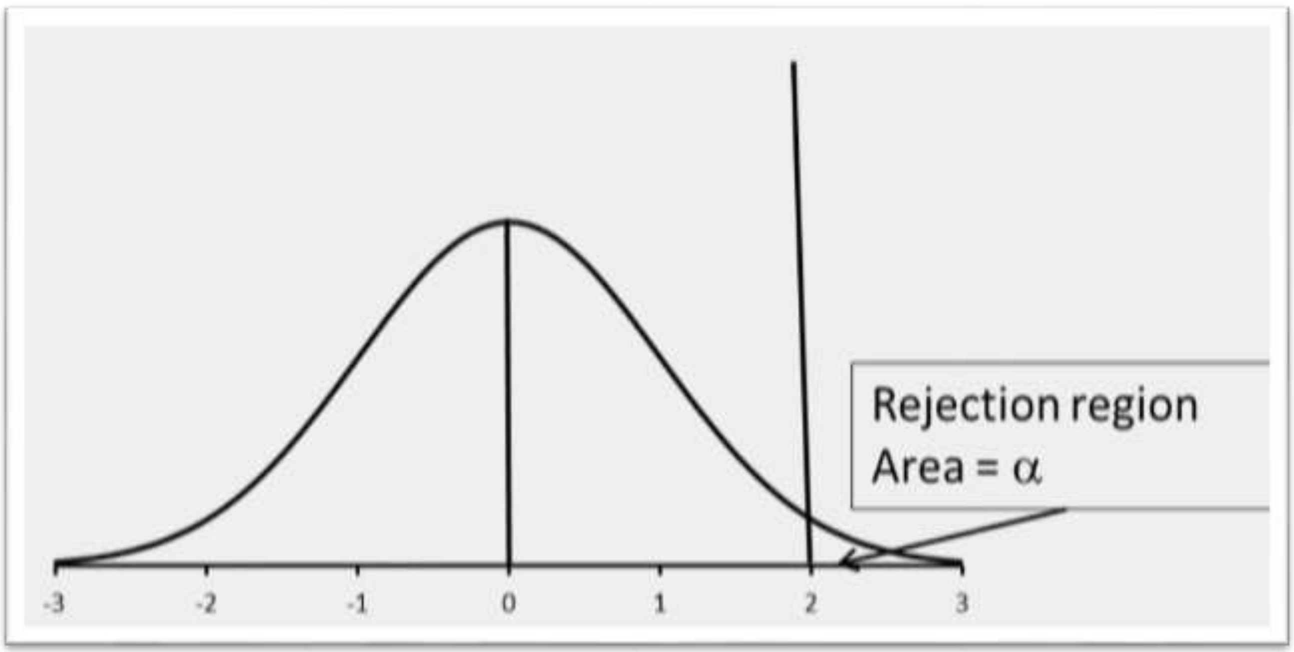
The null and alternative hypotheses in this case are given by

H_0: $\mu_m \leq 100{,}000$
H_A: $\mu_m > 100{,}000$

where μ_m is the average annual salary of machine learning experts. Note that the equality symbol is always part of the null hypothesis since we have to measure the difference between estimated value from the sample and the hypothesis value. In this case, reject or retain decision will depend on the direction of deviation of the estimated parameter from the sample from hypothesis value.

Solution

Below figure shows the rejection region on the right side of the distribution. Since the rejection region is only on one side this is a one-tailed test (right tailed test). Specifically, since the alternative hypothesis in this case is $\mu_m > 100{,}000$, this is called right-tailed test.



