GOSSIP: decomposition software for the Global Optimization of nonconvex two-Stage Stochastic mixed-Integer nonlinear Programs

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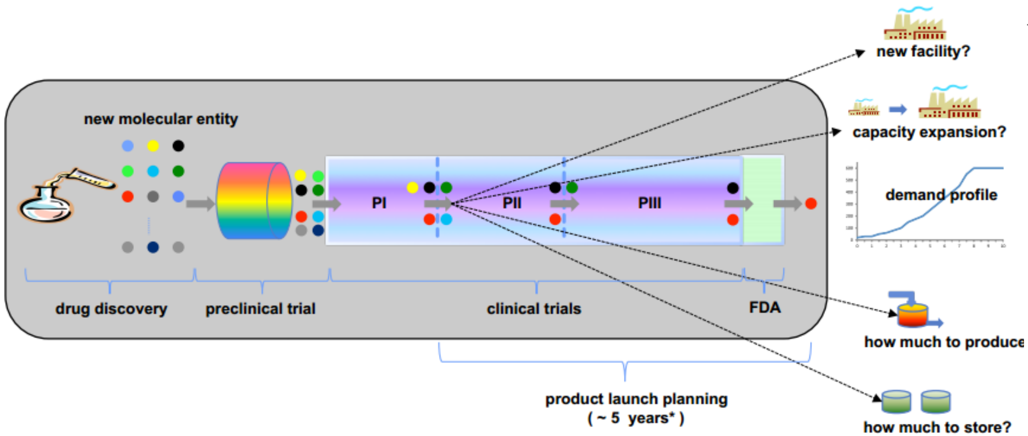
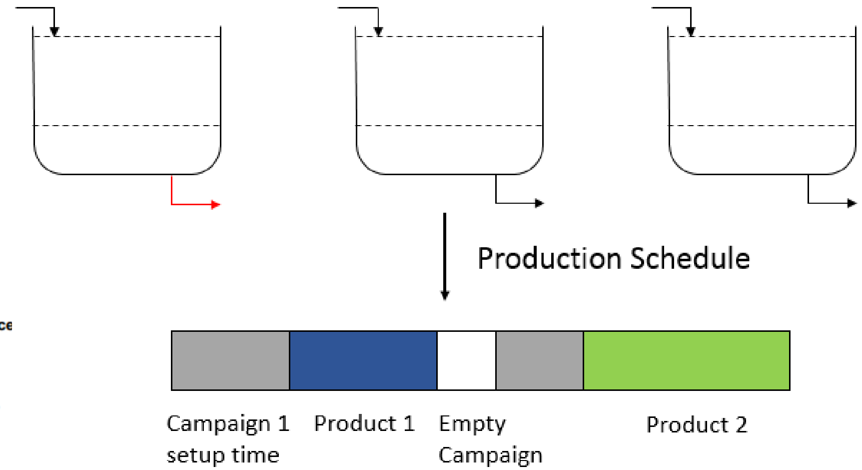
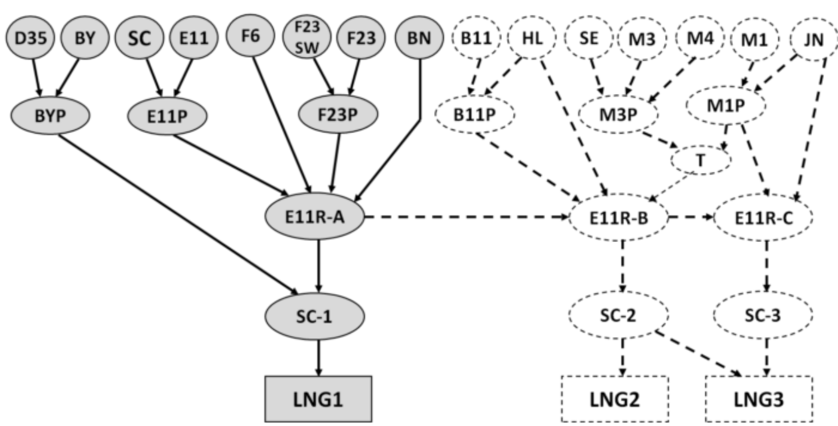
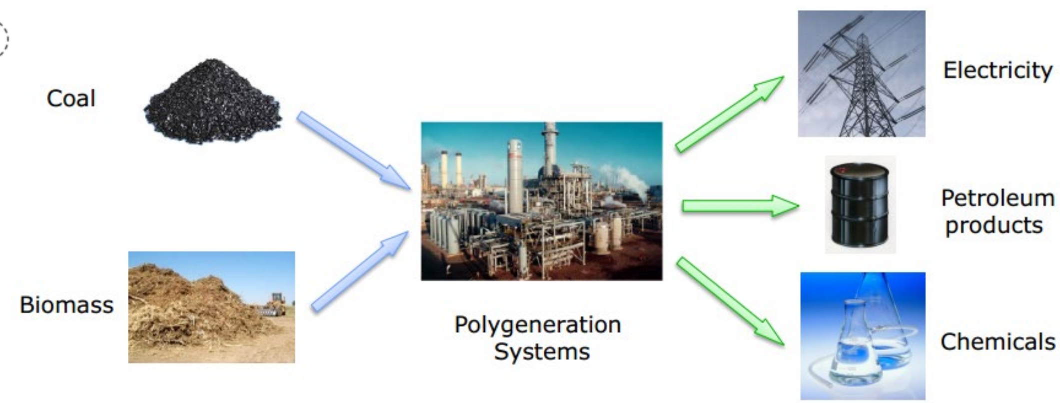
Motivation

Engineering Applications









Li, X. et al., AIChE Journal, 2011.

Sundaramoorthy, A. et al., Ind. Eng. Chem. Res., 2012.

Rebennack, S. et al., Comput. Chem. Eng., 2011.

Li, X. et al., Ind. Eng. Chem. Res., 2011.

Two-Stage Stochastic MINLP Framework

- Scenario-based formulation

Probability of scenario h

$$\min_{x_1,\cdots,x_s,y,z} \sum_{h=1}^{s} p_h f_h(x_h, y, z)$$ Minimize the expected cost

$$\text{s.t.} \quad g_h(x_h, y, z) \le 0, \ \forall h \in \{1, \cdots, s\},$$ Constraints for all scenarios

$$x_h \in X_h \subset \{0,1\}^{n_{x_b}} \times \mathbb{R}^{n_{x_c}}, \ \forall h \in \{1, \cdots, s\},$$ Second-stage decisions for all scenarios

$$y \in Y \subset \{0,1\}^{n_y}, \ z \in Z \subset \mathbb{R}^{n_z}.$$ First-stage decisions

- The solution times of algorithms implemented in commercial general-purpose global optimization software are worst-case exponential in the number of scenarios

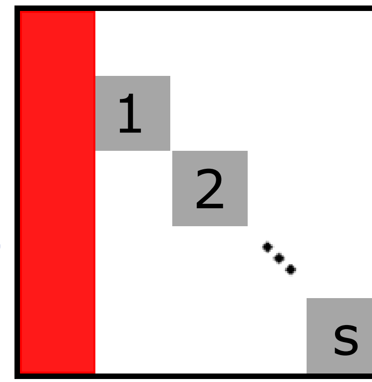
Decomposition Strategy #1
Complicating Variables Viewpoint

$$\min_{x_1,\cdots,x_s,y,z} \sum_{h=1}^{s} p_h f_h(x_h, y, z)$$
$$\text{s.t.} \quad g_h(x_h, y, z) \le 0, \ \forall h \in \{1,\cdots,s\},$$
$$x_h \in X_h, \ \forall h \in \{1,\cdots,s\},$$
$$y \in Y, \ z \in Z.$$

Complicating variables

Variable

Scenario (Constraint)



• • • • • •

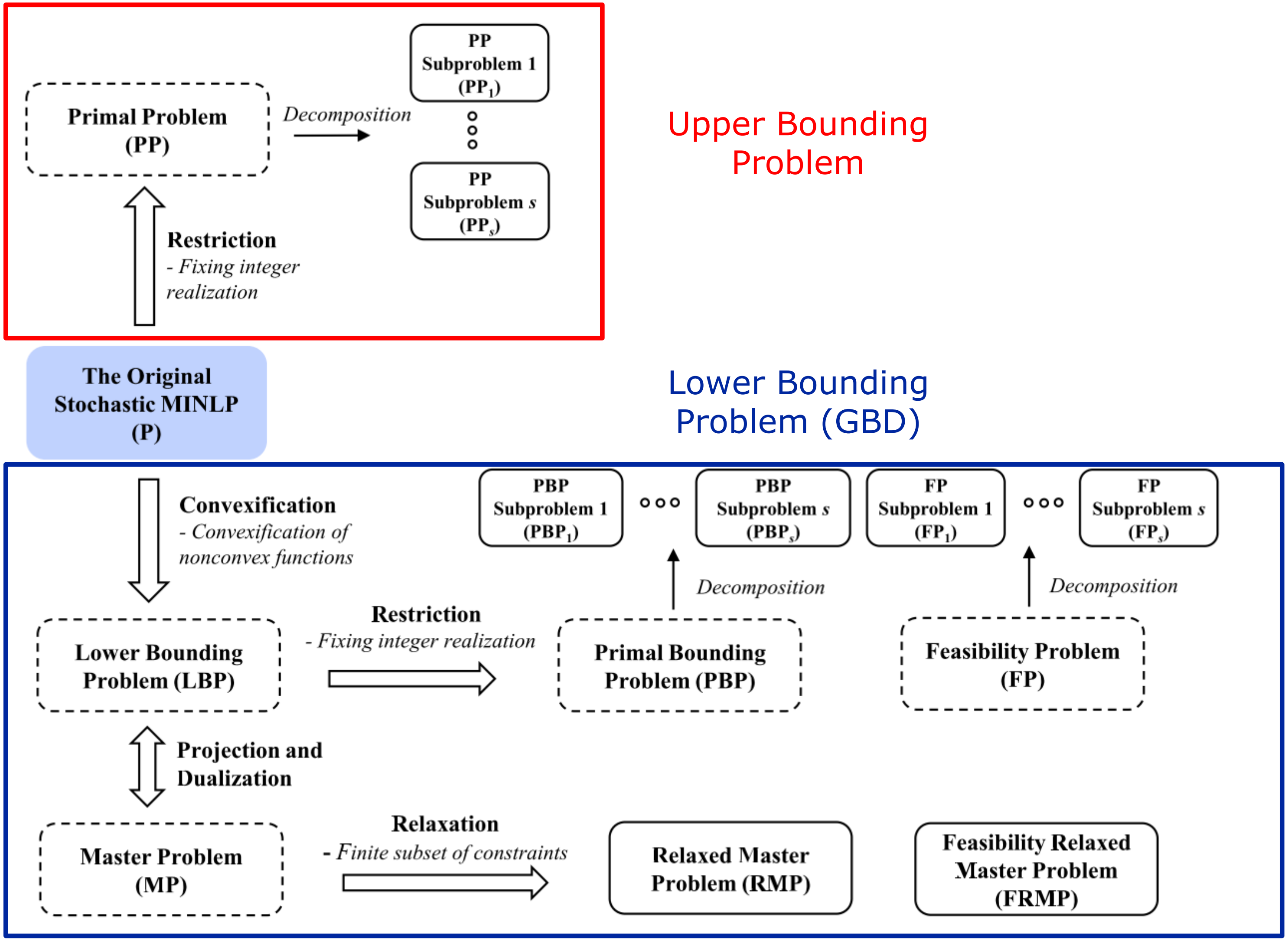
$$\min_{x_1} \ p_1 f_1(x_1, y, z)$$
$$\text{s.t.} \ g_1(x_1, y, z) \le 0,$$
$$x_1 \in X_1.$$

$$\min_{x_h} \ p_h f_h(x_h, y, z)$$
$$\text{s.t.} \ g_h(x_h, y, z) \le 0,$$
$$x_h \in X_h.$$

$$\min_{x_s} \ p_s f_s(x_s, y, z)$$
$$\text{s.t.} \ g_s(x_s, y, z) \le 0,$$
$$x_s \in X_s.$$

Decomposition Algorithms

Nonconvex Generalized Benders Decomposition (NGBD)



Li, X. et al., J. Optimiz. Theory. App., 2011.

