Homework #1
Introduction to Algorithms/Algorithms 1
600.463
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1 Problem 1 (25 points)

1.1 (15 points)

For each statement below explain if it is true or false and prove your answer. Be as precise as you can. The base of $\log$ is 2 unless stated otherwise.

1. $\frac{n}{\log n} = \Theta(n)$
   ANSWER: False
   For $\frac{n}{\log n} = \Theta(n)$ to be true it should satisfy:
   $0 \leq C_1 n \leq \frac{n}{\log n} \leq C_2 n \, \forall n \geq n_0$
   As n is always $\geq \log n$ and $\log n$ is always positive
   $\implies \frac{n}{\log n} \leq C_2 n$ is always true but $C_1 n \leq \frac{n}{\log n}$ may not always be true
   For example: for $C_1 = 2,\ C_2 = 2,\ n = 3:\ 6 \nleq \frac{n}{\log n} = 1.89 \leq 6$

2. $2^n = o(3^n)$
   ANSWER: True
   For $2^n = o(3^n)$ to be true it should satisfy:
   $0 \leq 2^n < C3^n \, \forall n \geq n_0$
   We can prove this by induction

$$Basis\ Step : Let\, n = 1 : 2 < C.3 \tag{1}$$

$$Induction\ Hypothesis : Let\, n = i : 2^i < C3^i \tag{2}$$

Induction Step:
$2^{(i+1)} = 2.2^i < C.2.3^i$ Using (2)
$\implies 2^{(i+1)} < C.3.3^i$ Using (1)
$\implies 2^{(i+1)} < C3^{(i+1)}$
Hence Proved as our hypothesis is true
Alternatively we can prove $0 \leq 2^n < C3^n$by dividing both sides by 3^n
$\frac{2^n}{3^n} < c$
For any given value of c,if the value of n is made sufficiently large, the above statement holds true

3. $2^n = \Theta(\left(\frac{3}{2}\right)^n)$
ANSWER: False
For $2^n = \Theta(\left(\frac{3}{2}\right)^n)$ to be true it should satisfy:
$0 \leq C_1(\left(\frac{3}{2}\right)^n) \leq 2^n \leq C_2(\left(\frac{3}{2}\right)^n) \, \forall n \geq n_0$
As though $2 > \frac{3}{2}$ $0 \leq C_1(\left(\frac{3}{2}\right)^n) \leq 2^n$ may be true for all positive n and suitable C_1(where $C_1 \leq 2$)
$but 2^n \leq C_2(\left(\frac{3}{2}\right)^n)$ may not be true(unless $C_2 \geq 2$).
For Example:
For $C_1 = 3, C_2 = 1 \, and \, n = 3$:0$\leq$ $3.(\left(\frac{3}{2}\right)^3) \nleq 2^3 \nleq 1.(\left(\frac{3}{2}\right)^3)$
$\implies$ 0$\leq$ 10.125 $\nleq$ 8 $\nleq$ 3.375
Hence False

4. $2^n = O(e^{n+\log n})$
ANSWER: True
For $2^n = O(e^{n+\log n})$ to be true it should satisfy:
$0 \leq 2^n \leq C(e^{n+\log n}) \, \forall n \geq n_0$
As C$e > 2$ (1) let us look at the powers
As $\log n \geq 0$
$\implies$ n+$\log n \geq n$
$\implies$ n $\leq n + \log n$
$\implies 2^n \leq 2^(n + \log n)$
$\implies$ $(using1)$0$\leq$ $2^n \leq Ce^{(n+\log n)}$ for all positive C
Hence proved true

5. $\log\log n = O(\log(\frac{\sqrt{n}}{3\log n}) + \log(n - \log n))$
ANSWER: True
For $\log\log n = O(\log(\frac{\sqrt{n}}{3\log n}) + \log(n - \log n))$ it should satisfy that:
$loglogn \leq c.(\log(\frac{\sqrt{n}}{3\log n}) + \log(n - \log n)$