Example 2

- **Statement 2 – Average waiting time at the London Heathrow airport security check is less than 30 minutes:**

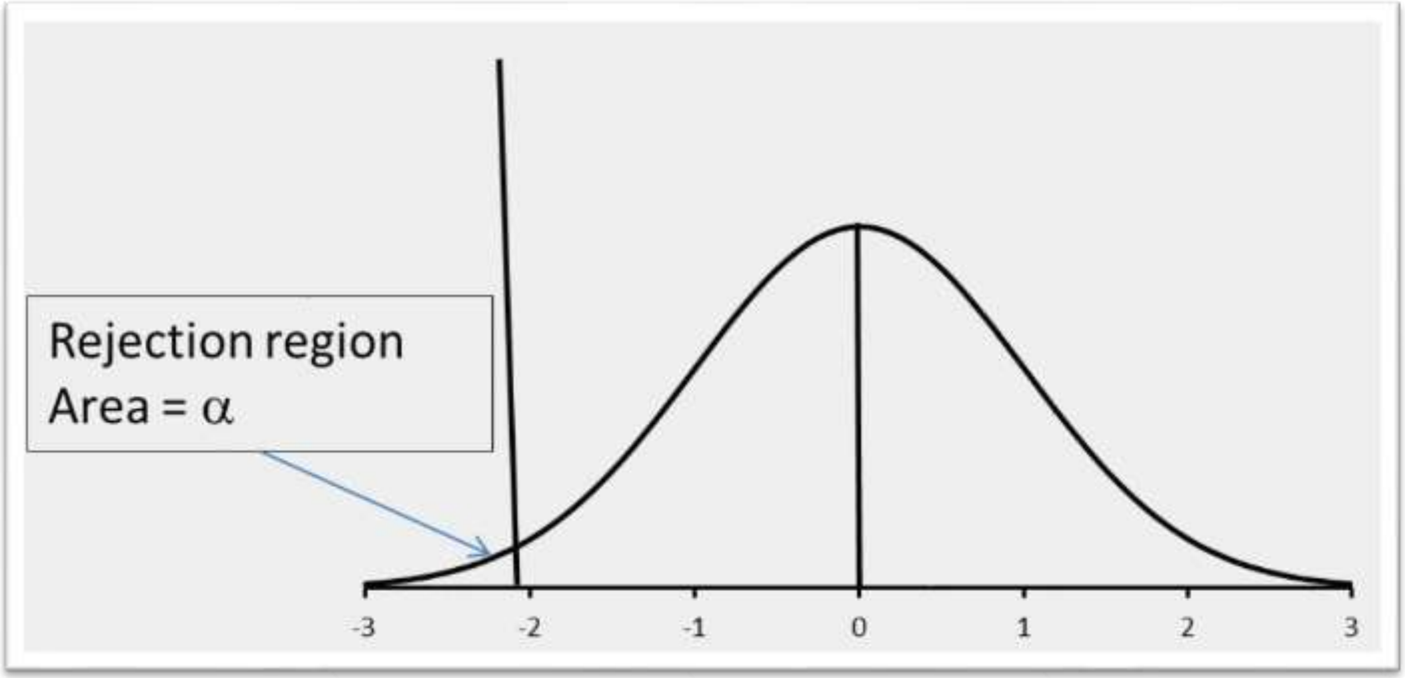
The null and alternative hypotheses in this case are given by

$$H_0: \mu_w \geq 30$$
$$H_A: \mu_w < 30$$

where μ_w is the average waiting time at London Heathrow security check. In this case, reject region will on the left side (known as left tailed test) of the distribution as shown in Figure

Solution

Rejection region in case of left-sided test



Example 3

Statement 3 – Average salary of male and female MBA students at graduation is different:

The null and alternative hypotheses in this case are given by

H_0: $\mu_m = \mu_f$

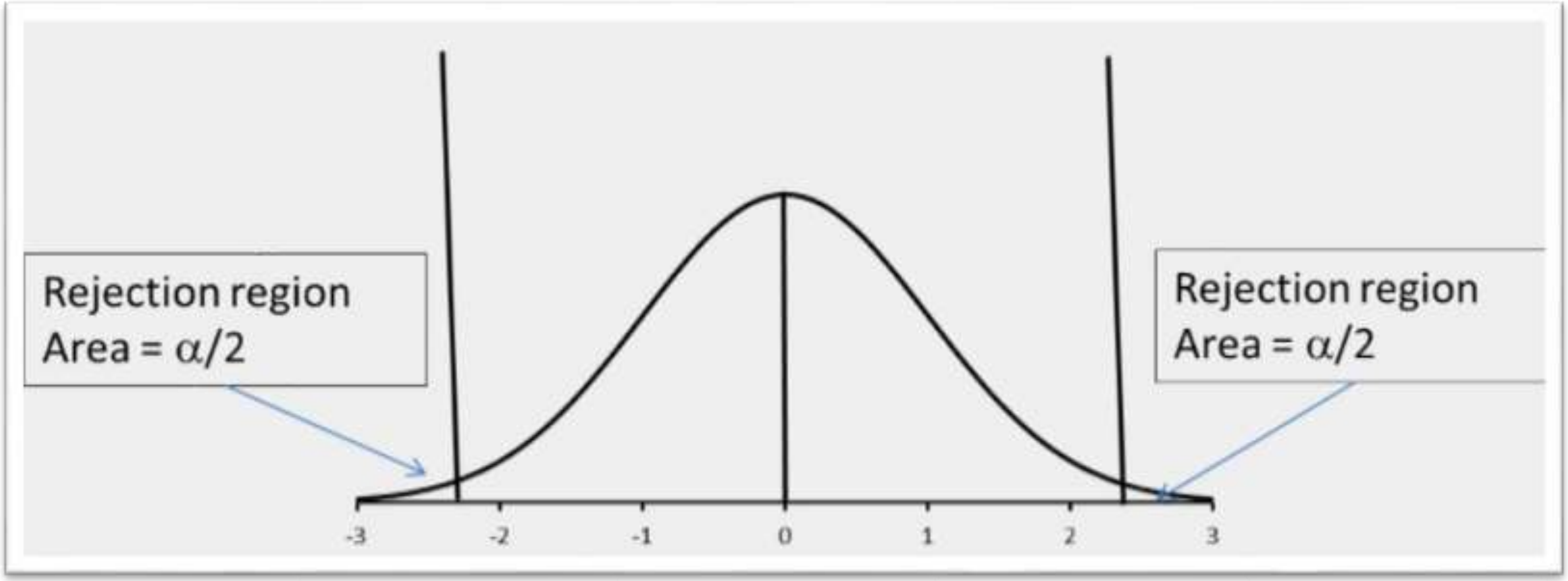
H_A: $\mu_m \neq \mu_f$

Where μ_m and μ_f are the average salaries of male and female MBA students, respectively, at the time of graduation.

In this case, the rejection region will be on either side of the distribution and if the significance level is α then the rejection region will be $\alpha/2$ on either side of the distribution. Since the rejection region is on either side of the distribution, it will be a two-tailed test.

Solution

Rejection region in case of two-tailed test



Decisions of Hypothesis Test

In hypothesis test we end up with the following two decisions:

- Reject null hypothesis.
- Fail to reject (or retain) null hypothesis.

Type I Error, Type II Error, and Power of the Hypothesis Test

TYPE I ERROR

Type I Error: Conditional probability of rejecting a null hypothesis when it is true is called Type I Error or False Positive (falsely believing claim made in alternative hypothesis is true). The significance value α is the value of Type I error.

Type I Error = $\alpha = P(\text{Rejecting null hypothesis} \mid H_0 \text{ is true})$

- Probability value (p-value) is the evidence against the null hypothesis whereas significance value α is the error based on repetitive sampling.
- Whereas the significance level α refers to incorrect rejection of null hypothesis when it is true under **repeated trials**.

