PROBABILITY AND STATISTICS

UNIT – I PROBABILITY

Introduction:

In random phenomena, past information, no matter how voluminous, will not allow to formulate a rule to determine precisely what will happen in the future. The theory of probability is the study of such random phenomena which are not deterministic.

In analyzing and interpreting data that involves an element of "chance" or "uncertainty". In the process of analyzing data, the expectation is based on one's present knowledge and belief about the event in question. Even though, these statements of expectation are by previous experience, present knowledge and analytical thinking, we need a quantitative measure to quantify the expectation.

Probability is a concept which numerically measures the degree of certainty or uncertainty of occurrence or nonoccurrence of events.

In short, the branch of mathematics which studies the influence of "chance" is the theory of probability.

Basic Terminology:

Experiment: An experiment is any physical action or process that is observed and the result noted.

Ex: (i) Tossing a coin (ii) Firing a missile (iii) getting up in the morning

We have two types of experiments

(i) Deterministic or Predictable

(ii) Undeterministic or unpredictable or Random experiments

Deterministic: An experiment is called deterministic if, the results can be predicted with certainty prior to the performance of the experiment.

Ex: Throwing a stone upwards where it is known that the stone will definitely fall to the ground due to the force of gravitation.

Random experiment: An experiment is called a random experiment if, when repeated under the same conditions, it is such that the outcome cannot be predicted with certainty but all possible outcomes can be determined prior to the performance of the experiment.

Trail: A single performance of an experiment is briefly called a trail.

Event: An event is a result of the experiment.

i) In throwing a die, getting 1 (or 2, 3 or … or 6) is an event.

ii) In drawing two cards from a pack of well shuffled cards, "getting a king" and "getting a queen" are events.

Sample Space: The collection of all possible outcomes of a random experiment is called the Sample Space, denoted by S. The elements of sample space are called sample points.

Example :

- Sample space of tossing a coin : S = {H, T} H-head, T-tail.
- Sample space of tossing 2 coins : S = {(H, H), (H, T), (T, H), (T, T)}
- Sample space of throwing a die : S {1, 2, 3, 4, 5, 6}
- Probability of an event is denoted by P(E) and $0 \leq P(E) \leq 1$
- Probability of sample space, P(S) = 1.

Def: A sample space is called finite sample space, if its sample points are finite in number.

Def: A sample space is called infinite sample space, if its sample points are infinite in number.

Def: Every non empty subset of a sample space of a random experiment is called an event.

Note: As sample space S is subset of itself and the empty set ϕ is also a subset of S, therefore, S and ϕ are also considered as events.

Def: The event S is called the Sure event or Certain event and the event ϕ is called an impossible event.

Def: The **complement** of an event is the event not occurring. The probability that Event A will not occur is denoted by $P(A^c)$.

Def: The probability that Events A and B *both* occur is the probability of the *intersection* of A and B. The probability of the intersection of Events A and B is denoted by $P(A \cap B)$.

Def: The probability that Events A or B occur is the probability of the *union* of A and B. The probability of the union of Events A and B is denoted by $P(A \cup B)$

Equally likely events: A set of events are said to equally likely, if no one of them is expected to occur in preference to other in any single trail of the random experiment.

Ex: (i) In tossing an unbiassed or uniform coin, getting head or tail are equally likely events.

(ii) In throwing an unbiased die, all the six faces are equally likely to occur.

Mutually exclusive events: Events of random experiment are said to be mutually exclusive, if the happening of one event, prevents the happening of all other events. i.e., if no two or more of them can happen simultaneously in the same trial.

i.e., If A and B are two events of a sample space, then i.e. $A \cap B = \emptyset$.

In general, the events E1, E2, … , En are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$

Ex: When two teams E1 and E2 are playing a game, the events "E1 winning the game" and "E2 winning the game" are mutually exclusive.

Ex: When A, B, C, D are appearing for an examination, the event "A passing in the examination" does not prevent the events B, C or D passing the examination. Hence these events are not mutually exclusive

Independent events: Two events are said to be independent if the occurrence or nonoccurrence of one event has no influence on the occurrence or non occurrence of the other events.