Expanding the Right Hand Side:
$\log\ \log n \leq c.(0.5 \log n - \log 3 - \log\ logn + \log(n - \log n))$
As $\log(n - \log n)$ is a polynomial term, it dominates $\log \log n$ which will be much smaller, and thus can be ignored
$\log\ \log n \leq c(0.5 \log n - \log 3)$
$\log n \leq c. \log \frac{\sqrt{n}}{3}$
$\log n \leq \frac{\sqrt{n}}{3}$
As the RHS is a polynomial function and the LHS is a polylogarithmic , the RHS always exceeds the LHS
Hence Proved

6. Let f, g, h be positive functions. Then $h(n)/(f(n)+g(n)) = O(h(n)/(f(n)g(n)))$
ANSWER: False
For $h(n)/(f(n) + g(n)) = O(h(n)/(f(n)g(n)))$ to hold true it should satisfy that:
$0 \leq h(n)/(f(n) + g(n)) \leq C(h(n)/(f(n)g(n)))$
As f(n) and g(n) are always increasing $f(n) + g(n) \leq f(n).g(n)$ holds true
$\implies$ f(n)+g(n)$\leq f(n).g(n)$
$\implies \frac{1}{f(n)+g(n)} \geq \frac{1}{f(n).g(n)}$
$\implies for\ constant\ C : \frac{1}{(f(n)+g(n))} \geq \frac{C}{f(n).g(n)}$
$\implies for\ increasing\ function\ h(n) : \frac{h(n)}{(f(n)+g(n))} \geq \frac{C.h(n)}{(f(n).g(n)}$
Hence the condition is not satisfied.
Hence False

7. Let f, g be positive functions. Then $|f(n) - g(n)| = \Theta(\min(f(n), g(n)))$
ANSWER: False
In order for the given condition to be true it should satisfy:
$0 \leq C_1 \min(f(n), g(n)) \leq |f(n) - g(n)| \leq C_2 \min(f(n), g(n))\ \forall n \geq n_0$
For f(n) and g(n) as increasing functions $|f(n) - g(n)| \leq C_2 \min(f(n), g(n))$ is violated
For f(n) and g(n) as decreasing functions $C_1 \min(f(n), g(n)) \leq |f(n) - g(n)|$ is violated
For Example:
Let $f(n) = n^4$ and let $g(n) = n^2$ (both are positive increasing functions)
$\implies Let \mathrm{C}_1 = 2, C_2 = 2$ and $n = 2$
$2 \min(16, 4) \leq |16 - 4| \nleq 2 \min(16, 4)$
$\implies$ 8$\leq$ 12 $\nleq$ 8
Even if we take sufficiently large values of the constants the value of n maybe picked in such a way that the condition is not satisfied.
Hence Proved False

8. Let f, g be positive functions. Then $f(n) + g(n) = \Theta(\max(f(n), g(n)))$
ANSWER:True
In order for the given condition to be true it should satisfy:
$0 \leq C_1 \max(f(n), g(n)) \leq f(n) + g(n) \leq C_2 \max(f(n), g(n)) \ \forall n \geq n_0$
Since both f(n) and g(n) are positive functions for any suitable value of C_1 :
$C_1 \max(f(n), g(n)) \leq f(n) + g(n)$ will be true
Also we can find a sufficiently large value for C_2 such that $f(n) + g(n) \leq C_2 \max(f(n), g(n))$ will be true.