TYPE II ERROR

- **Type II Error:** Conditional probability of failing to reject a null hypothesis (or retaining a null hypothesis) when the alternative hypothesis is true is called Type II Error or False Negative (falsely believing there is no relationship). Usually Type II error is denoted by the symbol β. Mathematically, Type II error can be defined as follows:

  Type II Error = β = P(Retain null hypothesis | H_0 is false)

Power of the Hypothesis Test

The value $(1 - \beta)$ is known as the power of hypothesis test. That is, the power of the test is given by

Power of the test = $1 - \beta = 1 - P$(Retain null hypothesis | H_0 is false)

Alternatively the power of test = 1 - β = P(Reject null hypothesis|H_0 is false)

Description of Type I error, Type II error, and the Power of Test

	Decision made about null hypothesis based on the hypothesis test	
Actual value of H_0	Reject H_0	Retain H_0
H_0 is true	Type I error P(Reject $H_0 \mid H_0$ = true) = α	Correct Decision P(Retain $H_0 \mid H_0$ = true) = $(1 - \alpha)$
H_0 is false	Correct Decision (Power of test) P(Reject $H_0 \mid H_0$ = false) = $1 - \beta$	Type II Error P(Retain $H_0 \mid H_0$ = false) = β

Hypothesis Testing for Population Mean with known Variance: *Z*-Test

- *Z*-test (also known as one-sample *Z*-test) is used when a claim (hypothesis) is made about the population parameter such as population mean or proportion when population variance is known.
- Since the hypothesis test is carried out with just one sample, this test is also known as **one-sample *Z*-test**.
- *Z*-test uses CLT to conduct a hypothesis test for population mean when the population variance is known; the test statistics for *Z*-test is given by

$$Z\text{-statistic} = \boxed{\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}}$$

- The critical value in this case will depend on the significance value α and whether it is a one-tailed or two-tailed test

Critical value for different values of α

	Approximate Critical Values		
α	Left-tailed test	Right-tailed test	Two-tailed test
0.1	–1.28	1.28	–1.64 and 1.64
0.05	–1.64	1.64	–1.96 and 1.96
0.01	–2.33	2.33	–2.58 and 2.58

Condition for rejection of null hypothesis H_0

Type of Test	Condition	Decision
Left-tailed test	Z-statistic < Critical value	Reject H_0
	Z-statistic $\geq$ Critical value	Retain H_0
Right-tailed test	Z-statistic > Critical value	Reject H_0
	Z-statistic $\leq$ Critical value	Retain H_0
Two-tailed test	|Z-statistic| > |Critical Value|	Reject H_0
	|Z-statistic| $\leq$ |Critical Value|	Retain H_0

Example

- An agency based out of Bangalore claimed that the average monthly disposable income of families living in Bangalore per month is greater than INR 4200 with a standard deviation of INR 3200. From a random sample of 40,000 families, the average disposable income was estimated as INR 4250. Assume that the population standard deviation is INR 3200. Conduct an appropriate hypothesis test at 95% confidence level ($\alpha = 0.05$) to check the validity of the claim by the agency.

Solution

Claim: Average disposable income is more than INR 4200.

Let μ and σ denote the mean and standard deviation in the population. The corresponding null and alternative hypotheses are

H_0: $\mu \leq 4200$

H_A: $\mu > 4200$

Since we know the population standard deviation, we can use the *Z*-test. The corresponding *Z*-statistic is given by