Decomposition Algorithms

Nonconvex Generalized Benders Decomposition (NGBD)

Original Problem: Nonconvex MINLP

$$
\begin{aligned}
\min_{x_1,\cdots,x_s,y} \quad & \sum_{h=1}^{s} p_h f_h(x_h, y) \\
\text{s.t.} \quad & g_h(x_h, y) \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& x_h \in X_h \subset \{0,1\}^{n_{x_b}} \times \mathbb{R}^{n_{x_c}}, \ \forall h \in \{1,\cdots,s\}, \\
& y \in Y \subset \{0,1\}^{n_y}.
\end{aligned}
$$

Convexification

Lower Bounding Problem: MILP/Convex MINLP

Solve using GBD!

$$
\begin{aligned}
\min_{\substack{x_1,\cdots,x_s,\\ q_1,\cdots,q_s,y}} \quad & \sum_{h=1}^{s} p_h \left[f_h^{\text{cv}}(x_h, q_h) + c_{y,h}^{\text{T}} y \right] \\
\text{s.t.} \quad & g_h^{\text{cv}}(x_h, q_h) + B_{y,h} y \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& (x_h, q_h) \in \text{conv}(X_h) \times Q_h, \ \forall h \in \{1,\cdots,s\}, \\
& y \in \{0,1\}^{n_y}.
\end{aligned}
$$

Li, X. et al., J. Optimiz. Theory. App., 2011.

Decomposition Strategy #2
Complicating Constraints Viewpoint

Formulation

$$\min_{x_1,\cdots,x_s,y,z} \sum_{h=1}^{s} p_h f_h(x_h, y, z)$$

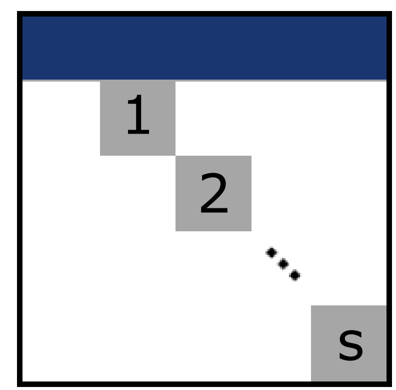
$$\text{s.t.} \quad g_h(x_h, y, z) \le 0, \ \forall h \in \{1,\cdots,s\},$$

$$x_h \in X_h, \ \forall h \in \{1,\cdots,s\},$$

$$y \in Y, \ z \in Z.$$

Scenario
(Variable)

Constraint



Equivalent Formulation

$$\min_{\substack{x_1,\cdots,x_s,\\ y_1,\cdots,y_s,\\ z_1,\cdots,z_s}} \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h)$$

$$\text{s.t.} \quad g_h(x_h, y_h, z_h) \le 0, \ \forall h \in \{1,\cdots,s\},$$

Complicating constraints

$$y_h - y_{h+1} = 0, \ \forall h \in \{1,\cdots,s-1\},$$

$$z_h - z_{h+1} = 0, \ \forall h \in \{1,\cdots,s-1\},$$

$$x_h \in X_h, \ y_h \in Y, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}.$$

Decomposition Strategy #2
Complicating Constraints Viewpoint

$$\min_{x_1,y_1,z_1} \ p_1 f_1(x_1,y_1,z_1) \quad \text{s.t. } g_1(x_1,y_1,z_1) \le 0, \quad x_1 \in X_1, y_1 \in Y, z_1 \in Z.$$

$$\cdots$$

$$\min_{x_h,y_h,z_h} \ p_h f_h(x_h,y_h,z_h) \quad \text{s.t. } g_h(x_h,y_h,z_h) \le 0, \quad x_h \in X_h, y_h \in Y, z_h \in Z.$$

$$\cdots$$

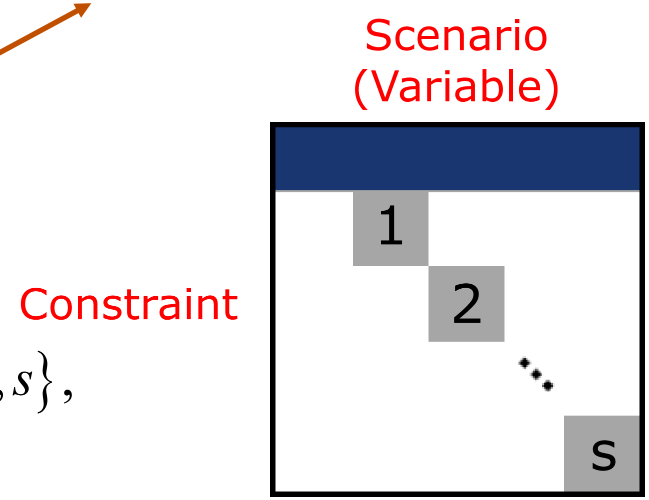
$$\min_{x_s,y_s,z_s} \ p_s f_s(x_s,y_s,z_s) \quad \text{s.t. } g_s(x_s,y_s,z_s) \le 0, \quad x_s \in X_s, y_s \in Y, z_s \in Z.$$

Equivalent Formulation

$$\min_{\substack{x_1,\cdots,x_s,\\ y_1,\cdots,y_s,\\ z_1,\cdots,z_s}} \ \sum_{h=1}^{s} p_h f_h(x_h,y_h,z_h)$$

$$\text{s.t.} \quad g_h(x_h,y_h,z_h) \le 0, \ \forall h \in \{1,\cdots,s\},$$

$$x_h \in X_h, \ y_h \in Y, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}.$$



Decomposition Algorithms

Lagrangian Relaxation (LR)

$$\min_{\substack{x_1,\cdots,x_s,\\ y_1,\cdots,y_s,\\ z_1,\cdots,z_s}} \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h)$$

$$\begin{aligned}
\text{s.t.}\quad & g_h(x_h, y_h, z_h) \le 0,\ \forall h \in \{1,\cdots,s\}, \\
& \left.\begin{aligned} & y_h - y_{h+1} = 0,\ \forall h \in \{1,\cdots,s-1\}, \\ & z_h - z_{h+1} = 0,\ \forall h \in \{1,\cdots,s-1\}, \end{aligned}\right\} \text{Non-anticipativity constraints} \\
& x_h \in X_h,\ y_h \in Y,\ z_h \in Z,\ \forall h \in \{1,\cdots,s\}.
\end{aligned}$$

$\downarrow$ Dualize the nonanticipativity constraints

$$\boxed{\begin{aligned}
\sup_{\substack{\mu_1,\cdots,\mu_{s-1},\\ \lambda_1,\cdots,\lambda_{s-1}}} \min_{\substack{x_1,\cdots,x_s,\\ y_1,\cdots,y_s,\\ z_1,\cdots,z_s}} \quad & \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h) + \sum_{h=1}^{s-1} \mu_h^{\mathrm{T}} \left(y_h - y_{h+1}\right) + \sum_{h=1}^{s-1} \lambda_h^{\mathrm{T}} \left(z_h - z_{h+1}\right) \\
\text{s.t.}\quad & g_h(x_h, y_h, z_h) \le 0,\ \forall h \in \{1,\cdots,s\}, \\
& x_h \in X_h,\ y_h \in Y,\ z_h \in Z,\ \forall h \in \{1,\cdots,s\}.
\end{aligned}}$$

The inner minimization can be decomposed into independent scenario problems

Guignard, M. et al., Math. Program., 1987.

Our proposed approach (MLR) aims to leverage the advantages of both NGBD and LR

- Upper bounds are generated using efficient local optimization techniques that exploit the near-decomposable structure

- Lower bounds are generated by relaxing the complicating constraints corresponding to the continuous first-stage variables z

$$\max_{\lambda_1,\cdots,\lambda_{s-1}} \boxed{\begin{aligned} \min_{\substack{x_1,\cdots,x_s,\\ y,z_1,\cdots,z_s}} \ & \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \lambda_h^T (z_h - z_{h+1}) \\ \text{s.t.} \ & g_h(x_h, y, z_h) \le 0, \ \forall h \in \{1,\cdots,s\}, \\ & x_h \in X_h, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}, \\ & y \in Y. \end{aligned}}$$

Inner minimization can be solved efficiently using NGBD

- Convergence is guaranteed by B&B, where it is sufficient to branch on the continuous first-stage variables z to converge
  - Convergence is accelerated potentially by using tailored decomposable bounds tightening techniques

MLR = Modified Lagrangian Relaxation

GOSSIP
Overview and Motivation

- Software for the Global Optimization of nonconvex two-Stage Stochastic mixed-Integer nonlinear Programs
  - More than 100,000 lines of source code (primarily in C++)
  - Links to state-of-the-art solvers, e.g., CPLEX, IPOPT, ANTIGONE
  - Enables solution of large-scale scenario cases studies from the literature
- Motivation