Ex: If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

Note: In many cases the wording of a problem identifies the set operation that is appropriate in writing its equivalent mathematical expression. When the word 'OR' is used, the *union* of two or more events is involved. The word 'AND' entails a set *intersection*. The word 'NOT' calls for a set *complement*.

Permutation: If 'r' objects are chosen from a set of n distinct objects, any particular arrangement or order of these objects is called a permutation.

The number of permutation of r objects chosen from a set of n objects is

$$n_{P_r} = \frac{n!}{(n-r)!}$$

Combination: The way of selecting r objects from a set of n distinct objects is called a combination.

The number of combinations in which r objects can be selected from a set of n objects is $n_{C_r} = \frac{n!}{r!(n-r)!}$

Classical Definition of Probability

If there are n mutually exclusive and equally likely events of random experiment, out of which 'm' are favorable to a particular event E, then we define the probability of E as, $P(E) = \frac{m}{n} = = \frac{\text{No. of favorable events to E}}{\text{Total no. of events of the experiment}}$

Note: The probability of E not happening or failure of E is denoted by $P(E^C)$ or $P(\overline{E})$

Note: In any experiment if m outcomes are favorable to the event E then remaining n-m outcomes are not favorable to E. This set of unfavorable events is denoted by $\overline{E}$ or E^C.

Note: $P(E) + P(E^C) = 1$

Example 1

Suppose we draw a card from a deck of playing cards. What is the probability that we draw a spade?

Solution: The sample space of this experiment consists of 52 cards, and the probability of each sample point is 1/52. Since there are 13 spades in the deck, the probability of drawing a spade is

P(Drawing a spade) = (13)(1/52) = 1/4

Example 2

Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head?

Solution: For this experiment, the sample space consists of 8 sample points.

S = {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}

Each sample point is equally likely to occur, so the probability of getting any particular sample point is 1/8. The event "getting two tails and one head" consists of the following subset of the sample space.

A = {TTH, THT, HTT}

The probability of Event A is the sum of the probabilities of the sample points in A. Therefore,

P(A) = no. of favorable outcomes **/** total number of outcomes = 3/8

Axiomatic definition of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties.

If S is the sample space and E is any event in a random experiment,

(i) $0 \leq P(E) \leq 1$ for each event E in S.

(ii) P(S) = 1 (sum of all probabilities)

(iii) If E1 and E2 are any mutually exclusive events in S, then $P(E1 \cup E2) = P(E1) + P(E2)$

Elementary Theorems:

1. Probability of complementary event A^C is $P(A^C) = 1 - P(A)$
2. For any two events A and B: $P(A^C \cap B) = P(B) - P(A \cap B)$
3. For any two events A and B: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   Addition Law of Probability: The probability that Event A or Event B occurs is equal to the probability that Event A occurs plus the probability that Event B occurs minus the probability that both Events A and B occur simultaneously.
4. If $A \subseteq B$ then (i) $P(A^C \cap B) = P(B) - P(A)$ (ii) $P(A) \leq P(B)$

Example

A student goes to the library. The probability that she checks out (a) a work of fiction is 0.40, (b) a work of non-fiction is 0.30, and (c) both fiction and non-fiction is 0.20. What is the probability that the student checks out a work of fiction, non-fiction, or both?

Solution:

Let F be the event that the student checks out fiction;

Let N be the event that the student checks out non-fiction.

Then, based on the rule of addition: $P(F \cup N) = P(F) + P(N) - P(F \cap N)$

i.e., $P(F \cup N) =$ 0.40+0.30-0.20 = 0.50

Conditional probability

If A and E are any two events of a sample space S, then the event of "happening of E after the happening of A" is called conditional event and is denoted by (E/A)

If E and A are any events in S, P(A) > 0, then the conditional probability of E given A is, $P(E / A) = \dfrac{P(E \cap A)}{P(A)} = = \dfrac{P(\text{Both events E and A occur})}{P(\text{given event A occur})}$

Note: If A and B are any two independent events in a sample space 'S' then, P(A/B) = P(A) or P(B/A) = P(B)

Note: By conditional probability, $P(A / B) = \dfrac{P(A \cap B)}{P(A)}$. If A and B are independent events, then $P(A \cap B) = P(A)P(B)$. This is known as special multiplication rule for independent events.