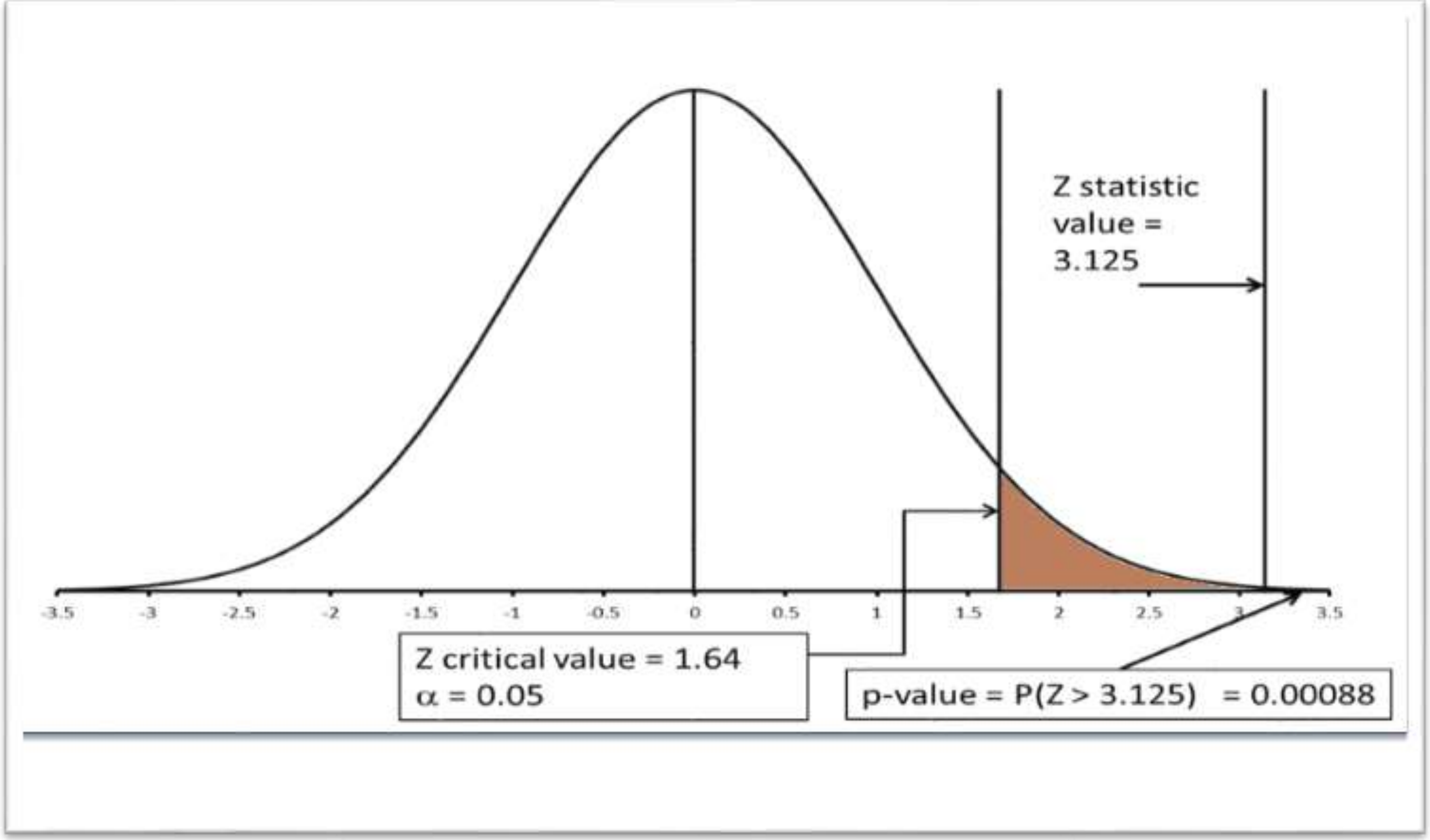
$$\boxed{Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{4250 - 4200}{3200 / \sqrt{40000}} = 3.125}$$

Solution Continued…

This is a right-tailed test.

Since the calculated Z-statistic value is greater than the Z-critical value, we reject the null hypothesis.

Critical value, *Z*-statistic value, and corresponding *p*-value.



Example

A passport office claims that the passport applications are processed within 30 days of submitting the application form and all necessary documents. Table 6.6 shows processing time of 40 passport applicants. The population standard deviation of the processing time is 12.5 days.

Conduct a hypothesis test at significance level α = 0.05 to verify the claim made by the passport office.

16	**16**	**30**	**37**	**25**	**22**	**19**	**35**	**27**	**32**
34	28	24	35	24	21	32	29	24	35
28	29	18	31	28	33	32	24	25	22
21	27	41	23	23	16	24	38	26	28

Solution

Null and alternative hypotheses in this case are given by

H_0: $\mu \geq 30$

H_A: $\mu < 30$

From the data in Table, the estimated sample mean is 27.05 days.