9. $n^{n \log n} = \Omega(e^{n^2 - n \log n})$
ANSWER: False
For $n^{n \log n} = \Omega(e^{n^2 - n \log n})$ to be true it should satisfy:
$0 \leq Ce^{n^2 - n \log n n \log n}$
Taking log on both sides
$\log c + log\ e^{2 - nlogn} \leq \log n^{n \log n}$
$(n^2 - n \log n) \log e \leq n \log n \log n$

$$\lim_{n \to \infty} \frac{n \log n \log n}{(n^2 - n \log n) \log e} = 0$$

In this case the numerator is a poly-logarithmically bounded function while the denominator is a polynomial. Thus the limit can be evaluated to 0 when n tends to infinity.
NOTE: This is derived from the standard identity

$$\lim n \to \infty \frac{(\log n)^b}{n^a} = 0$$

10. Let f, g, h be a positive functions. If $g(n) = o(f(n))$ and $f(n) = O(h(n))$, then $g(n) = o(h(n))$.
ANSWER: True
As $g(n) = o(f(n))$
$\implies$ g(n)<C$_1 f(n)$ (1)
As $f(n) = O(h(n))$
$\implies$ f(n)$\leq C_2 h(n)$ (2)
From (1) and (2)
$g(n) < C_1 f(n) \leq C_1 C_2 h(n)$
$\implies$ g(n)<C$_1 C_2 h(n)$
$\implies$ g(n)=o(h(n))
$Hence Proved$

1.2 (10 points)

1. Prove that
$$\sum_{i=1}^{n} i^2 \ln\left(e + \frac{1}{i}\right) = \Theta(n^3).$$

ANSWER:
Taking out the e term on both sides:
$c_1 n^3 \leq \sum_{i=0}^{n} i^2 \ln e(1 + \frac{1}{ei}) \leq c_2 n^3$
Since $ln(e) = 1$
$c_1 n^3 \leq \sum_{i=0}^{n} i^2 + i^2 \ln(1 + \frac{1}{ei}) \leq c_2 n^3$
Expanding $\ln(1 + \frac{1}{ei})$ we get
$\ln(1 + \frac{1}{ei}) = \frac{1}{ie} - \frac{1}{2i^2e^2} + \frac{1}{3i^3e^3} \ldots$
However, $0 < \frac{1}{ie} < 1$ we can ignore the succeeding terms as they are negligible for large values of i because they are additive and are lesser than $0 < \frac{1}{ie} < 1$ as both i and e are >1
$c_1 n^3 \leq \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} i^2 \frac{1}{ie} \leq c_2 n^3$
Evaluating the summation
$c_1(n^3) \leq \frac{n^3-1}{3} + \frac{n^2-1}{2e} \leq c_2(n^3)$
$c_1(n^3) \leq \frac{2en^3+3n^2}{6e} \leq c_2(n^3)$
$c_1 \leq \frac{2en^3+3n^2}{6en^3} \leq c_2$
Thus, we can always find such values of $c_1 and c_2$ that satisfy the above equation
Hence Proved

2 Problem 2(25 Points)

2.1 (10 points)

Prove by induction that $\binom{2n}{n} > \frac{2^n}{\sqrt{\pi n}}$ for all positive n.