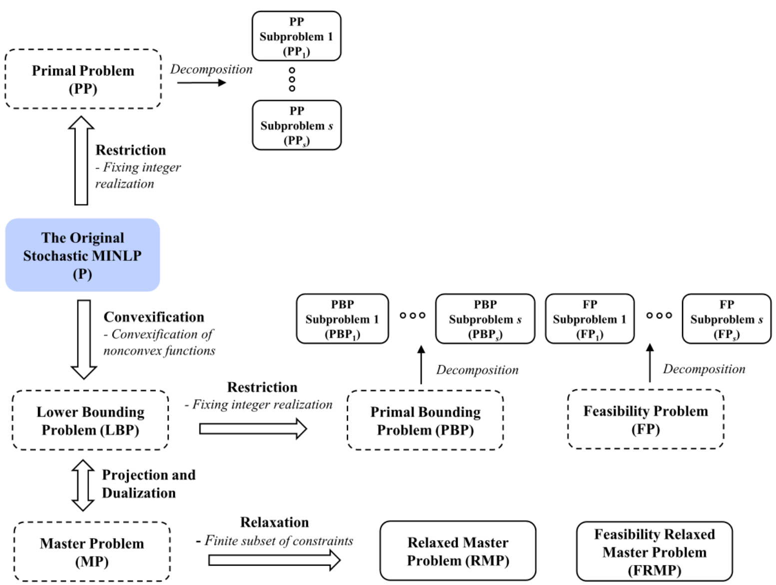
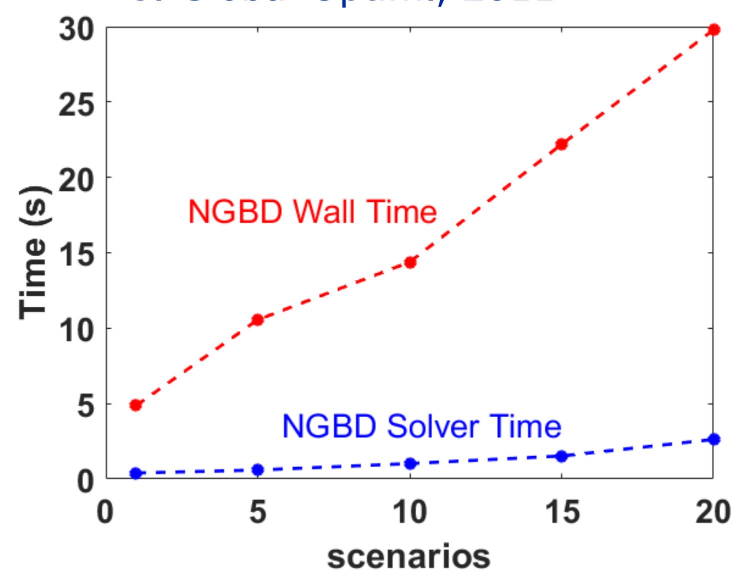


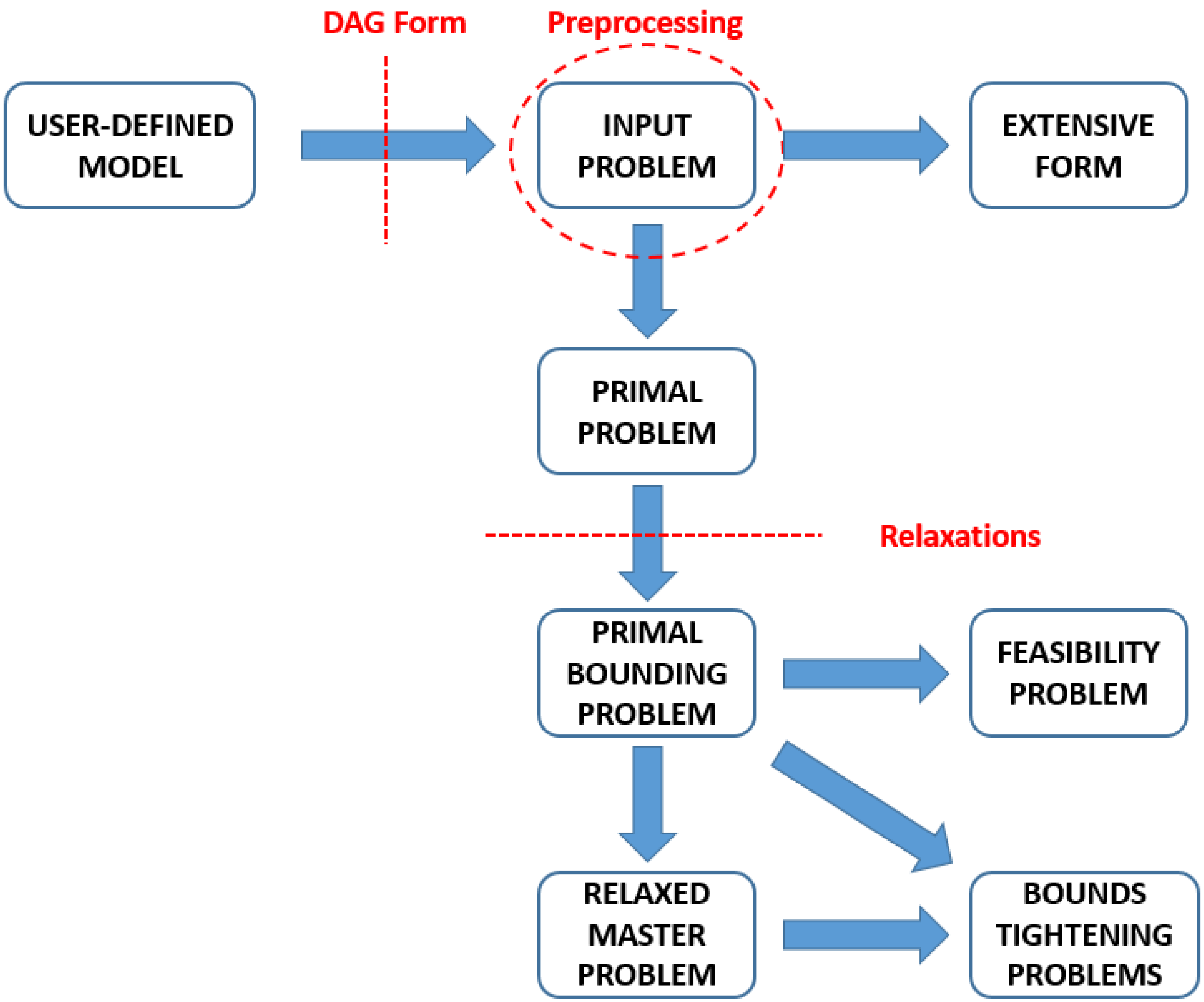
Image adapted from Li et al., J. Global Optim., 2011



Implementing decomposition algorithms such as NGBD is a nontrivial task

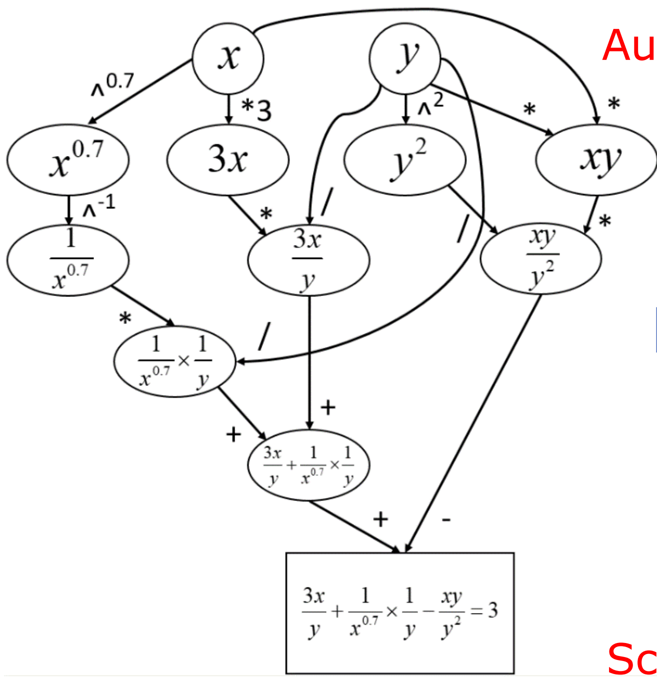
Naïve implementations may result in significant overhead

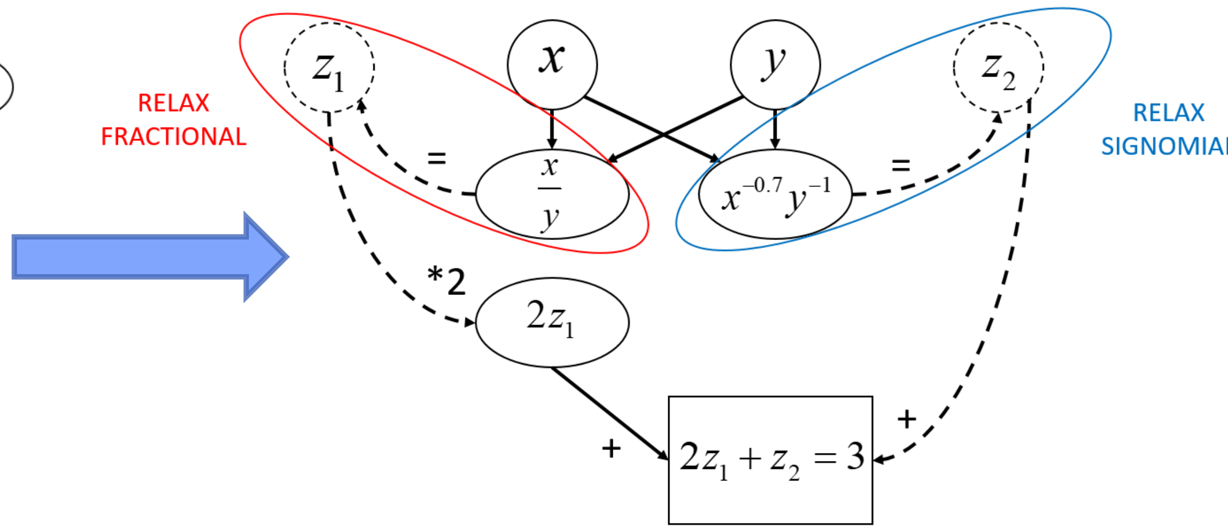
GOSSIP
Model Reformulation



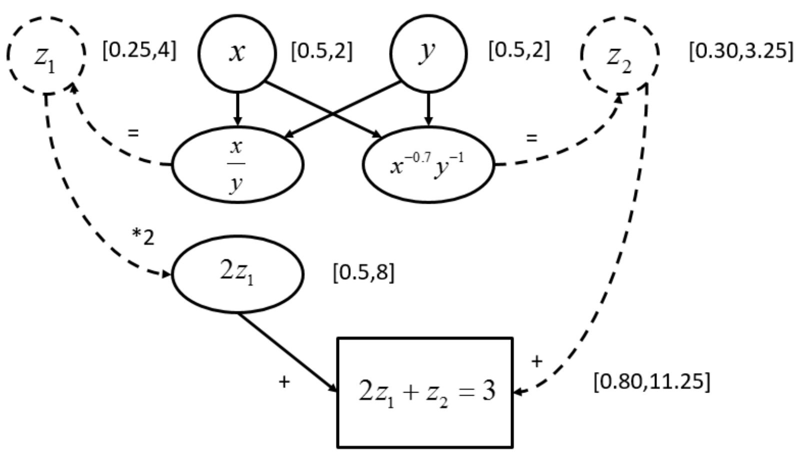
Some of GOSSIP's features

Automatic Structure Detection





Scalable Bounds Tightening Techniques



$$z^{j,\mathrm{lo}} = \max_{h\in\{1,\cdots,s\}} \boxed{\begin{aligned} \min_{x_h,y,z_h}\ & z_h^j \\ \text{s.t.}\ & g_h^{\mathrm{cv}}(x_h,y,z_h) \le 0, \\ & x_h \in \operatorname{conv}(X_h), \\ & y\in Y,\ z_h \in Z. \end{aligned}}$$

Computational Studies
Implementation Details

- Platform
  - CPU 3.5 GHz, Memory 6.0 GB, VMWare Workstation, GAMS 24.7.1, GCC 4.8.1, GFortran 4.8.1