The standard deviation of the sampling distribution

The value of *Z*-statistic is given by

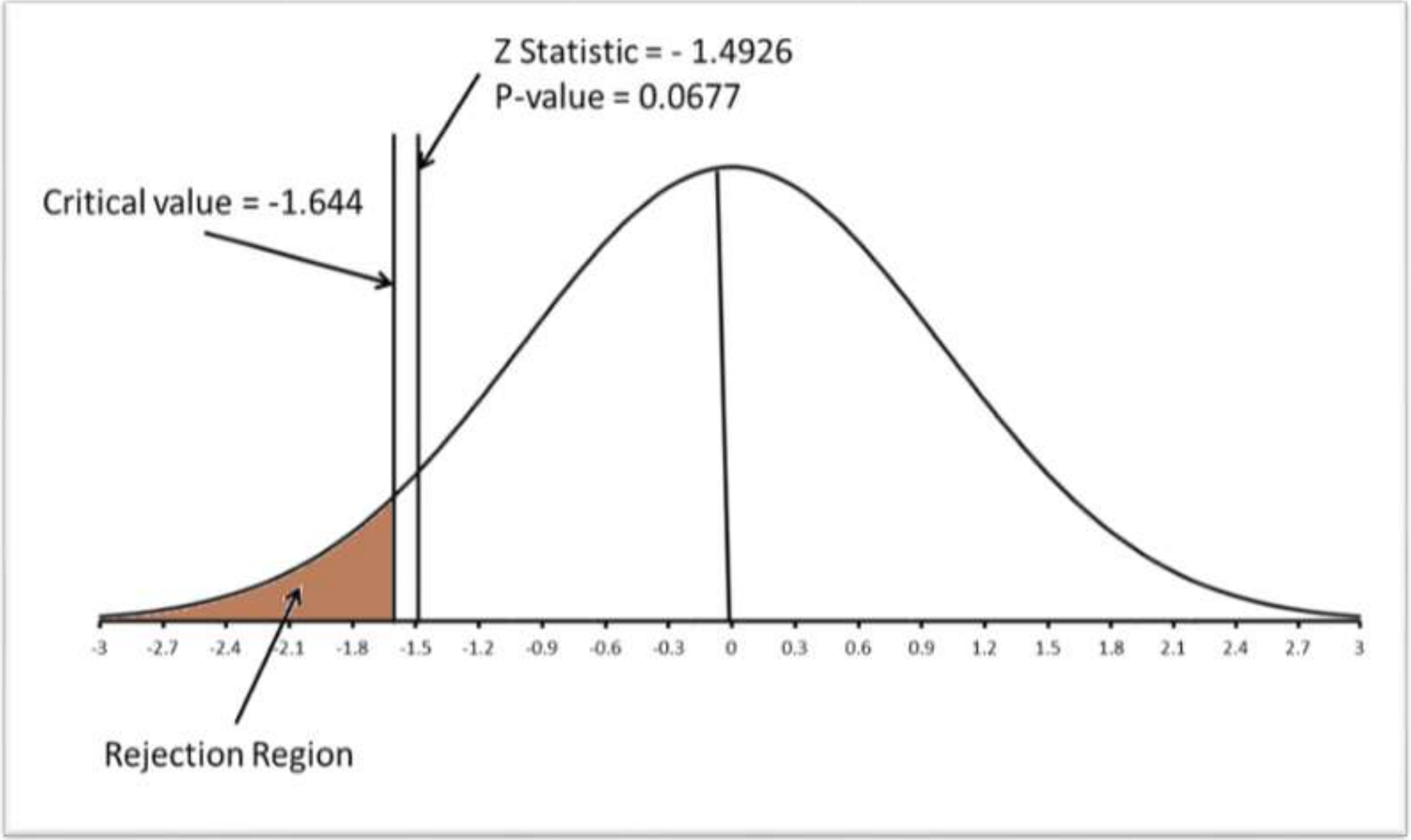
$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{27.05 - 30}{12.5 / \sqrt{40}} = -1.4926$$