ANSWER:
Proving by induction that $\binom{2n}{n} > \frac{2^n}{\sqrt{\pi n}}$ for all positive n
$\binom{2n}{n}$ is a Binomial Distribution.
Binomial Distribution $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
In this case n = 2n and k = n, Hence, $\binom{2n}{n} = \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!}$
Base Case: Let n = 1 $\binom{2}{1} = \frac{2!}{1!1!} = \frac{2}{1} = 2 > \frac{2}{\sqrt{\pi}}$
Induction Hypothesis: Assume n=i and
$\binom{2i}{i} = \frac{2i!}{i!i!} > \frac{2^i}{\sqrt{(\pi i)}}$ is true
Induction Step: Assume n=i+1:$\binom{2(i+1)}{(i+1)} = \frac{2(i+1)!}{(i+1)!(i+1)!}$
$\implies \frac{2(i+1)!}{(i+1)!(i+1)!} = \frac{2(i+1).2i!}{(i+1)(i+1)i!i!}$
$\implies \frac{2(i+1)!}{(i+1)!(i+1)!} = \frac{2(i+1).2i!}{(i+1)(i+1)i!i!} = \frac{2(i+1)}{(i+1)(i+1)} . \frac{2i!}{i!i!}$
From the Induction Hypothesis: $\frac{2(i+1)!}{(i+1)!(i+1)!} > \frac{2(i+1)}{(i+1)(i+1)} . \frac{2^i}{\sqrt{(\pi i)}}$
$\implies \frac{2(i+1)!}{(i+1)!(i+1)!} > \frac{(i+1)}{(i+1)(i+1)} . \frac{2^{(i+1)}}{\sqrt{(\pi i)}}$
As $(i+1)^2 > \sqrt{(i+1)}$: $\frac{(i+1)}{(i+1)(i+1)} . \frac{2^{(i+1)}}{\sqrt{(\pi i)}} > \frac{2^{(i+1)}}{\sqrt{(\pi(i+1))}}$
$\implies \frac{2(i+1)!}{(i+1)!(i+1)!} > \frac{2^{(i+1)}}{\sqrt{(\pi(i+1))}}$
Hence our hypothesis is true
$\implies HenceProved$

2.2 (15 points)

1. Let A, B, C be sets. Prove that

   (a) $A \setminus (A \setminus B) = A \cap B$
   ANSWER:
   Proving from LHS:
   Assume $x \in \setminus(A \setminus B)$
   $\implies$ x $\in A \wedge x \notin A \setminus B$
   $\implies$ x $\in A \wedge (x \notin A \vee x \in B)$
   Applying Distributive Law:
   $\implies$ (x $\in A \wedge x \notin A) \vee (x \in A \wedge x \in B)$
   $\implies$ (x $\in \emptyset) \vee (x \in (A \cap B))$
   $\implies$ x $\in (A \cap B)$
   Hence Proved

   Conversely proving from RHS:
   Let's assume $x \in (A \cap B)$
   $\implies$ x $\in A \wedge x \in B$
   This means that x cannot belong to A without belonging to B
   $\implies$ x $\in A \wedge x \notin (A \setminus B)$
   $\implies$ x $\in A \setminus (A \setminus B)$
   Hence Proved

   (b) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
   ANSWER:
   Conversely proving from LHS:
   let's assume $x \in (A \cup B)$
   $\implies$ x $\in A \cup x \in B$
   x can fall into the following cases:
   $x \in A \wedge x \notin B$
   $x \in B \wedge x \notin A$
   $x \in A \wedge x \in B$
   $\implies$ (x $\in A \wedge x \notin B) \vee$(x $\in B \wedge x \notin A) \vee$(x $\in A \wedge x \in B)$
   $\implies$ (x $\in (A \setminus B)) \vee$(x $\in B \cap A) \vee$(x $\in (B \setminus A))$
   $\implies$ $x \in (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
   $Hence Proved$
   $Conversely proving from RHS:$
   $Let's assume$x $\in (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
   $\implies$ x $\in (A \setminus B) \vee x \in (A \cap B) \vee x \in (B \setminus A)$

$\implies$ (x $\in A \wedge x \notin B) \vee (x \in A \wedge x \in B) \vee (x \in B \wedge x \notin A)$
$\implies$ x $\in (A \cup B)$
Hence Proved

(c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
ANSWER:
Proving from LHS:
Let's assume $x \in (A \setminus (B \setminus C))$
$\implies$ x $\in A \wedge x \notin (B \setminus C)$
$\implies$ x $\in A \wedge (x \notin B \vee x \in C)$
Applying Distributive Law
$\implies$ (x $\in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$
$\implies$ (x $\in (A \setminus B)) \vee (x \in (A \cap C))$
$\implies$ x $\in (A \setminus B) \cup (A \cap C)$
Hence Proved
Conversely proving from RHS:
Let's assume $x \in ((A \setminus B) \cup (A \cap C))$
$\implies$ x $\in (A \setminus B) \vee x \in (A \cap C)$
$\implies$ (x $\in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$
Applying inverse Distributive Law
$\implies$ x $\in A \wedge (x \notin B \vee x \in C)$
$\implies$ x $\in A \wedge x \notin (B \setminus C)$
$\implies$ x $\in A \setminus (B \setminus C)$
Hence Proved