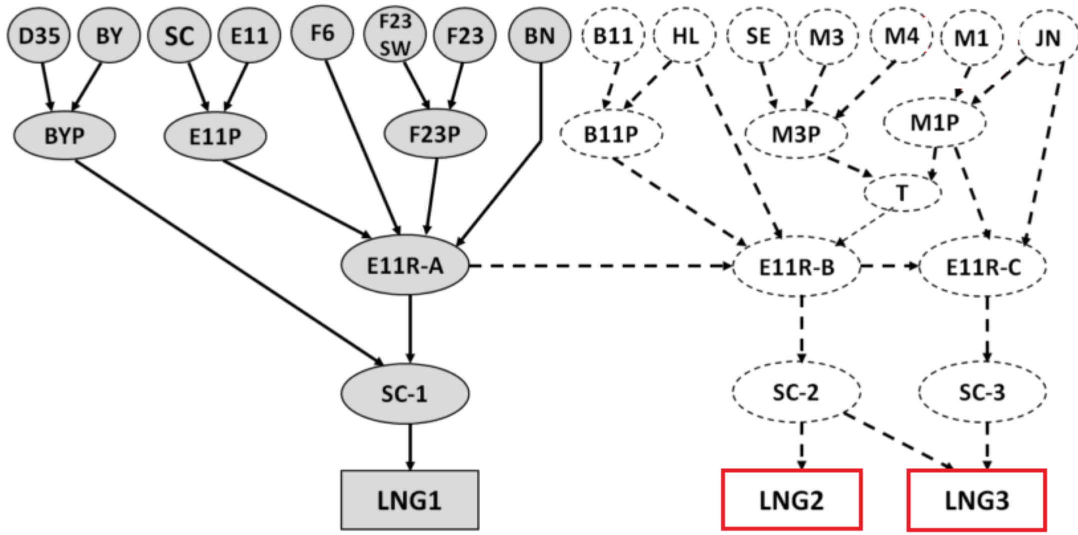
- GOSSIP Solvers
  - LP and MILP solver: CPLEX 12.6 (C library)
  - Global NLP solver: ANTIGONE 1.1 (C++ library)
  - Local NLP solver: IPOPT 3.12.8 (C++ library)
  - Bundle solver: MPBNGC 2.0 (Fortran library)

- Methods for comparison
  - ANTIGONE 1.1, BARON 16.3.4, COUENNE 0.5, SCIP 3.2
  - Nonconvex generalized Benders decomposition (NGBD)
  - Lagrangian relaxation (LR)
  - Modified Lagrangian relaxation (MLR)

- Relative tolerance: 10^{-3}, Absolute tolerance: 10^{-9}
- Time Limit: 10,000 seconds

Computational Study

Design and Operation of a Natural Gas Network



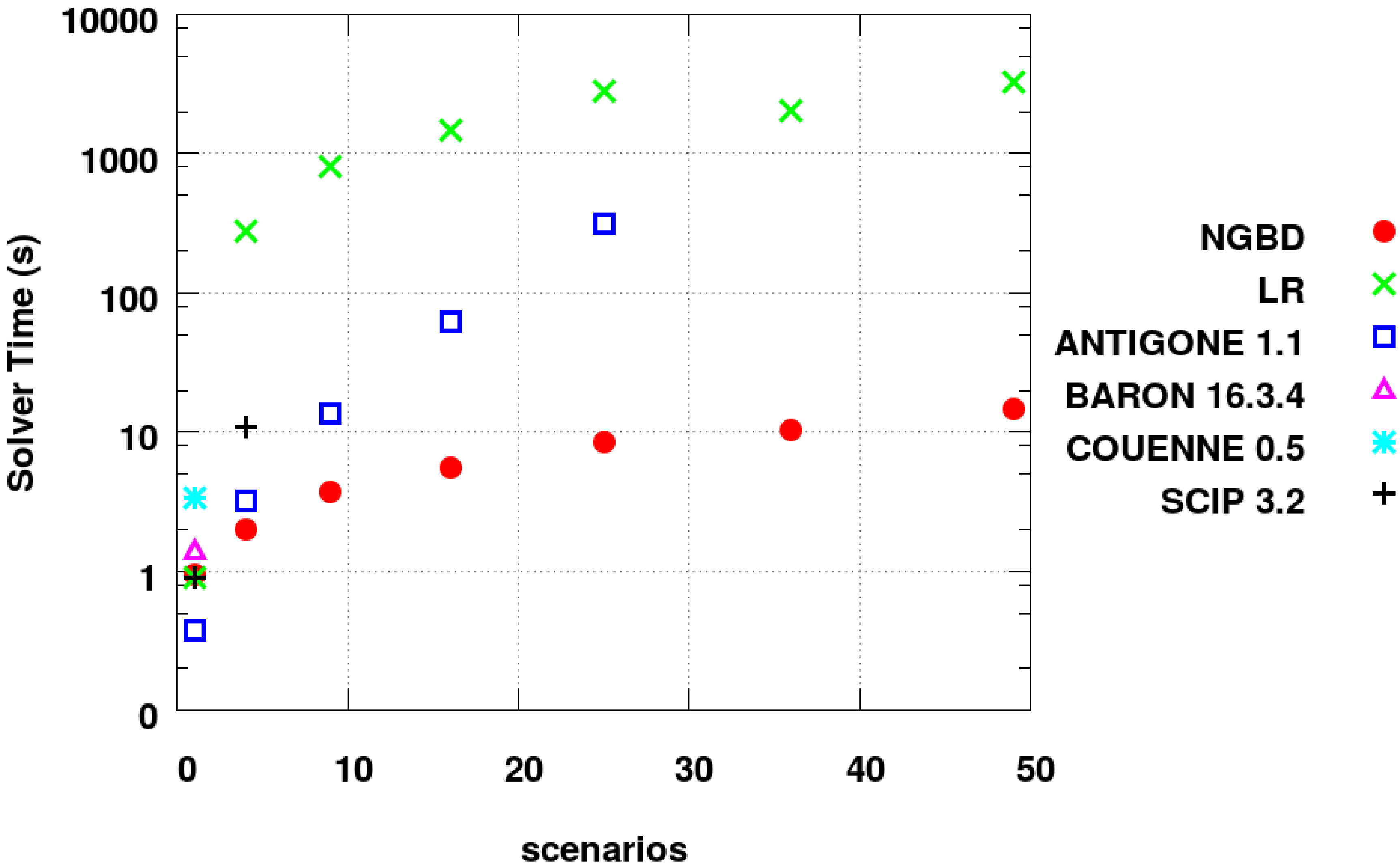
38 binary first-stage variables,
0 continuous first-stage variables,
93s continuous second-stage variables,
34s bilinear terms.

(s denotes the number of scenarios)

Li et al., AIChE Journal, 2011.

Computational Study

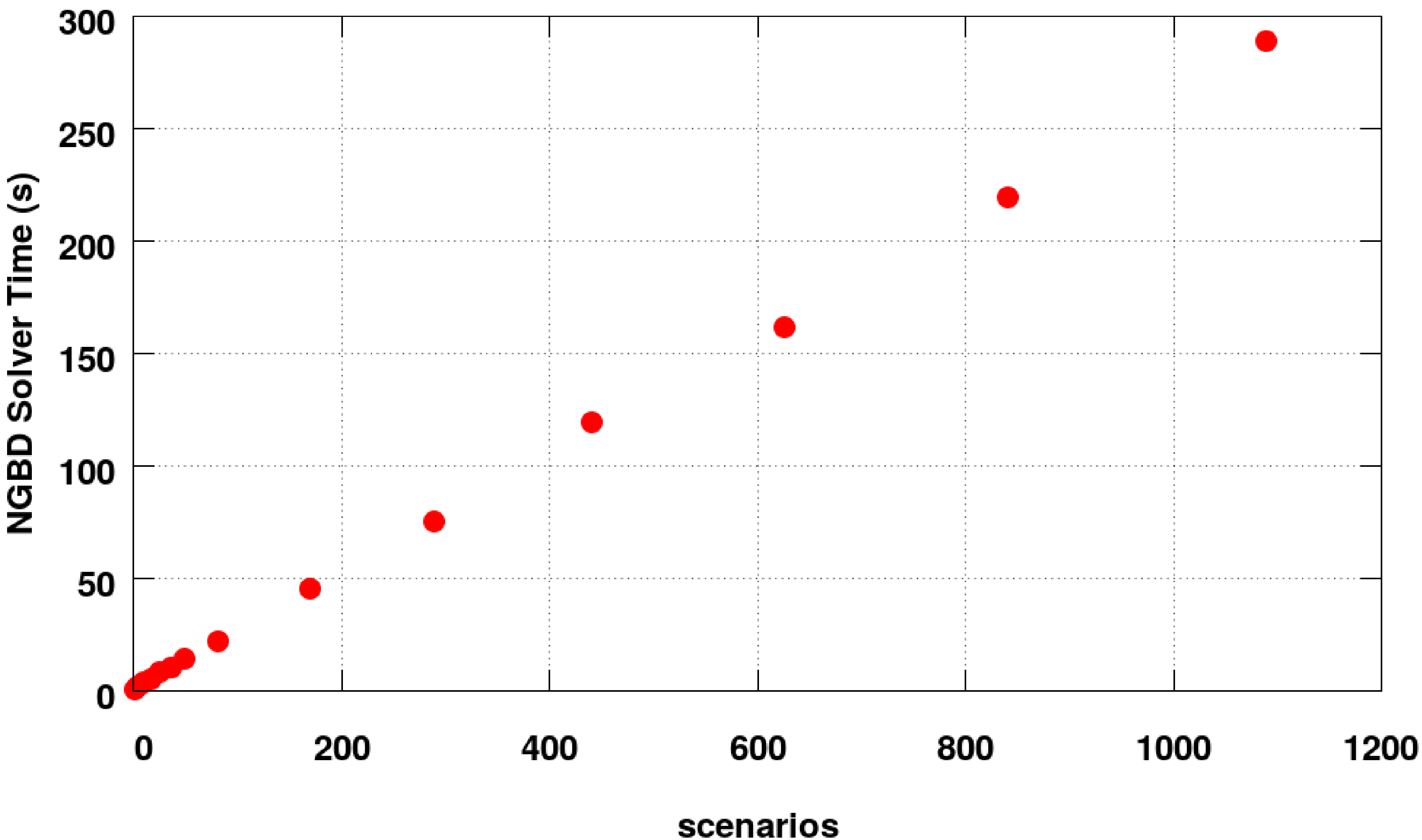
Design and Operation of a Natural Gas Network