$$\sigma / \sqrt{n} = 12.5 / \sqrt{40} = 1.9764$$

Solution Continued…

- The critical value of left-tailed test for $\alpha = 0.05$ is -1.644.
- Since the critical value is less than the *Z*-statistic value, we fail to reject the null hypothesis. The *p*-value for $Z = -1.4926$ is 0.06777 which is greater than the value of α.
- That is, there is no strong evidence against null hypothesis so we retain the null hypothesis, which is $\mu \geq 30$. Figure 6.6 shows the calculated *Z*-statistic value and the rejection region.

Left-tailed test



Example

According to the company IQ Research, the average Intelligence Quotient (IQ) of Indians is 82 derived based on a research carried out by Professor Richard Lynn, a British Professor of Psychology, using the data collected from 2002 to 2006 (Source: IQ Research). The population standard deviation of IQ is estimated as 11.03. Based on a sample of 100 people from India, the sample IQ was estimated as 84.

(a) Conduct an appropriate hypothesis test at α = 0.05 to validate the claim of IQ Research (that average IQ of Indians is 82).

(b) Ministry of education believes that the IQ is more than 82. If the actual IQ (population mean) of Indians is 86, calculate the Type II error and the power of hypothesis test.

Solution

a)Hypothesis test: It is given that μ = 82, σ = 11.03, n = 100, and
$\bar{X}$=84.

The null and alternative hypotheses in this case are:

H_0: μ = 82

H_A: $\mu \neq$ 82

Since the direction of alternative hypothesis is both ways, we have a two-tailed t-test. The test statistics is given by

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{84 - 82}{11.03 / \sqrt{100}} = 1.8132$$

Solution Continued…

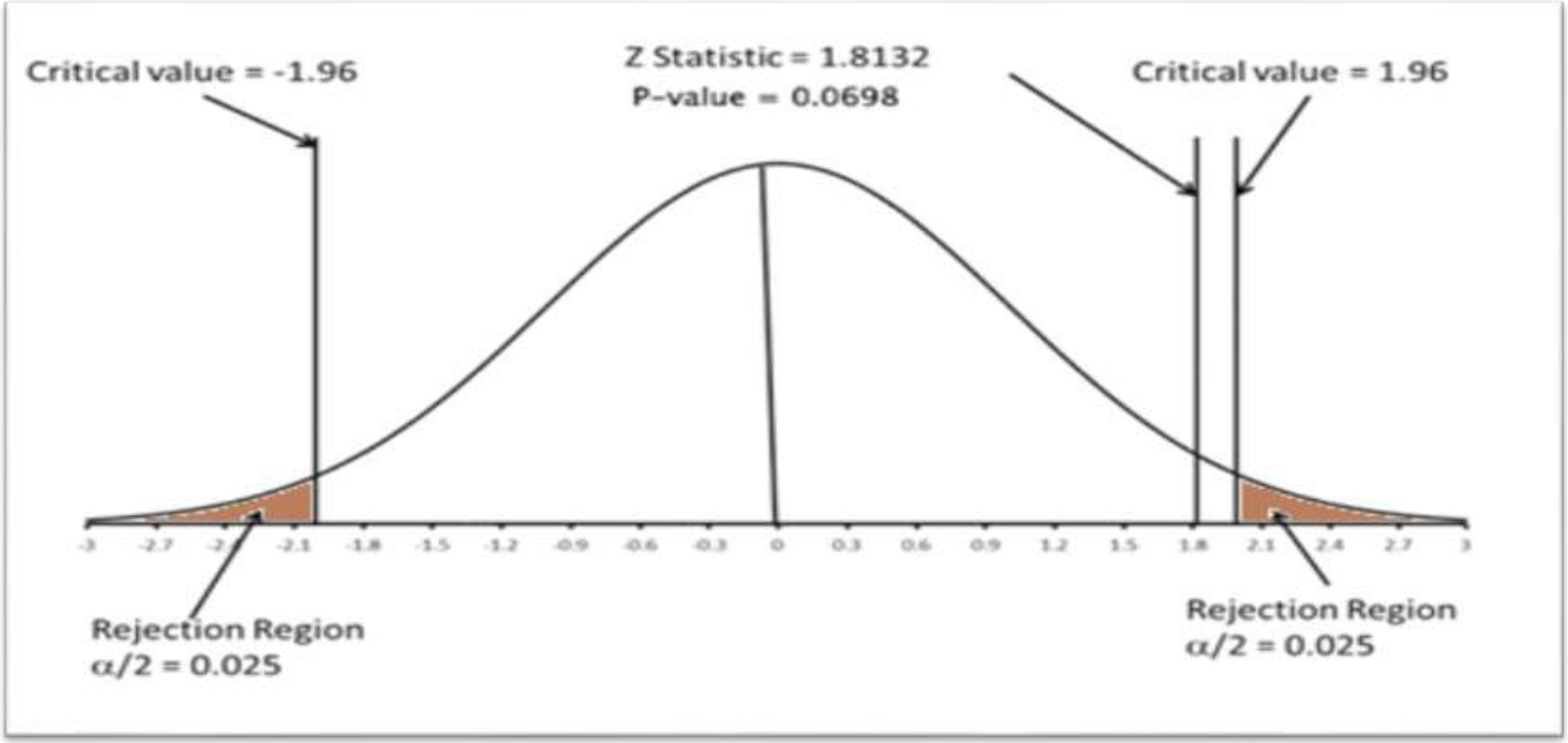
- For a two-tailed test, the critical values at $\alpha/2$ = 0.025 are -1.96 and 1.96.