2. There are n identical apples in the basket. What is the number of different ways to divide the apples between Alice, Bob and John? What is the answer if we request that Alice and John get at most $k < n/2$ apples each?
   ANSWER:
   Since the apples are identical the problem of dividing the apples between Alice, Bob and John is equivalent to the problem of counting the number of distinct 3-tuples of non-negative integers whose sum is n for which we can use the Stars and Bars theorem of combinatorics.
   The Stars and Bars Theorem says: For any pair of positive integers n and k, the number of k-tuples of non-negative integers whose sum is n is equal to the number of multisets of cardinality k 1 taken from a set of size n + 1. Both numbers are given by the multiset number $\left(\!\binom{n+1}{k-1}\!\right)$, or equivalently by the binomial coefficient $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ or multiset number $\left(\!\binom{k}{n}\!\right)$.
   NOTE: Stars and Bars Definition taken from Wikipedia
   In this case, n = n and k = 3, then there are $\left(\!\binom{n+1}{3-1}\!\right) = \binom{n+3-1}{n}$.
   The number of ways to divide n apples are $\left(\!\binom{n+2}{2}\!\right) = \binom{n+2}{n}$.
   For the second part if Alice and John get at most $k < n/2$ apples each.
   Combinations where Alice and John get at most $k < n/2$ apples=All Combinations - Alice and John get at least $k < n/2$ apples
   Combinations of Alice getting at least k apples
   $\implies$ Alice gets$\leq k + 1$ apples
   $\implies$ Total Left $= n - k + 1$
   $\implies$ Using Stars and bars now we get: $\binom{n-k+1}{2}$
   Similarly combinations of John getting at least k apples is the same as above
   Thus we subtract both the cases from $\binom{n+2}{n}$

   We also take into consideration cases where both of them getting more than k apples
   This leads to the solution:
   $(\binom{n+2}{2} - 2.\binom{n-k+1}{2} + \binom{n-2k}{2})$

3. Suppose that a random machine outputs each number from 1 to $x - 1$ with equal probability. What is probability that the output is coprime with x, where $x = 3^n 5^m 7^k$ and n, m, k are non-negative integers.
   ANSWER:
   As $x = 3^n 5^m 7^k$ the smallest value of $x = 105$ with $n = 1, m = 1 and k = 1$
   We can use Euler's Totient Function(particularly Euler's product formula) to find the probability of a random number being chosen from (x-1) possible

values is coprime with x.
Euler's Product Formula for n and $p_1, p_2, ...pr$ are prime factors of n:
$\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_r})$
NOTE: Definition of Euler's Totient Function from Wikipedia
This probability is:
$x(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})$
$\Longrightarrow 3^n 5^m 7^k (1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})$
$\Longrightarrow 3^n 5^m 7^k (\frac{2}{3})(\frac{4}{5})(\frac{6}{7})$
The probability = $3^{n-1} 5^{m-1} 7^{k-1} (2.4.6)$ = $48.3^{n-1} 5^{m-1} 7^{k-1}$

4. Assume there are n students in a class. How many different ways are there to form n/k teams with k students in each?
   ANSWER:
   Since there are n students and n/k teams with k students in each, this means that each student will be assigned to any one of the n/k teams
   So we can choose any one of the n/k value as team for a student.
   Choosing k students from n students to form the 1st team, k students from the remaining n-k for the 2nd and so on. The last one team will have the remaining k students
   Hence, the solution is given by $\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}.....\binom{k}{k}$