Absolute Tolerance: 1E-09, Relative Tolerance: 1E-03, Time Limit: 10,000 seconds

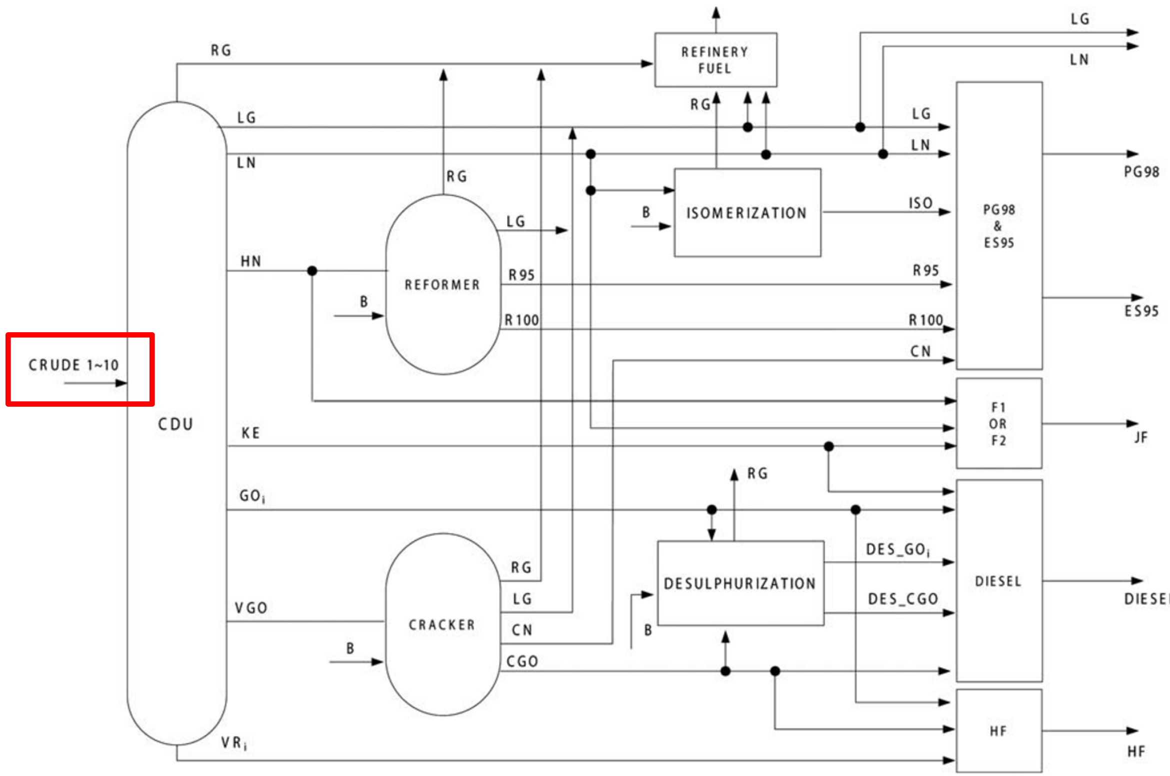
Computational Study

Design and Operation of a Natural Gas Network



Computational Study

Integrated Crude Selection and Refinery Operation



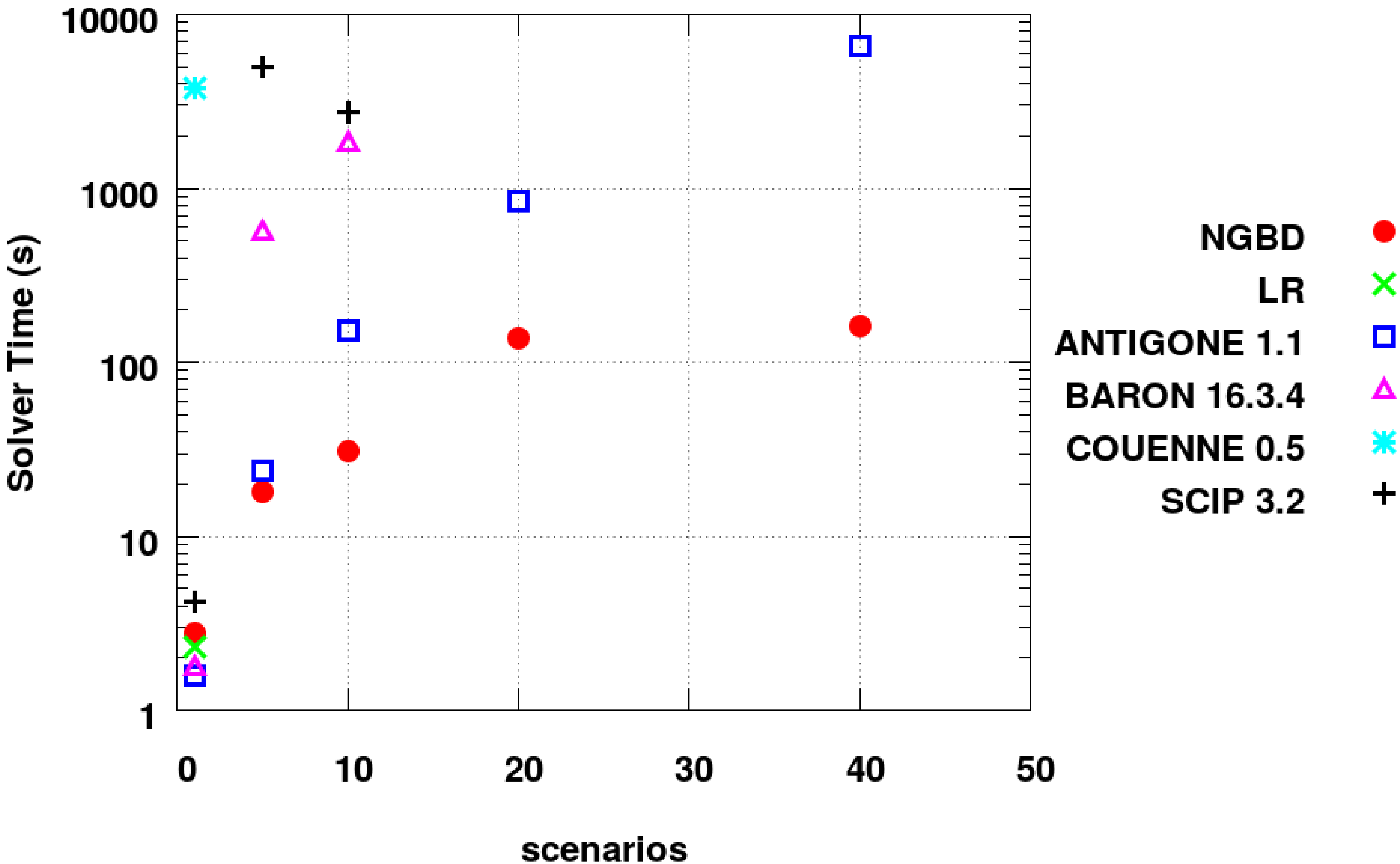
100 binary first-stage variables,
0 continuous first-stage variables,
122s continuous second-stage variables,
26s bilinear terms.

(s denotes the number of scenarios)

Yang and Barton, AIChE Journal, 2016.

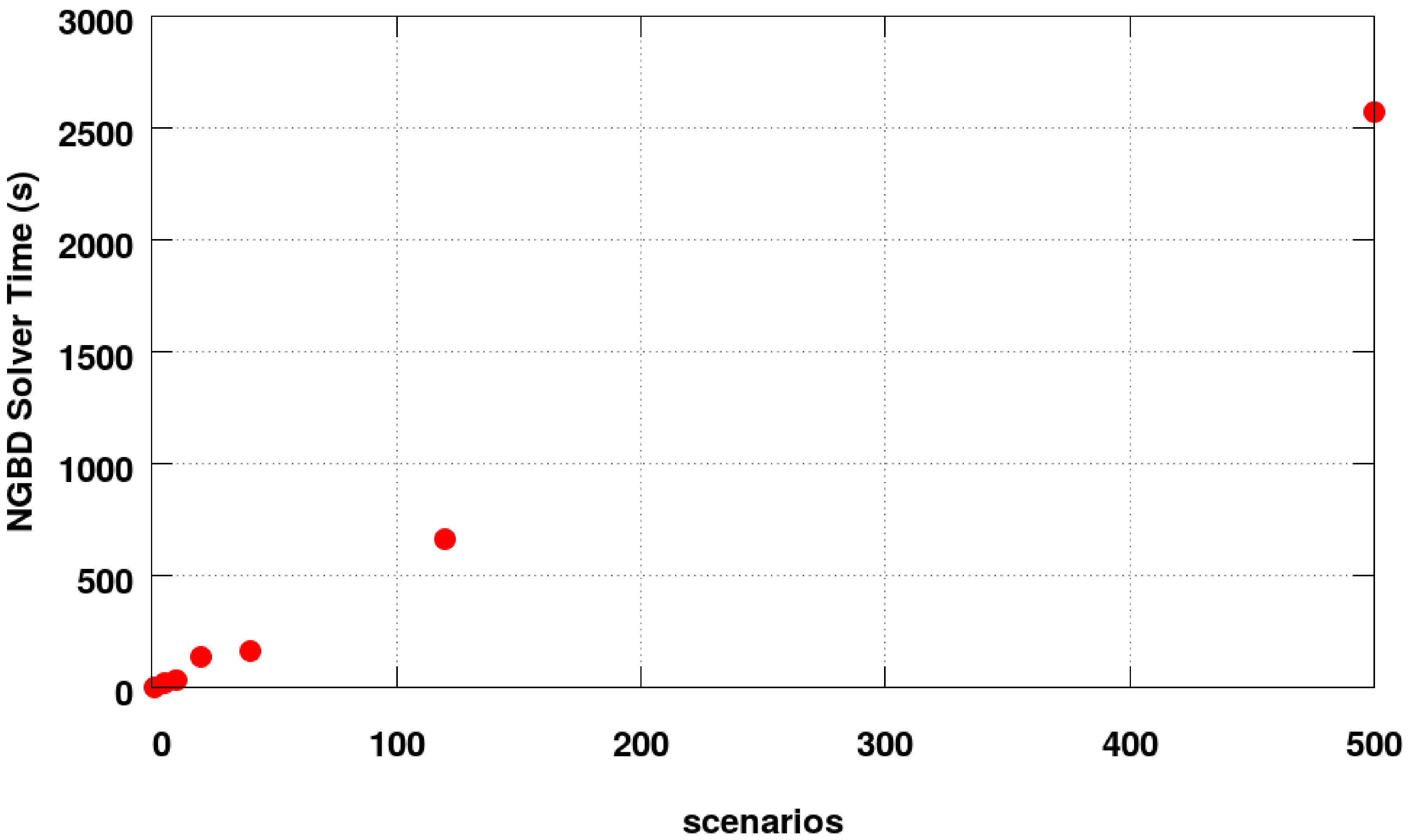
Computational Study

Integrated Crude Selection and Refinery Operation



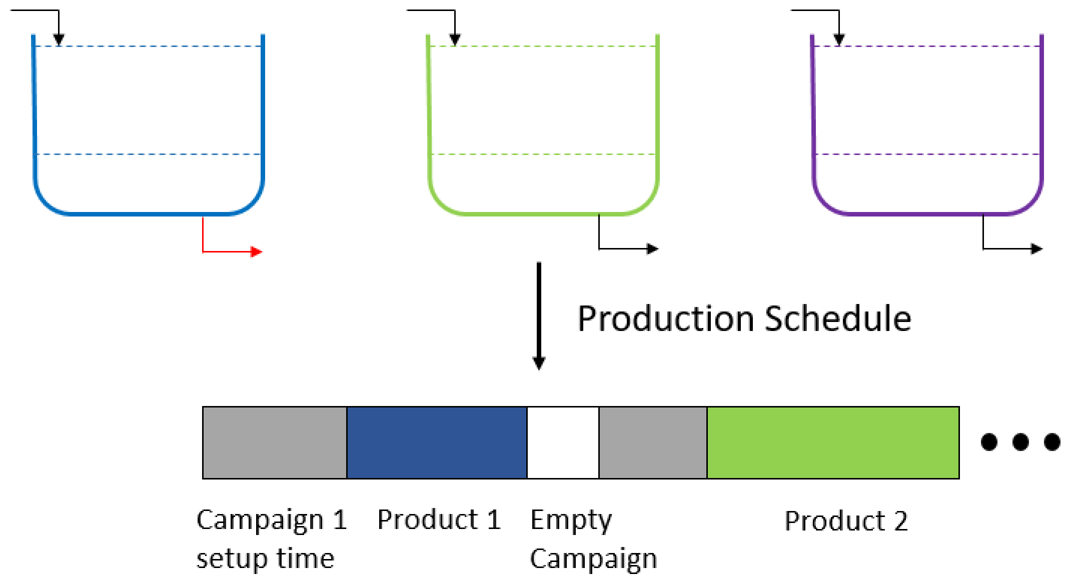
Absolute Tolerance: 1E-09, Relative Tolerance: 1E-03, Time Limit: 10,000 seconds

