LINEAR ALGEBRA HOMEWORK - 1

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1) $x + y + 2z = 8$
$-2y + 3z = 1$
$3x + 4z = 10$

$$[R \quad d]$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -2 & 3 & 1 \\ 3 & 0 & 4 & 10 \end{bmatrix}$$

Interchanging R_3 & R_1, $\begin{bmatrix} 3 & 0 & 4 & 10 \\ 0 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \end{bmatrix}$

$R_3' \to R_3 - \frac{R_1}{3}$, $\begin{bmatrix} 3 & 0 & 4 & 10 \\ 0 & -2 & 3 & 1 \\ 0 & 1 & (2-\frac{4}{3}) & (8-\frac{10}{3}) \end{bmatrix}$

$R_3'' \to R_3' + \frac{R_2}{2}$, $\begin{bmatrix} 3 & 0 & 4 & 10 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & (\frac{2}{3}+\frac{3}{2}) & (\frac{14}{3}+\frac{1}{2}) \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 4 & 10 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 13/6 & 31/6 \end{bmatrix}$$

$\Rightarrow \frac{13}{6} z = \frac{31}{6}$ $\quad \therefore \boxed{z = \frac{31}{13}}$

$-2y + 3z = 1$
$\Rightarrow y = \frac{3\left(\frac{31}{13}\right) - 1}{2}$ $\quad \boxed{\therefore y = \frac{40}{13}}$

$3x + 4z = 10$
$\Rightarrow x = \frac{10 - 4\left(\frac{31}{13}\right)}{3}$ $\quad \therefore \boxed{x = 2/13}$

2) $A^{-1} = ?$ $\quad A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 5 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

Gauss Jordan method, $[A \mid I]$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 3 & 5 & -1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_3' \to R_3 - 2R_1 \\ R_1' \to R_1/3 \\ R_2' \to R_2 - 2R_1' \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 5/3 & -1/3 & 1/3 & 0 & 0 \\ 0 & 5/3 & 5/3 & -2/3 & 1 & 0 \\ 0 & -9 & 2 & -2 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2'' \to \frac{3}{5} R_2' \\ R_1'' \to R_1' + R_2'(-1) \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & -1/3 & 1 & 0 \\ 0 & 1 & 1 & -2/5 & 3/5 & 0 \\ 0 & -9 & 2 & -2 & 0 & 1 \end{array}\right] \qquad R_3'' \to R_3 + 9R_2''$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & -1/3 & 1 & 0 \\ 0 & 1 & 1 & -2/5 & 3/5 & 0 \\ 0 & 0 & 11 & 28/5 & -27/5 & 1 \end{array}\right] \qquad \begin{array}{l} R_3''' \to R_3''/11 \\ R_2''' \to R_2'' - R_3''' \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & -1/3 & 1 & 0 \\ 0 & 1 & 0 & 6/55 & 6/55 & -1/11 \\ 0 & 0 & 1 & -28/55 & 27/55 & -1/11 \end{array}\right] \qquad R_1''' \to R_1'' - \frac{4}{3} R_3'''$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/55 & -1/55 & 2/11 \\ 0 & 1 & 0 & 6/55 & 6/55 & -1/11 \\ 0 & 0 & 1 & -28/55 & 27/55 & -1/11 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/55 & -1/55 & 2/11 \\ 6/55 & 6/55 & -1/11 \\ -28/55 & 27/55 & -1/11 \end{bmatrix}$$

3) $A^T = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 5 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $\qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4) Simplify $(R^T R)^T$

$$\Rightarrow \underline{\underline{(R^T R)^T}}$$

5) Null space of $A = \begin{bmatrix} 5 & -1 & 1 \\ 1 & -2 & 2 \\ 6 & 1 & 3 \\ 9 & 1 & 5 \end{bmatrix}$

$R_1' \to R_1/5$
$R_2' \to R_2 - R_1'$
$R_3' \to R_3 - 6R_1'$
$R_4' \to R_4 - 9R_1'$

$$\Rightarrow A = \begin{bmatrix} 1 & -1/5 & 1/5 \\ 0 & -9/5 & 9/5 \\ 0 & 11/5 & 9/5 \\ 0 & 14/5 & 16/5 \end{bmatrix}$$

$\Rightarrow R_2'' \to -\frac{5}{9} R_2'$
$R_3'' \to R_3' - \frac{11}{5} R_2''$
$R_4'' \to R_4' - \frac{14}{5} R_2''$

$$\Rightarrow A = \begin{bmatrix} 1 & -1/5 & 1/5 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

$\Rightarrow R_3''' \to R_3''/4, \quad R_4''' \to R_4'' - 6R_3''', \quad R_1''' \to R_1'' + \frac{1}{5} R_2''$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2''' \to R_3 + R_2$$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Here, the rows are independent.

Hence null space of $A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6) $A = \begin{bmatrix} 5 & -1 & 1 & 3 \\ 6 & 1 & 3 & 4 \\ 9 & 1 & 5 & 2 \end{bmatrix}$

$R_1' \to R_1/5$; $R_2' \to R_2 - 6R_1'$, $R_3' \to R_3 - 9R_1'$

$\Rightarrow A = \begin{bmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 11/5 & 9/5 & 2/5 \\ 0 & 14/5 & 16/5 & -17/5 \end{bmatrix}$

$R_2'' \to \frac{5}{11} R_2$ $\quad R_3'' \to R_2''\left(\frac{-14}{5}\right) + R_3'$

$\Rightarrow A = \begin{bmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 1 & 9/11 & 2/11 \\ 0 & 0 & 10/11 & -43/11 \end{bmatrix}$ $\quad R_3''' \to R_3''\left(\frac{11}{10}\right)$

$A = \begin{bmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 1 & 9/11 & 2/11 \\ 0 & 0 & 1 & -43/10 \end{bmatrix}$

$R_2''' \to R_2'' + R_3'''\left(\frac{-9}{11}\right)$ & $R_1'' \to R_1' + \frac{R_2}{5}$

$A = \begin{bmatrix} 1 & 0 & 1/5 & 67/50 \\ 0 & 1 & 0 & 37/10 \\ 0 & 0 & 1 & -43/10 \end{bmatrix}$

$R_1''' \to R_1'' - \frac{1}{5} R_3'''$ $\quad A = \begin{bmatrix} 1 & 0 & 0 & 11/5 \\ 0 & 1 & 0 & 37/10 \\ 0 & 0 & 1 & -43/10 \end{bmatrix}_{3 \times 4}$

There are 3 pivots columns $\to \therefore$ Rank = 3

$r = m = 3$ and $r < n$ $(n = 4)$ $\quad \therefore Ax = b$ has ∞ solutions

nullspace of $A = \begin{bmatrix} 11/5 \\ 37/10 \\ -43/10 \\ 1 \end{bmatrix}$ & has 3 free variables

7) $Ax = b$

when rank of $A = r$

size of $A = m \times n$

(a) when rank is full, $R = [I]$

$\Rightarrow r = m$ and $r = n$

$\therefore$ There are 3 pivots and each variable can be solved for.

→ There are no free variables.

Hence, $Ax = b$ has 1 solution.

(b) when $r < m$, $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

(i) if $r = n$, thin and long matrix A.

There exists rows with just zeros & b may exist

$\therefore$ $Ax = b$ has one solution or no solution

(c) when ~~if~~ $r < n$ & $r < m$, $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ There are zero rows & b exists

$\therefore$ $Ax = b$ has ∞ solutions or 0 solutions because $(n-r)$ unknowns can take arbitrary values