Note: If A and B are any events in S, then $P(A \cap B) = P(A)P(B / A)$, if *P(A)>0*

$= P(B)P(A / B)$, if *P(B)>0*

This is known as General multiplication rule.

Rule of Multiplication

The rule of multiplication applies to the situation when we want to know the probability of the intersection of two events; that is, we want to know the probability that two events (Event A and Event B) both occur.

Example

A die is rolled. If the outcome is an odd number, what is the probability that it is prime?

Solution: When a die is rolled, the sample space S={1,2,3,4,5,6}

Let A be the event of getting an odd number. The chances for the occurrence of A are {1,3,5} and P(A)=3/6=1/2.

Let E be the event of getting a prime number. The chances for the occurrence of E are {2,3,5} and P(E)=3/6=1/2.

Now $E \cap A$ is the event of getting both prime and odd number together. Chances for the occurrence of $E \cap A$ are {3,5}. P($E \cap A$)=2/6=1/3.

P(getting a prime number which is an odd number also) =

P(getting prime/getting an odd number)=

= P(E/A)=P($E \cap A$)/P(A)

=(1/3)/(1/2)=2/3

Theorems on Independent events:

1. If A and B are independent then (i) A and B^C are independent (ii) A^C and B^C are independent
2. If A, B, C are mutually independent events then $A \cup B$ and C are also independent.

Baye's Theorem (Bayes Rule)

Bayes' theorem (also known as Bayes' rule) is a useful tool for calculating conditional probabilities. Bayes' theorem can be stated as follows:

Bayes' theorem. Let E_1, E_2, ..., E_n be a set of mutually exclusive events that together form the sample space S. Let A be any event from the same sample space, such that P(A) > 0. Then, $P(E_i / A) = \dfrac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)}$

When to Apply Bayes' Theorem

Bayes' theorem is considered when the following conditions exist.

- The sample space is partitioned into a set of mutually exclusive events $\{E_1, E_2, \ldots, E_n\}$.
- Within the sample space, there exists an event A, for which P(A) > 0.
- The analytical goal is to compute a conditional probability of the form: $P(E_i \mid A)$.
- You know at least one of the two sets of probabilities described below.
  - $P(E_i \cap A)$ for each E_i
  - $P(E_i)$ and $P(A \mid E_i)$ for each E_i

Example

Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

Solution: The sample space is defined by two mutually-exclusive events - it rains or it does not rain. Additionally, a third event occurs when the weatherman predicts rain. Notation for these events appears below.

- Event A_1. It rains on Marie's wedding.
- Event A_2. It does not rain on Marie's wedding.
- Event B. The weatherman predicts rain.

In terms of probabilities, we know the following:

- $P(A_1) = 5/365 = 0.0136985$ [It rains 5 days out of the year.]
- $P(A_2) = 360/365 = 0.9863014$ [It does not rain 360 days out of the year.]
- $P(B \mid A_1) = 0.9$ [When it rains, the weatherman predicts rain 90% of the time.]
- $P(B \mid A_2) = 0.1$ [When it does not rain, the weatherman predicts rain 10% of the time.]

We want to know $P(A_1 \mid B)$, the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$P(A_1 \mid B) = \frac{P(A_1) P(B \mid A_1)}{P(A_1) P(B \mid A_1) + P(A_2) P(B \mid A_2)}$$

$$P(A_1 \mid B) = (0.014)(0.9) / [(0.014)(0.9) + (0.986)(0.1)]$$

$$P(A_1 \mid B) = 0.111$$

Note the somewhat unintuitive result. Even when the weatherman predicts rain, it rains only about 11% of the time. Despite the weatherman's gloomy prediction, there is a good chance that Marie will get married with out any disturbance of raining.