Computational Study
Integrated Crude Selection and Refinery Operation



Computational Study

Tank Sizing and Scheduling for a Chemical Plant



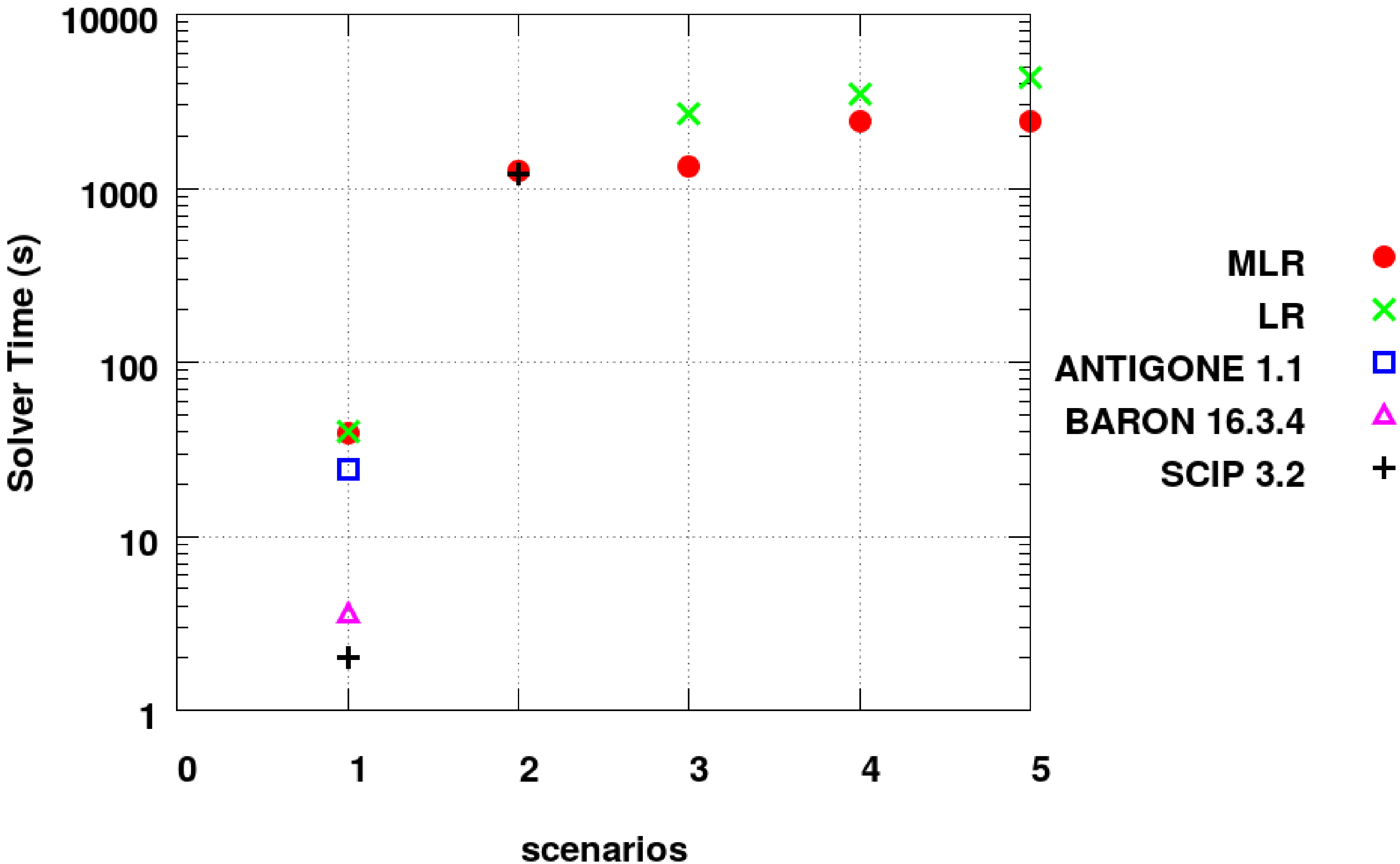
0 binary first-stage variables,
3 continuous first-stage variables,
9s binary second-stage variables,
38s continuous second-stage variables,
3 signomial terms,
47s bilinear terms.

(s denotes the number of scenarios)

Rebennack et al., Comput. Chem. Eng., 2011.

Computational Study

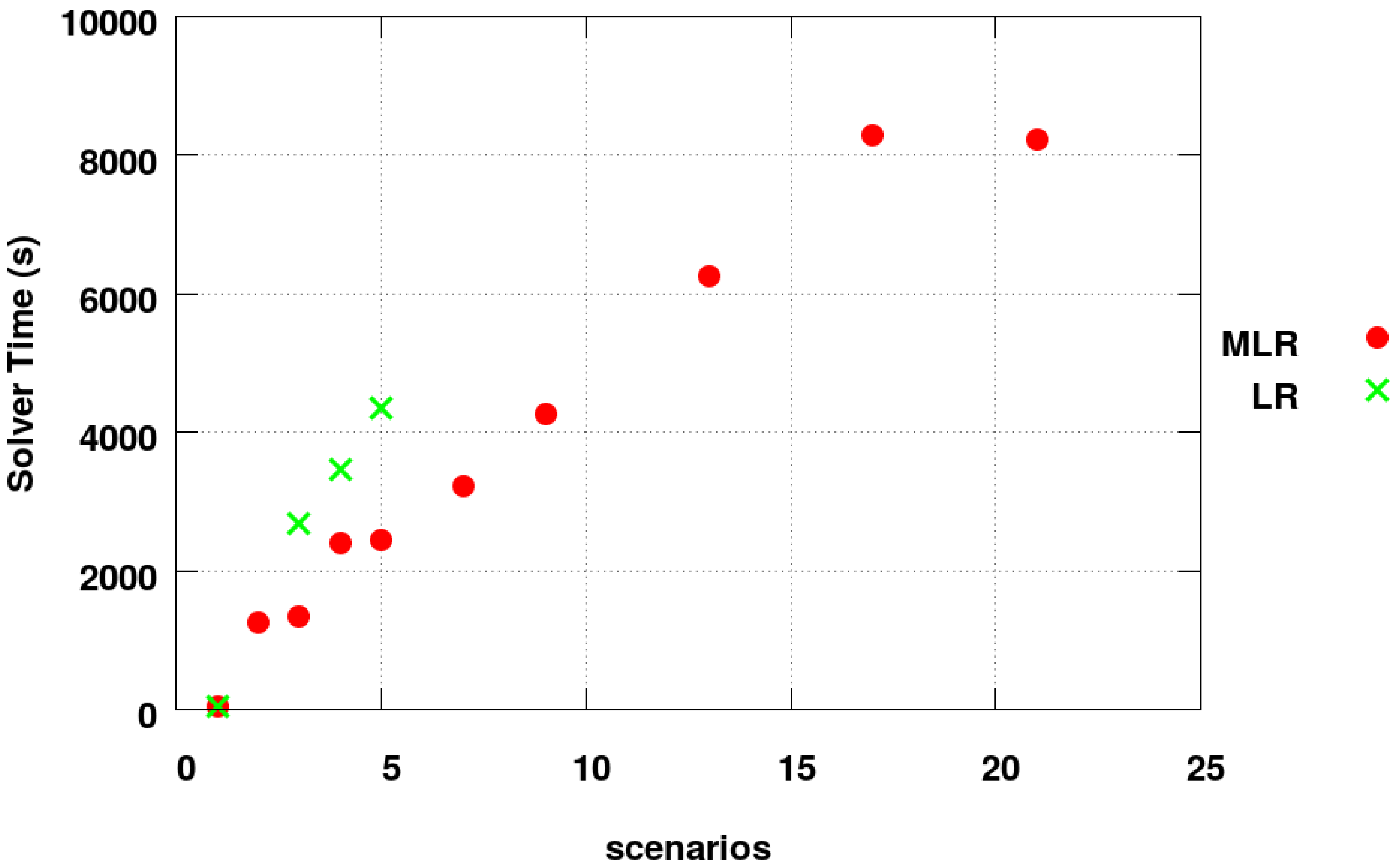
Tank Sizing and Scheduling for a Chemical Plant



Absolute Tolerance: 1E-09, Relative Tolerance: 1E-03, Time Limit: 10,000 seconds

Computational Study

Tank Sizing and Scheduling for a Chemical Plant



Summary and Future Work

- GOSSIP implements state-of-the-art decomposition techniques for nonconvex stochastic programs

- Case studies demonstrate the advantages of the software framework for solving large-scale problems

- Future work:
  - Additional features such as polyhedral relaxations, piecewise-convex relaxations, edge concave relaxations, RLT cuts
  - Incorporate alternate decomposition techniques such as nonconvex outer-approximation

- Please contribute to the test library (rohitk@alum.mit.edu). Contributions will be acknowledged

Acknowledgements

- Prof. Ruth Misener & Prof. Chris Floudas
- Prof. Yu Yang

Decomposition Approaches

Formulation

$$\min_{x_1,\cdots,x_s,y,z} \sum_{h=1}^{s} p_h\left[f_h(x_h) + c_{y,h}^{\mathrm{T}} y + c_{z,h}^{\mathrm{T}} z \right]$$

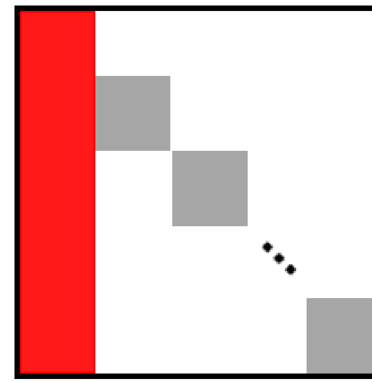
$$\text{s.t.} \quad g_h(x_h) + B_{y,h} y + B_{z,h} z \le 0, \ \forall h \in \{1,\cdots,s\},$$

$$A_y y + A_z z \le d_{y,z},$$

$$x_h \in X_h, \ \forall h \in \{1,\cdots,s\},$$

$$y \in Y, \ z \in Z.$$

Complicating variables



Equivalent Formulation

$$\min_{\substack{x_1,\cdots,x_s,\\ y_1,\cdots,y_s,\\ z_1,\cdots,z_s}} \sum_{h=1}^{s} p_h\left[f_h(x_h) + c_{y,h}^{\mathrm{T}} y_h + c_{z,h}^{\mathrm{T}} z_h \right]$$

$$\text{s.t.} \quad g_h(x_h) + B_{y,h} y_h + B_{z,h} z_h \le 0, \ \forall h \in \{1,\cdots,s\},$$