- Since the calculated *Z*-statistic value is less than the critical value, we fail to reject the null hypothesis (retain the null hypothesis).

- Since the *Z*-statistic value is 1.8132 and falls on the right tail, we first calculate normal distribution beyond 1.8132 which is equal to 0.0348.

- Since this is a two-tailed test, the *p*-value is twice the area to the right side of the *Z*-statistic value, which is = 0.0698, that is the *p*-value in this case is 0.0698

Statistic, critical values, and the rejection region



Solution Continued…

(b) **Calculating Type II Error and Power of Test**: In this case, the null and alternative hypotheses are

$$H_0: \mu \leq 82$$
$$H_A: \mu > 82$$

Note that ministry of education believes that the average IQ is 86 (thus we have to carry out a right tailed test).

Type II error is the conditional probability of retaining a null hypothesis when it is false, that is P(retaining H_0 | H_0 is false).

Solution Continued…

The mean and standard deviation of Z-statistic in null hypothesis are 82 and 1.103, respectively. For the standard normal distribution the critical value for a right tailed test when $\alpha = 0.05$ is 1.644. The corresponding critical value for the normal distribution $N(82, 1.103)$ is

$$X_{\text{critical}} = \mu + Z_{\alpha} \times \sigma/\sqrt{n} = 82 + 1.644 \times 1.103 = 83.8133$$

That is, under normal distribution $N(82, 1.103)$, the region beyond 83.8133 is the rejection region (rejection of null hypothesis).

Solution Continued…

Now consider the normal distribution $N(86, 1.103)$. Area under this normal distribution may take values below 83.8133 which is region of retaining the null hypothesis, although the actual mean in this case is 86. Thus, we will be retaining the null hypothesis when it is incorrect resulting in Type II error, β.

For the normal distribution $N(86, 1.103)$, the probability of the variable taking value less than 83.8133 (the critical value) is given by

$$P(X \leq 83.8133) = \quad P\left(Z \leq \frac{83.8133 - 86}{1.103}\right) = 0.0237$$

That is, the Type II error $\beta = 0.0237$

The power of test, $1-\beta = 1 - 0.0237 = 0.9763$