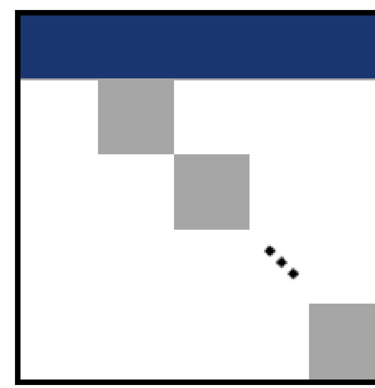
$$A_y y_h + A_z z_h \le d_{y,z}, \ \forall h \in \{1,\cdots,s\},$$

$$y_h - y_{h+1} = 0, \ \forall h \in \{1,\cdots,s-1\},$$

$$z_h - z_{h+1} = 0, \ \forall h \in \{1,\cdots,s-1\},$$

$$x_h \in X_h, \ y_h \in Y, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}.$$

Complicating constraints



Decomposition Algorithms

Nonconvex Generalized Benders Decomposition

$$
\begin{aligned}
\min_{x_1,\cdots,x_s,y} \quad & \sum_{h=1}^{s} p_h \left[f_h(x_h) + c_{y,h}^{\mathrm{T}} y \right] \\
\text{s.t.} \quad & g_h(x_h) + B_{y,h} y \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& A_y y \le d_y, \\
& x_h \in X_h \subset \{0,1\}^{n_{xb}} \times \mathbb{R}^{n_{xc}}, \ \forall h \in \{1,\cdots,s\}, \\
& y \in Y \subset \{0,1\}^{n_y}.
\end{aligned}
$$

Original Problem: Nonconvex MINLP

Fix the first-stage variables

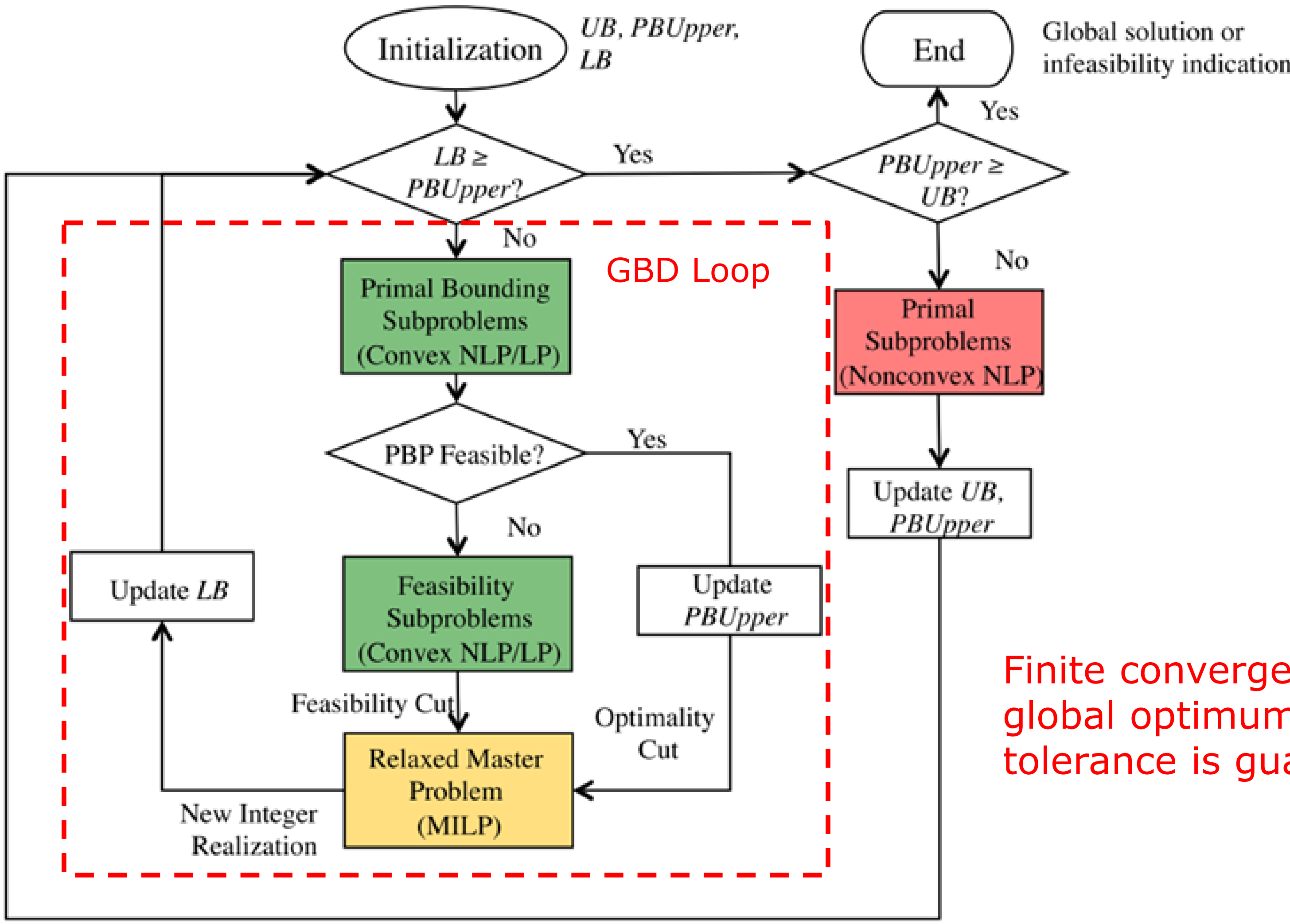
$$
\boxed{
\begin{aligned}
\min_{x_1,\cdots,x_s} \quad & \sum_{h=1}^{s} p_h \left[f_h(x_h) + c_{y,h}^{\mathrm{T}} \bar{y} \right] \\
\text{s.t.} \quad & g_h(x_h) + B_{y,h} \bar{y} \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& x_h \in X_h \subset \{0,1\}^{n_{xb}} \times \mathbb{R}^{n_{xc}}, \ \forall h \in \{1,\cdots,s\}.
\end{aligned}
}
$$

Solve the scenario primal problems independently

Primal Problem: Nonconvex NLP/MINLP

Decomposition Algorithms

Nonconvex Generalized Benders Decomposition



Finite convergence to a global optimum with a given tolerance is guaranteed

Decomposition Algorithms

Modified Lagrangian Relaxation

$$
\begin{aligned}
\min_{\substack{x_1,\cdots,x_s,y,\\ z_1,\cdots,z_s}} \quad & \sum_{h=1}^{s} p_h \left[f_h(x_h) + c_{y,h}^{\mathrm{T}} y + c_{z,h}^{\mathrm{T}} z_h \right] \\
\text{s.t.} \quad & g_h(x_h) + B_{y,h} y + B_{z,h} z_h \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& A_y y + A_z z_h \le d_{y,z}, \ \forall h \in \{1,\cdots,s\}, \\
& z_h = z_{h+1}, \ \forall h \in \{1,\cdots,s-1\}, \\
& x_h \in X_h, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}, \\
& y \in Y \subset \{0,1\}^{n_y}.
\end{aligned}
$$

Non-anticipativity constraints

Dualize the nonanticipativity constraints

$$
\begin{aligned}
\sup_{\lambda_1,\cdots,\lambda_{s-1}} \ \min_{\substack{x_1,\cdots,x_s,y,\\ z_1,\cdots,z_s}} \quad & \sum_{h=1}^{s} \left(p_h \left[f_h(x_h) + c_{y,h}^{\mathrm{T}} y + c_{z,h}^{\mathrm{T}} z_h \right] \right) + \sum_{h=1}^{s-1} \lambda_h^{\mathrm{T}} \left(z_h - z_{h+1} \right) \\
\text{s.t.} \quad & g_h(x_h) + B_{y,h} y + B_{z,h} z_h \le 0, \ \forall h \in \{1,\cdots,s\}, \\
& A_y y + A_z z_h \le d_{y,z}, \ \forall h \in \{1,\cdots,s\}, \\
& x_h \in X_h, \ z_h \in Z, \ \forall h \in \{1,\cdots,s\}, \\
& y \in Y.
\end{aligned}
$$

The inner minimization can be solved in a decomposable manner using NGBD

GOSSIP
Model Formulation

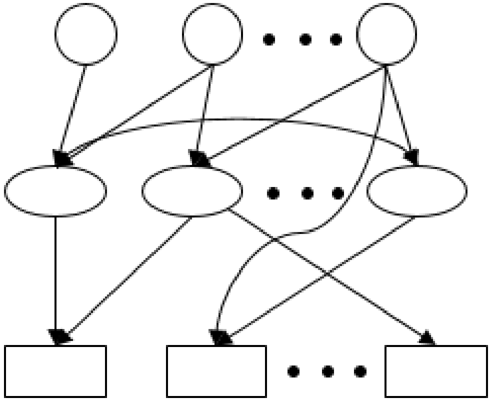
```
for(int j=0;j<NUM_POOLS;++j)
    for(int j2=0;j2<NUM_POOLS;++j2)
        if(T_PP[j][j2])
            for(int h=0;h<NUM_SCEN;++h)
            {
                sprintf(clabel,"s_PP[%d][%d][%d]", j+1, j2+1, h+1);
                s_PP[j][j2][h].setIndependentVariable(
                                    ++varcount,
                                    compgraph::CONTINUOUS,
                                    I(0,1),
                                    0.,
                                    h+1,
                                    clabel);
            }

for(int j=0;j<NUM_POOLS;++j)
    for(int h=0;h<NUM_SCEN;++h)
    {
        Split_bal[j][h] = 1;
        for(int j2=0;j2<NUM_POOLS;++j2)
            if(T_PP[j][j2])
                Split_bal[j][h] -= s_PP[j][j2][h];
        for(int k=0;k<NUM_TERMINALS;++k)
            if(T_PT[j][k])
                Split_bal[j][h] -= s_PT[j][k][h];
        Split_bal[j][h].setDependentVariable(++concount,compgraph::EQUALITY);
    }
```

DAG Representation

Variables

Constraints



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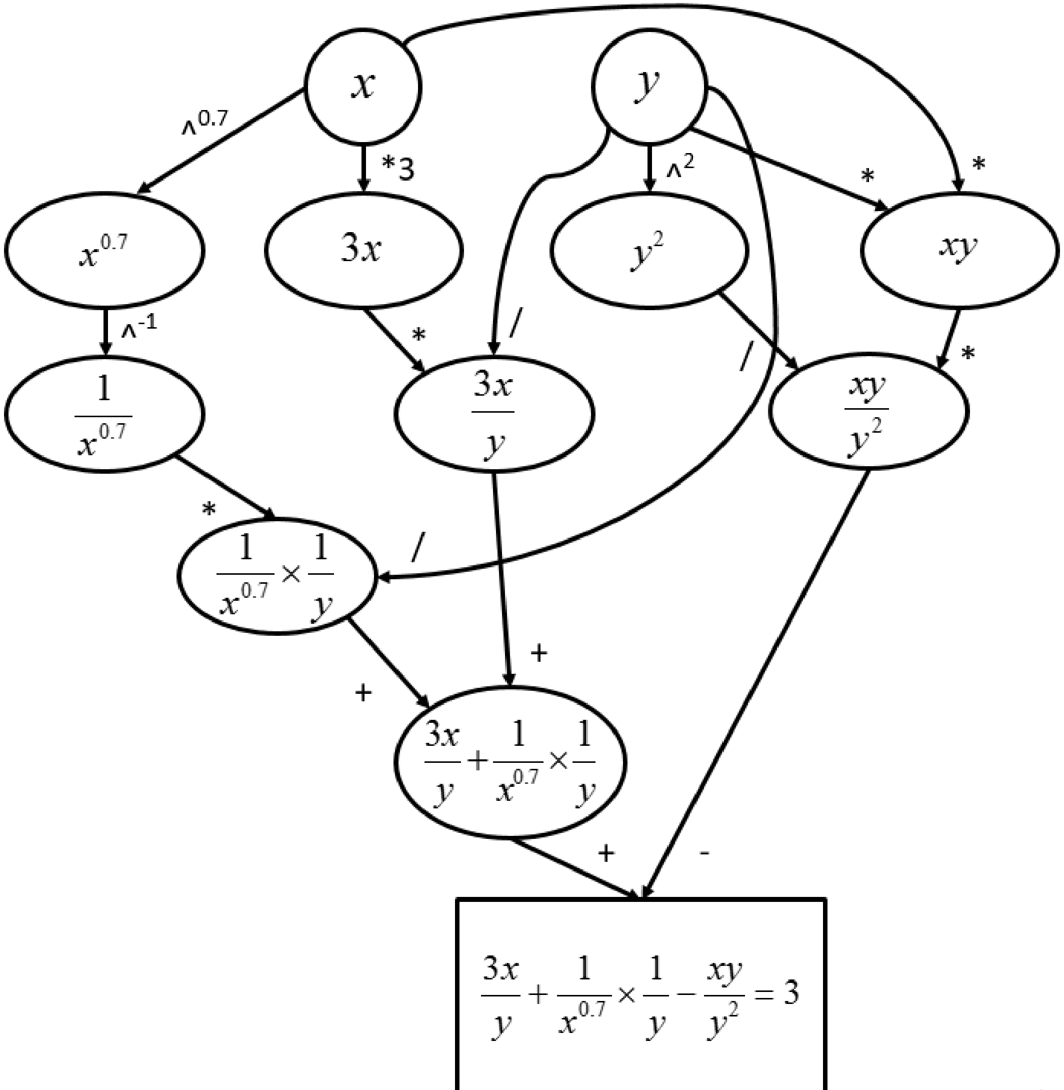
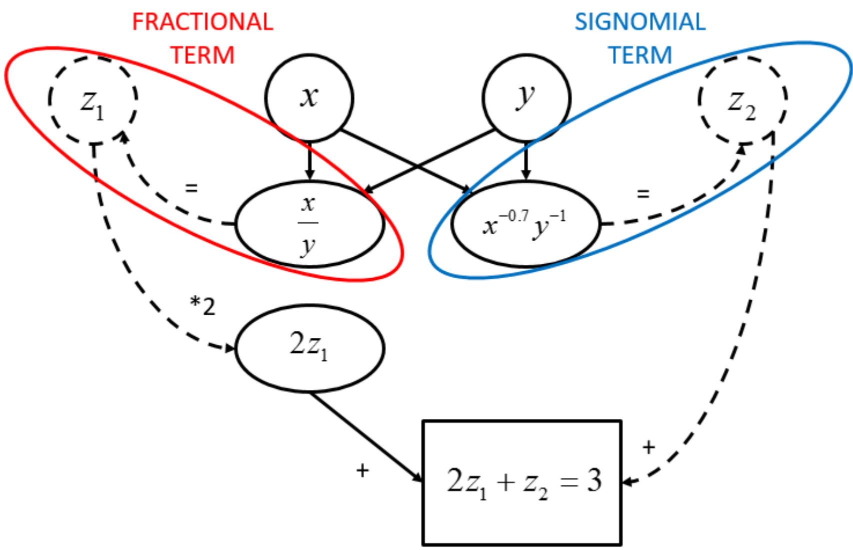
Automatic Structure Detection





Smith, E.M.B. et al., Comput. Chem. Eng., 1999.

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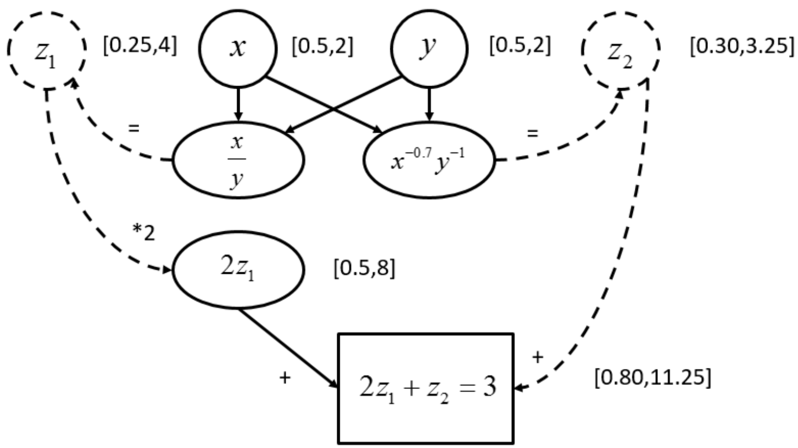
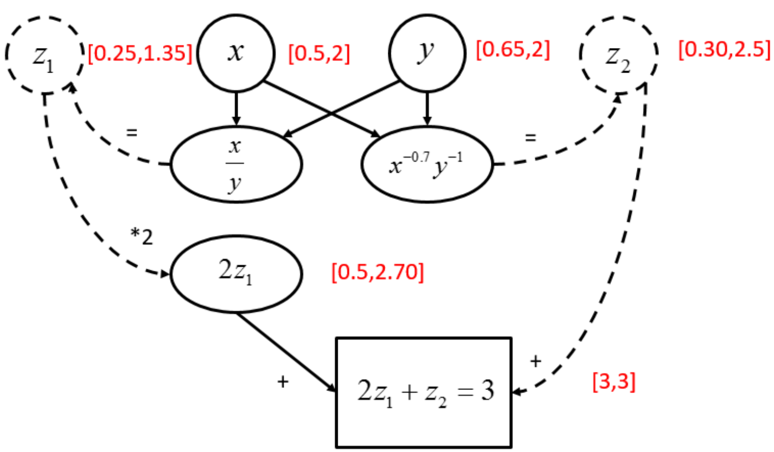
Bounds Tightening Techniques



Recourse Variables

$$x_h^{i,\mathrm{L}} = \min_{x_h, y, z} \; x_h^i$$
$$\text{s.t.} \;\; g_h^{\mathrm{cv}}(x_h) + B_{y,h} y + B_{z,h} z \le 0,$$
$$A_y y + A_z z \le d_{y,z},$$
$$x_h \in \mathrm{conv}(X_h),$$
$$y \in Y, \; z \in Z.$$



Complicating Variables

$$z^{i,\mathrm{L}} = \max_h \min_{x_h, y, z_h} \; z_h^i$$
$$\text{s.t.} \;\; g_h^{\mathrm{cv}}(x_h) + B_{y,h} y + B_{z,h} z_h \le 0,$$
$$A_y y + A_z z_h \le d_{y,z},$$
$$x_h \in \mathrm{conv}(X_h),$$
$$y \in Y, \; z_h \in Z.$$

Vu, X.-H. et al., J. Global. Optim., 2009.

GOSSIP
Relaxation Strategies

Term	Relaxation
xy	McCormick envelope
$\frac{x}{y}$	Bilinear reformulation, Quesada and Grossmann envelope
x^c	Secant, Liberti and Pantelides linearization
$\log(x)$	Secant
$\exp(x)$	Secant
x^y	Reformulate as $\exp(y \log(x))$
$\lvert x \rvert$	MIP reformulation
$\min(x, y)$	Reformulate as $\frac{1}{2}(x + y - \lvert x - y \rvert)$
$\max(x, y)$	Reformulate as $\frac{1}{2}(x + y + \lvert x - y \rvert)$
$x \log(x)$	Secant
$x \exp(x)$	Bilinear reformulation, Secant
xyz	Meyer and Floudas envelope
$xyzw$	Cafieri et al. relaxations
${x_1}^{c_1} \cdot {x_2}^{c_2} \cdots {x_n}^{c_n}$	Bilinear reformulation, Secant, Transformation-based relaxations

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Upper Bounding Techniques

Lower Bounding Problem

$$
\begin{aligned}
\sup_{\lambda_1,\cdots,\lambda_{s-1}} \min_{\substack{x_1,\cdots,x_s,y,\\ z_1,\cdots,z_s}} \; & \sum_{h=1}^{s}\left(p_h\left[f_h(x_h)+c_{y,h}^{\mathrm{T}}y+c_{z,h}^{\mathrm{T}}z_h\right]\right)+\sum_{h=1}^{s-1}\lambda_h^{\mathrm{T}}\left(z_h-z_{h+1}\right) \\
\text{s.t.} \; & g_h(x_h)+B_{y,h}y+B_{z,h}z_h \le 0,\ \forall h\in\{1,\cdots,s\}, \\
& A_y y+A_z z_h \le d_{y,z},\ \forall h\in\{1,\cdots,s\}, \\
& x_h\in X_h,\ z_h\in Z,\ \forall h\in\{1,\cdots,s\}, \\
& y\in Y.
\end{aligned}
$$

Upper Bounding Problem

$$
\begin{aligned}
\min_{\substack{x_1,\cdots,x_s,y,\\ z_1,\cdots,z_s}} \; & \sum_{h=1}^{s} p_h\left[f_h(x_h)+c_{y,h}^{\mathrm{T}}y+c_{z,h}^{\mathrm{T}}z_h\right] \\
\text{s.t.} \; & g_h(x_h)+B_{y,h}y+B_{z,h}z_h \le 0,\ \forall h\in\{1,\cdots,s\}, \\
& A_y y+A_z z_h \le d_{y,z},\ \forall h\in\{1,\cdots,s\}, \\
& z_h = z_{h+1},\ \forall h\in\{1,\cdots,s-1\}, \\
& x_h\in X_h,\ z_h\in Z,\ \forall h\in\{1,\cdots,s\}, \\
& y\in Y.
\end{aligned}
$$

- Fix the binary variables in the upper bounding problem to the lower bounding solution
- Initialize the continuous second-stage variables to the lower bounding solution
- Initialize the continuous first-stage variables to the average lower bounding solution